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Calculus of Variations

Nonrelativistic limit of solitary waves for nonlinear Maxwell–Klein–Gordon equations

Sangdon Jin[1] · Jinmyoung Seok[2]

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Abstract

We study the nonrelativistic limit of solitary waves from Nonlinear Maxwell–Klein–Gordon equations (NMKG) to Nonlinear Schrödinger–Poisson equations (NSP). It is known that the existence or multiplicity of positive solutions depends on the choices of parameters the equations contain. In this paper, we prove that for a given positive solitary wave of NSP, which is found in Ruiz's work (J Funct Anal 237(2):655–674, 2006), there corresponds a family of positive solitary waves of NMKG under the nonrelativistic limit. Notably, our results contain a new result of existence of positive solutions to (NMKG) with lower order nonlinearity.

Keywords Maxwell–Klein–Gordon · Schrodinger–Poisson · Nonrelativistic limit · Solitary wave

1 Introduction

Nonlinear Maxwell–Klein–Gordon equations are written by

$$\begin{cases} D_\alpha D^\alpha \phi = (mc)^2\phi - |\phi|^{p-2}\phi, \\ \partial^\beta F_{\alpha\beta} = \dfrac{q}{c}\mathrm{Im}(\phi\overline{D_\alpha\phi}), \end{cases} \quad \text{in } \mathbb{R}^{1+3}. \tag{NMKG}$$

where $D_\alpha := \partial_\alpha + \frac{q}{c} i A_\alpha, \alpha = 0, 1, 2, 3$ and $F_{\alpha\beta} := \partial_\alpha A_\beta - \partial_\beta A_\alpha$. Here, $m > 0$ represents the mass of a particle, $q > 0$ is a unit charge and $c > 0$ is the speed of light. We write $\partial_0 = \frac{\partial}{c\partial t}$, $\partial_i = \frac{\partial}{\partial x_j}$, $j = 1, 2, 3$. Indices are raised under the Minkowski metric

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✉ Jinmyoung Seok
jmseok@kgu.ac.kr

Sangdon Jin
sdjin@kaist.ac.kr

1 Department of Mathematical Sciences, KAIST, Daejeon 34141, Republic of Korea

2 Department of Mathematics, Kyonggi University, Suwon 16227, Republic of Korea

$g_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$, i.e., $X^\alpha := g_{\alpha\beta} X_\beta$. If we pay attention to the electrostatic situation, that is, $A_1 = A_2 = A_3 = 0$, then NMKG is reduced to

$$\begin{cases} \left(-\frac{\partial^2}{c^2\partial t^2} + \Delta\right)\phi - \frac{2q}{c^2} i A_0 \frac{\partial\phi}{\partial t} - \frac{q}{c^2} i \frac{\partial A_0}{\partial t}\phi + \left(\frac{q}{c}\right)^2 A_0^2\phi = (mc)^2\phi - |\phi|^{p-2}\phi, \\ -\Delta A_0 = \frac{q}{c^2}\mathrm{Im}\left(\phi\overline{\frac{\partial\phi}{\partial t}}\right) - \left(\frac{q}{c}\right)^2 A_0|\phi|^2, \end{cases} \quad \text{in } \mathbb{R}^{1+3}. \tag{1}$$

This paper is concerned with the nonrelativistic limit for NMKG in electrostatic case. By modulating the solution as $\phi(t, x) = e^{imc^2 t}\psi(t, x)$, the system of equations (1) transforms into

$$\begin{cases} -\frac{\partial^2\psi}{c^2\partial t^2} - 2mi\frac{\partial\psi}{\partial t} + \Delta\psi + 2qmA_0\psi - \frac{2q}{c^2} i A_0\frac{\partial\psi}{\partial t} - \frac{q}{c^2} i\frac{\partial A_0}{\partial t}\psi + \left(\frac{q}{c}\right)^2 A_0^2\psi = -|\psi|^{p-2}\psi, \\ -\Delta A_0 + \left(\frac{q}{c}\right)^2 |\psi|^2 A_0 = \frac{q}{c^2}\mathrm{Im}\left(\psi\overline{\frac{\partial\psi}{\partial t}}\right) - qm|\psi|^2. \end{cases} \tag{2}$$

Then, taking so-called nonrelativistic limit $c \to \infty$, the relativistic system (2) formally converges to nonlinear equations of Schrödinger type, called the nonlinear Schrödinger–Poisson equations

$$\begin{cases} -2mi\frac{\partial\psi}{\partial t} + \Delta\psi + 2qmA_0\psi = -|\psi|^{p-2}\psi, \\ -\Delta A_0 = -qm|\psi|^2, \end{cases} \quad \text{in } \mathbb{R}^{1+3}. \tag{NSP}$$

When the nonlinear potential term $|\psi|^{p-2}\psi$ is absent, the rigorous justifications of this limit are carried out by Masmoudi-Nakanishi [17] and Bechouche-Mauser-Selberg [4]. As for the stuides on the nonlinear Klein-Gordon equations without the Maxwell gauge terms ($A_\mu = 0$, $\mu = 0, 1, 2, 3$), we refer to a series for works [15,16,18].

The main interest of this paper lies in investigating the correspondence between solitary waves of NMKG and NSP under the nonrelativistic limit $c \to \infty$. During recent two decades, existence theories for solitary waves of NMKG and NSP have been well developed. Inserting the standing wave ansatz $\psi(t, x) = e^{-i\mu t}u(x)$, $u \in \mathbb{R}$ into (2), we get

$$\begin{cases} -\Delta u + \left(m^2c^2 - \left(\frac{mc^2 - \mu}{c} + \frac{q\Phi}{c}\right)^2\right)u - |u|^{p-2}u = 0, \\ -\Delta\Phi + \frac{q^2}{c^2}u^2\Phi = -\frac{q}{c}\left(\frac{mc^2 - \mu}{c}\right)u^2, \end{cases} \quad \text{in } \mathbb{R}^3. \tag{3}$$

Lax-Milgram theorem implies that for each $u \in H^1(\mathbb{R}^3)$, there exists a unique solution $\Phi_u \in D^{1,2}(\mathbb{R}^3)$ of

$$-\Delta\Phi + \frac{q^2}{c^2}u^2\Phi = -q\left(m - \frac{\mu}{c^2}\right)u^2 \text{ in } \mathbb{R}^3. \tag{4}$$

Then, by [6, Proposition 3.5], $(u, \Phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ is a solution of (3) if and only if $u \in H^1(\mathbb{R}^3)$ is a critical point of I_c, and $\Phi = \Phi_u$, where

$$I_c(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \left(2m\mu - \frac{\mu^2}{c^2}\right)u^2 - q\left(m - \frac{\mu}{c^2}\right)u^2\Phi_u dx - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p dx,$$

which is a C^1 functional on $H^1(\mathbb{R}^3)$. We note that the system of equations (3) is equivalent to the single nonlocal equation

$$-\Delta u+\Big(m^2c^2-\big(\frac{mc^2-\mu}{c}+\frac{q\Phi_u}{c}\big)^2\Big)u-|u|^{p-2}u=0 \text{ in } \mathbb{R}^3. \tag{5}$$

Before stating the existence results for (5), we simplify the parameters by denoting $\bar{m}=mc$, $e=q/c$ and $\omega=(mc^2-\mu)/c$ to rewrite (5) as

$$-\Delta u+\left(\bar{m}^2-(\omega+e\varphi_u)^2\right)u-|u|^{p-2}u=0 \text{ in } \mathbb{R}^3, \tag{6}$$

where $e>0$, $0<\omega<\bar{m}$ and φ_u is a unique solution of $-\Delta\varphi+e^2u^2\varphi=-e\omega u^2$. The corresponding action functional is given by

$$I_{\bar{m},e,\omega}(u)=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2+(\bar{m}^2-\omega^2)u^2-e\omega u^2\varphi_u dx-\frac{1}{p}\int_{\mathbb{R}^3}|u|^p dx.$$

For fixed $e>0$, Benci and Fortunato [6] first proved by applying critical point theory to $I_{\bar{m},e,\omega}$ that there exist infinitely many solutions of (6) for $4<p<6$ and $0<\omega<\bar{m}$. This result is extended by D'Aprile and Mugnai [12] to the cases $4\le p<6$ and $0<\omega<\bar{m}$ or $2<p<4$ and $0<\sqrt{2}\omega<\bar{m}\sqrt{p-2}$. They also proved in [13] that there exist no nontrivial solutions if $p\le 2$ or $p\ge 6$ and $0<\omega\le\bar{m}$. In [3], Azzollini, Pisani and Pomponio widened the existence range of $\bar{m},\omega$ for the case $2<p<4$ by showing that (6) admits a nontrivial solution when $0<\omega<\bar{m}g(p)$, where

$$g(p):=\begin{cases}\sqrt{(p-2)(4-p)} & \text{if } 2<p<3,\\ 1 & \text{if } 3\le p<4.\end{cases}$$

Azzollini and Pomponio also focused on the existence of a ground state solution of (6). A critical point of $I_{\bar{m},e,\omega}$ is said to be a ground state solution to (6) if it minimizes the value of $I_{\bar{m},e,\omega}$ among all nontrivial critical points of $I_{\bar{m},e,\omega}$. In [2], they showed (6) admits a ground state solution if $4\le p<6$ and $0<\omega<\bar{m}$ or $2<p<4$ and $\bar{m}\sqrt{p-1}>\omega\sqrt{5-p}$. Wang [23] established the same result to the range of parameters that $2<p<4$ and $0<\sqrt{h(p)}\omega<\bar{m}$, where

$$h(p):=1+\frac{(4-p)^2}{4(p-2)}.$$

We now turn to the standing wave solutions for NSP. We again insert the same ansatz $\psi(t,x)=e^{-i\mu t}u(x),\ u\in\mathbb{R}$ into NSP to obtain

$$\begin{aligned}&-\Delta u+2m\mu u-2qmu\phi-|u|^{p-2}u=0 \text{ in } \mathbb{R}^3,\\ &-\Delta\phi=-qmu^2 \text{ in } \mathbb{R}^3.\end{aligned} \tag{7}$$

For any $u\in H^1(\mathbb{R}^3)$, there exists a unique $\phi_u\in D^{1,2}(\mathbb{R}^3)$ satisfying

$$-\Delta\phi_u=-qmu^2 \text{ in } \mathbb{R}^3, \tag{8}$$

by Lax-Milgram theorem (note that actually $\phi_u=-\frac{qm}{4\pi|x|}*u^2$). We define the corresponding action integral as

$$I_\infty(u)=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2+2m\mu u^2-qmu^2\phi_u dx-\frac{1}{p}\int_{\mathbb{R}^3}|u|^p dx. \tag{9}$$

Then, by [12, Lemma 3.2], $(u,\phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ is a solution of (7) if and only if $u \in H^1(\mathbb{R}^3)$ is a critical point of I_∞, and $\phi = \phi_u$. It is also standard to show that $I_\infty \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$ and a critical point u of I_∞ satisfies

$$-\Delta u + 2m\mu u - 2qmu\phi_u - |u|^{p-2}u = 0 \text{ in } \mathbb{R}^3. \tag{10}$$

We summarize some existence results for problem (10). D'Aprile-Mugnai [12] and Coclite [7] proved the existence of a radial positive solution of (10) for $4 \le p < 6$. On the other hand, using a Pohozaev equality, D'Aprile-Mugnai [13] showed that there exists no nontrivial solutions of (10) for $p \le 2$ or $p \ge 6$. By a new approach, Ruiz [21] fills a gap for the range $2 < p < 4$. More precisely, he proved the following results:

(i) ($3 < p < 6$ and $q > 0$) $\exists$ a nontrivial solution, which is a ground state in radial class;
(ii) ($2 < p < 3$ and small $q > 0$) $\exists$ a nontrivial solution, which is a minimizer of I_∞;
(iii) ($2 < p \le 3$ and small $q > 0$) $\exists$ a nontrivial solution emanating from a ground state solution of

$$-\Delta u + 2m\mu u - |u|^{p-2}u = 0 \text{ in } \mathbb{R}^3; \tag{11}$$

(iv) ($2 < p \le 3$ and large $q > 0$) $\nexists$ nontrivial solution of (10).

In [1], Azzollini and Pomponio constructed a ground state solution of (10) for $3 < p < 6$, which is possibly non-radial. It was shown by Colin and Watanabe [8] that a ground state is unique and radial up to a translation for small $q > 0$. This result implies that the solution found by Ruiz coincides with the ground state constructed by Azzollini and Pomponio for small $q > 0$ if $3 < p < 6$. As far as we know, it is unknown whether the ground states is radial when $q > 0$ is arbitrary.

Concerning the nonrelativistic limit between solitary waves, one can naturally ask is the following:

Question: For any positive solution u of (10), is there a corresponding family of positive solutions u_c of (5), which converges to u as $c \to \infty$?

In this paper, we not only give a complete answer to this question, but also construct blow up solutions to NMKG for $2 < p < 3$. Our first theorem states the convergence of nonrelativistic limit of ground states between (5) and (10) for $3 < p < 6$. The theorem contains the existence of a ground state to (5) for $3 < p < 4$ with arbitrary parameters $m, q, \mu, c > 0$ and $c > \sqrt{\mu/m}$, which is not covered by the aforementioned results of Azzollini-Pomponio [2] or Wang [23] (see Proposition 3).

Theorem 1 (Existence and nonrelativistic limit of ground states) *Fix arbitrary $\mu, m, q > 0$ and $3 < p < 6$. Then there holds the following:*

(i) There exists a ground state solution of (5) for any $c > \sqrt{\mu/m}$.
(ii) Any ground state solution u_c of (5) belongs to $H^2(\mathbb{R}^3)$, and there exists a sequence $\{x_c\} \in \mathbb{R}^3$ such that $\{u_c(\cdot + x_c)\}$ converges to a ground state solution of (10) in $H^2(\mathbb{R}^3)$ as $c \to \infty$, after choosing a subsequence.

Based on the strategies proposed in [10,11], we shall prove the convergence of nonrelativistic limit in Theorem 1 by establishing the following steps:

1. Uniform upper estimate of ground energy levels for (5) by the ground energy level for (10), i.e.,

$$\limsup_{c\to\infty} E_c \le E_\infty, \tag{12}$$

where

$$E_c = \inf\{I_c(u) \mid u \neq 0,\ I_c'(u) = 0\} \quad \text{and} \quad E_\infty = \inf\{I_\infty(u) \mid u \neq 0,\ I_\infty'(u) = 0\};$$

2. Uniform H^1 bounds for ground states $\{u_c\}$ of (5) and solvability of its weak limit u_∞ to (10);
3. Energy estimates for establishing u_∞ to be a ground state;
4. H^1 convergence of u_c to u_∞ and its upgrade to H^2.

A new difficulty arises when we prove the step 1 in the case $3 < p < 4$. It is worth to point out that we couldn't construct a ground state of (5) by using a constrained minimization method for $3 < p < 4$. It seems not possible to find a suitable constraint working for every admissible parameters μ, m, q, c. As a consequence, we couldn't compare ground states energy levels between (5) and (10). To bypass the obstacle, we directly construct a ground state that satisfies the upper estimate (12). That is, we first show the existence of a family of nontrivial solutions to (5) satisfying the upper estimate (12) by applying a deformation argument developed in [5]. Then, by the compactness of a sequence of solutions to (5), we prove that aforementioned nontrivial solutions to (5) is ground state solutions to (5) (see Proposition 3).

The next theorem covers the case that $2 < p < 3$ and q is small. We recall the aforementioned results by Ruiz [21], which say the existence of at least two positive radial solutions u_∞ and v_∞ of (10); u_∞ is a perturbation of the ground state to (11) and v_∞ is a global minimizer of I_∞. In Theorem 2, we show the existence of two radial positive solutions u_c and v_c to (5) such that u_c and v_c converges to u_∞ and v_∞, respectively.

Theorem 2 (Correspondence of two positive solutions for $2 < p < 3$) *Assume* $2 < p < 3$. *Fix arbitrary but sufficiently small* $q > 0$ *that guarantees the existence of at least two positive radial solutions* u_∞ *and* v_∞ *to* (10) *mentioned above. If* $c > 0$ *is sufficiently large, then there exist two distinct radially symmetric positive solutions* u_c *and* v_c *of* (5) *such that*

$$\text{(i)} \lim_{c\to\infty} \|u_c - u_\infty\|_{H^1(\mathbb{R}^3)} = 0, \qquad \text{(ii)} \lim_{c\to\infty} \|v_c - v_\infty\|_{H^1(\mathbb{R}^3)} = 0.$$

In [21], Ruiz proved that a global minimizer v_∞ of I_∞ blows up in H^1 as $q \to 0$, which implies that the solution v_c constructed in Theorem 2 blows up in H^1 as $q \to 0$ and $c \to \infty$. We point out that Theorem 2 not only proves the correspondence between solitary waves but also establishes a new existence result to (5) for $2 < p < 3$. As we have seen above, the previous approaches [2,3,12,23] doesn't cover the case that $\omega > 0$ is less than but sufficiently close to $\bar{m}$. In this respect, one family of solutions u_c is actually not brand new because it is a simple consequence of implicit function theorem, which relies on nondegeneracy of the solution u_∞. However, the other family of solutions v_c is brand new because v_c bifurcates from a global minimizer of I_∞, which blows up in H^1. As for the construction of v_c, it seems not easy to show whether the global minimum of I_c is finite, unlike I_∞. This prevents us from simply adopting the minimization argument. To overcome this difficulty, we develop a new deformation argument, which strongly depends on the fact that the global minimum level of I_∞ is bounded below. We conjecture that if c is sufficiently large, there exists a global minimizer of I_c, which converges to v_∞.

We organize the paper as follows: In sect. 2, we give variational settings for NSP and NMKG, and a simple proof for the existence of a ground state to (6) for $3 < p < 6$. Section 3 is devoted to construct nontrivial solutions to (5) with the energy bound E_∞ when $3 < p < 6$. In Sect. 4, we prove Theorem 1 by combining the results in Sect. 3. In Sect. 5, we deal with the case $2 < p < 3$. We construct two radial positive solutions of (5) and prove the convergence of their nonrelativistic limit. Finally, in Appendix, we give basic estimates, which are used in the proofs of main theorems.

2 Preliminaries

This preliminary section introduces basic functional and variational settings for NMKG and NSP. In addition, we provide a simple proof for the existence of a ground state to (6) for every $3 < p < 6$ and every $e, \bar{m}, \omega > 0$ such that $\bar{m} > \omega$.

2.1 Function spaces

The space $D^{1,2}(\mathbb{R}^3)$ is defined by the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{D^{1,2}(\mathbb{R}^3)} = \Big(\int_\Omega |\nabla u|^2 dx\Big)^{1/2}.$$

For an open set $\Omega \subset \mathbb{R}^3$ and $r \in [1, \infty)$, let us denote the norms

$$\|u\|_{L^r(\Omega)} = \Big(\int_\Omega |u|^r dx\Big)^{1/r}, \quad \|u\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega}|u(x)|, \quad \|u\|_{H^1(\Omega)} = \Big(\int_\Omega |\nabla u|^2 + u^2 dx\Big)^{1/2}.$$

We also use the following abbreviations,

$$\|u\|_{L^r} = \|u\|_{L^r(\mathbb{R}^3)}, \quad \|u\|_{D^{1,2}} = \|u\|_{D^{1,2}(\mathbb{R}^3)} \quad \text{and} \quad \|u\|_{H^1} = \|u\|_{H^1(\mathbb{R}^3)}.$$

We denote by H_r^1 the Sobolev space of radial functions u such that $u, \nabla u$ are in $L^2(\mathbb{R}^3)$.

2.2 Variaional settings for NSP

Recall the action functional for (10),

$$\begin{aligned} I_\infty(u) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + 2m\mu u^2 + |\nabla\phi_u|^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p dx \\ &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + 2m\mu u^2 - qmu^2\phi_u dx - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p dx. \end{aligned}$$

The map $\lambda : u \in H^1 \to \phi_u \in D^{1,2}$ is continuously differentiable, where ϕ_u satisfies (8) (see [12]). Since $\lambda'(u)[v]$ satisfies

$$-\Delta(\lambda'(u)[v]) = -2qmuv \text{ in } \mathbb{R}^3 \qquad \text{for } v \in H^1,$$

we have

$$\int_{\mathbb{R}^3} \nabla(\lambda'(u)[v]) \cdot \nabla\phi_u dx = -2qm\int_{\mathbb{R}^3} uv\phi_u dx.$$

Then we see that

$$\begin{aligned} I'_\infty(u)v &= \int_{\mathbb{R}^3} \nabla u \cdot \nabla v + 2m\mu uv + \nabla(\lambda'(u)[v]) \cdot \nabla\phi_u dx - \int_{\mathbb{R}^3} |u|^{p-2}uv dx \\ &= \int_{\mathbb{R}^3} \nabla u \cdot \nabla v + 2m\mu uv - 2qmuv\phi_u dx - \int_{\mathbb{R}^3} |u|^{p-2}uv dx, \end{aligned}$$

which shows that a critical point of I_∞ is a weak solution to (10). We define the Nehari and Pohozaev functionals for (10) by

$$J_\infty(u) \equiv I'_\infty(u)u = \int_{\mathbb{R}^3} |\nabla u|^2 + 2m\mu u^2 - 2qmu^2\phi_u - |u|^p dx,$$
$$P_\infty(u) \equiv \int_{\mathbb{R}^3} \frac{1}{2}|\nabla u|^2 + 3m\mu u^2 - \frac{5}{2}qmu^2\phi_u - \frac{3}{p}|u|^p dx.$$

We note that the values of J_∞ and P_∞ should be zero at every critical point of I_∞ (see [21]). By defining $G_\infty(u) \equiv 2J_\infty(u) - P_\infty(u)$, we denote

$$M_\infty \equiv \left\{u \in H^1 \setminus \{0\} \;\middle|\; G_\infty(u) \equiv \int_{\mathbb{R}^3} \frac{3}{2}|\nabla u|^2 + m\mu u^2 - \frac{3}{2}qmu^2\phi_u - \frac{2p-3}{p}|u|^p dx = 0\right\}$$

and

$$E_\infty \equiv \inf_{u\in M_\infty} I_\infty(u). \tag{13}$$

It is proved in [21] that for $3 < p < 6$, E_∞ equals to the ground energy level for (10), i.e.

$$E_\infty = \inf\{I_\infty(u) \mid u \neq 0,\ I'_\infty(u) = 0\}.$$

2.3 Variational settings for NMKG

The action functional for (5) is given by

$$\begin{aligned} I_c(u) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \Big(2m\mu - \frac{\mu^2}{c^2}\Big)u^2 + |\nabla\Phi_u|^2 + \Big(\frac{q}{c}\Big)^2 u^2\Phi_u^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p dx \\ &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \Big(2m\mu - \frac{\mu^2}{c^2}\Big)u^2 - q\Big(m - \frac{\mu}{c^2}\Big)u^2\Phi_u dx - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p dx. \end{aligned}$$

The map $\Lambda : u \in H^1 \to \Phi_u \in D^{1,2}$ is continuously differentiable, where Φ_u satisfies (4) (see [12]). For $v \in H^1$, since $\Lambda'(u)[v]$ satisfies

$$-\Delta(\Lambda'(u)[v]) + \Big(\frac{q}{c}\Big)^2 u^2(\Lambda'(u)[v]) = -2\Big(\frac{q}{c}\Big)^2 uv\Phi_u - 2q\Big(m - \frac{\mu}{c^2}\Big)uv,$$

we have

$$\int_{\mathbb{R}^3} \nabla(\Lambda'(u)[v]) \cdot \nabla\Phi_u + \Big(\frac{q}{c}\Big)^2 u^2(\Lambda'(u)[v])\Phi_u dx = \int_{\mathbb{R}^3} -2\Big(\frac{q}{c}\Big)^2 uv\Phi_u^2 - 2q\Big(m - \frac{\mu}{c^2}\Big)uv\Phi_u dx.$$

Then we see that for $v \in H^1$,

$$\begin{aligned} I'_c(u)v &= \int_{\mathbb{R}^3} \nabla u \cdot \nabla v + \Big(2m\mu - \frac{\mu^2}{c^2}\Big)uv + \nabla\Phi_u \cdot \nabla(\Lambda'(u)[v]) + \Big(\frac{q}{c}\Big)^2 uv\Phi_u^2 \\ &\quad + \Big(\frac{q}{c}\Big)^2 u^2\Phi_u(\Lambda'(u)[v]) - |u|^{p-2}uv dx \\ &= \int_{\mathbb{R}^3} \nabla u \cdot \nabla v + \Big(2m\mu - \frac{\mu^2}{c^2}\Big)uv - \Big(\frac{q}{c}\Big)^2 uv\Phi_u^2 - 2q\Big(m - \frac{\mu}{c^2}\Big)uv\Phi_u - |u|^{p-2}uv dx. \end{aligned}$$

In particular, we have

$$J_c(u) \equiv I'_c(u)u = \int_{\mathbb{R}^3} |\nabla u|^2 + \Big(2m\mu - \frac{\mu^2}{c^2}\Big)u^2 - \Big(\frac{q}{c}\Big)^2 u^2\Phi_u^2 - 2q\Big(m - \frac{\mu}{c^2}\Big)u^2\Phi_u - |u|^p dx.$$

For any critical point w_c of I_c, it is clear that $J_c(w_c) = 0$ and it is shown in [13] that the Pohozaev's identity $P_c(w_c) = 0$ holds true, where

$$P_c(u) \equiv \int_{\mathbb{R}^3} \frac{1}{2}|\nabla u|^2 + \frac{3}{2}\Big(2m\mu - \frac{\mu^2}{c^2}\Big)u^2 - \frac{q^2}{c^2}\Phi_u^2 u^2 - \frac{5}{2}q\Big(m - \frac{\mu}{c^2}\Big)u^2\Phi_u - \frac{3}{p}|u|^p dx.$$

2.4 Existence of a ground state for $3 < p < 6$

We recall the equation (6)

$$-\Delta u + \big(\bar{m}^2 - (e\varphi_u + \omega)^2\big)u = |u|^{p-2}u \text{ in } \mathbb{R}^3$$

where $e > 0$, $0 < \omega < \bar{m}$ and φ_u is a unique solution of

$$-\Delta\varphi + e^2\varphi u^2 = -e\omega u^2.$$

Here we point out that by the maximum principle, we have the uniform bound

$$-\frac{\omega}{e} \le \varphi_u \le 0.$$

Proposition 3 *Assume that* $3 < p < 6$*,* $e > 0$ *and* $0 < \omega < \bar{m}$*. If there exists a non-trivial solution of* (6)*, then there exists a non-trivial ground state solution of* (6).

Proof Suppose that there exists a non-trivial solution solution of (6). We recall the action functional of (6)

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + (\bar{m}^2 - \omega^2)u^2 - e\omega\varphi_u u^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p dx.$$

and consider the minimization problem

$$\mathcal{S} = \inf\{I(u) \mid u \in \mathcal{B}\},$$

where

$$\mathcal{B} \equiv \{u \in H^1 \mid u \text{ is a non-trivial solution solution of (6)}\}.$$

By the definition, a ground state solution u of (6) is a nontrivial critical point of I satisfying $I(u) = \mathcal{S}$. Let us define

$$\begin{cases} T(u) := I'(u)u = \displaystyle\int_{\mathbb{R}^3} |\nabla u|^2 + (\bar{m}^2 - \omega^2)u^2 - 2e\omega\varphi_u u^2 - e^2\varphi_u^2 u^2 - |u|^p dx \\ Q(u) := \displaystyle\int_{\mathbb{R}^3} \frac{1}{2}|\nabla u|^2 + \frac{3}{2}(\bar{m}^2 - \omega^2)u^2 - \frac{5}{2}e\omega\varphi_u u^2 - e^2\varphi_u^2 u^2 - \frac{3}{p}|u|^p dx. \end{cases}$$

Since $T(v) = Q(v) = 0$ for any $v \in \mathcal{B}$, (see [13]), one has

$$\begin{aligned} \frac{5p-12}{2}I(v) &= \frac{5p-12}{2}I(v) - T(v) + \frac{4-p}{2}Q(v) \\ &= \int_{\mathbb{R}^3} (p-3)|\nabla v|^2 + \frac{p-2}{2}(\bar{m}^2 - \omega^2)v^2 + \frac{p-2}{2}e^2 v^2 \varphi_v^2 dx \end{aligned}$$

for $v \in \mathcal{B}$. This implies that $\mathcal{S} \ge 0$.

Let $\{u_n\}$ be a minimizing sequence of $\mathcal{S}$. From the estimates

$$\frac{5p-12}{2}\mathcal{S}+o(1)=\int_{\mathbb{R}^3}(p-3)|\nabla u_n|^2+\frac{p-2}{2}(\bar{m}^2-\omega^2)u_n^2+\frac{p-2}{2}e^2u_n^2\varphi_{u_n}^2dx \quad (14)$$

and

$$\begin{aligned}0=T(u_n)&=\int_{\mathbb{R}^3}|\nabla u_n|^2+(\bar{m}^2-\omega^2)u_n^2-\varphi_{u_n}(2e\omega+e^2\varphi_{u_n})u_n^2-|u_n|^pdx\\&\geq\int_{\mathbb{R}^3}|\nabla u_n|^2+(\bar{m}^2-\omega^2)u_n^2-|u_n|^pdx\geq C\|u_n\|_{L^p}^{2/p}-\|u_n\|_{L^p}^p,\end{aligned}$$

we deduce that (u_n) is bounded in H^1 and $\|u_n\|_{L^p}\geq C_1$ for some positive constant C_1. Then we see from Lemma 1.1 in [14],

$$\sup_{x\in\mathbb{R}^3}\int_{B_1(x)}|u_n|^2dx=\int_{B_1(x_n)}|u_n|^2dx\geq C_2>0,$$

where $x_n\in\mathbb{R}^3$ and C_2 is a positive constant. Then we may assume that $u_n(\cdot+x_n)$ converges to $u\not\equiv 0$ weakly in H^1. It is standard to show that u is a non-trivial critical point of I. Moreover, by (14) and the fact that u is a non-trivial critical point of I, we see that

$$\begin{aligned}\frac{5p-12}{2}\mathcal{S}&=\liminf_{n\to\infty}\int_{\mathbb{R}^3}(p-3)|\nabla u_n|^2+\frac{p-2}{2}(\bar{m}^2-\omega^2)u_n^2+\frac{p-2}{2}e^2u_n^2\varphi_{u_n}^2dx\\&\geq\int_{\mathbb{R}^3}(p-3)|\nabla u|^2+\frac{p-2}{2}(\bar{m}^2-\omega^2)u^2+\frac{p-2}{2}e^2u^2\varphi_u^2dx=\frac{5p-12}{2}I(u),\end{aligned}$$

which implies that u is a non-trivial ground state solution of (6). □

Observe that Proposition 3 implies the existence of a ground state to (6) for any $e,\bar{m},\omega>0$ such that $0<\omega<\bar{m}$ since there exists a nontrivial solution at those ranges of parameters by [3].

3 Construction of nontrivial solutions to NKGM with the energy bound E_∞

In this section, based on the idea of [5], we shall construct a family of nontrivial solutions w_c to (5) satisfying

$$\limsup_{c\to\infty}I_c(w_c)\leq E_\infty.$$

Before proceeding further, we first introduce a modified functional $\tilde{I}_c$ as

$$\tilde{I}_c(u)=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2+\Big(2m\mu-\frac{\mu^2}{c^2}\Big)u^2-q\Big(m-\frac{\mu}{c^2}\Big)u^2\Phi_udx-\frac{1}{p}\int_{\mathbb{R}^3}u_+^pdx,$$

where $c>0$ and $u_+=\max\{u,0\}$. A critical point of $\tilde{I}_c$ corresponds to a solution of

$$\begin{aligned}&-\Delta u+\Big(2m\mu-\Big(\frac{\mu}{c}\Big)^2\Big)u-\Big(\frac{q}{c}\Big)^2u\Phi^2-2q\Big(m-\frac{\mu}{c^2}\Big)u\Phi-u_+^{p-1}=0\text{ in }\mathbb{R}^3,\\&-\Delta\Phi+\frac{q^2}{c^2}u^2\Phi=-q\Big(m-\frac{\mu}{c^2}\Big)u^2\text{ in }\mathbb{R}^3.\end{aligned} \quad (15)$$

It is possible to show from the maximum principle that a critical point u of $\tilde{I}_c$ is positive everywhere in $\mathbb{R}^3$ for $c \geq \sqrt{\frac{2m}{\mu}}$. Indeed, since $-\frac{c^2}{q}\Big(m - \frac{\mu}{c^2}\Big) \leq \Phi_u \leq 0$, multiplying u_- to the equation

$$-\Delta u + \Big(2m\mu - (\frac{\mu}{c})^2\Big)u - \Big(\frac{q}{c}\Big)^2 u\Phi_u^2 - 2q\Big(m - \frac{\mu}{c^2}\Big)u\Phi_u - u_+^{p-1} = 0 \text{ in } \mathbb{R}^3$$

and then integrating over $\mathbb{R}^3$, we have

$$\begin{aligned}&\int_{\mathbb{R}^3} |\nabla u_-|^2 + \Big(2m\mu - \frac{\mu^2}{c^2}\Big)u_-^2\,dx \\ &\quad\leq \int_{\mathbb{R}^3} |\nabla u_-|^2 + \Big(2m\mu - \frac{\mu^2}{c^2}\Big)u_-^2 - u_-^2\,\Phi_u\Big[\Big(\frac{q}{c}\Big)^2\Phi_u + 2q\Big(m - \frac{\mu}{c^2}\Big)\Big]dx = 0,\end{aligned}$$

where $u_- = \min\{u, 0\}$. Therefore a nontrivial critical point of $\tilde{I}_c$ gives a positive solution to (5). We also define

$$\begin{aligned}\tilde{I}_\infty(u) &:= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + 2m\mu u^2 - qmu^2\phi_u\,dx - \frac{1}{p}\int_{\mathbb{R}^3} u_+^p\,dx, \\ \tilde{J}_\infty(u) &:= I_\infty'(u)u = \int_{\mathbb{R}^3} |\nabla u|^2 + 2m\mu u^2 - 2qmu^2\phi_u - u_+^p\,dx, \\ \tilde{P}_\infty(u) &:= \int_{\mathbb{R}^3} \frac{1}{2}|\nabla u|^2 + 3m\mu u^2 - \frac{5}{2}qmu^2\phi_u - \frac{3}{p}u_+^p\,dx.\end{aligned}$$

Let $\mathcal{A} \equiv \{u \in H^1 \mid \tilde{I}_\infty'(u) = 0, \tilde{I}_\infty(u) = E_\infty, \text{ and } \max_{\mathbb{R}^3} u = u(0)\}$. We note that $\mathcal{A} \neq \emptyset$. Indeed, if $u \in \mathcal{M}_\infty$ satisfies $I_\infty(u) = E_\infty$, we see that $|u|$ satisfies $\tilde{I}_\infty(|u|) = E_\infty$ and $\tilde{I}_\infty'(|u|) = 0$.

Proposition 4 *For $3 < p < 6$, there exist positive constants C_1 and C_2 independent of $U \in \mathcal{A}$ such that for $U \in \mathcal{A}$,*

$$U(x) + |\nabla U(x)| \leq C_1 \exp(-C_2|x|).$$

Moreover, $\inf_{U\in\mathcal{A}} \|U\|_{L^\infty} > 0$.

Proof Let $U \in \mathcal{A}$. It follows from

$$\begin{aligned}E_\infty = \tilde{I}_\infty(U) &= \tilde{I}_\infty(U) - \frac{2}{5p-12}\tilde{J}_\infty(U) - \frac{p-4}{5p-12}\tilde{P}_\infty(U) \\ &= \int_{\mathbb{R}^3} \frac{2(p-3)}{5p-12}|\nabla U|^2 + \frac{2(p-2)}{5p-12}m\mu U^2\,dx\end{aligned} \tag{16}$$

where $U \in \mathcal{A}$, that $\mathcal{A}$ is bounded in H^1 if $3 < p < 6$. Then, since

$$\begin{aligned}\|\phi_U + |U|^{p-2}\|_{L^{\frac{6}{p-2}}(\Omega)} &\leq \|\phi_U\|_{L^{\frac{6}{p-2}}(\Omega)} + \|U\|_{L^6(\Omega)}^{p-2} \leq |\Omega|^{\frac{p-2}{6}-\frac{1}{6}}\|\phi_U\|_{L^6(\Omega)} + \|U\|_{L^6(\Omega)}^{p-2} \\ &\leq C\big(|\Omega|^{\frac{p-2}{6}-\frac{1}{6}}\|U\|_{H^1}^2 + \|U\|_{H^1}^{p-2}\big),\end{aligned}$$

where $3 < p < 6$, $U \in \mathcal{A}$, Ω is a bounded domain in $\mathbb{R}^3$ and C is a positive constant independent of $U \in \mathcal{A}$, we see that $\mathcal{A}$ is bounded in L^∞ (see [22, Theorem 4.1]).

We claim that $\lim_{|x|\to\infty} U(x) = 0$ uniformly for $U \in \mathcal{A}$. Indeed, contrary to our claim, suppose that there exist $\{U_i\}_{i=1}^\infty \subset \mathcal{A}$ and $\{x_i\}_{i=1}^\infty \subset \mathbb{R}^N$ satisfying $\lim_{i\to\infty} |x_i| = \infty$ and

$\liminf_{i\to\infty} U_i(x_i) > 0$. Denote $V_i \equiv U_i(\cdot + x_i)$. We note that if $u_i \rightharpoonup u$ in H^1, $\phi_{u_i} \rightharpoonup \phi_u$ in $D^{1,2}$. Then if $u_i \rightharpoonup u$ in H^1, for $\psi \in C_0^\infty(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} (u_i\phi_{u_i} - u\phi_u)\psi\, dx = \int_{\mathbb{R}^3} (u_i - u)\phi_{u_i}\psi + u(\phi_{u_i} - \phi_u)\psi\, dx = o(1) \tag{17}$$

as $i \to \infty$. By (17) and the fact that $\{U_i, V_i\}_{i=1}^\infty$ is bounded in H^1, we see that U_i and V_i converge to U and V weakly in H^1 as $i \to \infty$, up to a subsequence, respectively, where U and V are non-trivial solutions of (10). It follows from (16) that for $2R \le |x_i|$,

$$\begin{aligned} E_\infty &= \liminf_{i\to\infty} \tilde{I}_\infty(U_i) = \liminf_{i\to\infty} \int_{\mathbb{R}^3} \frac{2(p-3)}{5p-12}|\nabla U_i|^2 + \frac{2(p-2)}{5p-12} m\mu U_i^2 dx \\ &\ge \liminf_{i\to\infty} \int_{B(0,R)} \frac{2(p-3)}{5p-12}|\nabla U_i|^2 + \frac{2(p-2)}{5p-12} m\mu U_i^2 dx \\ &\quad + \liminf_{i\to\infty} \int_{B(x_i,R)} \frac{2(p-3)}{5p-12}|\nabla U_i|^2 + \frac{2(p-2)}{5p-12} m\mu U_i^2 dx \\ &\ge \int_{B(0,R)} \frac{2(p-3)}{5p-12}|\nabla U|^2 + \frac{2(p-2)}{5p-12} m\mu U^2 dx \\ &\quad + \int_{B(0,R)} \frac{2(p-3)}{5p-12}|\nabla V|^2 + \frac{2(p-2)}{5p-12} m\mu V^2 dx. \end{aligned} \tag{18}$$

Since

$$\tilde{I}_\infty(U), \tilde{I}_\infty(V) \ge \tilde{I}_\infty(W) \text{ for any } W \in \mathcal{A},$$

if we take large $R > 0$ in (18), we deduce a contradiction. This implies that $\lim_{|x|\to\infty} U(x) = 0$ uniformly for $U \in \mathcal{A}$.

We note that for large $|x|$,

$$\begin{aligned} \phi_U(x) &= -\frac{qm}{4\pi}\int_{\mathbb{R}^3} \frac{U^2(y)}{|x-y|}dy = -\frac{qm}{4\pi}\int_{B(x,R)} \frac{U^2(y)}{|x-y|}dy - \frac{qm}{4\pi}\int_{\mathbb{R}^3\setminus B(x,R)} \frac{U^2(y)}{|x-y|}dy \\ &= o(1)R^2 + O(1)\frac{1}{R} = o(1) \end{aligned}$$

uniformly in $U \in \mathcal{A}$. Then, by the comparison principle and the elliptic estimates, we see that for $U \in \mathcal{A}$,

$$U(x) + |\nabla U(x)| \le C_1 \exp(-C_2|x|),$$

where C_1 and C_2 are positive constants independent of $U \in \mathcal{A}$.

To show $\inf_{U\in\mathcal{A}} \|U\|_{L^\infty} > 0$, we assume that there exists $\{U_i\}_{i=1}^\infty \subset \mathcal{A}$ such that $\|U_i\|_{L^\infty} \to 0$ as $i \to \infty$. Then, since U_i satisfies

$$-\Delta U_i + 2m\mu U_i - U_i^{p-1} \le -\Delta U_i + 2m\mu U_i - 2qmU_i\phi_{U_i} - U_i^{p-1} = 0 \text{ in } \mathbb{R}^3,$$

we see that $\|U_i\|_{H^1} \to 0$ as $i \to \infty$, which is a contradiction to (16). □

For a fixed $U_0 \in \mathcal{A}$, we define $\gamma(t)(x) = t^2U_0(tx)$. It follows from

$$\tilde{I}_\infty(\gamma(t)) = \frac{1}{2}\int_{\mathbb{R}^3} t^3|\nabla U_0|^2 + 2m\mu t U_0^2 - qmt^3U_0^2\phi_{U_0} dx - \frac{t^{2p-3}}{p}\int_{\mathbb{R}^3} U_0^p dx$$

that for $3 < p < 6$, there exists $t_0 > 1$ such that $\tilde{I}_\infty(\gamma(t)) < 0$ for $t \geq t_0$. Moreover, by [21, Lemma 3.3] and the fact that U_0 is a critical point of $\tilde{I}_\infty$, we see that for $3 < p < 6$, $t = 1$ is a unique critical point of $\tilde{I}_\infty(\gamma(t))$, corresponding to its maximum.

We define

$$\hat{e}_c := \max_{t\in[0,t_0]} \tilde{I}_c(\gamma(t)), \quad \text{and} \quad e_c := \inf_{\Gamma\in\mathcal{W}} \max_{s\in[0,1]} \tilde{I}_c(\Gamma(s)),$$

where $\mathcal{W} \equiv \{\Gamma \in C([0,1], H^1) \mid \Gamma(0) = 0, \Gamma(1) = \gamma(t_0)\}$.

Proposition 5 *Let $3 < p < 6$. Then we have*

$$\limsup_{c\to\infty} \hat{e}_c \leq E_\infty.$$

Proof We see from Lemma 21 and the scaling $\phi_{t^2U_0(t\cdot)} = t^2\phi_{U_0}(t\cdot)$, that for $t \in [0, t_0]$,

$$\begin{aligned}
\tilde{I}_c(\gamma(t)) &= \frac{1}{2}\int_{\mathbb{R}^3} |t^3(\nabla U_0)(tx)|^2 + \left(2m\mu - \frac{\mu^2}{c^2}\right)t^4U_0^2(tx) - q\left(m - \frac{\mu}{c^2}\right)t^4U_0^2(tx)\Phi_{t^2U_0(t\cdot)}dx \\
&\quad - \frac{t^{2p}}{p}\int_{\mathbb{R}^3}(U_0(tx))^p dx \\
&= \frac{1}{2}\int_{\mathbb{R}^3} |t^3(\nabla U_0)(tx)|^2 + 2m\mu t^4U_0^2(tx) - qmt^4U_0^2(tx)\phi_{t^2U_0(t\cdot)}dx \\
&\quad - \frac{t^{2p}}{p}\int_{\mathbb{R}^3}(U_0(tx))^p dx + o(1) \\
&= \frac{1}{2}\int_{\mathbb{R}^3} t^3|\nabla U_0|^2 + 2m\mu tU_0^2 - qmt^3U_0^2\phi_{U_0}dx - \frac{t^{2p-3}}{p}\int_{\mathbb{R}^3}(U_0)^p dx + o(1) \\
&= \tilde{I}_\infty(\gamma(t)) + o(1),
\end{aligned} \tag{19}$$

where $o(1)$ is uniform in $t \in [0, t_0]$ as $c \to \infty$. Thus, since $t = 1$ is a unique maximum point of $\tilde{I}_\infty(\gamma(t))$ for $3 < p < 6$, we deduce that

$$\hat{e}_c = \max_{s\in[0,1]} \tilde{I}_c(\gamma(t_0 s)) = \tilde{I}_\infty(U_0) + o(1) = E_\infty + o(1)$$

as $c \to \infty$. □

Proposition 6 *Let $3 < p < 6$. Then we have*

$$\liminf_{c\to\infty} e_c \geq E_\infty.$$

Proof We note that for $\Gamma \in \mathcal{W}$,

$$\begin{aligned}
\tilde{I}_c(\Gamma(t)) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\Gamma(t)|^2 + 2m\mu\Gamma^2(t) - qm\Gamma^2(t)\phi_{\Gamma(t)}dx - \frac{1}{p}\int_{\mathbb{R}^3}(\Gamma(t))_+^p dx \\
&\quad - \frac{1}{c^2}\int_{\mathbb{R}^3} \mu^2\Gamma^2(t) - q\mu\Gamma^2(t)\Phi_{\Gamma(t)}dx - \frac{1}{2}qm\int_{\mathbb{R}^3}\Gamma^2(t)(\Phi_{\Gamma(t)} - \phi_{\Gamma(t)})dx \\
&= \tilde{I}_\infty(\Gamma(t)) + G_c(t),
\end{aligned}$$

where $G_c(t) \equiv -\frac{1}{c^2}\int_{\mathbb{R}^3} \mu^2\Gamma^2(t) - q\mu\Gamma^2(t)\Phi_{\Gamma(t)}dx - \frac{1}{2}qm\int_{\mathbb{R}^3}\Gamma^2(t)(\Phi_{\Gamma(t)} - \phi_{\Gamma(t)})dx$. By Lemma 21, we have

$$|G_c(t)| = o(1) \text{ uniformly in } t \in [0, 1] \text{ as } c \to \infty.$$

Then, since

$$\max_{t\in[0,1]} \tilde{I}_\infty(\Gamma(t)) \geq E_\infty,$$

where $\Gamma \in \mathcal{W}$ (see [1, Lemma 2.4]), we have

$$e_c \geq E_\infty + \inf_{\Gamma\in\mathcal{W}} \max_{t\in[0,1]} G_c(t) \geq E_\infty - \inf_{\Gamma\in\mathcal{W}} \max_{t\in[0,1]} |G_c(t)| = E_\infty + o(1)$$

as $c \to \infty$. □

We define

$$\mathcal{X} \equiv \{U(\cdot - y) \mid U \in \mathcal{A}, y \in \mathbb{R}^3\}$$

and

$$N_d(\mathcal{X}) \equiv \{u \in H^1 \mid \inf_{v\in\mathcal{X}} \|u - v\|_{H^1} \leq d\},$$

where $d > 0$ is a constant and $\mathcal{A} \equiv \{u \in H^1 \mid \tilde{I}'_\infty(u) = 0, \tilde{I}_\infty(u) = E_\infty, \text{ and } \max_{\mathbb{R}^3} u = u(0)\}$.

Proposition 7 *Let* $3 < p < 6$*. For large* $c > 0$*, for small* $d > 0$*, and for any* $d' \in (0, d)$*, there exists* $\nu \equiv \nu(d, d') > 0$ *independent of* $c > 0$ *such that*

$$\inf\{\|\tilde{I}'_c(u)\|_{H^{-1}} \mid \tilde{I}_c(u) \leq \hat{e}_c, u \in N_d(\mathcal{X}) \setminus N_{d'}(\mathcal{X})\} \geq \nu > 0.$$

Proof Let $\{c_i\}_{i=1}^\infty$ be such that $\lim_{i\to\infty} c_i = \infty$. It suffices to show that for small $d > 0$, if

$$u_{c_i} \in N_d(\mathcal{X}), \quad \tilde{I}_{c_i}(u_{c_i}) \leq \hat{e}_{c_i}, \quad \text{and} \quad \|\tilde{I}'_{c_i}(u_{c_i})\|_{H^{-1}} \to 0$$

as $i \to \infty$, then

$$\inf_{v\in\mathcal{X}} \|u_{c_i} - v\|_{H^1} \to 0 \text{ as } i \to \infty.$$

For the sake of simplicity of notation, we write c for c_i. Since $u_c \in N_d(\mathcal{X})$, we have

$$\|u_c(x) - U_c(x - y_c)\|_{H^1} \leq d, \tag{20}$$

where $U_c \in \mathcal{A}$ and $y_c \in \mathbb{R}^3$. We define $\eta \in C_0^\infty(\mathbb{R}^3)$ such that $0 \leq \eta \leq 1$, $\eta(x) = 1$ for $|x| \leq 1$, $\eta(x) = 0$ for $|x| \geq 2$, and $|\nabla\eta| \leq 2$. Also, we set $\tilde{\eta}_c(x) = \eta(\frac{x-y_c}{c})$. We divide the proof into three steps.

Step 1. $\tilde{I}_c(u_c) \geq \tilde{I}_\infty(v_c) + \tilde{I}_\infty(w_c) + o(1)$ as $c \to \infty$, where $v_c = \tilde{\eta}_c u_c$ and $w_c = (1 - \tilde{\eta}_c)u_c$.

We claim first that for $\alpha \in (2, 6)$,

$$\lim_{c\to\infty} \int_{B(y_c,2c)\setminus B(y_c,c)} |u_c|^\alpha dx = 0. \tag{21}$$

Suppose that there exist $z_c \in B(y_c, 2c) \setminus B(y_c, c)$ and $R > 0$ such that

$$\liminf_{c\to\infty} \int_{B(z_c,R)} |u_c|^2 dx > 0. \tag{22}$$

Denote $\tilde{u}_c = u_c(\cdot + z_c)$. We note that, by Lemma 21 and the fact that $\|u_c\|_{H^1}$ is bounded, for $\psi \in C_0^\infty(\mathbb{R}^3)$,

$$\begin{aligned}
&\tilde{I}_c'(\tilde{u}_c)\psi \\
&= \int_{\mathbb{R}^3} \nabla \tilde{u}_c \cdot \nabla \psi + \Big(2m\mu - \frac{\mu^2}{c^2}\Big)\tilde{u}_c \psi - \Big(\frac{q}{c}\Big)^2 \tilde{u}_c \psi \Phi_{\tilde{u}_c}^2 - 2q\Big(m - \frac{\mu}{c^2}\Big)\tilde{u}_c \psi \Phi_{\tilde{u}_c} - (\tilde{u}_c)_+^{p-1}\psi dx \\
&= \int_{\mathbb{R}^3} \nabla \tilde{u}_c \cdot \nabla \psi + 2m\mu \tilde{u}_c \psi - 2qm\tilde{u}_c \psi \phi_{\tilde{u}_c} - (\tilde{u}_c)_+^{p-1}\psi dx \\
&+ \int_{\mathbb{R}^3} -\frac{\mu^2}{c^2}\tilde{u}_c\psi - \Big(\frac{q}{c}\Big)^2 \tilde{u}_c \psi \Phi_{\tilde{u}_c}^2 + 2q\frac{\mu}{c^2}\tilde{u}_c \psi \Phi_{\tilde{u}_c} - 2qm\tilde{u}_c\psi(\Phi_{\tilde{u}_c} - \phi_{\tilde{u}_c})dx \\
&= \tilde{I}_\infty'(\tilde{u}_c)\psi + o(1)
\end{aligned} \tag{23}$$

as $c \to \infty$. By (17) and the assumption that $\|\tilde{I}_c'(u_c)\|_{H^{-1}} \to 0$ as $c \to \infty$, we have $u_c(\cdot + z_c) \rightharpoonup \tilde{U} \not\equiv 0$ in H^1, where $\tilde{U}$ satisfies $\tilde{I}_\infty'(\tilde{U}) = 0$. By (16), we have

$$\int_{\mathbb{R}^3} |\nabla \tilde{U}|^2 + \tilde{U} dx \geq E_\infty\Big(\max\Big\{\frac{2(p-3)}{5p-12}, \frac{2(p-2)}{5p-12}m\mu\Big\}\Big)^{-1}. \tag{24}$$

Then, by Proposition 4 and the fact that $|z_c - y_c| \geq c$, we see that for $R > 0$,

$$\begin{aligned}
d^2 &\geq \|u_c(x) - U_c(x - y_c)\|_{H^1}^2 = \|\tilde{u}_c(x) - U_c(x + z_c - y_c)\|_{H^1}^2 \\
&\geq \|\tilde{u}_c(x) - U_c(x + z_c - y_c)\|_{H^1(B(0,R))}^2 = \|\tilde{u}_c(x)\|_{H^1(B(0,R))}^2 + o(1) \geq \|\tilde{U}\|_{H^1(B(0,R))}^2
\end{aligned}$$

as $c \to \infty$. If we take small $d > 0$, by (24), we deduce a contradiction. Since there does not exists such a sequence $\{z_c\}$ satisfying (22), by [14, Lemma 1.1], we deduce (21). Then, by (21), we have

$$\int_{\mathbb{R}^3} (u_c)_+^p - (v_c)_+^p - (w_c)_+^p dx = o(1) \tag{25}$$

as $c \to \infty$, where v_c and w_c are given in (21) above. By (21) and Lemma 17,

$$\begin{aligned}
\int_{B(y_c,2c)\setminus B(y_c,c)} u_c^2|\phi_{u_c}| dx &\leq \|\phi_{u_c}\|_{L^6(B(y_c,2c)\setminus B(y_c,c))}\|u_c^2\|_{L^{6/5}(B(y_c,2c)\setminus B(y_c,c))} \\
&\leq C_1\|u_c\|_{H^1}^2\|u_c\|_{L^{12/5}(B(y_c,2c)\setminus B(y_c,c))}^2 \to 0
\end{aligned}$$

as $c \to \infty$, where C_1 is a positive constant. From this and the fact that $|\nabla \eta_c| \leq 2/c$, we see that

$$\begin{aligned}
&\int_{\mathbb{R}^3} v_c^2\phi_{v_c} + w_c^2\phi_{w_c} - u_c^2\phi_{u_c} dx \\
&= \int_{B(y_c,c)\cup(\mathbb{R}^3\setminus B(y_c,2c))} v_c^2\phi_{v_c} + w_c^2\phi_{w_c} - u_c^2\phi_{u_c} dx + o(1) \\
&= \frac{qm}{4\pi}\int_{B(y_c,c)\cup(\mathbb{R}^3\setminus B(y_c,2c))}\int_{\mathbb{R}^3} \frac{u_c^2(x)u_c^2(y) - v_c^2(x)v_c^2(y) - w_c^2(x)w_c^2(y)}{|x-y|} dydx + o(1) \\
&= \frac{qm}{4\pi}\int_{B(y_c,c)}\int_{\mathbb{R}^3} \frac{u_c^2(x)(u_c^2(y) - v_c^2(y))}{|x-y|} dydx \\
&\quad + \frac{qm}{4\pi}\int_{\mathbb{R}^3\setminus B(y_c,2c)}\int_{\mathbb{R}^3} \frac{u_c^2(x)(u_c^2(y) - w_c^2(y))}{|x-y|} dydx + o(1) \geq o(1)
\end{aligned} \tag{26}$$

as $c \to \infty$. Thus, by (25), (26), Lemma 21 and the fact that $|\nabla \eta_c| \le 2/c$, we have

$$\begin{aligned}\tilde{I}_c(u_c) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_c|^2 + 2m\mu u_c^2 - qmu_c^2\phi_{u_c}dx - \frac{1}{p}\int_{\mathbb{R}^3}(u_c)_+^p dx \\ &\quad - \frac{1}{2c^2}\int_{\mathbb{R}^3} \mu^2 u_c^2 - q\mu u_c^2 \Phi_{u_c}dx - \frac{1}{2}qm\int_{\mathbb{R}^3} u_c^2(\Phi_{u_c} - \phi_{u_c})dx \\ &\ge \tilde{I}_\infty(v_c) + \tilde{I}_\infty(w_c) + \int_{\mathbb{R}^3} \nabla v_c \cdot \nabla w_c + 2m\mu v_c w_c dx + o(1) \\ &= \tilde{I}_\infty(v_c) + \tilde{I}_\infty(w_c) + \int_{\mathbb{R}^3} (1-\tilde{\eta}_c)\tilde{\eta}_c|\nabla u_c|^2 + 2m\mu(1-\tilde{\eta}_c)\tilde{\eta}_c u_c^2 dx + o(1) \\ &\ge \tilde{I}_\infty(v_c) + \tilde{I}_\infty(w_c) + o(1)\end{aligned}$$

as $c \to \infty$.

Step 2. $\tilde{I}_\infty(w_c) \ge 0$ for large c, where $w_c = (1-\tilde{\eta}_c)u_c$.

We note that, by Lemma 17,

$$\left|\int_{\mathbb{R}^3} w_c^2 \phi_{w_c} dx\right| \le \|\phi_{w_c}\|_{L^6}\|w_c^2\|_{L^{6/5}} \le C_2\|w_c\|_{H^1}^4,$$

where C_2 is a positive constant independent of c. Moreover, by (20) and Proposition 4, $\|w_c\|_{H^1} \le 2d$ for large $c > 0$. Then we have

$$\begin{aligned}\tilde{I}_\infty(w_c) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla w_c|^2 + 2m\mu w_c^2 - qmw_c^2\phi_{w_c}dx - \frac{1}{p}\int_{\mathbb{R}^3}(w_c)_+^p dx \\ &\ge \|w_c\|_{H^1}^2\Big(\min\Big\{\frac{1}{2}, m\mu\Big\} - qmC_2(\|w_c\|_{H^1}^2 + \|w_c\|_{H^1}^{p-2})\Big).\end{aligned} \tag{27}$$

Taking $d > 0$ small, we deduce that $\tilde{I}_\infty(w_c) \ge 0$ for large c.

Step 3. $v_c \to \tilde{V}(\cdot - z)$ in H^1, where $\tilde{V} \in \mathcal{A}$, $z \in \mathbb{R}^3$ and $v_c = \tilde{\eta}_c u_c$.

Let $W_c \equiv v_c(\cdot + y_c)$. We can assume that $W_c \rightharpoonup W \not\equiv 0$ in H^1, up to a subsequence, as $c \to \infty$. Since $W_c - u_c(\cdot + y_c) \rightharpoonup 0$ in H^1, $\phi_{W_c} - \phi_{u_c(\cdot+y_c)} \rightharpoonup 0$ in $D^{1,2}$. Then for any $\psi \in C_0^\infty(\mathbb{R}^3)$,

$$\begin{aligned}\int_{\mathbb{R}^3}(W_c\phi_{W_c} - u_c(\cdot+y_c)\phi_{u_c(\cdot+y_c)})\psi dx &= \int_{\mathbb{R}^3}(W_c - W)\big(\phi_{W_c} - \phi_{u_c(\cdot+y_c)}\big)\psi + W\big(\phi_{W_c} - \phi_{u_c(\cdot+y_c)}\big)\psi \\ &\quad + (W_c - u_c(\cdot+y_c))\phi_{u_c(\cdot+y_c)}\psi dx \to 0\end{aligned}$$

as $c \to \infty$. From this, (17), (23) and the assumption that $\|\tilde{I}_c'(u_c)\|_{H^{-1}} \to 0$ as $c \to \infty$, we can see that W satisfies $\tilde{I}_\infty'(W) = 0$. By the maximum principle, W is positive. Suppose that there exist $R > 0$ and a sequence $\tilde{z}_c \in B(y_c, 2c)$ satisfying

$$\liminf_{c\to\infty}|\tilde{z}_c - y_c| = \infty \text{ and } \liminf_{c\to\infty}\int_{B(\tilde{z}_c,R)}|v_c|^2 dx > 0.$$

Then $v_c(\cdot + z_c)$ converges weakly to $\tilde{W}$ in H^1, where $I_\infty'(\tilde{W}) = 0$. By the same arguments in Step 1, we deduce a contradiction. By [14, Lemma 1.1], we have

$$\lim_{c\to\infty}\int_{\mathbb{R}^3}(W_c)_+^p dx = \int_{\mathbb{R}^3} W^p dx. \tag{28}$$

We note that

$$\liminf_{c\to\infty}\Big(-\int_{\mathbb{R}^3} W_c^2\phi_{W_c}dx\Big)=\liminf_{c\to\infty}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{W_c^2(x)W_c^2(y)}{|x-y|}dydx \\ \geq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{W^2(x)W^2(y)}{|x-y|}dydx=-\int_{\mathbb{R}^3}W^2\phi_W dx. \tag{29}$$

Then, by (28), (29) and Lemma 21, we have

$$\liminf_{c\to\infty}\tilde{I}_\infty(W_c)=\liminf_{c\to\infty}\frac{1}{2}\int_{\mathbb{R}^3}|\nabla W_c|^2+2m\mu W_c^2-qmW_c^2\phi_{W_c}dx-\frac{1}{p}\int_{\mathbb{R}^3}(W_c)_+^p dx \\ \geq \tilde{I}_\infty(W). \tag{30}$$

By (30), the results of Step1 and Step 2, and the assumption that $\tilde{I}_c(u_c)\leq \hat{e}_c$, we see that $\tilde{I}_\infty(W)=E_\infty$. By (28), (29) and (30), we have

$$\limsup_{c\to\infty}\int_{\mathbb{R}^3}|\nabla W_c|^2+2m\mu W_c^2-qmW_c^2\phi_{W_c}dx=\int_{\mathbb{R}^3}|\nabla W|^2+2m\mu W^2-qmW^2\phi_W dx \\ \leq \int_{\mathbb{R}^3}|\nabla W|^2+2m\mu W^2+\limsup_{c\to\infty}\Big(-\int_{\mathbb{R}^3}qmW_c^2\phi_{W_c}dx\Big),$$

which implies that $W_c\to W$ in H^1. By (27), the result of Step 1 and the fact that $\hat{e}_c\to E_\infty$, we have for small $d>0$,

$$\hat{e}_c\geq \tilde{I}_c(u_c)\geq \tilde{I}_\infty(v_c)+\frac{1}{2}\min\Big\{\frac{1}{2},m\mu\Big\}\|w_c\|_{H^1}^2+o(1) \\ \geq E_\infty+\frac{1}{2}\min\Big\{\frac{1}{2},m\mu\Big\}\|w_c\|_{H^1}^2+o(1)$$

as $c\to\infty$, which implies that $\|w_c\|_{H^1}\to 0$ as $c\to\infty$. Thus, letting $W=\tilde{V}(\cdot-z)$, where $\tilde{V}\in\mathcal{A}$ and $z\in\mathbb{R}^3$, we have

$$\|u_c-\tilde{V}(\cdot-y_c-z)\|_{H^1}\leq\|v_c(\cdot+y_c)-\tilde{V}(\cdot-z)\|_{H^1}+\|w_c\|_{H^1}\to 0$$

as $c\to\infty$. □

Proposition 8 *Let $3<p<6$. For a fixed $c\in(\sqrt{\frac{\mu}{m}},\infty)$, suppose that for some $b\in\mathbb{R}$, there exists a sequence $\{u_j\}\subset H^1$ satisfying*

$$u_j\in N_d(\mathcal{X}), \\ \|\tilde{I}_c'(u_j)\|_{H^{-1}}\to 0, \\ \tilde{I}_c(u_j)\to b \text{ as } j\to\infty,$$

where $d>0$ is a constant. Then for small $d>0$, b is a critical value of $\tilde{I}_c$, and the sequence $\{u_j(\cdot+x_j)\}_{j=1}^\infty\subset H^1$ has a strongly convergent subsequence in H^1, where $x_j\in\mathbb{R}^3$.

Proof Since $u_j\in N_d(\mathcal{X})$, $\{u_j\}_{j=1}^\infty$ is bounded in H^1. Then we can extract a subsequence such that $\tilde{u}_{j_k}\equiv u_{j_k}(\cdot+x_{j_k})$ converges to $u_0\not\equiv 0$ weakly in H^1 as $k\to\infty$, where $x_{j_k}\in\mathbb{R}^3$. It is standard to show that u_0 is a critical point of I_c.

Next, we show $\tilde{u}_{j_k}\to u_0$ in H^1 as $k\to\infty$. By Proposition 4, there exists $R_0>0$ such that

$$\|\tilde{u}_{j_k}\|_{H^1(\mathbb{R}^3\setminus B(0,R_0))}\leq 2d. \tag{31}$$

We choose a function $\zeta \in C^\infty(\mathbb{R}^3)$ such that

$$\zeta(x) = \begin{cases} 1 & \text{for } |x| \geq 2R_0, \\ 0 & \text{for } |x| \leq R_0. \end{cases}$$

Since $\tilde{I}_c'(\tilde{u}_{j_k})(\zeta(\tilde{u}_{j_k} - u_0)) - \tilde{I}_c'(u_0)(\zeta(\tilde{u}_{j_k} - u_0)) \to 0$ as $k \to \infty$, we deduce that

$$\begin{aligned} &\int_{\mathbb{R}^3 \setminus B(0,2R)} |\nabla(\tilde{u}_{j_k} - u_0)|^2 + \Big(2m\mu - \frac{\mu^2}{c^2}\Big)(\tilde{u}_{j_k} - u_0)^2 dx \\ &\leq \int_{\mathbb{R}^3 \setminus B(0,2R)} \Big(\frac{q}{c}\Big)^2 (\tilde{u}_{j_k} - u_0)(\tilde{u}_{j_k}\Phi^2_{\tilde{u}_{j_k}} - u_0\Phi^2_{u_0}) + 2q\Big(m - \frac{\mu}{c^2}\Big)(\tilde{u}_{j_k} - u_0)(\tilde{u}_{j_k}\Phi_{\tilde{u}_{j_k}} - u_0\Phi_{u_0}) \\ &+ (\tilde{u}_{j_k} - u_0)((\tilde{u}_{j_k})_+^{p-1} - (u_0)_+^{p-1})dx + o(1) \end{aligned} \tag{32}$$

as $k \to \infty$. We note that, by Lemma 18,

$$\begin{aligned} &\int_{\mathbb{R}^3 \setminus B(0,2R)} (v - w)(v\Phi_v^2 - w\Phi_w^2)dx \\ &\leq \big(\|\Phi_v\|^2_{L^6}\|v\|_{L^3(\mathbb{R}^3 \setminus B(0,2R))} + \|\Phi_w\|^2_{L^6}\|w\|_{L^3(\mathbb{R}^3 \setminus B(0,2R))}\big)\|v - w\|_{L^3(\mathbb{R}^3 \setminus B(0,2R))} \\ &\leq C_1\big(\|v\|^4_{H^1}\|v\|_{H^1(\mathbb{R}^3 \setminus B(0,2R))} + \|w\|^4_{H^1}\|w\|_{H^1(\mathbb{R}^3 \setminus B(0,2R))}\big)\|v - w\|_{H^1(\mathbb{R}^3 \setminus B(0,2R))}, \end{aligned} \tag{33}$$

$$\begin{aligned} &\int_{\mathbb{R}^3 \setminus B(0,2R)} (v - w)(v\Phi_v - w\Phi_w)dx \\ &\leq \big(\|\Phi_v\|_{L^6}\|v\|_{L^3(\mathbb{R}^3 \setminus B(0,2R))} + \|\Phi_w\|_{L^6}\|w\|_{L^3(\mathbb{R}^3 \setminus B(0,2R))}\big)\|v - w\|_{L^2(\mathbb{R}^3 \setminus B(0,2R))} \\ &\leq C_2\big(\|v\|^2_{H^1}\|v\|_{H^1(\mathbb{R}^3 \setminus B(0,2R))} + \|w\|^2_{H^1}\|w\|_{H^1(\mathbb{R}^3 \setminus B(0,2R))}\big)\|v - w\|_{H^1(\mathbb{R}^3 \setminus B(0,2R))}, \end{aligned} \tag{34}$$

and

$$\begin{aligned} &\int_{\mathbb{R}^3 \setminus B(0,2R)} ((v)_+^{p-1} - (w)_+^{p-1})(v - w)dx = (p-1)\int_{\mathbb{R}^3 \setminus B(0,2R)} (tv + (1-t)w)_+^{p-2}(v - w)^2 dx \\ &\leq (p-1)\|tv + (1-t)w\|^{p-2}_{L^p(\mathbb{R}^3 \setminus B(0,2R))}\|v - w\|^2_{L^p(\mathbb{R}^3 \setminus B(0,2R))} \\ &\leq C_3\big(\|v\|^{p-2}_{H^1(\mathbb{R}^3 \setminus B(0,2R))} + \|w\|^{p-2}_{H^1(\mathbb{R}^3 \setminus B(0,2R))}\big)\|v - w\|^2_{H^1(\mathbb{R}^3 \setminus B(0,2R))}, \end{aligned} \tag{35}$$

where $t \in [0, 1]$. Then, by (31)–(35), we see that for small $d > 0$,

$$\|\tilde{u}_{j_k} - u_0\|_{H^1(\mathbb{R}^3 \setminus B(0,2R))} \to 0 \tag{36}$$

as $k \to \infty$. Thus, by (36) and the Rellich-Kondrachov compactness theorem, we see that $\tilde{u}_{j_k} \to u_0$ in H^1 as $k \to \infty$. □

Proposition 9 *For $3 < p < 6$, there exist $\bar{c}_0 > 0$ and $\bar{d}_0 > 0$ such that for $c > \bar{c}_0$ and for $0 < d < \bar{d}_0$, $\tilde{I}_c$ has a critical point u in $N_d(\mathcal{X})$ with $\tilde{I}_c(u) \leq \hat{e}_c$.*

Proof Arguing indirectly, suppose $\tilde{I}_c'(u) \neq 0$ for $u \in N_d(\mathcal{X})$ with $\tilde{I}_c(u) \leq \hat{e}_c$. By Proposition 7 and Proposition 8, we can take positive constants $\bar{c}_0$ and $\bar{d}_0$ such that for $c > \bar{c}_0$ and for $0 < d < \bar{d}_0$,

$$\|\tilde{I}_c'(u)\|_{H^{-1}} \geq \nu$$

for $u \in N_d(\mathcal{X}) \setminus N_{d/2}(\mathcal{X})$ with $\tilde{I}_c(u) \le \hat{e}_c$, and

$$\|\tilde{I}_c'(u)\|_{H^{-1}} \ge \sigma_c$$

for $u \in N_d(\mathcal{X})$ with $\tilde{I}_c(u) \le \hat{e}_c$, where $\nu > 0$ is a constant independent of c, and $\sigma_c > 0$ is a constant depending on c. Then, by a deformation argument using Proposition 5 and Proposition 6 (see Proposition 7 in [5] for a detailed argument), we get a contradiction. □

4 Nonrelativistic limit of ground states for 3 < *p* < 6

In this section, we complete the proof of Theorem 1. By Proposition 3, Proposition 5 and Proposition 9, we see that for every $3 < p < 6$, there exists a ground state solutions u_c to (5) such that

$$\limsup_{c\to\infty} I_c(u_c) \le E_\infty. \tag{37}$$

Proposition 10 *Let $3 < p < 6$ and u_c be a ground state solution of (5). Then we have*

$$\sup_{c>\sqrt{\frac{\mu}{m}}} \|u_c\|_{H^1} \le C \text{ and } \inf_{c>\sqrt{\frac{\mu}{m}}} \|u_c\|_{L^p} \ge \frac{1}{C},$$

where $C > 0$ is a constant independent of c.

Proof We note by (37) that

$$\begin{aligned} C_1 &\ge \frac{5p-12}{2} I_c(u_c) - J_c(u_c) + \frac{4-p}{2} P_c(u_c) \\ &= \int_{\mathbb{R}^3} (p-3)|\nabla u_c|^2 + \frac{p-2}{2}\Big(2m\mu - \frac{\mu^2}{c^2}\Big)u_c^2 + \frac{p-2}{2}\Big(\frac{q}{c}\Big)^2 u_c^2 \Phi_{u_c}^2\, dx, \end{aligned} \tag{38}$$

where $C_1 > 0$ is a constant independent of c. This implies $\|u_c\|_{H^1}$ is bounded uniformly in $c > \sqrt{\frac{\mu}{m}}$. Moreover, since $J_c(u_c) = 0$ and $-\frac{1}{q}(c^2 m - \mu) \le \Phi_{u_c} \le 0$, we have for $c > \sqrt{\frac{\mu}{m}}$,

$$\begin{aligned} \int_{\mathbb{R}^3} |u_c|^p &= \int_{\mathbb{R}^3} |\nabla u_c|^2 + \Big(2m\mu - \frac{\mu^2}{c^2}\Big)u_c^2 - \Big(\frac{q}{c}\Big)^2 \Phi_{u_c} u_c^2 \Big(\Phi_{u_c} + \Big(\frac{c}{q}\Big)^2 2q\Big(m - \frac{\mu}{c^2}\Big)\Big) dx \\ &\ge \int_{\mathbb{R}^3} |\nabla u_c|^2 + m\mu u_c^2 dx + \Big(\frac{q}{c}\Big)^2 |\Phi_{u_c}| u_c^2 \Big(\Phi_{u_c} + 2\frac{1}{q}(c^2 m - \mu)\Big) dx \\ &\ge \int_{\mathbb{R}^3} |\nabla u_c|^2 + m\mu u_c^2 dx \ge C_2 \Big(\int_{\mathbb{R}^3} |u_c|^p dx\Big)^{2/p}, \end{aligned} \tag{39}$$

where C_2 is a positive constant indendent of c. Then we have $\int_{\mathbb{R}^3} |u_c|^p dx \ge \frac{1}{C}$, where C is a positive constant indendent of c. □

Proposition 11 *For $3 < p < 6$, let $\{u_c\}_{c>\sqrt{\frac{\mu}{m}}} \subset H^1$ be a ground state solution of (5). Then there exists a sequence $\{x_c\} \in \mathbb{R}^3$ such that $\bar{u}_c(\cdot) \equiv u_c(\cdot + x_c)$ converges to u_∞ in $H^1(\mathbb{R}^3)$ as $c \to \infty$, up to a subsequence, where u_∞ is a ground state solution of (10).*

Proof By Proposition 10 and [14, Lemma 1.1], we have

$$\sup_{x\in\mathbb{R}^3} \int_{B_1(x)} |u_c|^2 dx = \int_{B_1(x_c)} |u_c|^2 dx \ge \bar{C} > 0,$$

where $\bar{C}$ is a constant indepnedent of c.

It follows from Proposition 10 that $\{u_c\}_{c>\sqrt{\frac{\mu}{m}}}$ is bounded in H^1 uniformly in c. Then we may assume $\bar{u}_c \equiv u_c(\cdot + x_c)$ converges to $u_\infty \not\equiv 0$ weakly in H^1 and strongly in $L^q_{loc}(\mathbb{R}^3)$, where $0 < q < 6$. Let $\Phi_{\bar{u}_c}$ be the solution of

$$-\Delta\Phi + \frac{q^2}{c^2}\bar{u}_c^2\Phi = -q(m - \frac{\mu}{c^2})\bar{u}_c^2 \text{ in } \mathbb{R}^3.$$

Since $\|\Phi_{\bar{u}_c}\|_{D^{1,2}} \le C_1 q(m - \frac{\mu}{c^2})\|\bar{u}_c\|^2_{H^1} \le C_2$, where $C_1, C_2 > 0$ are constants independent of c, we may assume that

$$\Phi_{\bar{u}_c} \rightharpoonup \phi_{u_\infty} \text{ weakly in } D^{1,2} \text{ and } \Phi_{\bar{u}_c} \to \phi_{u_\infty} \text{ in } L^q_{loc}(\mathbb{R}^3),$$

as $c \to \infty$, where $0 < q < 6$ and ϕ_{u_∞} is a weak solution of $-\Delta\phi + qmu_\infty^2 = 0$. Then it is standard to show that u_∞ is a non-trivial weak solution of (10).

Next, we claim that u_∞ is a ground state solution of (10). We note that, since u_∞ is a non-trivial weak solution of (10), we have

$$J_\infty(u_\infty) = P_\infty(u_\infty) = 0$$

and

$$\frac{5p-12}{2}I_\infty(u_\infty) - J_\infty(u_\infty) + \frac{4-p}{2}P_\infty(u_\infty) = \int_{\mathbb{R}^3}(p-3)|\nabla u_\infty|^2 + (p-2)m\mu u_\infty^2. \tag{40}$$

Then, by (37), (38) and (40), we have

$$\begin{aligned}
\frac{5p-12}{2}E_\infty &\ge \frac{5p-12}{2}\liminf_{c\to\infty} I_c(u_c) \\
&= \liminf_{c\to\infty}\left(\int_{\mathbb{R}^3}(p-3)|\nabla u_c|^2 + \frac{p-2}{2}\left(2m\mu - \frac{\mu^2}{c^2}\right)u_c^2 + \frac{p-2}{2}\left(\frac{q}{c}\right)^2 u_c^2\Phi_{u_c}^2 dx\right) \\
&\ge \int_{\mathbb{R}^3}(p-3)|\nabla u_\infty|^2 + (p-2)m\mu u_\infty^2 = \frac{5p-12}{2}I_\infty(u_\infty),
\end{aligned}$$

which proves the claim.

Finally, to prove the strong convergence in $H^1(\mathbb{R}^3)$, we note that, by (37), (38), (40), Proposition 10 and the fact that $\bar{u}_c$ converges to $u_\infty \not\equiv 0$ weakly in H^1,

$$\begin{aligned}
\frac{5p-12}{2}E_\infty &\ge \frac{5p-12}{2}\lim_{c\to\infty} I_c(\bar{u}_c) \\
&= \lim_{c\to\infty}\int_{\mathbb{R}^3}(p-3)|\nabla \bar{u}_c|^2 + \frac{p-2}{2}\left(2m\mu - \frac{\mu^2}{c^2}\right)\bar{u}_c^2 + \frac{p-2}{2}\left(\frac{q}{c}\right)^2 \bar{u}_c^2\Phi_{\bar{u}_c}^2 dx \\
&= \int_{\mathbb{R}^3}(p-3)|\nabla u_\infty|^2 + (p-2)m\mu u_\infty^2 dx \\
&\quad + \lim_{c\to\infty}\int_{\mathbb{R}^3}(p-3)|\nabla(\bar{u}_c - u_\infty)|^2 + (p-2)m\mu(\bar{u}_c - u_\infty)^2 dx \\
&= \frac{5p-12}{2}E_\infty + \lim_{c\to\infty}\int_{\mathbb{R}^3}(p-3)|\nabla(\bar{u}_c - u_\infty)|^2 + (p-2)m\mu(\bar{u}_c - u_\infty)^2 dx.
\end{aligned}$$

From this, we deduce that $\bar{u}_c \to u_\infty$ in H^1 as $c \to \infty$, up to a subsequence. This completes the proof. □

Proof of Theorem 1 It is sufficient to show H^2 convergence of $\bar{u}_c$ to u_∞. We may rewrite $\bar{u}_c$ as u_c. We note that, by Lemma 18 and [22, Theorem 4.1], for $u \in H^1$,

$$\sup_{x\in\Omega} |\Phi_u(x)| \le C_1 \|u\|_{H^1}^2 \quad \text{and} \quad \||u|^{p-2}\|_{L^{\frac{6}{p-2}}(\Omega)} = \|u\|_{L^6(\Omega)}^{p-2} \le C_2 \|u\|_{H^1}^{p-2},$$

where Ω is bounded domain in $\mathbb{R}^3$, and C_1 and C_2 are positive constants independent of u and Ω. Then, since $\{\|u_c\|_{H^1}\}_c$ is bounded, we see that $\{\|u_c\|_{L^\infty}\}_c$ is bounded (see [22, Theorem 4.1]).

Since u_∞ and u_c are solutions of (10) and (5) respectively, we have

$$\begin{aligned} -\Delta(u_c - u_\infty) = -2m\mu(u_c - u_\infty) + \left(\frac{\mu}{c}\right)^2 u_c + \left(\frac{q}{c}\right)^2 u_c \Phi_{u_c}^2 - 2q\frac{\mu}{c^2} u_c \Phi_{u_c} \\ + 2qm(u_c\Phi_{u_c} - u_\infty \phi_{u_\infty}) + |u_c|^{p-2}u_c - |u_\infty|^{p-2}u_\infty. \end{aligned} \tag{41}$$

We note that, by Lemma 17, Lemma 19, Lemma 21 and Proposition 11,

$$\begin{aligned} &\|u_c\Phi_{u_c} - u_\infty\phi_{u_\infty}\|_{L^2} \\ &\quad = \|u_c(\Phi_{u_c} - \phi_{u_c}) + (u_c - u_\infty)\phi_{u_c} + (\phi_{u_c} - \phi_{u_\infty})u_\infty\|_{L^2} \\ &\quad \le \|u_c\|_{L^3}\|\Phi_{u_c} - \phi_{u_c}\|_{L^6} + \|u_c - u_\infty\|_{L^3}\|\phi_{u_c}\|_{L^6} + \|\phi_{u_c} - \phi_{u_\infty}\|_{L^6}\|u_\infty\|_{L^3} \to 0 \end{aligned} \tag{42}$$

as $c \to \infty$, and by the fact that $\{\|u_c\|_{L^\infty}\}_c$ is bounded,

$$\left\||u_c|^{p-2}u_c - |u_\infty|^{p-2}u_\infty\right\|_{L^2} = (p-1)\left\||u_\infty + t(u_c - u_\infty)|^{p-2}(u_c - u_\infty)\right\|_{L^2} \to 0 \tag{43}$$

as $c \to \infty$, where $t \in [0, 1]$. Thus, by (41)-(43) and the Calderón–Zygmund inequality, we have

$$\|u_c - u_\infty\|_{H^2(\mathbb{R}^3)} = \| - \Delta(u_c - u_\infty)\|_{L^2} + o(1) = o(1)$$

as $c \to \infty$. □

5 Nonrelativistic limit of two positive solutions for 2 < p < 3

In this section, we will construct two radially symmetric positive solutions of NMKG for $2 < p < 3$. We prove first the existence of a radially symmetric positive solution $v_{c,q}$ of (5) satisfying

$$\lim_{c\to\infty} \|v_{c,q} - v_\infty\|_{H^1} = 0,$$

where v_∞ is a global minimizer of I_∞.

We assume $2 < p < 3$ and denote

$$e_\infty \equiv \inf_{u\in H_r^1} \tilde{I}_\infty(u), \quad \mathcal{X}_r \equiv \{u \in H_r^1 \mid \tilde{I}_\infty(u) = e_\infty\}$$

and

$$N_d(\mathcal{X}_r) \equiv \{u \in H_r^1 \mid \inf_{v\in\mathcal{X}_r} \|u - v\|_{H^1} \le d\},$$

where $d > 0$ is a constant. We remark that, by [21, Theorem 4.3, Corollary 4.4], $\mathcal{X}_r$ is bounded in H^1, and for small $q > 0$, $e_\infty < 0$ and $\mathcal{X}_r \ne \emptyset$. Moreover, since $e_\infty < 0$ for small $q > 0$, and for $u \in \mathcal{X}_r$,

$$e_\infty = \tilde{I}_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + 2m\mu u^2 - qmu^2\phi_u dx - \frac{1}{p}\int_{\mathbb{R}^3} (u)_+^p dx$$

$$\geq \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + 2m\mu u^2 dx - \frac{C_1}{p}\Big(\int_{\mathbb{R}^3} |\nabla u|^2 + u^2 dx\Big)^{p/2},$$

where $C_1 > 0$ is a constant independent of $u \in \mathcal{X}_r$, we see that there exists $\hat{q}_0 > 0$ such that for $0 < q < \hat{q}_0$, $\mathcal{X}_r \neq \emptyset$ and

$$\inf_{u\in\mathcal{X}_r} \|u\|_{H^1} > \hat{d}_0 > 0, \tag{44}$$

where $\hat{d}_0$ is a positive constant. Taking $d \in (0, \frac{\hat{d}_0}{2})$, we deduce that for $0 < q < \hat{q}_0$, $0 \notin N_d(\mathcal{X}_r)$. For $d \in (0, \frac{\hat{d}_0}{2})$ and $0 < q < \hat{q}_0$, take $V_0 \in \mathcal{X}_r$ and set

$$\alpha_c = \inf_{u\in N_d(\mathcal{X}_r)} \tilde{I}_c(u) \text{ and } m_c = \tilde{I}_c(V_0).$$

Clearly, we have $m_c \geq \alpha_c$. We try to find a critical point of $\tilde{I}_c$ in $N_d(\mathcal{X}_r)$.

Proposition 12 *For* $2 < p < 3$, $0 < q < \hat{q}_0$ *and* $d \in (0, \frac{\hat{d}_0}{2})$, *we have*

$$\liminf_{c\to\infty} \alpha_c \geq e_\infty.$$

Proof It is standard to show that there exists $v_c \in N_d(\mathcal{X}_r)$ such that

$$\alpha_c = \tilde{I}_c(v_c),$$

because $\mathcal{X}_r$ is bounded in H^1. Since v_c is bounded in H^1_r uniformly in c, we assume that v_c converges to v in L^s and weakly in H^1 as $c \to \infty$, where $s \in (2, 6)$ and $v \in N_d(\mathcal{X}_r)$. Then, by Lemma 21, we have

$$\begin{aligned}
\liminf_{c\to\infty} \alpha_c &= \liminf_{c\to\infty} \tilde{I}_c(v_c)\\
&= \liminf_{c\to\infty} \Big[\frac{1}{2}\int_{\mathbb{R}^3} |\nabla v_c|^2 + \Big(2m\mu - \frac{\mu^2}{c^2}\Big)v_c^2 - q\Big(m - \frac{\mu}{c^2}\Big)v_c^2\Phi_{v_c} dx - \frac{1}{p}\int_{\mathbb{R}^3} (v_c)_+^p dx\Big]\\
&\geq \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 + 2m\mu v^2 - qmv^2\phi_v dx - \frac{1}{p}\int_{\mathbb{R}^3} (v)_+^p dx = \tilde{I}_\infty(v) \geq e_\infty.
\end{aligned}$$

Proposition 13 *For* $2 < p < 3$ *and* $0 < q < \hat{q}_0$, *we have*

$$m_c \to e_\infty$$

uniformly in q *as* $c \to \infty$.

Proof By Lemma 21,

$$\begin{aligned}
\tilde{I}_c(V_0) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla V_0|^2 + \Big(2m\mu - \frac{\mu^2}{c^2}\Big)V_0^2 - q\Big(m - \frac{\mu}{c^2}\Big)V_0^2\Phi_{V_0} dx - \frac{1}{p}\int_{\mathbb{R}^3} (V_0)_+^p dx\\
&= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla V_0|^2 + 2m\mu V_0^2 - qmV_0^2\phi_{V_0} dx - \frac{1}{p}\int_{\mathbb{R}^3} (V_0)_+^p dx + o(1)\\
&= \tilde{I}_\infty(V_0) + o(1) = e_\infty + o(1)
\end{aligned}$$

as $c \to \infty$. □

Proposition 14 *Let* $2 < p < 3$, $0 < q < \hat{q}_0$ *and* $d \in (0, \frac{\hat{d}_0}{2})$. *For large* $c > 0$ *and for any* $d' \in (0, d)$, *there exists* $\nu_0 \equiv \nu_0(d, d') > 0$ *independent of* $c > 0$ *such that*

$$\inf\{\|\tilde{I}_c'(u)\|_{H^{-1}} \mid \tilde{I}_c(u) \leq m_c, u \in N_d(\mathcal{X}_r) \setminus N_{d'}(\mathcal{X}_r)\} \geq \nu_0 > 0.$$

Proof Let $\{c_i\}_{i=1}^{\infty}$ be such that $\lim_{i\to\infty} c_i = \infty$. It suffices to show that if

$$u_{c_i} \in N_d(\mathcal{X}_r), \quad \tilde{I}_{c_i}(u_{c_i}) \le m_{c_i}, \quad \text{and} \quad \|\tilde{I}'_{c_i}(u_{c_i})\|_{H^{-1}} \to 0$$

as $i \to \infty$, then

$$\inf_{v\in\mathcal{X}_r} \|u_{c_i} - v\|_{H^1} \to 0 \text{ as } i \to \infty.$$

For the sake of simplicity of notation, we write c for c_i. Since $\{u_c\} \subset H_r^1$ is bounded in H^1, we see that u_c converges to u in L^s and weakly in H^1 as $c \to \infty$, up to a subsequence, where $s \in (2, 6)$. Then, by Lemma 21 and Proposition 13, we have

$$\begin{aligned} e_\infty &= \liminf_{c\to\infty} m_c \ge \liminf_{c\to\infty} \tilde{I}_c(u_c) \\ &= \liminf_{c\to\infty} \Big[\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_c|^2 + \Big(2m\mu - \frac{\mu^2}{c^2}\Big)u_c^2 - q\Big(m - \frac{\mu}{c^2}\Big)u_c^2\Phi_{u_c}dx - \frac{1}{p}\int_{\mathbb{R}^3}(u_c)_+^p dx\Big] \\ &\ge \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + 2m\mu u^2 - qmu^2\phi_u dx - \frac{1}{p}\int_{\mathbb{R}^3}(u)_+^p dx = \tilde{I}_\infty(u), \end{aligned}$$

which implies that $e_\infty = \tilde{I}_\infty(u)$.

We claim that $u_c \to u$ in H^1. Indeed, by Lemma 21 and the fact that $\|\tilde{I}'_c(u_c)\|_{H^{-1}} \to 0$ as $c \to \infty$, we see that

$$\begin{aligned} o(1) &= \tilde{I}'_c(u_c)u \\ &= \int_{\mathbb{R}^3} \nabla u_c \cdot \nabla u + \Big(2m\mu - \frac{\mu^2}{c^2}\Big)u_c u - \Big(\frac{q}{c}\Big)^2 u_c u \Phi_{u_c}^2 - 2q\Big(m - \frac{\mu}{c^2}\Big)u_c u\Phi_{u_c} - (u_c)_+^{p-1}u dx \\ &= \int_{\mathbb{R}^3} |\nabla u|^2 + 2m\mu u^2 - 2qmu^2\phi_u - (u)_+^p dx + o(1) \end{aligned} \tag{45}$$

as $c \to \infty$, and

$$\begin{aligned} o(1) &= \tilde{I}'_c(u_c)u_c \\ &= \int_{\mathbb{R}^3} |\nabla u_c|^2 + \Big(2m\mu - \frac{\mu^2}{c^2}\Big)u_c^2 - \Big(\frac{q}{c}\Big)^2 u_c^2 \Phi_{u_c}^2 - 2q\Big(m - \frac{\mu}{c^2}\Big)u_c^2\Phi_{u_c} - (u_c)_+^p dx \\ &= \int_{\mathbb{R}^3} |\nabla u_c|^2 + 2m\mu u_c^2 + 2qmu^2\phi_u - (u)_+^p dx + o(1) \end{aligned} \tag{46}$$

as $c \to \infty$. Thus, by (45) and (46), we have $u_c \to u$ in H^1. □

Proposition 15 *Let $2 < p < 3$, $0 < q < \hat{q}_0$ and $d \in (0, \frac{\hat{d}_0}{2})$. For a fixed $c \in (\sqrt{\frac{\mu}{m}}, \infty)$, suppose that for some $b \in \mathbb{R}$, there exists a sequence $\{u_j\} \subset H_r^1$ satisfying*

$$\begin{aligned} &u_j \in N_d(\mathcal{X}_r), \\ &\|\tilde{I}'_c(u_j)\|_{H^{-1}} \to 0, \\ &\tilde{I}_c(u_j) \to b \text{ as } j \to \infty. \end{aligned}$$

Then b is a critical value of $\tilde{I}_c$, and the sequence $\{u_j\}_{j=1}^{\infty} \subset H_r^1$ has a strongly convergent subsequence in H^1.

Proof Since $\{u_j\} \subset N_d(\mathcal{X}_r)$ is bounded in H^1, we see that u_j converges to u in L^s and weakly in H^1 as $c \to \infty$, up to a subsequence, where $s \in (2, 6)$. It is standard to show that u is a critical point of $\tilde{I}_c$.

We claim that $u_j \to u$ in H^1. Indeed, by Lemma 20 and the fact that $\|\tilde{I}_c'(u_j)\|_{H^{-1}} \to 0$ as $j \to \infty$, we have

$$\begin{aligned} o(1) &= \tilde{I}_c'(u_j)u_j \\ &= \int_{\mathbb{R}^3} |\nabla u_j|^2 + \Big(2m\mu - \frac{\mu^2}{c^2}\Big)u_j^2 - \Big(\frac{q}{c}\Big)^2 u_j^2\Phi_{u_j}^2 - 2q\Big(m - \frac{\mu}{c^2}\Big)u_j^2\Phi_{u_j} - |u_j|^p dx \\ &= \int_{\mathbb{R}^3} |\nabla u_j|^2 + \Big(2m\mu - \frac{\mu^2}{c^2}\Big)u_j^2 - \Big(\frac{q}{c}\Big)^2 u^2\Phi_u^2 - 2q\Big(m - \frac{\mu}{c^2}\Big)u^2\Phi_u - |u|^p dx + o(1) \end{aligned}$$

as $j \to \infty$ and

$$0 = \tilde{I}_c'(u)u = \int_{\mathbb{R}^3} |\nabla u|^2 + \Big(2m\mu - \frac{\mu^2}{c^2}\Big)u^2 - \Big(\frac{q}{c}\Big)^2 u^2\Phi_u^2 - 2q\Big(m - \frac{\mu}{c^2}\Big)u^2\Phi_u - |u|^p dx.$$

Thus, we deduce that $u_j \to u$ in H^1 as $j \to \infty$. □

Proposition 16 *Let* $2 < p < 3$, $0 < q < \hat{q}_0$ *and* $d \in (0, \frac{\hat{d}_0}{2})$. *Then there exists* $\hat{c}_0 > 0$ *such that for* $c > \hat{c}_0$, $\tilde{I}_c$ *has a non-trivial critical point* u *in* $N_d(\mathcal{X}_r)$ *with* $\tilde{I}_c(u) \le m_c$.

Proof Assume that $2 < p < 3$, $0 < q < \hat{q}_0$ and $d \in (0, \frac{\hat{d}_0}{2})$. Suppose $\tilde{I}_c'(u) \neq 0$ for $u \in N_d(\mathcal{X}_r)$ with $\tilde{I}_c(u) \le m_c$. By Proposition 12–15, we can take a positive constant $\hat{c}_0$ such that for $c > \hat{c}_0$ and for $0 < q < \hat{q}_0$,

$$\alpha_c \ge e_\infty - \epsilon_1, \quad |m_c - e_\infty| \le \epsilon_1, \tag{47}$$

$$\|\tilde{I}_c'(u)\|_{H^{-1}} \ge \nu_0 \tag{48}$$

for $u \in N_{\frac{2}{3}d}(\mathcal{X}_r) \setminus N_{\frac{1}{3}d}(\mathcal{X}_r)$ with $\tilde{I}_c(u) \le m_c$, and

$$\|\tilde{I}_c'(u)\|_{H^{-1}} \ge \hat{\sigma}_c \tag{49}$$

for $u \in N_d(\mathcal{X}_r)$ with $\tilde{I}_c(u) \le m_c$, where $d \in (0, \frac{\hat{d}_0}{2})$, $\epsilon_1 \in (0, \frac{d\nu_0}{6})$, and $\hat{\sigma}_c > 0$ is a constant depending on c. For $u \in N_d(\mathcal{X}_r)$ with $\tilde{I}_c(u) \le m_c$, we consider the following ODE:

$$\begin{cases} \frac{d\eta}{d\tau} = -\varphi_1(\tilde{I}_c(\eta))\varphi_2(dist_{H^1}(\eta, \mathcal{X}_r))\frac{\tilde{I}_c'(\eta)}{\|\tilde{I}_c'(\eta)\|_{H^{-1}}}, \\ \eta(0, u) = u, \end{cases}$$

where

$$dist_{H^1}(w, \mathcal{X}_r) = \inf\{\|w - v\|_{H^1} \mid v \in \mathcal{X}_r\}$$

for $w \in H^1$, and $\varphi_1, \varphi_2 : \mathbb{R} \to [0, 1]$ are Lipschitz continuous functions such that

$$\varphi_1(\xi) = \begin{cases} 1 & \text{if } \xi \ge e_\infty - \epsilon_1, \\ 0 & \text{if } \xi \le e_\infty - 2\epsilon_1, \end{cases} \qquad \varphi_2(\xi) = \begin{cases} 1 & \text{if } \xi \le \frac{2}{3}d, \\ 0 & \text{if } \xi \ge d. \end{cases}$$

Let $T = 3\epsilon_1/\hat{\sigma}_c$ and $V_0 \in \mathcal{X}_r$. Since $\tilde{I}_c(\eta(\tau, V_0)) \ge \alpha_c \ge e_\infty - \epsilon_1$ for $\tau \in [0, T]$, we deduce that there exists $t_0 \in [0, T]$ such that

$$dist_{H^1}(\eta(t_0, V_0)) = \frac{2}{3}d. \tag{50}$$

Indeed, if $dist_{H^1}(\eta(\tau, V_0)) < \frac{2}{3}d$ for $\tau \in [0, T]$, by (47) and (49),

$$\tilde{I}_c(\eta(T, V_0)) = \tilde{I}_c(V_0) + \int_0^T \frac{d}{d\tau}\tilde{I}_c(\eta(\tau, V_0))d\tau \le e_\infty + \epsilon_1 - T\hat{\sigma}_c = e_\infty - 2\epsilon_1,$$

which is a contradiction. Assume that t_0 is the first time that satisfies (50). Since $\|\frac{d}{d\tau}\eta\|_{H^1} \le 1$, we see that $t_0 \ge \frac{2}{3}d$ and

$$\eta(\tau, V_0) \in N_{\frac{2}{3}d}(\mathcal{X}_r) \setminus N_{\frac{1}{3}d}(\mathcal{X}_r) \text{ for } \tau \in [t_0 - \frac{1}{3}d, t_0].$$

Then, by (47) and (48), we have

$$\begin{aligned}\tilde{I}_c(\eta(T, V_0)) &= \tilde{I}_c(V_0) + \int_0^T \frac{d}{d\tau}\tilde{I}_c(\eta(\tau, V_0))d\tau \le e_\infty + \epsilon_1 + \int_{t_0-\frac{1}{3}d}^{t_0} \frac{d}{d\tau}\tilde{I}_c(\eta(\tau, V_0))d\tau \\ &= e_\infty + \epsilon_1 - \frac{1}{3}d\nu_0 < e_\infty - \epsilon_1,\end{aligned}$$

which is a contradiction. □

Proof of Theorem 2 Let $2 < p < 3$. By Proposition 16 and the proof of Proposition 14, we prove the existence of a radially symmetric positive solution $v_{c,q}$ of (5) satisfying

$$\limsup_{c\to\infty} \tilde{I}_c(v_{c,q}) \le \inf_{u\in H_r^1} \tilde{I}_\infty(u).$$

By repeating the same procedure in the proof of Proposition 14, we can prove Theorem 2 (ii).

On the other hand, it is known that the ground state solution w_0 of the equation

$$-\Delta u + 2m\mu - |u|^{p-2}u = 0 \text{ in } \mathbb{R}^3 \tag{51}$$

is positive, radially symmetric, up to a translation. It is also non-degenerate in the radial class, i.e., $\mathrm{Ker}L_0 = \{0\}$, where $L_0 : H_r^1 \to H_r^{-1}$ is the linearized operator of (51) at w_0, given by $L_0(w) \equiv -\Delta w + 2m\mu w - (p-1)|u_0|^{p-2}w$.

Exploiting the non-degeneracy of w_0, we see from the implicit function theorem that there exists of a family of radially symmetric solutions $w_{\infty,q}$ of (10) for small $q > 0$ such that $w_{\infty,q} \to w_0$ as $q \to 0$ in H^1 (refer to [20, Proposition 2.1] for detail). As a consequence, one can easily see that $w_{\infty,q}$ is also non-degenerate in the radial class for any small fixed $q > 0$ (see [9, Proposition 3.2]). Then one can once more invoke the implicit function theorem to find a family of nontrivial radial solutions $w_{c,q}$ of (5) for large value $c > 0$ and small $q > 0$, which converges in H^1 to $w_{\infty,q}$ as $c \to \infty$. This proves Theorem 2 (i). □

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Appendix A Basic estimates

Here, we provide with several basic estimates, which are repeatedly invoked in the proofs of main theorems.

Lemma 17 *Let $u \in H^1$. Then we have*

$$\|\phi_u\|_{D^{1,2}} \le Cqm\|u\|_{H^1}^2,$$

where C is a positive constant.

Proof Let $u \in H^1$. Since ϕ_u satisfies

$$-\Delta\phi_u = -qmu^2 \text{ in } \mathbb{R}^3,$$

we have

$$\int_{\mathbb{R}^3} |\nabla\phi_u|^2 dx = -qm \int_{\mathbb{R}^3} u^2 \phi_u dx \le qm\|\phi_u\|_{L^6}\|u^2\|_{L^{6/5}} \le Cqm\|\phi_u\|_{D^{1,2}}\|u\|_{H^1}^2,$$

where C is a positive constant. This implies the result.

Lemma 18 *Let $u \in H^1$. For $c > \sqrt{\frac{\mu}{m}}$, we have*

$$\|\Phi_u\|_{D^{1,2}} \le Cq\Big(m - \frac{\mu}{c^2}\Big)\|u\|_{H^1}^2,$$

where C is a positive constant.

Proof Let $u \in H^1$. Since Φ_u satisfies

$$-\Delta\Phi_u + \Big(\frac{q}{c}\Big)^2 u^2\Phi_u = -q\Big(m - \frac{\mu}{c^2}\Big)u^2 \text{ in } \mathbb{R}^3,$$

and

$$\|u^2\Phi_u\|_{L^1} \le \|\Phi_u\|_{L^6}\|u^2\|_{L^{6/5}} = \|\Phi_u\|_{L^6}\|u\|_{L^{12/5}}^2 \le C\|\Phi_u\|_{D^{1,2}}\|u\|_{H^1}^2, \tag{52}$$

we have for $c > \sqrt{\frac{\mu}{m}}$,

$$\begin{aligned}\|\Phi_u\|_{D^{1,2}}^2 = \int_{\mathbb{R}^3} |\nabla\Phi_u|^2 dx &\le -q\Big(m - \frac{\mu}{c^2}\Big)\int_{\mathbb{R}^3} u^2\Phi_u \\ &\le Cq\Big(m - \frac{\mu}{c^2}\Big)\|u\|_{H^1}^2\|\Phi_u\|_{D^{1,2}},\end{aligned}$$

where C is a positive constant. This implies the result.

Lemma 19 *Let $v, w \in H^1$. Then we have*

$$\|\phi_v - \phi_w\|_{D^{1,2}} \le C\|v + w\|_{H^1}\|v - w\|_{H^1},$$

where $C = C(q, m)$ is a positive constant.

Proof We note that for $v, w \in H^1$,

$$-\Delta(\phi_v - \phi_w) = -qm(v - w)(v + w) \text{ in } \mathbb{R}^3.$$

Then we have

$$\|\phi_v - \phi_w\|_{D^{1,2}} \le C\|v + w\|_{H^1}\|v - w\|_{H^1},$$

where $C = C(q, m)$ is a positive constant.

Lemma 20 *Let $v, w \in H^1$.Then for $c > \sqrt{\frac{\mu}{m}}$, we have*

$$\|\Phi_v - \Phi_w\|_{D^{1,2}} \le C(\|v\|_{H^1}^2 + 1)\|v + w\|_{H^1}\|v - w\|_{L^3},$$

where $C = C(q, m, \mu)$ is a positive constant.

Proof Since Φ_u satisfies

$$-\Delta\Phi_u + \frac{q^2}{c^2}u^2\Phi_u = -q\Big(m - \frac{\mu}{c^2}\Big)u^2 \text{ in } \mathbb{R}^3,$$

we have

$$-\Delta(\Phi_v - \Phi_w) + \frac{q^2}{c^2}w^2(\Phi_v - \Phi_w) = -\frac{q^2}{c^2}(v^2 - w^2)\Phi_v - q\Big(m - \frac{\mu}{c^2}\Big)(v^2 - w^2) \text{ in } \mathbb{R}^3.$$

Multiplying $(\Phi_v - \Phi_w)$ to the above equation and then integrating over $\mathbb{R}^3$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} |\nabla(\Phi_v - \Phi_w)|^2 dx \\
&\le \int_{\mathbb{R}^3} -\frac{q^2}{c^2}(v^2 - w^2)\Phi_v(\Phi_v - \Phi_w) - q\Big(m - \frac{\mu}{c^2}\Big)(v^2 - w^2)(\Phi_v - \Phi_w)dx \\
&\le \frac{q^2}{c^2}\|v + w\|_{L^3}\|v - w\|_{L^3}\|\Phi_v\|_{L^6}\|\Phi_v - \Phi_w\|_{L^6} \\
&+ q(m - \frac{\mu}{c^2})\|v + w\|_{L^2}\|v - w\|_{L^3}\|\Phi_v - \Phi_w\|_{L^6} \\
&\le C_1(\|\Phi_v\|_{D^{1,2}} + 1)\|\Phi_v - \Phi_w\|_{D^{1,2}}\|v + w\|_{H^1}\|v - w\|_{L^3}.
\end{aligned}$$

where $C_1 = C_1(q, m, \mu)$ is a positive constant. Then, by Lemma 18, for $c > \sqrt{\frac{\mu}{m}}$,

$$\|\Phi_v - \Phi_w\|_{D^{1,2}} \le C(\|v\|_{H^1}^2 + 1)\|v + w\|_{H^1}\|v - w\|_{L^3},$$

where $C = C(q, m, \mu)$ is a positive constant.

Lemma 21

$$\|\Phi_v - \phi_w\|_{D^{1,2}} \le C\Big(\frac{1}{c^2}(\|v\|_{H^1}^2 + 1)\|v\|_{H^1}^2 + \|v + w\|_{H^1}\|v - w\|_{L^3}\Big),$$

where $C = C(q, m, \mu)$ is a positive constant.

Proof Since ϕ_w and Φ_v satisfy

$$-\Delta\phi_w = -qmw^2 \text{ in } \mathbb{R}^3 \quad \text{and} \quad -\Delta\Phi_v = -\frac{q^2}{c^2}v^2\Phi_v - q\Big(m - \frac{\mu}{c^2}\Big)v^2 \text{ in } \mathbb{R}^3$$

respectively, we have

$$-\Delta(\Phi_v - \phi_w) = -\frac{q^2}{c^2}v^2\Phi_v + q\frac{\mu}{c^2}v^2 - qm(v^2 - w^2) \text{ in } \mathbb{R}^3.$$

We multiply $(\Phi_v - \phi_w)$ to the above equation and integrate over $\mathbb{R}^3$ to deduce

$$\begin{aligned}
&\int_{\mathbb{R}^3} |\nabla(\Phi_v - \phi_w)|^2 dx \\
&= \frac{1}{c^2}\int_{\mathbb{R}^3}(-q^2v^2\Phi_v + q\mu v^2)(\Phi_v - \phi_w)dx - qm\int_{\mathbb{R}^3}(v^2 - w^2)(\Phi_v - \phi_w)dx \\
&\le \frac{1}{c^2}\|\Phi_v - \phi_w\|_{L^6}(q^2\|\Phi_v\|_{L^6}\|v^2\|_{L^{3/2}} + q\mu\|v^2\|_{L^{6/5}}) + qm\|\Phi_v - \phi_w\|_{L^6}\|v + w\|_{L^2}\|v - w\|_{L^3} \\
&\le C_1\|\Phi_v - \phi_w\|_{D^{1,2}}\Big(\frac{1}{c^2}(\|\Phi_v\|_{D^{1,2}}\|v\|_{H^1}^2 + \|v\|_{H^1}^2) + \|v + w\|_{H^1}\|v - w\|_{L^3}\Big),
\end{aligned}$$

where $C_1 = C_1(q, m, \mu)$ is a positive constant. Then, by Lemma 18, we have

$$\|\Phi_v - \phi_w\|_{D^{1,2}} \le C\Big(\frac{1}{c^2}(\|v\|^2_{H^1} + 1)\|v\|^2_{H^1} + \|v + w\|_{H^1}\|v - w\|_{L^3}\Big),$$

where $C = C(q, m, \mu)$ is a positive constant. □

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