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Calculus of Variations

Dynamics of nearly parallel vortex filaments for the Gross–Pitaevskii equation

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Abstract
Klein et al. (J Fluid Mech 288:201–248, 1995) have formally derived a simplified asymptotic motion law for the evolution of nearly parallel vortex filaments in the context of the three dimensional Euler equation for incompressible fluids. In the present work, we rigorously derive the corresponding asymptotic motion law in the context of the Gross–Pitaevskii equation.

Mathematics Subject Classification 35Q55 · 35B40 · 35Q35 · 76Y05

1 Introduction

The mathematical analysis of the evolution of vortex filaments within the framework of the classical equations for fluids is a challenging problem that dates back to the second half of the nineteenth century with the works of Kelvin and Helmholtz. Some "simplified" flows have long been considered as potential candidates for the description of the asymptotic regime of small vortex cores, the most well-known being the binormal curvature flow of Da Rios over a century ago, but the convergence proofs in all these cases are missing, and the validity of the convergence is sometimes questioned too in the literature.

Klein et al. [20] have proposed the system

$$\partial_t X_j = J\alpha_j \Gamma_j \partial_{zz} X_j + J \sum_{k\neq j} 2\Gamma_k \frac{X_j - X_k}{|X_j - X_k|^2}, \qquad j = 1, \ldots, n \tag{1}$$

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as a simplified candidate model for the evolution of n nearly parallel vortex filaments in perfect incompressible fluids. This model extends a remark by Zakharov [26] for pairs of anti-parallel filaments, and is expected to be valid only when

(i) the wavelength of the filaments perturbations are large with respect to the filaments mutual distances,
(ii) the latter are large with respect to the size of the filaments cores, and
(iii) the Reynolds number is sufficiently large.

In the above formulation, the filaments are assumed to be nearly parallel to the z-axis, and after rescaling[1] each of them is described by a function $z \mapsto (X_j(z,t), z)$, where $X_j(\cdot, t)$ takes values in $\mathbb{R}^2$, which represents the horizontal displacement of the filament. The canonical two by two symplectic matrix is denoted by J, the constants $\Gamma_j \in \mathbb{R}$ are the circulations associated to each vortex filament, and the constants $\alpha_j \in \mathbb{R}$ are derived from assumptions on the vortex core profiles prior to passing in the limit.

From the fluid mechanics point of view, the case $n = 1$ in (1) is already highly interesting and corresponds to a single weakly curved vortex filament. In that case, system (1) reduces to the free Schrödinger equation in one variable, and as a matter of fact this is also the linearized equation for the binormal curvature flow around a straight filament.

From a mathematical point of view, system (1) has been studied for his own (see e.g. [1,2,19,21]) when $n > 1$, in particular its well-posedness and the possibility of colliding filaments under (1). Nevertheless, as mentioned already, the justification of the model itself as a limit from a classical fluid mechanics model (such as the Euler equation or the Navier–Stokes equation in a vanishing viscosity limit) has so far only been obtained formally through matched asymptotic, even for $n = 1$.

The goal the present work is to rigorously derive system (1), for arbitrary $n \geq 1$, as a limit from (yet another) PDE model whose relation to fluid mechanics is not new. In that framework, all the limiting circulations Γ_j will end up being equal. Our object of study in this paper is indeed the Gross–Pitaevskii equation

$$i\partial_t u_\varepsilon - \Delta u_\varepsilon + \frac{1}{\varepsilon^2}(|u_\varepsilon|^2 - 1)u_\varepsilon = 0 \qquad \text{in } (0,T) \times \Omega, \tag{2}$$

with initial data $u_\varepsilon(\cdot, 0) = u_\varepsilon^0(\cdot)$. Here $0 < \varepsilon \ll 1$ is a real parameter, $\Omega = \omega \times \mathbb{T}_L$ where $\omega \subset \mathbb{R}^2$ is a bounded open set with smooth boundary[2] and $\mathbb{T}_L = \mathbb{R}/\mathrm{L}\mathbb{Z}$ for some $L > 0$. Without loss of generality, we shall assume that $0 \in \omega$. We also consider Neumann boundary conditions on $\partial\omega \times \mathbb{T}_L$:

$$\nu \cdot \nabla u_\varepsilon = 0 \text{ on } \partial\omega \times \mathbb{T}_L.$$

Our main result will describe solutions of (2) associated to initial data u_ε^0 for vanishing families of ε, and corresponding in a sense to be described in detail below to n nearly parallel vortex filaments clustered around the vertical axis $\{0\} \times (0, L)$.

1 Described further down, otherwise they wouldn't be anything close to parallel!

2 Since a rescaling will eventually be made in the description that sends the lateral boundary to infinity, the exact shape of ω is of limited impact on the analysis, and the limit flow for the filaments does *not* depend at all on ω. Still, some of our later assumptions for establishing convergence do depend on ω, see e.g. (9).

1.1 Statement of main result

We consider the system

$$i\partial_t f_j - \partial_{zz} f_j - 2\sum_{k\neq j} \frac{f_j - f_k}{|f_j - f_k|^2} = 0, \qquad j = 1, \dots, n \tag{3}$$

for $f \equiv (f_1, \dots, f_n) : \mathbb{T}_L \times \mathbb{R} \to \mathbb{C}^n$. This is the Klein Majda and Damodaran system (1) in the special case where all constants are equal and normalized to unity.

For $f \in H^1(\mathbb{T}_L, \mathbb{C}^n)$, we define

$$G_0(f) := \pi \int_0^L \left(\frac{1}{2} \sum_{i=1}^n |f_i'|^2 - \sum_{i\neq j} \log |f_i - f_j| \right) dz,$$

it is the Hamiltonian associated to the Eq. (3). We also set

$$\rho_f := \inf_{z\in(0,L),\ j\neq k} |f_j(z) - f_k(z)|.$$

A sufficient condition for the Hamiltonian $G_0(f)$ to be finite is that $\rho_f > 0$. For $f^0 \in H^1(\mathbb{T}_L, \mathbb{C}^n)$ such that $\rho_{f^0} > 0$, system (3) possesses a unique solution $f \in \mathcal{C}((-T, T), H^1(\mathbb{T}_L, \mathbb{C}^n))$ for some $T > 0$, and which satisfies $\rho_{f(\cdot,t)} > 0$ for all $t \in (-T, T)$. Moreover, f can be approximated by (arbitrarily) smooth solutions of (3). If $\liminf_{t\to\pm T} \rho_{f(\cdot,t)} = 0$, corresponding to a collision between filaments, the possibility to extend the solution past $\pm T$ is a delicate question, a situation which we won't consider in this work.

Regarding the Ginzburg–Landau energy, we write points in Ω in the form $(x, z) \in \omega \times \mathbb{T}_L$, and define

$$e_\varepsilon(u) := \frac{1}{2} \left(|\nabla_x u|^2 + |\partial_z u|^2 \right) + \frac{1}{4\varepsilon^2} (|u|^2 - 1)^2 ,$$

and

$$G_\varepsilon(u) := \int_\Omega e_\varepsilon(u)\, dx\, dz - L\kappa(n, \varepsilon, \omega) \tag{4}$$

where $\kappa(n, \varepsilon, \omega) = n\pi|\log \varepsilon| + n(n - 1)\pi|\log h_\varepsilon| + O(1)$ is defined more precisely in (9) below. The Cauchy problem for the Gross–Pitaevskii equation is globally well posed for initial data with finite Ginzburg–Landau energy (i.e. in $H^1(\Omega)$ here), and solutions can be approximated by smooth ones too.

The quantity which will define and locate the vorticity of a solution u_ε is the (horizontal[3]) Jacobian

$$Ju_\varepsilon := \nabla_x^\perp \cdot \mathrm{Re}(u_\varepsilon \nabla_x \overline{u}_\varepsilon),$$

it is therefore a real function of (x, z, t).

In order to measure the discrepancy between vorticity and an indefinitely thin filament, we will integrate in z some norms on the slices $\omega \times \{z\}$. For $\mu \in W^{-1,1}(\omega)$ we let

$$\|\mu\|_{W^{-1,1}(\omega)} := \sup \left\{ \int \phi\, d\mu\ : \phi \in W_0^{1,\infty}(\omega),\ \max\{\|\phi\|_\infty, \|D\phi\|_\infty\} \leq 1 \right\}.$$

[3] The other two components of the 3D Jacobian also have interpretations, see e.g. Proposition 2 below, but they do not enter in the statement of our main theorem.

Among the various equivalent norms that induce the $W^{-1,1}(\omega)$ topology, this choice has the property that there exists $r(\omega) > 0$ such that if $a_1, \ldots, a_n$ and $b_1, \ldots b_n$ are points in $B_r \subset \omega$, then

$$\| \sum_{i=1}^{n} \delta_{a_i} - \sum_{i=1}^{n} \delta_{b_i} \|_{W^{-1,1}(\omega)} = \min_{\sigma \in S_n} \sum_{i=1}^{n} |a_i - b_{\sigma(i)}| \tag{5}$$

where S_n denotes the group of permutations on n elements, see [4]. Indeed, this property holds whenever $r(\omega) \leq \min\{\frac{1}{2}\mathrm{dist}(0, \partial\omega), 1\}$, as then any 1-Lipschitz function on B_r that equals zero at the origin can be extended to a function ϕ such that $\phi = 0$ on $\partial\omega$ and $\max\{\|\phi\|_\infty, \|D\phi\|_\infty\} \leq 1$.

Finally, we introduce the scale

$$h_\varepsilon := \frac{1}{\sqrt{|\log \varepsilon|}}.$$

It will correspond to the amount of deformation of the filaments with respect to perfectly straight ones, and is also the typical separation distance between distinct filaments. At the same time, the scale ε corresponds to the typical core size of the filaments, and therefore since $h_\varepsilon \gg \varepsilon$ as $\varepsilon \to 0$, the displacements and mutual distances of filaments are much larger in this asymptotic regime than their core size.

Our main result is

Theorem 1 *Let $f = (f_1, \ldots, f_n) \in \mathcal{C}((-T, T), H^1(\mathbb{T}_L, \mathbb{C}^n))$ be solution of the vortex filament system (3) with initial data f^0 and such that $\rho_{f(t)} \geq \rho_0 > 0$ for all $t \in (-T, T)$.*

For $\varepsilon \in (0, 1]$, let u_ε solve the Gross–Pitaevskii equation (2) for initial data such that

$$\int_0^L \left\| J_x u_\varepsilon^0(\cdot, z) - \pi \sum_{j=1}^{n} \delta_{h_\varepsilon f_j^0(z)} \right\|_{W^{-1,1}(\omega)} dz \;=\; o(h_\varepsilon) \tag{6}$$

and

$$G_\varepsilon(u_\varepsilon^0) \to G_0(f^0) \tag{7}$$

as $\varepsilon \to 0$. Then for every $t \in (-T, T)$,

$$\int_0^L \left\| J_x u_\varepsilon(\cdot, z, h_\varepsilon^2 t) - \pi \sum_{j=1}^{n} \delta_{h_\varepsilon f_j(z,t)} \right\|_{W^{-1,1}(\omega)} dz \;=\; o(h_\varepsilon), \tag{8}$$

as $\varepsilon \to 0$.

Comments. The positivity of ρ_0 in Theorem 1 is essential, it implies that no collision between filaments occured over time, and the corresponding conclusion would very likely be incorrect without assuming it. Indeed, filaments collisions in superfluids experiments was observed to lead to highly complex reconnection dynamics, see for example [11], which exit the case of graph-like filaments considered here. Assumption (6) is responsible for the concentration of the initial vorticity of u_ε around the filaments parametrized by (rescalings) of f^0. Assumption (7) can be understood as requiring that the former concentration holds in the most energy efficient way (at least asymptotically as $\varepsilon \to 0$); this follows from results in [7], building on earlier work of [10]. Below we will recall these results in detail and refine some of them. The conclusion (8) implies that the concentration of vorticity is preserved in time, and its location follows (after appropriate rescalings) the model of Klein Majda and Damodaran.

The periodicity assumption which we make on the vertical variable is probably only technical, but at the level of the Gross–Pitaevskii equation the framework needed to deal with local perturbations of straight filaments would involve some further renormalization process of the (otherwise infinite) energy. Periodic perturbations of the limit system (1) have been studied in particular in [8].

In the context of the 3D Gross–Pitaevskii equation, there are very few available mathematical results which rigorously derive a motion law for vortex filaments. Besides Theorem 1, the only one we are aware of which does not require a symmetry assumption reducing the actual problem to 2D is [15], where the case of a single vortex ring was treated (the limiting filament is symmetric but the field u_ε is not assumed to be so). The situation is slightly better understood in the axisymmetric setting, in particular the case of a finite number of vortex rings was analized in [16], where the so-called leapfrogging phenomenon was established. In 2D the situation is of course brighter, and since vortex filaments are for the most part tensored versions of 2D vortex points, it is not surprising that the analysis of the latter is at the basis of all the 3D works we were referring to so far.

Vortex points and approximations of in 2D evolve according to the so-called point vortex system. That was established in [6] in the context of the Gross–Pitaevskii equation, but parallel results were also obtained (and actually earlier) in the framework of the incompressible 2D Euler equation [22,23].

The analogy between Euler and Gross–Pitaevskii equations is expected to be valid not only in 2D, and as stated at the beginning of this introduction a common open challenge in both frameworks is to rigorously derive the binormal curvature flow equation for general vortex filament shapes. In this context, we emphasize the $n = 1$ case of Theorem 1 establishes a linearized version of this so-called self-induction approximation for (2); the general case of the theorem describes evolution governed by a combination of the linearized self-induction of filaments and interaction with other filaments.

Contrary to the Euler equation, the Gross–Pitaevskii equation has a fixed "core length" ε in its very definition: this simplifies some of the analysis and may explain why in particular the equivalent of the nonlinear 3D stability for one vortex ring or the leapfrogging phenomenon have not yet been proved in that context.[4] On the other hand, there is no equivalent of the Biot–Savart law in the context of the Gross–Pitaveskii equation, the field is complex and the analysis often involves tricky controls of the phases. Partial results in the context of Euler in 3D include [12,13] for the 3D spectral stability of a columnar vortex, [5] for the evolution of a finite number of axisymmetric vortex rings in a regime where they do not interact, and [9] for the existence of travelling helices.

Theorem 1 does not cover the case of anti-parallel vortex filaments, a situation which in (1) would correspond to constants $\Gamma_j \in \pm 1$ that do not all share the same sign. This is something that we wish to consider in the future.

In the remaining subsections of this introduction, after fixing a number of notations which we use throughout, we describe in details the strategy followed to prove Theorem 1 and we state the key intermediate lemmas and propositions. The proofs of the latter are presented latter in Sect. 2, for the key arguments related to the dynamics, in Sect. 3, for the results which do not depend on a time variable and which are for the most part extensions or variations of results in [7], and in Sect. 4, for those related to a priori compactness in time.

[4] After this work was completed, Dávila, del Pino, Musso and Wei have announced the construction of solutions to the Euler equation exhibiting the leapfrogging phenomenon.

1.2 Further notations

In addition to the scale $h_\varepsilon := |\log \varepsilon|^{-1/2}$, we will always write $\omega_\varepsilon := h_\varepsilon^{-1}\omega$ and $\Omega_\varepsilon := \omega_\varepsilon \times \mathbb{T}_L$ to denote the rescaled versions of ω and Ω respectively. Given $u_\varepsilon \in H^1(\Omega, \mathbb{C})$ we will always let v_ε denote the function in $H^1(\Omega_\varepsilon, \mathbb{C})$ defined by

$$v_\varepsilon(x, z) = u_\varepsilon(h_\varepsilon x, z), \qquad (x, z) \in \Omega_\varepsilon.$$

We will write

$$jv_\varepsilon := iv_\varepsilon \cdot \nabla_x v_\varepsilon,$$

where here and throughout, a dot product of complex numbers denotes the *real* inner product:

$$\text{for } v, w \in \mathbb{C}, \qquad v \cdot w = \mathrm{Re}(v\bar{w}).$$

Observe once more that jv_ε contains only the *horizontal* components of the momentum vector $iv_\varepsilon \cdot Dv_\varepsilon = (iv_\varepsilon \cdot \nabla_x v_\varepsilon, iv_\varepsilon \cdot \partial_z v_\varepsilon)$.

In many places, we implicitly identify $\mathbb{C}^n$ with $(\mathbb{R}^2)^n$ when no complex products are involded. We fix $\chi \in C^\infty(\mathbb{R})$ to be a nonnegative nonincreasing function such that

$$\chi(s) = 1 \text{ if } s < 1, \qquad \chi(s) = 0 \text{ if } s \geq 2,$$

and for arbitrary $r > 0$ we set $\chi_r(s) := \chi(s/r)$. For $f \in H^1((0, L), (\mathbb{R}^2)^n)$ such that $\rho_f > 0$, and for $0 < r < \rho_f/4$, we also set

$$\chi_r^f(x, z) := \sum_{i=1}^n \chi_r(|x - f_i(z)|)\, |x - f_i(z)|^2.$$

$$\chi_{r,\varepsilon}^f(x, z) := \frac{1}{h_\varepsilon^2}\chi_{h_\varepsilon r}^{h_\varepsilon f}(x, z) = \sum_{i=1}^n \chi_r\Big(\frac{|x - h_\varepsilon f_i(z)|}{h_\varepsilon}\Big) \left|\frac{x - h_\varepsilon f_i(z)}{h_\varepsilon}\right|^2.$$

Repeated indices $a, b, c, \ldots$ are implicitly summed from 1 to 2; these correspond to the horizontal x variables. We will also write ε_{ab} to denote the usual antisymmetric symbol, with components

$$\varepsilon_{12} = -\varepsilon_{21} = 1, \qquad \varepsilon_{11} = \varepsilon_{22} = 0.$$

For $v = (v_1, v_2) \in \mathbb{R}^2$, we will write $v^\perp := (-v_2, v_1)$. Thus $(v^\perp)_b = \varepsilon_{ab} v_a$. We will similarly write $\nabla_x^\perp := (-\partial_y, \partial_x)$. In the same spirit,

$$v^\perp := (v_1^\perp, \ldots, v_n^\perp) \quad \text{for } v = (v_1, \ldots, v_n) \in (\mathbb{R}^2)^n,$$

with a similar convention for $\nabla^\perp W$, for $W : (\mathbb{R}^2)^n \to \mathbb{R}$.

If μ_z is a family of signed measures on an open set $U \subset \mathbb{R}^2$, depending (measurably) on a parameter $z \in (0, L)$, then $\mu_z \otimes dz$ denotes the measure on $U \times (0, L)$ defined by

$$\int_{U\times(0,L)} f\, d\mu_z \otimes dz = \int_0^L \Big(\int_U f(x, z)\, d\mu_z(x)\Big) dz.$$

For a smooth bounded $A \subset \mathbb{R}^2$ (typically ω or ω_ε) and $a \in A^n$ we will write

$$j_A^*(x; a) := -\nabla_x^\perp \psi_A^*,$$

where $\psi_A^* = \psi_A^*(x; a)$ solves

$$\begin{cases} -\Delta_x \psi_A(\cdot; a)^* &= 2\pi \sum_{i=1}^n \delta_{a_i} \quad \text{in } A \\ \quad \psi_A^* &= 0 \qquad\qquad \text{on } \partial A\ . \end{cases}$$

Equivalently, $j_A^*(x; a) : A \to \mathbb{R}^2$ is the unique solution of

$$\nabla_x \cdot j_A^* = 0, \qquad \nabla_x^\perp \cdot j_A^* = 2\pi \sum_{i=1}^n \delta_{a_i}, \qquad j_A^*(\cdot, a) \cdot \nu = 0 \text{ on } \partial A$$

where ν denotes the outer unit normal to A. It is straightforward to check that

$$j_{\omega_\varepsilon}^*(x; a) = h_\varepsilon j_\omega^*(h_\varepsilon x; h_\varepsilon a)$$

and that

$$\lim_{\varepsilon \to 0} j_{\omega_\varepsilon}^*(x; a) = \sum_{i=1}^n \frac{(x - a_i)^\perp}{|x - a_i|^2} =: j_{\mathbb{R}^2}^*(x; a).$$

Given $g : (0, L) \to A^n$, we will write $j_A^*(g)$ to denote the function $A \times (0, L) \to \mathbb{R}^2$ defined by

$$j_A^*(g)(x, z) = j_A^*(x; g(z)).$$

We define a couple of other auxiliary functions related to ψ_A. First, note that

$$\psi_A(x; a) = -\sum_{i=1}^n (\log |x - a_i| + H_A(x, a_i))$$

where for $a_i \in \Omega$, we define $H_A(\cdot, a_i)$ to be the solution of

$$-\Delta_x H_A(x, a_i) = 0 \text{ for } x \in A, \qquad H_A(x, a_i) = -\log |x - a_i| \text{ for } x \in \partial A.$$

We define

$$W_A(a) = -\pi \Big(\sum_{i \neq j} \log |a_i - a_j| + \sum_{i,j} H_A(a_i, a_j) \Big).$$

The constant $\kappa(n, \varepsilon, \omega)$ appearing in (4) is defined by

$$\kappa(n, \varepsilon, \omega) = n(\pi |\log \varepsilon| + \gamma) + n(n-1)\pi |\log h_\varepsilon| - \pi n^2 H_\omega(0, 0) \tag{9}$$

where γ is a universal constant[5] introduced in the pioneering work of Béthuel, Brezis and Hélein [3], see Lemma IX.1.

1.3 Variational aspects of nearly parallel vortex filaments

In this section we first collect some information about the behaviour of nearly parallel vortex filaments under energy and localisation constraints, but without introducing any time dependence. Most of these results are contained in Contreras and Jerrard [7], or can be obtained by adapting and combining results in [7]. The necessary details are given in Sect. 3.

Our first result follows directly from arguments in [7], although it does not appear there in exactly this form.

[5] We will not need the exact definition of $\kappa(n, \varepsilon, \omega)$ or γ in this paper, but these constants will appear in various formulas.

Proposition 1 *Assume that* $(u_\varepsilon) \subset H^1(\Omega, \mathbb{C})$ *is a sequence satisfying*

$$\int_0^L \|J_x u_\varepsilon(\cdot, z) - n\pi\delta_0\|_{W^{-1,1}(\omega)} dz \le c_1 h_\varepsilon, \tag{10}$$

$$G_\varepsilon(u_\varepsilon) \le c_2. \tag{11}$$

Then

$$\int_\Omega |\partial_z u_\varepsilon|^2 dx\, dz \le C(c_1, c_2) \tag{12}$$

and there exists some $f = (f_1, \ldots, f_n) \in H^1(\mathbb{T}_L, \mathbb{C}^n)$ *such that after passing to a subsequence if necessary:*

$$\int_0^L \|J_x u_\varepsilon(\cdot, z) - \pi \sum_{j=1}^n \delta_{h_\varepsilon f_j(z)}\|_{W^{-1,1}(\omega)} dz = o(h_\varepsilon) \qquad \text{as } \varepsilon \to 0. \tag{13}$$

Finally, f *satisfies*

$$G_0(f) \le \liminf_{\varepsilon \to 0} G_\varepsilon(u_\varepsilon), \qquad \|f\|_{H^1} \le C(c_1, c_2), \tag{14}$$

where the lim inf refers to the subsequence for which (13) *holds.*

The arguments needed to extract Proposition 1 from facts established in [7] are presented in Sect. 3.2. Next we describe weak limits of products of derivatives of v_ε.

Proposition 2 *Assume that* $(u_\varepsilon) \subset H^1(\Omega, \mathbb{C})$ *satisfies* (11) *and* (13) *(and hence* (10)*), and let* $v_\varepsilon(x, z) = u_\varepsilon(h_\varepsilon x, z)$*. Then the following hold, in the weak sense of measures on* Ω

$$\frac{1}{|\log\varepsilon|} \partial_{x_k} v_\varepsilon \cdot \partial_{x_l} v_\varepsilon \rightharpoonup \pi\delta^{kl} \sum_{i=1}^n \delta_{f_i(z)} \otimes dz, \tag{15}$$

$$\frac{1}{|\log\varepsilon|} \nabla_x v_\varepsilon \cdot \partial_z v_\varepsilon \rightharpoonup -\pi \sum_{i=1}^n \partial_z f_i(z) \delta_{f_i(z)} \otimes dz, \tag{16}$$

for all k, l *in* $\{1, 2\}$*. Moreover, for any nonnegative* $\phi \in C_c(\mathbb{R}^2 \times \mathbb{T}_L)$*,*

$$\liminf_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \phi \frac{|\partial_z v_\varepsilon|^2}{|\log\varepsilon|} dx\, dz \ge \pi \sum_{i=1}^n \int_0^L |f_i'(z)|^2 \phi(f_i(z), z)\, dz. \tag{17}$$

The proof of Proposition 2 is given in Sect. 3.3. Briefly, (15) and (17) are deduced by combining results from [7] with facts established in [14,15,24], and (16) is obtained via a short argument whose starting point is (15) and (17).

Finally we will need a refinement of a Γ-limit lower bound from [7]. The proof is given in Sect. 3.4.

Proposition 3 *Let* $r > 0$ *and* $f \in H^1((0, L), \mathbb{C}^n)$ *be given such that* $r < \rho_f/4$*. Then given* $\delta > 0$*, there exist* $c_3, \varepsilon_3 > 0$*, depending only on* $\|f\|_{H^1}$ *and* r*, such that for all* $\Sigma \in (0, 1]$ *and any* $\varepsilon \le \varepsilon_3$*, if* $u_\varepsilon \in H^1(\Omega, \mathbb{C})$ *and*

$$\int_0^L \|J u_\varepsilon(\cdot, z) - \pi \sum_{i=1}^n \delta_{h_\varepsilon f_i(z)}\|_{W^{-1,1}(\omega)} dz \le c_3 h_\varepsilon, \tag{18}$$

$$G_\varepsilon(u_\varepsilon) - G_0(f) \le \Sigma, \tag{19}$$

then

$$\int_0^L \int_{\omega\setminus\cup_{i=1}^n B(h_\varepsilon f_i(x),h_\varepsilon r)} e_\varepsilon(|u_\varepsilon|) + \frac{1}{4}\left|\frac{ju_\varepsilon}{|u_\varepsilon|} - j^*_\omega(h_\varepsilon f)\right|^2 \le K_3\Sigma + \delta, \tag{20}$$

where K_3 *depends only on* r, n, *and* $\|f\|_{H^1}$. *Moreover, if*

$$T^f_{r,\varepsilon}(u_\varepsilon) := \int_\Omega J_x u_\varepsilon(x,z)\chi^f_{r,\varepsilon}\, dx\, dz \le \frac{c_3^2}{4n\pi L} \tag{21}$$

then

$$\frac{1}{h_\varepsilon}\int_0^L \|J_x u_\varepsilon(\cdot,z) - \pi\sum_{i=1}^n \delta_{h_\varepsilon f_i(z)}\|_{W^{-1,1}(\omega)} dz \le \left(n\pi L T^f_{r,\varepsilon}(u_\varepsilon)\right)^{\frac{1}{2}} + o(1) \le \frac{1}{2}c_3. \tag{22}$$

1.4 Compactness in time

In this section we now assume that u_ε is a solution of the Gross–Pitaevskii equation and we shall obtain sufficient compactness in time to pass to the limit as $\varepsilon \to 0$ on intervals of time of positive length.

Proposition 4 *Let* $r > 0$ *and* $g \in W^{1,\infty}(\mathbb{T}_L, \mathbb{C}^n)$ *be given such that* $r \le \rho_g/4$. *There exist* ε_4, $c_4 > 0$, *depending only on* $\|g\|_{H^1}$ *and* r, *and there exist* C_4, *depending only on* $\|g\|_{Lip}$ *and* r, *with the following properties. If* u_ε *solves the Gross–Pitaevskii equation* (2) *for some* $0 < \varepsilon \le \varepsilon_4$ *for initial data* u^0_ε *satisfying*

$$G_\varepsilon(u^0_\varepsilon) \le G_0(g) + 1, \tag{23}$$

$$\int_0^L \|Ju^0_\varepsilon(\cdot,z) - \pi\sum_{i=1}^n \delta_{h_\varepsilon g_i(z)}\|_{W^{-1,1}(\omega)} dz \le c_4 h_\varepsilon, \tag{24}$$

and

$$T^g_{r,\varepsilon}(u^0_\varepsilon) \le \frac{c_4^2}{4n\pi L}, \tag{25}$$

then for every $0 \le t \le t_4 := 3c_4^2/(4C_4 n\pi L)$,

$$T^g_{r,\varepsilon}(u_\varepsilon(\cdot,\cdot,h^2_\varepsilon t)) \le T^g_{r,\varepsilon}(u^0_\varepsilon) + C_4 t, \tag{26}$$

$$\frac{1}{h_\varepsilon}\int_0^L \|J_x u_\varepsilon(\cdot,z,h^2_\varepsilon t) - \pi\sum_{i=1}^n \delta_{h_\varepsilon g_i(z)}\|_{W^{-1,1}(\omega)} dz \le \left(n\pi L(T^g_{r,\varepsilon}(u^0_\varepsilon) + C_4 t)\right)^{\frac{1}{2}} + o(1), \tag{27}$$

and in particular

$$\int_0^L \|J_x u_\varepsilon(\cdot,z,h^2_\varepsilon t) - \pi\sum_{i=1}^n \delta_{h_\varepsilon g_i(z)}\|_{W^{-1,1}(\omega)} dz \le (c_4 + o(1))h_\varepsilon. \tag{28}$$

The proof is given in Sect. 4, as is the proof of the following.

Corollary 1 *Under the assumptions of Theorem* 1, *there exists* $t_0 > 0$, *depending only on* ρ_{f^0} *and* $\|f^0\|_{H^1}$, f^* *in* $\mathcal{C}([0,t_0], L^1(\mathbb{T}_L,\mathbb{C}^n)) \cap L^\infty([0,t_0], H^1(\mathbb{T}_L,\mathbb{C}^n))$, *and a common sequence* $\varepsilon \to 0$, *such that for every* $0 \le t \le t_0$

$$\int_0^L \|J_x u_\varepsilon(\cdot,z,h^2_\varepsilon t) - \pi\sum_{j=1}^n \delta_{h_\varepsilon f^*_j(z,t)}\|_{W^{-1,1}(\omega)} dz = o(h_\varepsilon) \qquad \text{as } \varepsilon \to 0$$

and in addition the equivalent of (28) *holds for all* $t \in [0, t_0]$, *for every* ε *in the sequence.*
Moreover, we have $f^*(0) = f(0)$ *and*

$$\sup_{s,t\in[0,t_0]} \max_{i,z} |f_i^*(z,t) - f_i(z,s)| \le \frac{\rho_0}{8}, \qquad \textit{and hence } \inf_{t\in[0,t_0]} \rho_{f^*(t)} \ge \frac{3}{4}\rho_0. \tag{29}$$

Our main goal in the sequel is to show that f and f^* coincide on $[0, t_0]$, from which Theorem 1 will follow by a straightforward continuation argument.

Proposition 5 *In addition to the statements in Corollary* 1, *we have*

$$\frac{j(v_\varepsilon)}{|v_\varepsilon|} \rightharpoonup j^*_{\mathbb{R}^2}(f^*) \quad \textit{weakly in } L^2(O)$$

for every open $O \subset\subset \{(t, x, z) \in [0, t_0] \times \mathbb{R}^2 \times \mathbb{T}_L : x \ne f_k^*(z, t),\ k = 1, \ldots, n\}$.

1.5 Proof of the main theorem

For points $a = (a_1, \ldots, a_n) \in (\mathbb{R}^2)^n$ such that $a_i \ne a_j$ for $i \ne j$, we will write

$$\mathcal{W}(a) = -\sum_{i\ne j} \log|a_i - a_j|. \tag{30}$$

With this notation,

$$G_0(g) = \pi \int_0^L \frac{1}{2}|g'(z)|^2 + \mathcal{W}(g(z))\, dz \quad \text{for } g : \mathbb{T}_L \to (\mathbb{R}^2)^n.$$

For $0 \le t \le t_0$ (where t_0 appears in Corollary 1), we define

$$I_1(t) := \pi \int_0^L |f(z,t) - f^*(z,t)|^2\, dz$$
$$I_2(t) := \pi \int_0^L (-\partial_{zz} f(z,t) + \nabla\mathcal{W}(f(z,t)) \cdot (f(z,t) - f^*(z,t))dz$$
$$I_3(t) := G_0(f(\cdot,t)) - G_0(f^*(\cdot,t)).$$

Note that, as a consequence of conservation of energy for both (2) and (3),

$$G_0(f(\cdot,t)) = G_0(f^0) = \lim_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon^0) = \lim_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon(t)) \ge G_0(f^*(t)).$$

The last inequality follows from (14), as discussed following the statement of Proposition 1. Thus $I_3(t) \ge 0$ for all $t \in [0, t_0]$. In addition, $I_3(0) = 0$, due to Corollary 1.

We aim to apply Proposition 3 to control quantities such as $\frac{ju_\varepsilon}{|u_\varepsilon|}(t) - j^*_\omega(h_\varepsilon f^*(t))$ for a range of t. To this end, we will need

$$\Sigma_\varepsilon(t) := G_\varepsilon(u_\varepsilon(t)) - G_0(f^*(t)) \le 1. \tag{31}$$

Arguing as above, we see that $\lim_{\varepsilon\to 0} \Sigma_\varepsilon(t) = I_3(t)$. Thus $\Sigma_\varepsilon(t) \le 1$ if ε is sufficiently small and $I_3(t) \le \frac{1}{2}$. We therefore define

$$t^* := \sup\{t \in [0, t_0] : 0 \le I_3(s) \le \frac{1}{2} \text{ for all } s \in [0, t]\}.$$

The positivity of t^* is a consequence of the weak H^1 lowersemicontinuity of $f \mapsto G_0(f)$ and the continuity properties of f^* as stated in Corollary 1. (The other hypothesis of Proposition 3 follows directly from Corollary 1.)

Theorem 1 will be an easy consequence of the following three lemmas.

Lemma 1 *There exists a constant C_2 such that for every $t \in [0, t^*]$,*

$$I_3(t) \leq I_2(t) + C_2 I_1(t).$$

Proof First, it follows from (29) that for every $z \in [0, L]$ and $t \in [0, t^*]$,

$$\mathcal{W}(f(z,t)) - \mathcal{W}(f^*(z,t)) \leq \nabla\mathcal{W}(f(z,t)) \cdot (f(z,t) - f^*(z,t)) + C|f(z,t) - f^*(z,t)|^2,$$

for C depending only on $\rho_{f(0)}$. The conclusion of the lemma follows by integrating this inequality with respect to z and combining the result with the estimate

$$\begin{aligned}\frac{\pi}{2}\int_0^L |\partial_z f|^2 - |\partial_z f^*|^2\, dz &= \frac{\pi}{2}\int_0^L 2\partial_z f \cdot \partial_z(f - f^*) - |\partial_z(f - f^*)|^2\, dz \\ &\leq -\pi \int_0^L \partial_{zz} f \cdot (f - f^*)\, dz.\end{aligned}$$

□

The proofs of the next two lemmas are presented in Sect. 2 below.

Lemma 2 *For every $\tau \in [0, t^*]$,*

$$I_1(\tau) \leq I_1(0) + C\int_0^\tau (I_1(t) + I_3(t))\, dt.$$

Lemma 3 *For every $\tau \in [0, t^*]$,*

$$I_2(\tau) \leq I_2(0) + C\int_0^\tau (I_1(t) + I_3(t))\, dt.$$

With these, we can complete the

Proof of Theorem 1 Let $I_4(t) = I_2(t) + (1 + C_2)I_1(t)$. It follows from Lemma 1 that $I_4(t) \geq I_3(t) + I_1(t) \geq 0$ for all $t \in [0, t^*]$, moreover $I_4(0) = 0$ by Corollary 1 and Lemmas 1–3 imply that

$$I_4(\tau) \leq (1 + C + C_2)\int_0^\tau I_4(t)\, dt \qquad \text{for all } \tau \in [0, t^*].$$

It follows by Grönwall's inequality that $I_4(\tau) = 0$ for all $\tau \in [0, t^*]$, and therefore also that $I_1(\tau) = 0$ for all $\tau \in [0, t^*]$, in other words, that $f = f^*$ on $[0, t^*]$. A straightforward continuation argument now shows that this equality holds on $(0, T)$, and then by reversibility on $(-T, T)$, thus completing the proof. □

2 Dynamics

The object of this section is to present the proofs of Lemmas 2 and 3, from which (together with Lemma 1) our main Theorem was derived in the Introduction. We will find it useful to rescale the Gross–Pitaevskii equation (2), setting

$$v_\varepsilon(x, z, t) := u_\varepsilon(h_\varepsilon x, z, h_\varepsilon^2 t), \tag{32}$$

where

$$h_\varepsilon := |\log\varepsilon|^{-1/2}.$$

Thus

$$i\partial_t v_\varepsilon - \Delta_x v_\varepsilon - \frac{\partial_{zz} v_\varepsilon}{|\log\varepsilon|} + \frac{1}{|\log\varepsilon|\varepsilon^2}(|v_\varepsilon|^2 - 1)v_\varepsilon = 0. \tag{33}$$

We will write

$$j_x v_\varepsilon := i v_\varepsilon \cdot \nabla_x v_\varepsilon,$$
$$j_z v_\varepsilon := i v_\varepsilon \cdot \partial_z v_\varepsilon.$$

For the rescaled equation (33), the equation for conservation of mass takes the form

$$\frac{1}{2}\partial_t |v_\varepsilon|^2 = \nabla_x \cdot j_x v_\varepsilon + h_\varepsilon^2\, \partial_z j_z v_\varepsilon. \tag{34}$$

We will rely mainly on the equation for vorticity, and in fact only for the z component of the vorticity vector, which is precisely $J_x v_\varepsilon$. By rescaling standard identities we have

$$\partial_t J_x v_\varepsilon = \varepsilon_{ab}\partial_{ac}(\partial_b v_\varepsilon \cdot \partial_c v_\varepsilon) + \varepsilon_{ab}\partial_{az}(\frac{\partial_b v_\varepsilon \cdot \partial_z v_\varepsilon}{|\log\varepsilon|}).$$

Thus,

$$\begin{aligned}\frac{d}{dt}\int \varphi J_x v_\varepsilon dx\, dz &= \int \partial_t\varphi\ J_x v_\varepsilon dx\, dz + \int \varepsilon_{ab}\partial_{ac}\varphi\ \partial_b v_\varepsilon \cdot \partial_c v_\varepsilon\, dx\, dz \\ &\quad + \int \varepsilon_{ab}\partial_{az}\varphi \frac{\partial_b v_\varepsilon \cdot \partial_z v_\varepsilon}{|\log\varepsilon|}\, dx\, dz, \end{aligned} \tag{35}$$

for smooth $\varphi : \Omega_\varepsilon \times (0,T) \to \mathbb{R}$ for some $T > 0$, with compact support in $\Omega_\varepsilon = \omega_\varepsilon \times \mathbb{T}_L$. (That is, test functions are only required to have compact support with respect to the horizontal x variables, not the periodic z variable.)

Lemma 4 *Assume that $\varphi \in C_c^2(\Omega_\varepsilon \times [0,t^*])$ is a function such that for some $k \in \{1,\ldots,n\}$,*

$$supp(\varphi) \subset \{(x,z,t) : |x - f_k(z,t)| \le \frac{\rho_0}{2}\},$$

and

$$\partial_{ac}\varphi(x,z,t) = c(z,t)\delta^{ac} \qquad in\ \{(z,t) : |x - f_k(z,t)| \le \frac{\rho_0}{4}\} \tag{36}$$

for some continuous $c(z,t)$. Then for any $\tau \in [0,t^*]$,

$$\begin{aligned}&\int_0^L \varphi(f_k^*(z,t),z,t)\, dz\Big|_{t=0}^{t=\tau} \\ &\le C\int_0^\tau I_3(t)\, dt \\ &\quad + \int_0^\tau\int_0^L \partial_t\varphi(f_k^*(z,t),z,t)\, dz\, dt \\ &\quad - \int_0^\tau\int_0^L \nabla^\perp\partial_z\varphi(f_k^*(z),z,t)\cdot\partial_z f_k^*(z,t)\, dz\, dt \\ &\quad + \int_0^\tau\int_0^L \nabla\varphi(f_k^*(z,t),z,t)\cdot\nabla_k^\perp\mathcal{W}(f^*(z,t))\, dz\, dt\ ,\end{aligned}$$

where C depends only ρ_0, $\|f\|_{L^\infty H^1}$ and $\|\nabla_x^2\varphi\|_{L^\infty}$.

Proof We apply (35) to φ, integrate both sides from 0 to τ, and send $\varepsilon \to 0$. We consider the various terms that arise.

1. Corollary 1 and properties of the support of φ imply that

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} \varphi(x,z,t)\, J_x v_\varepsilon(x,z,t)dx\, dz = \pi \int_0^L \varphi(f_k^*(z,t),z,t)\, dz \tag{37}$$

for every $t \in [0, t^*]$, and in particular for $t = 0, \tau$.

2. Similarly, (37) holds with φ replaced by $\partial_t\varphi$. In addition, it follows from (28) that $|\int_{\Omega_\varepsilon} \partial_t\varphi(x,z,t)\, J_x v_\varepsilon(x,z,t)dx\, dz|$ is bounded uniformly in t. Thus

$$\lim_{\varepsilon\to 0}\int_0^\tau \int_{\Omega_\varepsilon} \partial_t\varphi\, J_x v_\varepsilon\, dx\, dz\, dt = \pi \int_0^\tau \int_0^L \partial_t\varphi(f_k^*(z,t),z,t)\, dz\, dt.$$

3. The last term on the right-hand side of (35) is similar. First note that there exists some C such that

$$\int_{\Omega_\varepsilon} \varepsilon_{ab}\partial_{az}\varphi \frac{\partial_b v_\varepsilon \cdot \partial_z v_\varepsilon}{|\log\varepsilon|}\, dx\, dz\ \le C$$

for every t. This is a consequence of (12) (which is available for all $t \in [0, t^*]$ by Corollary 1) and (7), since

$$\int_{\Omega_\varepsilon} |\partial_z v_\varepsilon(y,z,t)|^2\, dy\, dz\ =\ \int_\Omega \frac{|\partial_z u_\varepsilon(x,z,h_\varepsilon^2 t)|^2}{|\log\varepsilon|}\, dx\, dz$$

and $\int_{\Omega_\varepsilon} \frac{1}{2}|\nabla_x v_\varepsilon(y,z,t)|^2\, dy \le G_\varepsilon(u_\varepsilon(\cdot,\cdot,h_\varepsilon^2 t)) = G_\varepsilon(u_\varepsilon^0)$. Also,

$$\int_{\Omega_\varepsilon} \varepsilon_{ab}\partial_{az}\varphi \frac{\partial_b v_\varepsilon \cdot \partial_z v_\varepsilon}{|\log\varepsilon|}\, dx\, dz \to -\pi \int_0^L \nabla^\perp\partial_z\varphi(f_k^*(z),z,t)\cdot\partial_z f_k^*(z,t)\, dz$$

for every t, due to (16). It follows that

$$\int_0^\tau\int_{\Omega_\varepsilon} \varepsilon_{ab}\partial_{az}\varphi \frac{\partial_b v_\varepsilon \cdot \partial_z v_\varepsilon}{|\log\varepsilon|}\, dx\, dz\, dt \to -\pi \int_0^\tau\int_0^L \nabla^\perp\partial_z\varphi(f_k^*(z),z,t)\cdot\partial_z f_k^*(z,t)\, dz\, dt.$$

4. To describe the limit of the remaining term coming from (35), first note that (36), together with our assumptions on the support of φ, implies that

$$\text{supp}(\varepsilon_{ab}\partial_{ac}\varphi\partial_b v_\varepsilon\cdot\partial_c v_\varepsilon)(\cdot,t) \subset \Omega_{\varepsilon,k}(t) := \{(x,z)\in\Omega_\varepsilon\, :\, |x - f_k(z,t)| \in [\frac{\rho_0}{4}, \frac{\rho_0}{2}]\}.$$

Next, we follow standard arguments and write

$$\partial_b v_\varepsilon\cdot\partial_c v_\varepsilon = \partial_c|v_\varepsilon|\,\partial_c|v_\varepsilon| + \frac{j_b(v_\varepsilon)j_c(v_\varepsilon)}{|v_\varepsilon|^2}.$$

For the rest of this proof we will write j_ε^* as an abbreviation for $j_{\omega_\varepsilon}^*(f^*)$, and $j^* := \lim_{\varepsilon\to 0} j_\varepsilon^* = j_{\mathbb{R}^2}^*(f^*)$. With this notation, we further decompose the last term above as

$$\begin{aligned}\frac{j_b(v_\varepsilon)j_c(v_\varepsilon)}{|v_\varepsilon|^2} &= j_{\varepsilon,b}^*\, j_{\varepsilon,c}^* + \left(\frac{j(v_\varepsilon)}{|v_\varepsilon|} - j_\varepsilon^*\right)_b\left(\frac{j(v_\varepsilon)}{|v_\varepsilon|} - j_\varepsilon^*\right)_c \\ &+ j_{\varepsilon,b}^*\left(\frac{j(v_\varepsilon)}{|v_\varepsilon|} - j_\varepsilon^*\right)_c + j_{\varepsilon,c}^*\left(\frac{j(v_\varepsilon)}{|v_\varepsilon|} - j_\varepsilon^*\right)_b.\end{aligned}$$

Thus,

$$\int_0^\tau\int_{\Omega_\varepsilon} \varepsilon_{ab}\partial_{ac}\varphi\,\partial_b v_\varepsilon\cdot\partial_c v_\varepsilon\,dz\,dz\,dt \le \int_0^\tau\int_{\Omega_{\varepsilon,k}(t)} \varepsilon_{ab}\partial_{ac}\varphi\, j^*_{\varepsilon,b} j^*_{\varepsilon,c}$$
$$+\int_0^\tau\int_{\Omega_{\varepsilon,k}(t)} \varepsilon_{ab}\partial_{ac}\varphi\left[j^*_{\varepsilon,b}\left(\frac{j(v_\varepsilon)}{|v_\varepsilon|}-j^*_\varepsilon\right)_c + j^*_{\varepsilon,c}\left(\frac{j(v_\varepsilon)}{|v_\varepsilon|}-j^*_\varepsilon\right)_b\right]$$
$$+\int_0^\tau\int_{\Omega_{\varepsilon,k}(t)} |\nabla_x^2\varphi|\left(|\nabla_x|v_\varepsilon||^2+\left|\frac{j(v_\varepsilon)}{|v_\varepsilon|}-j^*_\varepsilon\right|^2\right).$$

It follows from Proposition 5 that the second term on the right-hand side converges to 0 as $\varepsilon\to 0$.

Using (19) and (20) of Proposition 3 for a sequence $\delta_n\to 0$ and recalling that $\Sigma_\varepsilon(t)$, as defined in (31), satisfies $\Sigma_\varepsilon(t)\to I_3(t)$ as $\varepsilon\to 0$, we find that

$$\limsup_{\varepsilon\to 0}\int_0^\tau\int_{\Omega_\varepsilon}|\nabla_x^2\varphi|\left(|\nabla_x|v_\varepsilon||^2+\left|\frac{j(v_\varepsilon)}{|v_\varepsilon|}-j^*_\varepsilon\right|^2\right)\le C\int_0^\tau I_3(t)\,dt.$$

Since $j^*_\varepsilon\to j^*$ locally uniformly on $\mathbb{R}^2$, it is clear that

$$\int_0^\tau\int_{\Omega_{\varepsilon,k}(t)}\varepsilon_{ab}\partial_{ac}\varphi\, j^*_{\varepsilon,b}j^*_{\varepsilon,c}\to\int_0^\tau\int_{\Omega_{\varepsilon,k}(t)}\varepsilon_{ab}\partial_{ac}\varphi\, j^*_b j^*_c$$

as $\varepsilon\to 0$. Finally, we claim that

$$\int_0^\tau\int_{\Omega_{\varepsilon,k}(t)}\varepsilon_{ab}\partial_{ac}\varphi\, j^*_b j^*_c=\pi\int_0^\tau\int_0^L\nabla\varphi(f^*_k,z,t))\cdot\nabla_k^\perp\mathcal{W}(f^*(z,t))\,dz\,dt.$$

This is a small variant of a classical fact. We recall the proof for the reader's convenience. First note that for every t and every $z\in(0,L)$,

$$\int_{\{x\in\omega:|x-f_k(z,t)|\in[\frac{\rho_0}{4},\frac{\rho_0}{2}]\}}\varepsilon_{ab}\partial_{ac}\varphi\, j^*_b j^*_c\,dx=\lim_{s\to 0^+}\int_{\omega\setminus B_s(f_k(z,t))}\varepsilon_{ab}\partial_{ac}\varphi\, j^*_b j^*_c\,dx$$

(where all integrands are evaluated at the fixed value of t). Indeed, the right-hand side is independent of s for $0<s<\rho_0/4$, since the integrand vanishes identically in $B_{\rho_0/4}(f_k(z,t))$. For every $s<\rho_0/4$,

$$\begin{aligned}\int_{\omega\setminus B_s(f_k(z,t))}\varepsilon_{ab}\partial_{ac}\varphi\, j^*_b j^*_c\,dx&=\int_{\omega\setminus B_s(f_k(z,t))}\varepsilon_{ab}\partial_{ac}\varphi\,(j^*_b j^*_c-\frac12\delta^{bc}|j^*|^2)\,dx\\&=-\int_{\partial B_s(f_k(z,t))}\varepsilon_{ab}\partial_a\varphi\,(j^*_b j^*_c-\frac12\delta^{bc}|j^*|^2)\nu_c\\&=-\int_{\partial B_s(f_k(z,t))}(\nabla^\perp\varphi\cdot j^*)(\nu\cdot j^*)-\frac12\nabla^\perp\varphi\cdot\nu|j^*|^2.\end{aligned}\quad(38)$$

Note that

$$j^*(x,z,t)=\frac{(x-f_k(z,t))^\perp}{|x-f_k(z,t)|^2}+\tilde{j}(x;k),\quad\text{where }\tilde{j}(x;k)=\sum_{\ell\ne k}\frac{(x-f_\ell(z,t))^\perp}{|x-f_\ell(z,t)|^2}.$$

We decompose j^* in this way on the right-hand side of (38), then expand and let s tend to zero. This leads to

$$\int_{\{x\in\omega:|x-f_k(z,t)|\in[\frac{\rho_0}{4},\frac{\rho_0}{2}]\}}\varepsilon_{ab}\partial_{ac}\varphi\, j^*_b j^*_c\,dx=-2\pi\nabla\varphi(f^*_k(z,t),z,t)\cdot\tilde{j}(f^*_k(z,t);k).$$

Since

$$\nabla_k^\perp \mathcal{W}(a) := -2 \sum_{\ell \neq k} \frac{(a_k - a_\ell)^\perp}{|a_k - a_\ell|^2} = -2\widetilde{j}(f_k^*(z,t); k),$$

this implies the claim, and the proof of Lemma 4 is completed. □

Proof of Lemma 2 We first assume that f is of class $\mathcal{C}^2$ and we apply Lemma 4 with

$$\varphi(x,z,t) = \chi_{\rho_0/4}(|x - f_k(z,t)|)\, |x - f_k(z,t)|^2,$$

and then sum the resulting inequalities over k. This leads to the estimate

$$I_1(\tau) \leq I_1(0) + \int_0^\tau \int_0^L (f - f^*) \cdot \partial_t f + \partial_z f^\perp \cdot \partial_z f^* \, dz\, dt$$
$$- \int_0^\tau \int_0^L (f - f^*) \cdot \nabla^\perp \mathcal{W}(f^*)\, dz\, dt + C \int_0^\tau I_3(t)\, dt.$$

The equation (3) satisfied by f may be written

$$\partial_t f^\perp = \partial_{zz} f - \nabla \mathcal{W}(f). \tag{39}$$

Substituting this into the above inequality and integrating by parts, we obtain

$$I_1(\tau) \leq I_1(0) + \int_0^\tau \int_0^L (f - f^*) \cdot (\nabla^\perp \mathcal{W}(f) - \nabla^\perp \mathcal{W}(f^*))\, dz\, dt + C \int_0^\tau I_3(t)\, dt.$$

It follows from the definition of t_0 that

$$|\nabla^\perp \mathcal{W}(f) - \nabla^\perp \mathcal{W}(f^*)| \leq C|f - f^*|,$$

and the conclusion follows immediately.
It remains to remove the smoothness assumption on f. For that purpose, it suffices to replace f, in the definition of φ above, by $\mathcal{C}^2$ solutions f^δ of (1) which converge towards f in $L^\infty H^1$ as $\delta \to 0$ and then to send δ to zero in the resulting inequality. The key point is that in the statement of Lemma 4, the constant C only depends on ρ_0, $\|f\|_{L^\infty H^1}$ and bounds on the second derivatives of φ with respect to the variable x only. □

Proof of Lemma 3 As for the proof of Lemma 2 we may assume without loss of generality that f is regular, the general case can then be obtained by approximation in $L^\infty H^1$. We apply Lemma 4 with

$$\varphi(x,z,t) = \chi_{\rho_0/4}(|x - f_k(z,t)|)\, (-\partial_{zz} f_k(z,t) + \nabla_k \mathcal{W}(f(z,t)) \cdot (f(z,t) - x)_k,$$

and then (implicitly) sum the resulting inequalities over k. This leads to the estimate

$$I_2(\tau) \leq I_2(0) + \int_0^\tau \int_0^L \partial_t\, (-\partial_{zz} f_k + \nabla_k \mathcal{W}(f)) \cdot (f - f^*)_k \, dz\, dt$$
$$+ \int_0^\tau \int_0^L \partial_z\, (-\partial_{zz} f_k + \nabla_k \mathcal{W}(f))^\perp \cdot \partial_z f_k^* \, dz\, dt$$
$$+ \int_0^\tau \int_0^L (\partial_{zz} f_k - \nabla_k \mathcal{W}(f)) \cdot \nabla_k^\perp \mathcal{W}(f^*)\, dz\, dt + C \int_0^\tau I_3(t)\, dt.$$

The middle integral on the right-hand side can be rewritten

$$\int_0^\tau \int_0^L \partial_z \partial_t f_k \cdot \partial_z f_k^* \, dz \, dt = - \int_0^\tau \int_0^L \partial_{tzz} f_k \cdot f_k^* \, dz \, dt,$$

and hence cancels out part of the first integral. We then integrate by parts and expand $\partial_t \nabla_k \mathcal{W}(f)$ to obtain

$$\begin{aligned} I_2(\tau) \leq I_2(0) &- \int_0^\tau \int_0^L \partial_t f_j \cdot \partial_{zz} f_j \, dz \, dt \\ &+ \int_0^\tau \int_0^L \partial_t f_j \cdot \nabla_j \nabla_k \mathcal{W}(f) \cdot (f - f^*)_k \, dz \, dt \\ &+ \int_0^\tau \int_0^L (\partial_{zz} f_k - \nabla_k \mathcal{W}(f)) \cdot \nabla_k^\perp \mathcal{W}(f^*) \, dz \, dt + C \int_0^\tau I_3(t) \, dt. \end{aligned}$$

Using the PDE (39) to eliminate $\partial_{zz} f$, we rewrite this as

$$\begin{aligned} I_2(\tau) \leq I_2(0) &+ C \int_0^\tau I_3(t) \, dt \\ &+ \int_0^\tau \int_0^L \partial_t f_j \cdot \left[\nabla_j \mathcal{W}(f^*) - \nabla_j \mathcal{W}(f) - \nabla_k \nabla_j \mathcal{W}(f) \cdot (f^* - f)_k \right] dz \, dt. \end{aligned}$$

Finally, it follows from the definition of t_0 that

$$\left| \nabla_j \mathcal{W}(f^*) - \nabla_j \mathcal{W}(f) - \nabla_k \nabla_j \mathcal{W}(f) \cdot (f^* - f)_k \right| \leq C |f^* - f|^2.$$

The conclusion of the lemma follows immediately. □

3 Proofs of variational results

In this section we present the proofs of Propositions 1, 2, and 3.

3.1 Tools

We start by assembling some tools that give information about the vortex structure of a function satisfying (10), (11) for small but fixed $\varepsilon > 0$, rather than in the limit $\varepsilon \to 0$. All of these are established in [7], but in some cases our presentation here differs a little. We therefore give short proofs that sketch the arguments needed to obtain the precise statements given here from those in [7].

Our first result of this sort states that under assumptions (10), (11), for *every* $z \in (0, L)$, if ε is small enough then $u_\varepsilon(\cdot, z)$ has either n distinct, well-localized vortices clustered near the vertical axis, or a certain amount of "extra energy". We will write

$$e_\varepsilon^{2d}(u) := \frac{1}{2} |\nabla_x u_\varepsilon|^2 + \frac{1}{4\varepsilon^2} (|u_\varepsilon|^2 - 1)^2$$

the Ginzburg–Landau energy density with respect to horizontal variables.

Lemma 5 *Assume that* $u_\varepsilon \in H^1(\Omega, \mathbb{C})$ *satisfies* (10) *and* (11).

There exist positive numbers θ, a, b, C and ε_0 depending on n, c_1, c_2 such that $b < a$, and if $0 < \varepsilon < \varepsilon_0$, then for every $z \in (0, L)$ such that

$$\int_{\omega\times\{z\}} e_\varepsilon^{2d}(u_\varepsilon)\,dx \le \pi(n+\theta)|\log\varepsilon|\ , \tag{40}$$

there exist $g_j^\varepsilon(z) \in \mathbb{R}^2$ for $j = 1, \ldots, n$ such that

$$\|J_x u_\varepsilon(\cdot, z) - \pi\sum_{j=1}^{n}\delta_{g_j^\varepsilon(z)}\|_{F(\omega)} \le \varepsilon^a\ , \tag{41}$$

$$|g_j^\varepsilon(z) - g_k^\varepsilon(z)| \ge \varepsilon^b \quad \textit{for all } j \ne k, \qquad \textit{dist}(g_j^\varepsilon(z), \partial\omega) \ge C^{-1} \quad \textit{for all } j, \tag{42}$$

$$|g_j^\varepsilon(z)| \le Ch_\varepsilon \quad \textit{for all } j, \tag{43}$$

$$\int_{\omega\times\{z\}} e_\varepsilon^{2d}(w)dx \ \ge\ n(\pi|\log\varepsilon| + \gamma) + W_\omega(g_1^\varepsilon(z), \ldots, g_n^\varepsilon(z)) - C\varepsilon^{(a-b)/2}, \tag{44}$$

where W_ω is the renormalized energy defined in Sect. 1.2.

Proof of Lemma 5, excluding estimate (43) Given a sequence of functions $u_\varepsilon \in H^1(\Omega, \mathbb{C})$ satisfying (10) and (11), a set $\mathcal{G}_1^\varepsilon = \mathcal{G}_1^\varepsilon(u_\varepsilon) \subset (0, L)$ is defined in equation (3.11) of [7] with the following properties. First, if $z \notin \mathcal{G}_1^\varepsilon$ then

$$\int_\omega e_\varepsilon^{2d}(u_\varepsilon)(x, z)\,dx \ge \varepsilon^{-1/2},$$

for all sufficiently small ε (where “sufficiently small” may depend on the given sequence). And second, if $z \in \mathcal{G}_1^\varepsilon$ and (40) holds, then there exist $g_j^\varepsilon(z) \in \omega$, for $j = 1, \ldots, n$, satisfying (41), (44) and (42). These are proved in [7], Proposition 1 and Lemma 3 respectively, which actually assume a somewhat weaker condition in place of (10).

The conclusions of the lemma, apart from (43) (proved below), follow directly from these facts. □

We will henceforth write

$$\mathcal{G}(u_\varepsilon) := \{z \in (0, L) : (40) \text{ holds}\}, \qquad \mathcal{B}(u_\varepsilon) := (0, L) \setminus \mathcal{G}(u_\varepsilon)\ . \tag{45}$$

Thus, for every $z \in \mathcal{G}(u_\varepsilon)$, Lemma 5 provides a detailed description of the vorticity of $u_\varepsilon(\cdot, z)$.

For $z \in \mathcal{G}(u_\varepsilon)$ we will write

$$f_j^\varepsilon(z) := g_j^\varepsilon(z)/h_\varepsilon. \tag{46}$$

Rescaling (41), we find that $\|J_x v_\varepsilon(\cdot, z) - \pi\sum_{j=1}^n \delta_{f_j^\varepsilon(z)}\|_{W^{-1,1}(\omega_\varepsilon)} \le \varepsilon^a/h_\varepsilon$, where $v_\varepsilon(x, z) = u_\varepsilon(h_\varepsilon x, z)$ as usual.

Remark 1 It is clear from the proof in [7] that $z \mapsto \chi_{\mathcal{G}(u_\varepsilon)} g_j^\varepsilon(z)$ may be taken to be measurable.

We next collect some conclusions that follow rather easily from Lemma 5.

Lemma 6 *Assume that* $0 < \varepsilon < 1/2$ *and that* $u_\varepsilon \in H^1(\Omega, \mathbb{C})$ *satisfies* (10) *and* (11)*. Then there exists a positive constant* $C = C(c_1, c_2, n)$ *such that*

$$\int_\Omega e_\varepsilon^{2d}(u_\varepsilon) \ge n\pi L|\log\varepsilon| + \pi n(n-1)L|\log h_\varepsilon| - C, \tag{47}$$

$$|\mathcal{B}(u_\varepsilon)| \le C|\log\varepsilon|^{-1}, \tag{48}$$

$$\int_{z\in\mathcal{B}(u_\varepsilon)} \int_\omega e_\varepsilon^{2d}(u_\varepsilon)\, dx\, dz \le C, \tag{49}$$

$$\int_\Omega |\partial_z u_\varepsilon|^2 dx\, dz \le C. \tag{50}$$

We will later improve on some of these estimates under the hypotheses of our main theorem.

Proof of Lemma 6 Conclusions (47) and (50) are proved in Lemma 9 of [7]. The proof relies on the parts of Lemma 5 proved above, together with properties of the renormalized energy W_ω (see Lemma 4 of [7]) and a short argument using Jensen's inequality. The proof also easily yields the other conclusions (48), (49) stated here. Indeed, the proof of Lemma 9 in [7] actually shows[6] that

$$\int_{z\in\mathcal{G}(u_\varepsilon)} \int_\omega e_\varepsilon^{2d}(u_\varepsilon)\, dx\, dz \ge \Big(n\pi|\log\varepsilon| + n(n-1)\pi(|\log h_\varepsilon| - C\Big)|\mathcal{G}(u_\varepsilon)|.$$

On the other hand it is clear from the definitions that

$$\int_{z\in\mathcal{B}(u_\varepsilon)} \int_\omega e_\varepsilon^{2d}(u_\varepsilon)\, dx\, dz \ge (n\pi + \theta)|\log\varepsilon|\,|\mathcal{B}(u_\varepsilon)|.$$

Since $e_\varepsilon(u_\varepsilon) = e_\varepsilon^{2d}(u_\varepsilon) + \frac{1}{2}|\partial_z u_\varepsilon|^2$ and $|\mathcal{G}(u_\varepsilon)| + |\mathcal{B}(u_\varepsilon)| = L$, by comparing these estimates with the hypothesis (11), we easily obtain (48) and (49). □

We now state a result that establishes a sort of approximate equicontinuity of the map $z \in \mathcal{G}(u_\varepsilon) \mapsto \pi \sum \delta_{f_j^\varepsilon(z)}$ for finite $\varepsilon > 0$.

Lemma 7 *Assume that* (10)*,* (11) *hold. Then for every* $\delta > 0$*, there exists positive constants* ε_0, C *such that if* $0 < \varepsilon < \varepsilon_0$*, then the following holds:*

Assume that z_1, z_2 *are points in* $\mathcal{G}(u_\varepsilon)$ *such that* $|z_1 - z_2| > \delta$*, and let* $g_j^\varepsilon(z_\ell)$ *denote the points provided by Lemma 5 for* $\ell = 1, 2$*. Then for* $f_j^\varepsilon(z_\ell) := g_j^\varepsilon(z_\ell)/h_\varepsilon$*,*

$$\pi \min_{\sigma\in S_n} \sum_{j=1}^n \frac{|f_{\sigma(j)}^\varepsilon(z_2) - f_j^\varepsilon(z_1)|^2}{|z_2 - z_1|} \le C. \tag{51}$$

Proof of conclusion (43) of Lemma 5 and of Lemma 7 Estimate (43) is shown to hold in Step 3 of the proof of Lemma 12 in [7], via a compactness argument based on Lemma 8, see below.

Lemma 7 then follows from Lemma 8 by almost exactly the same compactness argument. The constant C appearing in (51) may be chosen to be a multiple of the uniform bound for $\int_\Omega |\partial_z u_\varepsilon|^2$, established in Lemma 6 and depending only on c_1, c_2 from (10), (11). □

6 Note that the sets $\mathcal{G}_2^\varepsilon$ and $\mathcal{B}_2^\varepsilon$ from [7] coincide exactly with our sets $\mathcal{G}(u_\varepsilon)$ and $\mathcal{B}(u_\varepsilon)$; compare our definitions (45) with [7], equation (3.16).

The last result in this section is the lemma used in the compactness arguments described above. It will be used again in the proof of Proposition 3. In [7] it provides the basic estimate that eventually implies that $z \mapsto f(z) = (f_1(z), \dots, f_n(z))$ belongs to $H^1((0, L), (\mathbb{R}^2)^n)$, see Proposition 1.

Lemma 8 *Assume that* (u_ε) *satisfies* (10), (11). *Let* $v_\varepsilon(x, z) := u_\varepsilon(h_\varepsilon x, z)$.

Assume that $\{z_1^\varepsilon\}$ *and* $\{z_2^\varepsilon\}$ *are sequences in* $[0, L]$ *such that* $z_j^\varepsilon \to z_j$ *for* $j = 1, 2$, *with* $0 \le z_1 < z_2 \le L$, *and that the following conditions hold for* $j = 1, 2$ *(perhaps after passing to a subsequence):*

$$J_x v_\varepsilon(\cdot, z_j^\varepsilon) \to \pi \sum_{i=1}^{n(z)} \delta_{p_i(z_j)} \qquad \text{in } W^{-1,1}(B(R)), \quad \text{for all } R > 0,$$

(for certain points $\{p_i(z_j)\}_{i=1}^{n(z_j)}$, *not necessarily distinct) and*

$$\limsup_{\varepsilon \to 0} |\log \varepsilon|^{-1} \int_\omega e_\varepsilon^{2d}(u_\varepsilon(x, z_j^\varepsilon)) dx \le M\pi$$

for some $M > 0$. *Then* $n(z_1) = n(z_2) =: m$, *and*

$$\frac{\pi}{2} \min_{\sigma \in S_m} \sum_{i=1}^{m} \frac{|p_i(z_1) - p_{\sigma(i)}(z_2)|^2}{z_2 - z_1} \le \liminf_{\varepsilon \to 0} \int_{z_1}^{z_2} \int_{\omega_\varepsilon} \frac{1}{2} |\partial_z u_\varepsilon|^2 \, dx \, dz.$$

Proof This is essentially Lemma 10 of [7]. Apart from some notational changes, the main difference is that Lemma 10 of [7] is proved under an assumption that is somewhat weaker than (10). As a result, it is stated there for a rescaling $v_\varepsilon(x, z) := u_\varepsilon(\ell_\varepsilon x, z)$ using a scaling factor ℓ_ε that is shown only later to equal h_ε. With the stronger assumption (10), the proof can be simplified, and one can work directly with the $\ell_\varepsilon = h_\varepsilon$. □

3.2 Proof of Proposition 1

Proof With a couple of exceptions, everything in Proposition 1 is taken directly from the statement of Theorem 3 in [7].

The first exception is the compactness assertion (13); in [7], compactness is proved to hold only with respect to a weaker topology. To prove (13), we argue as follows. First note that

$$\begin{aligned}
&\int_{z \in \mathcal{B}(u_\varepsilon)} \| J_x u_\varepsilon(\cdot, z) - \pi \sum_{j=1}^{n} \delta_{h_\varepsilon f_j(z)} \|_{W^{-1,1}(\omega)} dz \\
&\le n\pi |\mathcal{B}(u_\varepsilon)| + \int_{z \in \mathcal{B}(u_\varepsilon)} \| J_x u_\varepsilon(\cdot, z) \|_{W^{-1,1}(\omega)} dz \\
&\le n\pi |\mathcal{B}(u_\varepsilon)| + C |\log \varepsilon|^{-1} \int_{z \in \mathcal{B}(u_\varepsilon)} e_\varepsilon^{2d}(u_\varepsilon)(x, z) dz \\
&\le C |\log \varepsilon|^{-1} = C h_\varepsilon^2
\end{aligned} \tag{52}$$

by standard Jacobian estimates (see for example [17] or [25]) and Lemma 6, for $C = C(c_1, c_2, n)$. On the other hand, by (41) and (46),

$$\int_{z\in\mathcal{G}(u_\varepsilon)} \|J_x u_\varepsilon(\cdot,z) - \pi\sum_{j=1}^n \delta_{h_\varepsilon f_j(z)}\|_{W^{-1,1}(\omega)} dz \leq \int_{z\in\mathcal{G}(u_\varepsilon)} \|\pi\sum_{j=1}^n \delta_{h_\varepsilon f_j^\varepsilon(z)} - \pi\sum_{j=1}^n \delta_{h_\varepsilon f_j(z)}\|_{W^{-1,1}(\omega)} dz + C\varepsilon^a. \tag{53}$$

It is also shown in [7], Lemmas 13 and 14 that after passing to a suitable subsequence $\varepsilon_k \to 0$, there is a set $H_G \subset (0, L)$ of full measure, such that if $z \in H_G$, then there exists $\ell = \ell(z)$ such that $z \in \mathcal{G}(u_{\varepsilon_k})$ for all $k \geq \ell$, and

$$\|\pi\sum_{j=1}^n \delta_{f_j^{\varepsilon_k}(z)} - \pi\sum_{j=1}^n \delta_{f_j(z)}\|_{W^{-1,1}(B(R))} \to 0 \qquad \text{for every } R > 0$$

as $k \to \infty$. This implies that

$$\|\pi\sum_{j=1}^n \delta_{h_{\varepsilon_k} f_j^{\varepsilon_k}(z)} - \pi\sum_{j=1}^n \delta_{h_{\varepsilon_k} f_j(z)}\|_{W^{-1,1}(\omega)} = o(h_{\varepsilon_k}) \qquad \text{for every } z \in H_G$$

as $k \to \infty$. It also follows from (43) that

$$\|\pi\sum_{j=1}^n \delta_{h_{\varepsilon_k} f_j^{\varepsilon_k}(z)} - \pi\sum_{j=1}^n \delta_{h_{\varepsilon_k} f_j(z)}\|_{W^{-1,1}(\omega)} \leq Ch_{\varepsilon_k} \quad \text{for } z \in \mathcal{G}(u_{\varepsilon_k}) \setminus H_G,$$

so the conclusion follows from the dominated convergence theorem, together with (52) and (53).

The other assertion that is not taken directly from the statement of Theorem 3 in [7] is the estimate $\|f\|_{H^1} \leq C(c_1, c_2)$. To prove this, we use (5) to deduce that for $z \in H_G$,

$$\sum_i |f_i(z)| = \sum_i |f_i(z) - 0| = \lim_{k\to\infty} \frac{1}{h_\varepsilon} \|\sum_i \delta_{h_{\varepsilon_k} f_i(z)} - n\pi\delta_0\|_{W^{-1,1}(\omega)} = \lim_{k\to\infty} \frac{1}{\pi h_\varepsilon} \|J_x u_\varepsilon(\cdot,z) - n\pi\delta_0\|_{W^{-1,1}(\omega)}.$$

Thus Fatou's Lemma and (10) imply that

$$\|f\|_{L^1} \leq C(c_1).$$

We may then use Jensen's inequality and the fact from [7] that $G_0(f) \leq c_2$ to estimate

$$\begin{aligned} \frac{\pi}{2}\int_0^L \sum_j |f_j'|^2 dz &= G_0(f) + \pi\sum_{i\neq j}\int_0^L \log|f_i - f_j| dz \\ &\leq c_2 + L\pi\sum_{i\neq j} \log\left(\frac{1}{L}\int_0^L |f_i - f_j| dz\right) \\ &\leq C(c_1, c_2). \end{aligned}$$

Finally, $\|f\|_{L^2}$ is controlled by interpolating between $\|f\|_{L^1}$ and $\|f'\|_{L^2}$. □

3.3 Proof of Proposition 2

Proof of (15) It suffices to show, given any subsequence satisfying (11), (13) for which

$$|\log \varepsilon|^{-1}\partial_a v_\varepsilon \cdot \partial_b v_\varepsilon \rightharpoonup \text{some limit, weakly as measures}$$

that this limit can only equal $\pi\delta^{ab}\sum_i \delta_{f_i(z)} \otimes dz$. For $z \in (0, L)$, let

$$E_\varepsilon^{2d}(z) := \frac{1}{|\log \varepsilon|}\int_{\omega\times\{z\}} e_\varepsilon^{2d}(u_\varepsilon)\,dx = \frac{1}{|\log \varepsilon|}\int_{\omega_\varepsilon\times\{z\}} e_{\varepsilon'}^{2d}(v_\varepsilon)\,dx$$

where $\varepsilon' = \varepsilon/h_\varepsilon$. It follows from the definition of $\mathcal{B}(u_\varepsilon)$ that $E_\varepsilon^{2d}(z) \geq n\pi + \theta$ for $z \in \mathcal{B}(u_\varepsilon)$, and since (43) implies that $W_\omega(g_1^\varepsilon, \ldots, g_n^\varepsilon) \geq n\pi|\log h_\varepsilon| - C$, we deduce from (44) that $E_\varepsilon^{2d}(z) \geq n\pi - o(1)$ uniformly for $z \in \mathcal{G}(u_\varepsilon)$, as $\varepsilon \to 0$. On the other hand, the assumed energy scaling (11) implies that $\int_0^L E_\varepsilon^{2d}(z)\,dz \to n\pi L$ as $\varepsilon \to 0$. In view of these facts, after passing to a further subsequence if necessary, we may assume that

$$\frac{1}{|\log \varepsilon|}\int_{\omega_\varepsilon\times\{z\}} e_{\varepsilon'}^{2d}(v_\varepsilon)\,dx \to n\pi \qquad \text{for a.e. } z \in (0, L). \tag{54}$$

Next, upon rescaling (13) and passing to a further subsequence,

$$\|Jv_\varepsilon - \pi\sum_{i=1}^n \delta_{f_i(z)}\|_{W^{-1,1}(\omega_\varepsilon)} \to 0 \qquad \text{for a.e. } z \in (0, L). \tag{55}$$

It follows from Theorem 5 in [15] or Corollary 4 in [24] that whenever the above two conditions hold (*i.e.* a.e.),

$$\frac{1}{|\log \varepsilon|}\partial_a v_\varepsilon \cdot \partial_b v_\varepsilon(\cdot, z) \rightharpoonup \delta^{ab}\pi\sum_{i=1}^n \delta_{f_i(z)} \qquad \text{weakly as measures.}$$

Now fix any $\phi \in C_c(\mathbb{R}^2 \times [0, L])$, and let

$$\Phi_\varepsilon(z) := \frac{1}{|\log \varepsilon|}\int_{\omega_\varepsilon\times\{z\}} \phi(x, z)\partial_a v_\varepsilon \cdot \partial_b v_\varepsilon(x, z)\,dx.$$

We write $\Phi_\varepsilon = \Phi_{\mathcal{G},\varepsilon} + \Phi_{\mathcal{B},\varepsilon}$, where $\Phi_{\mathcal{G},\varepsilon} = \chi_{z\in\mathcal{G}(u_\varepsilon)}\Phi_\varepsilon(z)$. It follows immediately from (49) that $\Phi_{\mathcal{B},\varepsilon} \to 0$ in $L^1((0, L))$. We may assume after passing to a subsequence that $\chi_{\mathcal{B}(u_\varepsilon)} \to 0$ a.e.. It then follows that

$$\Phi_{\mathcal{G},\varepsilon}(z) \to \delta^{ab}\pi\sum_{i=1}^n \phi(f_i(z), z) \qquad \text{for a.e. } z.$$

The definition of $\mathcal{G}(u_\varepsilon)$ implies that $\sup_z |\Phi_{\mathcal{G},\varepsilon}(z)| \leq (n\pi + \theta)\sup_{(x,z)}|\phi(x, z)| \leq C$. Thus the dominated convergence theorem implies that

$$\lim_{\varepsilon\to 0}\int_0^L \Phi_\varepsilon(z)dz = \delta^{ab}\pi\sum_{i=1}^n \phi(f_i(z), z)\,dz.$$

This is (15). □

Proof of (17) For $\delta > 0$, let

$$\mathcal{I}_\delta := \{z \in (0, L) : \min_{i\neq j}|f_i(z) - f_j(z)| > \delta\}.$$

We know from (14) that $G_0(f) < \infty$, which implies that $|\mathcal{I}_\delta| \to L$ as $\delta \to 0$. It thus suffices to prove that for any nonnegative $\phi \in C_c(\mathbb{R}^2 \times [0, L])$ and for every $\delta > 0$,

$$\liminf_{\varepsilon \to 0} \int_{\omega_\varepsilon \mathcal{I}_\delta} \phi \frac{|\partial_z v_\varepsilon|^2}{|\log \varepsilon|}\, dx\, dz \geq \pi \sum_{i=1}^{n} \int_{\mathcal{I}_\delta} |f_i'(z)|^2 \phi(f_i(z), z)\, dz. \tag{56}$$

We may write $\mathcal{I}_\delta$ as a disjoint union of open intervals. Let I denote one such interval. In view of arguments in the proof of (15), it suffices to prove that if $f \in H^1(I, (\mathbb{R}^2)^n)$ is such that (54), (55) hold for a.e. $z \in I$ and $\min_{z \in I} \min_{i \neq h} |f_i(z) - f_j(z)| \geq \delta > 0$, then (56) is satisfied (with $\mathcal{I}_\delta$ replaced by I).

There are a number of proofs of this fact[7] when $\phi \equiv 1$; see for example [14] Proposition 3 or [24], Corollary 7. These proofs proceed by considering separately the energetic contributions associated to each trajectory $z \mapsto (f_i(z), z)$, and they show that for any $r > 0$, and every $i \in \{1, \ldots, n\}$, and every interval $J \subset I$

$$\liminf_{\varepsilon \to 0} \int_{z \in J} \int_{B_r(f_i(z))} \frac{|\partial_z v_\varepsilon|^2}{|\log \varepsilon|}\, dx\, dz \geq \pi \int_J |f_i'(z)|^2\, dz.$$

This easily implies the desired estimate. □

Proof of (16) First, recalling that $v_\varepsilon(x, z) = u_\varepsilon(h_\varepsilon x, z)$ and using (12),

$$\int_\Omega |\partial_z u_\varepsilon|^2\, dx\, dz = \int_\Omega \frac{|\partial_z v_\varepsilon|^2}{|\log \varepsilon|} dx\, dz \leq C(c_1, c_2, n).$$

We may thus assume that $|\log \varepsilon|^{-1} \partial_z v_\varepsilon \cdot \nabla_x v_\varepsilon$ converges weakly to a limiting $\mathbb{R}^2$-valued measure, say λ on $\mathbb{R}^2 \times [0, L]$.

Now fix some $g \in C^1((0, L), \mathbb{R}^2)$, and let

$$\tilde{u}_\varepsilon(x, z) := u_\varepsilon(x - h_\varepsilon g(z), z), \qquad \tilde{v}_\varepsilon(x, z) := \tilde{u}_\varepsilon(h_\varepsilon x, z) = v_\varepsilon(x - g(z), z).$$

If we fix some $\tilde{\omega} \subset\subset \omega$ such that $0 \in \tilde{\omega}$, we may then take the domain of $\tilde{u}_\varepsilon$ to be $\tilde{\Omega} := \tilde{\omega} \times (0, L)$, for all sufficiently small ε. (We remark that although we are ultimately interested in u_ε that is periodic in the z variable, here we do not assume that g is periodic.)

It is straightforward to check from (13) and the definition of $\tilde{u}_\varepsilon$ that

$$\int_0^L \| J_x \tilde{u}_\varepsilon(\cdot, z) - \pi \sum_{j=1}^{n} \delta_{h_\varepsilon(f_j(z) + g(z))} \|_{W^{-1,1}(\tilde{\omega})} dz = o(h_\varepsilon) \qquad \text{as } \varepsilon \to 0.$$

Also, since $h_\varepsilon = |\log \varepsilon|^{-1/2}$ and extending the definition (4) of G_ε to include a dependence in the domain, we have

$$\begin{aligned} G_\varepsilon(\tilde{u}_\varepsilon; \tilde{\Omega}) &\leq G_\varepsilon(u_\varepsilon; \Omega) + \int_\Omega \frac{|g'(z) \cdot \nabla_x u_\varepsilon|}{\sqrt{|\log \varepsilon|}} |\partial_z u_\varepsilon| + \frac{1}{2} \frac{|g'(z) \cdot \nabla_x u_\varepsilon|^2}{|\log \varepsilon|}\, dx\, dz \\ &\leq c_2 + C \int_\Omega |\partial_z u_\varepsilon|^2 + \frac{|\nabla_x u_\varepsilon|^2}{|\log \varepsilon|} dx\, dz \\ &\leq \tilde{K}_1 \end{aligned}$$

[7] These results assume that (54), (55) hold for *every* $z \in I$, but the proofs extend to our situation with essentially no change.

for some suitable $\tilde{K}_1$, whenever ε is sufficiently small. Thus (17) implies that for any continuous $\tilde{\phi} \geq 0$,

$$\liminf_{\varepsilon\to 0} \int \tilde{\phi}(x,z) \frac{|\partial_z \tilde{v}_\varepsilon(x,z)|^2}{|\log\varepsilon|}\, dx\, dz \geq \sum_i \pi \int_0^L |\partial_z(f_i+g)(z)|^2 \tilde{\phi}(f_i(z)+g(z),z)\, dz.$$

Taking $\tilde{\phi}$ of the form $\tilde{\phi}(x,z) = \phi(x-g(z),z)$, we get the more convenient expression

$$\liminf_{\varepsilon\to 0} \int \phi(x-g(z),z) \frac{|\partial_z \tilde{v}_\varepsilon(x,z)|^2}{|\log\varepsilon|}\, dx\, dz \geq \sum_i \pi \int_0^L |\partial_z(f_i+g)(z)|^2 \phi(f_i(z),z)\, dz.$$

On the other hand, by using the definition of $\tilde{v}_\varepsilon$ and making the change of variables $(x - g(x), z) \mapsto (x,z)$, we obtain

$$\begin{aligned}\int \phi(x-g(z),z)|\partial_z \tilde{v}_\varepsilon(x,z)|^2\, dx\, dz &= \int \phi(x,z)|\partial_z v_\varepsilon(x,z)|^2\, dx\, dz \\ &+ \int \phi(x,z)\left(-2g'(z)\cdot\nabla_x v_\varepsilon(x,z)\cdot\partial_z v_\varepsilon(x,z) + |g'(z)\cdot\nabla_x v_\varepsilon(x,z)|^2\right) dx\, dz.\end{aligned}$$

Dividing by $|\log\varepsilon|$, letting $\varepsilon\to 0$, and invoking (12) and (15), we find that

$$\begin{aligned}&\limsup_{\varepsilon\to 0} \int \phi(x-g(z),z) \frac{|\partial_z \tilde{v}_\varepsilon(x,z)|^2}{|\log\varepsilon|}\, dx\, dz \\ &\leq C - 2\int_{\mathbb{R}^2\times(0,L)} \phi(x,z) g'(z)\cdot d\lambda + \sum_i \pi \int_0^L |\partial_z g(z)|^2 \phi(f_i,z)\, dz.\end{aligned}$$

Combining this with the previous inequality and rewriting, we conclude that

$$\int_{\mathbb{R}^2\times(0,L)} \phi(x,z)g'(z)\cdot d\lambda + \pi\int_0^L \phi(x,z) g'(z)\cdot d\left(\sum_i f_i'(z)\delta_{f_i(z)}\otimes dz\right) \leq C$$

for g, ϕ as above, with C depending on c_1, c_2, f, n, ϕ but independent of g . Since we may multiply a given g by an arbitrary real constant, it follows that in fact

$$\int \phi(x,z)g'(z)\cdot d\lambda + \pi\int \phi(x,z)g'(z)\cdot d\left(\sum_i f_i'(z)\delta_{f_i(z)}\otimes dz\right) = 0$$

and hence that

$$\lambda = -\pi\sum_i f_i'(z)\delta_{f_i(z)}\otimes dz.$$

This is (16). □

3.4 Proof of Proposition 3

Define

$$\sigma_\varepsilon^{2d}(z) = \sigma_\varepsilon^{2d}(z; u_\varepsilon, h_\varepsilon f) = \int_\omega e_\varepsilon^{2d}(u_\varepsilon(x,z))dx - W_\varepsilon(h_\varepsilon f(z);\omega)$$

where for $a\in\omega^n$,

$$W_\varepsilon(a;\omega) = n(\pi|\log\varepsilon|+\gamma) - \pi\sum_{i\neq j}\log|a_i(z)-a_j(z)| + \pi\sum_{i,j} H_\omega(a_i,a_j)\ .$$

Recall that H_ω is defined in Sect. 1.2. We interpret $\sigma_\varepsilon^{2d}(z)$ as the surplus 2d (horizontal) energy of u_ε at height z, with respect to the vortex positions $h_\varepsilon f(z)$. Further define

$$\Sigma_\varepsilon^{2d} = \Sigma_\varepsilon^{2d}(u_\varepsilon, h_\varepsilon f) = \int_0^L \sigma_\varepsilon^{2d}(z)dz.$$

Proof of estimate (20) Assume toward a contradiction that there exists a sequence $(u_\varepsilon)_{\varepsilon\in(0,1]}$ in $H^1(\Omega, \mathbb{C})$ such that

$$\int_0^L \|Ju_\varepsilon(\cdot, z) - \pi \sum_{i=1}^n \delta_{h_\varepsilon f_i(z)}\|_{W^{-1,1}(\omega)} dz = o(h_\varepsilon)$$

and $G_\varepsilon(u_\varepsilon) - G_0(f) \le \Sigma_\varepsilon \le 1$, but

$$\limsup_{\varepsilon\to 0} \int_0^L \int_{\omega\setminus\cup_{i=1}^n B(h_\varepsilon f_i(x), h_\varepsilon r)} e_\varepsilon(|u_\varepsilon|) + \frac{1}{4}\left|\frac{ju_\varepsilon}{|u_\varepsilon|} - j^*_{h_\varepsilon f}\right|^2 - K_3\Sigma_\varepsilon > 0 \qquad (57)$$

for K_3 to be chosen in a moment, and depending only on $\|f\|_{H^1}$ and $r < \frac{1}{4}\rho_f$.

This sequence satisfies the hypotheses (10), (11) of Lemma 5 with and $c_1 = 1+n\pi L\|f\|_\infty$ and $c_2 = G_0(f) + 1$, which are both controlled by $\|f\|_{H^1}$ and r. Let $\theta = \theta(n, c_1, c_2)$ be the constant found in Lemma 5. We will obtain a contradiction to (57) with $K_3 = \frac{4}{\theta}n\pi + 4$, thereby proving (20) for that value of K_3.

For this choice of θ, we define sets $\mathcal{G}(u_\varepsilon)$ and $\mathcal{B}(u_\varepsilon)$ as in (45). For $z \in \mathcal{G}(u_\varepsilon)$, Lemma 5 provides points $g_j^\varepsilon(z)$ satisfying (41), (42) for $0 < \varepsilon < \varepsilon_0(n, \|f\|_{H^1}, \rho_f, \Sigma)$, with constants such as a in (41) depending on the same quantities.

Setting $f_j^\varepsilon(z) = h_\varepsilon^{-1} g_j^\varepsilon(z)$, it follows from (41) that

$$\int_{z\in\mathcal{G}(u_\varepsilon)} \|\sum_{i=1}^n \delta_{h_\varepsilon f_j^\varepsilon(z)} - \sum_{i=1}^n \delta_{h_\varepsilon f_i(z)}\|_{W^{-1,1}(\omega)} dz = o(h_\varepsilon) \quad \text{as } \varepsilon \to 0. \qquad (58)$$

Our first goal is to strengthen this to read

$$\sup_{z\in\mathcal{G}(u_\varepsilon)} \|\sum_{i=1}^n \delta_{h_\varepsilon f_j^\varepsilon(z)} - \sum_{i=1}^n \delta_{h_\varepsilon f_i(z)}\|_{W^{-1,1}(\omega)} = o(h_\varepsilon) \quad \text{as } \varepsilon \to 0. \qquad (59)$$

In brief, this follows from a compactness argument based on (58) and Lemma 8. Here are the details:

Assume toward a contradiction that (59) fails. Then there exists a (sub)sequence $\varepsilon \to 0$ and points $z_\varepsilon \in \mathcal{G}(u_\varepsilon)$ such that

$$\|\sum_{i=1}^n \delta_{h_\varepsilon f_j^\varepsilon(z_\varepsilon)} - \sum_{i=1}^n \delta_{h_\varepsilon f_i(z_\varepsilon)}\|_{W^{-1,1}(\omega)} \ge ch_\varepsilon > 0 \quad \text{for all } \varepsilon. \qquad (60)$$

It follows from (48) and (58) that for all sufficiently small terms in the same subsequence, we may find points $\zeta_\varepsilon \in \mathcal{G}(u_\varepsilon)$ such that

$$\|\sum_{i=1}^n \delta_{h_\varepsilon f_j^\varepsilon(\zeta_\varepsilon)} - \sum_{i=1}^n \delta_{h_\varepsilon f_i(\zeta_\varepsilon)}\|_{W^{-1,1}(\omega)} = o(h_\varepsilon), \quad \text{and} \quad \alpha < |z_\varepsilon - \zeta_\varepsilon| < 2\alpha$$

for some α to be fixed below. Extracting a further subsequence we may assume that $z_\varepsilon \to z$ and $\zeta_\varepsilon \to \zeta$, and that there exist $m \le n$ and $p_1, \ldots, p_m \in \mathbb{R}^2$ such that

$$\sum_{i=1}^n \delta_{f_i^\varepsilon(\zeta_\varepsilon)} \to \sum_{i=1}^n \delta_{f_i(\zeta)}, \quad \text{and} \quad \sum_{i=1}^n \delta_{f_i^\varepsilon(z_\varepsilon)} \to \sum_{i=1}^m \delta_{p_i(z)}$$

in $W^{-1,1}(B(R))$ for every $R > 0$. (In fact both limits hold in stronger topologies as well.) These facts and (41) imply that for $v_\varepsilon(x, z) := u_\varepsilon(h_\varepsilon x, z)$,

$$J_x v(\cdot, \zeta_\varepsilon) \to \pi \sum_{i=1}^n \delta_{f_i(\zeta)}, \qquad J_x v(\cdot, z_\varepsilon) \to \pi \sum_{i=1}^m \delta_{p_i(z)}$$

in the same topology. Then Lemma 8 and conclusion (12) from Proposition 1 imply that $m = n$ and that

$$\min_{\sigma \in S_n} \sum_{i=1}^n |f_i(\zeta) - p_{\sigma(i)}(z)|^2 \le |z - \zeta| C \le 2\alpha C.$$

(Here and below, the constant depends on f and Σ.) On the other hand, since f is Hölder continuous, it follows from (60) that

$$\min_{\sigma \in S_n} \sum_{i=1}^n |f_i(\zeta) - p_{\sigma(i)}(z))| \ge \min_{\sigma \in S_n} \sum_{i=1}^n |f_i(z) - p_{\sigma(i)}(z))| - nC|z - \zeta|^{1/2} \ge c - nC\alpha^{1/2}.$$

A contradiction is reached by choosing α sufficiently small, depending only on f, Σ, and c. This completes the proof of (59).

Next, we remark that in view of the fact that $\rho_f > 0$, it follows from (59) and (5) that the labels on f_i^ε may be chosen so that

$$\sup_{z \in \mathcal{G}(u_\varepsilon)} |f_i^\varepsilon(z) - f_i(z)| \to 0 \quad \text{as } \varepsilon \to 0. \tag{61}$$

We will write

$$\omega(z, \varepsilon, f) := \omega \setminus \cup_{i=1}^n B(h_\varepsilon f_i(z), h_\varepsilon r).$$

For $z \in \mathcal{G}(u_\varepsilon)$, Theorem 2 of [18], for which the main hypothesis is a consequence of (41), provides certain integral estimates on $\omega \setminus \cup_{i=1}^n B(h_\varepsilon f_i^\varepsilon(z), C\varepsilon^{a/2})$, where $a > 0$ comes from (41) and C depends on various ingredients that are fixed. It follows from (59) and (5) that if ε is sufficiently small, then for every $z \in \mathcal{G}(u_\varepsilon)$, this set contains $\omega(z, \varepsilon, f)$. Theorem 2 of [18] thus implies that for every $z \in \mathcal{G}(u_\varepsilon)$,

$$\int_{\omega(z,\varepsilon,f)\times\{z\}} e_\varepsilon^{2d}(|u_\varepsilon|) + \frac{1}{4}\left|\frac{ju_\varepsilon}{|u_\varepsilon|} - j_\omega^*(h_\varepsilon f^\varepsilon(z))\right|^2 dx$$
$$\le \int_{\omega\times\{z\}} e_\varepsilon^{2d}(u)\, dx - \left[n(\pi|\log\varepsilon| + \gamma) + W_\omega(h_\varepsilon f^\varepsilon(z))\right] + C\varepsilon^{a/2}.$$

We recall that W_ω is defined in Sect. 1.2. It is easy to check from the definition there that

$$n(\pi|\log\varepsilon| + \gamma) + W_\omega(h_\varepsilon f^\varepsilon(z)) = \pi\mathcal{W}(f^\varepsilon(z)) + \kappa(n, \varepsilon, \omega) + O(h_\varepsilon)$$

where $\mathcal{W}$ is introduced in (30). Thus

$$\int_{z\in\mathcal{G}(u_\varepsilon)}\int_{\omega(z,\varepsilon,f)\times\{z\}} e_\varepsilon^{2d}(|u_\varepsilon|)+\frac{1}{8}\left|\frac{ju_\varepsilon}{|u_\varepsilon|}-j^*_\omega(h_\varepsilon f(z))\right|^2 dx\,dz$$
$$\le \int_{z\in\mathcal{G}(u_\varepsilon)}\int_{\omega(z,\varepsilon,f)\times\{z\}} \frac{1}{4}\left|j^*_\omega(h_\varepsilon f(z))-j^*_\omega(h_\varepsilon f^\varepsilon(z))\right|^2 dx\,dz$$
$$+\int_{z\in\mathcal{G}(u_\varepsilon)}\left(\int_{\omega\times\{z\}} e_\varepsilon^{2d}(u)\,dx-\kappa(n,\varepsilon,\omega)-\pi\mathcal{W}(f^\varepsilon(z))\right)dz+O(h_\varepsilon).$$

It follows from (61) and Lemma 9 below that the first term on the right-hand side vanishes as $\varepsilon\to 0$. Using this, we add and subtract various terms to rewrite the above inequality as

$$\int_{z\in\mathcal{G}(u_\varepsilon)}\int_{\omega(z,\varepsilon,f)\times\{z\}} e_\varepsilon^{2d}(|u_\varepsilon|)+\frac{1}{8}\left|\frac{ju_\varepsilon}{|u_\varepsilon|}-j^*_\omega(h_\varepsilon f(z))\right|^2 dx\,dz$$
$$\le G_\varepsilon(u_\varepsilon)-G_0(f)-\left(\int_\Omega\frac{|\partial_z u_\varepsilon|^2}{2}\,dx\,dz-\frac{\pi}{2}\int_0^L|f'(z)|^2\,dz\right) \tag{62}$$
$$-\int_{z\in\mathcal{B}(u_\varepsilon)}\left(\int_{\omega\times\{z\}} e_\varepsilon^{2d}(u)\,dx-\kappa(n,\varepsilon,\omega)-\pi\mathcal{W}(f(z))\right)dz+o(1).$$

Clearly $|\mathcal{W}(f)|$ is bounded by a constant depending on n, ρ_0 and $\|f\|_{H^1}$, and it follows that $\kappa(n,\varepsilon,\omega)+\pi\mathcal{W}(f(z))\le(\pi n+\frac{\theta}{2})|\log\varepsilon|$ for all sufficiently small ε. Then the definition of $\mathcal{B}(u_\varepsilon)$ implies that $\int_{\omega\times\{z\}}e_\varepsilon^{2d}(u)\,dx-\kappa(n,\varepsilon,\omega)-\pi\mathcal{W}(f(z))\ge\frac{\theta}{2}|\log\varepsilon|$ when $z\in\mathcal{B}(u_\varepsilon)$. Taking ε smaller, if necessary, we may assume by (17) that

$$\int_\Omega\frac{|\partial_z u_\varepsilon|^2}{2}\,dx\,dz-\frac{\pi}{2}\int_0^L|f'(z)|^2\,dz\ge-\varpi\delta$$

for $\varpi>0$ to be chosen. Employing this in (62) and discarding the left-hand side, we deduce that

$$|\mathcal{B}(u_\varepsilon)|\le\frac{4}{\theta}(\Sigma_\varepsilon+\varpi\delta)|\log\varepsilon|^{-1}$$

for all sufficiently small $\varepsilon>0$. Returning to (62) with this new information, we deduce that

$$\int_{z\in\mathcal{B}(u_\varepsilon)}\int_{\omega\times\{z\}}e_\varepsilon^{2d}(u)\,dx\,dz\le\Sigma_\varepsilon+\varpi\delta+\frac{4}{\theta}(\Sigma_\varepsilon+\varpi\delta)(n\pi+\frac{\theta}{2})$$
$$\le(3+\frac{4n\pi}{\theta})\Sigma_\varepsilon+\frac{\delta}{4}+o(1)$$

provided $\varpi\le\frac{1}{4}$ is chosen small enough, depending only on n and θ, which itself is universal. Then, since

$$e_\varepsilon^{2d}(|u_\varepsilon|)+\frac{1}{8}\left|\frac{ju_\varepsilon}{|u_\varepsilon|}-j^*_\omega(h_\varepsilon f(z))\right|^2\le e_\varepsilon^{2d}(u)+\frac{1}{4}|j^*_\omega(h_\varepsilon f(z))|^2,$$

we use (62) and the above estimate of $|\mathcal{B}(u_\varepsilon)|$ to find that

$$\int_0^L\int_{\omega(z,\varepsilon,f)\times\{z\}}e_\varepsilon^{2d}(|u_\varepsilon|)+\frac{1}{8}\left|\frac{ju_\varepsilon}{|u_\varepsilon|}-j^*_\omega(h_\varepsilon f(z))\right|^2dx\,dz$$
$$\le(4+\frac{4n\pi}{\theta})\Sigma_\varepsilon+\frac{\delta}{2}+\int_{z\in\mathcal{B}(u_\varepsilon)}\int_{\omega(z,\varepsilon,f)\times\{z\}}\frac{1}{4}|j^*_\omega(h_\varepsilon f(z))|^2dx\,dz+o(1).$$

Finally,

$$\int_{\omega(z,\varepsilon,f)\times\{z\}} \frac{1}{4}|j^*_\omega(h_\varepsilon f(z))|^2 dx\, dz \le C|\log h_\varepsilon| = o(|\log\varepsilon|)$$

for a constant that depends only on n and $\|f\|_{H^1}$ and r; this can be verified by arguments similar to those in Lemma 9 below. Using this in the above inequality, we conclude that

$$\int_0^L \int_{\omega(z,\varepsilon,f)\times\{z\}} e^{2d}_\varepsilon(|u_\varepsilon|) + \frac{1}{8}\left|\frac{ju_\varepsilon}{|u_\varepsilon|} - j^*_\omega(h_\varepsilon f(z))\right|^2 dx\, dz \le \left(\frac{4}{\theta}n\pi + 4\right)\Sigma_\varepsilon + \frac{3}{4}\delta$$

for all sufficiently small ε. This contradicts (57) and completes the proof of (20). □

Note that one can repeat the above proof with essentially no change, after replacing f in (57) and the two preceding assumptions by a sequence $\tilde f^\varepsilon$ with a uniform upper bound on $\|\tilde f^\varepsilon\|_{H^1}$ and the uniform lower bound on $\rho_{\tilde f^\varepsilon} \ge 4r$, for r fixed. Then essentially[8] the same argument as above leads to the same contradiction, establishing (20) with ε_3, c_3 that depend only on $\|f\|_{H^1}$ and r.

Next is the lemma that was used above.

Lemma 9 *Assume that $a, a' \in \omega^n$ and that there exist $r_0 \ge r_1 > 0$ such that*

$$\operatorname{dist}(a_i, \partial\omega) > r_0 \qquad \text{and} \qquad |a_i - \tilde a_i| \le \frac{1}{2}r_1 \le \frac{1}{4}\rho_a \quad \text{for all } i.$$

Then

$$\int_{\omega\setminus\cup B_{r_1}(a_i)} |j^*_\omega(a) - j^*_\omega(a')|^2\, dx \le C(n, r_0, \omega)|a-a'|^2 + C(n)\left(\frac{|a-a'|}{r_1}\right)^2.$$

In particular, the above constants are independent of r_1.

Proof Using notation from Sect. 1.2,

$$|j^*_\omega(x;a) - j^*_\omega(x;a')|^2 \le 2n\sum_i \left|\frac{x-a_i}{|x-a_i|^2} - \frac{x-a'_i}{|x-a'_i|^2}\right|^2 + 2n\sum_i |\nabla H_\omega(x,a_i) - \nabla H_\omega(x,a'_i)|^2.$$

The definition of H_ω and the maximum principle imply that

$$|\nabla H_\omega(x,a_i) - \nabla H_\omega(x,a'_i)| \le C(r_0)|a_i - a'_i|,$$

and a short computation shows that if $|x-a| \ge 2|a-a'|$, then

$$\left|\frac{x-a_i}{|x-a_i|^2} - \frac{x-a'_i}{|x-a'_i|^2}\right|^2 \le 4\frac{|a_i-a'_i|^2}{|x-a_i|^4}.$$

Thus

$$\int_{\omega\setminus\cup B_{r_1}(a_i)} |j^*_\omega(x;a) - j^*_\omega(x;a')|^2 \le 2n|a-a'|^2 \int_{\mathbb{R}^2\setminus B_{r_1}(0)} |x|^{-4}dx + C(n, r_0, \omega)|a-a'|^2$$

from which the conclusion of the lemma follows.

[8] after extracting a uniformly convergent subsequence of $\{\tilde f^\varepsilon\}$

Proof of (22) Assume toward a contradiction that there is a subsequence along which (18), (19) and (21) hold for every ε, but there exists $\eta_1 > 0$ such that

$$\lim_{\varepsilon\to 0} h_\varepsilon^{-1} \int_0^L \|J_x u_\varepsilon(\cdot,z) - \pi\sum_{i=1}^h \delta_{h_\varepsilon f_i(z)}\|_{F(\omega)} dz \geq \lim_{\varepsilon\to 0} \left(\pi n L(T^f_{r,\varepsilon}(u_\varepsilon)+\eta_1)\right)^{\frac{1}{2}} \\ =: (\pi n L(T_{lim}+\eta_1))^{1/2}. \tag{63}$$

Clearly (18), (19) imply that the hypotheses of Proposition 1 are satisfied (with a larger constant in (10) than in (18)), so we may use the proposition to find a subsequence, still denoted (u_ε), and a function $f^0 \in H^1((0,L),(\mathbb{R}^2)^n)$ such that

$$\int_0^L \|J_x u_\varepsilon(\cdot,z) - \pi\sum_{j=1}^n \delta_{h_\varepsilon f_j^0(z)}\|_{W^{-1,1}(\omega)} dz = o(h_\varepsilon) \tag{64}$$

as $\varepsilon \to 0$.

We will first show that, after choosing c_3 suitably small and possibly relabelling,

$$\|f_j - f_j^0\|_{L^\infty((0,L))} \leq r \quad \text{for } j = 1,\ldots,n. \tag{65}$$

We start by noting from (18), (63), and (64) that

$$(\pi n L(T_{lim}+\eta_1))^{\frac{1}{2}} \leq \lim_{\varepsilon\to 0} \frac{1}{h_\varepsilon} \int_0^L \|\pi \sum_j (\delta_{h_\varepsilon f_j(z)} - \delta_{h_\varepsilon f_j^0(z)})\|_{W^{-1,1}(\omega)} \leq c_3.$$

It follows from (5) that for all sufficiently small ε and all z,

$$\|\pi \sum_j (\delta_{h_\varepsilon f_j^0(z)} - \delta_{h_\varepsilon f_j(z)})\|_{W^{-1,1}(\omega)} = \pi h_\varepsilon \min_{\sigma\in S_n} \sum_j |f_j(z) - f^0_{\sigma(j)}(z)|.$$

Thus

$$(\pi n L T_{lim})^{\frac{1}{2}} + \eta_1 \leq \pi \int_0^L \min_{\sigma\in S_n} \sum_j |f_j(z) - f^0_{\sigma(j)}(z)|\, dz \leq c_3. \tag{66}$$

In particular, this implies that

$$\|f\|_{L^1} \leq C(f^0, c_3).$$

It follows from a Sobolev embedding and (14) that there exists $C = C(f^0, c_2, c_3)$ such that

$$[f]_{C^{0,1/2}} \leq \|f'\|_{L^2} \leq C, \qquad \text{and thus } [f - f^0]_{C^{0,1/2}} \leq C. \tag{67}$$

Next, we deduce from (66) and Chebyshev's inequality that

$$\left|\left\{ z \in (0,L) : \min_{\sigma\in S_n} \sum_j |f_j(z) - f^0_{\sigma(j)}(z)| > r/2 \right\}\right| \leq \frac{2c_3}{r}.$$

If $\min_{\sigma\in S_n} \sum_j |f_j(z_0) - f^0_{\sigma(j)}(z_0)| > r$ for any $z_0 \in (0,L)$, then it follows from (67) that

$$\min_{\sigma\in S_n} \sum_j |f_j(z) - f^0_{\sigma(j)}(z)| > r/2 \quad \text{for all } z \in (0,L) \text{ such that } |z - z_0| < r^2/C.$$

Fixing c_3 small enough (which only decreases the constant $C(f_0, c_2, c_3)$ in (67)), we can arrange that the two above estimates are incompatible. (This adjustment to c_3 again depends only on $\rho_f \geq 4r$ and $\|f\|_{H^1}$.) It follows that for this choice of c_3,

$$\min_{\sigma \in S_n} \sum_j |f_j(z) - f^0_{\sigma(j)}(z)| \leq r \qquad \text{for every } z \in (0, L).$$

As a result, we can find a single permutation π, independent of z, such that $\sum_j |f_j(z) - f^0_{\pi(j)}(z)| = \min_\sigma \sum_j |f_j(z) - f^0_{\sigma(j)}(z)| \leq r$ for all z. Using this permutation π to relabel the indices, we obtain (65).

If we write $\varphi(x) := \chi_r(\frac{|x|}{h_\varepsilon})(\frac{|x|}{h_\varepsilon})^2$, then since $\|\nabla_x \varphi\|_\infty \leq C/h_\varepsilon$, it follows from (18), (64) that

$$T_{lim} = \pi \sum_{i,j} \int_0^L \chi_r(|f_j(z) - f^0_i(z)|)|f_j(z) - f^0_i(z)|^2 \, dz.$$

However, since $|f^0_i - f^0_j| \geq 4r$, we see from (65) that

$$\chi_r(|f_j(z) - f^0_i(z)|) = \delta^{ij} \quad \text{for all } i, j \text{ and all } z \in (0, L).$$

So we obtain

$$\pi \|f - f^0\|^2_{L^2} = T_{lim}.$$

On the other hand, since we have by now arranged that

$$\min_\sigma \sum_j |f_j(z) - f^0_{\sigma(j)}(z)| = \sum_j |f_j(z) - f^0_j(z)| \leq \sqrt{n}|f(z) - f^0(z)| \quad \text{for all } z,$$

we pass to the limit in (63) to find that

$$\sqrt{n\pi L}(\sqrt{\pi}\|f - f^0\|_{L^2} + \eta_1) \leq \sqrt{n\pi}\|f - f^0\|_{L^1},$$

in contradiction to the Cauchy-Schwarz inequality. Thus (22) holds. □

4 Compactness in time

In this last section we present the proofs of Proposition 4, Corollary 1 and Proposition 5.

4.1 Proof of Proposition 4

Proof We only need to prove (26), since all other conclusions follow from that and Proposition 3.

To prove (26), we define the stopping time

$$t^* := \sup\{t > 0 : u_\varepsilon(\cdot, \cdot, h^2_\varepsilon s) \text{ satisfies (18), (21) for all } s \in (0, t)\}$$

where f should be replaced by g in (18), (21). By a change of variables,

$$T^g_{r,\varepsilon}(u_\varepsilon(\cdot, \cdot, h^2_\varepsilon t)) = T^g_r(v_\varepsilon(\cdot, \cdot, t)),$$

where $T_r^g := T_{r,1}^g$ and $u_\varepsilon, v_\varepsilon$ are related by (32). We use (35) with $\varphi(x,z,t) = \chi_r^g(x,z)$ to find that

$$\frac{d}{dt} T_r^g(v_\varepsilon(\cdot,\cdot,t)) \le \left| \int \varepsilon_{ab} \partial_{ac} \chi_r^g \, \partial_b v_\varepsilon \cdot \partial_c v_\varepsilon \, dx\, dz \right| + \left| \int \varepsilon_{ab} \partial_{az} \chi_r^g \frac{\partial_b v_\varepsilon \cdot \partial_z v_\varepsilon}{|\log \varepsilon|} \, dx\, dz \right|.$$

The definition of χ_r^g and assumption $r \le \rho_r/4$ implies that $\partial_{ac}\chi_r^g(x,z) = 2\delta^{ac}$ when $|x - g_i(z)| < r$ for some i, and hence that

$$\varepsilon_{ab} \partial_{ac} \chi_r^g \, \partial_b v_\varepsilon \cdot \partial_c v_\varepsilon = 0 \text{ in } \cup_i B(g_i(z), r).$$

In addition,

$$|\nabla_x v_\varepsilon|^2 \le 2e_\varepsilon(|v_\varepsilon|) + \frac{|j(v_\varepsilon)|^2}{|v_\varepsilon|^2} \le 2e_\varepsilon(|v_\varepsilon|) + 2\left| \frac{ju_\varepsilon}{|u_\varepsilon|} - j_\omega^*(h_\varepsilon g) \right|^2 + 2\left| j_\omega^*(h_\varepsilon g)\right|^2.$$

The definition of t^* allows us to apply estimates from Proposition 1 (with $c_1 = c_4 + n\pi L\|g\|_\infty$ and $c_2 = G_0(g) + 1$) and Proposition 3 (with $\delta = \Sigma = 1$ for example) to $v_\varepsilon(\cdot,\cdot,t)$, for any $t \in (0, t^*)$, as long as c_4, ε_4 are taken to be small enough, depending only on $\|g\|_{H^1}$, n and r. We may therefore deduce from (20) that

$$\left| \int \varepsilon_{ab} \partial_{ac} \chi_r^g \, \partial_b v_\varepsilon \cdot \partial_c v_\varepsilon \, dx\, dz \right| \le C(K_3 + 1)\|\nabla_x^2 \chi_r^g\|_\infty = C(r, n, g).$$

The remaining integral on the right-hand side is estimated by using (12) (which after rescaling to v_ε acquires a factor of $|\log \varepsilon|^{-1}$) to find that

$$\begin{aligned} \left| \int \varepsilon_{ab} \partial_{az} \chi_r^g \frac{\partial_b v_\varepsilon \cdot \partial_z v_\varepsilon}{|\log \varepsilon|} \, dx\, dz \right| &\le \frac{1}{|\log \varepsilon|} \|\nabla_x \partial_z \chi_r^g\|_{L^\infty} \|\nabla_x v_\varepsilon\|_{L^2} \|\nabla_z v_\varepsilon\|_{L^2} \\ &\le \|g\|_{Lip} C(c_1, c_2, n). \end{aligned}$$

Thus

$$\frac{d}{dt} T_r^g(v_\varepsilon(\cdot,\cdot,t)) \le C(r, n, \|g\|_{H^1}) + \|g\|_{Lip} C(c_1, c_2, n) =: C_4.$$

It follows that (26) holds for all $t \in (0, t^*)$. Then, thanks to (27) and (28), we conclude that $t^* \ge t_4$, completing the proof of (26). □

4.2 Proof of Corollary 1

Proof Since $f(0)$ may not be a Lipschitz function, we first mollify it to a function which we call g and which we require to satisfy $\sup_{i,z} |f_i(0,z) - g_i(z)| < \alpha\rho_{f(0)}$ for some $\alpha < 1/8$ to be chosen, and thus $\rho_g > (1 - 2\alpha)\rho_{f(0)}$. Since $f(0)$ is already in H^1, we have that $|g|_{H^1} \le |f(0)|_{H^1}$. Proposition 4, applied to g, $r = \rho_g/4$, provides us with constants $\varepsilon_4, t_4, c_4, C_4$, the important point being that ε_4 and c_4 do *not* depend on the strength of the mollification. Without loss of generality, we may also assume that $c_4 \le \frac{1}{8}\rho_{f(0)}$. In view of the assumptions of Theorem 1, we may assume, decreasing the value of ε_4 if necessary, that (23) and (24) hold for every $\varepsilon \le \varepsilon_4$. Finally, it is clear that $||\chi_{r,\varepsilon}^g(\cdot, z)\|_{W^{1,\infty}(\omega)} \le C(r)h_\varepsilon^{-1}$ for every $z \in (0, L)$, so assumption (6) implies that $\limsup_{\varepsilon \to 0} T_{r,\varepsilon}^g(u_\varepsilon^0) \le \pi\|f(0) - g\|_{L^2}^2$. We may therefore assume, decreasing ε_4 further if necessary, that $T_{r,\varepsilon}^g(u_\varepsilon^0) \le 2\pi\|f(0) - g\|_{L^2}^2 \le 2n\pi^2\alpha^2 L\rho_{f(0)}^2$ for every $\varepsilon \le \varepsilon_4$, and in particular that (25) holds. In view of (23) and (28), we may then apply Proposition 1 for each fixed time $t \in [0, t_4]$ and derive some limiting $f^*(t)$ after passing to a possible subsequence.

The potential difficulty at this level is that the subsequence may depend on the value of t; to overcome this we will rely on the form of continuity in time provided by estimate (26). We first derive some estimates that apply to any limit $f^*(t)$ produced by the above argument. Note that (27) and (13) imply that

$$\frac{1}{h_\varepsilon}\int_0^L \|\pi\sum_{i=1}^n \delta_{h_\varepsilon f_i^*(z,t)} - \pi\sum_{i=1}^n \delta_{h_\varepsilon g_i(z)}\|_{W^{-1,1}(\omega)}dz \le \left(n\pi L(T^g_{r,\varepsilon}(u^0_\varepsilon) + C_4 t)\right)^{\frac{1}{2}},$$

and (14) implies that $\|f^*(t)\|_{H^1} \le C(G_0(g))$. Using (5),

$$\int_0^L \min_{\sigma\in S_n}|f^*_{\sigma(i)}(z,t) - g_i(z)|dz \ \le \left(n\pi L(T^g_{r,\varepsilon}(u^0_\varepsilon) + C_4 t)\right)^{\frac{1}{2}}.$$

Since $f^*(t) - g$ is uniformly bounded in H^1, by choosing $t_0 \le t_4$ and α sufficiently small, we conclude that

$$\begin{aligned}\max_z \min_{\sigma\in S_n}|f^*_{\sigma(i)}(z,t) - f^0_i(z)| &< \max_z \min_{\sigma\in S_n}|f^*_{\sigma(i)}(z,t) - g_i(z)| + |f^0_i(z) - g_i(z)|\\ &\le \frac{1}{16}\rho_{f(0)}\end{aligned}$$

for all $t \in [0, t_0]$. It follows that there is a single permutation σ that attains the min for all z. After relabelling f^* if necessary, we deduce that (29) holds when $s = 0$. Finally, using the L^∞ continuity of $s \mapsto f(\cdot, s)$ and decreasing t_0 as needed, we deduce that (29) holds for all $s, t \in [0, t_0]$.

To prove continuity in time, we start by using a Cantor diagonal argument to fix a subsequence $\varepsilon \to 0$ such that

$$\int_0^L \|J_x u_\varepsilon(\cdot, z, h_\varepsilon^2 t) - \pi\sum_{j=1}^n \delta_{h_\varepsilon f_j^*(z,t)}\|_{W^{-1,1}(\omega)}dz = o(h_\varepsilon) \qquad \text{as } \varepsilon \to 0$$

for every time t in $\mathbb{Q}\cap[0, t_0]$. We claim that the mapping $t \mapsto f^*(t)$ is uniformly continuous from $\mathbb{Q}\cap[0, t_0]$ into $L^1([0, L])$. Indeed, let $\eta > 0$ be given, and let $s_0, s_1 \in \mathbb{Q}\cap[0, t_0]$ be arbitrary. We write

$$\sum_i \|f_i^*(s_0) - f_i^*(s_1)\|_{L^1} \le \sum_i \|f_i^*(s_0) - g_i^*(s_0)\|_{L^1} + \sum_i \|g_i^*(s_0) - f_i^*(s_1)\|_{L^1} \tag{68}$$

where $g^*(s_0)$ is a mollification of $f^*(s_0)$. It follows from (14) that $t \mapsto f^*(t)$ is uniformly bounded with values into H^1 , so we may fix the mollification parameter sufficiently fine, but independently of s_0, such that

$$\sum_i \|f_i^*(s_0) - g_i^*(s_0)\|_{L^1} \le \eta/2. \tag{69}$$

Next, we pass to the limit in the conclusions of Proposition 4 applied this time to $g = g^*(s_0)$ and conclude that

$$\begin{aligned} \pi \sum_i \|g_i^*(s_0) - f_i^*(s_1)\|_{L^1} \\ &= \lim_{\varepsilon \to 0} h_\varepsilon^{-1} \int_0^L \|J_x u_\varepsilon(\cdot, z, h_\varepsilon^2 s_1) - \pi \sum_{i=1}^n \delta_{h_\varepsilon g_i^*(z,s_0)}\|_{W^{-1,1}(\omega)} dz \\ &\le \lim_{\varepsilon \to 0} \left(n\pi L (T_{r,\varepsilon}^{g^*(s_0)}(u_\varepsilon^0) + C_4 |s_1 - s_0|) \right)^{\frac{1}{2}} \\ &\le \left(n\pi L (\pi \|g^*(s_0) - f^*(s_0)\|_{L^2}^2 + C_4 |s_1 - s_0|) \right)^{\frac{1}{2}}, \end{aligned} \tag{70}$$

where C_4 depends only on the mollification parameter. (We have implicitly used the fact that components of f^* have been labelled correctly, as reflected in (29).) We therefore further decrease the mollification parameter if necessary, yet independently of s_0, so that $n\pi^2 L \|g^*(s_0) - f^*(s_0)\|_{L^2}^2 \le \eta^2/32$. Once this, and hence C_4 are fixed, we require $|s_0 - s_1|$ to be small enough so that $n\pi L C_4 |s_1 - s_0| \le \eta^2/32$. Combining (69) and (70) in (68) yields the uniform continuity of f^*. In the sequel we denote still by f^* the unique continuous extension of f^* to the whole interval $[0, t_0]$. We claim that the conclusion of Corollary 1 holds for any $t \in [0, t_0]$, with no need of further subsequences. Indeed, this follows from the fact that for each fixed t in $[0, t_0]$ there exist at least some further subsequence for which the convergence to some $f^{**}(t)$ holds (this is by Proposition 1 as we already saw it), and on the other hand by our previous argument (equally applied to the countable set $(\mathbb{Q} \cap [0, t_0]) \cup \{t\}$) the only possible limit along *any* such subsequence is necessarily equal to $f^*(t)$. □

4.3 Proof of Proposition 5

Proof For $r, R > 0$, define

$$\mathcal{G}_{r,R} := \{(t, x, z) \in [0, t_0] \times B(R) \times [0, L] : |x - f_k^*(z, t)| \ge r, \ k = 1, \dots, n\}.$$

Given $\mathcal{O}$ as in the statement of the Proposition, we may fix $r, R > 0$ such that $\mathcal{O} \subset \mathcal{G}_{r,R}$. We will only consider ε small enough that $B(R) \subset \omega_\varepsilon$. It is then rather clear that

$$j_{\omega_\varepsilon}^*(f^*(t)) \to j_{\mathbb{R}^2}^*(f^*(t)) \quad \text{locally uniformly on } \mathcal{G}_{r,R} \text{ for every } r > 0.$$

It thus follows from Proposition 3 (with $\Sigma = \delta = 1$, rewritten in terms of v_ε) that

$$\left\| \frac{j v_\varepsilon}{|v_\varepsilon|} - j_{\mathbb{R}^2}^* \right\|_{\mathcal{G}_{r,R}} \le C$$

for all sufficiently small ε, where C is independent of r and R. By extracting weak limits and employing a Cantor diagonal argument, we conclude that there exists a vector field $H \in L^2([0, t_0] \times \mathbb{R}^2 \times \mathbb{T}_L)$ such that

$$\frac{j v_\varepsilon}{|v_\varepsilon|} - j_{\mathbb{R}^2}^* \rightharpoonup H \text{ weakly in } L^2(\mathcal{G}_{r,R}) \text{ for every } r, R > 0.$$

Now fix $\varphi \in \mathcal{D}((0, t_0) \times \mathbb{R}^2 \times \mathbb{T}_L)$ and compute, for ε sufficiently small,

$$\left| \int \nabla_x^\perp \varphi \cdot \frac{j v_\varepsilon}{|v_\varepsilon|} - \int \nabla_x^\perp \varphi \cdot j v_\varepsilon \right| \le \int |\nabla_x^\perp \varphi| \left| \frac{j v_\varepsilon}{|v_\varepsilon|} \right| |1 - |v_\varepsilon|| = o(1) \tag{71}$$

as $\varepsilon \to 0$, in view of the pointwise inequality $\left|\frac{jv_\varepsilon}{|v_\varepsilon|}\right| |1-|v_\varepsilon|| \le \varepsilon e_\varepsilon(v_\varepsilon)$ and the energy bound on v_ε. Next, integrating by parts and using Corollary 1 and the definition of $j^*_{\mathbb{R}^2}$,

$$\int \nabla_x^\perp \varphi \cdot jv_\varepsilon = 2\int \varphi J v_\varepsilon \to \int_0^{t_0}\int_0^L \sum_{i=1}^n \varphi(f_i(z),z)\,dz\,dt = \int \nabla_x^\perp \varphi \cdot j^*_{\mathbb{R}^2}(f^*).$$

By combining these and using the fact that $H \in L^2$, which implies that the singularities along $\{(t, f_i(z), z) : t \in [0,t_0], z \in [0,L], i = 1,\dots n\}$ are removable, we infer that $\nabla^\perp \cdot H = 0$ on $\mathbb{R}\times\mathbb{R}^2\times\mathbb{T}_L$. Similarly, by (34),

$$\int \nabla_x\varphi \cdot jv_\varepsilon = -\int \partial_t\varphi(|v_\varepsilon|^2-1) + h_\varepsilon^2\partial_z\varphi\, j_z v_\varepsilon \to 0,$$

since $(v_\varepsilon|^2-1)^2 \le 4\varepsilon^2 e_\varepsilon(v_\varepsilon)$ and $\left|h_\varepsilon^2\partial_z\varphi\, j_z v_\varepsilon\right| \le h_\varepsilon|\partial_z\varphi|(\frac{|\partial_z v_\varepsilon|^2}{|\log\varepsilon|} + |v_\varepsilon|^2)$, together with (12), rescaled to read $\|\nabla v_\varepsilon(t)\|^2_{L^2(dx\,dz)} \le C|\log\varepsilon|$ for every $t \in [0,t_0]$. Arguing as in (71) to eliminate the factor of $|v_\varepsilon|$ in the denominator and recalling that $\nabla_x \cdot j^*_{\mathbb{R}^2}(f^*) = 0$ by definition, we conclude that

$$\int \nabla_x \cdot (\frac{jv_\varepsilon}{|v_\varepsilon|} - j^*_{\mathbb{R}^2}) \to 0,$$

and hence that $\nabla_x \cdot H = 0$ in $\mathcal{D}'$. We conclude by applying Lemma 10 below to the vector field $w(t,x,z) = \zeta(t)H(t,x,z)$, where ζ is an arbitrary function with compact support in $[0,t_0]$. □

The proof of Proposition 5 used the following

Lemma 10 *Assume that $w \in L^2(\mathbb{R}\times\mathbb{R}^2\times\mathbb{T}_L)$ satisfies*

$$\nabla_x \cdot w = 0, \qquad \nabla_x^\perp \cdot w = 0 \qquad \textit{in } \mathcal{D}'. \tag{72}$$

Then $w = 0$.

Proof If w is smooth, then since $\nabla_x^\perp \cdot w = 0$, we may write $w = \nabla_x f$ for some scalar function f. Then the fact that $\nabla \cdot w = 0$ implies that f is harmonic, and hence that w is harmonic. For *a.e.* $t \in \mathbb{R}$ and $z \in \mathbb{T}_L$,

$$\int_{\mathbb{R}^2} |w(t,x,z)|^2\,dx = 0,$$

so Liouville's Theorem implies that $w(t,\cdot,z) = 0$ for such (t,z),and therefore everywhere in $\mathbb{R}\times\mathbb{R}^2\times\mathbb{T}_L$.

If w is not smooth, then we fix an approximate identity (η_ε), and we write $w_\varepsilon := \eta_\varepsilon * w$. Then w_ε satisfies conditions (72), with $\|w_\varepsilon\|_{L^2} \le \|w\|_{L^2} < \infty$ for every $\varepsilon > 0$, and $w_\varepsilon \to w$ in L^2, so it follows that $w = 0$ *a.e.* □

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