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Calculus of Variations

Nonlocal semilinear elliptic problems with singular nonlinearity

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Abstract
We consider a Lazer-Mckenna-type problem involving the fractional Laplacian and singular nonlinearity. We investigate existence, regularity and uniqueness of solutions in light of the interplay between the nonlinearities and the summability of the datum.

Mathematics Subject Classification 35R11 · 35J75 · 35S15 · 47G20 · 35B51

1 Introduction

In this paper, we are interested in the existence, regularity and uniqueness of solutions for the following nonlocal problem

$$\begin{cases} (-\Delta)^s u = \frac{f(x)}{u^\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N > 2s$, of class $\mathcal{C}^{1,1}$, $s \in (0, 1)$, $\gamma > 0$, $f \in L^m(\Omega)$, $m \geq 1$, is a non-negative function and $(-\Delta)^s$ is the fractional Laplacian operator defined by

$$(-\Delta)^s u = a(N, s) P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy,$$

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where "$P.V.$" stands for the principal value and $a(N, s)$ is a positive renormalizing constant, depending only on N and s, given by

$$a(N, s) = \frac{4^s \Gamma(\frac{N}{2} + s)}{\pi^{\frac{N}{2}}} \frac{s}{\Gamma(1-s)}$$

to ensure that

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F}u), \quad \xi \in \mathbb{R}^N, s \in (0,1) \text{ and } u \in \mathcal{S}(\mathbb{R}^N),$$

where $\mathcal{F}u$ stands for the Fourier transform of u belonging to the Schwartz class $\mathcal{S}(\mathbb{R}^N)$. More details on the operator $(-\Delta)^s$ and the asymptotic behaviour of $a(N, s)$ can be found in [26].

In the case of semilinear local problem corresponding to $s = 1$, the study of singular elliptic equations was initiated in the pioneering work [22] which constitutes the starting point of a wide literature about singular semilinear elliptic equations. Let us start recalling the important result of Lazer-McKenna [36]. Under regularity assumptions on Ω and if $0 < f \in \mathcal{C}^\alpha(\overline{\Omega})$, the authors obtained an optimal power related to the existence of finite energy solutions. In fact, a solution lying in $H_0^1(\Omega)$ should exists if and only if $\gamma < 3$ while it is not in $\mathcal{C}^1(\overline{\Omega})$ if $\gamma > 1$. The threshold 3 is analysed in [51] when the datum f is a positive L^1 function defined on Ω. In that paper [51], the authors provide an extension of the classical Lazer-McKenna obstruction. Existence and uniqueness results for (1.1) are obtained in [19] while in [16,24] the authors showed that (1.1) has a solution u for every f in $L^1(\Omega)$ and for every $\gamma > 0$ and how the regularity of this solution u depends on the summability of f and on γ. In the case where the function f belongs to $L^m(\Omega)$ with $m \geq 1$, Boccardo and Orsina [15] proved the existence and regularity of a distributional solution $u \in W_0^{1,q}(\Omega)$ where $q = \frac{Nm(\gamma+1)}{N-m(1-\gamma)}$ if $0 < \gamma < 1$ and $f \in L^m(\Omega)$, $1 \leq m < \left(\frac{2^*}{1-\gamma}\right)'$, while $u \in H_0^1(\Omega)$ if $f \in L^m(\Omega)$ with $m = \left(\frac{2^*}{1-\gamma}\right)'$. In the case where $f \in L^1(\Omega)$, if $\gamma = 1$ then $u \in H_0^1(\Omega)$, while if $\gamma > 1$ then $u \in H_{loc}^1(\Omega)$ and $u^{\frac{\gamma+1}{2}} \in H_0^1(\Omega)$. In connection with the problem studied in [15], uniqueness of finite energy solutions was established in [14] where the main ingredient is the extension of the set of admissible test functions. We will use the same idea in this case of fractional Laplacian. In [9] the authors proved that if the non-negative function $f \in L^m(\Omega)$, $m > 1$, is strictly far away from zero on Ω (that is there exists a positive constant f_0 such that $f \geq f_0 > 0$ a.e. $x \in \Omega$) then $u^\alpha \in H_0^1(\Omega)$ for every $\alpha \in \left(\frac{(m+1)(\gamma+1)}{4m}, \frac{\gamma+1}{2}\right]$ if $1 < \gamma < \frac{3m-1}{m+1}$. Some related existence and regularity results for local problems with singular nonlinearity involving reaction or absorption terms are proved in [21,40,41]. Let us also mention the contributions in [2,17,32,35,42,43,50] where related problems involving singular nonlinearities are considered. It is worth recalling here that singular local semilinear elliptic problems such as (1.1) arise in various contexts of chemical heterogeneous catalysts [10], non-Newtonian fluids [28] as well as heat conduction in electrically conducting materials (the term u^γ describes the resistivity of the material), see for instance [30,39].

Let us now discuss the nonlocal problem (1.1). Recall first that a rich amount of research work has been done on nonlocal problems of either elliptic or parabolic types, we refer for instance to [3–5,7,37,52]. Starting with the case $\gamma = 0$, the problem (1.1) with L^1-data was studied in [1,18,38] where a general fractional Laplacian operator including $(-\Delta)^s$ is involved, while for bounded Radon measure data it was investigated in [33,44]. In the case where $\gamma > 0$, existence and regularity results of solutions to (1.1) were established in [7] when the datum f is a Hölder continuous function and behaviours basically as $\frac{1}{dist^\beta(x,\partial\Omega)}$ for some β such that $0 \leq \beta < 2s$. Existence and uniqueness results for positive solutions of the problem (1.1) have been also obtained in [11,18]. It has been shown in [18] that (1.1) has a weak solution $u \in X_0^s(\Omega)$ when $0 < \gamma \leq 1$ and $f \in L^{\overline{m}}(\Omega)$ with $\overline{m} := \frac{2N}{N+2s+\gamma(N-2s)}$, while if $\gamma > 1$ and $f \in L^1(\Omega)$ then (1.1) has a weak solution $u \in H_{loc}^s(\Omega)$ with $u^{\frac{\gamma+1}{2}} \in X_0^s(\Omega)$. In the same spirit, the existence of positive solutions have been also established in [11] according to the range of $\gamma > 0$ and to the summability of f. Precisely, in that paper [11] it has been proven that if $\gamma \leq 1$ and $f \in L^{(2_s^*)'}(\Omega)$, $2_s^* := \frac{2N}{N-2s}$ and $(2_s^*)' := \frac{2N}{N+2s}$, then (1.1) has a solution $u \in X_0^s(\Omega) \cap L^{(\gamma+1)2_s^*}(\Omega)$, while if $\gamma > 1$ and $f \in L^1(\Omega)$ then (1.1) has a solution u such that $u^{\frac{\gamma+1}{2}} \in X_0^s(\Omega)$.

It is worth pointing out that the interest brought to the fractional Laplacian operator is due to the wide range of its applications, for instance in thin obstacle problems [23], in crystal dislocation [27] and in phase transition [49].

In the present paper, our aim is to lead investigations about the existence and regularity of positive solutions to (1.1) establishing some missing results in [11,18]. The case where $\gamma = 1$ is treated in [11,18]. We study the case where $0 < \gamma < 1$ and $f \in L^m(\Omega)$ with $1 \leq m < \overline{m}$ which provides infinite energy solutions (see Theorem 3.1 bellow) and we prove the existence of finite energy solutions to problem (1.1) in the case $\gamma > 1$ under some suitable assumptions on the datum f. Further, to show the accuracy of our results we highlight the relationship with the Lazer-Mckenna condition. We also provide some regularity results for solutions as well as the uniqueness of finite energy solutions.

The plan of the paper is organized as follows : in Sect. 2 we give some basic notations and tools that we will need in this paper, as well as the meaning of solution for the problem (1.1) and some useful algebraic inequalities. In Sect. 3 we present the main results of the paper i.e. Theorems 3.1, 3.2, 3.3 and 3.4. Comments and comparisons with previous results known in the topic are also provided. In Sect. 4 we prove some a priori estimates for the approximate solutions which we use to prove the main results. In Sect. 5 we prove some regularity results. At the end, we give an appendix.

2 Basic notations and useful tools

In this section we give some basic facts about fractional Sobolev spaces. For a detailed expository, we refer to [13,25,26]. Let Ω be an open subset in $\mathbb{R}^N$. For any $0 < s < 1$ and for any $1 \leq q < +\infty$, the fractional Sobolev space $W^{s,q}(\Omega)$ is defined as the set of all functions (equivalence classes) u in $L^q(\Omega)$ such that

$$\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^q}{|x - y|^{N+qs}} dydx < \infty.$$

$W^{s,q}(\Omega)$, also known as *Aronszajn*, *Gagliardo* or *Slobodeckij* spaces, is a Banach space when equipped with the natural norm

$$\|u\|_{W^{s,q}(\Omega)} = \|u\|_{L^q(\Omega)} + \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^q}{|x - y|^{N+qs}} dydx \right)^{\frac{1}{q}}. \tag{2.1}$$

It can be regarded as an intermediate space between $L^q(\Omega)$ and $W^{1,q}(\Omega)$. Recall that the space $W^{s,q}(\Omega)$ is reflexive for all $q > 1$ (see [34, Theorem 6.8.4]). We point out that if $0 < s \leq s' < 1$ then $W^{s',q}(\Omega)$ is continuously embedded in $W^{s,q}(\Omega)$ (see [26, Proposition 2.1]).

Throughout the paper, we will make use of the notations $supp(f)$ to designate the support of the function f and $\omega \subset\subset \Omega$ that means ω is a compact subset of Ω.

Let us define $W_0^{s,q}(\Omega)$ as the closure of the set $\mathcal{C}_0^\infty(\Omega)$ in $W^{s,q}(\mathbb{R}^N)$ with respect to the norm $\|\cdot\|_{W^{s,q}(\mathbb{R}^N)}$ defined in (2.1) where

$$\mathcal{C}_0^\infty(\Omega) := \Big\{ f : \mathbb{R}^N \to \mathbb{R} / f \in \mathcal{C}^\infty(\mathbb{R}^N),\, supp(f) \subset\subset \Omega \Big\}.$$

$W_0^{s,q}(\Omega)$ is a Banach space under the norm $\|\cdot\|_{W^{s,q}(\Omega)}$. Let us recall the following Fractional Poincaré-type inequality.

Lemma 2.1 *([6]) Let Ω be a bounded open subset of $\mathbb{R}^N$ of class $\mathcal{C}^{0,1}$, $q \geq 1$ and let $0 < s < 1$. Then there exists a constant $C(N, s, \Omega)$ such that for any $f \in W_0^{s,q}(\Omega)$ one has*

$$\|f\|^q_{L^q(\Omega)} \leq C(N, s, \Omega) \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^q}{|x - y|^{N+qs}} dydx.$$

Under the same assumptions of Lemma 2.1, the Banach space $W_0^{s,q}(\Omega)$ can be also endowed with the norm

$$\|u\|_{W_0^{s,q}(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^q}{|x - y|^{N+qs}} dydx \right)^{\frac{1}{q}}$$

which is equivalent to $\|u\|_{W^{s,q}(\Omega)}$. In the case where $q = 2$, we note $W^{s,2}(\Omega) = H^s(\Omega)$ and $W_0^{s,2}(\Omega) = H_0^s(\Omega)$. Endowed with the inner product

$$\langle u, v \rangle_{H_0^s(\Omega)} = \int_\Omega \int_\Omega \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dydx.$$

$(H_0^s(\Omega), \|\cdot\|_{H_0^s(\Omega)})$ is a Hilbert space. Now, we define the following spaces

$$H^s_{loc}(\Omega) = \Big\{ u : \Omega \to \mathbb{R} : u \in L^2(K),\ \int_K \int_K \tfrac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dydx < \infty, \text{ for every } K \subset\subset \Omega \Big\}$$

and

$$X_0^s(\Omega) = \Big\{ f \in H^s(\mathbb{R}^N) / f = 0 \text{ a.e. in } \mathcal{C}\Omega \Big\},$$

where from now on $\mathcal{C}\Omega := \mathbb{R}^N \setminus \Omega$ stands for the complementary of Ω in $\mathbb{R}^N$. Observe that if Ω has a continuous boundary, by [29, Theorem 6] (see also [31, Theorem 1.4.2.2]) we can infer that $X_0^s(\Omega) \subset H_0^s(\Omega)$. Indeed, if $f \in X_0^s(\Omega)$ then, by [29, Theorem 6] there exists a sequence $\{\rho_n\}_n$ that belongs to $\mathcal{C}_0^\infty(\Omega)$ satisfying

$$\|\rho_n - f\|_{H^s(\mathbb{R}^N)} \to 0 \text{ as } n \to +\infty$$

and in particular we obtain

$$\|\rho_n - f\|_{H^s(\Omega)} \to 0 \text{ as } n \to +\infty,$$

which yields $f \in H_0^s(\Omega)$. Under the same assumptions of Lemma 2.1, the following quantity

$$\|u\|_{X_0^s(\Omega)} = \left(\int_Q \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dydx \right)^{\frac{1}{2}},$$

where $Q = \mathbb{R}^{2N} \backslash (\mathcal{C}\Omega \times \mathcal{C}\Omega)$, is a norm on $X_0^s(\Omega)$. It is well known that the pair $(X_0^s(\Omega), \|\cdot\|_{X_0^s(\Omega)})$ is a Hilbert space (see [47, Lemma 7]). It is worth recalling that for any u and φ belonging to $H^s(\mathbb{R}^N)$, we have the following duality product

$$\int_{\mathbb{R}^N} (-\Delta)^s u\varphi dx = \frac{a(N,s)}{2} \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}} dydx.$$

Thus, it can be seen that

$$(-\Delta)^s : H^s(\mathbb{R}^N) \to H^{-s}(\mathbb{R}^N)$$

is a continuous and symmetric operator defined on $H^s(\mathbb{R}^N)$. In the particular case, if u and φ belong to $X_0^s(\Omega)$, we have

$$\int_{\mathbb{R}^N} (-\Delta)^s u\varphi dx = \frac{a(N,s)}{2} \int_{Q} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}} dydx.$$

For $N > 2s$ we define the fractional Sobolev critical exponent $2_s^* = \frac{2N}{N-2s}$. The following result is a fractional version of the Sobolev inequality which provides a continuous embedding of $H_0^s(\Omega)$ in the critical Lebesgue space $L^{2_s^*}(\Omega)$. The proof can be found, for example, in [26,45].

Theorem 2.1 *(Fractional Sobolev embedding) Let $0 < s < 1$ be such that $N > 2s$. Then, there exists a constant $S(N,s)$ depending only on N and s, such that for all $f \in \mathcal{C}_0^\infty(\mathbb{R}^N)$*

$$\|f\|^2_{L^{2_s^*}(\mathbb{R}^N)} \leq S(N,s) \iint_{\mathbb{R}^{2N}} \frac{|f(x)-f(y)|^2}{|x-y|^{N+2s}} dydx. \tag{2.2}$$

We now define the meaning we will give to the solution of the problem(1.1).

Definition 2.1 Let $f \in L^1(\Omega)$ be a non-negative function. By a weak solution of the problem (1.1), we mean a measurable function u satisfying

$$\forall \omega \subset\subset \Omega,\ \exists c_\omega > 0 \ :\ u(x) \geq c_\omega > 0, \text{ in } \omega \tag{2.3}$$

and

$$\frac{a(N,s)}{2} \int_Q \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}} dydx = \int_\Omega \frac{f\varphi}{u^\gamma} dx, \tag{2.4}$$

for any $\varphi \in \mathcal{C}_0^\infty(\Omega)$.

Definition 2.2 We say that $u \in X_0^s(\Omega)$ is a finite energy solution of (1.1) if it is a weak solution u of problem (1.1) which further satisfies (2.4) for every $\varphi \in X_0^s(\Omega)$.

Remark 2.1 By Lemma 5.4, if $u \in X_0^s(\Omega)$ is a weak solution of problem (1.1) (in the sense Definition 2.1), then u is a finite energy solution. In other words if $u \in X_0^s(\Omega)$ the two definitions 2.1 and 2.2 are equivalent.

We will also need the following technical algebraic inequalities (See [5, Lemma 2.22]).

Lemma 2.2 *i)- Let $\alpha > 0$. For every $x, y \geq 0$ one has*

$$(x-y)(x^\alpha - y^\alpha) \geq \frac{4\alpha}{(\alpha+1)^2}(x^{\frac{\alpha+1}{2}} - y^{\frac{\alpha+1}{2}})^2.$$

ii)- Let $0 < \alpha < 1$. For every $x,\ y \geq 0$ with $x \neq y$ one has

$$\frac{x-y}{x^\alpha - y^\alpha} \leq \frac{1}{\alpha}(x^{1-\alpha} + y^{1-\alpha}).$$

iii)- Let $0 < \alpha \leq 1$, then for every $x,\ y \geq 0$ one has

$$|x^\alpha - y^\alpha| \leq |x-y|^\alpha.$$

iv)- Let $\alpha \geq 1$, then for every $x,\ y \geq 0$ one has

$$|x^\alpha - y^\alpha| \leq \alpha(x^{\alpha-1} + y^{\alpha-1})|x-y|.$$

v)- Let $\alpha \geq 1$, then for every $x,\ y \geq 0$ one has

$$|x+y|^{\alpha-1}|x-y| \leq C_\alpha |x^\alpha - y^\alpha|,$$

where C_α is a constant depending only on α.

3 Main results

3.1 The case $0 < \gamma < 1$: Infinite energy solutions

We consider the problem (1.1) under the assumption $0 < \gamma < 1$. We recall that in this case it is proved in [18] that (1.1) has energy solutions when $f \in L^{\overline{m}}(\Omega)$, where $\overline{m}$ stands for the Hölder conjugate exponent of $\frac{2_s^*}{1-\gamma}$, that is $\overline{m} := \left(\frac{2_s^*}{1-\gamma}\right)' = \frac{2N}{N+2s+\gamma(N-2s)}$. It is in our purpose here to investigate the remaining range of summability of source terms corresponding to the data $f \in L^m(\Omega)$ with $1 \leq m < \overline{m}$. We show that the problem (1.1) has solutions lying in a fractional Sobolev space larger than $H_0^s(\Omega)$.

Theorem 3.1 *Let $0 < \gamma < 1$ and let $f \in L^m(\Omega)$, with $1 \leq m < \overline{m}$. Then the problem (1.1) admits a weak solution $u \in W_0^{s_1,\overline{q}}(\Omega)$ for all $s_1 < s$ with $\overline{q} = \frac{Nm(1+\gamma)}{N-sm(1-\gamma)}$. Furthermore, $u \in L^\sigma(\Omega)$ where $\sigma = \frac{Nm(1+\gamma)}{N-2sm}$.*

Remark 3.1 Note that $\overline{q} < 2$ since $m < \overline{m}$. Moreover, the exponent σ is well defined. Indeed, since $N > 2s$ we have

$$4ms < m(N+2s) < m\Big(N+2s+\gamma(N-2s)\Big).$$

As $m < \overline{m} := \frac{2N}{N+2s+\gamma(N-2s)}$, we get $4ms < 2N$.

Remark 3.2 Observe that the inclusion $W_0^{s_1,q}(\Omega) \subset W_0^{s_2,q}(\Omega)$ holds for any $s_2 < s_1$ (see [26]). So we infer that it is sufficient to choose s_1 very close to s that is $\frac{s}{2-s} \leq s_1 < s$ which implies that the results in Theorem 3.1 covers that obtained in [15, Theorem 5.6] when $s \to 1$.

Remark 3.3 Notice that if $\gamma = 0$ the problem (1.1) reduces to

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{3.1}$$

In [38] the authors proved the existence of a unique weak solution u of the problem (3.1) such that

1. If $f \in L^1(\Omega)$ then $u \in L^q(\Omega)$ for every $q < \frac{N}{N-2s}$.
2. If $f \in L^m(\Omega)$, with $1 < m < \frac{2N}{N+2s}$, then $u \in L^{\frac{Nm}{N-2sm}}(\Omega)$.

We point out that when $1 < m < \overline{m}$ we have a kind of 'continuity' of the summability of the solution with respect to γ. If we let $\gamma \to 0$, the value of $\sigma = \frac{Nm(1+\gamma)}{N-2sm}$ tends to $\frac{Nm}{N-2sm}$ which is exactly the summability of solutions obtained in [38]. However, this 'continuity' fails to hold when $m = 1$ since $\sigma = \frac{N(1+\gamma)}{N-2s}$ tends to $\frac{N}{N-2s}$ but the solutions obtained in [38] belong to $L^q(\Omega)$ for every $q < \frac{N}{N-2s}$. In fact, the case where $\gamma = 0$ can not be considered, this is mainly due to the inequality (4.13) where we divide by γ.

3.2 The case $\gamma > 1$: finite energy solutions

Let us recall that Lazer and McKenna [36] proved that the problem

$$\begin{cases} -\Delta u = \frac{f(x)}{u^\gamma} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

where the datum f is regular enough (say Hölder continuous) and bounded away from zero on Ω, admits a unique solution $u \in H_0^1(\Omega)$ if and only if $\gamma < 3$. In the case where f is a non-negative function such that $f \in L^m(\Omega)$ with $m > 1$ and strictly far away from zero on Ω, the authors [9] proved that if $1 < \gamma < \frac{3m-1}{m+1}$ then $u \in H_0^1(\Omega)$. As regards the case where the datum $f \in L^1(\Omega)$, the problem 3.2 has only a local solution $u \in H^1_{loc}(\Omega)$ which does not belong to $H_0^1(\Omega)$ (see [15, Theorem 4.2]). In the case of the fractional Laplacian operator, J.Giacomoni et al.[7] studied the following problem

$$\begin{cases} (-\Delta)^s u = \frac{f(x)}{u^\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.3}$$

where f is a Hölder continuous function such that $f \simeq \frac{1}{dist^\beta(x, \partial\Omega)}$, with $0 \le \beta < 2s$. They proved that if $\frac{\beta}{s} + \gamma > 1$ then the problem (3.3) admits a unique solution $u \in X_0^s(\Omega)$ if and only if $2\beta + \gamma(2s-1) < 2s+1$. This last inequality implies $\gamma(2s-1) < 2s+1$. So that letting s tends to 1^- one can find $\gamma < 3$ which is exactly the Lazer-Mckenna condition.

In this section, we investigate the existence of finite energy solutions for (1.1) when $\gamma > 1$ and $f \in L^m(\Omega)$, with $m \ge 1$. We impose some assumptions on the datum f and γ that provide solutions for (1.1) in $X_0^s(\Omega)$. The first result deals with data f strictly far away from zero.

Theorem 3.2 *Let $\gamma > 1$ and $s \in (0,1)$. Assume that $f \in L^m(\Omega)$, $m > 1$, is such that there exists a positive constant f_0 satisfying $f(x) \ge f_0 > 0$ a.e. $x \in \Omega$. Then the problem (1.1) admits a weak solution $u \in H^s_{loc}(\Omega)$ such that $u^\alpha \in X_0^s(\Omega)$ for every $\alpha \in \left(\max\left(\frac{1}{2}, \frac{(\gamma+1)(2sm-m+1)}{4sm}\right), \frac{\gamma+1}{2}\right]$. In particular if γ satisfies*

$$(m(2s-1)+1)\gamma < m(2s+1)-1, \tag{3.4}$$

then $u \in X_0^s(\Omega)$.

Remark 3.4 Observe that from (3.4) we get $\max\left(\frac{1}{2}, \frac{(\gamma+1)(2sm-m+1)}{4sm}\right) < 1 < \frac{\gamma+1}{2}$, so that $\alpha = 1$ can be chosen to obtain $u \in X_0^s(\Omega)$. Furthermore, notice that for every $m > 1$ (3.4) reads as

$$\gamma(2s-1)+\frac{\gamma}{m}<2s+1-\frac{1}{m},$$

which implies $\gamma(2s-1)<2s+1$ and this is exactly the necessary and sufficient condition for the existence of the unique solution in $X_0^s(\Omega)$ obtained in [7, Theorem 1.2 ii)] when $\beta=0$. We also observe that when s tends to 1^-, the condition (3.4) yields $1<\gamma<\frac{3m-1}{m+1}$ and therefore Theorem 3.2 reduces to the same result stated in [9, Theorem 3]. Furthermore, letting m tends to $+\infty$ in the last inequality we get $1<\gamma<3$, which can be seen as an extension of the Lazer-Mckenna condition [36] for obtaining finite energy solutions to strictly positive L^∞-data.

Remark 3.5 In the local case corresponding to $s=1$, it is known that the threshold $\frac{3m-1}{m+1}$ obtained in [9, Theorem 3] is not the optimal one. Using [51, Theorem 1], Oliva and Petitta [42] proved that the optimal threshold is $3-\frac{2}{m}$. For the nonlocal problem (1.1), the situation is somehow different. Notice that for $m>1$ if $\frac{m-1}{2m}<s<1$ then (3.4) reads as

$$\gamma<h(s):=\frac{m(2s+1)-1}{m(2s-1)+1}.$$

The optimality is lost since s is varying, however we can obtain more information. Observe that the function h decreases from infinity to $\frac{3m-1}{m+1}$ as $\frac{m-1}{2m}<s<1$. Setting $\bar{s}:=1-\frac{1}{2m}$, one has $\frac{m-1}{2m}<\bar{s}<1$ and $h(\bar{s})=3-\frac{2}{m}$. Thus, for $s<\bar{s}$ we have $h(\bar{s})=3-\frac{2}{m}<h(s)$. On the other hand, if $0<s\le\frac{m-1}{2m}$ then (3.4) is satisfied for every $\gamma>1$. We conclude that the range of γ is wide than the one of the local case.

We point out that we can avoid the hypothesis that the source term f is far from zero and we continue to obtain energy solutions. This is stated in the following theorem.

Theorem 3.3 *Let $\gamma>1$ and $s\in(0,1)$. Suppose that $f\in L^m(\Omega)$ with $m>1$. Then the problem (1.1) admits a weak solution $u\in H^s_{loc}(\Omega)$ such that $u^\alpha\in X_0^s(\Omega)$ for every $\alpha\in\left(\max\left(\frac{1}{2},\frac{sm(\gamma+1)-m+1}{2sm}\right),\frac{\gamma+1}{2}\right]$. In particular, if $1<\gamma<1+\frac{m-1}{sm}$ then $u\in X_0^s(\Omega)$.*

Here again, letting s tends to 1^- and m tends to $+\infty$ we obtain $1<\gamma<2$ which is a restriction of the Lazer-Mckenna condition to positive L^m-data, $m>1$. Notice that the case where $m=1$ can not be considered in the two last theorems, since the range of α will be empty. However, if we consider data $f\in L^1(\Omega)$ with compact support in Ω we can also obtain an energy solution. This is stated in the following theorem.

Theorem 3.4 *Let $\gamma>1$ and $s\in(0,1)$. Suppose that $f\in L^1(\Omega)$ with compact support in Ω. Then the problem (1.1) admits a weak solution $u\in H^s_{loc}(\Omega)$ such that $u^\alpha\in X_0^s(\Omega)$ for every $\alpha\in\left(\frac{1}{2},\frac{\gamma+1}{2}\right]$. In particular, $u\in X_0^s(\Omega)$.*

We point out that the Lazer-Mckenna condition vanishes when we deal with positive L^1-data having compact support.

3.3 Uniqueness of finite energy solutions

As mentioned in the introduction, the existence of weak solutions for the problem (1.1) lying $X_0^s(\Omega)$ has been proved in [18, Theorem 3.2] when $0<\gamma\le 1$ and $f\in L^{\overline{m}}(\Omega)$. In the case where $\gamma > 1$, the existence of a weak solution $u \in X_0^s(\Omega)$ to the problem (1.1) is obtained in the previous theorems 3.2, 3.3 and 3.4. In the following theorem we prove the uniqueness of finite energy solutions to the problem (1.1).

Theorem 3.5 *Let $\gamma > 0$ and $s \in (0, 1)$. Let $0 < f \in L^1(\Omega)$ be such that the problem (1.1) admits a finite energy solution $u \in X_0^s(\Omega)$ (in the sense of Definition 2.2). Then u is unique.*

4 Proof of main results

4.1 Approximated problems

Consider the sequence of approximate problems

$$\begin{cases} (-\Delta)^s u_n = \frac{f_n}{(u_n+\frac{1}{n})^\gamma} & \text{in } \Omega, \\ u_n > 0 & \text{in } \Omega, \\ u_n = 0 & \text{on } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{4.1}$$

where $f_n = \min(f, n)$. The following results are proved in [11].

Lemma 4.1 *([11, Lemma 3.1]) For each integer $n \in \mathbb{N}$, the problem (4.1) admits a non-negative solution $u_n \in X_0^s(\Omega) \cap L^\infty(\Omega)$ in the sense*

$$\frac{a(N,s)}{2}\int_Q \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx = \int_\Omega \frac{f_n\varphi}{(u_n+\frac{1}{n})^\gamma}dx,$$

for every $\varphi \in X_0^s(\Omega)$.

Lemma 4.2 *([11, Lemma 3.2]) The sequence $\{u_n\}_{n\in\mathbb{N}}$ is an increasing and for every subset $\omega \subset\subset \Omega$, there exists a positive constant c_ω, not depending on n, such that*

$$u_n(x) \geq c_\omega > 0, \quad \textit{for every } x \in \omega \textit{ and for every } n \in \mathbb{N}.$$

Lemma 4.3 *Let $\gamma > 1$, $f \in L^1(\Omega)$ and let $u_n \in X_0^s(\Omega) \cap L^\infty(\Omega)$ be a solution of the problem (4.1). Then the sequence $\{u_n\}$ is uniformly bounded in $H^s_{loc}(\Omega)$.*

Proof Taking u_n^γ a test function in (4.1), we obtain

$$\int_Q \frac{(u_n(x)-u_n(y))(u_n^\gamma(x)-u_n^\gamma(y))}{|x-y|^{N+2s}}dydx \leq \frac{2\|f\|_{L^1(\Omega)}}{a(N,s)}. \tag{4.2}$$

An application of the item $i)$ in Lemma 2.2 yields

$$\int_Q \frac{\left|u_n^{\frac{\gamma+1}{2}}(x)-u_n^{\frac{\gamma+1}{2}}(y)\right|^2}{|x-y|^{N+2s}}dydx \leq \frac{(\gamma+1)^2}{2\gamma a(N,s)}\|f\|_{L^1(\Omega)}.$$

Then by the Sobolev inequality (2.2) we get

$$\int_\Omega |u_n(x)|^{\frac{(\gamma+1)}{2}2_s^*}dx \leq \left(S(N,s)\frac{(\gamma+1)^2}{2\gamma a(N,s)}\right)^{\frac{N}{N-2s}}\|f\|_{L^1(\Omega)}^{\frac{N}{N-2s}}.$$

As $\frac{(\gamma+1)}{2}2_s^* > 2$, the sequence $\{u_n\}_n$ is uniformly bounded in $L^2(\Omega)$. On the other hand, let ω be a compact subset of Ω. Applying the item $v)$ in Lemma 2.2 (recall that $\gamma > 1$) and

Lemma 4.2 in the left-hand side of the inequality (4.2), we obtain

$$
\begin{aligned}
&\int_Q \frac{|u_n(x)-u_n(y)||u_n^\gamma(x)-u_n^\gamma(y)|}{|x-y|^{N+2s}}dydx \\
&\geq \frac{1}{C_\gamma}\int_\Omega\int_\Omega \frac{|u_n(x)-u_n(y)|^2|u_n(x)+u_n(y)|^{\gamma-1}}{|x-y|^{N+2s}}dydx \\
&\geq \frac{1}{C_\gamma}\int_\omega\int_\omega \frac{|u_n(x)-u_n(y)|^2|u_n(x)+u_n(y)|^{\gamma-1}}{|x-y|^{N+2s}}dydx \\
&\geq \frac{1}{C_\gamma}(2c_\omega)^{\gamma-1}\int_\omega\int_\omega \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}dydx.
\end{aligned}
$$

This shows that $\{u_n\}_n$ is uniformly bounded in $H^s_{loc}(\Omega)$. □

Now, let $\phi \in X_0^s(\Omega)\cap L^\infty(\Omega)$ be the solution (see [38]) of the following problem

$$
\begin{cases} (-\Delta)^s\phi = 1 & \text{in } \Omega, \\ \phi = 0 & \text{on } \mathbb{R}^N\backslash\Omega. \end{cases} \tag{4.3}
$$

In order to prove Theorem 3.2, we shall prove the following comparison result for the approximate solutions u_n. In the proof of this comparison result, we use Lemma 2.7 and Lemma 2.9 of [46], which require that Ω is a bounded domain which satisfies the condition of the ball. Such a condition is equivalent (see [8, Lemma 2.2]) to say that Ω is a bounded domain of class $\mathcal{C}^{1,1}$.

Lemma 4.4 *(Comparison result) Let $\gamma > 1$, $\theta \in (1,2)$ and let u_n be a solution of the problem (4.1). Then there exists a positive constant T not depending on n such that*

$$
u_n \geq \underline{u}_n := \left[T\phi^\theta + \frac{1}{n^{\frac{1+\gamma}{2}}}\right]^{\frac{2}{1+\gamma}} - \frac{1}{n}. \tag{4.4}
$$

Proof We shall prove that there exists a sub-solution $\underline{u}_n$ of the approximate problem (4.1), that is

$$
\begin{cases} (-\Delta)^s\underline{u}_n \leq \frac{f_n}{(\underline{u}_n+\frac{1}{n})^\gamma} & \text{in } \Omega, \\ \underline{u}_n > 0 & \text{in } \Omega, \\ \underline{u}_n = 0 & \text{on } \mathbb{R}^N\backslash\Omega, \end{cases} \tag{4.5}
$$

such that $u_n \geq \underline{u}_n$.

Let $\underline{u}_n := \psi_n^{\frac{2}{1+\gamma}}(x) - \frac{1}{n}$, where we have set $\psi_n = T\phi^\theta + \frac{1}{n^{\frac{1+\gamma}{2}}}$ and $T > 0$ is a constant not depending on n and that will be chosen later. We will show that $\underline{u}_n$ satisfies (4.5). Applying the inequality (5.1) with $F(t) = t^{\frac{2}{1+\gamma}}$ yields

$$
\begin{aligned}
(-\Delta)^s\underline{u}_n(x) &= (-\Delta)^s\left(\psi_n^{\frac{2}{\gamma+1}} - \frac{1}{n}\right)(x) = (-\Delta)^s(F\circ\psi_n)(x) \\
&\leq F'(\psi_n(x))(-\Delta)^s\psi_n(x) - \frac{a(N,s)(\gamma+1)T^2}{2}F''(\psi_n(x))\int_{\mathbb{R}^N}\frac{|\phi^\theta(x)-\phi^\theta(y)|^2}{|x-y|^{N+2s}}dy \\
&= \frac{2T}{1+\gamma}\psi_n^{\frac{1-\gamma}{1+\gamma}}(x)(-\Delta)^s(\phi^\theta(x)) + \frac{(\gamma-1)T^2}{(\gamma+1)\psi_n^{\frac{2\gamma}{1+\gamma}}(x)}a(N,s)\int_{\mathbb{R}^N}\frac{|\phi^\theta(x)-\phi^\theta(y)|^2}{|x-y|^{N+2s}}dy.
\end{aligned}
$$

Since $\theta > 1$, the function $g(t) = t^\theta, t > 0$, is convex so that one has the identity $g(t) - g(t') \leq g'(t)(t - t')$ which holds true for every t', t. Using the fact that ϕ solves (4.3), we get

$$(-\Delta)^s(\phi^\theta(x)) \leq \theta\phi^{\theta-1}(x)(-\Delta)^s(\phi(x)) = \theta\phi^{\theta-1}(x), \text{ for every } x \in \Omega.$$

Then, for every $x \in \Omega$ we get

$$(-\Delta)^s \underline{u}_n(x) \leq \frac{T}{\psi_n^{\frac{2\gamma}{1+\gamma}}(x)}\left(\frac{2\theta}{1+\gamma}\psi_n(x)\phi^{\theta-1}(x) + \frac{(\gamma-1)T}{\gamma+1}a(N,s)\int_{\mathbb{R}^N}\frac{|\phi^\theta(x)-\phi^\theta(y)|^2}{|x-y|^{N+2s}}dy\right). \tag{4.6}$$

On the other hand, let B_R be an open ball with radius $R > 0$ such that $\Omega \subset B_R$ and set $d_1 := dist(\partial\Omega, \partial B_R) > 0$. For every $x \in \Omega$, we can write

$$\begin{aligned}\int_{\mathbb{R}^N}\frac{|\phi^\theta(x)-\phi^\theta(y)|^2}{|x-y|^{N+2s}}dy &= \int_{B_R\setminus\Omega}\frac{|\phi^\theta(x)-\phi^\theta(y)|^2}{|x-y|^{N+2s}}dy + \int_{\mathbb{R}^N\setminus B_R}\frac{|\phi^\theta(x)-\phi^\theta(y)|^2}{|x-y|^{N+2s}}dy \\ &\quad + \int_\Omega\frac{|\phi^\theta(x)-\phi^\theta(y)|^2}{|x-y|^{N+2s}}dy \\ &= I_1(x) + I_2(x) + I_3(x).\end{aligned}$$

We start by estimating the first integral I_1. Since Ω is a bounded domain of class $\mathcal{C}^{1,1}$, by [46, Lemma 2.7] there exists a positive constant C_1, depending only on Ω and s, such that $|\phi(x)| \leq C_1\delta^s(x)$ for all $x \in \Omega$, where $\delta(x) := dist(x, \partial\Omega)$. Whence, we get

$$I_1(x) = \int_{B_R\setminus\Omega}\frac{|\phi^\theta(x)|^2}{|x-y|^{N+2s}}dy \leq C_1^{2\theta}\int_{B_R\setminus\Omega}\frac{|\delta^{s\theta}(x)|^2}{|x-y|^{N+2s}}dy.$$

Note that for $(x, y) \in \Omega \times B_R\setminus\Omega$, we have $\delta(x) \leq |x - y|$. Thus, we can write passing to the polar coordinates

$$\begin{aligned}I_1(x) &\leq C_1^{2\theta}\int_{B_R\setminus\Omega}\frac{dy}{|x-y|^{N-2s(\theta-1)}}dy \\ &\leq C_1^{2\theta}\int_{\{0\leq|z|\leq 2R\}}\frac{dz}{|z|^{N-2s(\theta-1)}} \\ &= C_1^{2\theta}|S^{N-1}|\int_0^{2R} r^{2s(\theta-1)-1}dr = C_1',\end{aligned}$$

with $C_1' = \frac{(2R)^{2s(\theta-1)}C_1^{2\theta}|S^{N-1}|}{2s(\theta-1)}$, where from now on $|S^{N-1}|$ stands for the Lebesgue measure of the unit sphere in $\mathbb{R}^N$. For the second integral $I_2(x)$, noticing that

$$|x-y| \geq d_1 := dist(\partial\Omega, \partial B_R) > 0 \text{ for every } (x, y) \in \Omega \times (\mathbb{R}^N\setminus B_R),$$

we can estimate I_2 as follows

$$\begin{aligned}I_2(x) &= \int_{\mathbb{R}^N\setminus B_R}\frac{|\phi^\theta(x)|^2}{|x-y|^{N+2s}}dy \\ &\leq \|\phi\|_{L^\infty(\Omega)}^{2\theta}\int_{|z|\geq d_1}\frac{dz}{|z|^{N+2s}}dy \\ &= \|\phi\|_{L^\infty(\Omega)}^{2\theta}|S^{N-1}|\int_{d_1}^{+\infty}\frac{dr}{r^{2s+1}} = C_2',\end{aligned}$$

where $C_2' = \|\phi\|^{2\theta}_{L^\infty(\Omega)} \frac{|S^{N-1}|}{2sd_1^{2s}}$. We now turn to estimate $I_3(x)$. Combining $iii)$ et $iv)$ of Lemma 2.2, we obtain

$$|\phi^\theta(x) - \phi^\theta(y)|^2 \leq 2\theta^2 |\phi(x) - \phi(y)|^{2\theta} + 8\theta^2 \phi^{2(\theta-1)}(x)|\phi(x) - \phi(y)|^2. \tag{4.7}$$

By [46, Lemma 2.9] the function ϕ is $C^\beta(\Omega)$ for all $\beta \in (0, 2s)$. In particular and in what follows we make the choice $\beta \in (s, \min(1, s\theta))$. Furthermore, there exists a constant $C_3 > 0$, depending on Ω, s and β, such that for every $x \in \Omega$

$$|\phi(x) - \phi(y)| \leq C_3 |x - y|^\beta \left(\frac{\delta(x)}{2}\right)^{s-\beta}, \tag{4.8}$$

for every $y \in B_{\frac{\delta(x)}{2}}(x)$, where $B_{\frac{\delta(x)}{2}}(x)$ stands for the open ball of radius $\frac{\delta(x)}{2}$ centered at x with $\delta(x) := dist(x, \partial\Omega)$. Now, using (4.7) we can write for every $x, y \in \Omega$

$$I_3(x) = \int_\Omega \frac{|\phi^\theta(x) - \phi^\theta(y)|^2}{|x-y|^{N+2s}} dy \leq 2\theta^2 \int_\Omega \frac{|\phi(x) - \phi(y)|^{2\theta}}{|x-y|^{N+2s}} dy$$
$$+ 8\theta^2 \int_\Omega \frac{\phi^{2(\theta-1)}(x)|\phi(x) - \phi(y)|^2}{|x-y|^{N+2s}} dy.$$

Splitting the second integral on the right-hand side, we obtain

$$\begin{aligned} I_3(x) \leq\ & 2\theta^2 \int_\Omega \frac{|\phi(x) - \phi(y)|^{2\theta}}{|x-y|^{N+2s}} dy \\ & + 8\theta^2 \int_{\{y\in\Omega:|x-y|\geq \frac{\delta(x)}{2}\}} \frac{\phi^{2(\theta-1)}(x)|\phi(x) - \phi(y)|^2}{|x-y|^{N+2s}} dy \\ & + 8\theta^2 \int_{\{y\in\Omega:|x-y|< \frac{\delta(x)}{2}\}} \frac{\phi^{2(\theta-1)}(x)|\phi(x) - \phi(y)|^2}{|x-y|^{N+2s}} dy \\ :=\ & J_1(x) + J_2(x) + J_3(x). \end{aligned}$$

We shall estimate $J_1(x)$, $J_2(x)$ and $J_3(x)$. For $J_1(x)$, we note that by [46, Proposition 1.1] we have $\phi \in \mathcal{C}^s(\mathbb{R}^N)$. In addition, there exists a positive constant c_3 such that for every x, $y \in \mathbb{R}^N$, $|\phi(x) - \phi(y)| \leq c_3 |x - y|^s$. Thus,

$$J_1(x) \leq 2\theta^2 c_3^{2\theta} \int_\Omega \frac{dy}{|x-y|^{N-2s(\theta-1)}} dy.$$

We calculate the integral using the change of variable $z = x - y$. We have

$$\begin{aligned} & \int_\Omega \frac{dy}{|x-y|^{N-2s(\theta-1)}} \\ & = \int_{\Omega\cap|x-y|>1} \frac{dy}{|x-y|^{N-2s(\theta-1)}} + \int_{\Omega\cap|x-y|\leq 1} \frac{dy}{|x-y|^{N-2s(\theta-1)}} \\ & \leq |\Omega| + \int_{|z|\leq 1} \frac{dz}{|z|^{N-2s(\theta-1)}} = |\Omega| + \frac{|S^{N-1}|}{2s(\theta-1)}. \end{aligned} \tag{4.9}$$

Thus, we obtain

$$J_1(x) \leq 2\theta^2 c_3^{2\theta} \left(|\Omega| + \frac{|S^{N-1}|}{2s(\theta-1)}\right).$$

For J_2 we use the fact that $\phi \in \mathcal{C}^s(\mathbb{R}^N)$ and $|\phi(x)| \le C_1\delta^s(x)$ for all $x \in \Omega$. By (4.9) we get

$$J_2(x) \le 8\theta^2 c_3^2\big(2^s C_1\big)^{2(\theta-1)} \int_\Omega \frac{dy}{|x-y|^{N-2s(\theta-1)}}$$

$$\le 8\theta^2 c_3^2\big(2^s C_1\big)^{2(\theta-1)}\Big(|\Omega| + \frac{|S^{N-1}|}{2s(\theta-1)}\Big).$$

While for $J_3(x)$ we use (4.8) and $|\phi(x)| \le C_1\delta^s(x)$ for all $x \in \Omega$. We arrive at

$$J_3(x) \le 8\theta^2\big(2^{\beta-s}C_1^{\theta-1}C_3\big)^2 \int_{\{y\in\Omega:|x-y|<\frac{\delta(x)}{2}\}} \frac{\delta^{2(s\theta-\beta)}(x)}{|x-y|^{N-2(\beta-s)}}dy.$$

The fact that $\beta \in (s, \min(1, s\theta))$ and that Ω is bounded, enables us to get

$$J_3(x) \le$$
$$8\theta^2\big(2^{\beta-s}C_1^{\theta-1}C_3\big)^2\big(diam(\Omega)\big)^{2(s\theta-\beta)} \int_{\{y\in\Omega:|x-y|<\frac{\delta(x)}{2}\}} \frac{dy}{|x-y|^{N-2(\beta-s)}}$$

$$\le 4\theta^2\big(2^{\beta-s}C_1^{\theta-1}C_3\big)^2\big(diam(\Omega)\big)^{2s(\theta-1)}\frac{|S^{N-1}|}{\beta-s},$$

where $diam(\Omega)$ stands for the diameter of Ω. Finally, there exists a constant $C_3' > 0$ depending on Ω, R, N, s, θ and β, such that

$$I_3(x) \le C_3'.$$

Let $T_0 = \min(1, f_0)$ and let us choose T small enough such that

$$0 < T\Big[\frac{2\theta}{1+\gamma}\big(T\|\phi\|^\theta_{L^\infty(\Omega)} + 1)\big)\|\phi\|^{\theta-1}_{L^\infty(\Omega)} + \frac{3(\gamma-1)T}{\gamma+1}a(N,s)\max(C_1', C_2', C_3')\Big]$$
$$\le T_0.$$

Going back to (4.6), we deduce that for every $x \in \Omega$

$$(-\Delta)^s\underline{u}_n(x) \le \frac{T_0}{\psi_n^{\frac{2\gamma}{1+\gamma}}(x)},$$

which yields

$$(-\Delta)^s\underline{u}_n(x) \le \frac{f_n(x)}{(\underline{u}_n + \frac{1}{n})^\gamma}.$$

Thus, $\underline{u}_n$ is a sub-solution of (4.1). Now, we prove that $u_n(x) \ge \underline{u}_n(x)$ for every $x \in \Omega$. Assume by contradiction that there exists $\xi \in \Omega$ such that

$$u_n(\xi) < \underline{u}_n(\xi). \tag{4.10}$$

Then we have

$$(-\Delta)^s(u_n - \underline{u}_n)(\xi) = (-\Delta)^s u_n(\xi) - (-\Delta)^s\underline{u}_n(\xi)$$
$$\ge f_n(\xi)\Big[\frac{1}{(u_n(\xi)+\frac{1}{n})^\gamma} - \frac{1}{(\underline{u}_n(\xi)+\frac{1}{n})^\gamma}\Big] > 0.$$

It follows from the weak maximum principle [48] that $(u_n - \underline{u}_n)(\xi) \ge 0$, which contradicts (4.10). Therefore, we have

$$u_n(x) + \frac{1}{n} \ge \psi_n^{\frac{2}{1+\gamma}}(x) = \Big[T\phi^\theta(x) + \frac{1}{n^{\frac{1+\gamma}{2}}}\Big]^{\frac{2}{1+\gamma}}.$$

□

4.2 The case $0 < \gamma < 1$: Proof of Theorem 3.1

In order to prove the existence of solutions for the problem (1.1), we first need to prove some a priori estimates on u_n.

4.2.1 A priori estimates

Lemma 4.5 *Let $f \geq 0$, $f \in L^m(\Omega)$, with $1 \leq m < \overline{m} := \frac{2N}{N+2s+\gamma(N-2s)}$, and u_n be a solution of the problem (4.1). If $0 < \gamma < 1$, then $\{u_n\}$ is uniformly bounded in $W_0^{s_1,\overline{q}}(\Omega)$ for all $s_1 < s$, where $\overline{q} = \frac{Nm(1+\gamma)}{N-sm(1-\gamma)}$. Moreover, $\{u_n\}$ is uniformly bounded in $L^\sigma(\Omega)$, where $\sigma = \frac{Nm(1+\gamma)}{N-2sm}$.*

Proof Let $n \in \mathbb{N}$, $n \geq 1$, and let $\gamma \leq \theta < 1$ to be chosen later. Let $0 < \varepsilon < \frac{1}{n}$. By [38, Proposition 3.], the function $(u_n + \varepsilon)^\theta - \varepsilon^\theta$ is an admissible test function in (4.1). Taking it so, it yields

$$\begin{aligned}&\int_\Omega \int_\Omega \frac{(u_n(x) - u_n(y))((u_n(x) + \varepsilon)^\theta - (u_n(y) + \varepsilon)^\theta)}{|x-y|^{N+2s}} dydx \\ &\leq \frac{2}{a(N,s)} \int_\Omega f_n(u_n(x) + \varepsilon)^{\theta-\gamma} dx.\end{aligned}$$

Passing to the limit as ε tends to 0, we obtain

$$\int_\Omega \int_\Omega \frac{(u_n(x) - u_n(y))(u_n^\theta(x) - u_n^\theta(y))}{|x-y|^{N+2s}} dydx \leq \frac{2}{a(N,s)} \int_\Omega f_n u_n(x)^{\theta-\gamma} dx. \tag{4.11}$$

By the item $i)$ of Lemma 2.2, we can minimize the term in the left-hand side of (4.11) as follows

$$\int_\Omega \int_\Omega \frac{\left|u_n^{\frac{\theta+1}{2}}(x) - u_n^{\frac{\theta+1}{2}}(y)\right|^2}{|x-y|^{N+2s}} dydx \leq \frac{(\theta+1)^2}{2a(N,s)\theta} \int_\Omega f_n u_n^{\theta-\gamma} dx.$$

Applying the fractional Sobolev inequality, we obtain

$$\int_\Omega |u_n(x)|^{\frac{N(\theta+1)}{N-2s}} dx \leq \left[\frac{S(N,s)(\theta+1)^2}{2a(N,s)\theta}\right]^{\frac{N}{N-2s}} \left[\int_\Omega f_n u_n^{\theta-\gamma} dx\right]^{\frac{N}{N-2s}}. \tag{4.12}$$

• If $m = 1$, then the choice $\theta = \gamma$ gives

$$\int_\Omega |u_n(x)|^{\frac{N(\gamma+1)}{N-2s}} dx \leq \left[\frac{S(N,s)(\gamma+1)^2}{2a(N,s)\gamma}\right]^{\frac{N}{N-2s}} \|f\|_{L^1(\Omega)}^{\frac{N}{N-2s}}. \tag{4.13}$$

• While if $1 < m < \overline{m}$ and $\gamma < \theta < 1$, an application of Hölder's inequality in the right-hand side term of (4.12) with the exponents m and $m' := \frac{m}{m-1}$, gives

$$\begin{aligned}&\int_\Omega |u_n(x)|^{\frac{N(\theta+1)}{N-2s}} dx \leq \\ &\left[\frac{S(N,s)(\theta+1)^2}{2a(N,s)\theta}\right]^{\frac{N}{N-2s}} \|f\|_{L^m(\Omega)}^{\frac{N}{N-2s}} \left(\int_\Omega |u_n|^{(\theta-\gamma)m'} dx\right)^{\frac{N}{m'(N-2s)}}.\end{aligned} \tag{4.14}$$

We now choose θ to be such that $\frac{N(\theta+1)}{N-2s} = (\theta-\gamma)m'$, that is

$$\theta = \frac{N(m-1)+\gamma m(N-2s)}{N-2sm}.$$

Observe that the assumption $m < \overline{m}$ implies $\theta < 1$ and since $\gamma > 0$ we have $\gamma < \theta$. This choice of θ yields

$$\frac{N(\theta+1)}{N-2s} = \frac{Nm(1+\gamma)}{N-2sm} = \sigma.$$

Noticing that $\frac{N}{m'(N-2s)} < 1$ and using (4.14) we deduce the following inequality

$$\int_\Omega |u_n(x)|^{\frac{Nm(1+\gamma)}{N-2sm}} dx \le \left[\frac{S(N,s)(\theta+1)^2}{2a(N,s)\theta}\right]^{\frac{Nm}{N-2sm}} \|f\|_{L^m(\Omega)}^{\frac{Nm}{N-2sm}}. \tag{4.15}$$

Thus, from (4.13) and (4.15) we conclude that the sequence $\{u_n\}_n$ is uniformly bounded in $L^\sigma(\Omega)$ for $\sigma = \frac{Nm(1+\gamma)}{N-2sm}$ and $1 \le m < \overline{m}$.

Now, going back to the inequality (4.11) and following exactly the same lines as above, that is if $m = 1$ we choose $\theta = \gamma$ while if $1 \le m < \overline{m}$ we choose $\theta = \frac{N(m-1)+\gamma m(N-2s)}{N-2sm} < 1$. In both cases, applying the Hölder inequality we obtain

$$\int_\Omega\int_\Omega \frac{(u_n(x)-u_n(y))(u_n^\theta(x)-u_n^\theta(y))}{|x-y|^{N+2s}} dydx \le C, \tag{4.16}$$

where C is a positive constant not depending on n. Let $s_1 \in (0,s)$ be fixed and let $\overline{q} = \frac{Nm(1+\gamma)}{N-sm(1-\gamma)}$. We set $\theta = \frac{N(m-1)+\gamma m(N-2s)}{N-2sm}$ for $1 \le m < \overline{m}$ (we note that $\theta = \gamma$ if $m = 1$). We note that $\overline{q} \ge m(1+\gamma) > 1$ and the assumption $m < \overline{m}$ implies $\overline{q} < 2$. Thus, observe that $N + \overline{q}s_1$ can be splitted as follows

$$N + \overline{q}s_1 = \frac{\overline{q}}{2}N + \overline{q}s + \frac{2-\overline{q}}{2}N - \overline{q}(s-s_1).$$

Hence, setting $\tilde{\Omega} := \left\{y \in \Omega : u_n(y) \ne u_n(x)\right\}$ we can write

$$\int_\Omega\int_\Omega \frac{|u_n(x)-u_n(y)|^{\overline{q}}}{|x-y|^{N+\overline{q}s_1}} dydx = \int_\Omega\int_{\tilde{\Omega}} \frac{|u_n(x)-u_n(y)|^{\overline{q}}}{|x-y|^{\frac{\overline{q}}{2}N+\overline{q}s}} \times \frac{(u_n^\theta(x)-u_n^\theta(y))}{(u_n(x)-u_n(y))} \times \frac{(u_n(x)-u_n(y))}{(u_n^\theta(x)-u_n^\theta(y))} \times \frac{dydx}{|x-y|^{\frac{2-\overline{q}}{2}N-\overline{q}(s-s_1)}}.$$

Observe that the quantity in the middle of the product inside the integral can be written as follows

$$\frac{(u_n^\theta(x)-u_n^\theta(y))}{(u_n(x)-u_n(y))} = \left(\frac{(u_n^\theta(x)-u_n^\theta(y))}{(u_n(x)-u_n(y))}\right)^{\frac{\overline{q}}{2}} \times \left(\frac{(u_n^\theta(x)-u_n^\theta(y))}{(u_n(x)-u_n(y))}\right)^{\frac{2-\overline{q}}{2}},$$

we obtain

$$\int_\Omega\int_\Omega \frac{|u_n(x)-u_n(y)|^{\overline{q}}}{|x-y|^{N+\overline{q}s_1}}dydx$$
$$=\int_\Omega\int_{\{y\in\Omega:u_n(y)\neq u_n(x)\}}\left[\frac{|u_n(x)-u_n(y)|^{\overline{q}}}{|x-y|^{\frac{\overline{q}}{2}N+\overline{q}s}}\times\left(\frac{(u_n^\theta(x)-u_n^\theta(y))}{(u_n(x)-u_n(y))}\right)^{\frac{\overline{q}}{2}}\right]$$
$$\times\left[\left(\frac{(u_n^\theta(x)-u_n^\theta(y))}{(u_n(x)-u_n(y))}\right)^{\frac{2-\overline{q}}{2}}\times\frac{(u_n(x)-u_n(y))}{(u_n^\theta(x)-u_n^\theta(y))}\times\frac{1}{|x-y|^{\frac{2-\overline{q}}{2}N-\overline{q}(s-s_1)}}\right]dydx.$$

Now using Hölder's inequality with the exponents $\frac{2}{\overline{q}}$ and $\frac{2}{2-\overline{q}}$, we obtain

$$\int_\Omega\int_\Omega \frac{|u_n(x)-u_n(y)|^{\overline{q}}}{|x-y|^{N+\overline{q}s_1}}dydx$$
$$\leq\left[\int_\Omega\int_{\tilde{\Omega}}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\times\frac{|u_n^\theta(x)-u_n^\theta(y)|}{|u_n(x)-u_n(y)|}dydx\right]^{\frac{\overline{q}}{2}}$$
$$\times\left[\int_\Omega\int_{\tilde{\Omega}}\frac{(u_n^\theta(x)-u_n^\theta(y))}{(u_n(x)-u_n(y))}\times\left(\frac{(u_n(x)-u_n(y))}{(u_n^\theta(x)-u_n^\theta(y))}\right)^{\frac{2}{2-\overline{q}}}\times\frac{dydx}{|x-y|^{N-\beta}}\right]^{\frac{2-\overline{q}}{2}} \tag{4.17}$$
$$=\left[\int_\Omega\int_{\tilde{\Omega}}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\times\frac{|u_n^\theta(x)-u_n^\theta(y)|}{|u_n(x)-u_n(y)|}dydx\right]^{\frac{\overline{q}}{2}}$$
$$\times\left[\int_\Omega\int_{\tilde{\Omega}}\left(\frac{(u_n(x)-u_n(y))}{(u_n^\theta(x)-u_n^\theta(y))}\right)^{\frac{2}{2-\overline{q}}}\times\frac{(u_n^\theta(x)-u_n^\theta(y))}{(u_n(x)-u_n(y))}\times\frac{dydx}{|x-y|^{N-\beta}}\right]^{\frac{2-\overline{q}}{2}},$$

where we have set $\beta=\frac{2\overline{q}(s-s_1)}{2-\overline{q}}$. Then,

$$\int_\Omega\int_\Omega \frac{|u_n(x)-u_n(y)|^{\overline{q}}}{|x-y|^{N+\overline{q}s_1}}dydx\leq$$
$$\left(\int_\Omega\int_\Omega\frac{(u_n(x)-u_n(y))(u_n^\theta(x)-u_n^\theta(y))}{|x-y|^{N+2s}}dydx\right)^{\frac{\overline{q}}{2}}$$
$$\times\left(\int_\Omega\int_{\tilde{\Omega}}\left(\frac{u_n(x)-u_n(y)}{u_n^\theta(x)-u_n^\theta(y)}\right)^{\frac{\overline{q}}{2-\overline{q}}}\times\frac{dydx}{|x-y|^{N-\beta}}\right)^{\frac{2-\overline{q}}{2}}.$$

Using the item $ii)$ of Lemma 2.2 and the inequality (4.16), we obtain

$$\int_\Omega\int_\Omega \frac{|u_n(x)-u_n(y)|^{\overline{q}}}{|x-y|^{N+\overline{q}s_1}}dydx\leq$$
$$C_1\left(\int_\Omega\int_\Omega\left(u_n^{\frac{\overline{q}(1-\theta)}{2-\overline{q}}}(x)+u_n^{\frac{\overline{q}(1-\theta)}{2-\overline{q}}}(y)\right)\times\frac{dydx}{|x-y|^{N-\beta}}\right)^{\frac{2-\overline{q}}{2}},$$

where C_1 is a positive constant not depending on n. By x/y symmetry, there exists a constant C_2, not depending on n, such that

$$\int_\Omega\int_\Omega \frac{|u_n(x)-u_n(y)|^{\overline{q}}}{|x-y|^{N+\overline{q}s_1}}dydx\leq C_2\left(\int_\Omega u_n^{\frac{\overline{q}(1-\theta)}{2-\overline{q}}}(x)\left[\int_\Omega\frac{dy}{|x-y|^{N-\beta}}\right]dx\right)^{\frac{2-\overline{q}}{2}}.$$

Observing that $\frac{\overline{q}(1-\theta)}{2-\overline{q}} = \sigma := \frac{Nm(1+\gamma)}{N-2s}$ and having in mind (4.9) we get

$$\int_\Omega \int_\Omega \frac{|u_n(x) - u_n(y)|^{\overline{q}}}{|x-y|^{N+\overline{q}s_1}} dydx \le C_3,$$

where C_3 is a positive constant not depending on n. Thus, $\{u_n\}$ is uniformly bounded in $W_0^{s_1,\overline{q}}(\Omega)$ for every $s_1 < s$. □

Remark 4.1 Note that we can repeat the same lines as in the proof of Lemma 4.5 above with the exponent q instead of $\overline{q}$ in (4.17), with $1 \le q \le \overline{q}$. We obtain that $\{u_n\}$ is uniformly bounded in $W_0^{s_1,q}(\Omega)$ for all $1 \le q \le \overline{q}$ and for every $s_1 < s$ and $1 \le m < \overline{m}$.

4.2.2 Passage to the limit

Now, under the assumptions of Theorem 3.1, we are going to prove the existence of solution u to (1.1).

Proof of of Theorem 3.1 From Lemma 4.5 and by the compact embedding of $W_0^{s_1,\overline{q}}(\Omega)$ into $L^1(\Omega)$ (see [26, Corollary 7.2] or [25, Theorem 4.54]), there exist a subsequence of $\{u_n\}_n$, still indexed by n, and a measurable function $u \in W_0^{s_1,\overline{q}}(\Omega)$ such that

$$\begin{aligned} &u_n \rightharpoonup u \text{ weakly in } W_0^{s_1,\overline{q}}(\Omega),\\ &u_n \to u \text{ in norm in } L^1(\Omega),\\ &u_n \to u \text{ a.e. in } \mathbb{R}^N. \end{aligned}$$

Then

$$\frac{u_n(x) - u_n(y)}{|x-y|^{N+2s}} \to \frac{u(x) - u(y)}{|x-y|^{N+2s}} \text{ a.e. in } Q.$$

Let $\rho > 0$ be a small enough real number that we will choose later. For any $\varphi \in C_0^\infty(\Omega)$ we have

$$\begin{aligned} &\int_\Omega \int_\Omega \left[\frac{|(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))|}{|x-y|^{N+2s}} \right]^{1+\rho} dydx \\ &\le \int_\Omega \int_\Omega \frac{|u_n(x) - u_n(y)|^{1+\rho} (\|\nabla\varphi\|_{L^\infty(\Omega)} |x-y|)^{1+\rho}}{|x-y|^{N+(1+\rho)s_1}} \frac{dydx}{|x-y|^{\rho N + (1+\rho)(2s-s_1)}} \\ &\le \|\nabla\varphi\|^{1+\rho}_{L^\infty(\Omega)} \int_\Omega \int_\Omega \frac{|u_n(x) - u_n(y)|^{1+\rho} |x-y|^{(1+\rho)(1+s_1-2s)-\rho N}}{|x-y|^{N+(1+\rho)s_1}} dydx. \end{aligned}$$

We now choose ρ to be such that $(1+\rho)(1+s_1-2s) - \rho N \ge 0$. To do so, we consider s_1 to be very close of s. Precisely, we impose on s_1 the condition

$$\max(0, 1-3s) < s - s_1 < 1 - s.$$

We point out that with this range of values of s_1 and with the assumption $N > 2s$, we obtain

$$1 + s_1 - 2s > 0 \text{ and } N - 1 - s_1 + 2s > 0.$$

Thus, the fact that $(1+\rho)(1+s_1-2s)-\rho N \geq 0$ is equivalent to $0<\rho \leq \frac{1+s_1-2s}{N-1-s_1+2s}$. Therefore, we have

$$\begin{aligned}&\int_\Omega\int_\Omega\left[\frac{|(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))|}{|x-y|^{N+2s}}\right]^{1+\rho}dydx\\&\leq \|\nabla\varphi\|^{1+\rho}_{L^\infty(\Omega)}diam(\Omega)^{(1+\rho)(1+s_1-2s)-\rho N}\int_\Omega\int_\Omega\frac{|u_n(x)-u_n(y)|^{1+\rho}}{|x-y|^{N+(1+\rho)s_1}}dydx.\end{aligned}\tag{4.18}$$

Now we have to make a choice of ρ to prove that the right-hand integral in (4.18) is uniformly bounded. By Remark 4.1 we have the uniform boundedness of $\{u_n\}_n$ in $W_0^{s_1,q}(\Omega)$ for every $1 \leq q \leq \overline{q} = \frac{Nm(1+\gamma)}{N-sm(1-\gamma)}$. So it is sufficient to choose ρ such that $1+\rho \leq \overline{q} = \frac{Nm(1+\gamma)}{N-sm(1-\gamma)}$. Thus, the choice we need for ρ is the following

$$0<\rho\leq \min\Big(\frac{N(m-1)+m\gamma(N-s)+sm}{N-sm(1-\gamma)},\frac{1+s_1-2s}{N-1-s_1+2s}\Big).$$

Therefore, there is a constant $C>0$, not depending on n, such that

$$\sup_n\int_\Omega\int_\Omega\left[\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\right]^{1+\rho}dydx\leq C.$$

Finally, by De La Vallée Poussin and Dunford-Pettis theorems the sequence

$$\Big\{\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\Big\}$$

is equi-integrable in $L^1(\Omega\times\Omega)$. Now, inserting $\varphi \in C_0^\infty(\Omega)$ as a test function in (4.1) yields

$$\frac{a(N,s)}{2}\int_Q\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx=\int_\Omega\frac{f_n\varphi}{(u_n+\frac{1}{n})^\gamma}dx.\tag{4.19}$$

We split the integral in the left-hand side of (4.19) into three integrals as follows

$$\begin{aligned}&\int_Q\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx\\&=\int_\Omega\int_\Omega\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx\\&+\int_\Omega\int_{\mathcal{C}\Omega}\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx\\&+\int_{\mathcal{C}\Omega}\int_\Omega\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx\\&=I_1+I_2+I_3.\end{aligned}\tag{4.20}$$

By Vitali's lemma we have

$$\begin{aligned}&\lim_{n\to\infty}\int_\Omega\int_\Omega\frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx\\&=\int_\Omega\int_\Omega\frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx.\end{aligned}$$

For the second integral I_2 in (4.20), we start noticing that since $u_n(y) = \varphi(y) = 0$ for every $y \in \mathcal{C}\Omega$ we can write

$$\left|\frac{(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}}\right| = \frac{|u_n(x)\varphi(x)|}{|x - y|^{N+2s}} \text{ for every } (x, y) \in \Omega \times \mathcal{C}\Omega.$$

As a consequence of the convergence in norm of the sequence $\{u_n\}$ in $L^1(\Omega)$ there exist a subsequence of $\{u_n\}$ still indexed by n and a positive function g in $L^1(\Omega)$ such that

$$|u_n(x)| \le g(x) \text{ a.e. in } \Omega,$$

which enables us to get

$$\frac{|(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))|}{|x - y|^{N+2s}} \le \frac{|g(x)\varphi(x)|}{|x - y|^{N+2s}} \text{ a.e. in } (x, y) \in \Omega \times \mathcal{C}\Omega$$

and so we can write

$$\begin{aligned}\int_\Omega \int_{\mathcal{C}\Omega} \frac{|g(x)\varphi(x)|}{|x - y|^{N+2s}} dydx &= \int_{supp(\varphi)} \int_{\mathcal{C}\Omega} \frac{|g(x)\varphi(x)|}{|x - y|^{N+2s}} dydx \\ &\le \|\varphi\|_{L^\infty(\Omega)} \int_{supp(\varphi)} |g(x)| \left[\int_{\mathcal{C}\Omega} \frac{dy}{|x - y|^{N+2s}}\right] dx.\end{aligned}$$

Since $supp(\varphi)$ is a compact subset in Ω, we have

$$|x - y| \ge d_2 := dist(supp(\varphi), \partial\Omega) > 0 \text{ for every } (x, y) \in supp(\varphi) \times \mathcal{C}\Omega.$$

Hence passing to the polar coordinates, an easy computation leads to

$$\int_{\mathcal{C}\Omega} \frac{dy}{|x - y|^{N+2s}} = \int_{\{z \in \mathbb{R}^N : |z| \ge d_2\}} \frac{dz}{|z|^{N+2s}} = \int_{d_2}^{+\infty} \int_{v=1} \frac{dvdr}{r^{2s+1}} = \frac{|S^{N-1}|}{2sd_2^{2s}}.$$

This shows that the function $(x, y) \to \frac{|g(x)\varphi(x)|}{|x-y|^{N+2s}}$ belongs to $L^1(\Omega \times \mathcal{C}\Omega)$. Therefore, by the Lebesgue dominated convergence theorem we obtain

$$\begin{aligned}&\lim_{n\to\infty} \int_\Omega \int_{\mathcal{C}\Omega} \frac{(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} dydx \\ &= \int_\Omega \int_{\mathcal{C}\Omega} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} dydx.\end{aligned}$$

By x/y symmetry, the third integral I_3 in (4.20) can be treated in the similar way. Finally, we conclude that

$$\begin{aligned}&\lim_{n\to\infty} \int_Q \frac{(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} dydx \\ &= \int_Q \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} dydx,\end{aligned}$$

for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$. Now, for what concerns the right-hand side of (4.19), by virtue of Lemma 4.2, for any $\varphi \in \mathcal{C}_0^\infty(\Omega)$ with $supp(\varphi) = \omega$, there exists a constant $c_\omega > 0$ not depending on n such that

$$0 \le \left|\frac{f_n\varphi}{(u_n + \frac{1}{n})^\gamma}\right| \le \frac{|f\|\varphi|}{c_\omega^\gamma} \in L^1(\Omega).$$

So that by the Lebesgue dominated convergence theorem we get

$$\lim_{n\to\infty}\int_\Omega \frac{f_n\varphi}{(u_n+\frac{1}{n})^\gamma}dx = \int_\Omega \frac{f\varphi}{u^\gamma}dx.$$

Finally, passing to the limit in (4.19) as $n \to +\infty$ we obtain

$$\frac{a(N,s)}{2}\int_Q \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx = \int_\Omega \frac{f\varphi}{u^\gamma}dx,$$

for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$. That is u is a weak solution of (1.1). Furthermore, from (4.13) and (4.15) we conclude by Fatou's lemma that $u \in L^\sigma(\Omega)$ with $\sigma = \frac{Nm(1+\gamma)}{N-2sm}$ and $1 \le m < \overline{m}$. □

4.3 The case $\gamma > 1$: Proof of Theorem 3.2

4.3.1 A priori estimates

Lemma 4.6 *Let $0 < f_0 \le f \in L^m(\Omega)$, $m > 1$, where f_0 is a positive constant. Let $\gamma > 1$, $s \in (0,1)$ and let u_n be a solution of the problem (4.1). Then the sequence $\{u_n^\alpha\}_n$ is uniformly bounded in $X_0^s(\Omega)$ for every $\alpha \in \left(\max\left(\frac{1}{2}, \frac{(\gamma+1)(2sm-m+1)}{4sm}\right), \frac{\gamma+1}{2}\right]$. Furthermore, if γ satisfies*

$$\Big(m(2s-1)+1\Big)\gamma < m(2s+1)-1, \tag{4.21}$$

then $\{u_n\}_n$ is uniformly bounded in $X_0^s(\Omega)$.

Proof We shall prove a priori estimates on u_n^α in $X_0^s(\Omega)$ for every α such that $\max\left(\frac{1}{2}, \frac{(\gamma+1)(2sm-m+1)}{4sm}\right) < \alpha \le \frac{\gamma+1}{2}$. Let $n \ge 1$ and let $0 < \varepsilon < \frac{1}{n}$. For $\eta > 0$, taking $(u_n+\varepsilon)^\eta - \varepsilon^\eta$ as a test function in (4.1), we obtain

$$\frac{a(N,s)}{2}\int_Q \frac{(u_n(x)-u_n(y))((u_n(x)+\varepsilon)^\eta-(u_n(y)+\varepsilon)^\eta)}{|x-y|^{N+2s}}dydx \le \int_\Omega \frac{f_n}{(u_n(x)+\frac{1}{n})^{\gamma-\eta}}dx.$$

The passage to the limit in ε yields

$$\int_Q \frac{(u_n(x)-u_n(y))(u_n^\eta(x)-u_n^\eta(y))}{|x-y|^{N+2s}}dydx \le \frac{2}{a(N,s)}\int_\Omega \frac{f_n}{(u_n(x)+\frac{1}{n})^{\gamma-\eta}}dx.$$

An application of the item $i)$ in Lemma 2.2 and the Hölder inequality lead to

$$\int_Q \frac{|u_n^{\frac{\eta+1}{2}}(x)-u_n^{\frac{\eta+1}{2}}(y)|^2}{|x-y|^{N+2s}}dydx \le C(\eta,N,s)\|f\|_{L^m(\Omega)}\left(\int_\Omega \frac{dx}{\left(u_n(x)+\frac{1}{n}\right)^{(\gamma-\eta)m'}}\right)^{\frac{1}{m'}}.$$

Let η be such that $0 < \eta \le \gamma$. We can use (4.4) to get

$$\int_Q \frac{|u_n^{\frac{\eta+1}{2}}(x) - u_n^{\frac{\eta+1}{2}}(y)|^2}{|x-y|^{N+2s}} dydx$$
$$\le C(\eta, N, s)\|f\|_{L^m(\Omega)} \left(\int_\Omega \frac{dx}{\left(T\phi^\theta(x) + \frac{1}{n^{\frac{1+\gamma}{2}}}\right)^{\frac{2(\gamma-\eta)m'}{1+\gamma}}} \right)^{\frac{1}{m'}}.$$

From [12, Lemma 4.2] we know that there exists a positive constant $C > 0$, depending only on Ω and s, such that for every $x \in \Omega$, $\phi(x) \ge C\delta^s(x)$, where $\delta(x) := dist(x, \partial\Omega)$. Using this, the above inequality reads as

$$\int_Q \frac{|u_n^{\frac{\eta+1}{2}}(x) - u_n^{\frac{\eta+1}{2}}(y)|^2}{|x-y|^{N+2s}} dydx \le C\|f\|_{L^m(\Omega)} \left[\int_\Omega \frac{dx}{\delta^{\frac{2s(\gamma-\eta)m'}{\gamma+1}\theta}(x)} \right]^{\frac{1}{m'}}.$$

Choosing $\alpha = \frac{\eta+1}{2} > \frac{1}{2}$, we must seek for the range of α that ensures the convergence of the integral in the right-hand side in the above inequality. If $\alpha = \frac{\gamma+1}{2}$ the integral obviously converges. If $\alpha < \frac{\gamma+1}{2}$ it is sufficient to have $\frac{2s(\gamma+1-2\alpha)m'}{\gamma+1}\theta < 1$. If it is so, we get $\theta < \frac{\gamma+1}{2s(\gamma+1-2\alpha)m'}$. In order that $\theta \in (1, 2)$ exists, it suffices to have $1 < \frac{\gamma+1}{2s(\gamma+1-2\alpha)m'}$. This yields, $\frac{2sm-m+1}{4sm}(\gamma+1) < \alpha$. Finally, if $\max\left(\frac{1}{2}, \frac{(\gamma+1)(2sm-m+1)}{4sm}\right) < \alpha \le \frac{\gamma+1}{2}$ then the sequence $\{u_n^\alpha\}_n$ is uniformly bounded in $X_0^s(\Omega)$.

Furthermore, if the condition (4.21) holds then $\frac{(\gamma+1)(2sm-m+1)}{4sm} < 1$ and so we can chose $\alpha = 1$ obtaining the uniform boundedness of the sequence $\{u_n\}_n$ in $u \in X_0^s(\Omega)$. □

4.3.2 Passage to the limit

Proof of Theorem 3.2 By Lemma 4.6 the sequence $\{u_n^\alpha\}_n$ is uniformly bounded in $X_0^s(\Omega)$ and by the compact embedding in [26, Corollary 7.2] (see also [25, Theorem 4.54.]), there exists a subsequence of $\{u_n^\alpha\}_n$, still indexed by n, and a function $v_\alpha \in X_0^s(\Omega)$ such that $u_n^\alpha \to v_\alpha$ in $L^1(\Omega)$ and $u_n^\alpha \to v_\alpha$ a.e. in $\mathbb{R}^N$. In particular, the sequence $\{u_n\}$ is uniformly bounded in $L^{\frac{\gamma+1}{2}}(\Omega)$ and as $\frac{\gamma+1}{2} > 1$ it is also uniformly bounded in $L^1(\Omega)$. Thanks to Lemma 4.2, the sequence $\{u_n\}_n$ is increasing so that by Beppo-Levi's theorem the function $u(x) := \lim_{n\to\infty} u_n(x)$, for a.e. $x \in \Omega$, belongs to $L^1(\Omega)$. Since $u_n = 0$ on $\mathbb{R}^N \setminus \Omega$ we can extend u outside of Ω by setting $u = 0$ on $\mathbb{R}^N \setminus \Omega$ and then we obtain $u_n \to u$ a.e. in $\mathbb{R}^N$. By the uniqueness of the limit we get $v_\alpha = u^\alpha$ a.e. in $\mathbb{R}^N$. Therefore, $u^\alpha \in X_0^s(\Omega)$ for every $\max\left(\frac{1}{2}, \frac{(\gamma+1)(2sm-m+1)}{4sm}\right) < \alpha \le \frac{\gamma+1}{2}$. If the condition (4.21) holds, we can take $\alpha = 1$ obtaining $u \in X_0^s(\Omega)$.

Now, inserting $\varphi \in C_0^\infty(\Omega)$ as a test function in (4.1) we have

$$\frac{a(N,s)}{2} \int_Q \frac{(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} dydx = \int_\Omega \frac{f_n \varphi}{(u_n + \frac{1}{n})^\gamma} dx. \tag{4.22}$$

The fact that $u_n \to u$ a.e. in $\mathbb{R}^N$ implies

$$\frac{(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \to \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \text{ a.e. in } \mathbb{R}^N \times \mathbb{R}^N.$$

By Lemma 4.3, the sequence $\{u_n\}_n$ is uniformly bounded in $H^s_{loc}(\Omega)$ and so we have

$$\frac{u_n(x)-u_n(y)}{|x-y|^{\frac{N+2s}{2}}} \rightharpoonup \frac{u(x)-u(y)}{|x-y|^{\frac{N+2s}{2}}} \text{ weakly in } L^2(K\times K) \tag{4.23}$$

for every $K \subset\subset \Omega$. Now we choose the compact K to be such that $supp(\varphi) \subset K$ and set $d_3 := dist(supp(\varphi), \partial K)) > 0$. Using the fact that $u_n(x) = u_n(y) = 0$ for every $(x, y) \in \mathcal{C}\Omega \times \mathcal{C}\Omega$ and $\varphi(x) = \varphi(y) = 0$ for every $(x, y) \in \mathcal{C}K \times \mathcal{C}K$, we can split the integral in the left-hand side of (4.22) as follows

$$\begin{aligned}
&\int_Q \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx\\
&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx\\
&=\int_K\int_K \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx\\
&+\int_K\int_{\mathcal{C}K} \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx\\
&+\int_{\mathcal{C}K}\int_K \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx.\\
&= I_n^1 + I_n^2 + I_n^3.
\end{aligned}$$

In order to pass to the limit as $n \to +\infty$ in I_n^1, observe that for all $\varphi \in \mathcal{C}_0^\infty(\Omega) \subset H^s(\Omega)$, we have

$$\frac{\varphi(x)-\varphi(y)}{|x-y|^{\frac{N+2s}{2}}} \in L^2(\Omega\times\Omega).$$

Then, by (4.23) we get

$$\lim_{n\to\infty} I_n^1 = \int_K\int_K \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx.$$

For the integrals I_n^2 and I_n^3, we follow some ideas as in the the proof of Theorem 3.1 claiming that

$$\lim_{n\to\infty} I_n^2 = \int_K\int_{\mathcal{C}K} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx$$

and

$$\lim_{n\to\infty} I_n^3 = \int_{\mathcal{C}K}\int_K \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx.$$

Indeed, let us start with the second integral I_n^2. For every $(x, y) \in K \times \mathcal{C}K$, using the fact that $\varphi(y) = 0$ for every $y \in \mathcal{C}K$, we have

$$\begin{aligned}\frac{|(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))|}{|x-y|^{N+2s}} &\le \frac{|u_n(x)\varphi(x)|}{|x-y|^{N+2s}} + \frac{|u_n(y)\varphi(x)|}{|x-y|^{N+2s}}\\ &= |G_n(x,y)| + |H_n(x,y)|.\end{aligned} \tag{4.24}$$

We shall prove that the sequence $\{H_n(x,y)\}$ is uniformly bounded in $L^1(K\times \mathcal{C}K)$. Since $\varphi(x)=0$ on $K\backslash supp(\varphi)$ and $u_n(y)=0$ on $\mathcal{C}\Omega$, we obtain

$$\int_K\int_{\mathcal{C}K}|H_n(x,y)|dydx=\int_{supp(\varphi)}\int_{\Omega\backslash K}\frac{|u_n(y)\varphi(x)|}{|x-y|^{N+2s}}dydx.$$

Since for every $(x,y)\in supp(\varphi)\times \mathcal{C}K$, $|x-y|\geq d_3:=dist(supp(\varphi),\partial K)>0$, we obtain the following estimation

$$\int_K\int_{\mathcal{C}K}|H_n(x,y)|dydx\leq \frac{\|\varphi\|_{L^\infty(\Omega)}|supp(\varphi)|}{d_3^{N+2s}}\|u_n\|_{L^1(\Omega)}.$$

As the sequence $\{u_n\}$ is increasing, then so is $\{H_n(x,y)\}$ and by Beppo-Levi's theorem and the fact that $u_n\to u$ a.e. in $\mathbb{R}^N$, we obtain

$$H_n(x,y)\to \frac{u(y)\varphi(x)}{|x-y|^{N+2s}} \text{ in } L^1(K\times \mathcal{C}K).$$

We deduce that there exist a subsequence of $\{u_n\}$, still indexed by n, and a positive function $h\in L^1(K\times \mathcal{C}K)$ such that

$$|H_n(x,y)|\leq h(x,y) \text{ a.e. in } K\times \mathcal{C}K. \tag{4.25}$$

As regards the sequence $\{G_n(x,y)\}$, we write

$$\begin{aligned}\int_K\int_{\mathcal{C}K}|G_n(x,y)|dydx&=\int_{supp(\varphi)}|u_n(x)\varphi(x)|\int_{\mathcal{C}K}\frac{dy}{|x-y|^{N+2s}}dx\\&\leq \frac{|S^{N-1}|\|\varphi\|_{L^\infty(\Omega)}\|u_n\|_{L^1(\Omega)}}{d_3^{2s}2s}.\end{aligned}$$

As above, the sequence $\{G_n(x,y)\}$ is increasing and by Beppo-Levi's theorem and the fact that $u_n\to u$ a.e. in $\mathbb{R}^N$, we obtain

$$G_n(x,y)\to \frac{u(x)\varphi(x)}{|x-y|^{N+2s}} \text{ in } L^1(K\times \mathcal{C}K).$$

Again we deduce that there exist a subsequence of $\{u_n\}$, still indexed by n, and a positive function $g\in L^1(K\times \mathcal{C}K)$ such that

$$|G_n(x,y)|\leq g(x,y) \text{ a.e. in } K\times \mathcal{C}K. \tag{4.26}$$

Combining (4.24), (4.25) and (4.26), we obtain

$$\frac{|(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))|}{|x-y|^{N+2s}}\leq g(x,y)+h(x,y)\in L^1(K\times \mathcal{C}K),$$

for every $(x,y)\in K\times \mathcal{C}K$. So that by Lebesgue's dominated convergence theorem, we get

$$\lim_{n\to\infty}I_n^2=\int_K\int_{\mathcal{C}K}\frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx.$$

By x/y symmetry, one has

$$\lim_{n\to\infty}I_n^3=\int_{\mathcal{C}K}\int_K\frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx.$$

Then, we conclude that

$$\lim_{n\to\infty}\int_Q \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx$$
$$=\int_Q \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}dydx,$$

for all $\varphi\in C_0^\infty(\Omega)$. As regards the right-hand side of (4.22), we follow the same arguments as in Theorem 3.1 to obtain

$$\lim_{n\to\infty}\int_\Omega \frac{f_n\varphi}{(u_n+\frac{1}{n})^\gamma}dx=\int_\Omega \frac{f\varphi}{u^\gamma}dx.$$

Finally, the passage to the limit in (4.22), as $n\to+\infty$, shows that u is a weak solution of (1.1). □

4.4 The case $\gamma > 1$: Proof of Theorem 3.3

4.4.1 A priori estimates

Lemma 4.7 *Assume $\gamma>1$. Let $s\in(0,1)$ and $f\in L^m(\Omega)$ with $m>1$. Let u_n be a solution of the problem (4.1). Then the sequence $\{u_n^\alpha\}_n$ is uniformly bounded in $X_0^s(\Omega)$ for every $\alpha\in\left(\max\left(\frac{1}{2},\frac{sm(\gamma+1)-m+1}{2sm}\right),\frac{\gamma+1}{2}\right]$. Furthermore, if γ satisfies*

$$1<\gamma<1+\frac{m-1}{sm}, \tag{4.27}$$

then $\{u_n\}_n$ is uniformly bounded in $X_0^s(\Omega)$.

Proof Before estimating the sequence $\{u_n^\alpha\}_n$ in $X_0^s(\Omega)$, we need to prove that

$$u_n(x)\geq C_0\delta^s(x),\ \text{a.e. in }\Omega, \tag{4.28}$$

where $C_0>0$ is a constant not depending on n and $\delta(x):=dist(x,\partial\Omega)$. Observe that $0\leq\frac{f_1}{(u_1+1)^\gamma}\in L^\infty(\Omega)$. Thus, applying [12, Lemma 4.2] we get

$$\frac{u_1(x)}{\delta^s(x)}\geq C\int_\Omega\frac{f_1(y)}{(u_1+1)^\gamma}\delta^s(y)dy\geq C\int_\Omega\frac{f_1(y)}{(\|u_1\|_{L^\infty(\Omega)}+1)^\gamma}\delta^s(y)dy$$
$$\geq C_0:=\frac{C\delta^s(\partial K,\partial\Omega)}{(\|u_1\|_{L^\infty(\Omega)}+1)^\gamma}\int_K f_1(y)dy$$

where K is an arbitrary compact in Ω. By Lemma 4.2, the sequence $\{u_n\}_n$ is increasing and therefore the inequality (4.28) is satisfied.

Now, we shall prove a priori estimates on u_n^α in $X_0^s(\Omega)$ for every α such that

$$\max\left(\frac{1}{2},\frac{sm(\gamma+1)-m+1}{2sm}\right)<\alpha\leq\frac{\gamma+1}{2}.$$

Let $n\geq1$ and let $0<\varepsilon<\frac{1}{n}$. For $\eta>0$, taking $(u_n+\varepsilon)^\eta-\varepsilon^\eta$ as a test function in (4.1), we obtain

$$\frac{a(N,s)}{2}\int_Q\frac{(u_n(x)-u_n(y))((u_n(x)+\varepsilon)^\eta-(u_n(y)+\varepsilon)^\eta)}{|x-y|^{N+2s}}dydx$$
$$\leq\int_\Omega\frac{f_n}{(u_n(x)+\frac{1}{n})^{\gamma-\eta}}dx.$$

By Fatou's lemma we can pass to the limit in ε obtaining

$$\int_Q \frac{(u_n(x)-u_n(y))(u_n^\eta(x)-u_n^\eta(y))}{|x-y|^{N+2s}}dydx \le \frac{2}{a(N,s)}\int_\Omega \frac{f_n}{(u_n(x)+\frac{1}{n})^{\gamma-\eta}}dx.$$

Then, an application of the item i) in Lemma 2.2 and the Hölder inequality respectively yield

$$\int_Q \frac{\left|u_n^{\frac{\eta+1}{2}}(x)-u_n^{\frac{\eta+1}{2}}(y)\right|^2}{|x-y|^{N+2s}}dydx \le C(\eta,N,s)\|f\|_{L^m(\Omega)}\left(\int_\Omega \frac{dx}{u_n^{(\gamma-\eta)m'}(x)}\right)^{\frac{1}{m'}}.$$

Let us choose $0<\eta\le\gamma$. The inequality (4.28) implies

$$\int_Q \frac{\left|u_n^{\frac{\eta+1}{2}}(x)-u_n^{\frac{\eta+1}{2}}(y)\right|^2}{|x-y|^{N+2s}}dydx \le$$
$$C(\eta,N,s)C_0^{(\eta-\gamma)s}\|f\|_{L^m(\Omega)}\left(\int_\Omega \frac{dx}{\delta^{(\gamma-\eta)sm'}(x)}\right)^{\frac{1}{m'}}.$$

Now, choosing $\alpha=\frac{\eta+1}{2}$ one has $\frac{1}{2}<\alpha\le\frac{\gamma+1}{2}$ and then

$$\int_Q \frac{|u_n^\alpha(x)-u_n^\alpha(y)|^2}{|x-y|^{N+2s}}dydx \le$$
$$C(\eta,N,s)C_0^{(\eta-\gamma)s}\|f\|_{L^m(\Omega)}\left(\int_\Omega \frac{dx}{\delta^{(\gamma-2\alpha+1)sm'}(x)}\right)^{\frac{1}{m'}}.$$

Observe that the integral in the right-hand side of the above inequality converges if and only if $(\gamma-2\alpha+1)sm'<1$, that is $\frac{sm(\gamma+1)-m+1}{2sm}<\alpha$. Therefore, the sequence $\{u_n^\alpha\}$ is uniformly bounded in $X_0^s(\Omega)$, for every $\alpha\in\left(\max\left(\frac{1}{2},\frac{sm(\gamma+1)-m+1}{2sm}\right),\frac{\gamma+1}{2}\right)$.

In particular, if (4.27) holds then $\frac{sm(\gamma+1)-m+1}{2sm}<1$ and so $\{u_n\}$ is uniformly bounded in $X_0^s(\Omega)$. □

4.4.2 Passage to the limit

Proof of Theorem 3.3 We use similar arguments as in the proof of Theorem 3.2 obtaining that $u:=\lim_{n\to\infty}u_n$ is a weak solution to (1.1) and $u^\alpha\in X_0^s(\Omega)$ for every $\max\left(\frac{1}{2},\frac{sm(\gamma+1)-m+1}{2sm}\right)<\alpha\le\frac{\gamma+1}{2}$. Furthermore, if (4.27) holds then $\frac{sm(\gamma+1)-m+1}{2sm}<1$ and so $u\in X_0^s(\Omega)$. □

4.5 The case $\gamma>1$: Proof of Theorem 3.4

Proof of Theorem 3.4 Let $\gamma>1$ and let u_n be a solution of (4.1). Let $0<\varepsilon<\frac{1}{n}$, $n\ge 1$. For $\eta>0$, taking $(u_n+\varepsilon)^\eta-\varepsilon^\eta$ as a test function in (4.1), we follow the same lines in the proof of Lemma (4.7). We obtain

$$\int_Q \frac{\left|u_n^{\frac{\eta+1}{2}}(x)-u_n^{\frac{\eta+1}{2}}(y)\right|^2}{|x-y|^{N+2s}}dydx \le C(\eta,N,s)\int_{supp(f)}\frac{f}{u_n^{\gamma-\eta}}dx.$$

Now, let us choose $0 < \eta \le \gamma$ and set $\alpha = \frac{\eta+1}{2}$, we get

$$\int_Q \frac{\left|u_n^\alpha(x) - u_n^\alpha(y)\right|^2}{|x-y|^{N+2s}} dydx \le C(\eta, N, s) \int_{supp(f)} \frac{f}{u_n^{\gamma-(2\alpha-1)}} dx.$$

Applying Lemma 4.2, we obtain

$$\int_Q \frac{\left|u_n^\alpha(x) - u_n^\alpha(y)\right|^2}{|x-y|^{N+2s}} dydx \le \frac{C(\eta, N, s)}{c_{supp(f)}^{\gamma-(2\alpha-1)}} \|f\|_{L^1(\Omega)}.$$

It follows that $\{u_n^\alpha\}$ is uniformly bounded in $X_0^s(\Omega)$ for every $\alpha \in \left(\frac{1}{2}, \frac{\gamma+1}{2}\right]$.

Arguing as above, it's easy to see that $u := \lim_{n\to\infty} u_n$ is a weak solution of (1.1) and $u^\alpha \in X_0^s(\Omega)$ for every $\alpha \in \left(\frac{1}{2}, \frac{\gamma+1}{2}\right]$. □

4.6 Uniqueness : Proof of Theorem 3.5

Proof In order to prove the uniqueness of finite energy solutions, we assume that there exist two weak solutions u_1 and $u_2 \in X_0^s(\Omega)$ to (1.1). By Lemma 5.4 the weak solutions u_1 and u_2 both satisfy (5.3). By [38, Proposition 3] we have $(u_1 - u_2)^+ \in X_0^s(\Omega)$, hence $(u_1 - u_2)^+$ is an admissible test function in (5.3). Taking it so in the difference of formulations (5.3) solved by u_1 and u_2 we arrive at

$$\int_Q \frac{\Big((u_1(x) - u_2(x)) - (u_1(y) - u_2(y))\Big)\Big((u_1-u_2)^+(x) - (u_1-u_2)^+(y)\Big)}{|x-y|^{N+2s}} dydx$$
$$= \frac{2}{a(N,s)} \int_\Omega f(x)\left(\frac{1}{u_1^\gamma} - \frac{1}{u_2^\gamma}\right)(u_1 - u_2)^+(x)dx.$$

Observe that for any function $g : \mathbb{R}^N \to \mathbb{R}$ the following inequality

$$(g(x) - g(y))(g^+(x) - g^+(y)) \ge (g^+(x) - g^+(y))^2$$

holds true for every $x, y \in \mathbb{R}^N$. It follows that

$$\|(u_1 - u_2)^+\|^2_{X_0^s(\Omega)} = 0,$$

which gives $u_2 \ge u_1$. By the u_1/u_2 symmetry we obtain $u_1 = u_2$. □

5 Some regularity results

We point out that if $f \in L^m(\Omega)$ with $m \ge \overline{m} := \left(\frac{2_s^*}{1-\gamma}\right)' = \frac{2N}{N+2s+\gamma(N-2s)}$, then following the same lines as in the proof of [11, Lemma 3.4] we can prove that the sequence $\{u_n\}_n$ of non-negative solutions of the problem (4.1) is uniformly bounded in $X_0^s(\Omega)$. Furthermore, testing by a $\mathcal{C}_0^\infty(\Omega)$-function in (4.1) one can pass to the limit and obtain that $u := \lim_{n\to\infty} u_n$ is a weak solution for the problem (1.1) in the sense of Definition 2.1. In this section we give some further summability results of this weak solution u.

Lemma 5.1 *Suppose that* $0 < \gamma < 1$. *Let* u *be the weak solution of (1.1) corresponding to* $f \in L^m(\Omega)$ *with* $m \geq \left(\frac{2_s^*}{1-\gamma}\right)' = \frac{2N}{N+2s+\gamma(N-2s)}$. *If* $\left(\frac{2_s^*}{1-\gamma}\right)' \leq m < \frac{N}{2s}$, *then* $u \in L^\sigma(\Omega)$ *where* $\sigma = \frac{Nm(\gamma+1)}{N-2sm}$.

Proof Let $u_n \in X_0^s(\Omega) \cap L^\infty(\Omega)$ be a solution of the problem (4.1). Inserting u_n^θ, $\theta > 1$, as a test function in (4.1) we get

$$\int_Q \frac{(u_n(x) - u_n(y))(u_n^\theta(x) - u_n^\theta(y))}{|x-y|^{N+2s}} dydx \leq \frac{2}{a(N,s)} \int_\Omega f_n u_n^{\theta-\gamma}(x)dx.$$

Applying the item $i)$ in Lemma 2.2 in the right-hand side and Hölder's inequality in the left hand-side, we get

$$\int_Q \frac{|u_n(x)^{\frac{\theta+1}{2}} - u_n(y)^{\frac{\theta+1}{2}}|^2}{|x-y|^{N+2s}} dydx \leq C_1 \|f\|_{L^m(\Omega)} \left(\int_\Omega u_n^{(\theta-\gamma)m'}(x)dx \right)^{\frac{1}{m'}}.$$

where $C_1 = \frac{(\theta+1)^2}{2\theta a(N,s)}$. Applying fractional Sobolev's inequality, we obtain

$$\int_\Omega |u_n(x)|^{\frac{N(\theta+1)}{N-2s}} dx \leq C_2 \|f\|_{L^m(\Omega)}^{\frac{N}{N-2s}} \left(\int_\Omega u_n^{(\theta-\gamma)m'}(x)dx \right)^{\frac{N}{m'(N-2s)}},$$

with $C_2 = (S(N,s)C_1)^{\frac{N}{N-2s}}$. Now we choose $\theta > 1$ in order to get $\frac{N(\theta+1)}{N-2s} = (\theta-\gamma)m'$, that is

$$\theta = \frac{N(m-1) + \gamma m(N-2s)}{N-2sm}.$$

Observe that $\theta > 1$ and

$$\frac{N(\theta+1)}{N-2s} = \frac{Nm(\gamma+1)}{N-2sm}.$$

In addition the assumption $m < \frac{N}{2s}$ implies $\frac{N}{m'(N-2s)} < 1$. Then it follows

$$\int_\Omega |u_n(x)|^{\frac{Nm(1+\gamma)}{N-2sm}} dx \leq C_2^{\frac{m(N-2s)}{N-2sm}} \|f\|_{L^m(\Omega)}^{\frac{Nm}{N-2sm}}.$$

By Fatou's Lemma, we obtain $u \in L^\sigma(\Omega)$ with $\sigma = \frac{Nm(\gamma+1)}{N-2sm}$. □

Remark 5.1 In the particular case where $m = \left(2_s^*\right)'$, we obtain $u \in L^{(1+\gamma)2_s^*}(\Omega)$ which is exactly the result stated in [11, Proposition 3.8]. While if $s = 1$ the exponent of summability $\sigma = \frac{Nm(\gamma+1)}{N-2sm}$ coincides with the one given [15, Lemma 5.5] in the local case.

Lemma 5.2 *(Limit case : Exponential summability) Assume that* $\gamma > 0$. *Let* $f \in L^{\frac{N}{2s}}(\Omega)$ *and let* u *be the weak solution of the problem (1.1) given by Theorem 3.3 if* $\gamma > 1$ *or given by [18, Theorem 3.2.] if* $0 < \gamma \leq 1$. *Then there exists* $\lambda > 0$ *such that* $e^{\lambda \frac{N(1+\gamma)}{N-2s} u} \in L^1(\Omega)$.

Proof Let us start with the case $\gamma > 1$. For $\lambda > 0$, we consider the locally Lipschitz function $t \to \psi(t) = (e^{\lambda t} - 1)^{\frac{\gamma+1}{2}}$. Let $u_n \in X_0^s(\Omega) \cap L^\infty(\Omega)$ be a non-negative solution of the problem (4.1). Since $\psi(0) = 0$ and we can take $\psi'(u_n)\psi(u_n)$ as a test function in (4.1). As $\gamma > 1$, the function ψ is convex so that according with [38, Proposition 4.] we arrive at

$$\begin{aligned}&\frac{a(N,s)}{2}\int_Q \frac{|\psi(u_n)(x) - \psi(u_n)(y)|^2}{|x-y|^{N+2s}}dydx\\&\le \int_\Omega \psi'(u_n)\psi(u_n)(-\Delta)^s u_n(x)dx\\&= \int_\Omega \frac{f_n}{(u_n + \frac{1}{n})^\gamma}\psi'(u_n)\psi(u_n)dx.\end{aligned}$$

Using the Sobolev inequality, we obtain

$$\|\psi(u_n)\|^2_{L^{2^*_s}(\Omega)} \le \frac{2S(N,s)}{a(N,s)}\int_\Omega \frac{f}{u_n^\gamma}\psi'(u_n)\psi(u_n)dx.$$

Using the elementary inequality $\frac{e^a-1}{a} \le e^a$ for every $a > 0$, we get

$$\frac{\psi'(u_n)\psi(u_n)}{u_n^\gamma} \le \frac{\gamma+1}{2}\lambda^{\gamma+1}e^{\lambda(\gamma+1)u_n} \le C(\gamma)\lambda^{\gamma+1}\psi^2(u_n) + C(\lambda,\gamma),$$

where we have set $C(\gamma) = 2^\gamma\frac{\gamma+1}{2}$ and $C(\lambda,\gamma) = \lambda^{\gamma+1}C(\gamma)$. Then, using Hölder's inequality we obtain

$$\begin{aligned}\|\psi(u_n)\|^2_{L^{2^*_s}(\Omega)} &\le \frac{2S(N,s)C(\gamma)\lambda^{\gamma+1}}{a(N,s)}\int_\Omega f\psi^2(u_n) + C(\lambda,\gamma,\Omega)\|f\|_{L^{\frac{N}{2s}}(\Omega)}\\&\le \frac{2S(N,s)C(\gamma)\lambda^{\gamma+1}}{a(N,s)}\|f\|_{L^{\frac{N}{2s}}(\Omega)}\|\psi(u_n)\|^2_{L^{2^*_s}(\Omega)}\\&+C(\lambda,\gamma,\Omega)\|f\|_{L^{\frac{N}{2s}}(\Omega)}.\end{aligned}$$

Choosing $\lambda > 0$ to be such that $\frac{2S(N,s)C(\gamma)\|f\|_{L^{\frac{N}{2s}}(\Omega)}\lambda^{\gamma+1}}{a(N,s)} < 1$, we deduce that

$$\int_\Omega e^{\lambda\frac{N(1+\gamma)}{N-2s}u_n}dx \le C,$$

where C is a constant not depending on n. Applying Fatou's lemma, we conclude the result.

We turn now to the case $\gamma \le 1$. We consider the convex and locally Lipschitz function $t \to \psi(t) = e^{\frac{\gamma+1}{2}\lambda t} - 1$ and we insert $\psi'(u_n)\psi(u_n)$ as a test function in (4.1). Again by [38, Proposition 4.] and the Sobolev inequality we obtain

$$\|\psi(u_n)\|^2_{L^{2^*_s}(\Omega)} \le \frac{2S(N,s)}{a(N,s)}\int_\Omega \frac{f}{u_n^\gamma}\psi'(u_n)\psi(u_n)dx.$$

Since $0 < \frac{\gamma+1}{2} \le 1$, we can apply the inequality in the item $iii)$ in Lemma 2.2 obtaining

$$\frac{\psi'(u_n)\psi(u_n)}{u_n^\gamma} \le \frac{\gamma+1}{2}\lambda\frac{e^{\frac{\gamma+1}{2}\lambda u_n}\left(e^{\lambda u_n} - 1\right)^{\frac{\gamma+1}{2}}}{u_n^\gamma}.$$

Noticing that $u_n^{\frac{\gamma+1}{2}} \leq u_n^{\gamma}$ on the subset $\{u_n \leq 1\} := \{x \in \Omega : u_n(x) \leq 1\}$, we can write

$$\int_\Omega \frac{f}{u_n^\gamma}\psi'(u_n)\psi(u_n)dx \leq \frac{\gamma+1}{2}\lambda \int_{\{u_n\leq 1\}} \frac{f e^{\frac{\gamma+1}{2}\lambda u_n}\left(e^{\lambda u_n}-1\right)^{\frac{\gamma+1}{2}}}{u_n^{\frac{\gamma+1}{2}}}dx$$
$$+\frac{\gamma+1}{2}\lambda \int_{\{u_n>1\}} f e^{\frac{\gamma+1}{2}\lambda u_n}\left(e^{\lambda u_n}-1\right)^{\frac{\gamma+1}{2}}dx.$$

Using the elementary inequality $\frac{e^a-1}{a} \leq e^a$, which holds for every $a > 0$, in the first integral in the right-hand side of the previous inequality, we obtain

$$\int_\Omega \frac{f}{u_n^\gamma}\psi'(u_n)\psi(u_n)dx \leq \frac{\gamma+1}{2}\lambda^{\frac{\gamma+3}{2}} \int_{\{u_n\leq 1\}} f e^{(\gamma+1)\lambda u_n}$$
$$+\frac{\gamma+1}{2}\lambda \int_{\{u_n>1\}} f e^{(\gamma+1)\lambda u_n}dx$$
$$\leq \frac{\gamma+1}{2}\lambda^{\frac{\gamma+3}{2}} e^{(\gamma+1)\lambda}\int_\Omega f dx$$
$$+\frac{\gamma+1}{2}\lambda \int_\Omega f(\psi(u_n)+1)^2 dx.$$

Using the fact that $(\psi(u_n)+1)^2 \leq 2(\psi(u_n)^2+1)$, we get

$$\int_\Omega \frac{f}{u_n^\gamma}\psi'(u_n)\psi(u_n)dx \leq \frac{\gamma+1}{2}\lambda^{\frac{\gamma+3}{2}} e^{(\gamma+1)\lambda}\int_\Omega f dx$$
$$+(\gamma+1)\lambda\int_\Omega f(\psi^2(u_n)+1)dx$$
$$\leq \left(\frac{\gamma+1}{2}\lambda^{\frac{\gamma+3}{2}} e^{(\gamma+1)\lambda} + (\gamma+1)\lambda\right)\int_\Omega f dx$$
$$+(\gamma+1)\lambda\int_\Omega f\psi^2(u_n)dx.$$

An application of Hölder's inequality with the exponents $\frac{N}{N-2s}$ and $\frac{N}{2s}$ gives

$$\|\psi(u_n)\|^2_{L^{2_s^*}(\Omega)} \leq \frac{S(N,s)}{a(N,s)}(\gamma+1)\left(\lambda^{\frac{\gamma+3}{2}} e^{(\gamma+1)\lambda} + 2\lambda\right)|\Omega|^{\frac{N-2s}{N}}\|f\|_{L^{\frac{N}{2s}}(\Omega)}$$
$$+\frac{2S(N,s)(\gamma+1)}{a(N,s)}\lambda\|f\|_{L^{\frac{N}{2s}}(\Omega)}\|\psi(u_n)\|^2_{L^{2_s^*}(\Omega)}.$$

Therefore, choosing $\lambda > 0$ such that $\lambda < \frac{a(N,s)}{2S(N,s)(\gamma+1)\|f\|_{L^{\frac{N}{2s}}(\Omega)}}$ we obtain

$$\int_\Omega e^{\lambda\frac{N(1+\gamma)}{N-2s}u_n}dx \leq C,$$

where C is a constant not depending on n, and by Fatou's lemma we conclude the result. □

Remark 5.2 Recall that the inequality $e^x \geq \frac{x^k}{k!}$ holds for every $x > 0$ and $k \in \mathbb{N}$. Thus, we conclude that $u \in L^r(\Omega)$ for every $r < \infty$.

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Appendix

We start by proving the following lemma which we have used in the proof of Lemma 4.4.

Lemma 5.3 *Let* $F(x) = x^r$, $0 < r < 1$, *for every* $x > 0$. *Then for every function* $v : \mathbb{R}^N \to]0, +\infty[$ *that satisfies*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}}dydx < \infty,$$

we have

$$(-\Delta)^s(F\circ v)(x) \leq F'(v(x))(-\Delta)^s v(x) - \frac{F''(v(x))}{r}a(N,s)\int_{\mathbb{R}^N}\frac{\big(v(x)-v(y)\big)^2}{|x-y|^{N+2s}}dy. \tag{5.1}$$

Proof Following [20, Lemma 2.3.], we can use Taylor's formula obtaining for every $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$

$$F(v(y)) - F(v(x)) = F'(v(x))(v(y)-v(x)) + R(F), \tag{5.2}$$

where

$$\begin{aligned} R(F) &= \int_{v(x)}^{v(y)} (v(y)-t)F''(t)dt \\ &= (v(y)-v(x))^2 \int_0^1 (1-s)F''(v(x)+s(v(y)-v(x)))ds. \end{aligned}$$

On other hand, since the function F'' is increasing we have

$$\begin{aligned} (1-s)v(x) &\leq v(x)+s(v(y)-v(x)) \\ &\Rightarrow F''((1-s)v(x)) \leq F''(v(x)+s(v(y)-v(x))). \end{aligned}$$

Hence, it follows

$$\begin{aligned} -R(F) &\leq -(v(y)-v(x))^2 \textstyle\int_0^1 (1-s)F''((1-s)v(x))ds \\ &= -(v(y)-v(x))^2 F''(v(x)) \textstyle\int_0^1 (1-s)^{r-1}ds. \end{aligned}$$

Then, from (5.2) we obtain

$$F(v(x)) - F(v(y)) \leq F'(v(x))(v(x)-v(y)) - \frac{F''(v(x))}{r}(v(y)-v(x))^2.$$

Dividing both sides of this inequality by $|x-y|^{N+2s}$ and then integrating with respect to the variable y we arrive at

$$\begin{aligned} a(N,s)P.V.\textstyle\int_{\mathbb{R}^N} \frac{F(v(x))-F(v(y))}{|x-y|^{N+2s}}dy &\leq F'(v(x))a(N,s)P.V.\textstyle\int_{\mathbb{R}^N}\frac{(v(x)-v(y))}{|x-y|^{N+2s}}dy \\ &\quad -\frac{F''(v(x))}{r}a(N,s)P.V.\textstyle\int_{\mathbb{R}^N}\frac{(v(y)-v(x))^2}{|x-y|^{N+2s}}dy, \end{aligned}$$

which proves (5.1). $\square$

In the following result we extend the space of admissible test functions in (2.4).

Lemma 5.4 *Let $u \in X_0^s(\Omega)$ be a solution of the problem (1.1) taken in the sense of Definition 2.1 with $f \in L^1(\Omega)$. Then for every $\phi \in X_0^s(\Omega)$ we get $\frac{f\phi}{u^\gamma} \in L^1(\Omega)$ and*

$$\frac{a(N,s)}{2}\int_Q \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}dydx = \int_\Omega \frac{f\phi}{u^\gamma}dx. \tag{5.3}$$

Proof Take an arbitrary $\phi \in X_0^s(\Omega)$. By [29, Theorem 6] there exists a sequence $\{\varphi_n\}_n \subset \mathcal{C}_0^\infty(\Omega)$ such that $\varphi_n \to \phi$ in norm in $H^s(\mathbb{R}^N)$. Writing (2.4) with $\varphi_n \in \mathcal{C}_0^\infty(\Omega)$ we obtain

$$\frac{a(N,s)}{2}\int_Q \frac{(u(x)-u(y))(\varphi_n(x)-\varphi_n(y))}{|x-y|^{N+2s}}dydx = \int_\Omega \frac{f\varphi_n}{u^\gamma}dx, \tag{5.4}$$

in which we shall pass to the limit as n tends to $+\infty$. Starting with the left-hand side of (5.4), we consider the following two functions

$$F_n(x,y) = \frac{(\varphi_n(x)-\varphi_n(y))}{|x-y|^{\frac{N+2s}{2}}} \text{ and } F(x,y) = \frac{(\phi(x)-\phi(y))}{|x-y|^{\frac{N+2s}{2}}}.$$

Notice that the convergence $\varphi_n \to \phi$ in norm in $H^s(\mathbb{R}^N)$ implies that the sequence $\{F_n(x,y)\}_n$ converges to $F(x,y)$ in $L^2(\mathbb{R}^{2N})$ and, up to a subsequence if necessary, we can assume that $\{F_n(x,y)\}_n$ converges almost everywhere in $\mathbb{R}^{2N}$.

As $u \in X_0^s(\Omega)$ we have $\frac{(u(x)-u(y))}{|x-y|^{\frac{N+2s}{2}}} \in L^2(\mathbb{R}^{2N})$ implying

$$\lim_{n\to\infty}\int_Q \frac{(u(x)-u(y))(\varphi_n(x)-\varphi_n(y))}{|x-y|^{N+2s}}dydx = \int_Q \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}dydx.$$

For the term in the right-hand side of (5.4), we first note that thanks to [38, Proposition 3.] the two functions $(\varphi_n-\varphi_k)^+$ and $(\varphi_n-\varphi_k)^-$ are both admissible test functions in (2.4). Taking them so we obtain

$$\int_\Omega \frac{f}{u^\gamma}(\varphi_n-\varphi_k)^+(x)dx = \frac{a(N,s)}{2}\int_Q \frac{(u(x)-u(y))\big((\varphi_n-\varphi_k)^+(x)-(\varphi_n-\varphi_k)^+(y)\big)}{|x-y|^{N+2s}}dydx$$

and

$$\int_\Omega \frac{f}{u^\gamma}(\varphi_n-\varphi_k)^-(x)dx = \frac{a(N,s)}{2}\int_Q \frac{(u(x)-u(y))\big((\varphi_n-\varphi_k)^-(x)-(\varphi_n-\varphi_k)^-(y)\big)}{|x-y|^{N+2s}}dydx.$$

Then, summing up both the two equalities we have

$$\begin{aligned}\int_\Omega \frac{f}{u^\gamma}\Big|\varphi_n-\varphi_k\Big|dx &= \frac{a(N,s)}{2}\int_Q \frac{(u(x)-u(y))\Big(|\varphi_n(x)-\varphi_k(x)|-|\varphi_n(y)-\varphi_k(y)|\Big)}{|x-y|^{N+2s}}dydx \\ &\le \frac{a(N,s)}{2}\int_Q \frac{|u(x)-u(y)|\Big|(\varphi_n(x)-\varphi_k(x))-(\varphi_n(y)-\varphi_k(y))\Big|}{|x-y|^{N+2s}}dydx\end{aligned}$$

and then the Hölder inequality implies

$$\int_\Omega \left|\frac{f\varphi_n}{u^\gamma} - \frac{f\varphi_k}{u^\gamma}\right| dx \le \frac{a(N,s)}{2}\|u\|_{X_0^s(\Omega)}\|\varphi_n - \varphi_k\|_{X_0^s(\Omega)}.$$

Thus, we deduce that $\left\{\frac{f\varphi_n}{u^\gamma}\right\}_n$ is a Cauchy sequence in $L^1(\Omega)$. Since φ_n converges to φ a.e. in Ω, the sequence $\left\{\frac{f\varphi_n}{u^\gamma}\right\}_n$ converges to $\frac{f\phi}{u^\gamma} \in L^1(\Omega)$ in norm in $L^1(\Omega)$. So that the passage to the limit as n tends to infinity in (5.4) yields

$$\frac{a(N,s)}{2}\int_Q \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}} dydx = \int_\Omega \frac{f\phi}{u^\gamma} dx,$$

for every $\phi \in X_0^s(\Omega)$. □

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