Calc. Var. (2021) 60:169
https://doi.org/10.1007/s00526-021-02020-7

Calculus of Variations

Existence of normalized ground states for the Sobolev critical Schrödinger equation with combined nonlinearities

Xinfu Li[1]

Received: 29 January 2021 / Accepted: 16 May 2021 / Published online: 25 July 2021

Abstract

We study the Sobolev critical Schrödinger equation with combined power nonlinearities

$$-\Delta u=\lambda u+|u|^{\frac{2N}{N-2}-2}u+\mu|u|^{q-2}u,\ \ x\in\mathbb{R}^N$$

having prescribed mass

$$\int_{\mathbb{R}^N}|u|^2dx=a^2.$$

For a L^2-critical or L^2-supercritical perturbation $\mu|u|^{q-2}u$, we prove existence of normalized ground states, by introducing the Sobolev subcritical approximation method to mass constrained problem. Our result settles a question raised by N. Soave [22]. Meanwhile, the Sobolev subcritical problem is treated again by using the Pohožaev constraint, Schwartz symmetrization rearrangements and various scaling transformations.

Mathematics Subject Classification 35J20 · 35Q55

1 Introduction and main results

In this paper, we study the existence of ground state standing waves with prescribed mass for the nonlinear Schrödinger equation with combined power nonlinearities

$$i\partial_t\psi+\Delta\psi+|\psi|^{p-2}\psi+\mu|\psi|^{q-2}\psi=0,\ \ (t,x)\in\mathbb{R}\times\mathbb{R}^N, \tag{1.1}$$

where $N\geq 1$, $\mu>0$ and $2<q<p\begin{cases}<2^*:=\infty, & N=1,2,\\ \leq 2^*:=2N/(N-2), & N\geq 3.\end{cases}$ Starting from the fundamental contribution by T. Tao, M. Visan and X. Zhang [23], the NLS equation with combined nonlinearities attracted much attention, see for example [1,6,7,11,12,15,18,19,26].

Communicated by A. Malchiodi.

✉ Xinfu Li
lxylxf@tjcu.edu.cn

1 School of Science, Tianjin University of Commerce, Tianjin 300134, People's Republic of China

Standing waves to (1.1) are solutions of the form $\psi(t,x) = e^{-i\lambda t}u(x)$, where $\lambda \in \mathbb{R}$ and $u : \mathbb{R}^N \to \mathbb{C}$. Then u satisfies the equation

$$-\Delta u = \lambda u + |u|^{p-2}u + \mu |u|^{q-2}u,\ x \in \mathbb{R}^N. \tag{1.2}$$

A possible choice is to fix $\lambda \in \mathbb{R}$ and to search for solutions to (1.2) as critical points of the action functional

$$J_{p,q}(u) := \int_{\mathbb{R}^N} \left(\frac{1}{2}|\nabla u|^2 - \frac{\lambda}{2}|u|^2 - \frac{1}{p}|u|^p - \frac{\mu}{q}|u|^q\right) dx,$$

see for example [2,17] and the references therein.

Alternatively, one can search for solutions to (1.2) having prescribed mass

$$\int_{\mathbb{R}^N} |u|^2 dx = a^2. \tag{1.3}$$

In this direction, define on $H := H^1(\mathbb{R}^N, \mathbb{C})$ the energy functional

$$E_{p,q}(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p dx - \frac{\mu}{q}\int_{\mathbb{R}^N} |u|^q dx.$$

It is standard to check that $E_{p,q} \in C^1$ and a critical point of $E_{p,q}$ constrained to

$$S_a = \{u \in H^1(\mathbb{R}^N, \mathbb{C}) : \int_{\mathbb{R}^N} |u|^2 = a^2\}$$

gives rise to a solution to (1.2), satisfying (1.3). Such solution is usually called a normalized solution of (1.2). In this method, the parameter $\lambda \in \mathbb{R}$ arises as a Lagrange multiplier, which depends on the solution and is not a priori given. In this paper, we will focus on the existence of normalized ground state of (1.2), defined as follows:

Definition 1.1 We say that u is a normalized ground state to (1.2) on S_a if

$$E_{p,q}(u) = z_{p,q} := \inf\{E_{p,q}(v) : v \in S_a,\ (E_{p,q}|_{S_a})'(v) = 0\}.$$

The set of the normalized ground states will be denoted by $\mathcal{Z}_{p,q}$.

In the study of (1.2-1.3) an important role is played by the so-called L^2-critical exponent

$$\bar{p} = 2 + \frac{4}{N}.$$

A very complete analysis of the various cases that may happen for (1.2-1.3), depending on the values of (p,q), has been provided recently in [4,9,10,21,22]. See [21] for the cases $N \geq 1$ and $p < 2^*$, [9,10,22] for the cases $N \geq 3$ and $p = 2^*$, and [4] for the cases $N = 1$, $p = +\infty$ and $q \leq 6$. See [20] for the Schrödinger equation with combined nonlinearities on metric graphs. For a L^2-critical or L^2-supercritical perturbation $q \geq \bar{p}$ and the Sobolev subcritical case $p < 2^*$, [21] obtained the following results to (1.2):

Theorem 1.2 *Let $N \geq 1$, $a > 0$, $\mu > 0$ and $\bar{p} \leq q < p < 2^*$. If $q = \bar{p}$, we further assume that $\mu a^{\frac{4}{N}} < (\bar{a}_N)^{\frac{4}{N}}$, where $\bar{a}_N$ is defined in (2.1). Then $E_{p,q}|_{S_a}$ has a critical point u at positive level $E_{p,q}(u) > 0$, with the following properties: u is a real-valued positive function in $\mathbb{R}^N$, is radially symmetric, is radially non-increasing, solves (1.2) for some $\lambda < 0$, and is a normalized ground state of (1.2) on S_a.*

Remark 1.3 In fact, [21] did not consider the case $q > \bar{p}$ of Theorem 1.2, while it also holds by repeating the proof for the case $q = \bar{p}$. In this paper, we will give Theorem 1.2 another proof, which is useful to the proof of Theorem 1.4, so we write it here in a unified form.

However, for the L^2-supercritical and Sobolev critical case $\bar{p} < q < p = 2^*$, a condition $\mu a^{N+q-Nq/2} < \alpha(N, q)$ is added to get similar results as to Theorem 1.2, where $\alpha(N, q)$ is finite for $N \geq 5$, see [22] for more details. Inspired by the results of the unconstrained problem considered in [14] and [17], we guess that the condition maybe can be removed when q is close to 2^*. Fortunately, we succeed to do it in the full interval $\bar{p} < q < 2^*$ and obtain similar results as Theorem 1.2 for the Sobolev critical problem. Our result settles an open question raised by N. Soave [22].

Theorem 1.4 *Let $N \geq 3$, $a > 0$, $\mu > 0$ and $\bar{p} \leq q < p = 2^*$. If $q = \bar{p}$, we further assume that $\mu a^{\frac{4}{N}} < (\bar{a}_N)^{\frac{4}{N}}$. Then $E_{p,q}|_{S_a}$ has a critical point u at positive level $0 < E_{p,q}(u) < \frac{1}{N} S^{\frac{N}{2}}$, with the following properties: u is a real-valued positive function in $\mathbb{R}^N$, is radially symmetric, is radially non-increasing, solves (1.2) for some $\lambda < 0$, and is a normalized ground state of (1.2) on S_a. Here S is defined in (3.2).*

Remark 1.5 In Theorem 1.4, we only improve the result of [22] for the case $q > \bar{p}$, while it is the same as that of [22] in the case $q = \bar{p}$. Since the proof will be done in a uniform way, we write it here.

Remark 1.6 When $q > \bar{p}$, similarly to [22], to prove Theorem 1.4, a key step is to show that $c_{2^*,q} < \frac{1}{N} S^{\frac{N}{2}}$, which will be obtained by choosing appropriate functions. To do this, in Lemma 6.4 of [22], they first constructed u_ϵ and $v_\epsilon := a\frac{u_\epsilon(x)}{\|u_\epsilon\|_2}$, and then estimated the maximum of $\Psi_{v_\epsilon}(\tau) := E_{2^*,q}((v_\epsilon)^\tau)$. In view of the expression of $\Psi_{v_\epsilon}(\tau)$ and the estimates of u_ϵ, the lower bound of the maximum point τ_{v_ϵ} of $\Psi_{v_\epsilon}(\tau)$ was needed and thus a condition $\mu a^{N+q-Nq/2} < \alpha(N, q)$ was added for $N \geq 5$. To remove this condition, in this paper, we will use a different transformation to define $v_\epsilon := (a^{-1}\|u_\epsilon\|_2)^{\frac{N-2}{2}} u_\epsilon(a^{-1}\|u_\epsilon\|_2 x)$ and subsequently obtain a different expression of $\Psi_{v_\epsilon}(\tau)$ (see (3.3)). In this case, by using the estimates of u_ϵ and the fact that $c_{2^*,q} > 0$, we can easily show that $\tau_{v_\epsilon} \in [\tau_0, \tau_1]$ with $\tau_0, \tau_1 > 0$ and then obtain the upper bound of $c_{2^*,q}$ without adding additional conditions, see Lemma 3.3.

Remark 1.7 Following the proof of Theorem 1.7 in [21] word by word, we can show that under the assumptions of Theorems 1.2 or 1.4,

$$\mathcal{Z}_{p,q} = \{e^{i\theta}|u| \text{ for some } \theta \in \mathbb{R} \text{ and } |u| > 0 \text{ in } \mathbb{R}^N\}$$

and for any $u \in \mathcal{Z}_{p,q}$, the standing wave $e^{-i\lambda t}u(x)$ is strongly unstable.

Remark 1.8 By Lemma 2.6, any normalized ground state u of (1.2) satisfies equation (1.2) with some $\lambda = \lambda(u) < 0$. For such fixed λ, it is natural to consider the ground state of (1.2), which is a solution $w \in H^1(\mathbb{R}^N, \mathbb{C})\backslash\{0\}$ of (1.2) satisfying

$$J_{p,q}(w) = \inf\{J_{p,q}(v) : v \in H^1(\mathbb{R}^N, \mathbb{C})\backslash\{0\},\ J'_{p,q}(v) = 0\}.$$

It is an open question whether a normalized ground state of (1.2) is a ground state of (1.2) with fixed $\lambda < 0$.

In the proofs of Theorems 1.2 and 1.4, the Pohožaev set

$$\mathcal{P}_{p,q} = \{u \in S_a : P_{p,q}(u) = 0\},$$

plays an important role, where

$$P_{p,q}(u)=\int_{\mathbb{R}^N}|\nabla u|^2dx-\gamma_p\int_{\mathbb{R}^N}|u|^pdx-\mu\gamma_q\int_{\mathbb{R}^N}|u|^qdx$$

and

$$\gamma_p=\frac{N(p-2)}{2p}=\frac{N}{2}-\frac{N}{p}.$$

It is well known that any critical point of $E_{p,q}|_{S_a}$ belongs to $\mathcal{P}_{p,q}$, as a consequence of the Pohožaev identity (we refer for instance to Lemma 2.7 in [8]). Moreover, $P_{p,q}$ is a natural constraint, see Lemma 2.6. So it is natural to consider the minimizing problem

$$c_{p,q}=\inf_{u\in\mathcal{P}_{p,q}}E_{p,q}(u)$$

and define

$$\mathcal{C}_{p,q}=\{u\in\mathcal{P}_{p,q}:E_{p,q}(u)=c_{p,q}\}.$$

For the Sobolev subcritical problem, we can show that $c_{p,q}$ is attained by using Schwartz symmetrization rearrangements. For the Sobolev critical problem, we can show that $c_{p,q}$ is attained, by introducing the Sobolev subcritical approximation method, which has already been used to deal with problems without mass constraint (see [13,14,17]). To our knowledge, it is the first time this method is used to discuss mass constrained problems. During the proofs, the following various expressions of $E_{p,q}(u)$ constrained on $\mathcal{P}_{p,q}$

$$\begin{aligned}E_{p,q}(u)&=\left(\frac{1}{2}-\frac{1}{p\gamma_p}\right)\int_{\mathbb{R}^N}|\nabla u|^2dx+\left(\frac{\gamma_q}{p\gamma_p}-\frac{1}{q}\right)\mu\int_{\mathbb{R}^N}|u|^qdx\\&=\left(\frac{1}{2}-\frac{1}{q\gamma_q}\right)\int_{\mathbb{R}^N}|\nabla u|^2dx+\left(\frac{\gamma_p}{q\gamma_q}-\frac{1}{p}\right)\int_{\mathbb{R}^N}|u|^pdx\\&=\left(\frac{\gamma_p}{2}-\frac{1}{p}\right)\int_{\mathbb{R}^N}|u|^pdx+\left(\frac{\gamma_q}{2}-\frac{1}{q}\right)\mu\int_{\mathbb{R}^N}|u|^qdx\end{aligned}$$

play an important role.

This paper is organized as follows. In Sect. 2, we cite some preliminaries and give the proof of Theorem 1.2. Section 3 is devoted to the proof of Theorem 1.4.

Notation: For $t\geq 1$, the L^t-norm of $u\in L^t(\mathbb{R}^N,\mathbb{C})$ (or of $L^t(\mathbb{R}^N,\mathbb{R})$) is denoted by $\|u\|_t$. We simply write H for $H^1(\mathbb{R}^N,\mathbb{C})$, and H^1 for the subspace of real valued functions $H^1(\mathbb{R}^N,\mathbb{R})$.

2 Preliminaries and proof of Theorem 1.2

The following Gagliardo-Nirenberg inequality can be found in [24].

Lemma 2.1 *Let $N\geq 1$ and $2<p<2^*$, then the following sharp Gagliardo-Nirenberg inequality*

$$\|u\|_p\leq C_{N,p}\|u\|_2^{1-\gamma_p}\|\nabla u\|_2^{\gamma_p}$$

holds for any $u\in H$, where the sharp constant $C_{N,p}$ is

$$C_{N,p}^p=\frac{2p}{2N+(2-N)p}\left(\frac{2N+(2-N)p}{N(p-2)}\right)^{\frac{N(p-2)}{4}}\frac{1}{\|Q_p\|_2^{p-2}}$$

and Q_p is the unique positive radial solution of equation

$$-\Delta Q + Q = |Q|^{p-2}Q.$$

In the special case $p = \bar{p}$, $C_{N,\bar{p}}^{\bar{p}} = \frac{\bar{p}}{2}\frac{1}{\|Q_{\bar{p}}\|_2^{4/N}}$, or equivalently,

$$\|Q_{\bar{p}}\|_2 = \left(\frac{\bar{p}}{2C_{N,\bar{p}}^{\bar{p}}}\right)^{N/4} =: \bar{a}_N. \tag{2.1}$$

The following lemma is useful in concerning the uniform bound of radial non-increasing functions, see [3] for its proof.

Lemma 2.2 *Let $N \geq 3$ and $1 \leq t < +\infty$. If $u \in L^t(\mathbb{R}^N)$ is a radial non-increasing function (i.e. $0 \leq u(x) \leq u(y)$ if $|x| \geq |y|$), then one has*

$$|u(x)| \leq |x|^{-N/t}\left(\frac{N}{|S^{N-1}|}\right)^{1/t}\|u\|_t, \ x \neq 0,$$

where $|S^{N-1}|$ is the area of the unit sphere in $\mathbb{R}^N$.

For any $u \in S_a$ and $\tau > 0$, we define

$$u^\tau(x) = \tau^{N/2}u(\tau x). \tag{2.2}$$

Then $u^\tau \in S_a$ and for any $\tau > 0$,

$$E_{p,q}(u^\tau) = \frac{1}{2}\tau^2\int_{\mathbb{R}^N}|\nabla u|^2dx - \frac{1}{p}\tau^{\frac{N}{2}p-N}\int_{\mathbb{R}^N}|u|^pdx - \frac{\mu}{q}\tau^{\frac{N}{2}q-N}\int_{\mathbb{R}^N}|u|^qdx \tag{2.3}$$

and

$$P_{p,q}(u^\tau) = \tau^2\int_{\mathbb{R}^N}|\nabla u|^2dx - \gamma_p\tau^{\frac{N}{2}p-N}\int_{\mathbb{R}^N}|u|^pdx - \mu\gamma_q\tau^{\frac{N}{2}q-N}\int_{\mathbb{R}^N}|u|^qdx.$$

The following lemma is about the properties of $E_{p,q}(u^\tau)$ and $P_{p,q}(u^\tau)$.

Lemma 2.3 *Let $N \geq 1$, $a > 0$, $\mu > 0$ and*

$$\bar{p} \leq q < p\begin{cases} < \infty, & N = 1, 2, \\ \leq 2^*, & N \geq 3. \end{cases}$$

If $q = \bar{p}$, we further assume that $\mu a^{\frac{4}{N}} < (\bar{a}_N)^{\frac{4}{N}}$. Then for any $u \in S_a$, there exists a unique $\tau_0 \in (0, \infty)$ such that $P_{p,q}(u^{\tau_0}) = 0$. Moreover, τ_0 is the unique critical point of $E_{p,q}(u^\tau)$ and $E_{p,q}(u^{\tau_0}) = \max_{\tau\in(0,\infty)} E_{p,q}(u^\tau)$. In particular, if $P_{p,q}(u) \leq 0$, then $\tau_0 \in (0, 1]$.

Proof Set $P_{p,q}(u^\tau) = \tau^2 g(\tau)$, where

$$g(\tau) = \int_{\mathbb{R}^N}|\nabla u|^2dx - \gamma_p\tau^{\frac{N}{2}p-N-2}\int_{\mathbb{R}^N}|u|^pdx - \mu\gamma_q\tau^{\frac{N}{2}q-N-2}\int_{\mathbb{R}^N}|u|^qdx.$$

When $\bar{p} < q < p$, we have $\frac{N}{2}p - N - 2 > \frac{N}{2}q - N - 2 > 0$. When $\bar{p} = q < p$ and $\mu a^{\frac{4}{N}} < (\bar{a}_N)^{\frac{4}{N}}$, we have $\frac{N}{2}p - N - 2 > \frac{N}{2}q - N - 2 = 0$ and by the Gagliardo-Nirenberg inequality,

$$\mu\gamma_q\int_{\mathbb{R}^N}|u|^qdx \leq \mu\gamma_q C_{N,q}^q a^{q(1-\gamma_q)}\|\nabla u\|_2^2 < \|\nabla u\|_2^2.$$

Hence, in both cases, $g(\tau) > 0$ for $\tau > 0$ small enough, $g(\tau) < 0$ for τ large enough, and $g'(\tau) < 0$ for $\tau \in (0, \infty)$. So $g(\tau)$ has a unique zero τ_0 as well as $P_{p,q}(u^\tau)$.

By direct calculations, we have $E'_{p,q}(u^\tau) = \tau^{-1} P_{p,q}(u^\tau)$, $E_{p,q}(u^\tau) > 0$ for $\tau > 0$ small enough and $\lim_{\tau\to\infty} E_{p,q}(u^\tau) = -\infty$. Thus, τ_0 is the unique critical point of $E_{p,q}(u^\tau)$ and $E_{p,q}(u^{\tau_0}) = \max_{\tau\in(0,\infty)} E_{p,q}(u^\tau)$. □

The following lemmas are about the properties of $c_{p,q}$ and $\mathcal{C}_{p,q}$.

Lemma 2.4 *Let $N \geq 1$, $a > 0$, $\mu > 0$ and*

$$\bar{p} \leq q < p \begin{cases} < \infty, & N = 1, 2, \\ \leq 2^*, & N \geq 3. \end{cases}$$

If $q = \bar{p}$, we further assume that $\mu a^{\frac{4}{N}} < (\bar{a}_N)^{\frac{4}{N}}$. Then $c_{p,q} > 0$.

Proof By Lemma 2.3, $\mathcal{P}_{p,q} \neq \emptyset$.

Case 1 ($p \neq 2^*$). For any $u \in \mathcal{P}_{p,q}$, by the Gagliardo-Nirenberg inequality (Lemma 2.1), we have

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u|^2 dx &= \gamma_p \int_{\mathbb{R}^N} |u|^p dx + \mu\gamma_q \int_{\mathbb{R}^N} |u|^q dx \\ &\leq \gamma_p C^p_{N,p} \|u\|_2^{p(1-\gamma_p)} \|\nabla u\|_2^{p\gamma_p} + \mu\gamma_q C^q_{N,q} \|u\|_2^{q(1-\gamma_q)} \|\nabla u\|_2^{q\gamma_q} \\ &= \mu\gamma_q C^q_{N,q} a^{q(1-\gamma_q)} \|\nabla u\|_2^{q\gamma_q} + \gamma_p C^p_{N,p} a^{p(1-\gamma_p)} \|\nabla u\|_2^{p\gamma_p}. \end{aligned} \tag{2.4}$$

If $\bar{p} < q < p$, then $p\gamma_p > q\gamma_q > 2$. (2.4) implies that there exists a constant $C > 0$ such that $\|\nabla u\|_2^2 \geq C$. Consequently,

$$\gamma_p \int_{\mathbb{R}^N} |u|^p dx + \mu\gamma_q \int_{\mathbb{R}^N} |u|^q dx \geq C.$$

If $\bar{p} = q < p$ and $\mu a^{\frac{4}{N}} < (\bar{a}_N)^{\frac{4}{N}}$, then $p\gamma_p > q\gamma_q = 2$, $\mu\gamma_q C^q_{N,q} a^{q(1-\gamma_q)} < 1$. (2.4) implies that there exists a constant $C > 0$ such that $\|\nabla u\|_2^2 \geq C$. Thus, it follows from (2.4) that

$$\gamma_p \int_{\mathbb{R}^N} |u|^p dx \geq \left(1 - \mu\gamma_q C^q_{N,q} a^{q(1-\gamma_q)}\right) \|\nabla u\|_2^2 \geq C \left(1 - \mu\gamma_q C^q_{N,q} a^{q(1-\gamma_q)}\right).$$

Any way, there always exists $C_1 > 0$ such that for any $u \in \mathcal{P}_{p,q}$,

$$E_{p,q}(u) = \left(\frac{\gamma_p}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^N} |u|^p dx + \left(\frac{\gamma_q}{2} - \frac{1}{q}\right) \mu \int_{\mathbb{R}^N} |u|^q dx \geq C_1, \tag{2.5}$$

which implies $c_{p,q} > 0$.

Case 2 ($p = 2^*$). Similarly to Case 1, just in (2.4), we estimate the term $\int_{\mathbb{R}^N} |u|^{2^*} dx$ by using

$$\int_{\mathbb{R}^N} |u|^{2^*} dx \leq \left(\frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{S}\right)^{\frac{N}{N-2}},$$

see (3.2). □

Lemma 2.5 *Let $N \geq 1$, $a > 0$, $\mu > 0$ and $\bar{p} \leq q < p < 2^*$. If $q = \bar{p}$, we further assume that $\mu a^{\frac{4}{N}} < (\bar{a}_N)^{\frac{4}{N}}$. Then $c_{p,q}$ is attained by a real-valued positive, radially symmetric and radially non-increasing function.*

Proof Let $\{u_n\}_{n=1}^{\infty} \subset \mathcal{P}_{p,q}$ be a minimizing sequence of $c_{p,q}$ and $|u_n|^*$ be the Schwartz symmetrization rearrangement of $|u_n|$. From Chapter 3 in [16], we have

$$\int_{\mathbb{R}^N} |\nabla(|u_n|^*)|^2 dx \le \int_{\mathbb{R}^N} |\nabla |u_n||^2 dx \le \int_{\mathbb{R}^N} |\nabla u_n|^2 dx$$

and

$$\int_{\mathbb{R}^N} ||u_n|^*|^t dx = \int_{\mathbb{R}^N} |u_n|^t dx, \ t \in [1, \infty) .$$

Hence $P_{p,q}(|u_n|^*) \le 0$.

Let $(|u_n|^*)^{\tau}(x)$ be defined as (2.2). By Lemma 2.3, there exists a unique $\tau_n \in (0, 1]$ such that $P_{p,q}((|u_n|^*)^{\tau_n}) = 0$. Hence $\{(|u_n|^*)^{\tau_n}\}_{n=1}^{\infty} \subset \mathcal{P}_{p,q}$. By direct calculations, we have

$$\begin{aligned} &E_{p,q}((|u_n|^*)^{\tau_n}) \\ &= \left(\frac{\gamma_p}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^N} |(|u_n|^*)^{\tau_n}|^p dx + \left(\frac{\gamma_q}{2} - \frac{1}{q}\right) \mu \int_{\mathbb{R}^N} |(|u_n|^*)^{\tau_n}|^q dx \\ &= \tau_n^{\frac{N}{2}p-N} \left(\frac{\gamma_p}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^N} ||u_n|^*|^p dx + \tau_n^{\frac{N}{2}q-N} \left(\frac{\gamma_q}{2} - \frac{1}{q}\right) \mu \int_{\mathbb{R}^N} ||u_n|^*|^q dx \\ &\le E_{p,q}(u_n). \end{aligned} \tag{2.6}$$

That is, $\{(|u_n|^*)^{\tau_n}\}_{n=1}^{\infty}$ is a minimizing sequence of $c_{p,q}$. Reversing the proof of Lemma 2.4, we can show that $\{(|u_n|^*)^{\tau_n}\}_{n=1}^{\infty}$ is bounded in $H^1(\mathbb{R}^N)$. Hence, there exists $u_0 \in H^1(\mathbb{R}^N)$ such that $(|u_n|^*)^{\tau_n} \rightharpoonup u_0$ weakly in $H^1(\mathbb{R}^N)$, $(|u_n|^*)^{\tau_n} \to u_0$ strongly in $L^t(\mathbb{R}^N)$ with $t \in (2, 2^*)$ and $(|u_n|^*)^{\tau_n} \to u_0$ a.e. in $\mathbb{R}^N$. Consequently,

$$\begin{aligned} &\int_{\mathbb{R}^N} |u_0|^2 dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^N} |(|u_n|^*)^{\tau_n}|^2 dx = a^2, \\ &\int_{\mathbb{R}^N} |\nabla u_0|^2 dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla (|u_n|^*)^{\tau_n}|^2 dx, \\ &E_{p,q}((|u_n|^*)^{\tau_n}) \to \left(\frac{\gamma_p}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^N} |u_0|^p dx + \left(\frac{\gamma_q}{2} - \frac{1}{q}\right) \mu \int_{\mathbb{R}^N} |u_0|^q dx = c_{p,q}, \end{aligned}$$

which imply that $u_0 \not\equiv 0$ and $P_{p,q}(u_0) \le 0$.

Set $\int_{\mathbb{R}^N} |u_0|^2 dx := c_0^2 \le a^2$ and define $\tilde{u}(x) = (c_0 a^{-1})^{\frac{2}{p-2}} u_0((c_0 a^{-1})^{\frac{2p}{N(p-2)}} x)$. Then

$$\begin{aligned} &\int_{\mathbb{R}^N} |\tilde{u}|^2 dx = a^2, \ \int_{\mathbb{R}^N} |\tilde{u}|^p dx = \int_{\mathbb{R}^N} |u_0|^p dx, \\ &\int_{\mathbb{R}^N} |\tilde{u}|^q dx = (c_0 a^{-1})^{\frac{2(q-p)}{p-2}} \int_{\mathbb{R}^N} |u_0|^q dx \ge \int_{\mathbb{R}^N} |u_0|^q dx, \\ &\int_{\mathbb{R}^N} |\nabla \tilde{u}|^2 dx = (c_0 a^{-1})^{\frac{2[2N+p(2-N)]}{N(p-2)}} \int_{\mathbb{R}^N} |\nabla u_0|^2 dx \le \int_{\mathbb{R}^N} |\nabla u_0|^2 dx. \end{aligned}$$

Hence $P_{p,q}(\tilde{u}) \le 0$. So there exists $\tau_0 \in (0, 1]$ such that $\tilde{u}^{\tau_0} \in \mathcal{P}_{p,q}$ and

$$\begin{aligned} E_{p,q}(\tilde{u}^{\tau_0}) &= \left(\frac{1}{2} - \frac{1}{q\gamma_q}\right) \int_{\mathbb{R}^N} |\nabla(\tilde{u}^{\tau_0})|^2 dx + \left(\frac{\gamma_p}{q\gamma_q} - \frac{1}{p}\right) \int_{\mathbb{R}^N} |(\tilde{u}^{\tau_0})|^p dx \\ &= \left(\frac{1}{2} - \frac{1}{q\gamma_q}\right) \tau_0^2 \int_{\mathbb{R}^N} |\nabla \tilde{u}|^2 dx + \left(\frac{\gamma_p}{q\gamma_q} - \frac{1}{p}\right) \tau_0^{\frac{N}{2}p-N} \int_{\mathbb{R}^N} |\tilde{u}|^p dx \\ &= \left(\frac{1}{2} - \frac{1}{q\gamma_q}\right) \tau_0^2 (c_0 a^{-1})^{\frac{2[2N+p(2-N)]}{N(p-2)}} \int_{\mathbb{R}^N} |\nabla u_0|^2 dx \\ &\quad + \left(\frac{\gamma_p}{q\gamma_q} - \frac{1}{p}\right) \tau_0^{\frac{N}{2}p-N} \int_{\mathbb{R}^N} |u_0|^p dx \\ &\le \left(\frac{1}{2} - \frac{1}{q\gamma_q}\right) \int_{\mathbb{R}^N} |\nabla u_0|^2 dx + \left(\frac{\gamma_p}{q\gamma_q} - \frac{1}{p}\right) \int_{\mathbb{R}^N} |u_0|^p dx \\ &\le \liminf_{n\to\infty} \left\{ \left(\frac{1}{2} - \frac{1}{q\gamma_q}\right) \int_{\mathbb{R}^N} |\nabla (|u_n|^*)^{\tau_n}|^2 dx \right. \\ &\qquad \left. + \left(\frac{\gamma_p}{q\gamma_q} - \frac{1}{p}\right) \int_{\mathbb{R}^N} |(|u_n|^*)^{\tau_n}|^p dx \right\} \\ &= c_{p,q}. \end{aligned} \tag{2.7}$$

By the definition of $c_{p,q}$, we obtain that $E_{p,q}(\tilde{u}^{\tau_0}) = c_{p,q}$, $\tau_0 = 1$ and $c_0 = a$. Hence, $u_0 \in \mathcal{P}_{p,q}$ is a real-valued nonnegative, radially symmetric and radially non-increasing minimizer of $c_{p,q}$. By the strong maximum principle, $u_0 > 0$ in $\mathbb{R}^N$. □

Lemma 2.6 *Let $N \ge 1$, $a > 0$, $\mu > 0$ and*

$$\bar{p} \le q < p \begin{cases} < \infty, & N = 1, 2, \\ \le 2^*, & N \ge 3. \end{cases}$$

If $\mathcal{C}_{p,q}$ is not empty, then for any $u \in \mathcal{C}_{p,q}$, there exists $\lambda < 0$ such that u satisfies equation (1.2). Moreover, $\mathcal{C}_{p,q} = \mathcal{Z}_{p,q}$ and $|u| \in \mathcal{C}_{p,q}$.

Proof For any $u \in \mathcal{C}_{p,q}$, there exist λ and η such that

$$-\Delta u - |u|^{p-2}u - \mu|u|^{q-2}u = \lambda u + \eta[-2\Delta u - p\gamma_p|u|^{p-2}u - \mu q\gamma_q|u|^{q-2}u], \tag{2.8}$$

or equivalently,

$$-(1-2\eta)\Delta u = \lambda u + (1-\eta p\gamma_p)|u|^{p-2}u + \mu(1-\eta q\gamma_q)|u|^{q-2}u.$$

Next we show $\eta = 0$. Similarly to the definition of $P_{p,q}(u)$, we obtain

$$(1-2\eta)\int_{\mathbb{R}^N} |\nabla u|^2 dx - (1-\eta p\gamma_p)\gamma_p \int_{\mathbb{R}^N} |u|^p dx - (1-\eta q\gamma_q)\mu\gamma_q \int_{\mathbb{R}^N} |u|^q dx = 0,$$

which combined with $P_{p,q}(u) = 0$ gives that

$$\eta\left(2\int_{\mathbb{R}^N} |\nabla u|^2 dx - p\gamma_p^2 \int_{\mathbb{R}^N} |u|^p dx - \mu q\gamma_q^2 \int_{\mathbb{R}^N} |u|^q dx\right) = 0.$$

If $\eta \ne 0$, then

$$2\int_{\mathbb{R}^N} |\nabla u|^2 dx - p\gamma_p^2 \int_{\mathbb{R}^N} |u|^p dx - \mu q\gamma_q^2 \int_{\mathbb{R}^N} |u|^q dx = 0,$$

which combined with $P_{p,q}(u)=0$ gives that

$$\int_{\mathbb{R}^N}|u|^p dx=\frac{2-q\gamma_q}{\gamma_p(p\gamma_p-q\gamma_q)}\int_{\mathbb{R}^N}|\nabla u|^2dx\le 0.$$

That is a contradiction. So $\eta=0$.

From (2.8), $P_{p,q}(u)=0$, $0<\gamma_q<\gamma_p\le 1$ and $\mu>0$, we obtain

$$\begin{aligned}\lambda a^2&=\int_{\mathbb{R}^N}|\nabla u|^2dx-\int_{\mathbb{R}^N}|u|^pdx-\mu\int_{\mathbb{R}^N}|u|^qdx\\&=(\gamma_p-1)\int_{\mathbb{R}^N}|u|^pdx+\mu(\gamma_q-1)\int_{\mathbb{R}^N}|u|^qdx<0.\end{aligned}$$

Hence $\lambda<0$.

Any normalized solution v of (1.2) satisfies $P_{p,q}(v)=0$. Hence $E_{p,q}(v)\ge c_{p,q}$ and then $c_{p,q}=z_{p,q}$, $\mathcal{C}_{p,q}=\mathcal{Z}_{p,q}$. Since $\int_{\mathbb{R}^N}|\nabla|u||^2dx\le\int_{\mathbb{R}^N}|\nabla u|^2dx$, we have $P_{p,q}(|u|)\le 0$. So there exists $\tau_0\in(0,1]$ such that $|u|^{\tau_0}\in\mathcal{P}_{p,q}$. Similarly to the proof of (2.6), we can show that $\tau_0=1$ and $|u|\in\mathcal{C}_{p,q}$. □

Proof of Theorem 1.2: It follows from Lemmas 2.4–2.6.

3 Proof of Theorem 1.4

In this section, we first study the properties of $c_{p,q}$ and then give the proof of Theorem 1.4.

Lemma 3.1 *Let $N\ge 3$, $a>0$, $\mu>0$ and $\bar p\le q<p<2^*$. If $q=\bar p$, we further assume that $\mu a^{\frac{4}{N}}<(\bar a_N)^{\frac{4}{N}}$. Then* $\limsup_{p\to 2^*}c_{p,q}\le c_{2^*,q}$.

Proof By the definition of $c_{2^*,q}$, for any fixed $\epsilon\in(0,1)$, there exists $u\in\mathcal{P}_{2^*,q}$ such that $E_{2^*,q}(u)<c_{2^*,q}+\epsilon$. By (2.3), there exists $\tau_0>0$ large enough such that $E_{2^*,q}(u^{\tau_0})\le -2$. By the Young inequality

$$|u|^p\le\frac{2^*-p}{2^*-q}|u|^q+\frac{p-q}{2^*-q}|u|^{2^*}\tag{3.1}$$

and the Lebesgue dominated convergence theorem, we know

$$\frac{1}{p}\tau^{\frac{N}{2}p-N}\int_{\mathbb{R}^N}|u|^pdx$$

is continuous on $p\in[\bar p,2^*]$ uniformly with $\tau\in[0,\tau_0]$. Hence, there exists $\delta>0$ such that $|E_{p,q}(u^\tau)-E_{2^*,q}(u^\tau)|<\epsilon$ for $2^*-\delta\le p\le 2^*$ and $0\le\tau\le\tau_0$, which implies that $E_{p,q}(u^{\tau_0})\le -1$ for all $2^*-\delta\le p\le 2^*$. In view of $E_{p,q}(u^\tau)>0$ for τ small enough for every $p\in[q,2^*]$, it follows from Lemma 2.3 that the unique critical (maximum) point $\tau_{p,q}$ of $E_{p,q}(u^\tau)$ belongs to $(0,\tau_0)$ and $P_{p,q}(u^{\tau_{p,q}})=0$. Since $u\in\mathcal{P}_{2^*,q}$, we deduce that $E_{2^*,q}(u)=\max_{\tau>0}E_{2^*,q}(u^\tau)$. Consequently,

$$c_{p,q}\le E_{p,q}(u^{\tau_{p,q}})\le E_{2^*,q}(u^{\tau_{p,q}})+\epsilon\le E_{2^*,q}(u)+\epsilon<c_{2^*,q}+2\epsilon$$

for any $2^*-\delta\le p\le 2^*$. Thus, $\limsup_{p\to 2^*}c_{p,q}\le c_{2^*,q}$. □

Lemma 3.2 *Let $N\ge 3$, $a>0$, $\mu>0$ and $\bar p\le q<p<2^*$. If $q=\bar p$, we further assume that $\mu a^{\frac{4}{N}}<(\bar a_N)^{\frac{4}{N}}$. Then* $\liminf_{p\to 2^*}c_{p,q}>0$.

Proof By Lemma 2.5, there exists a sequence $\{u_{p,q}\}_p \subset \mathcal{P}_{p,q}$ such that $E_{p,q}(u_{p,q}) = c_{p,q}$. By the Young inequality (3.1), we have

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u_{p,q}|^2 dx &= \gamma_p \int_{\mathbb{R}^N} |u_{p,q}|^p dx + \mu\gamma_q \int_{\mathbb{R}^N} |u_{p,q}|^q dx \\ &\le \left(\gamma_p \frac{2^*-p}{2^*-q} + \mu\gamma_q\right) \int_{\mathbb{R}^N} |u_{p,q}|^q dx + \gamma_p \frac{p-q}{2^*-q} \int_{\mathbb{R}^N} |u_{p,q}|^{2^*} dx.\end{aligned}$$

Letting $p \to 2^*$, similarly to the proof of Lemma 2.4, we can show that there exists $C > 0$ independent of p such that $\|\nabla u_{p,q}\|_2^2 > C$, subsequently, $\liminf_{p\to 2^*} c_{p,q} > 0$. □

Lemma 3.3 *Let $N \ge 3$, $a > 0$, $\mu > 0$ and $\bar{p} \le q < 2^*$. If $q = \bar{p}$, we further assume that $\mu a^{\frac{4}{N}} < (\bar{a}_N)^{\frac{4}{N}}$. Then $c_{2^*,q} < \frac{1}{N} S^{\frac{N}{2}}$, where S is defined by*

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N} |u|^{2^*} dx\right)^{\frac{N-2}{N}}}. \tag{3.2}$$

Proof For any $\epsilon > 0$, we define

$$u_\epsilon(x) = \varphi(x) U_\epsilon(x),$$

where

$$U_\epsilon(x) = \frac{\left(N(N-2)\epsilon^2\right)^{\frac{N-2}{4}}}{\left(\epsilon^2 + |x|^2\right)^{\frac{N-2}{2}}}$$

is the ground state of equation

$$-\Delta u = |u|^{2^*-2} u, \ x \in \mathbb{R}^N,$$

and $\varphi(x) \in C_c^\infty(\mathbb{R}^N)$ is a cut off function satisfying:

(a) $0 \le \varphi(x) \le 1$ for any $x \in \mathbb{R}^N$;
(b) $\varphi(x) \equiv 1$ in B_1, where B_s denotes the ball in $\mathbb{R}^N$ of center at origin and radius s;
(c) $\varphi(x) \equiv 0$ in $\mathbb{R}^N \setminus \overline{B_2}$.

By [5] (see also [25]), we have the following estimates.

$$\int_{\mathbb{R}^N} |\nabla u_\epsilon|^2 dx = S^{\frac{N}{2}} + O(\epsilon^{N-2}), \ N \ge 3,$$
$$\int_{\mathbb{R}^N} |u_\epsilon|^{2^*} dx = S^{\frac{N}{2}} + O(\epsilon^N), \ N \ge 3,$$

and

$$\int_{\mathbb{R}^N} |u_\epsilon|^2 dx = \begin{cases} K_2\epsilon^2 + O(\epsilon^{N-2}), & N \ge 5, \\ K_2\epsilon^2 |\ln \epsilon| + O(\epsilon^2), & N = 4, \\ K_2\epsilon + O(\epsilon^2), & N = 3, \end{cases}$$

where $K_2 > 0$. By direct calculations, for $t \in (2, 2^*)$, there exists $K_1 > 0$ such that

$$\begin{aligned}\int_{\mathbb{R}^N} |u_\epsilon|^t dx &\ge (N(N-2))^{\frac{N-2}{4}t} \epsilon^{N-\frac{N-2}{2}t} \int_{B_{\frac{1}{\epsilon}}(0)} \frac{1}{(1+|x|^2)^{\frac{N-2}{2}t}} dx \\ &\ge \begin{cases} K_1 \epsilon^{N-\frac{N-2}{2}t}, & (N-2)t > N, \\ K_1 \epsilon^{N-\frac{N-2}{2}t} |\ln \epsilon|, & (N-2)t = N, \\ K_1 \epsilon^{\frac{N-2}{2}t}, & (N-2)t < N. \end{cases}\end{aligned}$$

Define $v_\epsilon(x) = (a^{-1}\|u_\epsilon\|_2)^{\frac{N-2}{2}} u_\epsilon(a^{-1}\|u_\epsilon\|_2 x)$. Then

$$\int_{\mathbb{R}^N} |v_\epsilon|^2 dx = a^2, \ \int_{\mathbb{R}^N} |\nabla v_\epsilon|^2 dx = \int_{\mathbb{R}^N} |\nabla u_\epsilon|^2 dx, \ \int_{\mathbb{R}^N} |v_\epsilon|^{2^*} dx = \int_{\mathbb{R}^N} |u_\epsilon|^{2^*} dx,$$

and for $q \in [\bar{p}, 2^*)$,

$$\begin{aligned}\int_{\mathbb{R}^N} |v_\epsilon|^q dx &= (a^{-1}\|u_\epsilon\|_2)^{\frac{N-2}{2}q-N} \int_{\mathbb{R}^N} |u_\epsilon|^q dx \\ &\geq a^{N-\frac{N-2}{2}q} \|u_\epsilon\|_2^{\frac{N-2}{2}q-N} K_1 \epsilon^{N-\frac{N-2}{2}q} \\ &\geq \frac{1}{2} a^{N-\frac{N-2}{2}q} K_1 K_2^{\frac{N-2}{4}q-\frac{N}{2}} \times \begin{cases} 1, & N \geq 5, \\ |\ln \epsilon|^{\frac{N-2}{4}q-\frac{N}{2}}, & N = 4, \\ \epsilon^{\frac{N}{2}-\frac{N-2}{4}q}, & N = 3. \end{cases}\end{aligned}$$

Next we use v_ϵ to estimate $c_{2^*,q}$. By Lemma 2.3, there exists a unique τ_ϵ such that $P_{2^*,q}((v_\epsilon)^{\tau_\epsilon}) = 0$ and $E_{2^*,q}((v_\epsilon)^{\tau_\epsilon}) = \sup_{\tau \geq 0} E_{2^*,q}((v_\epsilon)^\tau)$. Thus, $c_{2^*,q} \leq \sup_{\tau \geq 0} E_{2^*,q}((v_\epsilon)^\tau)$. By direct calculations, one has

$$\begin{aligned}&E_{2^*,q}((v_\epsilon)^\tau) \\ &= \frac{1}{2}\tau^2 \int_{\mathbb{R}^N} |\nabla v_\epsilon|^2 dx - \frac{1}{2^*}\tau^{\frac{N}{2}2^*-N} \int_{\mathbb{R}^N} |v_\epsilon|^{2^*} dx - \frac{\mu}{q}\tau^{\frac{N}{2}q-N} \int_{\mathbb{R}^N} |v_\epsilon|^q dx \\ &\leq \frac{1}{2}\tau^2 \left(S^{\frac{N}{2}} + O(\epsilon^{N-2})\right) - \frac{1}{2^*}\tau^{2^*}\left(S^{\frac{N}{2}} + O(\epsilon^N)\right) \\ &\quad - \frac{\mu}{q}\tau^{\frac{N}{2}q-N} \frac{1}{2} a^{N-\frac{N-2}{2}q} K_1 K_2^{\frac{N-2}{4}q-\frac{N}{2}} \times \begin{cases} 1, & N \geq 5, \\ |\ln \epsilon|^{\frac{N-2}{4}q-\frac{N}{2}}, & N = 4, \\ \epsilon^{\frac{N}{2}-\frac{N-2}{4}q}, & N = 3. \end{cases}\end{aligned} \tag{3.3}$$

We claim that there exist $\tau_0, \tau_1 > 0$ independent of ϵ such that $\tau_\epsilon \in [\tau_0, \tau_1]$ for $\epsilon > 0$ small. Suppose by contradiction that $\tau_\epsilon \to 0$ or $\tau_\epsilon \to \infty$ as $\epsilon \to 0$. (3.3) implies that $\sup_{\tau \geq 0} E_{2^*,q}((v_\epsilon)^\tau) \leq 0$ as $\epsilon \to 0$ and then $c_{2^*,q} \leq 0$, which contradicts $c_{2^*,q} > 0$. Thus, the claim holds.

In (3.3), $O(\epsilon^{N-2})$ can be controlled by the last term for $\epsilon > 0$ small enough. Hence,

$$\sup_{\tau \geq 0} E_{2^*,q}((v_\epsilon)^\tau) < \sup_{\tau \geq 0} \left(\frac{1}{2}\tau^2 S^{\frac{N}{2}} - \frac{1}{2^*}\tau^{2^*} S^{\frac{N}{2}}\right) \leq \frac{1}{N} S^{\frac{N}{2}}.$$

The proof is complete. □

Lemma 3.4 *Let $N \geq 3$, $a > 0$, $\mu > 0$ and $\bar{p} \leq q < 2^*$. If $q = \bar{p}$, we further assume that $\mu a^{\frac{4}{N}} < (\bar{a}_N)^{\frac{4}{N}}$. Then $c_{2^*,q}$ is attained by a real-valued positive, radially symmetric and radially non-increasing function.*

Proof Let $p_n \to 2^{*-}$ as $n \to \infty$, by Lemmas 2.5 and 3.1, there exists a sequence of positive and radially non-increasing functions $\{u_n := u_{p_n,q}\} \subset \mathcal{P}_{p_n,q}$ such that $E_{p_n,q}(u_n) = c_{p_n,q} \leq c_{2^*,q} + 1$. If $q > \bar{p}$, we have

$$c_{2^*,q} + 1 \geq E_{p_n,q}(u_n) = \left(\frac{1}{2} - \frac{1}{q\gamma_q}\right) \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \left(\frac{\gamma_{p_n}}{q\gamma_q} - \frac{1}{p_n}\right) \int_{\mathbb{R}^N} |u_n|^{p_n} dx.$$

So $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. If $q = \bar{p}$, we have

$$c_{2^*,q} + 1 \geq E_{p_n,q}(u_n) = \left(\frac{\gamma_{p_n}}{2} - \frac{1}{p_n}\right) \int_{\mathbb{R}^N} |u_n|^{p_n} dx,$$

which implies that $\{\int_{\mathbb{R}^N} |u_n|^{p_n} dx\}$ is bounded. By the Young inequality

$$|u_n|^q \leq \frac{p_n - q}{p_n - 2}|u_n|^2 + \frac{q-2}{p_n - 2}|u_n|^{p_n},$$

we know that $\{\int_{\mathbb{R}^N} |u_n|^q dx\}$ is bounded. So it follows from the expression

$$E_{p_n,q}(u_n) = \left(\frac{1}{2} - \frac{1}{p_n\gamma_{p_n}}\right)\int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \left(\frac{\gamma_q}{p_n\gamma_{p_n}} - \frac{1}{q}\right)\mu\int_{\mathbb{R}^N} |u_n|^q dx$$

that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Thus, there exists a nonnegative and radially non-increasing function $u \in H^1(\mathbb{R}^N)$ such that up to a subsequence, $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$, $u_n \to u$ strongly in $L^t(\mathbb{R}^N)$ for $t \in (2, 2^*)$ and $u_n \to u$ a.e. in $\mathbb{R}^N$.

By Lemma 2.6, there exists $\lambda_n < 0$ such that u_n satisfies

$$-\Delta u_n = \lambda_n u_n + |u_n|^{p_n-2}u_n + \mu|u_n|^{q-2}u_n, \ x \in \mathbb{R}^N. \tag{3.4}$$

It follows from the expression

$$\lambda_n a^2 = (\gamma_{p_n} - 1)\int_{\mathbb{R}^N} |u_n|^{p_n} dx + \mu(\gamma_q - 1)\int_{\mathbb{R}^N} |u_n|^q dx$$

that $\{\lambda_n\}$ is bounded. So there exists $\lambda \leq 0$ such that up to a subsequence, $\lim_{n\to\infty} \lambda_n = \lambda$.

It follows from $N \geq 3$ that $\frac{N}{\frac{N-2}{2}(2-1)}$ and $\frac{N}{\frac{N-2}{2}(2^*-1)} \in (1, \infty)$. Since $p_n \to 2^*$ and $\psi \in L^r(\mathbb{R}^N)$ for $r \in (1, \infty)$, by the Young inequality, the Hölder inequality and Lemma 2.2 with $t = 2^*$, there exists a constant $C > 0$ independent of n such that

$$\begin{aligned} \left||u_n|^{p_n-2}u_n\psi\right| &\leq C\left(|u_n|^{2-1}|\psi| + |u_n|^{2^*-1}|\psi|\right) \\ &\leq C\left(|x|^{\frac{2-N}{2}(2-1)}|\psi| + |x|^{\frac{2-N}{2}(2^*-1)}|\psi|\right) \in L^1(\mathbb{R}^N). \end{aligned} \tag{3.5}$$

Passing to the limit in (3.4) and by using the Lebesgue dominated convergence theorem, we have for any $\psi \in C_c^\infty(\mathbb{R}^N)$,

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} (\nabla u_n \nabla\psi - \lambda_n u_n\psi)dx - \int_{\mathbb{R}^N} |u_n|^{p_n-2}u_n\psi dx - \mu\int_{\mathbb{R}^N} |u_n|^{q-2}u_n\psi dx \\ &\to \int_{\mathbb{R}^N} (\nabla u\nabla\psi - \lambda u\psi)dx - \int_{\mathbb{R}^N} |u|^{2^*-2}u\psi dx - \mu\int_{\mathbb{R}^N} |u|^{q-2}u\psi dx \end{aligned}$$

as $n \to \infty$. That is, u is a solution of

$$-\Delta u = \lambda u + |u|^{2^*-2}u + \mu|u|^{q-2}u, \ x \in \mathbb{R}^N.$$

Thus $P_{2^*,q}(u) = 0$.

We claim that $u \not\equiv 0$. Suppose by contradiction that $u \equiv 0$. By using $P_{p_n,q}(u_n) = 0$, $\int_{\mathbb{R}^N} |u_n|^q = o_n(1)$ and the Young inequality

$$|u_n|^{p_n} \leq \frac{2^* - p_n}{2^* - q}|u_n|^q + \frac{p_n - q}{2^* - q}|u_n|^{2^*},$$

we get that

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u_n|^2 dx &= \gamma_{p_n} \int_{\mathbb{R}^N} |u_n|^{p_n} dx + o_n(1)\\ &\le \gamma_{p_n} \frac{p_n - q}{2^* - q} \int_{\mathbb{R}^N} |u_n|^{2^*} dx + o_n(1)\\ &\le \gamma_{p_n} \frac{p_n - q}{2^* - q} \left(\frac{\int_{\mathbb{R}^N} |\nabla u_n|^2 dx}{S} \right)^{\frac{N}{N-2}} + o_n(1).\end{aligned}$$

Since $\liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 > 0$ (see the proof of Lemma 3.2), we obtain

$$\limsup_{n\to\infty} \|\nabla u_n\|_2^2 \ge S^{\frac{N}{2}}.$$

Consequently,

$$\begin{aligned}c_{2^*,q} &\ge \limsup_{n\to\infty} c_{p_n,q}\\ &= \limsup_{n\to\infty} \left\{ \left(\frac{1}{2} - \frac{1}{p_n \gamma_{p_n}} \right) \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \left(\frac{\gamma_q}{p_n \gamma_{p_n}} - \frac{1}{q} \right) \mu \int_{\mathbb{R}^N} |u_n|^q dx \right\}\\ &= \limsup_{n\to\infty} \left\{ \left(\frac{1}{2} - \frac{1}{p_n \gamma_{p_n}} \right) \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \right\}\\ &\ge \frac{1}{N} S^{\frac{N}{2}},\end{aligned}$$

which contradicts Lemma 3.3. Thus $u \not\equiv 0$.

Set $\int_{\mathbb{R}^N} |u|^2 dx = c^2 \le a^2$. Similarly to the proof of (2.7), we define $\tilde{u} \in S_a$. Then there exists $\tau_0 \in (0, 1]$ such that $P_{2^*,q}(\tilde{u}^{\tau_0}) = 0$ and by Fatou's lemma,

$$\begin{aligned}c_{2^*,q} &\le E_{2^*,q}(\tilde{u}^{\tau_0})\\ &= \left(\frac{1}{2} - \frac{1}{q\gamma_q} \right) \int_{\mathbb{R}^N} |\nabla(\tilde{u}^{\tau_0})|^2 dx + \left(\frac{\gamma_{2^*}}{q\gamma_q} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} |(\tilde{u}^{\tau_0})|^{2^*} dx\\ &\le \left(\frac{1}{2} - \frac{1}{q\gamma_q} \right) \int_{\mathbb{R}^N} |\nabla u|^2 dx + \left(\frac{\gamma_{2^*}}{q\gamma_q} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} |u|^{2^*} dx\\ &\le \liminf_{n\to\infty} \left\{ \left(\frac{1}{2} - \frac{1}{q\gamma_q} \right) \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \left(\frac{\gamma_{p_n}}{q\gamma_q} - \frac{1}{p_n} \right) \int_{\mathbb{R}^N} |u_n|^{p_n} dx \right\}\\ &= \liminf_{n\to\infty} c_{p_n,q} \le \limsup_{n\to\infty} c_{p_n,q} \le c_{2^*,q}.\end{aligned}$$

Hence, $E_{2^*,q}(\tilde{u}^{\tau_0}) = c_{2^*,q}$. That is $\tilde{u}^{\tau_0}$ is a real-valued nonnegative, radially symmetric and radially non-increasing minimizer of $c_{2^*,q}$. By the strong maximum principle, $\tilde{u}^{\tau_0} > 0$ in $\mathbb{R}^N$. □

Proof of Theorem 1.4: It follows from Lemmas 2.4, 2.6, 3.3 and 3.4.

Acknowledgements The authors would like to express sincere thanks to the anonymous referee for his or her carefully reading the manuscript and valuable comments and suggestions. This work is supported by the National Natural Science Foundation of China (No. 12001403).

References

1. Akahori, T., Ibrahim, S., Kikuchi, H., Nawa, H.: Existence of a ground state and blow-up problem for a nonlinear Schrödinger equation with critical growth. Differ. Integr. Equ. **25**(3–4), 383–402 (2012)
2. Alves, C.O., Souto, M.A.S., Montenegro, M.: Existence of a ground state solution for a nonlinear scalar field equation with critical growth. Calc. Var. Partial Diff. Equ. **43**(3–4), 537–554 (2012)
3. Berestycki, H., Lions, P.L.: Nonlinear scalar field equations, I existence of a ground state. Arch. Ration. Mech. Anal. **82**, 313–345 (1983)
4. Boni, F., Dovetta, S.: Prescribed mass ground states for a doubly nonlinear Schrödinger equation in dimension one. J. Math. Anal. Appl. **496**, (2021)
5. Brezis, H., Nirenberg, L.: Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. Commun. Pure Appl. Math. **36**, 437–477 (1983)
6. Cheng, X., Miao, C., Zhao, L.: Global well-posedness and scattering for nonlinear Schrödinger equations with combined nonlinearities inthe radial case. J. Diff. Equ. **261**, 2881–2934 (2016)
7. Feng, B.: On the blow-up solutions for the nonlinear Schrödinger equation with combined power-type nonlinearities. J. Evol. Equ. **18**(1), 203–220 (2018)
8. Jeanjean, L.: Existence of solutions with prescribed norm for semilinear elliptic equations. Nonlinear Anal. **28**(10), 1633–1659 (1997)
9. Jeanjean, L., Jendrej, J., Le, T.T., Visciglia, N.: Orbital stability of ground states for a Sobolev critical Schrödinger equation, arXiv: 2008.12084, 29 Aug (2020)
10. Jeanjean, L., Le, T.T.: Multiple normalized solutions for a Sobolev critical Schrödinger equation, arXiv: 2011.029452, 5 Nov (2020)
11. Killip, R., Oh, T., Pocovnicu, O., Visan, M.: Solitons and scattering for the cubic-quintic nonlinear Schrödinger equation on $\mathbb{R}^3$. Arch. Rational Mech. Anal. **225**, 469–548 (2017)
12. Le Coz, S., Martel, Y., Raphaël, P.: Minimal mass blow up solutions for a double power nonlinear Schrödinger equation. Rev. Mat. Iberoam. **32**(3), 795–833 (2016)
13. Li, X., Ma, S.: Ground states for Choquard equations with doubly critical exponents. Rocky Mt. J. Math. **49**(1), 153–170 (2019)
14. Li, X., Ma, S.: Choquard equations with critical nonlinearities. Commun. Contemp. Math. **22**(04), 1950023 (2020)
15. Li, X., Zhao, J.: Orbital stability of standing waves for Schrödinger type equations with slowly decaying linear potential. Comput. Math. Appl. **79**, 303–316 (2020)
16. Lieb, E.H., Loss, M.: Analysis, volume 14 of graduate studies in mathematics, American Mathematical Society, Providence, RI, (4) (2001)
17. Liu, J., Liao, J., Tang, C.: Ground state solution for a class of Schrödinger equations involving general critical growth term. Nonlinearity **30**, 899–911 (2017)
18. Miao, C., Xu, G., Zhao, L.: The dynamics of the 3D radial NLS with the combined terms. Commun. Math. Phys. **318**(3), 767–808 (2013)
19. Miao, C., Zhao, T., Zheng, J.: On the 4D nonlinear Schrödinger equation with combined terms under the energy threshold. Calc. Var. Partial Differ. Equ. **56**(6), 179 (2017)
20. Pierotti, D., Soave, N.: Ground states for the NLS equation with combined nonlinearities on non-compact metric graphs, arXiv:2011.00276, 3 Nov (2020)
21. Soave, N.: Normalized ground states for the NLS equation with combined nonlinearities. J. Diff. Equ. **269**(9), 6941–6987 (2020)
22. Soave, N.: Normalized ground states for the NLS equation with combined nonlinearities: The Sobolev critical case. J. Funct. Anal. **279**(6), (2020)
23. Tao, T., Visan, M., Zhang, X.: The nonlinear Schrödinger equation with combined power-type nonlinearities. Commun. Partial Differ. Equ. **32**(7–9), 1281–1343 (2007)
24. Weinstein, M.I.: Nonlinear Schrödinger equations and sharp interpolation estimates. Comm. Math. Phys. **87**, 567–576 (1983)
25. Willem, M.: Minimax Theorems. Birkhäuser, Boston (1996)
26. Zhang, X.: On the Cauchy problem of 3-D energy-critical Schrödinger equations with subcritical perturbations. J. Differ. Equ. **230**(2), 422–445 (2006)