**Calculus of Variations**

# Liouville theorems for harmonic map heat flow along ancient super Ricci flow via reduced geometry

**Keita Kunikawa[1] · Yohei Sakurai[2]**

## Abstract
We study harmonic map heat flow along ancient super Ricci flow, and derive several Liouville theorems with controlled growth from Perelman's reduced geometric viewpoint. For non-positively curved target spaces, our growth condition is sharp. For positively curved target spaces, our Liouville theorem is new even in the static case (i.e., for harmonic maps); moreover, we point out that the growth condition can be improved, and almost sharp in the static case. This fills the gap between the Liouville theorem of Choi and the example constructed by Schoen–Uhlenbeck.

## 1 Background

This is a continuation of [22] on Liouville theorems for heat equation along ancient super Ricci flow. The aim of this paper is to generalize the target spaces, and formulate Liouville theorems for harmonic map heat flow.

### 1.1 Ancient super Ricci flow

A smooth manifold $(M, g(t))_{t \in I}$ with a time-dependent Riemannian metric is called *Ricci flow* when

$$\partial_t g = -2 \operatorname{Ric},$$

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✉ Yohei Sakurai
   ysakurai@rimath.saitama-u.ac.jp

   Keita Kunikawa
   kunikawa@cc.utsunomiya-u.ac.jp

[1] Cooperative Faculty of Education, Utsunomiya University, 350 Mine-Machi, Utsunomiya 321-8505, Japan

[2] Department of Mathematics, Saitama University, 255 Shimo-Okubo, Sakura-ku, Saitama-City, Saitama 338-8570, Japan

which has been introduced by Hamilton [14]. A supersolution to this equation is called super Ricci flow. Namely, $(M, g(t))_{t \in I}$ is called *super Ricci flow* if

$$\partial_t g \geq -2 \operatorname{Ric},$$

which has been introduced by McCann-Topping [33] from the viewpoint of optimal transport theory. Recently, the super Ricci flow has begun to be investigated from various perspectives, especially metric measure geometry (see e.g., [3,4,16,19–21,25–30,39]). A Ricci flow $(M, g(t))_{t \in I}$ is said to be *ancient* when $I = (-\infty, 0]$, which is one of the crucial concepts in singular analysis of Ricci flow. In the present paper, we will focus on ancient super Ricci flow.

## 1.2 Liouville theorems for ancient solutions to heat equation

The celebrated Yau's Liouville theorem states that on a complete manifold of non-negative Ricci curvature, any positive harmonic functions must be constant. One of the natural research directions is to generalize his Liouville theorem for ancient solutions to heat equation

$$\partial_t u = \Delta u.$$

Souplet–Zhang [38] have proven the following parabolic analogue (see [38, Theorem 1.2]):

**Theorem 1.1** ([38]) *Let $(M, g)$ be a complete Riemannian manifold of non-negative Ricci curvature. Then we have the following:*

(1) *Let $u : M \times (-\infty, 0] \to (0, \infty)$ be a positive ancient solution to the heat equation. If*

$$u(x, t) = \exp\left[o\left(d(x) + \sqrt{|t|}\right)\right]$$

*near infinity, then u must be constant. Here $d(x)$ denotes the Riemannian distance from a fixed point;*

(2) *let $u : M \times (-\infty, 0] \to \mathbb{R}$ be an ancient solution to the heat equation. If*

$$u(x, t) = o\left(d(x) + \sqrt{|t|}\right)$$

*near infinity, then u is constant.*

The growth conditions in Theorem 1.1 are known to be sharp in the spatial direction (see [38], and cf. [9]). As mentioned in [22, Section 1], one of the next research directions is the following: For an ancient super Ricci flow $(M, g(t))_{t \in (-\infty, 0]}$, the problem is to find suitable growth conditions for a solution $u : M \times (-\infty, 0] \to \mathbb{R}$ to heat equation such that $u$ must become constant. In other words, for the reverse time parameter

$$\tau := -t,$$

and for a backward super Ricci flow $(M, g(\tau))_{\tau \in [0, \infty)}$, namely,

$$\operatorname{Ric} \geq \frac{1}{2} \partial_\tau g,$$

the problem is to find suitable conditions for a solution $u : M \times [0, \infty) \to \mathbb{R}$ to backward heat equation

$$(\Delta + \partial_\tau) u = 0$$

such that $u$ must become constant. Guo–Philipowski–Thalmaier [13] have provided an approach to this problem from stochastic analytic viewpoint, and obtained a Liouville theorem under a growth condition for entropy (see [13, Theorem 2]). On the other hand, the authors [22] have approached the problem from Perelman's reduced geometric viewpoint (cf. [35]), and established a Liouville theorem under a growth condition concerning reduced distance.

Now, let us recall the precise statement of the Liouville theorem in [22]. To do so, we fix some notations on a complete, time-dependent Riemannian manifold $(M, g(\tau))_{\tau \in [0,\infty)}$, which is not necessarily backward super Ricci flow. We put

$$h := \frac{1}{2} \partial_\tau g, \quad H := \operatorname{tr} h.$$

We begin with recalling the notion of reduced distance (more precisely, see Sect. 3). For $(x, \tau) \in M \times (0, \infty)$, let $L(x, \tau)$ be the *L-distance* from a space-time base point $(x_0, 0)$, which is defined as the infimum of the so-called *$\mathcal{L}$-length* over all curves $\gamma : [0, \tau] \to M$ with $\gamma(0) = x_0$ and $\gamma(\tau) = x$. Then the *reduced distance* $\ell(x, \tau)$ is defined as

$$\ell(x, \tau) := \frac{1}{2\sqrt{\tau}} L(x, \tau).$$

We say that $(M, g(\tau))_{\tau \in [0,\infty)}$ is *admissible* if for every $\tau > 0$ there is $c_\tau \geq 0$ depending only on $\tau$ such that $h \geq -c_\tau g$ on $[0, \tau]$. The admissibility guarantees that the $L$-distance is achieved by a minimal $\mathcal{L}$-geodesic.

Next, for a (time-dependent) vector field $V$, we recall the following *Müller quantity $\mathcal{D}(V)$* (see [34, Definition 1.3]), and *trace Harnack quantity $\mathcal{H}(V)$* (see [15], [34, Definition 1.5]):

$$\mathcal{D}(V) := -\partial_\tau H - \Delta H - 2\|h\|^2 + 4 \operatorname{div} h(V) - 2g(\nabla H, V) + 2 \operatorname{Ric}(V, V) - 2h(V, V),$$

$$\mathcal{H}(V) := -\partial_\tau H - \frac{H}{\tau} - 2g(\nabla H, V) + 2h(V, V).$$

The main result in [22] can be stated as follows (see [22, Theorem 2.2]):

**Theorem 1.2** ([22]) *Let $(M, g(\tau))_{\tau \in [0,\infty)}$ be an admissible, complete backward super Ricci flow. We assume*

$$\mathcal{D}(V) \geq 0, \quad \mathcal{H}(V) \geq -\frac{H}{\tau}, \quad H \geq 0$$

*for all vector fields $V$. Then we have the following:*

(1) *Let $u : M \times [0, \infty) \to (0, \infty)$ be a positive solution to backward heat equation. If*

$$u(x, \tau) = \exp \left[ o \left( \mathfrak{d}(x, \tau) + \sqrt{\tau} \right) \right]$$

*near infinity, then $u$ is constant. Here $\mathfrak{d}(x, \tau)$ is defined by*

$$\mathfrak{d}(x, \tau) := \sqrt{4\tau \, \ell(x, \tau)};$$

(2) *let $u : M \times [0, \infty) \to \mathbb{R}$ be a solution to backward heat equation. If*

$$u(x, \tau) = o \left( \mathfrak{d}(x, \tau) + \sqrt{\tau} \right)$$

*near infinity, then $u$ is constant.*

In the static case of $h = 0$, Theorem 1.2 is reduced to Theorem 1.1 (see [22, Remark 2.3]).

## 2 Main results

### 2.1 Liouville theorems for ancient solutions to harmonic map heat flow

One can now consider the following problem: For a backward super Ricci flow $(M, g(\tau))_{\tau \in [0,\infty)}$, and a manifold $(N, \mathfrak{g})$ with an upper sectional curvature bound, the problem is to find suitable conditions for a solution $u : M \times [0, \infty) \to N$ to backward harmonic map heat flow

$$(\Delta + \partial_\tau) u = 0 \tag{1}$$

such that $u$ must be constant. Here $\Delta$ is the tension field. Guo–Philipowski–Thalmaier [13] have approached this problem from stochastic analytic viewpoint, and produced various Liouville theorems (see [12, Section 4]). We here aim to approach the problem from Perelman's reduced geometric viewpoint. Our first main result is the following Liouville theorem of Cheng type (cf. [6]):

**Theorem 2.1** *Let* $(M, g(\tau))_{\tau \in [0,\infty)}$ *be an admissible, complete backward super Ricci flow. We assume*

$$\mathcal{D}(V) \geq 0, \quad \mathcal{H}(V) \geq -\frac{H}{\tau}, \quad H \geq 0 \tag{2}$$

*for all vector fields* $V$. *Let* $(N, \mathfrak{g})$ *be a complete, simply connected Riemannian manifold with* sec $\leq 0$. *Let* $u : M \times [0, \infty) \to N$ *be a solution to backward harmonic map heat flow. If*

$$\rho(u(x, \tau)) = o\left(\mathfrak{d}(x, \tau) + \sqrt{\tau}\right)$$

*near infinity, then* $u$ *is constant. Here* $\rho : N \to \mathbb{R}$ *is the Riemannian distance function from a fixed point* $y_0 \in N$.

When $N = \mathbb{R}$, Theorem 2.1 is nothing but Theorem 1.2. In the static case of $h = 0$, Theorem 2.1 has been proved by Wang [41] (see [41, Theorem 1.3]). Since growth conditions in these results are sharp in the spatial direction, so is the growth condition in Theorem 2.1.

We also prove the following result for positively curved target spaces:

**Theorem 2.2** *Let* $(M, g(\tau))_{\tau \in [0,\infty)}$ *be an admissible, complete backward super Ricci flow. We assume*

$$\mathcal{D}(V) \geq 0, \quad \mathcal{H}(V) \geq -\frac{H}{\tau}, \quad H \geq 0 \tag{3}$$

*for all vector fields* $V$. *Let* $(N, \mathfrak{g})$ *be a complete Riemannian manifold with* sec $\leq \kappa$ *for* $\kappa > 0$. *Assume that an open geodesic ball* $B_{\pi/2\sqrt{\kappa}}(y_0)$ *of radius* $\pi/2\sqrt{\kappa}$ *centered at* $y_0$ *in* $N$ *does not meet the cut locus* Cut $(y_0)$ *of* $y_0$. *Let* $u : M \times [0, \infty) \to N$ *be a solution to backward harmonic map heat flow. If the image of* $u$ *is contained in* $B_{\pi/2\sqrt{\kappa}}(y_0)$, *and if* $u$ *satisfies*

$$\frac{1}{\cos \sqrt{\kappa} \rho(u(x, \tau))} = o\left(\mathfrak{d}(x, \tau)^{1/2} + \tau^{1/4}\right)$$

*near infinity, then* $u$ *is constant.*

Theorems 2.1 and 2.2 follow from local gradient estimates (see Theorems 4.1 and 5.1).

### 2.2 Sharpness

Let us discuss the sharpness concerning Theorem 2.2. To do so, we recall the Liouville theorem of Choi [7] (see [7, Theorem]):

**Theorem 2.3** ([7]) *Let $(M, g)$ be a complete Riemannian manifold of non-negative Ricci curvature, and let $(N, \mathfrak{g})$ be a complete Riemannian manifold with $\sec \leq \kappa$ for $\kappa > 0$. Let $u : M \to N$ be a harmonic map (i.e., $\Delta u = 0$). We assume that $B_L(y_0)$ is a regular (i.e., $L \in (0, \pi/2\sqrt{\kappa})$ and $B_L(y_0) \cap \mathrm{Cut}\,(y_0) = \emptyset$), open geodesic ball. If the image of $u$ is contained in $B_L(y_0)$, then $u$ is constant.*

Theorem 2.2 enables us to improve Theorem 2.3 as follows:

**Corollary 2.4** *Let $(M, g)$ be a complete Riemannian manifold of non-negative Ricci curvature, and let $(N, \mathfrak{g})$ be a complete Riemannian manifold with $\sec \leq \kappa$ for $\kappa > 0$. Assume that $B_{\pi/2\sqrt{\kappa}}(y_0)$ does not meet $\mathrm{Cut}\,(y_0)$. Let $u : M \to N$ be a harmonic map. If the image of $u$ is contained in $B_{\pi/2\sqrt{\kappa}}(y_0)$, and if $u$ satisfies a growth condition*

$$\frac{1}{\cos \sqrt{\kappa} \rho(u(x))} = o(d(x)^{1/2}) \tag{4}$$

*near infinity, then $u$ is constant.*

The growth condition (4) controls the approach speed of $u$ to the boundary of $B_{\pi/2\sqrt{\kappa}}(y_0)$. Note that if the image of $u$ is contained in a regular ball $B_L(y_0)$, then the left hand side of (4) is bounded; in particular, (4) is trivially satisfied.

**Remark 2.5** In the literature of Liouville theorems for harmonic maps with positively curved targets, the results in the form of Theorem 2.3 have been examined (see e.g., [17, Theorem 1], [7, Theorem], [18, Theorem 6.1], [40, Theorem 1.4], [23, Example 3], [31, Theorem 3.2], [5, Theorem 2], [36, Theorem 2], [45, Corollary 1.8]). We emphasize that in Corollary 2.4, such a condition is relaxed to a growth condition (4) beyond the traditional form.

Although our formulation of Theorem 2.2 and Corollary 2.4 is new, the growth condition (4) is not sharp. Actually, we can further improve it as follows:

**Theorem 2.6** *Let $(M, g)$ be a complete Riemannian manifold of non-negative Ricci curvature, and let $(N, \mathfrak{g})$ be a complete Riemannian manifold with $\sec \leq \kappa$ for $\kappa > 0$. Assume that $B_{\pi/2\sqrt{\kappa}}(y_0)$ does not meet $\mathrm{Cut}\,(y_0)$. Let $u : M \to N$ be a harmonic map. If the image of $u$ is contained in $B_{\pi/2\sqrt{\kappa}}(y_0)$, and if $u$ satisfies a growth condition*

$$\frac{1}{\cos \sqrt{\kappa} \rho(u(x))} = o(d(x)) \tag{5}$$

*near infinity, then $u$ is constant.*

We can obtain Theorem 2.6 by adopting the technique for minimal hypersurfaces developed by Ecker-Huisken [10].

**Remark 2.7** According to Schoen–Uhlenbeck [37] (see also [11]), a harmonic map $u : \mathbb{R}^m \to \mathbb{S}^n_+$ is necessarily constant for $m \leq 6$, and for $m \geq 7$ such a map exists as a radial solution, where $\mathbb{S}^n_+$ is the open hemisphere. In Sect. 6.2, we observe that the growth of the radial solution is greater than the linear order. Moreover, it approaches the linear order as $m \to \infty$. In this sense, our growth condition (5) is almost sharp.

# 3 Preliminaries

We review some facts on Perelman's reduced geometry. The references are [8,22,34,35,42–44]. We mainly refer to [22, Section 3]. Throughout this subsection, let $(M, g(\tau))_{\tau \in [0, \infty)}$ be an $m$-dimensional, complete time-dependent Riemannian manifold.

For a curve $\gamma : [\tau_1, \tau_2] \to M$, its $\mathcal{L}$-*length* is defined as

$$\mathcal{L}(\gamma) := \int_{\tau_1}^{\tau_2} \sqrt{\tau} \left( H + \left\| \frac{d\gamma}{d\tau} \right\|^2 \right) d\tau.$$

It is well-known that its critical point over all curves with fixed endpoints is characterized by the following $\mathcal{L}$-*geodesic equation*:

$$X := \frac{d\gamma}{d\tau}, \quad \nabla_X X - \frac{1}{2}\nabla H + \frac{1}{2\tau}X + 2h(X) = 0.$$

For $(x, \tau) \in M \times (0, \infty)$, the $L$-*distance* $L(x, \tau)$ and *reduced distance* $\ell(x, \tau)$ from a space-time base point $(x_0, 0)$ are defined by

$$L(x, \tau) := \inf_{\gamma} \mathcal{L}(\gamma), \quad \ell(x, \tau) := \frac{1}{2\sqrt{\tau}}L(x, \tau), \tag{6}$$

where the infimum is taken over all curves $\gamma : [0, \tau] \to M$ with $\gamma(0) = x_0$ and $\gamma(\tau) = x$. A curve is called *minimal $\mathcal{L}$-geodesic* from $(x_0, 0)$ to $(x, \tau)$ if it attains the infimum of (6). We also set

$$\overline{L}(x, \tau) := 4\tau\,\ell(x, \tau).$$

We now assume that $(M, g(\tau))_{\tau \in [0,\infty)}$ is admissible (see Sect. 1.2). In this case, for every $(x, \tau) \in M \times (0, \infty)$, there exists at least one minimal $\mathcal{L}$-geodesic. Also, the functions $L(\cdot, \tau)$ and $L(x, \cdot)$ are locally Lipschitz in $(M, g(\tau))$ and $(0, \infty)$, respectively; in particular, they are differentiable almost everywhere.

Assume that $\ell$ is smooth at $(\overline{x}, \overline{\tau}) \in M \times (0, \infty)$. We have (see [22, Lemmas 3.5 and 3.6]):

**Lemma 3.1** *( [22]) Let $K \geq 0$. We assume*

$$\mathcal{D}(V) \geq -2K\left(H + \|V\|^2\right), \quad H \geq 0$$

*for all vector fields $V$. Then at $(\overline{x}, \overline{\tau})$ we have*

$$(\Delta + \partial_\tau)\overline{L} \leq 2m + 2K\overline{L}.$$

**Lemma 3.2** *( [22]) We assume*

$$\mathcal{H}(V) \geq -\frac{H}{\tau}, \quad H \geq 0$$

*for all vector fields $V$. Then at $(\overline{x}, \overline{\tau})$ we have*

$$\|\nabla\mathfrak{d}\|^2 \leq 3.$$

## 4 Proof of Theorem 2.1

In this section, we prove Theorem 2.1. For $K \in \mathbb{R}$, a time-dependent Riemannian manifold $(M, g(t))_{t \in I}$ is called $K$-*super Ricci flow* if

$$\frac{1}{2}\partial_t g + \mathrm{Ric} \geq Kg.$$

The key ingredient is the following local gradient estimate (cf. [22, Theorem 2.8]):

**Theorem 4.1** *For $K \geq 0$, let $(M, g(\tau))_{\tau \in [0,\infty)}$ be an $m$-dimensional, admissible, complete backward $(-K)$-super Ricci flow, namely,*

$$\text{Ric} \geq h - Kg.$$

*We assume*

$$\mathcal{D}(V) \geq -2K \left( H + \|V\|^2 \right), \quad \mathcal{H}(V) \geq -\frac{H}{\tau}, \quad H \geq 0 \tag{7}$$

*for all vector fields $V$. Let $(N, \mathfrak{g})$ stand for a complete, simply connected Riemannian manifold with $\sec \leq 0$. For a fixed $y_0 \in N$, let $\rho : N \to \mathbb{R}$ be the Riemannian distance function from $y_0$. Let $u : M \times [0, \infty) \to N$ be a solution to backward harmonic map heat flow. For $R, T > 0$ and $A > 0$, we suppose $2\rho \circ u \leq A$ on*

$$Q_{R,T} := \{ (x, \tau) \in M \times (0, T] \mid \mathfrak{d}(x, \tau) \leq R \}.$$

*Then there exists a positive constant $C_m > 0$ depending only on $m$ such that on $Q_{R/2,T/4}$,*

$$\frac{\|du\|}{A^2 - \rho^2 \circ u} \leq \frac{C_m}{A} \left( \frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{K} \right).$$

In the static case of $h = 0$, Wang [41] has obtained Theorem 4.1 (see [41, Theorem 1.2]). We will prove Theorem 4.1 along the line of the proof of [41, Theorem 1.2].

## 4.1 Backward harmonic map heat flows

In this and next section, let $(M, g(\tau))_{\tau \in [0,\infty)}$ denote an $m$-dimensional, admissible, complete time-dependent Riemannian manifold, and let $(N, \mathfrak{g})$ be a complete Riemannian manifold. Moreover, for a fixed $y_0 \in N$, let $\rho : N \to \mathbb{R}$ stand for the Riemannian distance function from $y_0$. We study properties of a solution $u : M \times [0, \infty) \to N$ to backward harmonic map heat flow. We start with the following:

**Lemma 4.2**

$$(\Delta + \partial_\tau)\|du\|^2 = 2\|\nabla du\|^2 + 2 \sum_{i=1}^m \mathfrak{g}(du(\mathcal{R}(e_i)), du(e_i))$$

$$- 2 \sum_{i,j=1}^m \mathfrak{g}(R(du(e_i), du(e_j))du(e_j), du(e_i)),$$

*where $\mathcal{R} := \text{Ric} - h$, and $\{e_i\}_{i=1}^m$ is an orthonormal frame on $M$ at some fixed time.*

***Proof*** By direct computations and backward harmonic map heat flow Eq. (1), we have the following (cf. [2, Lemma 4.5]):

$$\partial_\tau \|du\|^2 = -\sum_{i=1}^m \mathfrak{g}(du((\partial_\tau g)(e_i)), du(e_i)) + 2 \sum_{i=1}^m \mathfrak{g}(\nabla_{e_i}^{u^{-1}TN}(\partial_\tau u), du(e_i))$$

$$= -2 \sum_{i=1}^m \mathfrak{g}(du(h(e_i)), du(e_i)) - 2 \sum_{i=1}^m \mathfrak{g}(\nabla_{e_i}^{u^{-1}TN}\Delta u, du(e_i))$$

$$= -2 \sum_{i=1}^m \mathfrak{g}(du(\text{Ric}(e_i)), du(e_i)) - 2 \sum_{i=1}^m \mathfrak{g}(\nabla_{e_i}^{u^{-1}TN}\Delta u, du(e_i))$$

$$+ 2 \sum_{i=1}^{m} \mathfrak{g}(du(\mathcal{R}(e_i)), du(e_i)),$$

here $u^{-1}TN$ denotes the induced vector bundle from $TN$ by $u$, and $\nabla^{u^{-1}TN}$ is the canonical connection over $u^{-1}TN$. Combining the above equation and the Bochner formula of Eells-Sampson type tells us the following (see e.g., [1, Remark 1.15]):

$$\frac{1}{2} \Delta \|du\|^2 = \|\nabla du\|^2 + \sum_{i=1}^{m} \mathfrak{g}(\nabla_{e_i}^{u^{-1}TN} \Delta u, du(e_i)) + \sum_{i=1}^{m} \mathfrak{g}(du(\mathrm{Ric}(e_i)), du(e_i))$$

$$- \sum_{i,j=1}^{m} \mathfrak{g}(R(du(e_i), du(e_j))du(e_j), du(e_i))$$

$$= \|\nabla du\|^2 + \sum_{i=1}^{m} \mathfrak{g}(du(\mathcal{R}(e_i)), du(e_i)) - \frac{1}{2} \partial_\tau \|du\|^2$$

$$- \sum_{i,j=1}^{m} \mathfrak{g}(R(du(e_i), du(e_j))du(e_j), du(e_i)).$$

This completes the proof. □

We next show the following:

**Lemma 4.3** *Let $(N, \mathfrak{g})$ be simply connected, and $\sec \leq 0$. For $A > 0$, we assume $2\rho \circ u \leq A$. Set*

$$w := \frac{\|du\|^2}{(A^2 - \rho^2 \circ u)^2}. \tag{8}$$

*Then we have*

$$(\Delta + \partial_\tau)w - 2 \frac{g(\nabla w, \nabla(\rho^2 \circ u))}{A^2 - \rho^2 \circ u} \geq 4(A^2 - \rho^2 \circ u)w^2$$

$$+ \frac{2}{(A^2 - \rho^2 \circ u)^2} \sum_{i=1}^{m} \mathfrak{g}(du(\mathcal{R}(e_i)), du(e_i)).$$

**Proof** By straightforward computations,

$$\nabla w = \frac{\nabla \|du\|^2}{(A^2 - \rho^2 \circ u)^2} + 2 \frac{\|du\|^2 \nabla(\rho^2 \circ u)}{(A^2 - \rho^2 \circ u)^3},$$

$$\Delta w = \frac{\Delta \|du\|^2}{(A^2 - \rho^2 \circ u)^2} + \frac{4g(\nabla \|du\|^2, \nabla(\rho^2 \circ u))}{(A^2 - \rho^2 \circ u)^3} + \frac{2\|du\|^2 \Delta(\rho^2 \circ u)}{(A^2 - \rho^2 \circ u)^3}$$

$$+ \frac{6\|\nabla(\rho^2 \circ u)\|^2 \|du\|^2}{(A^2 - \rho^2 \circ u)^4}$$

$$= \frac{2g(\nabla w, \nabla(\rho^2 \circ u))}{A^2 - \rho^2 \circ u} + \frac{2\|du\|^2 \Delta(\rho^2 \circ u)}{(A^2 - \rho^2 \circ u)^3} + \frac{\Delta \|du\|^2}{(A^2 - \rho^2 \circ u)^2}$$

$$+ \frac{2g(\nabla \|du\|^2, \nabla(\rho^2 \circ u))}{(A^2 - \rho^2 \circ u)^3} + \frac{2\|\nabla(\rho^2 \circ u)\|^2 \|du\|^2}{(A^2 - \rho^2 \circ u)^4},$$

$$\partial_\tau w = \frac{\partial_\tau \|du\|^2}{(A^2 - \rho^2 \circ u)^2} + \frac{2\|du\|^2 \partial_\tau(\rho^2 \circ u)}{(A^2 - \rho^2 \circ u)^3}.$$

It follows that

$$(\Delta + \partial_\tau)w - \frac{2g(\nabla w, \nabla(\rho^2 \circ u))}{A^2 - \rho^2 \circ u} = \frac{2\|du\|^2 (\Delta + \partial_\tau)(\rho^2 \circ u)}{(A^2 - \rho^2 \circ u)^3} + \frac{(\Delta + \partial_\tau)\|du\|^2}{(A^2 - \rho^2 \circ u)^2}$$
$$+ \frac{2g(\nabla \|du\|^2, \nabla(\rho^2 \circ u))}{(A^2 - \rho^2 \circ u)^3} + \frac{2\|\nabla(\rho^2 \circ u)\|^2 \|du\|^2}{(A^2 - \rho^2 \circ u)^4}.$$

Since $(N, \mathfrak{g})$ is simply connected and sec $\leq 0$, the Greene-Wu Hessian comparison yields the following (see e.g., [1, (1.263)], and also [1, (1.181)]):

$$(\Delta + \partial_\tau)(\rho^2 \circ u) = \sum_{i=1}^m \nabla^2 \rho^2(du(e_i), du(e_i)) \geq 2\|du\|^2,$$

where we also used the backward harmonic map heat flow equation (1). Furthermore, in view of Lemma 4.2 and sec $\leq 0$,

$$(\Delta + \partial_\tau)\|du\|^2 \geq 2\|\nabla du\|^2 + 2\sum_{i=1}^m \mathfrak{g}(du(\mathcal{R}(e_i)), du(e_i)).$$

Combining the above estimates, we see

$$(\Delta + \partial_\tau)w - \frac{2g(\nabla w, \nabla(\rho^2 \circ u))}{A^2 - \rho^2 \circ u} \geq 4(A^2 - \rho^2 \circ u)w^2$$
$$+ \frac{2}{(A^2 - \rho^2 \circ u)^2} \sum_{i=1}^m \mathfrak{g}(du(\mathcal{R}(e_i)), du(e_i)) + 2\mathcal{F},$$

where

$$\mathcal{F} := \frac{\|\nabla du\|^2}{(A^2 - \rho^2 \circ u)^2} + \frac{\|\nabla(\rho^2 \circ u)\|^2 \|du\|^2}{(A^2 - \rho^2 \circ u)^4} + \frac{g(\nabla\|du\|^2, \nabla(\rho^2 \circ u))}{(A^2 - \rho^2 \circ u)^3}.$$

Now, it suffices to check that $\mathcal{F}$ is non-negative. For the first two terms, the inequality of arithmetic-geometric means, and the Kato inequality imply

$$\frac{\|\nabla du\|^2}{(A^2 - \rho^2 \circ u)^2} + \frac{\|\nabla(\rho^2 \circ u)\|^2 \|du\|^2}{(A^2 - \rho^2 \circ u)^4} \geq \frac{2\|\nabla du\| \|\nabla(\rho^2 \circ u)\| \|du\|}{(A^2 - \rho^2 \circ u)^3}$$
$$\geq \frac{\|\nabla\|du\|^2\| \|\nabla(\rho^2 \circ u)\|}{(A^2 - \rho^2 \circ u)^3}.$$

The Cauchy–Schwarz inequality tells us the desired conclusion.                    □

## 4.2 Cut-off arguments

Let us recall the following elementary fact:

**Lemma 4.4** *Let $R, T > 0$, $\alpha \in (0, 1)$. Then there is a smooth function $\psi : [0, \infty) \times [0, \infty) \to [0, 1]$ which is supported on $[0, R] \times [0, T]$, and a constant $C_\alpha > 0$ depending only on $\alpha$ such that the following hold:*

(1) $\psi \equiv 1$ *on* $[0, R/2] \times [0, T/4]$;
(2) $\partial_r \psi \leq 0$ *on* $[0, \infty) \times [0, \infty)$, *and* $\partial_r \psi \equiv 0$ *on* $[0, R/2] \times [0, \infty)$;

(3)  *we have*

$$\frac{|\partial_r \psi|}{\psi^\alpha} \leq \frac{C_\alpha}{R}, \quad \frac{|\partial_r^2 \psi|}{\psi^\alpha} \leq \frac{C_\alpha}{R^2}, \quad \frac{|\partial_\tau \psi|}{\psi^{1/2}} \leq \frac{C}{T},$$

*where $C > 0$ is a universal constant.*

We deduce the following:

**Proposition 4.5** *Let $K \geq 0$. We assume*

$$\mathcal{R}(V) \geq -K\|V\|^2, \quad \mathcal{D}(V) \geq -2K\left(H + \|V\|^2\right), \quad \mathcal{H}(V) \geq -\frac{H}{\tau}, \quad H \geq 0$$

*for all vector fields $V$. Let $(N, \mathfrak{g})$ be simply connected, and $\sec \leq 0$. Let $u : M \times [0, \infty) \to N$ be a solution to backward harmonic map heat flow. For $R, T > 0$ and $A > 0$, we suppose $2\rho \circ u \leq A$ on $Q_{R,T}$. We define $w$ as (8) on $Q_{R,T}$. We also take a function $\psi : [0, \infty) \times [0, \infty) \to [0, 1]$ in Lemma 4.4 with $\alpha = 3/4$, and set*

$$\psi(x, \tau) := \psi(\mathfrak{d}(x, \tau), \tau). \tag{9}$$

*Then we have*

$$(\psi w)^2 \leq \frac{1}{A^4}\left(\frac{\overline{C}_m}{R^4} + \frac{\widetilde{C}_1}{T^2} + \widetilde{C}_2 K^2\right) + \frac{1}{A^2}\Phi.$$

*at every point in $Q_{R,T}$ such that the reduced distance is smooth, where for the universal constants $C_{3/4}, C > 0$ given in Lemma 4.4, we put*

$$\overline{C}_m := 6C_{3/4}^2\left(m^2 + \frac{9}{4} + \frac{369}{32}C_{3/4}^2\right), \quad \widetilde{C}_1 := \frac{3}{2}C^2, \quad \widetilde{C}_2 := 6\left(1 + \frac{C_{3/4}^2}{4}\right), \tag{10}$$

$$\Phi := (\Delta + \partial_\tau)(\psi w) - \frac{2g(\nabla\psi, \nabla(\psi w))}{\psi} - \frac{2g(\nabla(\psi w), \nabla(\rho^2 \circ u))}{A^2 - \rho^2 \circ u}. \tag{11}$$

***Proof*** In virtue of Lemma 4.3,

$$\Phi = \psi(\Delta + \partial_\tau)w - \frac{2\psi g(\nabla w, \nabla(\rho^2 \circ u))}{A^2 - \rho^2 \circ u} + w(\Delta + \partial_\tau)\psi - \frac{2w\|\nabla\psi\|^2}{\psi}$$

$$- \frac{2wg(\nabla\psi, \nabla(\rho^2 \circ u))}{A^2 - \rho^2 \circ u}$$

$$\geq 4(A^2 - \rho^2 \circ u)\psi w^2 + \frac{2\psi}{(A^2 - \rho^2 \circ u)^2}\sum_{i=1}^{m} \mathfrak{g}(du(\mathcal{R}(e_i)), du(e_i))$$

$$+ w(\Delta + \partial_\tau)\psi - \frac{2w\|\nabla\psi\|^2}{\psi}$$

$$- 2\frac{wg(\nabla\psi, \nabla(\rho^2 \circ u))}{A^2 - \rho^2 \circ u}.$$

It follows that

$$4(A^2 - \rho^2 \circ u)\psi w^2 \leq \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 + \Phi \tag{12}$$

for

$$\Psi_1 := -\frac{2\psi}{(A^2 - \rho^2 \circ u)^2}\sum_{i=1}^{m} \mathfrak{g}(du(\mathcal{R}(e_i)), du(e_i)), \quad \Psi_2 := -w(\Delta + \partial_\tau)\psi,$$

$$\Psi_3 := \frac{2w\|\nabla\psi\|^2}{\psi}, \quad \Psi_4 := \frac{2wg(\nabla\psi, \nabla(\rho^2 \circ u))}{A^2 - \rho^2 \circ u}.$$

We provide upper bounds of $\Psi_1, \Psi_2, \Psi_3, \Psi_4$. The following Young inequality plays a crucial role: For all $p, q \in (1, \infty)$ with $p^{-1} + q^{-1} = 1$, $a, b \geq 0$, and $\varepsilon > 0$,

$$ab \leq \frac{\varepsilon a^p}{p} + \frac{b^q}{\varepsilon^{q/p} q}. \tag{13}$$

The inequality

$$\frac{\|\nabla\psi\|^2}{\psi^{3/2}} \leq \frac{3C_{3/4}^2}{R^2} \tag{14}$$

is also useful, which follows from Lemmas 3.2 and 4.4. We first study an upper bound of $\Psi_1$. By the assumption for $\mathcal{R}(V)$, the Young inequality (13) with $p, q = 2$, and $\psi \leq 1$,

$$\Psi_1 = -\frac{2\psi}{(A^2 - \rho^2 \circ u)^2} \sum_{i=1}^m g(du(\mathcal{R}(e_i)), du(e_i)) \leq 2K\psi w \leq \varepsilon\psi^2 w^2 + \frac{K^2}{\varepsilon} \leq \varepsilon\psi w^2 + \frac{K^2}{\varepsilon}. \tag{15}$$

We next produce an upper bound of $\Psi_2$. We see

$$\Psi_2 = -w(\Delta + \partial_\tau)\psi = -w\left(\partial_r\psi(\Delta + \partial_\tau)\mathfrak{d} + \partial_r^2\psi\|\nabla\mathfrak{d}\|^2 + \partial_\tau\psi\right)$$

$$= -w\left[\partial_r\psi\left(\frac{1}{2\mathfrak{d}}(\Delta + \partial_\tau)\overline{L} - \frac{\|\nabla\overline{L}\|^2}{4\mathfrak{d}^3}\right) + \partial_r^2\psi\|\nabla\mathfrak{d}\|^2 + \partial_\tau\psi\right]$$

$$= \frac{w|\partial_r\psi|}{2\mathfrak{d}}(\Delta + \partial_\tau)\overline{L} - w|\partial_r\psi|\frac{\|\nabla\overline{L}\|^2}{4\mathfrak{d}^3} - w\,\partial_r^2\psi\|\nabla\mathfrak{d}\|^2 - w\,\partial_\tau\psi$$

$$\leq \frac{w|\partial_r\psi|}{2\mathfrak{d}}(\Delta + \partial_\tau)\overline{L} + w|\partial_r^2\psi|\|\nabla\mathfrak{d}\|^2 + w\,|\partial_\tau\psi|.$$

Lemmas 3.1, 3.2 and $\overline{L} = \mathfrak{d}^2$ yield

$$\Psi_2 \leq m\frac{w|\partial_r\psi|}{\mathfrak{d}} + Kw|\partial_r\psi|\mathfrak{d} + 3w|\partial_r^2\psi| + w\,|\partial_\tau\psi|$$

$$\leq \frac{2m}{R}w|\partial_r\psi| + KRw|\partial_r\psi| + 3w|\partial_r^2\psi| + w\,|\partial_\tau\psi|,$$

where in the second inequality, we used the fact that $\partial_r\psi$ vanishes on $[0, R/2] \times [0, \infty)$. From the Young inequality (13) with $p, q = 2$, Lemma 4.4, and $\psi \leq 1$, we derive

$$\Psi_2 \leq \left(\varepsilon\psi w^2 + \frac{m^2}{R^2}\frac{|\partial_r\psi|^2}{\varepsilon\psi}\right) + \left(\varepsilon\psi w^2 + \frac{K^2 R^2}{4}\frac{|\partial_r\psi|^2}{\varepsilon\psi}\right)$$

$$+ \left(\varepsilon\psi w^2 + \frac{9}{4}\frac{|\partial_r^2\psi|^2}{\varepsilon\psi}\right) + \left(\varepsilon\psi w^2 + \frac{1}{4}\frac{|\partial_\tau\psi|^2}{\varepsilon\psi}\right)$$

$$\leq 4\varepsilon\psi w^2 + \frac{C_{3/4}^2}{\varepsilon}\left(m^2 + \frac{9}{4}\right)\frac{\psi^{1/2}}{R^4} + \frac{C^2}{4\varepsilon}\frac{1}{T^2} + \frac{C_{3/4}^2}{4\varepsilon}K^2\psi^{1/2}$$

$$\leq 4\varepsilon\psi w^2 + \frac{C_{3/4}^2}{\varepsilon}\left(m^2 + \frac{9}{4}\right)\frac{1}{R^4} + \frac{C^2}{4\varepsilon}\frac{1}{T^2} + \frac{C_{3/4}^2}{4\varepsilon}K^2. \tag{16}$$

We give an upper bound of $\Psi_3$. By the Young inequality (13) with $p, q = 2$, and (14),

$$\Psi_3 = \frac{2w\|\nabla\psi\|^2}{\psi} \leq \varepsilon\psi w^2 + \frac{\|\nabla\psi\|^4}{\varepsilon\psi^3} \leq \varepsilon\psi w^2 + \frac{9C_{3/4}^4}{\varepsilon}\frac{1}{R^4}. \tag{17}$$

We finally examine $\Psi_4$. The Cauchy–Schwarz inequality, the Young inequality (13) with $p = 4/3$, $q = 4$, $\varepsilon = 4/3$, and (14) lead us to

$$\Psi_4 = 2\frac{wg(\nabla\psi, \nabla(\rho^2 \circ u))}{A^2 - \rho^2 \circ u} \leq \frac{2w\|\nabla\psi\|\|\nabla(\rho^2 \circ u)\|}{A^2 - \rho^2 \circ u} \leq 2Aw^{3/2}\|\nabla\psi\|$$

$$\leq A^2\psi w^2 + \frac{27}{16}\frac{1}{A^2}\frac{\|\nabla\psi\|^4}{\psi^3} \leq A^2\psi w^2 + \frac{243C_{3/4}^4}{16}\frac{1}{A^2}\frac{1}{R^4}. \tag{18}$$

By summarizing (12), (15), (16), (17), (18),

$$3A^2\psi w^2 \leq 4(A^2 - \rho^2 \circ u)\psi w^2$$

$$\leq (6\varepsilon + A^2)\psi w^2 + \frac{C_{3/4}^2}{\varepsilon}\left(m^2 + \frac{9}{4} + 9C_{3/4}^2 + \frac{243\varepsilon C_{3/4}^2}{16}\frac{1}{A^2}\right)\frac{1}{R^4}$$

$$+ \frac{C^2}{4\varepsilon}\frac{1}{T^2} + \frac{1}{\varepsilon}\left(1 + \frac{C_{3/4}^2}{4}\right)K^2 + \Phi.$$

Letting $\varepsilon \to A^2/6$, we have

$$\psi w^2 \leq \frac{1}{A^4}\left(\frac{\overline{C}_m}{R^4} + \frac{\widetilde{C}_1}{T^2} + \widetilde{C}_2 K^2\right) + \frac{1}{A^2}\Phi.$$

Since $(\psi w)^2 \leq \psi w^2$, we arrive at the desired inequality. □

### 4.3 Proof of Theorems 2.1 and 4.1

Let us conclude Theorem 4.1.

***Proof of Theorem 4.1*** For $K \geq 0$, let $(M, g(\tau))_{\tau \in [0,\infty)}$ be backward $(-K)$-super Ricci flow satisfying (7) for all vector fields $V$. Let $(N, \mathfrak{g})$ be simply connected, and sec $\leq 0$. Let $u : M \times [0, \infty) \to N$ be a solution to backward harmonic map heat flow. For $R, T > 0$ and $A > 0$, we suppose $2\rho \circ u \leq A$ on $Q_{R,T}$. We define functions $w$ and $\psi$ as (8) and (9), respectively. For $\theta > 0$ we define a compact subset $Q_{R,T,\theta}$ of $Q_{R,T}$ by

$$Q_{R,T,\theta} := \{(x, \tau) \in Q_{R,T} \mid \tau \in [\theta, T]\}. \tag{19}$$

Fix a small $\theta \in (0, T/4)$, and take a maximum point $(\overline{x}, \overline{\tau})$ of $\psi w$ in $Q_{R,T,\theta}$. By virtue of the Calabi argument, we may assume that the reduced distance is smooth at $(\overline{x}, \overline{\tau})$ (cf. [22, Remark 3.3]). Using Proposition 4.5, we see

$$(\psi w)^2 \leq \frac{c_m}{A^4}\left(\frac{1}{R^4} + \frac{1}{T^2} + K^2\right) + \frac{1}{A^2}\Phi$$

at $(\overline{x}, \overline{\tau})$ for

$$c_m := \max\left\{\overline{C}_m, \widetilde{C}_1, \widetilde{C}_2\right\},$$

where $\overline{C}_m, \widetilde{C}_1, \widetilde{C}_2 > 0$ and $\Phi$ are defined as (10) and (11), respectively. On the other hand, since $(\overline{x}, \overline{\tau})$ is a maximum point,

$$\Delta(\psi w) \leq 0, \quad \partial_\tau(\psi w) \leq 0, \quad \nabla(\psi w) = 0$$

at $(\overline{x}, \overline{\tau})$; in particular, $\Phi(\overline{x}, \overline{\tau}) \leq 0$. Therefore,

$$(\psi w)(x, \tau) \leq (\psi w)(\overline{x}, \overline{\tau}) \leq \frac{c_m^{1/2}}{A^2} \left( \frac{1}{R^4} + \frac{1}{T^2} + K^2 \right)^{1/2} \leq \frac{c_m^{1/2}}{A^2} \left( \frac{1}{R^2} + \frac{1}{T} + K \right)$$

for all $(x, \tau) \in Q_{R,T,\theta}$. By $\psi \equiv 1$ on $Q_{R/2,T/4,\theta}$, and by the definition of $w$,

$$\frac{\|du\|}{A^2 - \rho^2 \circ u} \leq \frac{c_m^{1/4}}{A} \left( \frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{K} \right)$$

on $Q_{R/2,T/4,\theta}$. Letting $\theta \to 0$, we complete the proof of Theorem 4.1. $\qquad\square$

We are now in a position to show Theorem 2.1.

**Proof of Theorem 2.1** Let $(M, g(\tau))_{\tau \in [0,\infty)}$ be backward super Ricci flow satisfying (2) for all vector fields $V$. Let $(N, \mathfrak{g})$ be simply connected, and sec $\leq 0$. Let $u : M \times [0, \infty) \to N$ be a solution to backward harmonic map heat flow. For $R > 0$ we put

$$A_R := \sup_{Q_{R,R^2}} 2\rho \circ u.$$

In view of the growth condition, $A_R = o(R)$ as $R \to \infty$. For a fixed $(x, \tau) \in M \times (0, \infty)$, we possess $(x, \tau) \in Q_{R/2,R^2/4}$ for every sufficiently large $R > 0$, and fix such one. From Theorem 4.1 with $K = 0$, we derive

$$\frac{\|du\|}{A_R^2} \leq \frac{\|du\|}{A_R^2 - \rho^2 \circ u} \leq \frac{2C_m}{A_R R}$$

at $(x, \tau)$. Letting $R \to \infty$, we complete the proof of Theorem 2.1. $\qquad\square$

One can derive the following result from the Hamilton's trace Harnack inequality (see [15, Corollary 1.2], and cf. [22, Corollary 2.5]):

**Corollary 4.6** *Let $(M, g(\tau))_{\tau \in [0,\infty)}$ be a complete backward Ricci flow with bounded, non-negative curvature operator. Let $(N, \mathfrak{g})$ be a complete, simply connected Riemannian manifold with* sec $\leq 0$. *Let $u : M \times [0, \infty) \to N$ be a solution to backward harmonic map heat flow. If*

$$\rho(u(x, \tau)) = o\left( \mathfrak{d}(x, \tau) + \sqrt{\tau} \right)$$

*near infinity, then $u$ is constant.*

## 5 Proof of Theorem 2.2

We next prove Theorem 2.2. The key is the following:

**Theorem 5.1** *For $K \geq 0$, let $(M, g(\tau))_{\tau \in [0,\infty)}$ be an $m$-dimensional, admissible, complete backward $(-K)$-super Ricci flow. We assume*

$$\mathcal{D}(V) \geq -2K \left( H + \|V\|^2 \right), \quad \mathcal{H}(V) \geq -\frac{H}{\tau}, \quad H \geq 0 \tag{20}$$

*for all vector fields $V$. Let $(N, \mathfrak{g})$ denote a complete Riemannian manifold with* sec $\leq \kappa$ *for $\kappa > 0$. Assume that $B_{\pi/2\sqrt{\kappa}}(y_0)$ does not meet* Cut $(y_0)$. *Let $u : M \times [0, \infty) \to N$ be a*

*solution to backward harmonic map heat flow. Suppose that the image of u is contained in* $B_{\pi/2\sqrt{\kappa}}(y_0)$. *For R, T > 0, let*

$$\varphi := 1 - \cos \sqrt{\kappa}\rho, \quad A := \frac{1}{2}\left(1 + \sup_{Q_{R,T}} \varphi \circ u\right). \tag{21}$$

*Then there is a positive constant* $C_m > 0$ *depending only on m such that on* $Q_{R/2,T/4}$,

$$\frac{\|du\|}{A - \varphi \circ u} \leq \frac{C_m}{\sqrt{\kappa}}\left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{K}\right)\sup_{Q_{R,T}}\left(\frac{1}{\cos \sqrt{\kappa}\rho \circ u}\right)^2.$$

Unlike Theorem 4.1, this estimate seems to be new even in the context of Liouville theorems for harmonic maps (see Remark 2.5).

## 5.1 Backward harmonic map heat flows

Let us show the following:

**Lemma 5.2** *Let* $(N, \mathfrak{g})$ *be* sec $\leq \kappa$ *for* $\kappa > 0$. *Assume that* $B_{\pi/2\sqrt{\kappa}}(y_0)$ *does not meet* Cut $(y_0)$. *Let* $u : M \times [0, \infty) \to N$ *be a solution to backward harmonic map heat flow. Suppose that the image of u is contained in* $B_{\pi/2\sqrt{\kappa}}(y_0)$. *For R, T > 0, we define* $\varphi$ *and A as* (21). *Set*

$$w := \frac{\|du\|^2}{(A - \varphi \circ u)^2}. \tag{22}$$

*Then we have*

$$(\Delta + \partial_\tau)w - 2\frac{g(\nabla w, \nabla(\varphi \circ u))}{A - \varphi \circ u} \geq 2\kappa(1 - A)(A - \varphi \circ u)w^2$$

$$+ \frac{2}{(A - \varphi \circ u)^2}\sum_{i=1}^{m} \mathfrak{g}(du(\mathcal{R}(e_i)), du(e_i)).$$

**Proof** By similar computations to the proof of Lemma 4.3, we see

$$(\Delta + \partial_\tau)w = \frac{2g(\nabla w, \nabla(\varphi \circ u))}{A - \varphi \circ u} + \frac{2\|du\|^2 (\Delta + \partial_\tau)(\varphi \circ u)}{(A - \varphi \circ u)^3} + \frac{(\Delta + \partial_\tau)\|du\|^2}{(A - \varphi \circ u)^2}$$

$$+ \frac{2g(\nabla\|du\|^2, \nabla(\varphi \circ u))}{(A - \varphi \circ u)^3} + \frac{2\|\nabla(\varphi \circ u)\|^2 \|du\|^2}{(A - \varphi \circ u)^4}.$$

Due to the Hessian comparison,

$$(\Delta + \partial_\tau)(\varphi \circ u) = \sum_{i=1}^{m} \nabla^2\rho^2(du(e_i), du(e_i)) \geq \kappa \cos \sqrt{\kappa}\rho \circ u\|du\|^2.$$

Furthermore, Lemma 4.2 and sec $\leq \kappa$ lead us to

$$(\Delta + \partial_\tau)\|du\|^2 \geq 2\|\nabla du\|^2 + 2\sum_{i=1}^{m} \mathfrak{g}(du(\mathcal{R}(e_i)), du(e_i)) - 2\kappa\|du\|^4.$$

It holds that

$$(\Delta + \partial_\tau)w - \frac{2g(\nabla w, \nabla(\varphi \circ u))}{A - \varphi \circ u} \geq 2\kappa(A - \varphi \circ u)^2\left(\frac{\cos \sqrt{\kappa}\rho}{A - \varphi \circ u} - 1\right)w^2$$

$$+ \frac{2}{(A - \varphi \circ u)^2} \sum_{i=1}^{m} \mathfrak{g}(du(\mathcal{R}(e_i)), du(e_i)) + 2\mathcal{F}$$

$$= 2\kappa(1 - A)(A - \varphi \circ u)w^2$$

$$+ \frac{2}{(A - \varphi \circ u)^2} \sum_{i=1}^{m} \mathfrak{g}(du(\mathcal{R}(e_i)), du(e_i)) + 2\mathcal{F},$$

where

$$\mathcal{F} := \frac{\|\nabla du\|^2}{(A - \varphi \circ u)^2} + \frac{\|\nabla(\varphi \circ u)\|^2 \|du\|^2}{(A - \varphi \circ u)^4} + \frac{g(\nabla \|du\|^2, \nabla(\varphi \circ u))}{(A - \varphi \circ u)^3}.$$

By similar computations to the proof of Lemma 4.3, $\mathcal{F}$ is non-negative. □

## 5.2 Cut-off arguments

We have the following:

**Proposition 5.3** *Let $K \geq 0$. We assume*

$$\mathcal{R}(V) \geq -K\|V\|^2, \quad \mathcal{D}(V) \geq -2K\left(H + \|V\|^2\right), \quad \mathcal{H}(V) \geq -\frac{H}{\tau}, \quad H \geq 0$$

*for all vector fields $V$. Let $(N, \mathfrak{g})$ be sec $\leq \kappa$ for $\kappa > 0$. Assume that $B_{\pi/2\sqrt{\kappa}}(y_0)$ does not meet* Cut $(y_0)$. *Let $u : M \times [0, \infty) \to N$ be a solution to backward harmonic map heat flow. Suppose that the image of $u$ is contained in $B_{\pi/2\sqrt{\kappa}}(y_0)$. For $R, T > 0$, we define $\varphi$ and $A$ as (21). Furthermore, we define $w$ as (22). We also take a function $\psi : [0, \infty) \times [0, \infty) \to [0, 1]$ in Lemma 4.4 with $\alpha = 3/4$, and define $\psi$ as (9). Then for any $\varepsilon > 0$, we have*

$$2\kappa(1 - A)(A - \varphi \circ u)\psi w^2 \leq \frac{27\varepsilon}{4}\psi w^2 + \frac{C_{3/4}^2}{\varepsilon}\left(m^2 + \frac{9}{4} + 9C_{3/4}^2 + \frac{36\kappa^2 C_{3/4}^2}{\varepsilon^2}\right)\frac{1}{R^4}$$

$$+ \frac{C^2}{4\varepsilon}\frac{1}{T^2} + \frac{1}{\varepsilon}\left(1 + \frac{C_{3/4}^2}{4}\right)K^2 + \Phi$$

*at every point in $Q_{R,T}$ such that the reduced distance is smooth, where the universal constants $C_{3/4}, C > 0$ are given in Lemma 4.4, and put*

$$\Phi := (\Delta + \partial_\tau)(\psi w) - \frac{2g(\nabla \psi, \nabla(\psi w))}{\psi} - \frac{2g(\nabla(\psi w), \nabla(\varphi \circ u))}{A - \varphi \circ u}. \tag{23}$$

**Proof** Using Lemma 5.2, we see

$$\Phi = \psi(\Delta + \partial_\tau)w - \frac{2\psi g(\nabla w, \nabla(\varphi \circ u))}{A - \varphi \circ u} + w(\Delta + \partial_\tau)\psi$$

$$- \frac{2w\|\nabla \psi\|^2}{\psi} - \frac{2wg(\nabla \psi, \nabla(\varphi \circ u))}{A - \varphi \circ u}$$

$$\geq 2\kappa(1 - A)(A - \varphi \circ u)\psi w^2 + \frac{2\psi}{(A - \varphi \circ u)^2} \sum_{i=1}^{m} \mathfrak{g}(du(\mathcal{R}(e_i)), du(e_i))$$

$$+ w(\Delta + \partial_\tau)\psi$$

$$- \frac{2w\|\nabla \psi\|^2}{\psi} - 2\frac{wg(\nabla \psi, \nabla(\varphi \circ u))}{A - \varphi \circ u}.$$

We obtain

$$2\kappa(1-A)(A-\varphi \circ u)\psi w^2 \leq \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 + \Phi$$

for

$$\Psi_1 := -\frac{2\psi}{(A-\varphi \circ u)^2}\sum_{i=1}^{m}\mathfrak{g}(du(\mathcal{R}(e_i)), du(e_i)), \quad \Psi_2 := -w\left(\Delta + \partial_\tau\right)\psi,$$

$$\Psi_3 := \frac{2w\|\nabla\psi\|^2}{\psi}, \quad \Psi_4 := \frac{2wg(\nabla\psi, \nabla(\varphi \circ u))}{A-\varphi \circ u}.$$

For $\Psi_1$, the following holds:

$$\Psi_1 = -\frac{2\psi}{(A-\varphi \circ u)^2}\sum_{i=1}^{m}\mathfrak{g}(du(\mathcal{R}(e_i)), du(e_i)) \leq 2K\psi w \leq \varepsilon\psi^2 w^2 + \frac{K^2}{\varepsilon} \leq \varepsilon\psi w^2 + \frac{K^2}{\varepsilon}$$

in the same manner as in the proof of Proposition 4.5. For $\Psi_2$, $\Psi_3$, we possess the same upper estimates as in the proof of Proposition 4.5. For $\Psi_4$, the following holds:

$$\Psi_4 = \frac{2wg(\nabla\psi, \nabla(\varphi \circ u))}{A-\varphi \circ u} \leq \frac{2w\|\nabla\psi\|\|\nabla(\varphi \circ u)\|}{A-\varphi \circ u} \leq 2\sqrt{\kappa}w^{3/2}\|\nabla\psi\|$$

$$\leq \frac{3\varepsilon}{4}\psi w^2 + \frac{4\kappa^2}{\varepsilon^3}\frac{\|\nabla\psi\|^4}{\psi^3} \leq \frac{3\varepsilon}{4}\psi w^2 + \frac{36\kappa^2 C_{3/4}^4}{\varepsilon^3}\frac{1}{R^4}.$$

This proves the desired estimate. $\square$

### 5.3 Proof of Theorems 2.2 and 5.1

We are now in a position to prove Theorem 5.1.

***Proof of Theorem 5.1*** For $K \geq 0$, let $(M, g(\tau))_{\tau \in [0,\infty)}$ be backward $(-K)$-super Ricci flow satisfying (20) for all vector fields $V$. Let $(N, \mathfrak{g})$ be sec $\leq \kappa$ for $\kappa > 0$. Assume that $B_{\pi/2\sqrt{\kappa}}(y_0)$ does not meet Cut $(y_0)$. Let $u : M \times [0, \infty) \to N$ be a solution to backward harmonic map heat flow. Suppose that the image of $u$ is contained in $B_{\pi/2\sqrt{\kappa}}(y_0)$. For $R, T > 0$, we define $\varphi$ and $A$ as (21). Furthermore, we define $w$ as (22). Also, we define $\psi$ as in Proposition 5.3. For $\theta > 0$ we define $Q_{R,T,\theta}$ as (19). For a fixed small $\theta \in (0, T/4)$, we take a maximum point $(\overline{x}, \overline{\tau})$ of $\psi w$ in $Q_{R,T,\theta}$. We may assume that the reduced distance is smooth at $(\overline{x}, \overline{\tau})$.

We set

$$\delta := (1-A)(A - \varphi(u(\overline{x}, \overline{\tau}))).$$

Notice that

$$\frac{1}{1-A} = 2\sup_{Q_{R,T}}\frac{1}{\cos\sqrt{\kappa}\rho \circ u}, \quad \frac{1}{A - \varphi(u(\overline{x}, \overline{\tau}))} \leq \frac{2}{1 - \sup_{Q_{R,T}}\varphi \circ u}$$

$$= 2\sup_{Q_{R,T}}\frac{1}{\cos\sqrt{\kappa}\rho \circ u};$$

in particular,

$$\frac{1}{\delta} \leq 4\sup_{Q_{R,T}}\left(\frac{1}{\cos\sqrt{\kappa}\rho \circ u}\right)^2.$$

Letting $\varepsilon \to 4\kappa\delta/27$ in Proposition 5.3, and $\Phi(\overline{x}, \overline{\tau}) \leq 0$ tell us that

$$\kappa\delta\psi w^2 \leq \frac{27C_{3/4}^2}{4\kappa\delta}\left(m^2 + \frac{9}{4} + 9C_{3/4}^2 + \frac{6561C_{3/4}^2}{4\delta^2}\right)\frac{1}{R^4} + \frac{27C^2}{16\kappa\delta}\frac{1}{T^2} + \frac{27}{4\kappa\delta}\left(1 + \frac{C_{3/4}^2}{4}\right)K^2$$

at $(\overline{x}, \overline{\tau})$, where $\Phi$ is defined as (23). It follows that

$$(\psi w)^2(\overline{x}, \overline{\tau}) \leq \frac{27C_{3/4}^2}{4\kappa^2\delta^2}\left(m^2 + \frac{9}{4} + 9C_{3/4}^2 + \frac{6561C_{3/4}^2}{4\delta^2}\right)\frac{1}{R^4}$$
$$+ \frac{27C^2}{16\kappa^2\delta^2}\frac{1}{T^2} + \frac{27}{4\kappa^2\delta^2}\left(1 + \frac{C_{3/4}^2}{4}\right)K^2.$$

Since $\delta \in (0, 1)$, there is a positive constant $\overline{c}_m > 0$ depending only on $m$ such that

$$(\psi w)^2(\overline{x}, \overline{\tau}) \leq \frac{\overline{c}_m}{\kappa^2\delta^4}\left(\frac{1}{R^4} + \frac{1}{T^2} + K^2\right).$$

Thus,

$$(\psi w)(x, \tau) \leq (\psi w)(\overline{x}, \overline{\tau}) \leq \frac{\overline{c}_m^{1/2}}{\kappa\delta^2}\left(\frac{1}{R^2} + \frac{1}{T} + K\right)$$

for all $(x, \tau) \in Q_{R,T,\theta}$. By $\psi \equiv 1$ on $Q_{R/2,T/4,\theta}$,

$$\frac{\|du\|}{A - \varphi \circ u} \leq \frac{\overline{c}_m^{1/4}}{\sqrt{\kappa}\delta}\left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{K}\right) \leq \frac{4\overline{c}_m^{1/4}}{\sqrt{\kappa}}\left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{K}\right)\sup_{Q_{R,T}}\left(\frac{1}{\cos\sqrt{\kappa}\rho \circ u}\right)^2$$

on $Q_{R/2,T/4,\theta}$. Letting $\theta \to 0$, we complete the proof of Theorem 5.1. □

Let us conclude Theorem 2.2.

**Proof of Theorem 2.2** Let $(M, g(\tau))_{\tau \in [0,\infty)}$ be backward super Ricci flow satisfying (3) for all vector fields $V$. Let $(N, \mathfrak{g})$ be sec $\leq \kappa$ for $\kappa > 0$. Assume that $B_{\pi/2\sqrt{\kappa}}(y_0)$ does not meet Cut $(y_0)$. Let $u : M \times [0, \infty) \to N$ be a solution to backward harmonic map heat flow. Suppose that the image of $u$ is contained in $B_{\pi/2\sqrt{\kappa}}(y_0)$. For $R > 0$ we put

$$\mathcal{A}_R := \sup_{Q_{R,R^2}}\left(\frac{1}{\cos\sqrt{\kappa}\rho \circ u}\right)^2.$$

The growth condition says that $\mathcal{A}_R = o(R)$ as $R \to \infty$. We fix $(x, \tau) \in M \times (0, \infty)$, and a sufficiently large $R > 0$. Thanks to Theorem 5.1 with $K = 0$,

$$\frac{\|du\|}{A} \leq \frac{\|du\|}{A - \varphi \circ u} \leq \frac{2C_m\mathcal{A}_R}{R}$$

at $(x, \tau)$, where $\varphi$ and $A$ are defined as (21). Notice that $A \leq 1$. Thus, by letting $R \to \infty$, we complete the proof of Theorem 2.2. □

Similarly to Corollary 4.6, we obtain the following:

**Corollary 5.4** *Let $(M, g(\tau))_{\tau \in [0,\infty)}$ be a complete backward Ricci flow with bounded, non-negative curvature operator. Let $(N, \mathfrak{g})$ be a complete Riemannian manifold with sec $\leq \kappa$ for $\kappa > 0$. Assume that $B_{\pi/2\sqrt{\kappa}}(y_0)$ does not meet Cut $(y_0)$. Let $u : M \times [0, \infty) \to N$ be a*

*solution to backward harmonic map heat flow. If the image of u is contained in $B_{\pi/2\sqrt{\kappa}}(y_0)$, and if u satisfies a growth condition*

$$\frac{1}{\cos\sqrt{\kappa}\rho(u(x,\tau))} = o\left(\mathfrak{d}(x,\tau)^{1/2} + \tau^{1/4}\right)$$

*near infinity, then u is constant.*

## 6 Proof of Theorem 2.6 and Schoen–Uhlenbeck's example

Finally, we prove Theorem 2.6 and compare the result with Schoen–Uhlenbeck's example.

### 6.1 Proof of Theorem 2.6

In this subsection, let $(M, g)$ be an $m$-dimensional complete Riemannian manifold of non-negative Ricci curvature, and let $(N, \mathfrak{g})$ be an $n$-dimensional complete Riemannian manifold with sec $\leq \kappa$ for $\kappa > 0$. For harmonic maps, we can use the following refined Kato inequality:

**Lemma 6.1** *For a harmonic map $u : M \to N$, we have*

$$\|\nabla\|du\|\|^2 \leq \frac{m-1}{m}\|\nabla du\|^2. \tag{24}$$

We can find the proof of this inequality for a harmonic map between spheres in the paper by Lin-Wang [32]. However, their computation is pointwise and only uses properties of harmonic maps. Therefore it is also valid for harmonic maps between general Riemannian manifolds. Here, we give a proof for readers' convenience.

*Proof* It is enough to show the inequality at $x_0 \in M$ such that $\|du\|(x_0) \neq 0$. Let us fix such a point. We compute in normal coordinates $(x^i) = (x^1, \ldots, x^m)$ around $x_0 \in M$ and $(y^\alpha) = (y^1, \ldots, y^n)$ around $u(x_0) \in N$. Let $u(x) = (u^1(x^1, \ldots, x^m), \ldots, u^n(x^1, \ldots, x^m))$ be the local expression for $u : M \to N$ in these coordinates. We use the notations

$$u_i^\alpha = \frac{\partial u^\alpha}{\partial x^i}, \quad \text{and} \quad u_{ij}^\alpha = \frac{\partial^2 u^\alpha}{\partial x^i \partial x^j}.$$

Now we can write

$$\|\nabla du\|^2(x_0) = \sum_{\alpha=1}^n \sum_{i,j=1}^m (u_{ij}^\alpha)^2(x_0)$$

at $x_0 \in M$. For any $1 \leq \alpha \leq n$, let $\lambda_1^\alpha, \ldots, \lambda_m^\alpha$ be real eigenvalues of the symmetric matrix $(u_{ij}^\alpha(x_0))$ such that $|\lambda_1^\alpha| \leq \cdots \leq |\lambda_m^\alpha|$. Then we have

$$\|\nabla du\|^2(x_0) = \sum_{\alpha=1}^n \sum_{i=1}^m (\lambda_i^\alpha)^2.$$

On the other hand, since $u : M \to N$ is a harmonic map, we have

$$\sum_{i=1}^m u_{ii}^\alpha(x_0) = \sum_{i=1}^m \lambda_i^\alpha = 0 \quad \text{for all} \quad 1 \leq \alpha \leq n.$$

Using this, elementary computation yields

$$\sum_{i=1}^{m-1}(\lambda_i^\alpha)^2 \geq \frac{1}{m-1}\left(\sum_{i=1}^{m-1}\lambda_i^\alpha\right)^2 = \frac{1}{m-1}(\lambda_m^\alpha)^2 \quad \text{for all} \quad 1 \leq \alpha \leq n.$$

Adding $(\lambda_m^\alpha)^2$ to the both sides of this inequality, we have

$$\|\nabla^2 u^\alpha\|^2(x_0) = \sum_{i=1}^{m}(\lambda_i^\alpha)^2 \geq \frac{m}{m-1}(\lambda_m^\alpha)^2 \quad \text{for all} \quad 1 \leq \alpha \leq n.$$

For a general $m \times m$ symmetric matrix $A$ which has real eigenvalues $\lambda_1, \ldots, \lambda_m$ with $|\lambda_1| \leq \cdots \leq |\lambda_m|$, and a vector $v \in \mathbb{R}^m$, it holds that

$$\|Av\|^2 \leq |\lambda_m|^2\|v\|^2.$$

In our case, for each $1 \leq \alpha \leq n$, put $v = \nabla u^\alpha(x_0)$ and $A = (u_{ij}^\alpha(x_0))$, then we have

$$\|\nabla u^\alpha\|^2(x_0)\|\nabla^2 u^\alpha\|^2(x_0) = \|\nabla u^\alpha\|^2(x_0)\sum_{i=1}^{m}(\lambda_i^\alpha)^2$$

$$\geq \frac{m}{m-1}\|\nabla u^\alpha\|^2(x_0)|\lambda_m^\alpha|^2$$

$$\geq \frac{m}{m-1}\sum_{i=1}^{m}\left(\sum_{j=1}^{m}u_{ij}^\alpha(x_0)u_j^\alpha(x_0)\right)^2.$$

Therefore, using the Cauchy–Schwarz inequality and the Minkowski inequality, we have

$$\|du\|^2(x_0)\|\nabla du\|^2(x_0) = \left(\sum_{\alpha=1}^{n}\|\nabla u^\alpha\|^2(x_0)\right)\left(\sum_{\alpha=1}^{n}\|\nabla^2 u^\alpha\|^2(x_0)\right)$$

$$\geq \left(\sum_{\alpha=1}^{n}\|\nabla u^\alpha\|(x_0)\|\nabla^2 u^\alpha\|(x_0)\right)^2$$

$$\geq \frac{m}{m-1}\left[\sum_{\alpha=1}^{n}\left\{\sum_{i=1}^{m}\left(\sum_{j=1}^{m}u_{ij}^\alpha(x_0)u_j^\alpha(x_0)\right)^2\right\}^{\frac{1}{2}}\right]^2$$

$$\geq \frac{m}{m-1}\sum_{i=1}^{m}\left(\sum_{\alpha=1}^{n}\sum_{j=1}^{m}u_{ij}^\alpha(x_0)u_j^\alpha(x_0)\right)^2.$$

Note that

$$4\|du\|^2\|\nabla\|du\|\|^2 = \|\nabla\|du\|^2\|^2 = 4\sum_{i=1}^{m}\left(\sum_{\alpha=1}^{n}\sum_{j=1}^{m}u_{ij}^\alpha u_j^\alpha\right)^2.$$

Hence we obtain

$$\|du\|^2(x_0)\|\nabla du\|^2(x_0) \geq \frac{m}{m-1}\|du\|^2(x_0)\|\nabla\|du\|\|^2(x_0).$$

This completes the proof of Lemma 6.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now we are in a position to prove Theorem 2.6. We use the technique for minimal hypersurfaces developed by Ecker-Huisken in [10].

***Proof of Theorem 2.6*** The Bochner formula for a harmonic map $u : M \to N$ (i.e., harmonic map version of Lemma 4.2) combined with the refined Kato inequality (24) and curvature assumptions on $M, N$ tells us that

$$\Delta \|du\|^2 \geq -2\kappa \|du\|^4 + 2\|\nabla du\|^2 \geq -2\kappa \|du\|^4 + \frac{2m}{m-1}\|\nabla\|du\|\|^2. \tag{25}$$

Let $v(x) = 1/\cos \sqrt{\kappa} \rho(u(x))$. Note that this is well-defined when $u(M) \subset B_{\pi/2\sqrt{\kappa}}(y_0)$. The Hessian comparison theorem under the curvature assumption on $N$ implies

$$\Delta v = v^2 \Delta(\varphi \circ u) + \frac{2\|\nabla v\|^2}{v} \geq \kappa \|du\|^2 v + \frac{2\|\nabla v\|^2}{v}, \tag{26}$$

where $\varphi = 1 - \cos \sqrt{\kappa} \rho$. Using (25) and (26), a direct computation yields

$$\Delta(\|du\|^p v^q) \geq \kappa(q-p)\|du\|^{p+2} v^q$$
$$+ p\left(p - 1 + \frac{1}{m-1}\right) \|du\|^{p-2} v^q \|\nabla\|du\|\|^2 + q(q+1)\|du\|^p v^{q-2}\|\nabla v\|^2$$
$$+ 2pq\|du\|^{p-1} v^{q-1} g(\nabla\|du\|, \nabla v),$$

where $p, q$ are determined later. Using the Cauchy–Schwarz inequality and the Young inequality (with $\varepsilon > 0$) for the last term, we have

$$\Delta(\|du\|^p v^q) \geq \kappa(q-p)\|du\|^{p+2} v^q \tag{27}$$
$$+ p\left(p - 1 + \frac{1}{m-1} - \varepsilon q\right) \|du\|^{p-2} v^q \|\nabla\|du\|\|^2$$
$$+ q\left(q + 1 - \varepsilon^{-1} p\right) \|du\|^p v^{q-2}\|\nabla v\|^2.$$

For given $m \geq 2$, let $q = p > 2m - 3$ and $\varepsilon = q/(q+1)$. Then we have

$$\Delta(\|du\|^q v^q) \geq 0,$$

i.e., $\|du\|^q v^q$ is a subharmonic function on $M$. Therefore, we can use Li–Schoen's mean value inequality (see e.g., [24, Theorem 7.2]) to conclude

$$\sup_{B_{R/4}(x_0)} \|du\|^{2q} v^{2q} \leq \frac{C_m}{\mathrm{vol}(B_R(x_0))} \int_{B_R(x_0)} \|du\|^{2q} v^{2q}, \tag{28}$$

where $C_m$ is a positive constant depending only on $m$.

On the other hand, choosing $q = p + 1 > 2m - 3$ and $\varepsilon = (q-1)/(q+1)$ in (27), we have

$$\Delta(\|du\|^{q-1} v^q) \geq \kappa \|du\|^{q+1} v^q.$$

We multiply this by $\|du\|^{q-1} v^q \eta^{2q}$, where $\eta$ is a test function on $M$ with compact support, and then integrating by parts with the Cauchy–Schwarz inequality and the Young inequality yields

$$\kappa \int_M \|du\|^{2q} v^{2q} \eta^{2q} \leq \frac{1}{2} \int_M \|du\|^{2q-2} v^{2q} \eta^{2q-2}\|\nabla\eta\|^2.$$

Recall the generalized Young inequality for $a, b \geq 0$ with arbitrary $\varepsilon > 0$:

$$ab \leq \varepsilon \left( \frac{q-1}{q} \right) a^{q/(q-1)} + \frac{\varepsilon^{-(q-1)}}{q} b^q.$$

Putting $a = \|du\|^{2q-2} \eta^{2q-2}$ and $b = \|\nabla \eta\|^2$ in this inequality, we have

$$\left( \kappa - \frac{\varepsilon(q-1)}{2q} \right) \int_M \|du\|^{2q} v^{2q} \eta^{2q} \leq \frac{\varepsilon^{-(q-1)}}{2q} \int_M v^{2q} \|\nabla \eta\|^{2q}.$$

We take $\varepsilon = \kappa q / (q-1)$ to get

$$\int_M \|du\|^{2q} v^{2q} \eta^{2q} \leq \frac{(q-1)^{q-1}}{\kappa^q q^q} \int_M v^{2q} \|\nabla \eta\|^{2q}.$$

Choosing $\eta$ as the standard cut-off function in this inequality, we obtain

$$\int_{B_R(x_0)} \|du\|^{2q} v^{2q} \leq \frac{(q-1)^{q-1}}{\kappa^q q^q R^{2q}} \int_{B_{2R}(x_0)} v^{2q}$$

$$\leq \frac{(q-1)^{q-1}}{\kappa^q q^q R^{2q}} \operatorname{vol}(B_{2R}(x_0)) \sup_{B_{2R}(x_0)} v^{2q}. \tag{29}$$

Combining (28), (29) and the Bishop–Gromov volume comparison, it follows that

$$\sup_{B_{R/4}(x_0)} \|du\| v \leq \left( \frac{C_m (q-1)^{q-1}}{\kappa^q q^q R^{2q}} \right)^{1/2q} \left( \frac{\operatorname{vol}(B_{2R}(x_0))}{\operatorname{vol}(B_R(x_0))} \right)^{1/2q} \sup_{B_{2R}(x_0)} v \leq C_{m,\kappa,q} \frac{o(R)}{R},$$

where $C_{m,\kappa,q}$ is a positive constant depending only on $m$, $\kappa$ and $q$. We here notice that $q$ depends only on $m$. Letting $R \to \infty$, we complete the proof of Theorem 2.6. $\qquad \square$

## 6.2 Schoen–Uhlenbeck's radial solution

We examine our growth condition (5) in Theorem 2.6 by comparing with the known example. In [37, Example 2.2, Corollary 2.6], Schoen–Uhlenbeck showed that a smooth harmonic map $u : \mathbb{R}^m \to \mathbb{S}^n_+$ is necessarily constant for $m \leq 6$, and for $m \geq 7$ such a map exists as a radial solution.

Now we consider a radial solution, that is, a harmonic map $u : \mathbb{R}^m \to \mathbb{S}^m \subset \mathbb{R}^{m+1}$ of the form $u(r, \theta) = (\rho(r), \theta)$, where $(r, \theta) = (d(x), \theta)$ are polar coordinates in $\mathbb{R}^m$ and $(\rho, \theta)$ are polar coordinates in $\mathbb{S}^m$ centered at the north pole. Then the harmonic map equation can be reduced to the following second order nonlinear ODE of $\rho(r)$:

$$\frac{d^2 \rho}{dr^2} + \frac{m-1}{r} \frac{d\rho}{dr} - \frac{m-1}{2r^2} \sin(2\rho) = 0 \tag{30}$$

for $0 < r < \infty$ with initial conditions

$$\lim_{r \to 0} \rho(r) = 0, \quad \lim_{r \to 0} \frac{d\rho}{dr}(r) > 0.$$

According to Schoen–Uhlenbeck [37], if $m \geq 7$, $\rho(r)$ lies below the line $\rho = \pi/2$, is increasing and asymptotic to $\pi/2$. As a consequence, we have

$$\frac{1}{\cos(\rho(r))} \to \infty$$

as $r \to \infty$. Now we want to know the precise growth order of this near infinity. Following Schoen–Uhlenbeck [37], it is convenient to make the change of variables

$$\alpha = 2\rho, \quad t = \log r.$$

Then ODE (30) becomes

$$\frac{d^2\alpha}{dt^2} + (m-2)\frac{d\alpha}{dt} - (m-1)\sin\alpha = 0 \tag{31}$$

for $-\infty < t < \infty$ with

$$\lim_{t \to -\infty} \alpha(t) = 0, \quad \lim_{t \to -\infty} \frac{d\alpha}{dt}(t) = 0.$$

This is the nonlinear damped pendulum differential equation. Introducing $\beta = d\alpha/dt$ we get the first order autonomous system

$$\frac{d\alpha}{dt} = \beta, \quad \frac{d\beta}{dt} = (2-m)\beta + (m-1)\sin\alpha. \tag{32}$$

A standard way to analyze the behavior of nonlinear ODE near a critical point is to study the linearized equation at the point. In our case, we consider the linearization of the system (32) at the critical point $(\alpha, \beta) = (\pi, 0)$:

$$\frac{d\tilde{\alpha}}{dt} = \tilde{\beta}, \quad \frac{d\tilde{\beta}}{dt} = (2-m)\tilde{\beta} + (m-1)(\pi - \tilde{\alpha}).$$

or equivalently,

$$\frac{d^2\tilde{\alpha}}{dt^2} + (m-2)\frac{d\tilde{\alpha}}{dt} + (m-1)\tilde{\alpha} = (m-1)\pi. \tag{33}$$

The characteristic equation is $\lambda^2 + (m-2)\lambda + (m-1) = 0$ and its roots are

$$\lambda_1(m) = \frac{-(m-2) + \sqrt{m^2 - 8m + 8}}{2}, \quad \lambda_2(m) = \frac{-(m-2) - \sqrt{m^2 - 8m + 8}}{2}.$$

If $m \geq 7$, $\lambda_1(m)$ and $\lambda_2(m)$ are both negative real. In this case, it is known that the corresponding critical point $(\pi, 0)$ of the original nonlinear autonomous system (31) is asymptotically stable node. The general solution of the linearized ODE (33) is given by

$$\tilde{\alpha}(t) = \pi + C_1 e^{\lambda_1(m)t} + C_2 e^{\lambda_2(m)t},$$

where $C_1$ and $C_2$ are arbitrary constants. Putting $\tilde{\rho} = 2\tilde{\alpha}$ and $t = \log r$, we have

$$\frac{\pi}{2} - \tilde{\rho}(r) = C_1 r^{-N_1} + C_2 r^{-N_2} = \frac{C_1 r^{N_2 - N_1} + C_2}{r^{N_2}},$$

where $0 < N_1 := -\lambda_1(m) < -\lambda_2(m) =: N_2$. Therefore,

$$\frac{1}{\cos(\tilde{\rho}(r))} = \frac{1}{\sin\left(\frac{\pi}{2} - \tilde{\rho}(r)\right)} \sim \frac{r^{N_1}}{C_1} - \frac{C_2 r^{N_1}}{C_1(C_1 r^{N_2 - N_1} + C_2)} \tag{34}$$

as $r \to \infty$. Note that

$$N_1(m) \searrow 1 \quad \text{and} \quad N_2(m) \nearrow \infty \quad \text{as} \quad m \to \infty. \tag{35}$$

Since the solution $\rho(r)$ of the original nonlinear ODE (30) is approximated by the linearized one $\tilde{\rho}(r)$ as $r \to \infty$, $\rho(r)$ does not satisfy the growth condition (5) in our Liouville theorem. In addition, (34) and (35) tell us that our growth condition (5) is almost sharp.

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