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Calculus of Variations

The local behavior of positive solutions for higher order equation with isolated singularities

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Abstract

We use blow up analysis for local integral equations to provide a blow up rates of solutions of higher order Hardy–Hénon equation in a bounded domain with an isolated singularity, and show the asymptotic radial symmetry of the solutions near the singularity. This work generalizes the correspondence results of Jin–Xiong (in, Asymptotic symmetry and local behavior of solutions of higher order conformally invariant equations with isolated singularities. arXiv:1901.01678) on higher order conformally invariant equations with an isolated singularity.

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1 Introduction

This article aims to study the local behaviors of positive solutions for the higher order Hardy–Hénon equation

$$(-\Delta)^\sigma u = |x|^\tau u^p \qquad \text{in} \quad B_1 \backslash \{0\}, \tag{1}$$

where $1 \leq \sigma < \frac{n}{2}$ is an integer, $\tau > -2\sigma$, $p > 1$ and the punctured unit ball $B_1 \backslash \{0\} \subset \mathbb{R}^n$, $n \geq 2$.

In the special case of $\sigma = 1$, the local behavior of the positive solutions for (1) with isolated singularity has been very well understood. For $\tau > -2$, $1 < p \leq \frac{n+2}{n-2}$, the blow up rate of the solution

$$u(x) \leq C|x|^{-\frac{2+\tau}{p-1}}, \quad |\nabla u(x)| \leq C|x|^{-\frac{p+1+\tau}{p-1}} \quad \text{near} \ \ x = 0,$$

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is obtained by a number of authors, where ∇u denotes the gradient of u and C is the positive constant. See [1,2,5,9,12–14,17] for more precise estimates and details. In the classical paper [3], Caffarelli-Gidas-Spruck proved that every local solution of (1) is asymptotically radially symmetric

$$u(x) = \bar{u}(|x|)(1 + O(|x|)) \qquad \text{as } x \to 0,$$

where $\tau = 0$, $\frac{n}{n-2} \le p \le \frac{n+2}{n-2}$ and $\bar{u}(|x|) := \fint_{\mathbb{S}^n} u(|x|\theta)\mathrm{d}\theta$ is the spherical average of u. Li [10] improved their results for $\tau \le 0$, $1 < p \le \frac{n+2+\tau}{n-2}$, and simplified the proofs. Recently, Han et al. [6] studied the asymptotic behavior of solutions to the Yamabe equation with an asymptotically flat metric. For the fractional case $0 < \sigma < 1$, Caffarelli–Jin–Sire–Xiong [4] studied the blow up rate, asymptotically radially symmetric and removability of the positive solution for the fractional Yamabe equation with an isolated singularity

$$(-\Delta)^{\sigma} u = u^{\frac{n+2\sigma}{n-2\sigma}} \qquad \text{in} \quad B_1 \backslash \{0\}.$$

Motivated by this work, in our previous work [11], we have studied the fractional Hardy–Hénon equations and not only derived that there exists a positive constant C such that the blow up rates

$$u(x) \le C|x|^{-\frac{2\sigma+\tau}{p-1}}, \quad |\nabla u(x)| \le C|x|^{-\frac{2\sigma+\tau+p-1}{p-1}} \quad \text{near } x = 0,$$

for $\tau > -2\sigma$, $1 < p < \frac{n+2\sigma}{n-2\sigma}$, but also obtained the asymptotically radially symmetric

$$u(x) = \bar{u}(|x|)(1 + O(|x|)) \quad \text{as } x \to 0,$$

for $-2\sigma < \tau \le 0$, $\frac{n+\tau}{n-2\sigma} < p \le \frac{n+2\sigma+2\tau}{n-2\sigma}$, which is consistent with the classic case $\sigma = 1$.

Recently, by using blow up analysis Jin–Xiong [8] proved sharp blow up rates of the positive solutions of higher order conformally invariant equations with an isolated singularity

$$(-\Delta)^{\sigma} u = u^{\frac{n+2\sigma}{n-2\sigma}} \qquad \text{in} \quad B_1 \backslash \{0\},$$

where $1 \le \sigma < \frac{n}{2}$ is an integer, and showed the asymptotic radial symmetry of the solutions near the singularity. That is, they proved that there exists a positive constant C such that

$$u(x) \le C|x|^{-\frac{n-2\sigma}{2}} \quad \text{near } x = 0,$$

and

$$u(x) = \bar{u}(|x|)(1 + O(|x|)) \quad \text{as } x \to 0.$$

This is an extension of the celebrated theorem of Caffarelli–Gidas–Spruck [3] for the second order Yamabe equation and Caffarelli–Jin–Sire–Xiong [4] for the fractional Yamabe equation with isolated singularity to higher order equations.

Inspired by the above work, we are interested in the higher order Hardy–Hénon equation, that is, $1 \le \sigma < \frac{n}{2}$ is an integer, in a bounded domain with an isolated singularity in this paper. Our result provides a blow up rate estimate and show that the solution of (1) is asymptotically radially symmetric near an isolated singularity, which is consistent with $0 < \sigma \le 1$.

Theorem 1.1 *Suppose that* $1 \le \sigma < \frac{n}{2}$ *is an integer, and* $u \in C^{2\sigma}(B_1 \backslash \{0\})$ *is a positive solution of* (1).

(i) *If* $-2\sigma < \tau$, $\frac{n+\tau}{n-2\sigma} < p < \frac{n+2\sigma}{n-2\sigma}$ *and*

$$(-\Delta)^m u \geq 0 \quad \text{in} \quad B_1\backslash\{0\}, \quad m = 1, 2, \cdots, \sigma - 1, \tag{2}$$

then there exists a positive constant $C = C(n, \sigma, \tau, p,)$ *such that*

$$u(x) \leq C|x|^{-\frac{2\sigma+\tau}{p-1}}, \quad |\nabla u(x)| \leq C|x|^{-\frac{2\sigma+\tau+p-1}{p-1}} \quad \text{near} \ \ x = 0.$$

(ii) *If* $-2\sigma < \tau \leq 0$, $\frac{n+\tau}{n-2\sigma} < p \leq \frac{n+2\sigma+2\tau}{n-2\sigma}$ *and the solution satisfies* (2), *then*

$$u(x) = \bar{u}(|x|)(1 + O(|x|)) \qquad \text{as } x \to 0,$$

where $\bar{u}(|x|) := \fint_{\mathbb{S}^n} u(|x|\theta)\mathrm{d}\theta$ *is the spherical average of* u.

The main idea of our approach is to carry out blow up analysis to get the blow up rate estimate near the isolated singularity, and by the method of moving spheres to study the asymptotically radially symmetric as in Caffarelli–Jin–Sire–Xiong [4] for the fractional Yamabe equation $0 < \sigma < 1$. The method of moving spheres has become a very powerful tool in the study of nonlinear elliptic equations, i.e. the method of moving planes together with the conformal invariance, which fully exploits the conformal invariance of the problem. It is known that one of the conformal invariance, i.e. the Kelvin transform of u defined as

$$u_{x,\lambda}(y) := \left(\frac{\lambda}{|y-x|}\right)^{n-2\sigma} u\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right) \quad \text{in} \ \ \mathbb{R}^n,$$

with $\lambda > 0$, $x \in \mathbb{R}^n$, plays an important part in our proof. On the other hand, the sign conditions (2) will ensure the maximum principle and are essential for applying the moving spheres method. However, in our local situation (1), the sign conditions (2) may change when performing the Kelvin transforms. Inspired by a unified approach to solve the Nirenberg problem and its generalizations by the authors Jin–Li–Xiong in [7], we shall make use of integral representations. In details, we first prove $|x|^\tau u^p \in L^1(B_1)$ under the assumptions of Theorem 1.1, and then we can rewrite the differential equations (1) into the integral equation involving the Riesz potential

$$u(x) = \int_{B_1} \frac{|y|^\tau u^p(y)}{|x-y|^{n-2\sigma}} dy + h(x) \qquad \text{in} \quad B_1\backslash\{0\},$$

where $h \in C^1(B_1)$ is a positive function. As a result, we just need to study the integral equation.

This paper is organized as follows. In Sect. 2, we shall show that (1) can be written as the form of (3), and then give some results about the integral equation, which implies that Theorem 1.1 follows from these results. In Sect. 3, we prove the upper bound near the isolated singularity for the solution of (3), and the asymptotic radial symmetry will be obtained in Sect. 4.

2 Proof of the main results

For $0 < \sigma < \frac{n}{2}$, $-2\sigma < \tau$, $p > 1$, $u \in C(\overline{B_1}\backslash\{0\})$, and $|x|^\tau u^p(x) \in L^1(B_1)$, before that we consider the integral equation involving the Riesz potential

$$u(x) = \int_{B_1} \frac{|y|^\tau u^p(y)}{|x-y|^{n-2\sigma}} dy + h(x) \qquad \text{in} \quad B_1\backslash\{0\}, \tag{3}$$

where $h \in C^1(\overline{B_1})$ is a positive function, otherwise we consider the equation in a smaller ball. About the integral Eq. (3), we shall first show some results, which will recover our previous work [11] for the fractional Yamabe equation $0 < \sigma < 1$, and the proof will be given later in Sects. 3 and 4. Now we first introduce the upper bound of the positive solution near the singularity.

Theorem 2.1 *For* $-2\sigma < \tau$, $1 < p < \frac{n+2\sigma}{n-2\sigma}$, *suppose that* u *is a positive solution of* (3), *then there exists a positive constant* $C = C(n, \sigma, \tau, p)$ *such that*

$$u(x) \leq C|x|^{-\frac{2\sigma+\tau}{p-1}}, \quad |\nabla u(x)| \leq C|x|^{-\frac{2\sigma+\tau+p-1}{p-1}} \quad \text{near } x = 0. \tag{4}$$

One consequence of the upper bound of the solution near the singularity in Theorem 2.1 is the following Harnack inequality.

Corollary 2.2 *Assume as in Theorem* 2.1, *then for all* $0 < r < \frac{1}{4}$, *then there exists a positive constant* C *independent of* r *such that*

$$\sup_{B_{3r/2}\setminus B_{r/2}} u \leq C \inf_{B_{3r/2}\setminus B_{r/2}} u.$$

The following theorem shows the asymptotic radial symmetry of the positive solution near the singularity.

Theorem 2.3 *For* $-2\sigma < \tau \leq 0$, $\frac{n+\tau}{n-2\sigma} < p \leq \frac{n+2\sigma+2\tau}{n-2\sigma}$, *suppose that* u *is a positive solution of* (3), *then*

$$u(x) = \bar{u}(|x|)(1 + O(|x|)) \qquad \text{as } x \to 0,$$

where $\bar{u}(|x|) := \fint_{\mathbb{S}^n} u(|x|\theta)\mathrm{d}\theta$ *is the spherical average of* u.

Next we shall show that we can rewrite the differential Eq. (1) into the integral Eq. (3) involving the Riesz potential, which implies that Theorem 1.1 follows by Theorems 2.1 and 2.3.

2.1 Proof of Theorem 1.1

To prove Theorem 1.1, we first need the following proposition.

Proposition 2.4 *Suppose that* $1 \leq \sigma < \frac{n}{2}$ *is an integer,* $\tau > -2\sigma$, $p > \frac{n+\tau}{n-2\sigma}$, *and* $u \in C^{2\sigma}(\overline{B_1}\setminus\{0\})$ *is a positive solution of* (1), *then* $|x|^\tau u^p \in L^1(B_1)$.

Proof Let η be a smooth function defined in $\mathbb{R}$ satisfying $\eta(t) = 0$ for $|t| \leq 1$, $\eta(t) = 1$ for $|t| \geq 2$, and $0 \leq \eta(t) \leq 1$ for $1 \leq t \leq 2$. For small $\varepsilon > 0$, let $\varphi_\varepsilon(x) = \eta(\varepsilon^{-1}|x|)^q$ with

$q = \frac{2\sigma p}{p-1}$. Multiplying both sides by $\varphi_\varepsilon(x)$ and using integration by parts, we have

$$\begin{aligned}\int_{B_1} |x|^\tau u^p \varphi_\varepsilon &= \int_{B_1} u(-\Delta)^\sigma \varphi_\varepsilon + \int_{\partial B_1} \frac{\partial(-\Delta)^{\sigma-1}u}{\partial \nu} ds \\ &\le C\varepsilon^{-2\sigma} \int_{\varepsilon \le |x| \le 2\varepsilon} u\eta(\varepsilon^{-1}|x|)^{q-2\sigma} + C \\ &= C\varepsilon^{-2\sigma} \int_{\varepsilon \le |x| \le 2\varepsilon} u\varphi_\varepsilon^{\frac{1}{p}} + C \\ &= C\varepsilon^{-2\sigma} \int_{\varepsilon \le |x| \le 2\varepsilon} |x|^{\frac{\tau}{p}} u\varphi_\varepsilon^{\frac{1}{p}} |x|^{-\frac{\tau}{p}} + C \\ &\le C\varepsilon^{-2\sigma-\frac{\tau}{p}} \int_{\varepsilon \le |x| \le 2\varepsilon} |x|^{\frac{\tau}{p}} u\varphi_\varepsilon^{\frac{1}{p}} + C \\ &\le C\varepsilon^{-2\sigma-\frac{\tau}{p}+n-\frac{n}{p}} \left(\int_{B_1} |x|^\tau u^p \varphi_\varepsilon\right)^{\frac{1}{p}} + C.\end{aligned}$$

Since $p > \frac{n+\tau}{n-2\sigma}$, we have

$$\int_{2\varepsilon \le |x| \le 1} |x|^\tau u^p < \int_{B_1} |x|^\tau u^p \varphi_\varepsilon \le C.$$

By sending $\varepsilon \to 0$, we obtain

$$\int_{B_1} |x|^\tau u^p \le C.$$

Thus, we complete the proof. □

Next, we return to prove that if $u \in C^{2\sigma}(\overline{B_1}\backslash\{0\})$ is a positive solution of (1), then

$$u(x) = B(n,\sigma) \int_{B_r} \frac{|y|^\tau u^p(y)}{|x-y|^{n-2\sigma}} dy + h_1(x), \tag{5}$$

with

$$B(n,\sigma) := \frac{\Gamma\left(\frac{n-2\sigma}{2}\right)}{2^{2\sigma}\pi^{n/2}\Gamma(\sigma)},$$

where Γ is the Gamma function, and h_1 is smooth in B_r and satisfies $(-\Delta)^\sigma h_1 = 0$ in B_r. As a result, we can finish the proof of Theorem 1.1 by Theorems 2.1 and 2.3. For the purpose, we recall the green function of $-\Delta$ on the unit ball is

$$G_1(x,y) = \frac{1}{(n-2)w_{n-1}} \left(|x-y|^{2-n} - \left|\frac{x}{|x|} - |x|y\right|^{2-n}\right) \quad \text{for } x, y \in B_1,$$

and

$$H_1(x,y) := -\frac{\partial}{\partial \nu_y} G_1(x,y) = \frac{1-|x|^2}{w_{n-1}|x-y|^n} \quad \text{for } x \in B_1,\ y \in \partial B_1,$$

where w_{n-1} is the surface area of the unit sphere in $\mathbb{R}^n$. Define

$$G_\sigma(x,y) := \int_{B_1 \times \cdots \times B_1} G_1(x,y_1)G_1(y_1,y_2)\cdots G_1(y_{\sigma-1},y)dy_1\cdots dy_{\sigma-1},$$

then we have

$$G_\sigma(x,y) = B(n,\sigma)|x-y|^{2\sigma-n} + A_\sigma(x,y),$$

where $A_\sigma(\cdot,\cdot)$ is smooth in $B_1 \times B_1$. For $2 \le i \le \sigma$, define

$$H_i(x,y) := \int_{B_1\times\cdots\times B_1} G_1(x,y_1)G_1(y_1,y_2)\cdots G_1(y_{i-2},y_{i-1})H_1(y_{i-1},y)dy_1\cdots dy_{i-1}.$$

Proof of Theorem 1.1 We can suppose that $u \in C^{2\sigma}(\overline{B_1}\backslash\{0\})$ and $u > 0$ in $\overline{B_1}$, otherwise we just consider the equation in a smaller ball. By the above argument, we know that we only need to obtain (5), then we can finish the proof. To prove (5), let

$$v(x) := \int_{B_1} G_\sigma(x,y)|y|^\tau u^p(y)dy + \sum_{i=1}^m \int_{\partial B_1} H_i(x,y)(-\Delta)^{\sigma-i}u(y)dS_y,$$

and

$$w := u - v.$$

Then

$$(-\Delta)^\sigma w = 0 \qquad \text{in } \ B_1\backslash\{0\}.$$

Combining with $|y|^\tau u^p(y) \in L^1(B_1)$ from Proposition 2.4 and the fact that the Riesz potential $|y|^{2\sigma-n}$ is weak type $\left(1, \frac{n}{n-2\sigma}\right)$, $v \in L_{weak}^{\frac{n}{n-2\sigma}}(B_1) \cap L^1(B_1)$. Moreover, for every $\varepsilon > 0$ we can choose $\rho > 0$ such that $\int_{B_{2\rho}} |y|^\tau u^p(y)dy < \varepsilon$. Then for all sufficiently large λ, we have

$$\left|x \in B_\rho : |v(x)| > \lambda\right| \le \left|x \in B_\rho : \int_{B_{2\rho}} G_\sigma(x,y)|y|^\tau u^p(y)dy > \frac{\lambda}{2}\right| \le C(n,\sigma)\varepsilon\lambda^{-\frac{n}{n-2\sigma}}.$$

Hence, $w \in L_{weak}^{\frac{n}{n-2\sigma}}(B_1) \cap L^1(B_1)$ and for every $\varepsilon > 0$ there exist $\rho > 0$ such that for all sufficiently large λ,

$$\left|x \in B_\rho : |w(x)| > \lambda\right| \le \left|x \in B_\rho : |u(x)| > \frac{\lambda}{2}\right| + \left|x \in B_\rho : |v(x)| > \frac{\lambda}{2}\right|.$$

It follows that

$$\left|x \in B_\rho : |w(x)| > \lambda\right| \le C(n,\sigma)\varepsilon\lambda^{-\frac{n}{n-2\sigma}}.$$

By the generalized Bocher's Theorem for polyharmonic function, $(-\Delta)^\sigma w(x) = 0$ in B_1. Since $w = \Delta w = \cdots = \Delta^{\sigma-1}w = 0$ on ∂B_1, $w = 0$ and thus $u = v$. Since $-\Delta u \ge 0$ in $B_1\backslash\{0\}$, and $u > 0$ in $\overline{B_1}$, we know from the Maximum Principle that $c_1 := \inf_{B_1} u = \min_{\partial B_1} u > 0$. By $|y|^\tau u^p(y) \in L^1(B_1)$, we can find that $r < \frac{1}{4}$ such that for $x \in B_r$,

$$\int_{B_r} |A_\sigma(x,y)||y|^\tau u^p(y)dy \le \frac{c_1}{2},$$

then

$$u(x) = B(n,\sigma)\int_{B_r} \frac{|y|^\tau u^p(y)}{|x-y|^{n-2\sigma}}dy + h_1(x),$$

where

$$h_1(x) = \int_{B_r} A_\sigma(x, y)|y|^\tau u^p(y)dy + \int_{B_1\setminus B_r} G_\sigma(x, y)|y|^\tau u^p(y)dy$$
$$+ \sum_{i=1}^{\sigma} \int_{\partial B_1} H_i(x, y)(-\Delta)^{i-1}u(y)dS_y.$$

Hence, we have for $x \in B_r$,

$$h_1(x) \geq -\frac{c_1}{2} + \int_{\partial B_1} H_i(x, y)u(y)dS_y$$
$$\geq -\frac{c_1}{2} + \inf_{B_1} u = \frac{c_1}{2}.$$

On the other hand, h_1 is smooth in B_r and satisfies $(-\Delta)^\sigma h_1 = 0$ in B_r. We complete the proof. □

3 The upper bound near the isolated singularity

In this section, we shall give proofs of Theorem 2.1 and Corollary 2.2 respectively. The following we start our proof.

3.1 Proof of Theorem 2.1

First, we recall the Doubling Property [15, Lemma 5.1] and denote $B_R(x)$ as the ball in $\mathbb{R}^n$ with radius R and center x. For convenience, we write $B_R(0)$ as B_R for short.

Proposition 3.1 *Suppose that* $\emptyset \neq D \subset \Sigma \subset \mathbb{R}^n$, Σ *is closed and* $\Gamma = \Sigma \setminus D$. *Let* $M : D \to (0, \infty)$ *be bounded on compact subset of* D. *If for a fixed positive constant* k, *there exists* $y \in D$ *satisfying*

$$M(y)\mathrm{dist}(y, \Gamma) > 2k,$$

then there exists $x \in D$ *such that*

$$M(x) \geq M(y), \qquad M(x)\mathrm{dist}(x, \Gamma) > 2k,$$

and for all $z \in D \cap B_{kM^{-1}(x)}(x)$,

$$M(z) \leq 2M(x).$$

Next, in order to prove Theorem 2.1, we start with the following lemma.

Lemma 3.2 *Let* $1 < p < \frac{n+2\sigma}{n-2\sigma}$, $0 < \alpha \leq 1$ *and* $c(x) \in C^{2\sigma,\alpha}(\overline{B_1})$ *satisfy*

$$\|c\|_{C^{2,\alpha}(\overline{B_1})} \leq C_1, \quad c(x) \geq C_2 \quad \text{in} \quad \overline{B_1} \tag{6}$$

for some positive constants C_1, C_2. *Suppose that* $h \in C^1(B_1)$ *and* $u \in C^{2\sigma}(B_1)$ *is a nonnegative solution of*

$$u(x) = \int_{B_1} \frac{c(y)u^p(y)}{|x - y|^{n-2\sigma}}dy + h(x) \qquad \text{in} \quad B_1, \tag{7}$$

then there exists a positive constant C depending only on n, σ, p, C_1, C_2 such that

$$|u(x)|^{\frac{p-1}{2\sigma}} + |\nabla u(x)|^{\frac{p-1}{p+2\sigma-1}} \le C[\mathrm{dist}(x, \partial B_1)]^{-1} \quad \text{in} \quad B_1.$$

Proof Arguing by contradiction, for $k = 1, 2, \cdots$, we assume that there exist nonnegative functions u_k satisfying (7) and points $y_k \in B_1$ such that

$$|u_k(y_k)|^{\frac{p-1}{2\sigma}} + |\nabla u_k(y_k)|^{\frac{p-1}{p+2\sigma-1}} > 2k[\mathrm{dist}(y_k, \partial B_1)]^{-1}. \tag{8}$$

Define

$$M_k(x) := |u_k(x)|^{\frac{p-1}{2\sigma}} + |\nabla u_k(x)|^{\frac{p-1}{p+2\sigma-1}}.$$

Via Proposition 3.1, for $D = B_1$, $\Gamma = \partial B_1$, there exists $x_k \in B_1$ such that

$$M_k(x_k) \ge M_k(y_k), \quad M_k(x_k) > 2k[\mathrm{dist}(x_k, \partial B_1)]^{-1} \ge 2k, \tag{9}$$

and for any $z \in B_1$ and $|z - x_k| \le kM_k^{-1}(x_k)$,

$$M_k(z) \le 2M_k(x_k). \tag{10}$$

It follows from (9) that

$$\lambda_k := M_k^{-1}(x_k) \to 0 \qquad \text{as } k \to \infty, \tag{11}$$

$$\mathrm{dist}(x_k, \partial B_1) > 2k\lambda_k, \qquad \text{for } k = 1, 2, \cdots. \tag{12}$$

Consider

$$w_k(y) := \lambda_k^{\frac{2\sigma}{p-1}} u_k(x_k + \lambda_k y), \quad v_k(y) := \lambda_k^{\frac{2\sigma}{p-1}} h_k(x_k + \lambda_k y) \quad \text{in} \quad B_k.$$

Combining (12), we obtain that for any $y \in B_k$,

$$|x_k + \lambda_k y - x_k| \le \lambda_k |y| \le \lambda_k k < \frac{1}{2}\mathrm{dist}(x_k, \partial B_1),$$

that is,

$$x_k + \lambda_k y \in B_{\frac{1}{2}\mathrm{dist}(x_k, \partial B_1)}(x_k) \subset B_1.$$

Therefore, w_k is well defined in B_k and

$$|w_k(y)|^{\frac{p-1}{2\sigma}} = \lambda_k |u_k(x_k + \lambda_k y)|^{\frac{p-1}{2\sigma}},$$

$$|\nabla w_k(y)|^{\frac{p-1}{2\sigma+p-1}} = \lambda_k |\nabla u_k(x_k + \lambda_k y)|^{\frac{p-1}{2\sigma+p-1}}.$$

From (10), we find that for all $y \in B_k$,

$$|u_k(x_k + \lambda_k y)|^{\frac{p-1}{2\sigma}} + |\nabla u_k(x_k + \lambda_k y)|^{\frac{p-1}{2\sigma+p-1}} \le 2\left(|u_k(x_k)|^{\frac{p-1}{2\sigma}} + |\nabla u_k(x_k)|^{\frac{p-1}{p+2\sigma-1}}\right).$$

That is,

$$|w_k(y)|^{\frac{p-1}{2\sigma}} + |\nabla w_k(y)|^{\frac{p-1}{2\sigma+p-1}} \le 2\lambda_k M_k(x_k) = 2. \tag{13}$$

Moreover, w_k satisfies

$$w_k(x) = \int_{B_k} \frac{c_k(y) w_k^p(y)}{|x - y|^{n-2\sigma}} dy + v_k(x) \qquad \text{in} \quad B_k, \tag{14}$$

and

$$|w_k(0)|^{\frac{p-1}{2\sigma}} + |\nabla w_k(0)|^{\frac{p-1}{2\sigma+p-1}} = 1,$$

where $c_k(y) := c(x_k + \lambda_k y)$. By (11) it follows that

$$\|v_k\|_{C^1(B_k)} \to 0.$$

By condition (6), we obtain that $\{c_k\}$ is uniformly bounded in $\mathbb{R}^n$. For each $R > 0$, and for all $y, z \in B_R$, we have

$$|D^\beta c_k(y) - D^\beta c_k(z)| \le C_1 \lambda_k^{|\beta|} |\lambda_k(y-z)|^\alpha \le C_1 |y-z|^\alpha, \quad |\beta| = 0, 1, \cdots, 2\sigma$$

for k is large enough. Therefore, by Arzela-Ascoli's Theorem, there exists a function $c \in C^{2\sigma}(\mathbb{R}^n)$, after extracting a subsequence, $c_k \to c$ in $C^{2\sigma}_{\mathrm{loc}}(\mathbb{R}^n)$. Moreover, by (11), we obtain

$$|c_k(y) - c_k(z)| \to 0 \qquad \text{as } k \to \infty. \tag{15}$$

This implies that the function c actually is a constant C. By (6) again, $c_k \ge C_2 > 0$, we conclude that C is a positive constant.

On the other hand, applying the regularity results in Section 2.1 of [7], after passing to a subsequence, we have, for some nonnegative function $w \in C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$,

$$w_k \to w \qquad \text{in } C^\alpha_{\mathrm{loc}}(\mathbb{R}^n)$$

for some $\alpha > 0$. Moreover, w satisfies

$$w(x) = \int_{\mathbb{R}^n} \frac{C w^p(y)}{|x-y|^{n-2\sigma}} dy \qquad \text{in } \mathbb{R}^n \tag{16}$$

and

$$|w(0)|^{\frac{p-1}{2\sigma}} + |\nabla w(0)|^{\frac{p-1}{2\sigma+p-1}} = 1.$$

Since $p < \frac{n+2\sigma}{n-2\sigma}$, this contradicts the Liouville-type result [16, Theorem 1.4] that the only nonnegative entire solution of (16) is $w = 0$. Then we conclude the lemma. □

We now turn to prove Theorem 2.1.

Proof of Theorem 2.1 For $x_0 \in B_{1/2} \backslash \{0\}$, we denote $R := \frac{1}{2}|x_0|$. Then for any $y \in B_1$, we have $\frac{|x_0|}{2} < |x_0 + Ry| < \frac{3|x_0|}{2}$, and deduce that $x_0 + Ry \in B_1 \backslash \{0\}$. Define

$$w(y) := R^{\frac{2\sigma+\tau}{p-1}} u(x_0 + Ry), \quad v(y) := R^{\frac{2\sigma+\tau}{p-1}} h(x_0 + Ry).$$

Therefore, we obtain that

$$w(x) = \int_{B_1} \frac{c(y) w^p(y)}{|x-y|^{n-2\sigma}} dy + v(x) \qquad \text{in } B_1,$$

where $c(y) := |y + \frac{x_0}{R}|^\tau$. Notice that

$$1 < \left| y + \frac{x_0}{R} \right| < 3 \quad \text{in } \overline{B_1}.$$

Moreover,

$$\|c\|_{C^3(\overline{B_1})} \le C, \quad c(y) \ge 3^{-2\sigma} \quad \text{in } \overline{B_1}.$$

Applying Lemma 3.2, we obtain that

$$|w(0)|^{\frac{p-1}{2\sigma}} + |\nabla w(0)|^{\frac{p-1}{p+2\sigma-1}} \le C.$$

That is,

$$(R^{\frac{2\sigma+\tau}{p-1}} u(x_0))^{\frac{p-1}{2\sigma}} + (R^{\frac{2\sigma+\tau}{p-1}+1} |\nabla u(x_0)|)^{\frac{p-1}{p+2\sigma-1}} \le C.$$

Hence,

$$u(x_0) \le CR^{-\frac{2\sigma+\tau}{p-1}} \le C|x_0|^{-\frac{2\sigma+\tau}{p-1}},$$
$$|\nabla u(x_0)| \le CR^{-\frac{2\sigma+\tau+p-1}{p-1}} \le C|x_0|^{-\frac{2\sigma+\tau+p-1}{p-1}}.$$

Then Theorem 2.1 is proved by the fact that $x_0 \in B_{1/2} \setminus \{0\}$ is arbitrary. □

3.2 Proof of Corollary 2.2

Using the upper bound, we shall prove the Harnack inequality.

Proof of Corollary 2.2 Let

$$w(y) := r^{\frac{2\sigma+\tau}{p-1}} u(ry), \quad v(y) := r^{\frac{2\sigma+\tau}{p-1}} h(ry),$$

then

$$w(x) = \int_{B_{1/r}} \frac{|y|^\tau w^p(y)}{|x-y|^{n-2\sigma}} dy + v(x) \qquad \text{in } B_{1/r}\backslash\{0\}.$$

Theorem 2.1 gives that there exists a positive constant C such that

$$w(x) \le C \qquad \text{in } B_2\backslash B_{1/10}.$$

For $z \in \partial B_1$, let

$$g(x) = \int_{B_{1/r}\backslash B_{9/10}(z)} \frac{|y|^\tau w^p(y)}{|x-y|^{n-2\sigma}} dy.$$

For $x_1, x_2 \in B_{1/2}(z)$,

$$\begin{aligned} g(x_1) &= \int_{B_{1/r}\backslash B_{9/10}(z)} \frac{|y|^\tau w^p(y)}{|x_1-y|^{n-2\sigma}} dy \\ &= \int_{B_{1/r}\backslash B_{9/10}(z)} \frac{|x_2-y|^{n-2\sigma}}{|x_1-y|^{n-2\sigma}} \frac{|y|^\tau w^p(y)}{|x_2-y|^{n-2\sigma}} dy \\ &\le \left(\frac{7}{2}\right)^{n-2\sigma} \int_{B_{1/r}\backslash B_{9/10}(z)} \frac{|y|^\tau w^p(y)}{|x_2-y|^{n-2\sigma}} dy \\ &\le \left(\frac{7}{2}\right)^{n-2\sigma} g(x_2). \end{aligned}$$

Hence, g satisfies the Harnack inequality in $B_{1/2}(z)$. Since $h \in C^1(\overline{B_1})$ is a positive function, there exist a constant $C_0 \ge 1$ such that $\max_{\overline{B_{1/2}(z)}} v \le C_0 \min_{\overline{B_{1/2}(z)}} v$. On the other hand, we can write w as

$$w(x) = \int_{B_{9/10}(z)} \frac{|y|^\tau w^p(y)}{|x-y|^{n-2\sigma}} dy + g(x) + v(x) \qquad \text{in } B_{1/2}(z),$$

then from Proposition 2.2 in [7] we conclude that

$$\sup_{B_{1/2}(z)} w \leq C \inf_{B_{1/2}(z)} w.$$

A covering argument leads to

$$\sup_{B_{3/2}\setminus B_{1/2}} w \leq C \inf_{B_{3/2}\setminus B_{1/2}} w.$$

We complete the proof of Harnack inequality by rescaling back to u. □

4 Asymptotical radial symmetry

Last, we give a proof of the Theorem 2.3 for completely.

4.1 Proof of Theorem 2.3

Proof of Theorem 2.3 Assume that there exists some positive constant $\varepsilon \in (0,1)$ such that for all $0 < \lambda < |x| \leq \varepsilon$, $y \in B_{3/2}\setminus(B_\lambda(x) \cup \{0\})$,

$$u_{x,\lambda}(y) \leq u(y), \tag{17}$$

where

$$u_{x,\lambda}(y) := \left(\frac{\lambda}{|y-x|}\right)^{n-2\sigma} u\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right).$$

Let $r > 0$ and $x_1, x_2 \in \partial B_r$ be such that

$$u(x_1) = \max_{\partial B_r} u, \quad u(x_2) = \min_{\partial B_r} u,$$

and define

$$x_3 := x_1 + \frac{\varepsilon(x_1 - x_2)}{4|x_1 - x_2|}, \quad \lambda := \sqrt{\frac{\varepsilon}{4}\left(|x_1 - x_2| + \frac{\varepsilon}{4}\right)}.$$

Then

$$|x_3| = \left|x_1 + \frac{\varepsilon(x_1 - x_2)}{4|x_1 - x_2|}\right| \leq r + \frac{\varepsilon}{4}. \tag{18}$$

Via some direct computations and $|x_1|^2 = |x_2|^2 = r^2$, we find that

$$\begin{aligned}\lambda^2 - |x_3|^2 &= \frac{\varepsilon}{4}\left(|x_1 - x_2| + \frac{\varepsilon}{4}\right) - \left|x_1 + \frac{\varepsilon(x_1 - x_2)}{4|x_1 - x_2|}\right|^2 \\ &= \frac{\varepsilon(|x_2|^2 - |x_1|^2)}{4|x_1 - x_2|} - x_1^2 = -x_1^2 < 0,\end{aligned}$$

which follows from this and (18) that $\lambda < |x_3| < \varepsilon$ by choosing $r < \frac{3\varepsilon}{4}$.

It follows from (17) that

$$u_{x_3,\lambda}(x_2) \leq u(x_2).$$

Since

$$x_2 - x_3 = x_2 - x_1 + \frac{\varepsilon(x_2 - x_1)}{4|x_1 - x_2|} = \frac{x_2 - x_1}{|x_1 - x_2|}\left(|x_1 - x_2| + \frac{\varepsilon}{4}\right),$$

then

$$|x_2 - x_3| = |x_1 - x_2| + \frac{\varepsilon}{4},$$
$$\frac{x_2 - x_3}{|x_2 - x_3|^2} = \frac{x_2 - x_1}{|x_1 - x_2|\left(|x_1 - x_2| + \frac{\varepsilon}{4}\right)},$$

and

$$\frac{\lambda^2(x_2 - x_3)}{|x_2 - x_3|^2} = \frac{\varepsilon(x_2 - x_1)}{4|x_1 - x_2|}.$$

Hence,

$$\begin{aligned} u_{x_3,\lambda}(x_2) &= \left(\frac{\lambda}{|x_2 - x_3|}\right)^{n-2\sigma} u\left(x_3 + \frac{\lambda^2(x_2 - x_3)}{|x_2 - x_3|^2}\right) \\ &= \left(\frac{\lambda}{|x_1 - x_2| + \frac{\varepsilon}{4}}\right)^{n-2\sigma} u\left(x_3 + \frac{\varepsilon(x_2 - x_1)}{4|x_1 - x_2|}\right) \\ &= \left(\frac{\lambda}{|x_1 - x_2| + \frac{\varepsilon}{4}}\right)^{n-2\sigma} u(x_1). \end{aligned}$$

On the other hand,

$$u_{x_3,\lambda}(x_2) = \left(\frac{\lambda}{|x_1 - x_2| + \frac{\varepsilon}{4}}\right)^{n-2\sigma} u(x_1) = \frac{u(x_1)}{\left(\frac{4|x_1 - x_2|}{\varepsilon} + 1\right)^{\frac{n-2\sigma}{2}}} \geq \frac{u(x_1)}{\left(\frac{8r}{\varepsilon} + 1\right)^{\frac{n-2\sigma}{2}}},$$

then

$$u(x_1) \leq \left(\frac{8r}{\varepsilon} + 1\right)^{\frac{n-2\sigma}{2}} u_{x_3,\lambda}(x_2) \leq (1 + Cr)^{\frac{n-2\sigma}{2}} u(x_2),$$

for some $C = C(\varepsilon)$. That is,

$$\max_{\partial B_r} u \leq (1 + Cr) \min_{\partial B_r} u.$$

Hence for any $x \in \partial B_r$,

$$\frac{u(x)}{\bar{u}(|x|)} - 1 \leq \frac{\max_{\partial B_r} u}{\min_{\partial B_r} u} - 1 \leq Cr,$$
$$\frac{u(x)}{\bar{u}(|x|)} - 1 \geq \frac{\min_{\partial B_r} u}{\max_{\partial B_r} u} - 1 \geq \frac{1}{1 + Cr} - 1 > -Cr,$$

In conclusion, we have

$$\left|\frac{u(x)}{\bar{u}(|x|)} - 1\right| \leq Cr.$$

It follows that

$$u(x) = \bar{u}(|x|)(1 + O(r)) \quad \text{as } \; x \to 0.$$

Therefore, in order to complete the proof of Theorem 2.3, it suffices to prove (17). □

4.2 The proof of (17)

Replacing $u(x)$ by $r^{\frac{2\sigma+\tau}{p-1}}u(rx)$ and $h(x)$ by $r^{\frac{2\sigma+\tau}{p-1}}h(rx)$ for $r=\frac{2}{3}$, we can consider the Eq. (3) in $B_{3/2}$ for convenience, namely,

$$u(y)=\int_{B_{2/3}}\frac{|z|^{\tau}u^{p}(z)}{|y-z|^{n-2\sigma}}dz+h(y)\qquad \text{in}\quad B_{3/2}\backslash\{0\}, \tag{19}$$

with $h\in C^{1}(\overline{B_{3/2}})$ is positive and $|\nabla \ln h|\leq C$ in $\overline{B_{3/2}}$. Moreover, if we extend u to be identically 0 outside $B_{3/2}$, then (19) can be written as

$$u(y)=\int_{\mathbb{R}^n}\frac{|z|^{\tau}u^{p}(z)}{|y-z|^{n-2\sigma}}dz+h(y)\qquad \text{in}\quad B_{3/2}\backslash\{0\}.$$

For all $0<|x|<\frac{1}{16}$ and $\lambda>0$, it is a straightforward computation to show that

$$u_{x,\lambda}(y)=\int_{\mathbb{R}^n}\left(\frac{\lambda}{|z-x|}\right)^{p^*}\frac{\left|z_{x,\lambda}\right|^{\tau}u_{x,\lambda}^{p}(z)}{|y-z|^{n-2\sigma}}dz+h_{x,\lambda}(y)\qquad \text{in}\quad B_{3/2}^{x,\lambda},$$

where $z_{x,\lambda}:=x+\frac{\lambda^2(z-x)}{|z-x|^2}$, $p^*:=n+2\sigma-p(n-2\sigma)$, $B_{3/2}^{x,\lambda}:=\left\{y_{x,\lambda},\, y\in B_{3/2}\right\}$. It follows that

$$\begin{aligned}u(y)-u_{x,\lambda}(y)=&\int_{|z-x|\geq\lambda}K(x,\lambda;y,z)\left(|z|^{\tau}u^{p}(z)-\left(\frac{\lambda}{|z-x|}\right)^{p^*}\left|z_{x,\lambda}\right|^{\tau}u_{x,\lambda}^{p}(z)\right)\\&+h(y)-h_{x,\lambda}(y),\end{aligned}$$

where

$$K(x,\lambda;y,z):=\frac{1}{|y-z|^{n-2\sigma}}-\left(\frac{\lambda}{|y-x|}\right)^{n-2\sigma}\frac{1}{|y_{x,\lambda}-z|^{n-2\sigma}}.$$

On the other hand, since $h\in C^{1}(\overline{B_{3/2}})$ is positive and $|\nabla\ln h|\leq C$ in $B_{3/2}$, then by [8, Lemma 3.1], there exists $r_0\in(0,1/2)$ depending only on n, σ and C such that for every $x\in B_1$ and $0<\lambda\leq r_0$ there hold

$$h_{x,\lambda}(y)\leq h(y)\qquad \text{in}\quad B_{3/2}. \tag{20}$$

The aim is to show that there exists some positive constant $\varepsilon\in(0,r_0)$ such that for $|x|\leq\varepsilon$, $\lambda\in(0,|x|)$,

$$u_{x,\lambda}(y)\leq u(y)\quad \text{in}\quad B_{3/2}\backslash(B_{\lambda}(x)\cup\{0\}), \tag{21}$$

that is (17).

4.3 The proof of (21)

To prove (21), for fixed $x\in B_{1/16}\backslash\{0\}$, we first define

$$\bar{\lambda}(x):=\sup\left\{0<\mu\leq|x|\;\middle|\;u_{x,\lambda}(y)\leq u(y)\text{ in }\;B_{3/2}\backslash(B_{\lambda}(x)\cup\{0\}),\ \forall\, 0<\lambda<\mu\right\},$$

and then show $\bar{\lambda}(x)=|x|$.

For sake of clarity, the proof of (21) is divided into three steps. For the first step, we need the following Claim 1 to make sure that $\bar{\lambda}(x)$ is well defined.

Claim 1 There exists $\lambda_0(x) < |x|$ such that for all $\lambda \in (0, \lambda_0(x))$,

$$u_{x,\lambda}(y) \leq u(y) \quad \text{in } \ B_{3/2} \backslash (B_\lambda(x) \cup \{0\}).$$

Second, we give that

Claim 2 There exists a positive constant $\varepsilon \in (0, r_0)$ sufficiently small such that for all $|x| \leq \varepsilon$, $\lambda \in (0, |x|)$,

$$u_{x,\lambda}(y) < u(y) \quad \text{in } \ B_{3/2} \backslash B_{1/4}.$$

Last, we are going to prove that

Claim 3

$$\bar{\lambda}(x) = |x|.$$

Proof of Claim 1 First of all, we are going to show that there exist μ and $\lambda_0(x)$ satisfying $0 < \lambda_0(x) < \mu < |x|$ such that for all $\lambda \in (0, \lambda_0(x))$,

$$u_{x,\lambda}(y) \leq u(y) \quad \text{in } \ \overline{B_\mu(x)} \backslash B_\lambda(x). \tag{22}$$

Then we will prove that for all $\lambda \in (0, \lambda_0(x))$,

$$u_{x,\lambda}(y) \leq u(y) \quad \text{in } \ B_{3/2} \backslash \left(\overline{B_\mu(x)} \cup \{0\} \right). \tag{23}$$

Indeed, for every $0 < \lambda < \mu < \frac{1}{2}|x|$, we have

$$|\nabla \ln u| \leq C_0 \qquad \text{in } \overline{B_{|x|/2}(x)}.$$

Then for all $0 < r < \mu := \min \left\{ \frac{|x|}{4}, \frac{n-2\sigma}{2C_0} \right\}$, $\theta \in S^{n-1}$,

$$\begin{aligned} \frac{d}{dr} \left(r^{\frac{n-2\sigma}{2}} u(x + r\theta) \right) &= r^{\frac{n-2\sigma}{2} - 1} u(x + r\theta) \left(\frac{n - 2\sigma}{2} - r \frac{\nabla u \cdot \theta}{u} \right) \\ &\geq r^{\frac{n-2\sigma}{2} - 1} u(x + r\theta) \left(\frac{n - 2\sigma}{2} - C_0 r \right) > 0. \end{aligned}$$

For any $y \in B_\mu(x)$, $0 < \lambda < |y - x| \leq \mu$, let

$$\theta = \frac{y - x}{|y - x|}, \quad r_1 = |y - x|, \quad r_2 = \frac{\lambda^2}{|y - x|^2} r_1.$$

It follows that

$$r_2^{\frac{n-2\sigma}{2}} u(x + r_2\theta) < r_1^{\frac{n-2\sigma}{2}} u(x + r_1\theta).$$

That is (22). By Eq. (3), we have

$$u(x) \geq 4^{2\sigma - n} \int_{B_{3/2}} |y|^\tau u^p(y) dy =: C_1 > 0, \tag{24}$$

and thus we can find $0 < \lambda_0(x) \ll \mu$ such that, for every $\lambda \in (0, \lambda_0(x))$,

$$u_{x,\lambda}(y) \leq u(y) \quad \text{in } \ B_{3/2} \backslash \left(\overline{B_\mu(x)} \cup \{0\} \right),$$

that is (23). □

Proof of Claim 2 For $\frac{1}{4} \le |y| \le \frac{3}{2}$ and $0 < \lambda < |x| < \frac{1}{8}$, we have

$$|y-x| \ge |y| - |x| \ge \frac{1}{8} > |x|.$$

Hence

$$\left| x + \frac{\lambda^2(y-x)}{|y-x|^2} \right| \le |x| + \frac{|x|^2}{|y-x|} \le 2|x|,$$

and

$$\left| x + \frac{\lambda^2(y-x)}{|y-x|^2} \right| \ge |x| - \frac{|x|^2}{|y-x|} \ge \frac{|x|}{2}.$$

It follows from Theorem 2.1 that

$$u\left(x + \frac{\lambda^2(y-x)}{|y-x|^2} \right) \le C|x|^{-\frac{2\sigma+\tau}{p-1}},$$

Thus, for $0 < \lambda < |x| < \frac{1}{8}$, $\frac{1}{4} \le |y| \le \frac{3}{2}$, we conclude that

$$\begin{aligned} u_{x,\lambda}(y) &\le \left(\frac{\lambda}{|y-x|} \right)^{n-2\sigma} C|x|^{-\frac{2\sigma+\tau}{p-1}} \\ &\le C\lambda^{n-2\sigma}|x|^{-\frac{2\sigma+\tau}{p-1}} \\ &\le C|x|^{\frac{p(n-2\sigma)-n-\tau}{p-1}} \le C|\varepsilon|^{\frac{p(n-2\sigma)-n-\tau}{p-1}}. \end{aligned} \tag{25}$$

Since $\frac{n+\tau}{n-2\sigma} < p \le \frac{n+2\sigma+2\tau}{n-2\sigma}$, we have $\frac{p(n-2\sigma)-n-\tau}{p-1} > 0$. Then by (24), $\varepsilon > 0$ can be chosen sufficiently small to guarantee that for all $0 < \lambda < |x| \le \varepsilon < r_0$ and $\frac{1}{4} \le |y| \le \frac{3}{2}$,

$$u_{x,\lambda}(y) \le C|x|^{\frac{p(n-2\sigma)-n-\tau}{p-1}} < u(y). \tag{26}$$

□

Proof of Claim 3 We prove Claim 3 by contradiction. Assume $\bar{\lambda}(x) < |x| \le \varepsilon < r_0$ for some $x \neq 0$. We want to show that there exists a positive constant $\widetilde{\varepsilon} \in \left(0, \frac{|x|-\bar{\lambda}(x)}{2}\right)$ such that for $\lambda \in (\bar{\lambda}(x), \bar{\lambda}(x) + \widetilde{\varepsilon})$,

$$u_{x,\lambda}(y) \le u(y) \quad \text{in } B_{3/2} \backslash (B_\lambda(x) \cup \{0\}), \tag{27}$$

which contradicts the definition of $\bar{\lambda}(x)$, then we obtain $\bar{\lambda}(x) = |x|$.

By the Claim 2, it is obviously to obtain that (27) in $B_{3/2} \backslash B_{1/4}$. Next, we need to consider the region $B_{1/4} \backslash (B_\lambda(x) \cup \{0\})$.

It is a straightforward computation to show that for every $\bar{\lambda}(x) \le \lambda < |x| \le r_0$,

$$\begin{aligned} u(y) - u_{x,\lambda}(y) \ge &\int_{B_{1/2} \backslash B_\lambda(x)} K(x, \lambda; y, z) \left(|z|^\tau u^p(z) - \left(\frac{\lambda}{|z-x|} \right)^{p^*} |z_{x,\lambda}|^\tau u^p_{x,\lambda}(z) \right) \\ &+ J(x, \lambda, u, y), \end{aligned}$$

where (20) is used in the above inequality and

$$J(x,\lambda,u,y) := \int_{B_{3/2}\setminus B_{1/2}} K(x,\lambda;y,z)\left(|z|^{\tau}u^{p}(z) - \left(\frac{\lambda}{|z-x|}\right)^{p^{*}}\left|z_{x,\lambda}\right|^{\tau} u_{x,\lambda}^{p}(z)\right)dz$$
$$- \int_{B_{3/2}^{c}} K(x,\lambda;y,z)\left(\frac{\lambda}{|z-x|}\right)^{p^{*}}\left|z_{x,\lambda}\right|^{\tau} u_{x,\lambda}^{p}(z)dz.$$

It follows that

$$J(x,\lambda,u,y) \geq \int_{B_{3/2}\setminus B_{1/2}} K(x,\lambda;y,z)|z|^{\tau}\left(u^{p}(z) - u_{x,\lambda}^{p}(z)\right)dz$$
$$- \int_{B_{3/2}^{c}} K(x,\lambda;y,z)|z|^{\tau}u_{x,\lambda}^{p}(z)dz.$$

By (24) and (25), we have

$$J(x,\lambda,u,y) \geq \left(\frac{3}{2}\right)^{\tau}\int_{B_{3/2}\setminus B_{1/2}} K(x,\lambda;y,z)\left(C_1^{p} - \left(C|\varepsilon|^{\frac{p(n-2\sigma)-n-\tau}{p-1}}\right)^{p}\right)dz$$
$$- \left(\frac{3}{2}\right)^{\tau}\int_{B_{3/2}^{c}} K(x,\lambda;y,z)\left(\left(\frac{|x|}{|z-x|}\right)^{n-2\sigma}|x|^{-\frac{2\sigma+\tau}{p-1}}\right)^{p}dz.$$

Since $\frac{n+\tau}{n-2\sigma} < p \leq \frac{n+2\sigma+2\tau}{n-2\sigma}$, we have $\frac{p(n-2\sigma)-n-\tau}{p-1} > 0$. Then $\varepsilon > 0$ can be chosen sufficiently small to guarantee that

$$J(x,\lambda,u,y) \geq \frac{C_1^{p}}{2}\left(\frac{3}{2}\right)^{\tau}\int_{B_{3/2}\setminus B_{1/2}} K(x,\lambda;y,z)dz$$
$$- \left(\frac{3}{2}\right)^{\tau}|\varepsilon|^{\frac{p(n-2\sigma)-n-\tau}{p-1}}\int_{B_{3/2}^{c}} K(x,\lambda;y,z)\frac{1}{|z-x|^{p(n-2\sigma)}}dz$$
$$\geq \frac{C_1^{p}}{2}\left(\frac{3}{2}\right)^{\tau}\int_{B_{23/16\setminus 9/16}} K(0,\lambda;y-x,z)dz$$
$$- \left(\frac{3}{2}\right)^{\tau}\left(\frac{16}{7}\right)^{p(n-2\sigma)}|\varepsilon|^{\frac{p(n-2\sigma)-n-\tau}{p-1}}\int_{B_{23/16}^{c}} K(0,\lambda;y-x,z)dz,$$

Indeed, since for $|y-x| = \lambda < \frac{1}{16}$,

$$K(0,\lambda;y-x,z) = 0,$$

and for $|z| \geq \frac{3}{8}$, $|y-x| = \lambda$,

$$(y-x)\cdot\nabla_y K(0,\lambda;y-x,z) = (n-2\sigma)|y-x|^{2\sigma-n-2}(|z|^2 - |y-x|^2) > 0.$$

Using the positive and smoothness of K, we have

$$\frac{\delta_1(|y-x|-\lambda)}{|y-x-z|^{n-2\sigma}} \leq K(0,\lambda;y-x,z) \leq \frac{\delta_2(|y-x|-\lambda)}{|y-x-z|^{n-2\sigma}}, \tag{28}$$

for $\bar{\lambda}(x) \leq \lambda \leq |y-x| \leq |x| + \frac{1}{4} < \frac{5}{16}$, $\frac{3}{8} \leq |z| \leq M < +\infty$, where M and $0 < \delta_1 < \delta_2 < +\infty$ are positive constants. If M is large enough, then

$$0 < c_2 \leq (y-x)\cdot\nabla_y(|y-x|^{n-2\sigma}K(0,\lambda;y-x,z)) \leq c_3 < +\infty.$$

Thus, (28) holds for $|z| \geq M$, $\bar{\lambda}(x) \leq \lambda \leq |y-x| \leq |x| + \frac{1}{4}$.

With the help of it, for $y \in B_{1/4} \backslash (B_\lambda(x) \cup \{0\})$, there exists positive constants C_2 and C_3 such that

$$\begin{aligned} J(x, \lambda, u, y) \geq & \frac{C_1}{2}\left(\frac{3}{2}\right)^\tau \int_{B_{23/16 \backslash 9/16}} \frac{\delta_1(|y-x|-\lambda)}{|y-x-z|^{n-2\sigma}} dz \\ & - \left(\frac{3}{2}\right)^\tau \left(\frac{16}{7}\right)^{p(n-2\sigma)} |\varepsilon|^{\frac{p(n-2\sigma)-n-\tau}{p-1}} \int_{B_{23/16}^c} \frac{\delta_2(|y-x|-\lambda)}{|y-x-z|^{n-2\sigma}} dz \\ \geq & C_2(|y-x|-\lambda) - C_3(|y-x|-\lambda)|\varepsilon|^{\frac{p(n-2\sigma)-n-\tau}{p-1}}. \end{aligned}$$

For ε sufficiently small, we have

$$J(x, \lambda, u, y) \geq \frac{C_2}{2}(|y-x|-\lambda).$$

It follows that we can choose $\widetilde{\varepsilon} \in \left(0, \frac{|x|-\bar{\lambda}(x)}{2}\right)$ such that for every $\bar{\lambda}(x) \leq \lambda \leq \bar{\lambda}(x) + \widetilde{\varepsilon}$, and $y \in B_{1/4} \backslash (B_\lambda(x) \cup \{0\})$,

$$\begin{aligned} u(y) - u_{x,\lambda}(y) \geq & \int_{B_{1/2} \backslash B_\lambda(x)} K(x, \lambda; y, z) \left(|z|^\tau u^p(z) - \left(\frac{\lambda}{|z-x|}\right)^{p^*} \left|z_{x,\lambda}\right|^\tau u_{x,\lambda}^p(z)\right) dz \\ \geq & \int_{B_{1/2} \backslash B_\lambda(x)} K(x, \lambda; y, z)|z|^\tau \left(u^p(z) - u_{x,\lambda}^p(z)\right) dz. \end{aligned}$$

So Claim 2 gives that

$$\begin{aligned} u(y) - u_{x,\lambda}(y) \geq & \int_{B_{1/4} \backslash B_\lambda(x)} K(x, \lambda; y, z)|z|^\tau \left(u^p(z) - u_{x,\lambda}^p(z)\right) dz \\ & + \int_{B_{1/2} \backslash B_{5/16}} K(x, \lambda; y, z)|z|^\tau \left(u^p(z) - u_{x,\lambda}^p(z)\right) dz \\ \geq & \int_{B_{1/4} \backslash B_\lambda(x)} K(x, \lambda; y, z)|z|^\tau \left(u_{x,\bar{\lambda}(x)}^p(z) - u_{x,\lambda}^p(z)\right) dz \\ & + 2^\tau \int_{B_{1/2} \backslash B_{5/16}} K(x, \lambda; y, z) \left(u^p(z) - u_{x,\lambda}^p(z)\right) dz \\ \geq & - 4^{-\tau} \int_{B_{1/4} \backslash B_\lambda(x)} K(x, \lambda; y, z) \left|u_{x,\bar{\lambda}(x)}^p(z) - u_{x,\lambda}^p(z)\right| dz \\ & + 2^\tau \int_{B_{1/2} \backslash B_{5/16}} K(x, \lambda; y, z) \left(u^p(z) - u_{x,\lambda}^p(z)\right) dz. \end{aligned}$$

Since $\|u\|_{C(B_{\bar{\lambda}(x)+\widetilde{\varepsilon}}(x))} \leq C$, it follows that there exists some constant $C > 0$ such that for any $\bar{\lambda}(x) \leq \lambda \leq \bar{\lambda}(x) + \widetilde{\varepsilon}$, $z \in B_{1/4} \backslash B_\lambda(x)$,

$$|u_{x,\bar{\lambda}}^p(z) - u_{x,\lambda}^p(z)| \leq C(\lambda - \bar{\lambda}(x)) \leq C\widetilde{\varepsilon}.$$

Moreover, for $z \in \overline{B_{1/2}} \backslash B_{5/16}$, there exists some constant $C_1 > 0$ such that

$$u^p(z) - u_{x,\lambda}^p(z) \geq C_1.$$

Hence, we have

$$\begin{aligned} u(y) - u_{x,\lambda}(y) &\geq -C\widetilde{\varepsilon} \int_{B_{1/4}\backslash B_\lambda(x)} K(x, \lambda; y, z)dz + C_1 \int_{B_{1/2}\backslash B_{5/16}} K(x, \lambda; y, z)dz \\ &\geq -C\widetilde{\varepsilon} \int_{B_{1/4}\backslash B_\lambda(x)} K(x, \lambda; y, z)dz + C_1 \int_{B_{7/16}\backslash B_{3/8}} K(0, \lambda; y - x, z)dz. \end{aligned}$$

On the other hand, since

$$\begin{aligned} \int_{B_{1/4}\backslash B_\lambda(x)} K(x, \lambda; y, z)dz &\leq \int_{B_{5/16}\backslash B_\lambda} K(0, \lambda; y - x, z)dz \\ &\leq C(|y - x| - \lambda), \end{aligned}$$

and

$$\int_{B_{7/16}\backslash B_{3/8}} K(0, \lambda; y - x, z)dz \geq \frac{\delta_1(|y - x| - \lambda)}{|y - x - z|^{n-2\sigma}}.$$

Then we can choose $\widetilde{\varepsilon}$ sufficient small such that for $\bar{\lambda}(x) \leq \lambda \leq \bar{\lambda}(x) + \widetilde{\varepsilon}$,

$$u_{x,\lambda}(y) \leq u(y) \quad \text{ in } \quad B_{1/4}\backslash(B_\lambda(x) \cup \{0\}).$$

Combining Claim 2, we get a contradiction and then we finish the proof. □

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