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Calculus of Variations

Sharp profiles for periodic logistic equation with nonlocal dispersal

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Abstract
In this paper, we study the nonlocal dispersal logistic equation

$$\begin{cases} u_t = J * u - u + \lambda u - [b(x)q(t) + \delta]u^p & \text{in } \bar{\Omega} \times (0, \infty), \\ u(x,t) = 0 & \text{in } \mathbb{R}^N \setminus \bar{\Omega} \times (0, \infty), \\ u(x,t) = u(x, t+T) & \text{in } \bar{\Omega} \times [0, \infty), \end{cases}$$

here $\Omega \subset \mathbb{R}^N$ is a bounded domain, J is a nonnegative dispersal kernel, $p > 1$, λ is a fixed parameter and $\delta > 0$. The coefficients b, q are nonnegative and continuous functions, and q is periodic in t. We are concerned with the asymptotic profiles of positive solutions as $\delta \to 0$. We obtain that the temporal degeneracy of q does not make a change of profiles, but the spatial degeneracy of b makes a large change. We find that the sharp profiles are different from the classical reaction–diffusion equations. The investigation in this paper shows that the periodic profile has two different blow-up speeds and the sharp profile is time periodic in domain without spatial degeneracy.

Keywords Positive solution · Periodic profile · Nonlocal dispersal

Mathematics Subject Classification 35B40 · 35K57 · 35P05

1 Introduction and main results

Let $J : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative continuous function and $\Omega \subset \mathbb{R}^N$ be a bounded domain. We consider the periodic nonlocal dispersal equation

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$$\begin{cases} u_t = J * u - u + \lambda u - a(x,t)u^p & \text{in } \bar{\Omega} \times (0, +\infty), \\ u(x,t) = 0 & \text{in } (\mathbb{R}^N \setminus \bar{\Omega}) \times (0, +\infty), \\ u(x,0) = u_I(x) & \text{in } \bar{\Omega}, \end{cases} \tag{1.1}$$

where $p > 1$ and λ is a real parameter, the coefficient a is nonnegative, T-periodic in t and

$$Du(x,t) = J * u(x,t) - u(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)dy - u(x,t)$$

represents a nonlocal dispersal operator. It is known that the dispersal operator D and variations of it have been used to model different dispersal phenomena from applications as well as pure mathematics, see [1,2,4,10,29]. The nonlocal dispersal equation (1.1) arises typically in population dynamics [11,17,18]. Let $u(y,t)$ be the density of population at location y at time t, and $J(x-y)$ be the probability distribution of the population jumping from y to x, then $\int_{\mathbb{R}^N} J(x-y)u(y,t)dy$ denotes the rate at which individuals are arriving to location x from all other places and $-u(x,t) = -\int_{\mathbb{R}^N} J(y-x)u(x,t)dy$ is the rate at which they are leaving location x to all other places. Thus $Du(x,t)$ is the dispersal of population and (1.1) describes the change of population density $u(x,t)$ with initial value $u_I(x)$ and periodic logistic type growth rate. In (1.1), the dispersal takes place in $\mathbb{R}^N$, but we impose that u vanishes outside $\bar{\Omega}$, which is called homogeneous nonlocal Dirichlet boundary condition [17]. The operator D is a nonlocal operator since the dispersal of u at location x and time t does not only depend on u, but on all the values of u in a fixed spatial neighborhood of x through the term $J * u$. There is quite an extensive literature for the study of nonlocal problems recently, among others, the papers [5,6,14,23,24,26–28].

Since the coefficient $a(x,t)$ may have temporal or spatial degeneracies, the degenerate periodic logistic nonlinearity plays a great role on the dynamical behavior of (1.1), see [27]. In fact, the study of diffusion problems with refuge goes back to the classical works of Fraile et al. [12]. There is quite an extensive literature on the study of degenerate diffusion problems, for example, the papers [8,12–15,20–22,25] and the references therein. In this paper, we shall investigate the influence of degenerate heterogeneous environment on the nonlocal dispersal system (1.1). To this end, we take $a(x,t) = b(x)q(t)$, where q is T-periodic in t and consider the nonlocal dispersal equation

$$\begin{cases} u_t = J * u - u + \lambda u - b(x)q(t)u^p & \text{in } \bar{\Omega} \times (0, \infty), \\ u(x,t) = 0 & \text{in } \mathbb{R}^N \setminus \bar{\Omega} \times (0, \infty), \\ u(x,t) = u(x, t+T) & \text{in } \bar{\Omega} \times [0, \infty). \end{cases} \tag{1.2}$$

Throughout this paper, we make the following assumptions on $J(x)$, $b(x)$ and $q(t)$.

(H1) $J \in C(\mathbb{R}^N)$ is nonnegative, symmetric with unit integral and $J(0) > 0$.
(H2) $b \in C(\bar{\Omega})$ and $q \in C[0,\infty)$ satisfies $q(t) = q(t+T)$ in $[0,\infty)$ for some $T > 0$.

Our interest here is that the nonlinearity has degeneracies. That is, $b(x)$ or $q(t)$ vanishes in a proper subset. We shall distinguish the following two different cases.

$(A1)$ $b(x) > 0$ for all $x \in \bar{\Omega}$ and $q(t_q) > 0$ for some $t_q \in [0, T]$.
$(A2)$ $q(t) > 0$ for all $t \in [0, T]$ and $b(x) = 0$ on Ω_0, while

$$b(x) > 0 \text{ for all } x \in \bar{\Omega} \setminus \bar{\Omega}_0,$$

here $\Omega_0 \subset \Omega$ is a proper subdomain with positive measure.

The first case is that only the temporal degeneracy exists. We may assume that there exist $t_0, t_1 \in [0, T]$ such that $q(t) = 0$ for $t \in [t_0, t_1]$ and $b(x) > 0$ for $x \in \bar{\Omega}$. Then the assumption $(A1)$ holds and the positive solution of the periodic problem (1.2) is well studied, see [23,27]. Let $\lambda_P(\Omega)$ be the unique principle eigenvalue of nonlocal equation

$$\begin{cases} J * \phi - \phi = -\lambda\phi & \text{in } \bar{\Omega}, \\ \phi(x) = 0 & \text{in } \mathbb{R}^N \setminus \bar{\Omega}, \end{cases}$$

we know that (1.2) admits a unique positive solution if and only if $\lambda > \lambda_P(\Omega)$. If $b(x)$ has a spatial degeneracy, the results are different. If $(A2)$ holds, it follows from [27] that (1.2) admits a unique positive solution if and only if $\lambda_P(\Omega) < \lambda < \lambda_P(\Omega_0)$.

It is well known from [8,27] that the dynamical behavior of nonlocal equation (1.2) is different from the classical reaction–diffusion equation

$$\begin{cases} u_t = \Delta u + \lambda u - b(x)q(t)u^p & \text{in } \Omega \times (0, \infty), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x,t) = u(x, t+T) & \text{in } \Omega \times [0, \infty), \end{cases}$$

here we assume further that Ω is smooth. In order to find the sharp influence of complex environment on the nonlocal dispersal system, we consider the asymptotic profiles of positive periodic solutions. More precisely, we study the perturbed nonlocal dispersal equation

$$\begin{cases} u_t = J * u - u + \lambda u - [b(x)q(t) + \delta]u^p & \text{in } \bar{\Omega} \times (0, \infty), \\ u(x,t) = 0 & \text{in } \mathbb{R}^N \setminus \bar{\Omega} \times (0, \infty), \\ u(x,t) = u(x, t+T) & \text{in } \bar{\Omega} \times [0, \infty), \end{cases} \tag{1.3}$$

where $\delta > 0$ is a small parameter. In this case, we know that the degeneracy disappears and (1.3) admits a unique positive solution

$$u_\delta \in C^1([0, T]; C(\bar{\Omega}))$$

for $\lambda > \lambda_P(\Omega)$, see [23,27]. We want to obtain the sharp behavior of positive solutions when degeneracy appears. So we first establish the asymptotic profiles of positive solutions.

Theorem 1.1 *Assume that $(A1)$ holds. Let $u_\delta(x,t)$ be the unique positive solution of* (1.3) *for $\lambda > \lambda_P(\Omega)$ and $\delta > 0$. Then we have*

$$\lim_{\delta \to 0+} u_\delta(x,t) = u(x,t) \text{ uniformly in } \bar{\Omega} \times [0, T],$$

where $u(x,t)$ is the unique positive solution of (1.2).

Theorem 1.2 *Assume that $(A2)$ holds. Let $u_\delta(x,t)$ be the unique positive solution of* (1.3) *for $\lambda > \lambda_P(\Omega)$ and $\delta > 0$. Then the following hold.*

(i) If $\lambda_P(\Omega) < \lambda < \lambda_P(\Omega_0)$, then

$$\lim_{\delta \to 0+} u_\delta(x,t) = u(x,t) \text{ uniformly in } \bar{\Omega} \times [0, T],$$

where $u(x,t)$ is the unique positive solution of (1.2).

(ii) If $\lambda \geq \lambda_P(\Omega_0)$, then

$$\lim_{\delta \to 0+} u_\delta(x,t) = \infty \text{ uniformly in } \bar{\Omega} \times [0, T]. \tag{1.4}$$

Remark 1.3 If $b(x) > 0$ for $x \in \bar{\Omega}$ and $q(t) > 0$ for $t \in [0, T]$, we know that the assumption $(A1)$ still holds. In this case, there is no temporal degeneracy, the conclusion of Theorem 1.1 is also true. We show that only the temporal degeneracy of $q(t)$ does not make a change of the profiles. But if the spatial degeneracy appears, the profiles make a large change. In case of spatial degeneracy, the profiles are also different to the classical reaction–diffusion equation. Let $\lambda_L(\Omega)$ be the principal eigenvalue of

$$\begin{cases} \Delta u = -\lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then we know from [7,9,16,20] that the classical reaction–diffusion equation

$$\begin{cases} u_t = \Delta u + \lambda u - [b(x)q(t) + \delta]u^p & \text{in } \Omega \times (0, \infty), \\ u = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, t) = u(x, t + T) & \text{in } \Omega \times (0, \infty) \end{cases} \tag{1.5}$$

admits a unique positive periodic solution $u_\delta^L(x, t)$ for $\lambda > \lambda_L(\Omega)$ and the asymptotic profiles of $u_\delta^L(x, t)$ with respect to δ are well established. If $(A1)$ holds and $\lambda > \lambda_L(\Omega)$, then

$$\lim_{\delta \to 0+} u_\delta^L(x, t) = u^L(x, t) \text{ uniformly in } \bar{\Omega} \times [0, T],$$

where $u^L(x, t)$ is the unique positive solution of (1.5) for $\delta = 0$. Meanwhile, if $(A2)$ holds, then we have

$$\lim_{\delta \to 0+} u_\delta^L(x, t) = u^L(x, t) \text{ uniformly in } \bar{\Omega} \times [0, T]$$

for any $\lambda \in (\lambda_L(\Omega), \lambda_L(\Omega_0))$ and

$$\lim_{\delta \to 0+} u_\delta^L(x, t) = \infty \text{ uniformly in } \bar{\Omega}_0 \times [0, T]$$

for any $\lambda \geq \lambda_L(\Omega_0)$. In the later case, we know that $u_\delta^L(x, t)$ is still bounded as $\delta \to 0+$ in any compact subset of $\bar{\Omega} \setminus \bar{\Omega}_0 \times [0, T]$. However, from (1.4) we obtain that the profiles of nonlocal dispersal equation (1.3) are unbounded in $\bar{\Omega} \times [0, T]$ as $\delta \to 0+$. Thus we know from Theorems 1.1–1.2 that only the temporal degeneracy dose not change the profiles of positive solutions both for nonlocal and classical reaction–diffusion problems. However, the spatial degeneracy makes different changes.

To reveal the complex influence of spatial degeneracy environment on the nonlocal dispersal system (1.3), we investigate the sharp spatial pattern of positive periodic solutions.

Theorem 1.4 *Assume that $(A2)$ holds. Let $u_\delta(x, t)$ be the unique positive solution of (1.3) for $\lambda > \lambda_P(\Omega)$ and $\delta > 0$. Set $v_\delta(x, t) = \delta^{\frac{1}{p-1}} u_\delta(x, t)$, we have the following results.*

(i) *If $\lambda_P(\Omega) < \lambda \leq \lambda_P(\Omega_0)$, then*

$$\lim_{\delta \to 0+} v_\delta(x, t) = 0 \textit{ uniformly in } \bar{\Omega} \times [0, T].$$

(ii) *If $\lambda > \lambda_P(\Omega_0)$, then*

$$\lim_{\delta \to 0+} v_\delta(x, t) = \theta(x) \textit{ uniformly in } \bar{\Omega}_0 \times [0, T], \tag{1.6}$$

and

$$\lim_{\delta\to 0+} v_\delta(x,t) = 0 \text{ uniformly in any compact subset of } \bar{\Omega}\setminus\bar{\Omega}_0 \times [0,T], \tag{1.7}$$

where $\theta \in C(\bar{\Omega}_0)$ *satisfies* $\theta(x) > 0$ *in* $\bar{\Omega}_0$ *and*

$$\int_{\Omega_0} J(x-y)\theta(y)dy - \theta(x) = -\lambda\theta(x) + \theta^p(x) \text{ in } \bar{\Omega}_0. \tag{1.8}$$

Let us note that (1.8) exists a unique positive solution for any $\lambda > \lambda_P(\Omega_0)$ [14]. Since $\theta(x) > 0$ in $\bar{\Omega}_0$, the sharp pattern of $u_\delta(x,t)$ in $\bar{\Omega}_0 \times [0,T]$ is given by (1.6). Due to the effect of nonlocal effect, we know from (1.7) that the pattern is different in $\bar{\Omega}\setminus\bar{\Omega}_0 \times [0,T]$. We obtain the sharp profiles of $u_\delta(x,t)$ in $\bar{\Omega}\setminus\bar{\Omega}_0 \times [0,T]$ as follows.

Theorem 1.5 *Assume that* $(A2)$ *holds. Let* $u_\delta(x,t)$ *be the unique positive solution of* (1.3) *for* $\lambda > \lambda_P(\Omega_0)$ *and* $\delta > 0$. *Set* $\omega_\delta(x,t) = \delta^{\frac{1}{p(p-1)}} u_\delta(x,t)$, *we have*

$$\lim_{\delta\to 0+} \omega_\delta(x,t) = \infty \text{ uniformly in } \bar{\Omega}_0 \times [0,T],$$

and

$$\lim_{\delta\to 0+} \omega_\delta(x,t) = \eta(x,t) \text{ uniformly in any compact subset of } \bar{\Omega}\setminus\bar{\Omega}_0 \times [0,T],$$

where

$$\eta(x,t) = \left[\frac{\int_{\Omega_0} J(x-y)\theta(y)dy}{b(x)q(t)}\right]^{\frac{1}{p}}, \tag{1.9}$$

and $\theta(x) > 0$ *in* $\bar{\Omega}_0$ *is given by* (1.8).

Remark 1.6 In the above theorems, we obtain the sharp profiles of positive solutions to the nonlocal dispersal equation (1.3). If $(A2)$ holds, we establish that the sharp profiles in degeneracy domain are different from the domain without degeneracy. In fact, we prove that both the nonlocal effect and the degeneracy of $b(x)$ make the positive periodic solutions of (1.3) blow up, but have different blow-up speeds. Furthermore, we know from (1.9) that the sharp pattern of nonlocal dispersal equation (1.3) is time periodic in domain without degeneracy.

Comparing with the classical reaction–diffusion equation, the sharp pattern for nonlocal dispersal equation is quite different. Our main results reveal the following phenomena for nonlocal dispersal equation (1.3).
(i) The asymptotic profiles are unbounded in the whole domain Ω.
(ii) The asymptotic profiles have different blow-up speeds, depending on domain Ω_0.
(iii) The sharp profiles are time independent in degeneracy domain Ω_0, but time periodic in non-degeneracy domain.

The rest of this paper is organized as follows. In Sect. 2, we investigate the asymptotic profiles. The behavior of principal eigenfunction with respect to parameter is also obtained. Section 3 is devoted to the proofs of sharp profiles.

2 Asymptotic profiles and eigenvalue problems

In this section, we investigate the asymptotic profiles for positive solutions of (1.3). To begin with, we consider the case $(A1)$.

Lemma 2.1 *Assume that* $(A1)$ *holds. Let* $u_\delta(x,t)$ *be the unique positive solution of* (1.3) *for* $\lambda > \lambda_P(\Omega)$ *and* $\delta > 0$. *Then we have*

$$u_{\delta_2}(x,t) \le u_{\delta_1}(x,t) \le u(x,t) \text{ in } \bar{\Omega} \times [0,T] \tag{2.1}$$

for $\delta_2 \ge \delta_1 > 0$, *here* $u(x,t)$ *is the unique positive solution of* (1.2). *Moreover, we have*

$$\lim_{\delta \to 0+} u_\delta(x,t) = u(x,t) \text{ uniformly in } \bar{\Omega} \times [0,T]. \tag{2.2}$$

Proof Since $\delta_2 \ge \delta_1 > 0$, we can see that $u_{\delta_2}(x,t)$ is a lower-solution of (1.3) for $\delta = \delta_1$. Note that $u_{\delta_1}(x,t)$ is the unique solution of (1.3) for $\delta = \delta_1$, then by upper-lower solutions argument (see [2,27]), we get

$$u_{\delta_2}(x,t) \le u_{\delta_1}(x,t) \text{ in } \bar{\Omega} \times [0,T].$$

Similarly, we have

$$u_\delta(x,t) \le u(x,t) \text{ in } \bar{\Omega} \times [0,T]$$

for $\delta > 0$ and (2.1) holds.

Now by (2.1), we can find a bounded function $u_0(x,t)$ such that

$$\lim_{\delta \to 0+} u_\delta(x,t) = u_0(x,t)$$

for $(x,t) \in \bar{\Omega} \times [0,T]$. Thus we know from (1.3) that $u_0(x,0) = u_0(x,T)$ in $\bar{\Omega}$ and

$$\begin{aligned} &u_0(x,t) - u_0(x,0) \\ &\quad = \int_0^t \int_\Omega \left[J(x-y)u_0(y,s) - u_0(x,s) + \lambda u_0(x,s) - b(x)q(s)u_0^p(x,s)\right] dyds \end{aligned} \tag{2.3}$$

for $(x,t) \in \bar{\Omega} \times [0,T]$. Let ε be a small parameter, we have

$$\begin{aligned} &u_0(x,t+\varepsilon) - u_0(x,t) \\ &\quad = \int_t^{t+\varepsilon} \int_\Omega \left[J(x-y)u_0(y,s) - u_0(x,s) + \lambda u_0(x,s) - b(x)q(s)u_0^p(x,s)\right] dyds \end{aligned}$$

for $(x,t) \in \bar{\Omega}_0 \times [0,T]$. Since $u_0(x,t)$ is uniformly bounded in $\bar{\Omega} \times [0,T]$, we get

$$u_0 \in C([0,T]; L^\infty(\Omega)).$$

On the other hand, we have

$$\begin{aligned} &\lim_{\varepsilon \to 0} \frac{u_0(x,t+\varepsilon) - u_0(x,t)}{\varepsilon} \\ &\quad = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \left[\int_\Omega J(x-y)u_0(y,s)dy + (\lambda - 1)u_0(x,s) - b(x)q(s)u_0^p(x,s)\right] ds \\ &\quad = \int_\Omega \left[J(x-y)u_0(y,t) - u_0(x,t) + \lambda u_0(x,t) - b(x)q(s)u_0^p(x,t)\right] dy, \end{aligned}$$

and so

$$u_0 \in C^1([0,T]; L^\infty(\Omega)).$$

Then by (2.3) we know that $u_0(x,t)$ is a positive solution of (1.2) and the uniqueness shows that $u_0(x,t) = u(x,t)$ in $\bar{\Omega} \times [0,T]$. Thus we obtain (2.2) by Dini's theorem. □

If the spatial degeneracy appears, the case is quite different. To do this, we need to study the periodic nonlocal eigenvalue equation

$$\begin{cases} -\phi_t + J * \phi - \phi - \mu b(x)q(t)\phi = -\lambda\phi \text{ in } \bar{\Omega} \times (0,\infty), \\ \phi(x,t) = 0 \text{ in } (\mathbb{R}^N \setminus \bar{\Omega}) \times (0,\infty), \\ \phi(x,t) = \phi(x,t+T) \text{ in } \bar{\Omega} \times [0,\infty), \end{cases} \tag{2.4}$$

here $\mu \geq 0$. By the pioneering work of J. López-Gómez [19], we have the following lemma, one can see [27] for a similar proof.

Lemma 2.2 *Assume that* $(A2)$ *holds. If* $\mu \geq 0$*, then* (2.4) *admits a unique principal eigenvalue* $\lambda_P(\mu,\Omega)$*. Moreover,* $\lambda_P(\mu,\Omega)$ *is strictly increasing with respect to* μ*,* $\lambda_P(0,\Omega) = \lambda_P(\Omega)$ *and*

$$\lim_{\mu\to\infty} \lambda_P(\mu,\Omega) = \lambda_P(\Omega_0).$$

Now we give the asymptotic behavior of positive eigenfunctions associated with $\lambda_P(\mu,\Omega)$, which is a nonlocal version of the classical problem [3].

Theorem 2.3 *Assume that* $(A2)$ *holds. Let* $\phi_\mu(x,t)$ *and* $\psi(x)$ *be the positive eigenfunctions associated with* $\lambda_P(\mu,\Omega)$ *for* $\mu \geq 0$ *and* $\lambda_P(\Omega_0)$ *such that*

$$\frac{1}{T}\int_0^T \int_\Omega \phi_\mu(x,s)dxds = 1 \text{ and } \int_\Omega \psi(x)dx = 1, \tag{2.5}$$

respectively. Then we have

$$\lim_{\mu\to\infty} \phi_\mu(x,t) = \psi(x) \text{ uniformly in } \bar{\Omega}_0 \times [0,T],$$

and

$$\lim_{\mu\to\infty} \phi_\mu(x,t) = 0 \text{ uniformly in any compact subset of } \bar{\Omega} \setminus \bar{\Omega}_0 \times [0,T].$$

Proof We will prove the main results by the following four steps.
Step 1. We show that $\phi_\mu(x,t)$ is uniformly bounded in $\bar{\Omega} \times [0,T]$.

It follows from (2.4) that

$$(\phi_\mu)_t(x,t) \leq \int_\Omega J(x-y)\phi_\mu(y,t)dy - \phi_\mu(x,t) - \mu b(x)q_*\phi_\mu(x,t) + \lambda_P(\mu,\Omega)\phi_\mu(x,t),$$

where $q_* = \min_{[0,T]} q(t)$. Denote

$$J^* = \max_{\bar{\Omega}\times\bar{\Omega}} J(x-y),$$

since $\lambda_P(\mu,\Omega) \le \lambda_P(\Omega_0) < 1$, a direct comparison argument gives

$$\begin{aligned}\phi_\mu(x,t) \le\, & e^{[\lambda_P(\mu,\Omega)-1-\mu b(x)q_*]t}\phi_\mu(x,0)\\ & + \int_0^t e^{[\lambda_P(\mu,\Omega)-1-\mu b(x)q_*](t-s)}\int_\Omega J(x-y)\phi_\mu(y,s)dyds\\ \le\, & e^{[\lambda_P(\Omega_0)-1]t}\phi_\mu(x,0) + \int_0^t\int_\Omega J(x-y)\phi_\mu(y,s)dyds\\ \le\, & e^{[\lambda_P(\Omega_0)-1]t}\phi_\mu(x,0) + J^*T\end{aligned} \tag{2.6}$$

for $(x,t) \in \bar{\Omega}\times[0,T]$. This yields

$$\phi_\mu(x,0) = \phi_\mu(x,T) \le \frac{J^*T}{1-e^{[\lambda_P(\Omega_0)-1]T}}$$

for $x \in \bar{\Omega}$. Set

$$M = \frac{J^*T}{1-e^{[\lambda_P(\Omega_0)-1]T}} + J^*T,$$

again by (2.6) we get

$$0 < \phi_\mu(x,t) \le \phi_\mu(x,0) + J^*T \le M \tag{2.7}$$

for $(x,t) \in \bar{\Omega}\times[0,T]$.

Step 2. The eigenfunction $\phi_\mu(x,t)$ in $\bar{\Omega}_0\times[0,T]$.

Let $x_1, x_2 \in \bar{\Omega}_0$, we denote

$$v(t) = \phi_\mu(x_1,t) - \phi_\mu(x_2,t) \text{ in } [0,T].$$

Without loss of generality, we assume that $v(T) > 0$. By (2.4) we obtain that

$$\begin{aligned}v_t(t) =& \int_\Omega [J(x_1-y)-J(x_2-y)]\phi_\mu(y,t)dy + [\lambda_P(\mu,\Omega)-1]v(t)\\ \le& \int_\Omega |J(x_1-y)-J(x_2-y)|\phi_\mu(y,t)dy + [\lambda_P(\mu,\Omega)-1]v(t)\\ \le& G(x_1,x_2)\int_\Omega \phi_\mu(y,t)dy + [\lambda_P(\mu,\Omega)-1]v(t),\end{aligned}$$

where $t \in [0,T]$ and

$$G(x_1,x_2) = \max_{y\in\bar{\Omega}} |J(x_1-y)-J(x_2-y)|.$$

Since $\lambda_P(\Omega) \le \lambda_P(\mu,\Omega) < \lambda_P(\Omega_0) < 1$ for $\mu \ge 0$, we get

$$\begin{aligned}v(t) \le& e^{[\lambda_P(\mu,\Omega)-1]t}v(0) + G(x_1,x_2)\int_0^t e^{[\lambda_P(\mu,\Omega)-1](t-s)}\int_\Omega \phi_\mu(y,s)dyds\\ \le& e^{[\lambda_P(\mu,\Omega)-1]t}v(0) + G(x_1,x_2)\int_0^t\int_\Omega \phi_\mu(y,s)dyds\end{aligned} \tag{2.8}$$

for $t \in [0,T]$. But $v(0) = v(T)$, we have

$$|v(T)| \le \frac{G(x_1,x_2)T}{1-e^{[\lambda_P(\Omega_0)-1]T}}.$$

Meanwhile, we know from (2.8) that

$$\begin{aligned}|v(t)| =&|\phi_\mu(x_1,t)-\phi_\mu(x_2,t)|\\ \le&|v(T)|+G(x_1,x_2)T\\ \le&\frac{G(x_1,x_2)T}{1-e^{[\lambda_P(\Omega_0)-1]}}+G(x_1,x_2)T\end{aligned}$$

for $x_1, x_2 \in \bar{\Omega}_0$.

On the other hand, for $x \in \bar{\Omega}_0$ and $0 \le t_1 < t_2 \le T$, it follows from

$$\begin{aligned}\phi_\mu(x,t) &= e^{[\lambda_P(\mu,\Omega)-1]t}\phi_\mu(x,0)\\ &\quad+\int_0^t e^{[\lambda_P(\mu,\Omega)-1](t-s)}\int_\Omega J(x-y)\phi_\mu(y,s)dyds\end{aligned}$$

that there exist $t_1^*, t_2^* \in (t_1, t_2)$ such that

$$\begin{aligned}&\phi_\mu(x,t_2)-\phi_\mu(x,t_1)\\ &\quad=[e^{[\lambda_P(\mu,\Omega)-1]t_2}-e^{[\lambda_P(\mu,\Omega)-1]t_1}]\phi_\mu(x,0)\\ &\qquad+\int_0^{t_1}\left[e^{[\lambda_P(\mu,\Omega)-1](t_1-s)}-e^{[\lambda_P(\mu,\Omega)-1](t_2-s)}\right]\int_\Omega J(x-y)\phi_\mu(y,s)dyds\\ &\qquad-\int_{t_1}^{t_2} e^{[\lambda_P(\mu,\Omega)-1](t_2-s)}\int_\Omega J(x-y)\phi_\mu(y,s)dyds\\ &\quad\le[\lambda_P(\mu,\Omega)-1]e^{(\lambda_P(\mu,\Omega)-1)t_1^*}\phi_\mu(x,0)(t_2-t_1)\\ &\qquad+[\lambda_P(\mu,\Omega)-1](t_1-t_2)\int_0^{t_1}\left[e^{[\lambda_P(\mu,\Omega)-1](t_2^*-s)}\right]\int_\Omega J(x-y)\phi_\mu(y,s)dyds\\ &\qquad-\int_{t_1}^{t_2} e^{[\lambda_P(\mu,\Omega)-1](t_2-s)}\int_\Omega J(x-y)\phi_\mu(y,s)dyds\\ &\quad\le M|\lambda_P(\mu,\Omega)-1|(t_2-t_1)+J^*MT|\lambda_P(\mu,\Omega)-1|(t_2-t_1)+J^*M(t_2-t_1).\end{aligned}$$

Thus we have

$$|\phi_\mu(x,t_2)-\phi_\mu(x,t_1)|\le[M|\lambda_P(\mu,\Omega)-1|+J^*MT|\lambda_P(\mu,\Omega)-1|+J^*M]|t_2-t_1|$$

for $x \in \bar{\Omega}_0$ and $t_1, t_2 \in [0, T]$.

Accordingly, subject to a subsequence, we know that there exists $\hat{\phi} \in C(\bar{\Omega}_0 \times [0, T])$ such that

$$\lim_{\mu\to\infty}\phi_\mu(x,t)=\hat{\phi}(x,t)\text{ uniformly in }\bar{\Omega}_0\times[0,T]. \tag{2.9}$$

Step 3. The eigenfunction $\phi_\mu(x, t)$ in $\bar{\Omega} \setminus \bar{\Omega}_0 \times [0, T]$.

From (2.6)–(2.7), we know that

$$\begin{aligned}\phi_\mu(x,t)&\le e^{[\lambda_P(\Omega_0)-1-\mu b(x)q_*]t}\phi_\mu(x,0)\\ &\quad+\int_0^t e^{[\lambda_P(\Omega_0)-1-\mu b(x)q_*](t-s)}\int_\Omega J(x-y)\phi_\mu(y,s)dyds\\ &\le Me^{[\lambda_P(\Omega_0)-1-\mu b(x)q_*]t}+\frac{M-Me^{[\lambda_P(\Omega_0)-1-\mu b(x)q_*]t}}{1-\lambda_P(\Omega_0)+\mu b(x)q_*}.\end{aligned}$$

Thus we know that

$$\lim_{\mu\to\infty} \phi_\mu(x,t) = 0 \tag{2.10}$$

for $(x,t) \in \bar{\Omega} \setminus \bar{\Omega}_0 \times [0,T]$ and

$$\lim_{\mu\to\infty} \phi_\mu(x,t) = 0 \text{ uniformly in any compact subset of } \bar{\Omega} \setminus \bar{\Omega}_0 \times [0,T].$$

Step 4. We show $\hat{\phi}(x,t) = \psi(x)$ in $\bar{\Omega}_0 \times [0,T]$.

In view of (2.9) and (2.10), we get

$$\hat{\phi}(x,t) = \hat{\phi}(x,0) + \int_0^t \left[\int_{\Omega_0} J(x-y)\hat{\phi}(y,s)dy - \hat{\phi} + \lambda_P(\Omega_0)\hat{\phi} \right] ds$$

by the dominated convergence theorem. Then we have

$$\hat{\phi}(x,t+\varepsilon) - \hat{\phi}(x,t) = \int_t^{t+\varepsilon} \left[\int_{\Omega_0} J(x-y)\hat{\phi}(y,s)dy - \hat{\phi} + \lambda_P(\Omega_0)\hat{\phi} \right] ds$$

for $(x,t) \in \bar{\Omega}_0 \times [0,T]$, here ε is a small parameter. Thus we know from (2.7) that

$$|\hat{\phi}(x,t+\varepsilon) - \hat{\phi}(x,t)| \le [2 + \lambda_P(\Omega_0)]M\varepsilon.$$

This gives that $\hat{\phi}(x,\cdot) \in C[0,T]$. Furthermore, we have

$$\begin{aligned} \lim_{\varepsilon\to 0} \frac{\hat{\phi}(x,t+\varepsilon) - \hat{\phi}(x,t)}{\varepsilon} &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \left[\int_{\Omega_0} J(x-y)\hat{\phi}(y,s)dy - \hat{\phi} + \lambda_P(\Omega_0)\hat{\phi} \right] ds \\ &= \int_{\Omega_0} J(x-y)\hat{\phi}(y,s)dy - \hat{\phi} + \lambda_P(\Omega_0)\hat{\phi} \end{aligned}$$

and $\hat{\phi}(x,\cdot) \in C^1([0,T])$ for $x \in \bar{\Omega}$. Hence,

$$\begin{cases} -\hat{\phi}_t + J * \hat{\phi} - \hat{\phi} = -\lambda_P(\Omega_0)\hat{\phi} \text{ in } \bar{\Omega}_0 \times (0,\infty), \\ \hat{\phi}(x,t) = 0 \text{ in } (\mathbb{R}^N \setminus \bar{\Omega}_0) \times (0,\infty), \\ \hat{\phi}(x,t) = \hat{\phi}(x,t+T) \text{ in } \bar{\Omega}_0 \times [0,\infty). \end{cases}$$

In view of (2.5), we know that

$$\frac{1}{T} \int_0^T \int_{\Omega_0} \hat{\phi}(x,s)dxds = 1 \tag{2.11}$$

and the maximum principle shows that

$$\hat{\phi}(x,t) > 0 \text{ in } \bar{\Omega}_0 \times (0,\infty).$$

At last, as $\psi(x)$ is a positive eigenfunction associated with $\lambda_P(\Omega_0)$, by the uniqueness of principal eigenfunction we obtain

$$\hat{\phi}(x,s) = c\psi(x) \text{ in } \bar{\Omega}_0 \times (0,\infty)$$

for some constant $c > 0$. It follows from (2.5) and (2.11) that $c = 1$, this also shows that (2.9) holds for the entire sequences. □

By a similar argument as in the proof of Theorem 2.3, we have the following lemma.

Lemma 2.4 *Assume that (A2) holds. Let $\phi_\mu(x,t)$ and $\psi(x)$ be the positive eigenfunctions associated with $\lambda_P(\mu,\Omega)$ for $\mu \ge 0$ and $\lambda_P(\Omega_0)$ such that*

$$\frac{1}{T}\int_0^T\int_\Omega \phi_\mu(x,s)dxds = \int_\Omega \psi(x)dx = 1/M,$$

respectively, here

$$M = \frac{J^*T}{1-e^{[\lambda_P(\Omega_0)-1]T}} + \max_{\bar\Omega\times\bar\Omega} J(x-y)T.$$

Then we have $\phi_\mu(x,t) \le 1$ in $\bar\Omega \times [0,T]$,

$$\lim_{\mu\to\infty} \phi_\mu(x,t) = \psi(x) \textit{ uniformly in } \bar\Omega_0 \times [0,T],$$

and

$$\lim_{\mu\to\infty} \phi_\mu(x,t) = 0 \textit{ uniformly in any compact subset of } \bar\Omega \setminus \bar\Omega_0 \times [0,T].$$

For the time independent nonlocal eigenvalue equation

$$\begin{cases} J*\phi - \phi - \mu b(x)\phi = -\lambda\phi \text{ in } \bar\Omega, \\ \phi(x) = 0 \text{ in } \mathbb{R}^N \setminus \bar\Omega, \end{cases} \tag{2.12}$$

we know form [27] that (2.12) admits a unique principal eigenvalue $\sigma_P(\mu,\Omega)$ for $\mu \ge 0$ if $b(x)$ exists spatial degeneracy. Then we have the following result.

Corollary 2.5 *Assume that $b \in C(\bar\Omega)$ is nontrivial, nonnegative and $\Omega_0 = \{x \in \Omega : b(x) = 0\}$ has a positive measure. Let $\phi_\mu(x)$ and $\psi(x)$ be the positive eigenfunctions associated with $\sigma_P(\mu,\Omega)$ for $\mu \ge 0$ and $\lambda_P(\Omega_0)$ such that*

$$\int_\Omega \phi_\mu(x)dx = 1 \textit{ and } \int_\Omega \psi(x)dx = 1,$$

respectively. Then we have

$$\lim_{\mu\to\infty} \phi_\mu(x) = \psi(x) \textit{ uniformly in } \bar\Omega_0,$$

and

$$\lim_{\mu\to\infty} \phi_\mu(x) = 0 \textit{ uniformly in any compact subset of } \bar\Omega \setminus \bar\Omega_0.$$

Theorem 1.1 is followed by Lemma 2.1. At the end of this section, we prove Theorem 1.2.

Proof of Theorem 1.2 The conclusion (i) can be proved by the same way as in Lemma 2.1. We only show that claim (ii) is true.

Since $\lambda_P(\mu,\Omega) < \lambda_P(\Omega_0)$ for $\mu > 0$ and $\lambda \ge \lambda_P(\Omega_0)$, we can take δ small such that

$$\delta \le \frac{\lambda - \lambda_P(\mu,\Omega)}{\mu}.$$

Let $\phi_\mu(x,t)$ be a positive eigenfunction associated with $\lambda_P(\mu,\Omega)$ and

$$\frac{1}{T}\int_0^T\int_\Omega \phi_\mu(x,s)dxds = 1/M,$$

where M is given in Lemma 2.4. Then we know that $0 < \phi_\mu(x,t) \le 1$ in $\bar{\Omega} \times [0,T]$ and we can check that $\mu^{\frac{1}{p-1}}\phi_\mu(x,t)$ is a lower-solution to (1.3). Since $u_\delta(x,t)$ is monotone with respect to δ, by the uniqueness of positive solutions, we get

$$\mu^{\frac{1}{p-1}}\phi_\mu(x,t) \le \lim_{\delta\to 0} u_\delta(x,t) \text{ in } \Omega \times [0,T].$$

Letting $\mu \to \infty$, again by Lemma 2.4, we have

$$\lim_{\delta\to 0+} u_\delta(x,t) = \infty \text{ uniformly in } \bar{\Omega}_0 \times [0,T].$$

Now let $\hat{u}_\delta(x)$ be the unique positive solution of

$$\begin{cases} J * u - u = -\lambda u + [b(x)q^* + \delta]u^p & \text{in } \bar{\Omega}, \\ u(x) = 0 & \text{in } \mathbb{R}^N \setminus \bar{\Omega}, \end{cases}$$

for $\lambda \ge \lambda_P(\Omega_0)$, here $q^* = \max_{[0,T]} q(t)$. Similarly to the above argument, we know that

$$\lim_{\delta\to 0+} \hat{u}_\delta(x) = \infty \text{ uniformly in } \bar{\Omega}_0.$$

Since

$$\int_\Omega J(x-y)\hat{u}_\delta(y)dy = (1 - \lambda_P(\Omega_0) + [b(x)q^* + \delta]\hat{u}_\delta^{p-1})\hat{u}_\delta(x)$$

and

$$\int_\Omega J(x-y)\hat{u}_\delta(y)dy \ge \int_{\Omega_0} J(x-y)\hat{u}_\delta(y)dy,$$

we get

$$\lim_{\delta\to 0+} \hat{u}_\delta(x) = \infty \text{ uniformly in } \bar{\Omega}.$$

Then by the comparison principle we have

$$\lim_{\delta\to 0+} u_\delta(x,t) \ge \lim_{\delta\to 0+} \hat{u}_\delta(x) \text{ in } \bar{\Omega} \times [0,T]$$

and

$$\lim_{\delta\to 0+} v_\delta(x) = \infty \text{ uniformly in } \bar{\Omega} \times [0,T].$$

□

3 Sharp profiles

In this section, we establish the sharp profiles for positive solutions of (1.3). We first give some preliminaries and then prove the main theorems.

3.1 Preliminaries

To begin with, we give some estimates on the profiles of positive solutions to (1.3). Let $u_\delta \in C^1([0,T]; C(\bar{\Omega}))$ be the positive solution of (1.3) for $\lambda > \lambda_P(\Omega)$ and $\delta > 0$. Denote $v_\delta(x,t) = \delta^{\frac{1}{p-1}} u_\delta(x,t)$, then we have

$$\begin{cases} (v_\delta)_t = J * v_\delta - v_\delta + \lambda v_\delta - \left[\frac{b(x)q(t)}{\delta} + 1\right] v_\delta^p & \text{in } \bar{\Omega} \times (0,\infty), \\ v_\delta(x,t) = 0 & \text{in } \mathbb{R}^N \setminus \bar{\Omega} \times (0,\infty), \\ v_\delta(x,t) = v_\delta(x,t+T) & \text{in } \bar{\Omega} \times [0,\infty). \end{cases} \tag{3.1}$$

In order to obtain lower and upper bounds for $v_\delta(x,t)$, we consider the nonlocal dispersal equations

$$\begin{cases} J * u - u = -\lambda u + u^p & \text{in } \bar{\Omega}, \\ u(x) = 0 & \text{in } \mathbb{R}^N \setminus \bar{\Omega}, \end{cases} \tag{3.2}$$

and

$$\begin{cases} J * u - u = -\lambda u + u^p & \text{in } \bar{\Omega}_0, \\ u(x) = 0 & \text{in } \mathbb{R}^N \setminus \bar{\Omega}_0. \end{cases} \tag{3.3}$$

It follows from [14,26] that (3.2) exists a unique positive solution $\hat{u} \in C(\bar{\Omega})$ for $\lambda > \lambda_P(\Omega)$ and (3.3) exists a unique positive solution $\bar{u} \in C(\bar{\Omega})$ for $\lambda > \lambda_P(\Omega_0)$.

Lemma 3.1 *Assume that* (A2) *holds and* $\delta > 0$. *Let* $\hat{u}(x)$ *be the positive solution of* (3.2) *and* $\bar{u}(x)$ *be the positive solution of* (3.3), *respectively. Then we have*

$$0 < v_\delta(x,t) \le \hat{u}(x) \text{ in } \bar{\Omega} \times [0,T] \tag{3.4}$$

for $\lambda > \lambda_P(\Omega)$ *and*

$$v_\delta(x,t) \ge \bar{u}(x) \text{ in } \bar{\Omega}_0 \times [0,T] \tag{3.5}$$

for $\lambda > \lambda_P(\Omega_0)$.

Proof Since $b(x)$ and $q(t)$ are nonnegative, we can see that $\hat{u}(x)$ is an upper-solution of (3.1). The uniqueness gives that (3.4) holds.

On the other hand, we know that $v_\delta(x,t)$ satisfies the nonlocal dispersal equation

$$\begin{cases} u_t = J * u - u + \lambda u - u^p + f_\delta(x,t) & \text{in } \bar{\Omega}_0 \times (0,\infty), \\ u(x,t) = 0 & \text{in } \mathbb{R}^N \setminus \bar{\Omega}_0 \times (0,\infty), \\ u(x,t) = u(x,t+T) & \text{in } \bar{\Omega}_0 \times [0,\infty), \end{cases} \tag{3.6}$$

where

$$f_\delta(x,t) = \int_{\Omega \setminus \Omega_0} J(x-y) v_\delta(y,t)\,dy.$$

By a simple argument (see the proof of Theorem 5.4 in [27]), we know that $v_\delta(x,t)$ is the only continuous positive solution of (3.6). But $\bar{u}(x)$ is a lower-solution to (3.6), we get (3.5).

□

Lemma 3.2 *Assume that (A2) holds and* $\lambda \geq \lambda_P(\Omega_0)$. *Then there exists* $l > 0$ *which is independent of* δ *such that*

$$1 - \lambda + v_\delta^{p-1}(x,t) \geq l \text{ in } \bar{\Omega}_0 \times [0,T] \tag{3.7}$$

for $\delta > 0$.

Proof Since $\lambda_P(\Omega_0) \in [0, 1)$ and $v_\delta(x,t)$ is nonnegative, we can see that (3.7) holds for $\lambda = \lambda_P(\Omega_0)$.

If $\lambda > \lambda_P(\Omega_0)$, let $\bar{u}(x)$ be the unique solution of (3.3), we know from (3.5) that

$$1 - \lambda + v_\delta^{p-1}(x,t) \geq 1 - \lambda + \bar{u}^{p-1}(x) = \frac{\int_{\Omega_0} J(x-y)\bar{u}(y)dy}{\bar{u}(x)}.$$

Since $\bar{u}(x) > 0$ in $\bar{\Omega}_0$, we complete the proof. □

3.2 Proof of Theorems 1.4–1.5

In this subsection, we will prove the main theorems.

Proof of Theorems 1.4–1.5 The long discuss is divided into the following steps.
Step 1. The asymptotic profile for $\lambda_P(\Omega) < \lambda < \lambda_P(\Omega_0)$.

In this case, we know from Theorem 1.2 that

$$\lim_{\delta \to 0+} v_\delta(x,t) = \lim_{\delta \to 0+} \omega_\delta(x,t) = 0 \text{ uniformly in } \bar{\Omega} \times [0,T].$$

Step 2. The profile $v_\delta(x,t)$ in $\bar{\Omega}_0 \times [0,T]$ for $\lambda > \lambda_P(\Omega_0)$.

By (3.4) we can find $C > 0$ which is independent to δ such that

$$\max_{\bar{\Omega} \times [0,T]} v_\delta(x,t) \leq C.$$

For any given $x_1, x_2 \in \bar{\Omega}_0$, we denote

$$\omega(t) = v_\delta(x_1,t) - v_\delta(x_2,t) \text{ in } [0,T].$$

Without loss of generality, we assume that $\omega(0) = \omega(T) > 0$. Since $\omega(t)$ is continuous in $[0,T]$, we first show that $\omega(T)$ satisfies

$$|\omega(T)| = |v_\delta(x_1,T) - v_\delta(x_2,T)| \leq \frac{C}{l} \int_\Omega |J(x_1-y) - J(x_2-y)|dy, \tag{3.8}$$

where $l > 0$ is given by (3.7).

If $\omega(t)$ is not sign-changing in $[0,T]$, we assume that $\omega(t) \geq 0$ for $t \in [0,T]$. Then we know from (3.1) and (3.7) that

$$\begin{aligned}
\omega_t(t) &= \int_\Omega [J(x_1-y) - J(x_2-y)]v_\delta(y,t)dy + [\lambda - 1 - p\theta^{p-1}(t)]\omega(t) \\
&\leq [\lambda - 1 - v_\delta^{p-1}(x_2,t)]\omega(t) + C\int_\Omega |J(x_1-y) - J(x_2-y)|dy \\
&\leq -l\omega(t) + C\int_\Omega |J(x_1-y) - J(x_2-y)|dy,
\end{aligned}$$

where $\theta(t)$ is between $v_\delta(x_2, t)$ and $v_\delta(x_1, t)$. Thus we get

$$\omega(t) \le e^{-lt}\omega(0) + \frac{1 - e^{-lt}}{l} C \int_\Omega |J(x_1 - y) - J(x_2 - y)|dy \tag{3.9}$$

for $t \in [0, T]$. But $\omega(0) = \omega(T)$, we know (3.8) holds.

If $\omega(t)$ is sign-changing in $[0, T]$. In this case, (3.8) still true for $\omega(T) = 0$. If $\omega(T) > 0$, since $\omega \in C([0, T])$ is sign-changing in $[0, T]$, we can see that there exists $t_* \in (0, T)$ such that $\omega(T) \ge \omega(t_*)$ and

$$\omega(t) > 0 \text{ in } [t_*, T].$$

It follows from (3.1) that

$$\begin{cases} \omega_t(t) \le -l\omega(t) + C \int_\Omega |J(x_1 - y) - J(x_2 - y)|dy \text{ in } (t_*, T], \\ \omega(t_*) = \omega(t_*), \end{cases}$$

and so

$$\begin{aligned} \omega(t) &\le e^{-l(t-t_*)}\omega(t_*) + \frac{1 - e^{-l(t-t_*)}}{l} C \int_\Omega |J(x_1 - y) - J(x_2 - y)|dy \\ &\le e^{-l(t-t_*)}\omega(T) + \frac{1 - e^{-l(t-t_*)}}{l} C \int_\Omega |J(x_1 - y) - J(x_2 - y)|dy. \end{aligned}$$

We get $\omega(T)$ satisfies (3.8). For $\omega(T) < 0$, a similar argument from $-\omega(T)$ gives that $\omega(T)$ satisfies (3.8).

Now for any $x_1, x_2 \in \bar{\Omega}_0$, without loss of generality, we assume that

$$\omega(t) = v_\delta(x_1, t) - v_\delta(x_2, t) \ge 0 \text{ in } [0, T].$$

Then by (3.8) and (3.9), we have

$$|\omega(t)| \le \frac{2C}{l} \int_\Omega |J(x_1 - y) - J(x_2 - y)|dy \tag{3.10}$$

for $t \in [0, T]$.

Note that $v_\delta(x, t)$ satisfies

$$v_\delta(x, t) = \int_0^t \left[\int_\Omega J(x - y)v_\delta(y, s)dy - v_\delta + \lambda v_\delta - v_\delta^p \right] ds \text{ in } \bar{\Omega}_0 \times (0, \infty),$$

we have

$$|v_\delta(x, t_1) - v_\delta(x, t_2)| \le (2 + \lambda + C^{p-1})C|t_1 - t_2| \tag{3.11}$$

for $t_1, t_2 \in [0, T]$.

By a compactness argument from (3.10) and (3.11), subject to a subsequence, we know that there exists $v \in C(\bar{\Omega}_0 \times [0, T])$ such that

$$\lim_{\delta \to 0+} v_\delta(x, t) = v(x, t) \text{ uniformly in } \bar{\Omega}_0 \times [0, T]. \tag{3.12}$$

Step 3. The profiles $v_\delta(x, t)$ in $\bar{\Omega} \setminus \bar{\Omega}_0 \times [0, T]$ for $\lambda > \lambda_P(\Omega_0)$.

We consider the nonlocal dispersal equation

$$\begin{cases} J * u_\delta - u_\delta = -\lambda u_\delta + \left[\frac{b(x)q_*}{\delta} + 1 \right] u_\delta^p & \text{in } \bar{\Omega}, \\ u_\delta(x) = 0 & \text{in } \mathbb{R}^N \setminus \bar{\Omega}, \end{cases} \tag{3.13}$$

where $q_* = \min_{[0,T]} q(t)$. Let $u_\delta(x)$ be the unique solution of (3.13) for $\lambda > \lambda_P(\Omega)$, then the upper-lower solutions argument gives that

$$0 \le v_\delta(x,t) \le u_\delta(x) \le \hat{u}(x)$$

for any $(x,t) \in \bar{\Omega} \times [0,T]$, here $\hat{u}(x)$ is given by (3.4). Since

$$u_\delta(x) = \left[\frac{J * u_\delta - u_\delta + \lambda u_\delta}{\frac{b(x)q_*}{\delta} + 1}\right]^{1/p} \text{ in } \bar{\Omega} \setminus \bar{\Omega}_0,$$

we get

$$\lim_{\delta \to 0+} v_\delta(x,t) = 0 \text{ locally uniformly in } \bar{\Omega} \setminus \bar{\Omega}_0 \times [0,T] \tag{3.14}$$

and

$$\lim_{\delta \to 0+} v_\delta(x,t) = 0 \tag{3.15}$$

for any $(x,t) \in \bar{\Omega} \setminus \bar{\Omega}_0 \times [0,T]$.
Step 4. We show that $v(x,t) = \theta(x)$ in $\bar{\Omega}_0 \times [0,T]$.

In view of (3.12) and (3.15), by dominated convergence theorem, we know that

$$v(x,t) = \int_0^t \left[\int_{\Omega_0} J(x-y)v(y,s)dy - v + \lambda v - v^p\right] ds \text{ in } \bar{\Omega}_0 \times (0,\infty),$$

and $v(x,0) = v(x,T)$ in $\bar{\Omega}$. Meanwhile, we know from Lemma 3.1 that $v(x,t)$ is positive and boundned in $\bar{\Omega}_0 \times [0,T]$. Then a simple argument gives that

$$v \in C^1([0,T]; C(\bar{\Omega}_0)).$$

So we get

$$\begin{cases} v_t = J * v - v + \lambda v - v^p & \text{in } \bar{\Omega}_0 \times (0,\infty), \\ v(x,t) = 0 & \text{in } (\mathbb{R}^N \setminus \bar{\Omega}_0) \times (0,+\infty), \\ v(x,t) = v(x,t+T) & \text{in } \bar{\Omega}_0 \times [0,\infty). \end{cases} \tag{3.16}$$

Let $\theta(x)$ be the unique continuous positive solution of (1.8) for $\lambda > \lambda_P(\Omega_0)$, we can see that $\theta(x)$ satisfies (3.16). Since the positive solution is unique, we necessarily have

$$v(x,t) = \theta(x) \text{ in } \bar{\Omega}_0 \times [0,T].$$

This also implies that (3.12) holds for the entire original sequences.
Step 5. The profiles $\omega_\delta(x,t)$ in $\bar{\Omega} \times [0,T]$ for $\lambda > \lambda_P(\Omega_0)$.

Since $\omega_\delta(x,t) = \delta^{\frac{1}{p(p-1)}} u_\delta(x)$, we can see that $v_\delta(x,t) = \delta^{\frac{1}{p}} \omega_\delta(x,t)$ and so

$$\lim_{\delta \to 0+} \omega_\delta(x,t) = \infty \text{ uniformly in } \bar{\Omega}_0 \times [0,T].$$

Take f to be a smooth T-periodic function such that $f(0) = f(T) = 0$. Multiplying (3.1) by f and integrating in $[0,T]$, we obtain

$$\int_0^T v_\delta(x,t)f(t)dt = \int_0^T \left[J * v_\delta - v_\delta + \lambda v_\delta - (b(x)q(t)/\delta + 1)v_\delta^p\right] f(t)dt.$$

Letting $\delta \to 0+$, we know from (3.12) and (3.14) that

$$\lim_{\delta\to 0+}\int_0^T \left[\frac{b(x)q(t)}{\delta}+1\right] v_\delta^p f(t)dt = \int_0^T \int_{\Omega_0} J(x-y)\theta(y)f(t)dydt,$$

which is uniform in any compact subset of $\bar{\Omega}\setminus\bar{\Omega}_0$. Due to the arbitrariness of f, we necessary have

$$\lim_{\delta\to 0+}\left[\frac{b(x)q(t)}{\delta}+1\right] v_\delta^p(x,t) = \int_{\Omega_0} J(x-y)\theta(y)dy \tag{3.17}$$

for almost everywhere $t\in[0,T]$ and the convergence is uniform in any compact subset of $\bar{\Omega}\setminus\bar{\Omega}_0$. Let Ω_c be a compact subset of $\bar{\Omega}\setminus\bar{\Omega}_0$ and denote

$$H(x) = \left[\frac{b(x)q(t)}{\delta}+1\right] v_\delta^p(x,t)$$

for any given $t\in[0,T]$. Then we know from (3.17) that $H(x)$ is equicontinuous in Ω_c.

Now we know from (3.15) and (3.17) that there exists $\delta_0>0$ and such that

$$\begin{aligned}&|v_\delta(x,t)|\le 1 \text{ in } \Omega_c\times[0,T],\\ &\left|\left[\frac{b(x)q(t)}{\delta}+1\right] v_\delta^p(x,t)\right| \le C_1 = \max_{\Omega_c}\int_{\Omega_0} J(x-y)\theta(y)dy + 1 \text{ in } \Omega_c\times[0,T],\end{aligned}$$

and

$$\left|\left[\frac{b(x)}{\delta}\right] v_\delta^p(x,t)\right| \le C_2 = \frac{C_1}{\min_{[0,T]} q(t)} \text{ in } \Omega_c\times[0,T]$$

for $\delta\le\delta_0$. Since

$$v_\delta(x,t) = \int_0^t \left[\int_\Omega J(x-y)v_\delta(y,s)dy - v_\delta + \lambda v_\delta - (b(x)q(t)/\delta+1)v_\delta^p\right] ds$$

for $(x,t)\in\bar{\Omega}_c\times[0,T]$, we get

$$|v_\delta(x,t_1)-v_\delta(x,t_2)| \le (2+\lambda+C_1)|t_1-t_2|$$

for any $x\in\bar{\Omega}_c$ and $t_1,t_2\in[0,T]$. Then by (3.1) we obtain that

$$\begin{aligned}&|(v_\delta)_t(x,t_1)-(v_\delta)_t(x,t_2)|\\ &\le \left|\int_\Omega J(x-y)[v_\delta(y,t_1)-v_\delta(y,t_2)]dy + [\lambda+1][v_\delta(x,t_1)-v_\delta(x,t_2)]\right|\\ &\quad + \left[\frac{b(x)q(t_1)}{\delta}+1\right] p\theta_\delta^{p-1}(x,t)|v_\delta(x,t_1)-v_\delta(x,t_2)|\\ &\quad + \left[\frac{b(x)}{\delta}\right] v_\delta^p(x,t)|q(t_1)-q(t_2)|\\ &\le (2+\lambda+C_1)(2+\lambda+pC_1)|t_1-t_2| + C_2|q(t_1)-q(t_2)|\\ &\le (2+\lambda+pC_1)^2|t_1-t_2| + C_2|q(t_1)-q(t_2)|\end{aligned}$$

for any $x\in\bar{\Omega}_c$ and $t_1,t_2\in[0,T]$, here $\theta_\delta(x,t)$ is between $v_\delta^p(x,t_1)$ and $v_\delta^p(x,t_2)$.

Let us denote

$$V(t) = \left[\frac{b(x)q(t)}{\delta}+1\right] v_\delta^p(x,t)$$

for any given $x \in \bar{\Omega}_c$. Then by (3.1) we know that

$$|V(t_1) - V(t_2)| \le (2+\lambda)|v_\delta(x,t_1) - v_\delta(x,t_2)| + |(v_\delta)_t(x,t_1) - (v_\delta)_t(x,t_2)|$$
$$\le (2+\lambda+pC_1)^2|t_1 - t_2| + C_2|q(t_1) - q(t_2)|$$

for $t_1, t_2 \in [0, T]$.

Now it follows from (3.17) that

$$\lim_{\delta\to 0+}\left[\frac{b(x)q(t)}{\delta} + 1\right] v_\delta^p(x,t) = \int_{\Omega_0} J(x-y)\theta(y)dy, \tag{3.18}$$

which is uniform in any compact subset of $\bar{\Omega} \setminus \bar{\Omega}_0 \times [0, T]$.

At last, since $v_\delta(x,t) = \delta^{\frac{1}{p}}\omega_\delta(x,t)$, (3.18) yields that

$$\lim_{\delta\to 0+}[b(x)q(t) + \delta]\omega_\delta^p(x,t) = \int_{\Omega_0} J(x-y)\theta(y)dy$$

and so

$$\lim_{\delta\to 0+}\omega_\delta(x,t) = \left[\frac{\int_{\Omega_0} J(x-y)\theta(y)dy}{b(x)q(t)}\right]^{1/p},$$

which is uniform in any compact subset of $\bar{\Omega} \setminus \bar{\Omega}_0 \times [0, T]$.

Step 6. The profiles $v_\delta(x,t)$ in $\bar{\Omega} \times [0, T]$ for $\lambda = \lambda_P(\Omega_0)$.

In this case, we know that the only nonnegative solution of (3.16) is $u(x,t) = 0$ in $\bar{\Omega}_0 \times [0, T]$. A similar arguments as in the previous steps, we know that

$$\lim_{\delta\to 0+} v_\delta(x,t) = 0 \text{ uniformly in } \bar{\Omega}_0 \times [0, T]$$

and

$$\lim_{\delta\to 0+} v_\delta(x,t) = 0$$

for any $(x,t) \in \bar{\Omega} \setminus \bar{\Omega}_0 \times [0, T]$. Note that $v_\delta(x,t)$ is monotone with respect to δ, we end our proof by Dini's theorem. □

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