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Calculus of Variations

On the planar Schrödinger equation with indefinite linear part and critical growth nonlinearity

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Abstract

In the present paper, we develop a direct approach to find nontrivial solutions and ground state solutions for the following planar Schrödinger equation:

$$\begin{cases} -\Delta u + V(x)u = f(x,u), \quad x \in \mathbb{R}^2, \\ u \in H^1(\mathbb{R}^2), \end{cases}$$

where $V(x)$ is an 1-periodic function with respect to x_1 and x_2, 0 lies in a gap of the spectrum of $-\Delta + V$, and $f(x,t)$ behaves like $\pm e^{\alpha t^2}$ as $t \to \pm\infty$ uniformly on $x \in \mathbb{R}^2$. Our theorems extend and improve the results of de Figueiredo-Miyagaki-Ruf (Calc Var Partial Differ Equ, 3(2):139–153, 1995), of de Figueiredo-do Ó-Ruf (Indiana Univ Math J, 53(4):1037–1054, 2004), of Alves-Souto-Montenegro (Calc Var Partial Differ Equ 43: 537–554, 2012), of Alves-Germano (J Differ Equ 265: 444–477, 2018) and of do Ó-Ruf (NoDEA 13: 167–192, 2006).

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1 Introduction

This paper is concerned with the following planar Schrödinger equation:

$$\begin{cases} -\Delta u + V(x)u = f(x,u), \quad x \in \mathbb{R}^2, \\ u \in H^1(\mathbb{R}^2), \end{cases} \tag{1.1}$$

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where V and f satisfy the following basic assumptions:

(V1) $V \in \mathcal{C}(\mathbb{R}^2, \mathbb{R})$, $V(x)$ is 1-periodic in x_1 and x_2, and

$$\sup[\sigma(-\Delta + V) \cap (-\infty, 0)] < 0 < \inf[\sigma(-\Delta + V) \cap (0, \infty)];$$

(F1) $f \in \mathcal{C}(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R})$, $f(x,t)$ is 1-periodic in x_1 and x_2, and

$$\lim_{|t|\to\infty} \frac{|f(x,t)|}{e^{\alpha t^2}} = 0, \quad \text{uniformly on } x \in \mathbb{R}^2 \text{ for all } \alpha > 0; \tag{1.2}$$

or

(F1′) $f \in \mathcal{C}(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R})$, $f(x,t)$ is 1-periodic in x_1 and x_2, and there exists $\alpha_0 > 0$ such that

$$\lim_{|t|\to\infty} \frac{|f(x,t)|}{e^{\alpha t^2}} = 0, \quad \text{uniformly on } x \in \mathbb{R}^2 \text{ for all } \alpha > \alpha_0 \tag{1.3}$$

and

$$\lim_{|t|\to\infty} \frac{|f(x,t)|}{e^{\alpha t^2}} = +\infty, \quad \text{uniformly on } x \in \mathbb{R}^2 \text{ for all } \alpha < \alpha_0; \tag{1.4}$$

(F2) $f(x,t) = o(t)$ as $t \to 0$ uniformly on $x \in \mathbb{R}^2$.

As we all know, under (V1), the energy functional associated with (1.1) on $H^1(\mathbb{R}^2)$ is in general strongly indefinite near the origin. In this case, the generalized link theorem is a very effective tool to deal with this strongly indefinite problem, which was introduced by Kryszewski–Szulkin [21], and was improved by Li–Szulkin [23] and Ding [14,15] later. The generalized link theorem has been used extensively to study the periodic Schrödinger equation:

$$\begin{cases} -\Delta u + V(x)u = f(x,u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N) \end{cases} \tag{1.5}$$

with $N \geq 3$ and (V1), we would like to cite Ding–Lee [15], Tang [27], Tang–Lin–Yu [28], Tang–Chen–Lin–Yu [29], Zhang–Xu–Zhang [32] for the subcritical growth case:

$$\lim_{|t|\to\infty} \frac{|f(x,t)|}{|t|^{2^*-1}} = 0, \quad \text{uniformly on } x \in \mathbb{R}^N; \tag{1.6}$$

Chabrowski–Szulkin [9], Schechter–Zou [24], and Zhang–Xu–Zhang [31] for the critical growth case:

$$\lim_{|t|\to\infty} \frac{|f(x,t)|}{|t|^{2^*-1}} > 0, \quad \text{for every } x \in \mathbb{R}^N, \tag{1.7}$$

where $2^* = 2N/(N-2)$ is the critical exponent.

The case $N = 2$ is very special, as the corresponding Sobolev embedding yields $H^1(\mathbb{R}^2) \subset L^s(\mathbb{R}^2)$ for all $s \in [2, +\infty)$, but $H^1(\mathbb{R}^2) \not\subseteq L^\infty(\mathbb{R}^2)$. In dimension $N = 2$, the Trudinger–Moser inequality can be seen as a substitute of the Sobolev inequality. The first version of the Trundiger–Moser inequality in $\mathbb{R}^2$ was established by Cao in [7], see also [1,8], and reads as follows.

Lemma 1.1 i) *If* $\alpha > 0$ *and* $u \in H^1(\mathbb{R}^2)$, *then*

$$\int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1\right) \mathrm{d}x < \infty;$$

ii) *if* $u \in H^1(\mathbb{R}^2)$, $\|\nabla u\|_2^2 \le 1$, $\|u\|_2 \le M < \infty$, *and* $\alpha < 4\pi$, *then there exists a constant* $\mathcal{C}(M, \alpha)$, *which depends only on* M *and* α, *such that*

$$\int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1\right) \mathrm{d}x \le \mathcal{C}(M, \alpha).$$

Based on Lemma 1.1, we say that $f(x, t)$ has subcritical growth on $\mathbb{R}^2$ at $t = \pm\infty$ if (1.2) holds, and $f(x, t)$ has critical growth on $\mathbb{R}^2$ at $t = \pm\infty$ if (1.3) and (1.4) hold, which is the maximal growth on t that allows to treat the problem variationally in $H^1(\mathbb{R}^2)$. This notion of criticality was introduced by Adimurthi–Yadava [2], see also de Figueiredo–Miyagaki–Ruf [13].

Let us point out that the case when $N = 2$ and $f(x, t)$ has polynomial growth on t was in fact considered in the above mentioned papers, since it can be addressed similarly as the case when $N \ge 3$ and $f(x, t)$ is superlinear and subcritical at $t = \infty$. In particular, it is easy, in this case, to show that the functional $\Psi(u) = \int_{\mathbb{R}^2} F(x, u)\mathrm{d}x$ is weakly sequentially continuous in $H^1(\mathbb{R}^2)$, where and in the sequel $F(x, t) := \int_0^t f(x, s)\mathrm{d}s$, since the sequence $\{\int_{|u_n|\ge 1} |f(x, u_n)|^q \mathrm{d}x\}$ is still bounded for any constant $q > 1$ and any bounded sequence $\{u_n\} \subset H^1(\mathbb{R}^2)$. And so, the generalized link theorem can be applied to the functional associated with (1.1) to obtain a (PS) sequence or Cerami sequence. However, when $f(x, t)$ has exponential growth on t, on one hand, the embedding of the Sobolev space $H^1(\mathbb{R}^2)$ into the Orlicz space associated with the function $\varphi(s) = \exp(4\pi s^2) - 1$ is not compact, on the other hand, it is not standard to prove that $\Psi(u)$ is weakly sequentially continuous in $H^1(\mathbb{R}^2)$. But even worse, so far we have not found a method to show this conclusion when $f(x, t)$ has critical exponential growth on $\mathbb{R}^2$ at $t = \pm\infty$ (i.e.(1.3) and (1.4) hold). Therefore, the technical methods in proving the existence, boundedness and the non-vanishing of (PS) sequence or Cerami sequences for the energy functional associated with (1.5), used in aforementioned papers, do not work for (1.5) with $N = 2$. Also because of this, it is more complicated to deal with the case $N = 2$ than the case $N \ge 3$.

In the case $N = 2$ and $f(x, t)$ has exponential growth on t, when $V(x)$ is a positive potential bounded away from zero (i.e. the so-called definite case), motivated by the Moser–Trudinger inequality, the existence of nontrivial solutions to problem (1.1) has been studied by many authors; see, for example, Alves–Souto [4], Adimurthi–Yadava [2], Alves–Souto–Montenegro [5], Cao [7], de Figueiredo-do Ó-Ruf [11,12], de Figueiredo–Miyagaki–Ruf [13], Lam–Lu [22], Zhang-do Ó [33]. However, when (V1) holds, the operator $-\Delta + V$ on $L^2(\mathbb{R}^2)$ has a purely continuous spectrum consisting of closed disjoint intervals (i.e. the so-called indefinite case), to the best of our knowledge, it seems that there are only two papers [3,17] concerning the existence of nontrivial solutions for (1.1). To describe the existing results in [3,17], we first introduce the following conditions:

(F3) there exists $\bar{\mu} > 2$ such that

$$tf(x, t) \ge \bar{\mu} F(x, t) > 0, \quad \forall\, (x, t) \in \mathbb{R}^2 \times (\mathbb{R} \setminus \{0\});$$

(F4) there exist $M_0 > 0$ and $t_0 > 0$ such that for every $x \in \mathbb{R}^2$,

$$F(x, t) \le M_0 |f(x, t)|, \quad \forall\, |t| \ge t_0;$$

(F5′) $\lim_{|t|\to\infty} \frac{tf(x,t)}{e^{\alpha_0 t^2}} = \infty$ uniformly on $x \in \mathbb{R}^2$;

(F6) there exist constants $\Gamma, \lambda > 0$ and $q_0 > 2$ such that

$$|f(x, t)| \le \Gamma e^{4\pi t^2} \text{ and } F(x, t) \ge \lambda |t|^{q_0}, \quad \forall\, (x, t) \in \mathbb{R}^2 \times \mathbb{R};$$

(SQ) $\lim_{|t|\to\infty} \frac{F(x,t)}{|t|^2} = \infty$ for a.e. $x \in \mathbb{R}^2$;

(WN) $t \mapsto \frac{f(x,t)}{|t|}$ is non-decreasing on $(-\infty, 0) \cup (0, \infty)$ for every $x \in \mathbb{R}^2$.

Under (V1), (F1′), (F2), (F3), (F6) and (WN), Alves–Germano [3] proved that if λ is large enough, (1.1) has a ground state solution by using the method of generalized Nehari manifold developed by Szulkin–Weth [25,26]. They showed that the minimax-level is less than the threshold value under which (PS) sequences do not vanish in the same way as the case $N \geq 3$. Let us emphasize that the condition $F(x,t) \geq \lambda |t|^{q_0}$ with sufficiently large λ is very crucial in their arguments. Thanks to this condition, the minimax-level for the energy functional associated with (1.1) can be chosen to be small, and so ii) of Lemma 1.1 is available, thereby the obstacle arising from the critical growth of Trudinger–Moser type is easily overcome, see [3, Propositions 3.15, 3.16]. But this result has no relationship on the exponential growth velocity α_0 (see (F1′)), hence it does not reveal the essential characteristics for (1.1) with the critical growth of Trudinger–Moser type.

When V satisfies (V1), and $f(x,t) = f(t)$ satisfies (F1′), (F2)–(F4) and (F5′), based on an approximation technique of periodic function together with the linking theorem due to Bartolo-Benci-Fortunato [6], do Ó and Ruf [17] obtained the existence of a nontrivial solution of (1.1). To overcome the difficulties arising from lack of compactness of the corresponding energy functional, some of the ideas contained in [13,16] were used. More precisely, they first introduced a sequence of cubes $\{Q_k\} \subset \mathbb{R}^2$ with edge length $k \in \mathbb{N}$ and the orthogonal decomposition $H^1_{\text{per}}(Q_k) = Y_k \oplus Z_k$ with $\dim Y_k < \infty$ for every $k \in \mathbb{N}$, where $H^1_{\text{per}}(Q_k)$ denotes the space of $H^1(Q_k)$-functions which are k-periodic in x_1 and x_2, and then applied the link theorem to the approximation problem and yielded a (PS) sequence $\{u_{k,n}\}$ for every $k \in \mathbb{N}$; next proved that $\{u_{k,n}\}$ is bounded in $H^1_{\text{per}}(Q_k)$ and does not vanish; finally got a sequence of solutions $\{u_k\}$ of the approximation problems and then proved that it tends to a nontrivial solution of (1.1) as $k \to \infty$. In their arguments, they used many embedding inequalities on Q_k and upper or lower estimates for the functionals on $H^1(Q_k)$. Obviously, it is very crucial to verify that the embedding constants and the uppers or lowers are independent of $k \in \mathbb{N}$. However, it is quite difficult and complicated to do these works. For example, they used Schwarz symmetrization method to prove the following two claims:

Claim (i) ([17, Claim 2.5]) There exist constants $\rho_0 > 0$ and $C > 0$ independent of k such that

$$\int_{Q_k} |u|^q [\exp(u^2) - 1] \mathrm{d}x \leq C \|u\|^q_{H^1(Q_k)}$$

for all $u \in H^1(Q_k)$ with $\|u\|_{H^1(Q_k)} \leq \rho_0$.

Claim (ii) ([17, Claim 3.3]) The following conclusion holds:

$$\lim_{n\to\infty} \|u_n\|_q = 0 \Rightarrow \lim_{n\to\infty} \int_{Q_n} F(u_n) \mathrm{d}x = 0.$$

In the proof of Claim i), they established many embedding inequalities with embedding constants independent of k, such as $L^2(\mathbb{R}^2) \xrightarrow{P} L^2(B_{R_k}) \hookrightarrow L^2(Q_k) \hookrightarrow H^1(\mathbb{R}^2)$, see [17, Claim 2.5]. Claim ii) implies that the approach does not work any more for non-autonomous problem (1.1), since the Schwarz symmetrization method is only valid for autonomous function f.

In the present paper, motivated by [3,9,10,13,17], we will develop a direct approach which is different from [3,17] to find nontrivial solutions and ground state solutions of (1.1) in the subcritical and critical exponential growth cases. Particularly, employing some new techniques with a deep analysis and using an approaching argument and some detailed estimates, we succeed in overcoming four main difficulties: (1) looking for a Cerami sequence for the energy functional associated with (1.1); (2) showing the boundedness of the Cerami sequences; (3) showing that the minimax-level is less than the threshold value; (4) showing that the Cerami sequences do not vanish.

In particular, we will weaken (F5′) used in [17] to the following condition:

(F5) $\liminf_{|t|\to\infty} \frac{tf(x,t)}{e^{\alpha_0 t^2}} \geq \kappa > \frac{4}{\alpha_0 \rho^2} e^{16\pi \mathcal{C}_0^2}$ uniformly on $x \in \mathbb{R}^2$,

where $\rho > 0$ satisfies $4\pi(4+\rho)\rho\mathcal{C}_0^2 < 1$ and $\mathcal{C}_0 > 0$ is an embedding constant, see (4.15) and (4.16).

It deserves to be mentioned that an assumption similar to (F5) was introduced in [13] when $V(x)$ is positive periodic and $\mathbb{R}^2$ is replaced by a bounded domain $\Omega \subset \mathbb{R}^2$.

In detail, we have the following four results on the existence of nontrivial solutions.

Theorem 1.2 *Assume that V and f satisfy* (V1) *and* (F1)–(F3). *Then (1.1) has a nontrivial solution.*

Theorem 1.3 *Assume that V and f satisfy* (V1), (F1), (F2), (SQ) *and* (WN). *Then (1.1) has a ground state solution with positive energy.*

Theorem 1.4 *Assume that V and f satisfy* (V1), (F1′) *and* (F2)–(F5). *Then (1.1) has a nontrivial solution.*

Corollary 1.5 *Assume that V and f satisfy* (V1), (F1′), (F2)–(F4) *and* (F5′). *Then (1.1) has a nontrivial solution.*

Example 1.6 It is easy to check, using Taylor series, that the following two functions satisfy (F1)–(F3), (SQ) and (WN):

(i). $f(x,t) = a(2+\sin 2\pi x_1 \cos 2\pi x_2)\left(e^{b|t|^{3/2}} - 1\right)\operatorname{sign} t$ with $a, b > 0$;

(ii). $f(x,t) = a(2+\sin 2\pi x_1 \cos 2\pi x_2)\left(e^{bt} - 1 - bt - \frac{1}{2}b^2t^2\right)$ with $a, b > 0$;

and $f(x,t) = a\kappa t^{-1}\left(e^{t^2} - 1 - t^2\right)$ with $a \geq 1$ satisfies (F1′) and (F2)–(F5) with $\alpha_0 = 1$ and $\mu = 3$, but it does not satisfy (F5′).

The paper is organized as follows. In Sect. 2, we give the variational setting and preliminaries. We complete the proofs of Theorems 1.2, 1.3 and 1.4 in Sects. 3 and 4 respectively.

Throughout the paper, $C_1, C_2, \ldots$ denote positive constants possibly different in different places.

2 Variational setting

Let $\mathcal{A} = -\Delta + V$ with $V \in \mathcal{C}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Then $\mathcal{A}$ is self-adjoint in $L^2(\mathbb{R}^2)$ with domain $\mathfrak{D}(\mathcal{A}) = H^2(\mathbb{R}^2)$ (see [19, Theorem 4.26]). Let $\{\mathcal{E}(\lambda) : -\infty < \lambda < +\infty\}$ and $|\mathcal{A}|$ be the spectral family and the absolute value of $\mathcal{A}$, respectively, and $|\mathcal{A}|^{1/2}$ the square root of $|\mathcal{A}|$. Set $U = id - \mathcal{E}(0) - \mathcal{E}(0-)$. Then U commutes with $\mathcal{A}$, $|\mathcal{A}|$ and $|\mathcal{A}|^{1/2}$, and $\mathcal{A} = U|\mathcal{A}|$ is the polar decomposition of $\mathcal{A}$ (see [18, Theorem IV 3.3]). Let

$$E = \mathfrak{D}(|\mathcal{A}|^{1/2}), \quad E^- = \mathcal{E}(0-)E, \quad E^+ = [id - \mathcal{E}(0)]E. \tag{2.1}$$

By (V1), one has $E = E^- \oplus E^+$. For any $u \in E$, it is easy to see that $u = u^- + u^+$, where

$$u^- := \mathcal{E}(0-)u \in E^-, \quad u^+ := [id - \mathcal{E}(0)]u \in E^+ \tag{2.2}$$

and

$$\mathcal{A}u^- = -|\mathcal{A}|u^-, \quad \mathcal{A}u^+ = |\mathcal{A}|u^+, \quad \forall\, u \in E \cap \mathfrak{D}(\mathcal{A}). \tag{2.3}$$

On E, We can define an inner product

$$(u, v) = \left(|\mathcal{A}|^{1/2}u, |\mathcal{A}|^{1/2}v\right)_{L^2}, \quad u, v \in E \tag{2.4}$$

and the corresponding norm

$$\|u\| = \left\||\mathcal{A}|^{1/2}u\right\|_2, \quad u \in E, \tag{2.5}$$

where and in the sequel, $(\cdot, \cdot)_{L^2}$ denotes the inner product of $L^2(\mathbb{R}^2)$, $\|\cdot\|_s$ denotes the norm of $L^s(\mathbb{R}^2)$.

$E = H^1(\mathbb{R}^2)$ with equivalent norms (see [14,15]). Therefore, E embeds continuously in $L^s(\mathbb{R}^2)$ for all $2 \le s < \infty$, i.e. there exists $\gamma_s > 0$ such that

$$\|u\|_s \le \gamma_s \|u\|, \quad \forall\, u \in E. \tag{2.6}$$

In addition, one has the following orthogonal decomposition $E = E^- \oplus E^+$, where orthogonality is with respect to both $(\cdot, \cdot)_{L^2}$ and $(\cdot, \cdot)$. If $\sigma(-\Delta + V) \subset (0, \infty)$, then $E^- = \{0\}$, otherwise E^- is infinite-dimensional.

Under assumptions (V1), (F1) (or (F1′)) and (F2), the solutions of problem (1.1) are critical points of the functional

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^2} \left(|\nabla u|^2 + V(x)u^2\right) dx - \int_{\mathbb{R}^2} F(x, u) dx, \quad \forall\, u \in E. \tag{2.7}$$

In view of (2.3) and (2.5), we have

$$\Phi(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) - \int_{\mathbb{R}^2} F(x, u) dx, \quad \forall\, u = u^- + u^+ \in E^- \oplus E^+ = E. \tag{2.8}$$

By virtue of (F1) (or (F1′)) and (F2), we can choose $\alpha > 0$ such that for any given $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(x, t)| \le \varepsilon|t| + C_\varepsilon\left(e^{\alpha t^2} - 1\right), \quad \forall\, (x, t) \in \mathbb{R}^2 \times \mathbb{R}. \tag{2.9}$$

Consequently,

$$|F(x, t)| \le \varepsilon|t|^2 + C_\varepsilon|t|\left(e^{\alpha t^2} - 1\right), \quad \forall\, (x, t) \in \mathbb{R}^2 \times \mathbb{R}. \tag{2.10}$$

According to (2.10) and Lemma 1.1, we can demonstrate that Φ is of class $\mathcal{C}^1(E, \mathbb{R})$, and

$$\langle \Phi'(u), v\rangle = \int_{\mathbb{R}^2} (\nabla u \nabla v + V(x)uv)\, dx - \int_{\mathbb{R}^2} f(x, u)v dx, \quad \forall\, u, v \in E. \tag{2.11}$$

In particular, it follows from (2.3) and (2.5) that

$$\langle \Phi'(u), u\rangle = \|u^+\|^2 - \|u^-\|^2 - \int_{\mathbb{R}^2} f(x, u)u dx, \quad \forall\, u \in E. \tag{2.12}$$

Define

$$\mathcal{M} = \left\{u \in E \setminus E^- : \langle \Phi'(u), u\rangle = \langle \Phi'(u), v\rangle = 0, \ \forall\, v \in E^-\right\}. \tag{2.13}$$

Let X be a real Hilbert space with $X = X^- \oplus X^+$ and $X^- \perp X^+$. For a functional $\varphi \in \mathcal{C}^1(X, \mathbb{R})$, φ is said to be weakly sequentially lower semi-continuous if for any $u_n \rightharpoonup u$ in X one has $\varphi(u) \le \liminf_{n\to\infty} \varphi(u_n)$, and φ' is said to be weakly sequentially continuous if for any $u_n \rightharpoonup u$ in X one has $\lim_{n\to\infty}\langle \varphi'(u_n), v\rangle = \langle \varphi'(u), v\rangle$ for each $v \in X$.

Lemma 2.1 ([14,15]) *Let X be a real Hilbert space with $X = X^- \oplus X^+$ and $X^- \perp X^+$, and let $\varphi \in \mathcal{C}^1(X, \mathbb{R})$ of the form*

$$\varphi(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) - \psi(u), \quad u = u^- + u^+ \in X^- \oplus X^+.$$

Suppose that the following assumptions are satisfied:

(BD1) *$\psi \in \mathcal{C}^1(X, \mathbb{R})$ is bounded from below and weakly sequentially lower semi-continuous;*
(BD2) *ψ' is weakly sequentially continuous;*
(BD3) *there exists $\zeta > 0$ such that $\|u\| \le \zeta\|u^+\|$ for all $u \in \{v \in E : \varphi(v) \ge 0\}$;*
(BD4) *there exist $r > \rho > 0$ and $e \in X^+$ with $\|e\| = 1$ such that*

$$\hat{\kappa} := \inf \varphi(S_\rho^+) > \sup \varphi(\partial \hat{Q}),$$

where

$$S_\rho^+ = \left\{u \in X^+ : \|u\| = \rho\right\}, \quad \hat{Q} = \left\{v + se : v \in X^-,\ 0 \le s \le r,\ \|v\| \le r\right\}.$$

Then there exist a constant $c \in [\hat{\kappa}, \sup \varphi(\hat{Q})]$ and a sequence $\{u_n\} \subset X$ satisfying

$$\varphi(u_n) \to c, \quad \|\varphi'(u_n)\|(1 + \|u_n\|) \to 0. \tag{2.14}$$

We set

$$\Psi(u) := \int_{\mathbb{R}^2} F(x, u)\mathrm{d}x, \quad \forall\, u \in E. \tag{2.15}$$

Lemma 2.2 *Assume that* (V1),(F1) *and* (F2) *hold, and $F(x,t) \ge 0$ for all $(x,t) \in \mathbb{R}^2 \times \mathbb{R}$. Then Ψ is nonnegative, weakly sequentially lower semi-continuous, and Ψ' is weakly sequentially continuous in E.*

Proof We only prove that Ψ' is weakly sequentially continuous, the other is standard. Let $u_n \rightharpoonup u$ in E and let $v \in E$ be an any given function. Then $\|u_n\| \le C_1$ for some $C_1 > 0$. Since the norms $\|\cdot\|$ and $\|\cdot\|_{H^1}$ are equivalent, there exists $\vartheta_0 > 0$ such that

$$\|\nabla u\|_2 \le \vartheta_0\|u\|, \quad \forall\, u \in E. \tag{2.16}$$

Let $\alpha \in (0, 1/C_1^2\vartheta_0^2)$. Using (F1) and (F2), there exists $C_2 > 0$ such that

$$|f(x,t)| \le |t| + C_2\left(e^{\alpha t^2} - 1\right), \quad \forall\, (x,t) \in \mathbb{R}^2 \times \mathbb{R}. \tag{2.17}$$

For any $\varepsilon > 0$, we can choose $R > 0$ such that

$$\int_{\mathbb{R}^2 \setminus B_R} v^2 \mathrm{d}x < \varepsilon^2. \tag{2.18}$$

Then it follows from (2.17), (2.18) and Lemma 1.1 that

$$\begin{aligned}\int_{\mathbb{R}^2\setminus B_R}|f(x,u_n)v|\mathrm{d}x &\le \int_{\mathbb{R}^2\setminus B_R}|u_n v|\mathrm{d}x + C_2\int_{\mathbb{R}^2\setminus B_R}\left(e^{\alpha u_n^2}-1\right)|v|\mathrm{d}x\\ &\le \left\{\|u_n\|_2 + C_2\left[\int_{\mathbb{R}^2}\left(e^{\alpha u_n^2}-1\right)^2\mathrm{d}x\right]^{1/2}\right\}\left(\int_{\mathbb{R}^2\setminus B_R}v^2\mathrm{d}x\right)^{\frac{1}{2}}\\ &\le \left\{\|u_n\|_2 + C_2\left[\int_{\mathbb{R}^2}\left(e^{2\alpha u_n^2}-1\right)\mathrm{d}x\right]^{1/2}\right\}\varepsilon\\ &\le \left\{\|u_n\|_2 + C_2\left[\int_{\mathbb{R}^2}\left(e^{2\alpha\vartheta_0^2\|u_n\|^2(u_n/\vartheta_0\|u_n\|)^2}-1\right)\mathrm{d}x\right]^{1/2}\right\}\varepsilon\\ &\le C_3\varepsilon. \end{aligned} \tag{2.19}$$

Since $v \in L^2(B_R)$, it follows that there exists $\delta > 0$ such that

$$\int_A |v|^2\mathrm{d}x < \varepsilon^2 \ \text{ if meas}(A) \le \delta \tag{2.20}$$

for all measurable set $A \subset B_R$. Hence it follows from $\|u_n\| \le C_1$ that there exists $M > 0$ such that

$$\text{meas}(\{x \in B_R : |u_n(x)| \ge M\}) \le \delta, \quad \text{meas}(\{x \in B_R : |u(x)| \ge M\}) \le \delta. \tag{2.21}$$

Let $A_n := \{x \in B_R : |u_n(x)| \ge M\}$, $A_0 := \{x \in B_R : |u(x)| \ge M\}$ and $D_0 := \{x \in B_R : |u(x)| = M\}$. Then it follows from (2.17), (2.20), (2.21) and Lemma 1.1 that

$$\begin{aligned}\int_{A_n\cup D_0}|f(x,u_n)v|\mathrm{d}x &\le \int_{A_n\cup D_0}|u_n v|\mathrm{d}x + C_2\int_{A_n\cup D_0}\left(e^{\alpha u_n^2}-1\right)|v|\mathrm{d}x\\ &\le \left\{\|u_n\|_2 + C_2\left[\int_{\mathbb{R}^2}\left(e^{\alpha u_n^2}-1\right)^2\mathrm{d}x\right]^{1/2}\right\}\left(\int_{A_n\cup D_0}v^2\mathrm{d}x\right)^{\frac{1}{2}}\\ &\le 2\left\{\|u_n\|_2 + C_2\left[\int_{\mathbb{R}^2}\left(e^{2\alpha u_n^2}-1\right)\mathrm{d}x\right]^{1/2}\right\}\varepsilon\\ &\le 2\left\{\|u_n\|_2 + C_2\left[\int_{\mathbb{R}^2}\left(e^{2\alpha\vartheta_0^2\|u_n\|^2(u_n/\vartheta_0\|u_n\|)^2}-1\right)\mathrm{d}x\right]^{1/2}\right\}\varepsilon\\ &\le C_3\varepsilon. \end{aligned} \tag{2.22}$$

Similarly, we can show that

$$\int_{A_0}|f(x,u)v|\mathrm{d}x \le C_3\varepsilon. \tag{2.23}$$

Since $f(x,u_n)v\chi_{|u_n|\le M} \to f(x,u)v\chi_{|u|\le M}$ a.e. in $B_R \setminus D_0$, moreover,

$$|f(x,u_n)v|\chi_{|u_n|\le M} \le |v| \max_{x\in B_R,|t|\le M}|f(x,t)|, \quad \forall\, x \in B_R.$$

Then Lebesgue dominated convergence theorem leads to

$$\lim_{n\to\infty}\int_{B_R\setminus(A_n\cup D_0)} f(x,u_n)v\mathrm{d}x = \int_{B_R\setminus A_0} f(x,u)v\mathrm{d}x. \tag{2.24}$$

Let $\varepsilon \to 0$, it follows from (2.19), (2.22), (2.23) and (2.24) that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2} f(x,u_n)v\mathrm{d}x = \int_{\mathbb{R}^2} f(x,u)v\mathrm{d}x.$$

This shows that Ψ' is weakly sequentially continuous. □

The following lemma is very important and crucial, which has been proved in [9, Proposition 2.2 and Proposition 2.4]. Here, We give a different proof.

Lemma 2.3 *Assume that $V \in L^\infty(\mathbb{R}^2)$. Then for any $\mu > 0$ there exist two constant $\mathcal{K}_0 > 0$ and $\mathcal{K}_\mu > 0$ such that*

$$\|\nabla u\|_\infty + \|u\|_\infty \le \mathcal{K}_0\|u\|_2, \quad \forall\, u \in \mathcal{E}(0)E = E^- \tag{2.25}$$

and

$$\|u\|_\infty \le \mathcal{K}_\mu\|u\|_2, \quad \forall\, u \in \mathcal{E}(\mu)E. \tag{2.26}$$

Proof Let $b < \inf\sigma(\mathcal{A})$. Then we have

$$(\mathcal{A}^2u,u)_{L^2} = \int_b^\mu \lambda^2\mathrm{d}(\mathcal{E}(\lambda)u,u)_{L^2} \le (|b|+\mu)^2\|u\|_2^2, \quad \forall\, u \in \mathcal{E}(\mu)[H_0^2(\mathbb{R}^2)].$$

Consequently,

$$\|\mathcal{A}u\|_2 \le (|b|+\mu)\|u\|_2, \quad \forall\, u \in \mathcal{E}(\mu)[H_0^2(\mathbb{R}^2)]. \tag{2.27}$$

By virtue of (2.27) and the Hölder inequality, we obtain that

$$\begin{aligned}\left|(-\Delta u,v)_{L^2}\right| &= \left|(\mathcal{A}u,v)_{L^2} - \int_{\mathbb{R}^2} V(x)uv\mathrm{d}x\right|\\ &\le [\|\mathcal{A}u\|_2 + \|V\|_\infty\|u\|_2]\,\|v\|_2\\ &\le (|b|+\mu+\|V\|_\infty)\,\|u\|_2\|v\|_2,\\ &\qquad \forall\, u \in \mathcal{E}(\mu)[H_0^2(\mathbb{R}^2)],\ \ v \in L^2(\mathbb{R}^2),\end{aligned} \tag{2.28}$$

it leads to the following fact that

$$\|\Delta u\|_2 \le C_1\|u\|_2, \quad \forall\, u \in \mathcal{E}(\mu)[H_0^2(\mathbb{R}^2)]. \tag{2.29}$$

Employing the Calderon–Zygmund inequality (see [20, Theorem 9.9]) and Ehrling–Nirenberg–Gagliardo interpolation inequalities (see [20, Theorem 7.28]), we deduce that

$$\|u\|_{H^2(\mathbb{R}^2)} \le C_2\|u\|_2, \quad \forall\, u \in \mathcal{E}(\mu)[H_0^2(\mathbb{R}^2)], \tag{2.30}$$

which, together with the Sobolev embedding theorem, yields

$$\|u\|_\infty \le C_3\|u\|_{H^2(\mathbb{R}^2)} \le C_4\|u\|_2, \quad \forall\, u \in \mathcal{E}(\mu)[H_0^2(\mathbb{R}^2)]. \tag{2.31}$$

Since $\mathcal{E}(\mu)[H_0^2(\mathbb{R}^2)]$ is dense in $\mathcal{E}(\mu)L^2(\mathbb{R}^2)$ and $L^\infty(\mathbb{R}^2)$ is complete, it follows from (2.31) that

$$\|u\|_\infty \le C_5\|u\|_2, \quad \forall\, u \in \mathcal{E}(\mu)L^2(\mathbb{R}^2). \tag{2.32}$$

For any $u \in \mathcal{E}(0)[H_0^2(\mathbb{R}^2)]$, there exists $\tilde{u} \in H_0^2(\mathbb{R}^2)$ such that $u = \mathcal{E}(0)\tilde{u}$, we deduce that

$$[id - \mathcal{E}(0)]\mathcal{A}u = \mathcal{A}[id-\mathcal{E}(0)]u = \mathcal{A}[id-\mathcal{E}(0)]\mathcal{E}(0)\tilde{u} = 0.$$

This shows that $\mathcal{A}u \in \mathcal{E}(0)L^2(\mathbb{R}^2)$, $\forall\, u \in \mathcal{E}(0)[H_0^2(\mathbb{R}^2)]$. Hence, it follows from (2.27) and (2.32) that

$$\|\mathcal{A}u\|_\infty \le C_6\|\mathcal{A}u\|_2 \le |b|C_6\|u\|_2, \quad \forall\, u \in \mathcal{E}(0)[H_0^2(\mathbb{R}^2)]. \tag{2.33}$$

By virtue of (2.32), (2.33) and the Hölder inequality, we get

$$\begin{aligned}\left|(-\Delta u, v)_{L^2}\right| &= \left|(\mathcal{A}u, v)_{L^2} - \int_{\mathbb{R}^2} V(x)uv\mathrm{d}x\right| \\ &\leq (\|\mathcal{A}u\|_\infty + \|V\|_\infty\|u\|_\infty)\,\|v\|_1 \\ &\leq (|b|C_6\|u\|_2 + C_6\|V\|_\infty\|u\|_2)\,\|v\|_1, \\ &= C_7\|u\|_2\|v\|_1, \quad \forall\, u \in \mathcal{E}(0)[H_0^2(\mathbb{R}^2)], \quad v \in L^1(\mathbb{R}^2).\end{aligned} \tag{2.34}$$

Consequently,

$$\|\Delta u\|_\infty \leq C_8\|u\|_2, \quad \forall\, u \in \mathcal{E}(0)[H_0^2(\mathbb{R}^2)]. \tag{2.35}$$

Again applying the Calderon–Zygmund inequality and interpolation inequalities, one can get

$$\|\nabla u\|_\infty + \|u\|_\infty \leq C_9\|u\|_2, \quad \forall\, u \in \mathcal{E}(0)[H_0^2(\mathbb{R}^2)].$$

Now the conclusion follows by above inequality and the fact that $\mathcal{E}(0)[H_0^2(\mathbb{R}^2)]$ is dense in $\mathcal{E}(0)E$. □

Lemma 2.4 *Assume that* (V1), (F1) (*or* (F1′)), (F2) *and* (F3) *hold. Then there exists* $\bar{\rho} > 0$ *such that*

$$\kappa_0 := \inf\left\{\Phi(u) : u \in E^+, \|u\| = \bar{\rho}\right\} > 0. \tag{2.36}$$

Proof By (F1) (or (F1′) and (F2), one has for some constants $\alpha > 0$ and $C_{10} > 0$

$$|F(x,t)| \leq \frac{1}{4\gamma_2^2}t^2 + C_{10}\left(e^{\alpha t^2} - 1\right)|t|^3, \quad \forall\,(x,t) \in \mathbb{R}^2 \times \mathbb{R}. \tag{2.37}$$

In view of Lemma 1.1, (2.6) and (2.16), we have

$$\begin{aligned}\int_{\mathbb{R}^2}\left(e^{2\alpha u^2} - 1\right)\mathrm{d}x &= \int_{\mathbb{R}^2}\left(e^{2\alpha\vartheta_0^2\|u\|^2(u/\vartheta_0\|u\|)^2} - 1\right)\mathrm{d}x \\ &\leq \mathcal{C}(\gamma_2/\vartheta_0, 2\pi), \quad \forall\,\|u\| \leq \sqrt{\pi/\alpha\vartheta_0^2}.\end{aligned} \tag{2.38}$$

Then (2.37) and (2.38) give

$$\begin{aligned}\int_{\mathbb{R}^2} F(x,u)\mathrm{d}x &\leq \frac{1}{4\gamma_2^2}\|u\|_2^2 + C_{10}\int_{\mathbb{R}^2}\left(e^{\alpha u^2} - 1\right)|u|^3\mathrm{d}x \\ &\leq \frac{1}{4\gamma_2^2}\|u\|_2^2 + C_{10}\left[\int_{\mathbb{R}^2}\left(e^{2\alpha u^2} - 1\right)\mathrm{d}x\right]^{1/2}\|u\|_6^3 \\ &\leq \frac{1}{4}\|u\|^2 + C_{11}\|u\|^3, \quad \forall\,\|u\| \leq \sqrt{\pi/\alpha\vartheta_0^2}.\end{aligned} \tag{2.39}$$

Hence, it follows from (2.8) and (2.39) that

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^2} F(x,u)\mathrm{d}x \\ &\geq \frac{1}{4}\|u\|^2 - C_{11}\|u\|^3, \quad \forall\, u \in E^+, \ \|u\| \leq \sqrt{\pi/\alpha\vartheta_0^2}.\end{aligned}$$

Therefore, there exists $0 < \bar{\rho} < \sqrt{\pi/\alpha\vartheta_0^2}$ such that (2.36) holds. □

As in [27], we can prove the following three lemmas.

Lemma 2.5 *Assume that* (V1), (F1) (*or* (F1′)), (F2) *and* (F3) *hold. Let* $e \in E^+$. *Then there is* $r_0 > \rho$ *such that* $\sup \Phi(\partial Q) \le 0$, *where*

$$Q = \left\{v + se : v \in E^-, 0 \le s \le r_0, \|v\| \le r_0\right\}. \tag{2.40}$$

Lemma 2.6 *Assume that* (V1), (F1), (F2) *and* (WN) *hold. Then*

$$\Phi(u) \ge \frac{t^2}{2}\|u\|^2 + \int_{\mathbb{R}^2} F(x, tu^+)\mathrm{d}x + \frac{1-t^2}{2}\langle \Phi'(u), u^+\rangle + t^2\langle \Phi'(u), u^-\rangle, \quad \forall\, t \ge 0,\ u \in E. \tag{2.41}$$

Lemma 2.7 *Assume that* (V1), (F1), (F2), (SQ) *and* (WN) *hold. Then there exist a constant* $c^* \in [\kappa_0, m]$ *and a sequence* $\{u_n\} \subset E$ *satisfying*

$$\Phi(u_n) \to c^*, \quad \|\Phi'(u_n)\|(1 + \|u_n\|) \to 0, \tag{2.42}$$

where κ_0 *is defined by (2.36) and* $m = \inf_{u\in\mathcal{M}} \Phi(u)$.

By Lemmas 2.2, 2.4 and 2.5, one can get the following lemma.

Lemma 2.8 *Assume that* (V1), (F1), (F2) *and* (F3) *hold. Then there exist a constant* $\bar{c} \in [\kappa, \sup \Phi(Q)]$ *and a sequence* $\{u_n\} \subset E$ *satisfying*

$$\Phi(u_n) \to \bar{c}, \quad \|\Phi'(u_n)\|(1 + \|u_n\|) \to 0, \tag{2.43}$$

where Q *is defined by (2.40).*

3 Subcritical case

In this section, we study the subcritical exponential growth case and show Theorems 1.2 and 1.3. The first lemma is crucial when f has an exponential growth.

Lemma 3.1 *Assume that* (V1), (F1), (F2) *and* (F3) *hold. Then* $\{u_n\}$ *satisfying (2.43) is bounded in* E.

Proof From (F3), (2.8), (2.12) and (2.43), we have

$$\begin{aligned} \bar{c} + o(1) &= \Phi(u_n) - \frac{1}{2}\langle \Phi'(u_n), u_n\rangle \\ &= \int_{\mathbb{R}^2}\left[\frac{1}{2}f(x, u_n)u_n - F(x, u_n)\right]\mathrm{d}x \\ &\ge \frac{\bar{\mu}-2}{2\bar{\mu}}\int_{\mathbb{R}^2} f(x, u_n)u_n\mathrm{d}x. \end{aligned} \tag{3.1}$$

It follows from (2.11) and (2.43) that

$$o(1) = \langle \Phi'(u_n), u_n\rangle = \|u_n^+\|^2 - \|u_n^-\|^2 - \int_{\mathbb{R}^2} f(x, u_n)u_n\mathrm{d}x \tag{3.2}$$

and

$$o(1) = \langle \Phi'(u_n), u_n^-\rangle = -\|u_n^-\|^2 - \int_{\mathbb{R}^2} f(x, u_n)u_n^-\mathrm{d}x. \tag{3.3}$$

Combining (3.1) with (3.2), one obtains

$$\|u_n^+\|^2 - \|u_n^-\|^2 \le \frac{2\bar{\mu}\bar{c}}{\bar{\mu}-2} + o(1). \tag{3.4}$$

To prove the boundedness of $\{u_n\}$, arguing by contradiction, suppose that $\|u_n\| \to \infty$ as $n \to \infty$. Let $v_n = u_n/\|u_n\|$. Then $1 = \|v_n\|^2$. By (F2), we can choose $\delta_0 > 0$ such that

$$\left|\frac{f(x,t)}{t}\right| \le \frac{1}{4\gamma_2^2}, \quad \forall\, x \in \mathbb{R}^2,\ |t| \le \delta_0. \tag{3.5}$$

Then it follows from (2.25), (3.1), (3.3) and (3.5) that

$$\begin{aligned}
\|v_n^-\|^2 &= -\frac{1}{\|u_n\|}\int_{\mathbb{R}^2} f(x,u_n)v_n^- \mathrm{d}x + o(1)\\
&\le \frac{1}{\|u_n\|}\int_{\mathbb{R}^2} |f(x,u_n)||v_n^-|\mathrm{d}x + o(1)\\
&= \int_{|u_n|\le\delta_0} \frac{f(x,u_n)}{u_n}|v_n||v_n^-|\mathrm{d}x\\
&\quad + \frac{1}{\|u_n\|}\int_{|u_n|>\delta_0} |f(x,u_n)||v_n^-|\mathrm{d}x + o(1)\\
&\le \frac{1}{4\gamma_2^2}\|v_n\|_2\|v_n^-\|_2 + \frac{\|v_n^-\|_\infty}{\delta_0\|u_n\|}\int_{|u_n|>\delta_0} f(x,u_n)u_n\mathrm{d}x + o(1)\\
&\le \frac{1}{4\gamma_2^2}\|v_n\|_2\|v_n^-\|_2 + \frac{\mathcal{K}_0\|v_n^-\|_2}{\delta_0\|u_n\|}\int_{|u_n|>\delta_0} f(x,u_n)u_n\mathrm{d}x + o(1)\\
&\le \frac{1}{4} + o(1).
\end{aligned} \tag{3.6}$$

On the other hand, since $1 = \|v_n^+\|^2 + \|v_n^-\|^2$, then from (3.4) we obtain

$$\|v_n^-\|^2 \ge \frac{1}{2} + o(1), \tag{3.7}$$

which contradicts with (3.6). Thus $\{u_n\}$ is bounded in E. □

Lemma 3.2 *Assume that* (F1) (*or* (F1′)), (F2) *and* (F3) *hold. Let $u_n \rightharpoonup \bar{u}$ in E and*

$$\int_{\mathbb{R}^2} f(x,u_n)u_n\mathrm{d}x \le K_0 \tag{3.8}$$

for some constant $K_0 > 0$. Then for every $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$

$$\lim_{n\to\infty}\int_{\mathbb{R}^2} f(x,u_n)\phi\mathrm{d}x = \int_{\mathbb{R}^2} f(x,\bar{u})\phi\mathrm{d}x. \tag{3.9}$$

Lemma 3.2 is a direct consequence of [13, Lemma 2.1].

Proof of Theorem 1.2 Applying Lemmas 2.8 and 3.1, we deduce that there exists a bounded sequence $\{u_n\} \subset E$ satisfying (2.43) and $\|u_n\| \le C_1$ for some $C_1 > 0$. Thus there exists a constant $C_2 > 0$ such that $\|u_n\|_2 \le C_2$. If

$$\delta := \limsup_{n\to\infty}\sup_{y\in\mathbb{R}^2}\int_{B_1(y)} |u_n|^2\mathrm{d}x = 0,$$

then by Lions' concentration compactness principle [30, Lemma 1.21], one has $u_n \to 0$ in $L^s(\mathbb{R}^2)$ for $2 < s < \infty$. Let $\alpha \in (0, 1/C_1^2\vartheta_0^2)$, where ϑ_0 is defined by (2.16). Using (F1) and (F2), there exists $C_3 > 0$ such that

$$|f(x,t)| \le \frac{\bar{c}}{4C_2^2}|t| + C_3\left(e^{\alpha t^2} - 1\right), \quad \forall\,(x,t) \in \mathbb{R}^2\times\mathbb{R}. \tag{3.10}$$

Then (3.10) and Lemma 1.1 give

$$\begin{aligned}\int_{\mathbb{R}^2} f(x,u_n)u_n\mathrm{d}x &\le \frac{\bar{c}}{4C_2^2}\|u_n\|_2^2 + C_3\int_{\mathbb{R}^2}\left(e^{\alpha u_n^2}-1\right)|u_n|\mathrm{d}x\\ &\le \frac{\bar{c}}{4} + C_4\left[\int_{\mathbb{R}^2}\left(e^{\alpha u_n^2}-1\right)^{3/2}\mathrm{d}x\right]^{2/3}\|u_n\|_3\\ &\le \frac{\bar{c}}{4} + C_4\left[\int_{\mathbb{R}^2}\left(e^{3\alpha u_n^2/2}-1\right)\mathrm{d}x\right]^{2/3}\|u_n\|_3\\ &= \frac{\bar{c}}{4} + C_4\left[\int_{\mathbb{R}^2}\left(e^{\frac{3}{2}\alpha\vartheta_0^2\|u_n\|^2(u_n/\vartheta_0\|u_n\|)^2}-1\right)\mathrm{d}x\right]^{2/3}\|u_n\|_3\\ &\le \frac{\bar{c}}{4} + o(1).\end{aligned}\tag{3.11}$$

Now by (2.8), (2.12) and (3.11), we have

$$\begin{aligned}\bar{c}+o(1) &= \Phi(u_n) - \frac{1}{2}\langle\Phi'(u_n),u_n\rangle\\ &= \int_{\mathbb{R}^2}\left[\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\right]\mathrm{d}x \le \frac{\bar{c}}{8}+o(1).\end{aligned}\tag{3.12}$$

This contradiction shows that $\delta_0 > 0$.

Going if necessary to a subsequence, we may assume that there exists $\{k_n\}\subset\mathbb{Z}^2$ such that $\int_{B_{1+\sqrt{2}}(k_n)}|u_n|^2\mathrm{d}x > \frac{\delta}{2}$. Let us define $v_n(x) = u_n(x+k_n)$ so that

$$\int_{B_{1+\sqrt{2}}(0)}|v_n|^2\mathrm{d}x > \frac{\delta}{2}.\tag{3.13}$$

Since $V(x)$ and $f(x,u)$ are 1-periodic on x, we have $\|v_n\| = \|u_n\|$ and

$$\Phi(v_n)\to\bar{c},\quad \|\Phi'(v_n)\|(1+\|v_n\|)\to 0.\tag{3.14}$$

Passing to a subsequence, we have $v_n \rightharpoonup v$ in E, $v_n\to v$ in $L^s_{\text{loc}}(\mathbb{R}^2)$, $2\le s<\infty$ and $v_n\to v$ a.e. on $\mathbb{R}^2$. Thus, (3.13) implies that $v\ne 0$. Moreover, (2.11), (3.14) and Lemma 3.2 yield for every $\phi\in\mathcal{C}_0^\infty(\mathbb{R}^2)$,

$$\langle\Phi'(v),\phi\rangle = \lim_{n\to\infty}\langle\Phi'(v_n),\phi\rangle = 0.$$

Hence $\Phi'(v)=0$. This completes the proof. □

Lemma 3.3 *Assume that* (V1), (F1), (F2), (SQ) *and* (WN) *hold. Then any sequence* $\{u_n\}$ *satisfying (2.42) is bounded in* E.

Proof To prove the boundedness of $\{u_n\}$, arguing by contradiction, suppose that $\|u_n\|\to\infty$. Let $v_n = u_n/\|u_n\|$. Then $\|v_n\|=1$, and (2.6) gives $\|v_n\|_2\le\gamma_2$. Passing to a subsequence, we may assume that $v_n\rightharpoonup v$ in E, $v_n\to v$ in $L^s_{\text{loc}}(\mathbb{R}^2)$, $2\le s<\infty$, $v_n\to v$ a.e. on $\mathbb{R}^2$. If

$$\delta := \limsup_{n\to\infty}\sup_{y\in\mathbb{R}^2}\int_{B_1(y)}|v_n^+|^2\mathrm{d}x = 0,$$

then by Lions' concentration compactness principle [30, Lemma 1.21], $v_n^+\to 0$ in $L^s(\mathbb{R}^2)$ for $2<s<\infty$. By (WN), we obtain

$$f(x,t)t\ge 2F(x,t),\quad \forall\,(x,t)\in\mathbb{R}^2\times\mathbb{R}.\tag{3.15}$$

Let us fix $R > [2(1+c^*)]^{1/2}$, where c^* is given by Lemma 2.7. Set $\alpha \in (0, 1/(R\gamma_2\vartheta_0)^2)$. By (F1), (F2) and (3.15), there exists $C_6 > 0$ such that

$$|F(x,t)| \le \frac{1}{4(R\gamma_2)^2}t^2 + C_6|t|\left(e^{\alpha t^2}-1\right), \quad \forall\, (x,t) \in \mathbb{R}^2 \times \mathbb{R}. \tag{3.16}$$

Then (3.16) and Lemma 1.1-ii) lead to

$$\begin{aligned}
\int_{\mathbb{R}^2} F(x, Rv_n^+)\mathrm{d}x &\le \frac{1}{4\gamma_2^2}\|v_n^+\|_2^2 + C_6R\int_{\mathbb{R}^2}\left(e^{\alpha R^2(v_n^+)^2}-1\right)|v_n^+|\mathrm{d}x \\
&\le \frac{1}{4} + C_6R\left[\int_{\mathbb{R}^2}\left(e^{\alpha R^2(v_n^+)^2}-1\right)^{3/2}\mathrm{d}x\right]^{2/3}\|v_n^+\|_3 \\
&\le \frac{1}{4} + C_6R\left[\int_{\mathbb{R}^2}\left(e^{3\alpha R^2(v_n^+)^2/2}-1\right)\mathrm{d}x\right]^{2/3}\|v_n^+\|_3 \\
&= \frac{1}{4} + C_6R\left[\int_{\mathbb{R}^2}\left(e^{\frac{3}{2}\alpha R^2\vartheta_0^2\|v_n^+\|^2(v_n^+/\vartheta_0\|v_n^+\|)^2}-1\right)\mathrm{d}x\right]^{2/3}\|v_n^+\|_3 \\
&\le \frac{1}{4} + o(1).
\end{aligned} \tag{3.17}$$

Let $t_n = R/\|u_n\|$. Hence, from (2.42), (3.17) and Lemma 2.6, we derive

$$\begin{aligned}
c^* + o(1) &= \Phi(u_n) \\
&\ge \frac{t_n^2}{2}\|u_n\|^2 - \int_{\mathbb{R}^2} F(x, t_nu_n^+)\mathrm{d}x + \frac{1-t_n^2}{2}\langle \Phi'(u_n), u_n\rangle + t_n^2\langle \Phi'(u_n), u_n^-\rangle \\
&= \frac{R^2}{2}\|v_n\|^2 - \int_{\mathbb{R}^2} F(x, Rv_n^+)\mathrm{d}x + \left(\frac{1}{2} - \frac{R^2}{2\|u_n\|^2}\right)\langle \Phi'(u_n), u_n\rangle \\
&\quad + \frac{R^2}{\|u_n\|^2}\langle \Phi'(u_n), u_n^-\rangle \\
&= \frac{R^2}{2} - \int_{\mathbb{R}^2} F(x, Rv_n^+)\mathrm{d}x + o(1) \\
&\ge \frac{R^2}{2} - \frac{1}{4} + o(1) > c^* + \frac{3}{4} + o(1),
\end{aligned}$$

which is a contradiction. This shows that $\delta > 0$. The rest of the proof is standard, so we omit it. □

Proof of Theorem 1.3 Applying Lemmas 2.7 and 3.3, we can deduce that there exists a bounded sequence $\{u_n\} \subset E$ satisfying (2.42). Similar to the proof of Theorem 1.2, we have $u_n \rightharpoonup \bar{u} \in E \setminus \{0\}$ and $\Phi'(\bar{u}) = 0$. This shows that $\bar{u} \in \mathcal{M}$, and so $\Phi(\bar{u}) \ge m$. On the other hand, by using (2.42), (3.15) and Fatou's lemma, we have

$$\begin{aligned}
m \ge c_* &= \lim_{n\to\infty}\left[\Phi(u_n) - \frac{1}{2}\langle \Phi'(u_n), u_n\rangle\right] \\
&= \lim_{n\to\infty}\int_{\mathbb{R}^2}\left[\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\right]\mathrm{d}x \\
&\ge \int_{\mathbb{R}^2}\lim_{n\to\infty}\left[\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\right]\mathrm{d}x
\end{aligned}$$

$$= \int_{\mathbb{R}^2} \left[\frac{1}{2} f(x, \bar{u})\bar{u} - F(x, \bar{u})\right] \mathrm{d}x$$
$$= \Phi(\bar{u}) - \frac{1}{2}\langle \Phi'(\bar{u}), \bar{u}\rangle = \Phi(\bar{u}).$$

Hence, $\Phi(\bar{u}) \le m$ and so $\Phi(\bar{u}) = m = \inf_{\mathcal{M}} \Phi > 0$. This completes the proof. □

4 Critical case

In this section, we consider the critical exponential growth case and give the proof of Theorem 1.4.

Let $\{e_k\}$ be a total orthonormal sequence in E^-. Define $E_k^- := \mathrm{span}\{e_1, e_2, \dots, e_k\}$ and $E_k := E_k^- \oplus E^+$ for $k \in \mathbb{N}$.

Lemma 4.1 ([6]) *Let* $X = Y \oplus Z$ *be a Banach space with* $\dim Y < \infty$. *Let* $e \in \partial B_1(0) \cap Z$ *be fixed and let* $0 < \rho < R$ *be given positive real numbers. Let*

$$\tilde{Q} = \{v + se : v \in Y,\ 0 \le s \le R,\ \|v\| \le R\}.$$

Let $\varphi \in \mathcal{C}^1(X, \mathbb{R})$ *such that*

$$\inf_{Z \cap \partial B_\rho} \varphi > \sup_{\partial \tilde{Q}} \varphi.$$

Then there exists a sequence $\{u_n\} \subset X$ *satisfying*

$$\varphi(u_n) \to c, \quad \|\varphi'(u_n)\|(1 + \|u_n\|) \to 0 \tag{4.1}$$

with

$$c = \inf_{\gamma \in \Gamma} \sup_{u \in \tilde{Q}} I(\gamma(u)),$$

where

$$\Gamma = \{\gamma \in \mathcal{C}(\tilde{Q}, X) : \gamma|_{\partial \tilde{Q}} = \mathrm{id}\}.$$

Lemma 4.2 *Assume that* (V1), (F1′), (F2) *and* (F3) *hold. Let* $e \in \partial B_1(0) \cap E^+$. *Then there is* $r_0 > \bar{\rho}$ *such that* $\sup \Phi(\partial Q_k) \le 0$, *where* $\bar{\rho}$ *is given by* Lemma 2.4 *and*

$$Q_k = \left\{v + se : v \in E_k^-, 0 \le s \le r_0, \|v\| \le r_0\right\}, \quad k \in \mathbb{N}. \tag{4.2}$$

Proof By Lemma 2.5, there exists $r_0 > \bar{\rho}$ such that $\sup \Phi(\partial Q) \le 0$, where

$$Q = \left\{v + se : v \in E^-, 0 \le s \le r_0, \|v\| \le r_0\right\}. \tag{4.3}$$

Since $E_k^- \subset E^-$, then one has $\partial Q_k \subset \partial Q$ for all $k \in \mathbb{N}$. Thus, $\sup \Phi(\partial Q_k) \le 0$ for all $k \in \mathbb{N}$. □

For each $k \in \mathbb{N}$, let

$$\Gamma_k := \{\gamma \in \mathcal{C}(Q_k, E) : \gamma|_{\partial Q_k} = \mathrm{id}\} \tag{4.4}$$

and

$$c_k := \inf_{\gamma \in \Gamma_k} \sup_{u \in Q_k} I(\gamma(u)). \tag{4.5}$$

From Lemmas 2.4, 4.2 and the definition of c_k, one can show easily the following lemma.

Lemma 4.3 *Assume that* (V1), (F1′), (F2) *and* (F3) *hold. Then*

$$\kappa_0 \le c_k \le \frac{r_0^2}{2}, \quad \forall\, k \in \mathbb{N}. \tag{4.6}$$

where κ_0 *is given by* Lemma 2.4.

Applying Lemma 4.1 to Φ and E_k and using Lemmas 2.4 and 4.2, one can get the following lemma.

Lemma 4.4 *Assume that* (V1), (F1′), (F2) *and* (F3) *hold. Then for every* $k \in \mathbb{N}$, *there exists a sequence* $\{u_n^k\} \subset E_k$ *satisfying*

$$\Phi(u_n^k) \to c_k, \quad \|\Phi'(u_n^k)\|_{E_k^*}(1 + \|u_n^k\|) \to 0, \quad n \to \infty, \tag{4.7}$$

where c_k *is defined by (4.5).*

Lemma 4.5 *Assume that* (V1), (F1′)), (F2) *and* (F3) *hold. If* $\{u_n^k\}$ *satisfies (4.7), then*

$$\|u_n^k\| \le \max\left\{\frac{4\bar{\mu}c_k(\delta_0 + 2\mathcal{K}_0\gamma_2)}{(\bar{\mu}-2)\delta_0}, 1\right\} + o_n(1), \quad \forall\, k \in \mathbb{N}, \tag{4.8}$$

where γ_2 *and* δ_0 *are given by (2.6) and (3.5), respectively.*

Proof From (F3), (2.8), (2.12) and (4.7), we have

$$\begin{aligned} c_k + o_n(1) &= \Phi(u_n^k) - \frac{1}{2}\langle \Phi'(u_n^k), u_n^k\rangle \\ &= \int_{\mathbb{R}^2}\left[\frac{1}{2}f(x,u_n^k)u_n^k - F(x,u_n^k)\right]\mathrm{d}x \\ &\ge \frac{\bar{\mu}-2}{2\bar{\mu}}\int_{\mathbb{R}^2} f(x,u_n^k)u_n^k\mathrm{d}x. \end{aligned} \tag{4.9}$$

It follows from (2.11) and (4.7) that

$$o_n(1) = \langle \Phi'(u_n^k), u_n^k\rangle = \|(u_n^k)^+\|^2 - \|(u_n^k)^-\|^2 - \int_{\mathbb{R}^2} f(x,u_n^k)u_n^k\mathrm{d}x \tag{4.10}$$

and

$$o_n(1) = \langle \Phi'(u_n^k), (u_n^k)^-\rangle = -\|(u_n^k)^-\|^2 - \int_{\mathbb{R}^2} f(x,u_n^k)(u_n^k)^-\mathrm{d}x. \tag{4.11}$$

Combining (4.9) with (4.10), one obtain

$$\|(u_n^k)^+\|^2 - \|(u_n^k)^-\|^2 \le \frac{2\bar{\mu}c_k}{\bar{\mu}-2} + o_n(1). \tag{4.12}$$

Let $v_n = u_n^k/\|u_n^k\|$. Then $1 = \|v_n\|^2$ and $\|v_n^-\|_2 \le \gamma_2$. It follows from (2.25), (3.5), (4.9) and (4.11) that

$$\begin{aligned} \|v_n^-\|^2 &= -\frac{1}{\|u_n^k\|}\int_{\mathbb{R}^2} f(x,u_n^k)v_n^-\mathrm{d}x + o_n(1) \\ &\le \frac{1}{\|u_n^k\|}\int_{\mathbb{R}^2} |f(x,u_n^k)||v_n^-|\mathrm{d}x + o_n(1) \\ &= \int_{|u_n^k|\le\delta_0} \frac{f(x,u_n^k)}{u_n^k}|v_n||v_n^-|\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\|u_n^k\|}\int_{|u_n^k|>\delta_0}|f(x,u_n^k)||v_n^-|\mathrm{d}x+o_n(1)\\
&\le \frac{1}{4\gamma_2^2}\|v_n\|_2\|v_n^-\|_2+\frac{\|v_n^-\|_\infty}{\delta_0\|u_n^k\|}\int_{|u_n^k|>\delta_0}f(x,u_n^k)u_n^k\mathrm{d}x+o_n(1)\\
&\le \frac{1}{4\gamma_2^2}\|v_n\|_2\|v_n^-\|_2+\frac{\mathcal{K}_0\|v_n^-\|_2}{\delta_0\|u_n^k\|}\int_{|u_n^k|>\delta_0}f(x,u_n^k)u_n^k\mathrm{d}x+o_n(1)\\
&\le \frac{1}{4}+\frac{2\bar{\mu}c_k\mathcal{K}_0\gamma_2}{(\bar{\mu}-2)\delta_0\|u_n^k\|}+o_n(1).
\end{aligned}
\tag{4.13}
$$

On the other hand, since $1=\|v_n^+\|^2+\|v_n^-\|^2$, then from (4.12) we obtain

$$
\frac{\bar{\mu}c_k}{(\bar{\mu}-2)\|u_n^k\|^2}+\|v_n^-\|^2\ge\frac{1}{2}+o_n(1),
\tag{4.14}
$$

which, together with (4.13), implies that (4.8) holds. □

Applying Lemma 2.3, we deduce that

$$
\|\nabla v\|_\infty+\|v\|_\infty\le\mathcal{C}_0\|v\|,\quad \forall\, v\in E^-,
\tag{4.15}
$$

Without loss of generality, we may assume that $V(0)<0$. By (V1), we can choose a constant $\rho\in(0,1/2)\cap(0,4/\|V\|_\infty)$ such that

$$
4\pi\mathcal{C}_0^2(4+\rho)\rho<1\ \text{ and }\ V(x)\le 0,\quad |x|\le\rho.
\tag{4.16}
$$

As in [13], we define Moser type functions $w_n(x)$ supported in B_ρ as follows:

$$
w_n(x)=\frac{1}{\sqrt{2\pi}}\begin{cases}\sqrt{\log n}, & 0\le|x|\le\rho/n;\\ \frac{\log(\rho/|x|)}{\sqrt{\log n}}, & \rho/n\le|x|\le\rho;\\ 0, & |x|\ge\rho.\end{cases}
\tag{4.17}
$$

By a computation, one has

$$
\|w_n^+\|^2-\|w_n^-\|^2=\int_{\mathbb{R}^2}(|\nabla w_n|^2+V(x)w_n^2)\mathrm{d}x\le\int_{B_\rho}|\nabla w_n|^2\mathrm{d}x=1.
\tag{4.18}
$$

Lemma 4.6 *Assume that* (V1), (F1′), (F2), (F3) *and* (F5) *hold. Then there exists* $\bar{n}\in\mathbb{N}$ *such that*

$$
\max_{s\ge0,v\in E^-}\Phi(v+sw_{\bar{n}})<\frac{2\pi}{\alpha_0}.
\tag{4.19}
$$

Proof Assume by contradiction that this is not the case. So one has

$$
\max_{s\ge0,v\in E^-}\Phi(v+sw_n)\ge\frac{2\pi}{\alpha_0},\quad \forall\, n\in\mathbb{N}.
\tag{4.20}
$$

Let $v_n\in E^-$ and $s_n>0$ such that $\Phi(v_n+s_nw_n)=\max_{s\ge0,v\in E^-}\Phi(v+sw_n)$. Then we have $\Phi(v_n+s_nw_n)\ge2\pi/\alpha_0$ and $\langle\Phi'(v_n+s_nw_n),v_n+s_nw_n\rangle=0$, i.e.

$$
\frac{1}{2}\left(s_n^2\|w_n^+\|^2-\|v_n+s_nw_n^-\|^2\right)-\int_{\mathbb{R}^2}F(x,v_n+s_nw_n)\mathrm{d}x\ge\frac{2\pi}{\alpha_0}
\tag{4.21}
$$

and

$$s_n^2\|w_n^+\|^2-\|v_n+s_nw_n^-\|^2=\int_{\mathbb{R}^2}f(x,v_n+s_nw_n)(v_n+s_nw_n)\mathrm{d}x. \tag{4.22}$$

From (2.2), (2.4), (4.15) and (4.17), we have

$$\begin{aligned}|(w_n^-,v_n)|=|(w_n,v_n)|&=\left|\int_{\mathbb{R}^2}[\nabla w_n\nabla v_n+V(x)w_nv_n]\,\mathrm{d}x\right|\\ &\le\|\nabla v_n\|_\infty\int_{\mathbb{R}^2}|\nabla w_n|\mathrm{d}x+\|V\|_\infty\|v_n\|_\infty\int_{\mathbb{R}^2}|w_n|\mathrm{d}x\\ &\le\frac{\sqrt{2\pi}\mathcal{C}_0\rho}{\sqrt{\log n}}\|v_n\|. \end{aligned}\tag{4.23}$$

Hence it follows from (2.2), (2.4), (2.5), (4.18) and (4.23) that

$$\begin{aligned}s_n^2\|w_n^+\|^2-\|v_n+s_nw_n^-\|^2&=s_n^2\left(\|w_n^+\|^2-\|w_n^-\|^2\right)-\|v_n\|^2-2s_n(v_n,w_n^-)\\ &\le s_n^2-\|v_n\|^2+\frac{2\sqrt{2\pi}\mathcal{C}_0\rho s_n}{\sqrt{\log n}}\|v_n\|. \end{aligned}\tag{4.24}$$

Combining (4.21), (4.22) with (4.24), we have

$$\frac{4\pi}{\alpha_0}\le s_n^2-\|v_n\|^2+\frac{2\sqrt{2\pi}\mathcal{C}_0\rho s_n}{\sqrt{\log n}}\|v_n\|\le s_n^2\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right) \tag{4.25}$$

and

$$\begin{aligned}s_n^2\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right)&\ge s_n^2-\|v_n\|^2+\frac{2\sqrt{2\pi}\mathcal{C}_0\rho s_n}{\sqrt{\log n}}\|v_n\|\\ &\ge\int_{\mathbb{R}^2}f(x,v_n+s_nw_n)(v_n+s_nw_n)\mathrm{d}x. \end{aligned}\tag{4.26}$$

Moreover, (4.25) implies

$$s_n^2\ge\frac{4\pi}{\alpha_0}\left(1-\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right),\qquad\frac{\|v_n\|}{s_n}\le1+\frac{2\sqrt{2\pi}\mathcal{C}_0\rho}{\sqrt{\log n}}. \tag{4.27}$$

Let $M_n=\frac{1}{\sqrt{2\pi}}\sqrt{\log n}$. By (4.15), (4.17) and (4.27), we have

$$\begin{aligned}v_n(x)+s_nw_n(x)&\ge-\|v_n\|_\infty+s_nM_n\\ &\ge-\mathcal{C}_0\|v_n\|+s_nM_n\\ &\ge(1-2\mathcal{C}_0/M_n)s_nM_n,\quad\forall\,x\in B_{\rho/n}. \end{aligned}\tag{4.28}$$

By (F5), we can choose $\varepsilon>0$ such that

$$\frac{\kappa-\varepsilon}{1+\varepsilon}>>\frac{4e^{16\pi\mathcal{C}_0^2}}{\alpha_0\rho^2}. \tag{4.29}$$

Note that

$$\liminf_{|t|\to\infty}\frac{t^2F(x,t)}{e^{\alpha_0t^2}}\ge\liminf_{|t|\to\infty}\frac{\int_0^ts^2f(x,s)\mathrm{d}s}{e^{\alpha_0t^2}}=\liminf_{|t|\to\infty}\frac{tf(x,t)}{2\alpha_0e^{\alpha_0t^2}}=\frac{\kappa}{2\alpha_0}. \tag{4.30}$$

It follows from (F5) and (4.30) that there exists $t_\varepsilon > 0$ such that

$$tf(x,t) \geq (\kappa-\varepsilon)e^{\alpha_0 t^2}, \quad t^2F(x,t) \geq \frac{\kappa-\varepsilon}{2\alpha_0}e^{\alpha_0 t^2}, \quad \forall\, x\in\mathbb{R}^2,\ |t|\geq t_\varepsilon. \tag{4.31}$$

From now on, in the sequel, all inequalities hold for large $n\in\mathbb{N}$. By (4.26), (4.28) and (4.31), we have

$$\begin{aligned}
s_n^2\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right) &\geq \int_{\mathbb{R}^2} f(x, v_n+s_nw_n)(v_n+s_nw_n)\mathrm{d}x \\
&\geq (\kappa-\varepsilon)\int_{B_{\rho/n}} e^{\alpha_0(v_n+s_nw_n)^2}\mathrm{d}x \\
&\geq \frac{\pi(\kappa-\varepsilon)\rho^2}{n^2}e^{\alpha_0 s_n^2M_n^2(1-2\mathcal{C}_0/M_n)^2} \\
&\geq \frac{\pi(\kappa-\varepsilon)\rho^2}{n^2}\exp\left[\frac{\alpha_0 s_n^2\log n}{2\pi}\left(1-\frac{4\mathcal{C}_0}{M_n}\right)\right] \\
&= \pi(\kappa-\varepsilon)\rho^2\exp\left\{2\log n\left[\frac{\alpha_0 s_n^2}{4\pi}\left(1-\frac{4\mathcal{C}_0}{M_n}\right)-1\right]\right\},
\end{aligned}$$

which implies that there exists a constant $A>0$ such that

$$2\log n\left[\frac{\alpha_0 s_n^2}{4\pi}\left(1-\frac{4\mathcal{C}_0}{M_n}\right)-1\right]\leq A.$$

That is

$$s_n^2 \leq \frac{4\pi}{\alpha_0}\left(1-\frac{4\mathcal{C}_0}{M_n}\right)^{-1}\left(1+\frac{A}{2\log n}\right). \tag{4.32}$$

Hence, from (2.8), (4.17), (4.24), (4.28) and (4.31), we obtain

$$\begin{aligned}
&\Phi(v_n+s_nw_n) \\
&= \frac{1}{2}\left(s_n^2\|w_n^+\|^2-\|v_n+s_nw_n^-\|^2\right)-\int_{\mathbb{R}^2}F(x,v_n+s_nw_n)\mathrm{d}x \\
&\leq \frac{s_n^2}{2}-\frac{1}{2}\|v_n\|^2+\frac{\sqrt{2\pi}\mathcal{C}_0\rho s_n}{\sqrt{\log n}}\|v_n\|-\int_{\mathbb{R}^2}F(x,v_n+s_nw_n)\mathrm{d}x \\
&\leq \frac{s_n^2}{2}-\frac{1}{2}\|v_n\|^2+\frac{\sqrt{2\pi}\mathcal{C}_0\rho s_n}{\sqrt{\log n}}\|v_n\|-\frac{\kappa-\varepsilon}{2\alpha_0}\int_{B_{\rho/n}}\frac{e^{\alpha_0(v_n+s_nw_n)^2}}{(v_n+s_nw_n)^2}\mathrm{d}x \\
&\leq \frac{s_n^2}{2}-\frac{1}{2}\|v_n\|^2+\frac{\sqrt{2\pi}\mathcal{C}_0\rho s_n}{\sqrt{\log n}}\|v_n\|-\frac{(\kappa-\varepsilon)\pi\rho^2e^{\alpha_0(-\mathcal{C}_0\|v_n\|+s_nM_n)^2}}{2\alpha_0n^2(-\mathcal{C}_0\|v_n\|+s_nM_n)^2}.
\end{aligned} \tag{4.33}$$

Both (4.27) and (4.32) show that $\frac{4\pi}{\alpha_0}(1-\varepsilon)\leq s_n^2\leq\frac{4\pi}{\alpha_0}(1+\varepsilon)$. There are three cases to distinguish.

Case i) $\frac{4\pi}{\alpha_0}(1-\varepsilon)\leq s_n^2\leq\frac{4\pi}{\alpha_0}$. It follows from (4.25) that $\|v_n\|\leq 2\pi\mathcal{C}_0s_nM_n/\log n$. Then (4.33) leads to

$$\begin{aligned}
&\Phi(v_n+s_nw_n) \\
&\leq \frac{s_n^2}{2}-\frac{1}{2}\|v_n\|^2+\frac{\sqrt{2\pi}\mathcal{C}_0\rho s_n}{\sqrt{\log n}}\|v_n\|-\frac{(\kappa-\varepsilon)\pi\rho^2e^{\alpha_0(-\mathcal{C}_0\|v_n\|+s_nM_n)^2}}{2\alpha_0n^2(-\mathcal{C}_0\|v_n\|+s_nM_n)^2}
\end{aligned}$$

$$\leq \frac{s_n^2}{2}\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right)-\frac{(\kappa-\varepsilon)\rho^2 e^{\alpha_0(-\mathcal{C}_0\|v_n\|+s_nM_n)^2}}{8n^2(1+\varepsilon)M_n^2}$$
$$\leq \frac{s_n^2}{2}\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right)-\frac{(\kappa-\varepsilon)\rho^2 e^{\alpha_0 s_n^2M_n^2(1-2\mathcal{C}_0\|v_n\|/s_nM_n)}}{8n^2(1+\varepsilon)M_n^2}$$
$$\leq \frac{s_n^2}{2}\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right)-\frac{(\kappa-\varepsilon)\pi\rho^2 e^{\frac{\alpha_0 s_n^2}{2\pi}(\log n-4\pi\mathcal{C}_0^2)}}{4n^2(1+\varepsilon)\log n}. \tag{4.34}$$

Let us define a function $\varphi_n(s)$ as follows:

$$\varphi_n(s)=\frac{s^2}{2}\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right)-\frac{(\kappa-\varepsilon)\pi\rho^2 e^{\frac{\alpha_0 s^2}{2\pi}(\log n-4\pi\mathcal{C}_0^2)}}{4n^2(1+\varepsilon)\log n}. \tag{4.35}$$

Set $\hat{s}_n>0$ such that $\varphi_n'(\hat{s}_n)=0$. Then

$$\hat{s}_n^2=\frac{4\pi}{\alpha_0}\left[1+\frac{8\pi\mathcal{C}_0^2+\log 4(1+\varepsilon)-\log(\alpha_0(\kappa-\varepsilon)\rho^2)}{2(\log n-4\pi\mathcal{C}_0^2)}\right]+O\left(\frac{1}{\log^2 n}\right) \tag{4.36}$$

and

$$\varphi_n(s_n)\leq\varphi_n(\hat{s}_n)=\frac{1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}}{2}\hat{s}_n^2-\frac{\pi\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right)}{\alpha_0(\log n-4\pi\mathcal{C}_0^2)}. \tag{4.37}$$

Using (4.36), we have

$$\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right)\hat{s}_n^2$$
$$=\frac{4\pi}{\alpha_0}\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right)\left[1+\frac{8\pi\mathcal{C}_0^2+\log 4(1+\varepsilon)-\log(\alpha_0(\kappa-\varepsilon)\rho^2)}{2(\log n-4\pi\mathcal{C}_0^2)}\right]$$
$$+O\left(\frac{1}{\log^2 n}\right)$$
$$\leq\frac{4\pi}{\alpha_0}\left[1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}+\frac{8\pi\mathcal{C}_0^2+\log 4(1+\varepsilon)-\log(\alpha_0(\kappa-\varepsilon)\rho^2)}{2(\log n-4\pi\mathcal{C}_0^2)}\right]$$
$$+O\left(\frac{1}{\log^2 n}\right). \tag{4.38}$$

Hence, from (4.16), (4.29), (4.34), (4.37) and (4.38), we derive

$$\Phi(v_n+s_nw_n)\leq\varphi_n(s_n)$$
$$\leq\frac{1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}}{2}\hat{s}_n^2-\frac{\pi\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right)}{\alpha_0(\log n-4\pi\mathcal{C}_0^2)}$$
$$\leq\frac{4\pi}{\alpha_0}\left[\frac{1}{2}-\frac{1-4\pi\mathcal{C}_0^2\rho^2}{4\log n}+\frac{8\pi\mathcal{C}_0^2+\log 4(1+\varepsilon)-\log(\alpha_0(\kappa-\varepsilon)\rho^2)}{4(\log n-4\pi\mathcal{C}_0^2)}\right]$$
$$+O\left(\frac{1}{\log^2 n}\right)$$

$$\leq \frac{4\pi}{\alpha_0}\left[\frac{1}{2}-\frac{1-4\pi\mathcal{C}_0^2\rho^2}{4\log n}\right]+O\left(\frac{1}{\log^2 n}\right). \tag{4.39}$$

This contradicts with (4.20) due to (4.16).

Case ii) $\frac{4\pi}{\alpha_0}(1+2\mathcal{C}_0\|v_n\|/s_nM_n) \leq s_n^2 \leq \frac{4\pi}{\alpha_0}(1+\varepsilon)$. Then (4.25), (4.26), (4.28), (4.29), (4.31) and (4.32) yield

$$\begin{aligned}
\frac{4\pi}{\alpha_0}(1+\varepsilon) &\geq s_n^2\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right)\\
&\geq \int_{\mathbb{R}^2} f(x, v_n+s_nw_n)(v_n+s_nw_n)\mathrm{d}x\\
&\geq (\kappa-\varepsilon)\int_{B_{\rho/n}} e^{\alpha_0(v_n+s_nw_n)^2}\mathrm{d}x\\
&\geq \frac{\pi(\kappa-\varepsilon)\rho^2}{n^2}e^{\alpha_0(-\mathcal{C}_0\|v_n\|+s_nM_n)^2}\\
&\geq \frac{\pi(\kappa-\varepsilon)\rho^2}{n^2}e^{\alpha_0 s_n^2M_n^2(1-2\mathcal{C}_0\|v_n\|/s_nM_n)}\\
&\geq \frac{\pi(\kappa-\varepsilon)\rho^2}{n^2}e^{2\log n(1-\mathcal{C}_0^2\|v_n\|^2/s_n^2M_n^2)}\\
&\geq \pi(\kappa-\varepsilon)\rho^2e^{-16\pi\mathcal{C}_0^2\|v_n\|^2/s_n^2}\\
&\geq \frac{4\pi}{\alpha_0}(1+\varepsilon)e^{15\pi\mathcal{C}_0^2},
\end{aligned}$$

which yields a contradiction.

Case iii) $\frac{4\pi}{\alpha_0} \leq s_n^2 \leq \frac{4\pi}{\alpha_0}(1+2\mathcal{C}_0\|v_n\|/s_nM_n)$. Then it follows from (4.25) that

$$\|v_n\|^2-\frac{2\sqrt{2\pi}\mathcal{C}_0\rho s_n}{\sqrt{\log n}}\|v_n\| \leq \frac{8\pi\mathcal{C}_0\|v_n\|}{\alpha_0 s_nM_n}=\frac{8\pi\sqrt{2\pi}\mathcal{C}_0}{\alpha_0 s_n\sqrt{\log n}}\|v_n\|, \tag{4.40}$$

which, together with (4.27) and (4.32), implies that

$$\frac{\|v_n\|}{s_n} \leq \frac{2\sqrt{2\pi}(1+\rho)\mathcal{C}_0}{\sqrt{\log n}}. \tag{4.41}$$

It follows from (4.33) and (4.41) that

$$\begin{aligned}
&\Phi(v_n+s_nw_n)\\
&\leq \frac{s_n^2}{2}-\frac{1}{2}\|v_n\|^2+\frac{\sqrt{2\pi}\mathcal{C}_0\rho s_n}{\sqrt{\log n}}\|v_n\|-\frac{(\kappa-\varepsilon)\pi\rho^2e^{\alpha_0(-\mathcal{C}_0\|v_n\|+s_nM_n)^2}}{2\alpha_0n^2(-\mathcal{C}_0\|v_n\|+s_nM_n)^2}\\
&\leq \frac{s_n^2}{2}\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right)-\frac{(\kappa-\varepsilon)\rho^2e^{\alpha_0(-\mathcal{C}_0\|v_n\|+s_nM_n)^2}}{8n^2(1+\varepsilon)M_n^2}\\
&\leq \frac{s_n^2}{2}\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right)-\frac{(\kappa-\varepsilon)\rho^2e^{\alpha_0s_n^2M_n^2(1-2\mathcal{C}_0\|v_n\|/s_nM_n)}}{8n^2(1+\varepsilon)M_n^2}\\
&\leq \frac{s_n^2}{2}\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right)-\frac{(\kappa-\varepsilon)\pi\rho^2e^{\frac{\alpha_0s_n^2}{2\pi}[\log n-8\pi(1+\rho)\mathcal{C}_0^2]}}{4n^2(1+\varepsilon)\log n}.
\end{aligned} \tag{4.42}$$

Setting

$$\psi_n(s) = \frac{s^2}{2}\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right) - \frac{(\kappa-\varepsilon)\pi\rho^2 e^{\frac{\alpha_0 s^2}{2\pi}[\log n - 8\pi(1+\rho)\mathcal{C}_0^2]}}{4n^2(1+\varepsilon)\log n}. \tag{4.43}$$

Let $\tilde{s}_n > 0$ such that $\psi_n'(\tilde{s}_n) = 0$. Then

$$\tilde{s}_n^2 = \frac{4\pi}{\alpha_0}\left\{1+\frac{16\pi(1+\rho)\mathcal{C}_0^2 + \log 4(1+\varepsilon) - \log(\alpha_0(\kappa-\varepsilon)\rho^2)}{2[\log n - 8\pi(1+\rho)\mathcal{C}_0^2]}\right\} + O\left(\frac{1}{\log^2 n}\right) \tag{4.44}$$

and

$$\psi_n(s_n) \le \psi_n(\tilde{s}_n) = \frac{1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}}{2}\tilde{s}_n^2 - \frac{\pi\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right)}{\alpha_0[\log n - 8\pi(1+\rho)\mathcal{C}_0^2]}. \tag{4.45}$$

Combining (4.44) with (4.45), we have

$$\begin{aligned}
&\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right)\tilde{s}_n^2 \\
&= \frac{4\pi}{\alpha_0}\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right) \\
&\quad\times\left[1+\frac{16\pi(1+\rho)\mathcal{C}_0^2 + \log 4(1+\varepsilon) - \log(\alpha_0(\kappa-\varepsilon)\rho^2)}{2[\log n - 8\pi(1+\rho)\mathcal{C}_0^2]}\right] + O\left(\frac{1}{\log^2 n}\right) \\
&\le \frac{4\pi}{\alpha_0}\left\{1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}+\frac{16\pi(1+\rho)\mathcal{C}_0^2 + \log 4(1+\varepsilon) - \log(\alpha_0(\kappa-\varepsilon)\rho^2)}{2[\log n - 8\pi(1+\rho)\mathcal{C}_0^2]}\right\} \\
&\quad + O\left(\frac{1}{\log^2 n}\right).
\end{aligned} \tag{4.46}$$

Hence, from (4.16), (4.45) and (4.46), we deduce

$$\begin{aligned}
\psi_n(s_n) &\le \frac{1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}}{2}\tilde{s}_n^2 - \frac{\pi\left(1+\frac{2\pi\mathcal{C}_0^2\rho^2}{\log n}\right)}{\alpha_0[\log n - 8\pi(1+\rho)\mathcal{C}_0^2]} \\
&\le \frac{4\pi}{\alpha_0}\left\{\frac{1}{2} - \frac{1-4\pi\mathcal{C}_0^2\rho^2}{4\log n}\right. \\
&\quad \left. + \frac{16\pi(1+\rho)\mathcal{C}_0^2 + \log 4(1+\varepsilon) - \log(\alpha_0(\kappa-\varepsilon)\rho^2)}{4[\log n - 8\pi(1+\rho)\mathcal{C}_0^2]}\right\} \\
&\quad + O\left(\frac{1}{\log^2 n}\right) \\
&\le \frac{4\pi}{\alpha_0}\left[\frac{1}{2} - \frac{1-4\pi\mathcal{C}_0^2(4+\rho)\rho}{4\log n}\right] + O\left(\frac{1}{\log^2 n}\right).
\end{aligned} \tag{4.47}$$

It follows from (4.42) that

$$\Phi(v_n + s_n w_n) \le \psi_n(s_n) \le \frac{4\pi}{\alpha_0}\left[\frac{1}{2} - \frac{1 - 4\pi \mathcal{C}_0^2(4+\rho)\rho}{4\log n}\right] + O\left(\frac{1}{\log^2 n}\right). \tag{4.48}$$

This contradicts with (4.20) due to (4.16).

The above three cases show that there exists $\bar{n} \in \mathbb{N}$ such that (4.19) holds. □

Let $e = w_{\bar{n}}^{+}/\|w_{\bar{n}}^{+}\|$ in Lemma 4.2. Since $E_k^- \subset E^-$, then it follows from Lemma 4.6 that the following lemma.

Lemma 4.7 *Assume that* (V1), (F1′), (F2), (F3) *and* (F5) *hold. Then* $\sup_{k\in\mathbb{N}} c_k < 2\pi/\alpha_0$.

Proof of Theorem 1.4 By Lemmas 4.3 and 4.7, there exist a subsequence $\{c_{k_n}\}$ of $\{c_k\}$ and $\tilde{c} \in [\kappa_0, 2\pi/\alpha_0)$ such that

$$\lim_{n\to\infty} c_{k_n} = \tilde{c}. \tag{4.49}$$

By Lemma 4.4, we can choose a subsequence $\{u_{j_n}^{k_n}\}$ with $u_{j_n}^{k_n} \in E_{k_n}$ such that

$$\Phi(u_{j_n}^{k_n}) \to \tilde{c}, \quad \|\Phi'(u_{j_n}^{k_n})\|_{E_{k_n}^*}(1 + \|u_{j_n}^{k_n}\|) \to 0. \tag{4.50}$$

For the sake of simplicity, we let $\tilde{u}_n = u_{j_n}^{k_n}$. Then it follows from (4.50), Lemmas 4.3 and 4.5 that $\{\tilde{u}_n\}$ is bounded in E (i.e. $\|\tilde{u}_n\| \le C_1$ for some $C_1 > 0$) and

$$\Phi(\tilde{u}_n) \to \tilde{c}, \quad \|\Phi'(\tilde{u}_n)\|_{E_{k_n}^*}(1 + \|\tilde{u}_n\|) \to 0. \tag{4.51}$$

Thus there exists a constant $C_2 > 0$ such that $\|\tilde{u}_n\|_2 \le C_2$. By (4.6) and (4.9), one has

$$\int_{\mathbb{R}^2} f(x, \tilde{u}_n)\tilde{u}_n \mathrm{d}x \le C_3. \tag{4.52}$$

If

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_1(y)} |\tilde{u}_n|^2 \mathrm{d}x = 0,$$

then by Lions' concentration compactness principle [30, Lemma 1.21], $\tilde{u}_n \to 0$ in $L^s(\mathbb{R}^2)$ for $2 < s < \infty$. For any given $\varepsilon > 0$, we choose $M_\varepsilon > M_0 C_3/\varepsilon$, then it follows from (F4) and (4.52) that

$$\int_{|\tilde{u}_n|\ge M_\varepsilon} F(x, \tilde{u}_n)\mathrm{d}x \le M_0 \int_{|\tilde{u}_n|\ge M_\varepsilon} |f(x, \tilde{u}_n)|\mathrm{d}x \le \frac{M_0}{M_\varepsilon}\int_{|\tilde{u}_n|\ge M_\varepsilon} f(x, \tilde{u}_n)\tilde{u}_n \mathrm{d}x < \varepsilon. \tag{4.53}$$

Using (F2) and (F3), we can choose $N_\varepsilon \in (0, 1)$ such that

$$\int_{|\tilde{u}_n|\le N_\varepsilon} F(x, \tilde{u}_n)\mathrm{d}x \le \int_{|\tilde{u}_n|\le N_\varepsilon} f(x, \tilde{u}_n)\tilde{u}_n \mathrm{d}x \le \frac{\varepsilon}{C_2^2}\|\tilde{u}_n\|_2^2 < \varepsilon. \tag{4.54}$$

By (F1′), we have

$$\int_{N_\varepsilon\le|\tilde{u}_n|\le M_\varepsilon} F(x, \tilde{u}_n)\mathrm{d}x \le C_4\|\tilde{u}_n\|_3^3 = o(1), \quad \int_{N_\varepsilon\le|\tilde{u}_n|\le 1} f(x, \tilde{u}_n)\tilde{u}_n \mathrm{d}x \le C_5\|\tilde{u}_n\|_3^3 = o(1). \tag{4.55}$$

Due to the arbitrariness of $\varepsilon > 0$, from (4.53), (4.54) and (4.55), we obtain

$$\int_{\mathbb{R}^2} F(x, \tilde{u}_n)\mathrm{d}x = o(1). \tag{4.56}$$

Hence, it follows from (2.8), (4.51) and (4.56) that

$$\|\tilde{u}_n^+\|^2 - \|\tilde{u}_n^-\|^2 = 2\tilde{c} + o(1). \tag{4.57}$$

By (F1′), (F2), (2.11), (4.51), (4.52) and (4.54), we have

$$\begin{aligned}
\|\tilde{u}_n^-\|^2 &= -\int_{\mathbb{R}^2} f(x, \tilde{u}_n)\tilde{u}_n^- \mathrm{d}x + o(1) \\
&\le \int_{|\tilde{u}_n| \le N_\varepsilon} |f(x, \tilde{u}_n)||\tilde{u}_n^-|\mathrm{d}x + \int_{N_\varepsilon \le |\tilde{u}_n| \le M_\varepsilon} |f(x, \tilde{u}_n)||\tilde{u}_n^-|\mathrm{d}x \\
&\quad + \frac{\|\tilde{u}_n^-\|_\infty}{M_\varepsilon} \int_{|\tilde{u}_n| \ge M_\varepsilon} f(x, \tilde{u}_n)\tilde{u}_n \mathrm{d}x + o(1) \\
&\le \frac{\varepsilon}{C_2^2}\|\tilde{u}_n\|_2\|\tilde{u}_n^-\| + C_6\|\tilde{u}_n\|_3^{2/3}\|\tilde{u}_n^-\|_3 + \frac{\mathcal{C}_0}{M_0}\|\tilde{u}_n^-\|\varepsilon + o(1) \\
&\le C_7\varepsilon + o(1),
\end{aligned} \tag{4.58}$$

which implies

$$\|\tilde{u}_n^-\|^2 = o(1). \tag{4.59}$$

Then (4.57) and (4.59) give

$$\|\tilde{u}_n\|^2 = \|\tilde{u}_n^+\|^2 + \|\tilde{u}_n^-\|^2 = 2\tilde{c} + o(1) := \frac{4\pi}{\alpha_0}(1 - 3\bar{\varepsilon}) + o(1). \tag{4.60}$$

Inspired by [9], we choose $\mu > 0$ such that $\|V\|_\infty/(\mu - \|V\|_\infty) < \bar{\varepsilon}$. Let $\tilde{u}_n^+ = v_n + z_n$, where $v_n \in \mathcal{E}(\mu)E$ and $z_n \in [id - \mathcal{E}(\mu)]E$. Similarly to (4.58), from (F1′), (F2), (2.11), (4.51) and (4.52), we can obtain

$$\|v_n\|^2 = \langle \Phi'(\tilde{u}_n), v_n \rangle + \int_{\mathbb{R}^2} f(x, \tilde{u}_n)v_n \mathrm{d}x = o(1). \tag{4.61}$$

Hence, it follows from (4.59) and (4.61) that

$$\|\tilde{u}_n - z_n\|^2 = o(1), \quad \|\nabla \tilde{u}_n\|_2^2 = \|\nabla z_n\|_2^2 + o(1). \tag{4.62}$$

Since $z_n \in [id - \mathcal{E}(\mu)]E$, we have

$$\|z_n\|^2 = \int_{\mathbb{R}^2} \left[|\nabla z_n|^2 + V(x)z_n^2\right] \mathrm{d}x = (\mathcal{A}z_n, z_n)_{L^2} \ge \mu\|z_n\|_2^2. \tag{4.63}$$

It follows that

$$\|\nabla z_n\|_2^2 \ge (\mu - \|V\|_\infty)\|z_n\|_2^2. \tag{4.64}$$

Combining (4.63) with (4.64), one has

$$\begin{aligned}
\|z_n\|^2 &\ge \|\nabla z_n\|_2^2 - \|V\|_\infty\|z_n\|_2^2 \\
&\ge \left(1 - \frac{\|V\|_\infty}{\mu - \|V\|_\infty}\right)\|\nabla z_n\|_2^2 \ge (1 - \bar{\varepsilon})\|\nabla z_n\|_2^2.
\end{aligned} \tag{4.65}$$

From (4.62) and (4.65), we obtain

$$\|\tilde{u}_n\|^2 = \|z_n\|^2 + o(1) \ge (1 - \bar{\varepsilon})\|\nabla z_n\|_2^2 + o(1) = (1 - \bar{\varepsilon})\|\nabla \tilde{u}_n\|_2^2 + o(1) \tag{4.66}$$

Let us choose $q \in (1, 2)$ such that

$$\frac{(1+\bar{\varepsilon})(1-3\bar{\varepsilon})q}{1-\bar{\varepsilon}} < 1. \tag{4.67}$$

By (F1′), there exists $C_8 > 0$ such that

$$|f(x,t)|^q \le C_8\left[e^{\alpha_0(1+\bar{\varepsilon})qt^2} - 1\right], \quad \forall\, |t| \ge 1. \tag{4.68}$$

It follows from (4.67), (4.68) and Lemma 1.1-ii) that

$$\begin{aligned}\int_{|\tilde{u}_n|\ge 1} |f(x,\tilde{u}_n)|^q \mathrm{d}x &\le C_8 \int_{\mathbb{R}^2}\left[e^{\alpha_0(1+\bar{\varepsilon})q\tilde{u}_n^2} - 1\right]\mathrm{d}x \\ &= C_8 \int_{\mathbb{R}^2}\left[e^{\alpha_0(1+\bar{\varepsilon})q\|\tilde{u}_n\|^2(\tilde{u}_n/\|\tilde{u}_n\|)^2} - 1\right]\mathrm{d}x \le C_9.\end{aligned} \tag{4.69}$$

Let $q' = q/(q-1)$. Then we have

$$\int_{|\tilde{u}_n|\ge 1} f(x,\tilde{u}_n)\tilde{u}_n \mathrm{d}x \le \left[\int_{|\tilde{u}_n|\ge 1} |f(x,\tilde{u}_n)|^q \mathrm{d}x\right]^{1/q} \|\tilde{u}_n\|_{q'} = o(1). \tag{4.70}$$

Now from (2.8), (2.12), (4.51), (4.54), (4.55) and (4.70), we derive

$$\begin{aligned}\tilde{c} + o(1) &= \Phi(\tilde{u}_n) - \frac{1}{2}\langle \Phi'(\tilde{u}_n), \tilde{u}_n\rangle \\ &= \int_{\mathbb{R}^2}\left[\frac{1}{2}f(x,\tilde{u}_n)\tilde{u}_n - F(x,\tilde{u}_n)\right]\mathrm{d}x < \varepsilon + o(1).\end{aligned} \tag{4.71}$$

This contradiction shows that $\delta > 0$.

Going if necessary to a subsequence, we may assume that there exists $\{y_n\} \subset \mathbb{Z}^2$ such that $\int_{B_{1+\sqrt{2}}(y_n)} |\tilde{u}_n|^2 \mathrm{d}x > \frac{\delta}{2}$. Let us define $\tilde{v}_n(x) = \tilde{u}_n(x+y_n)$ so that

$$\int_{B_{1+\sqrt{2}}(0)} |\tilde{v}_n|^2 \mathrm{d}x > \frac{\delta}{2}. \tag{4.72}$$

Since $V(x)$ and $f(x,u)$ are 1-periodic on x, we have $\|\tilde{v}_n\| = \|\tilde{u}_n\|$ and

$$\Phi(\tilde{v}_n) \to \tilde{c}, \quad \|\Phi'(\tilde{v}_n)\|_{E^*_{k_n}}(1 + \|\tilde{v}_n\|) \to 0. \tag{4.73}$$

Passing to a subsequence, we have $\tilde{v}_n \rightharpoonup \tilde{v}$ in E, $\tilde{v}_n \to \tilde{v}$ in $L^s_{\mathrm{loc}}(\mathbb{R}^2)$, $2 \le s < \infty$ and $\tilde{v}_n \to \tilde{v}$ a.e. on $\mathbb{R}^2$. Thus, (4.72) implies that $\tilde{v} \ne 0$. Now for any $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, we have

$$\phi = \phi^+ + \sum_{j=1}^{\infty}(\phi, e_j)e_j, \quad \|\phi^-\|^2 = \sum_{j=1}^{\infty}|(\phi, e_j)|^2. \tag{4.74}$$

Let

$$\phi_n = \phi^+ + \sum_{j=1}^{k_n}(\phi, e_j)e_j, \quad \tilde{\phi}_n = \sum_{k_n+1}^{\infty}(\phi, e_j)e_j. \tag{4.75}$$

For any given $\varepsilon > 0$, we have

$$\int_{|\tilde{v}_n|\ge C_3\mathcal{K}_0\gamma_2\|\phi^-\|\varepsilon^{-1}} |f(x,\tilde{v}_n)\tilde{\phi}_n| \mathrm{d}x \le \frac{\varepsilon}{C_3}\int_{|\tilde{v}_n|\ge C_3\mathcal{K}_0\gamma_2\|\phi^-\|\varepsilon^{-1}} f(x,\tilde{v}_n)\tilde{v}_n \mathrm{d}x < \varepsilon. \tag{4.76}$$

On the other hand, it follows from (F1′), (F2), (F3),(4.74) and (4.75) that

$$\begin{aligned}\int_{|\tilde{v}_n|<C_3\mathcal{K}_0\gamma_2\|\phi^-\|\varepsilon^{-1}} |f(x,\tilde{v}_n)\tilde{\phi}_n|\mathrm{d}x &\leq \left(\int_{|\tilde{v}_n|<C_3\mathcal{K}_0\gamma_2\|\phi^-\|\varepsilon^{-1}} |f(x,\tilde{v}_n)|^2\mathrm{d}x\right)^{\frac{1}{2}} \|\tilde{\phi}_n\|_2 \\ &\leq C_{10}\left(\int_{|\tilde{v}_n|<C_3\mathcal{K}_0\gamma_2\|\phi^-\|\varepsilon^{-1}} f(x,\tilde{v}_n)\tilde{v}_n\mathrm{d}x\right)^{\frac{1}{2}} \|\tilde{\phi}_n\| \\ &\leq C_{10}\left(\int_{\mathbb{R}^2} f(x,\tilde{u}_n)\tilde{u}_n\mathrm{d}x\right)^{\frac{1}{2}} \|\tilde{\phi}_n\| \\ &\leq C_{11}\|\tilde{\phi}_n\| = o(1). \end{aligned} \tag{4.77}$$

From (4.76) and (4.77), one has

$$\lim_{n\to\infty}\int_{\mathbb{R}^2} f(x,\tilde{v}_n)\tilde{\phi}_n\mathrm{d}x = 0 \tag{4.78}$$

due to the arbitrariness of $\varepsilon > 0$. Therefore, (2.11), (4.73), (4.78) and Lemma 3.2 yield

$$\begin{aligned}\langle \Phi'(\tilde{v}),\phi\rangle &= \int_{\mathbb{R}^2}(\nabla\tilde{v}\nabla\phi + V(x)\tilde{v}\phi)\,\mathrm{d}x - \int_{\mathbb{R}^2} f(x,\tilde{v})\phi\mathrm{d}x \\ &= \lim_{n\to\infty}\left[\int_{\mathbb{R}^2}(\nabla\tilde{v}_n\nabla\phi + V(x)\tilde{v}_n\phi)\,\mathrm{d}x - \int_{\mathbb{R}^2} f(x,\tilde{v}_n)\phi\mathrm{d}x\right] \\ &= \lim_{n\to\infty}\left[\langle \Phi'(\tilde{v}_n),\phi_n\rangle - \int_{\mathbb{R}^2} f(x,\tilde{v}_n)\tilde{\phi}_n\mathrm{d}x\right] \\ &= -\lim_{n\to\infty}\int_{\mathbb{R}^2} f(x,\tilde{v}_n)\tilde{\phi}_n\mathrm{d}x = 0.\end{aligned}$$

This shows that $\tilde{v}$ is a nontrivial solution of (1.1). □

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