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Calculus of Variations

Stable standing waves of nonlinear Schrödinger equations with potentials and general nonlinearities

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Abstract

The existence and nonexistence of the minimizer of the L^2-constraint minimization problem $e(\alpha) := \inf\{E(u) | \ u \in H^1(\mathbb{R}^N), \ \|u\|^2_{L^2(\mathbb{R}^N)} = \alpha\}$ are studied. Here,

$$E(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)|u|^2 dx - \int_{\mathbb{R}^N} F(|u|)dx,$$

$V(x) \in C(\mathbb{R}^N)$, $0 \not\equiv V(x) \leq 0$, $V(x) \to 0$ $(|x| \to \infty)$ and $F(s) = \int_0^s f(t)dt$ is a rather general nonlinearity. We show that there exists $\alpha_0 \geq 0$ such that $e(\alpha)$ is attained for $\alpha > \alpha_0$ and $e(\alpha)$ is not attained for $0 < \alpha < \alpha_0$. We study differences between the cases $V(x) \not\equiv 0$ and $V(x) \equiv 0$, and obtain sufficient conditions for $\alpha_0 = 0$. In particular, if $N = 1, 2$, then $\alpha_0 = 0$, and hence $e(\alpha)$ is attained for all $\alpha > 0$.

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1 Introduction and main theorems

In this paper we are interested in the attainability of the L^2-constraint minimization problem

$$e(\alpha) := \inf_{u \in M(\alpha)} E(u),$$

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where $\alpha > 0$ is a constant,

$$E(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)|u|^2 dx - \int_{\mathbb{R}^N} F(|u|)dx, \quad F(s) := \int_0^s f(t)dt \text{ for } s \geq 0,$$

$$M(\alpha) := \left\{u \in H^1(\mathbb{R}^N) \mid \|u\|^2_{L^2(\mathbb{R}^N)} = \alpha\right\}, \quad H^1(\mathbb{R}^N) = H^1(\mathbb{R}^N, \mathbb{C}),$$

and $f(s)$ and $V(x)$ satisfy certain assumptions. This problem plays a role when we study the orbital stability of the standing wave of the nonlinear Schrödinger equation

$$iU_t = -\Delta U + V(x)U - f(U) \text{ for } (t, x) \in \mathbb{R} \times \mathbb{R}^N. \tag{1.1}$$

The standing wave is a solution of (1.1) of the special form $U(t, x) = e^{i\lambda t}u(x)$ and the orbital stability is defined in Theorem A. We impose the following assumptions (F1)–(F4) on $f(s)$:

(F1) $f \in C(\mathbb{C}, \mathbb{C})$, $f(0) = 0$.
(F2) $f(s) \in \mathbb{R}$ for $s \in \mathbb{R}$, $f(e^{i\theta}z) = e^{i\theta} f(z)$ for $\theta \in \mathbb{R}$ and $z \in \mathbb{C}$.
(F3) $\lim_{s\to 0} f(s)/s = 0$.
(F4) $\lim_{s\to\infty} f(s)/|s|^{p_c} = 0$, where $p_c := 1 + 4/N$.

We impose the following assumption (V1) on $V(x)$:

(V1) $V(x) \in C(\mathbb{R}^N)$, $0 \not\equiv V(x) \leq 0$ and $\lim_{|x|\to\infty} V(x) = 0$.

The assumptions (F1)–(F4) and (V1) are assumed throughout the present paper. In addition to (F1)–(F4) and (V1), we introduce the following conditions:

(F5) $f(s)$ is locally Hölder continuous with exponent $\nu \in (0, 1)$ in $\mathbb{R}$, $f(s) > 0$ for $s > 0$ and there exists $\delta_1 > 0$ such that $f(s)/s$ is nondecreasing in $(0, \delta_1)$.
(F6) If $N \geq 5$, then $\liminf_{s\to 0} f(s)/|s|^{p_{sg}} > 0$, where $p_{sg} := N/(N-2)$.
(V2) If $N \geq 5$, then

$$V \in W^{1,\infty}(\mathbb{R}^N) \text{ and } \nabla V(x) \cdot x \leq \frac{(N-2)^2}{2|x|^2} \text{ for a.e. } x \in \mathbb{R}^N \setminus \{0\}.$$

In order to obtain the orbital stability we further need the following:

(F7) There exist $K > 0$ and $1 < p < 2^* - 1$ such that $|f(z_1) - f(z_2)| \leq K(1 + |z_1| + |z_2|)^{p-1}|z_1 - z_2|$ for $z_1, z_2 \in \mathbb{C}$. Here, $2^* = 2N/(N-2)$ if $N \geq 3$, and $2^* = \infty$ if $N = 1, 2$.
(F8) There exist $L > 0$ and $1 < p < p_c$ such that $F(|s|) \leq L(|s|^2 + |s|^{p+1})$ for $s \in \mathbb{R}$.

It is known that the global well-posedness of (1.1) in $H^1(\mathbb{R}^N)$ holds if (F1), (F2), (F7) and (F8) hold and $V(x) \in L^\infty(\mathbb{R}^N)$. See [10, Corollary 6.1.2] for details.

To state our main theorems we recall related results. Lions [20] showed that every minimizing sequence for $e(\alpha)$ has a convergent subsequence in $H^1(\mathbb{R}^N)$ if and only if the strict subadditivity condition holds, i.e.,

$$e(\alpha) < e(\beta) + e_\infty(\alpha - \beta) \text{ for all } \beta \in \begin{cases} (0, \alpha) & \text{if } V(x) \equiv 0, \\ [0, \alpha) & \text{if } V(x) \not\equiv 0. \end{cases} \tag{1.2}$$

Here, $e_\infty(\alpha)$ is the problem at infinity, i.e.,

$$e_\infty(\alpha) := \inf_{u \in M(\alpha)} E_\infty(u),$$

where

$$E_\infty(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} F(|u|)dx.$$

The characterization (1.2) holds for rather wide class of functionals $E(u)$. However, it is not easy to check (1.2) for given f and V.

First, we consider the homogeneous case $V(x) \equiv 0$. Then, $E(u) = E_\infty(u)$ and $e(\alpha) = e_\infty(\alpha)$. In the model case $f(u) = |u|^{p-1}u$ $(1 < p < p_c)$, Cazenave–Lions [11] showed that (1.2) holds for all $\alpha > 0$ and that $e_\infty(\alpha) < 0$ for all $\alpha > 0$. In the case of a general nonlinear term f, the attainability for $e_\infty(\alpha)$ was mentioned in [11, Remark II.3]. However, in [11] the following condition was assumed:

$$\text{there exists } u_0 \in L^2(\mathbb{R}^N) \text{ such that } \|u_0\|_{L^2(\mathbb{R}^N)} \leq \alpha \text{ and } E_\infty(u_0) < 0. \tag{1.3}$$

The same attainability problem for $e_\infty(\alpha)$ was recently studied by [5,13,22]. In particular, Shibata [22] showed that there exists $\alpha_{0,\infty} \in [0,\infty)$ uniquely determined by f and N such that

$$e_\infty(\alpha) \begin{cases} = 0 & \text{if } 0 \leq \alpha \leq \alpha_{0,\infty}, \\ < 0 & \text{if } \alpha > \alpha_{0,\infty}. \end{cases} \tag{1.4}$$

Moreover, he showed that $e_\infty(\alpha)$ is not attained for $0 < \alpha < \alpha_{0,\infty}$ and $e_\infty(\alpha)$ is attained for $\alpha > \alpha_{0,\infty}$. See Proposition 2.1 of the present paper for details. It was shown in [22, Lemma 2.3] that $e_\infty(\alpha)$ is nonincreasing. Hence the assumption (1.3) leads to $e_\infty(\alpha) < 0$ for each $\alpha \geq \|u_0\|_{L^2(\mathbb{R}^N)}$.

Our result is about the attainability of the inhomogeneous problem $e(\alpha)$.

Theorem A *Suppose* (F1)–(F5) *and* (V1), *and suppose* (F6) *or* (V2). *Let* $\alpha_{0,\infty}$ *be given in* (1.4). *Then there exists* $\alpha_0 \in [0, \alpha_{0,\infty}]$ *such that the following hold:*

(i) *If* $\alpha > \alpha_0$, *then* $e(\alpha) < 0$ *and every minimizing sequence for* $e(\alpha)$ *has a strong convergent subsequence in* $H^1(\mathbb{R}^N)$. *Therefore,* $e(\alpha)$ *is attained, the set of all the minimizers, which is denoted by* S_α, *is precompact and* (1.2) *holds. Moreover, if* (F7) *and* (F8) *hold, then* S_α *is orbitally stable, i.e., for any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for any solution* U *of* (1.1) *with* $\mathrm{dist}_{H^1}(U(0,\cdot), S_\alpha) < \delta$ *satisfies*

$$\mathrm{dist}_{H^1}(U(t,\cdot), S_\alpha) < \varepsilon \text{ for all } t \in \mathbb{R}.$$

(ii) *If* $0 < \alpha < \alpha_0$, *then* $e(\alpha) = 0$ *and* $e(\alpha)$ *is not attained.*

Remark 1.1 (i) Notice that (F6) and (V2) are necessary only for $N \geq 5$. Therefore, when $1 \leq N \leq 4$, Theorem A holds under (F1)–(F5) and (V1) (for the orbital stability, we also need (F7) and (F8)).

(ii) If $\alpha_{0,\infty} = 0$, then $\alpha_0 = 0$ and Theorem A (i) always occurs. Remark that if $N \geq 5$, then $p_{sg} < p_c$. Hence, when $N \geq 5$ and (F6) hold, we have $\alpha_{0,\infty} = 0 = \alpha_0$ by [22, Theorem 1.3] (see also Proposition 2.2 below).

(iii) Compared to the conditions (F1)–(F4), the conditions (F5) and (F6) seem technical. The condition (F5) is used in interaction estimates in Lemmas 2.4 and 3.3 and (F6) is used to prove the nonexistence of the minimizer in Lemma 3.1.

(iv) If we assume $0 \not\equiv V(x) \geq 0$ and $\lim_{|x|\to\infty} V(x) = 0$ instead of (V1), then $e(\alpha)$ is not attained for all $\alpha > 0$, and $e(\alpha) = e_\infty(\alpha)$ for $\alpha \geq 0$. See Theorem A.1 in Appendix A.

As mentioned above, in [22, Theorem 1.3], Shibata observed whether $\alpha_{0,\infty} > 0$ or $\alpha_{0,\infty} = 0$. We also consider the same question: whether $\alpha_0 > 0$ or $\alpha_0 = 0$ under the presence of the potential term $V(x)$.

Theorem B *Suppose* (F1)–(F4) *and* (V1). *Then the following* (i) *and* (ii) *hold:*

(i) *In addition, assume that there exists an* $s_0 > 0$ *such that* $f(s) \geq 0$ *in* $[0, s_0]$ *and the following (V3) holds:*

$$\text{(V3)} \quad \inf_{\|\varphi\|_{L^2(\mathbb{R}^N)}=1} \int_{\mathbb{R}^N} \left(|\nabla\varphi|^2 + V(x)\varphi^2\right) dx < 0.$$

Then $\alpha_0 = 0$*. Moreover, when* $N = 1, 2$*, (V1) implies (V3) and* $\alpha_0 = 0$*.*

(ii) *Suppose* $N \geq 3$ *and the following condition (F9) in addition to (F1)–(F4) and (V1):*

(F9) $\limsup_{s\downarrow 0} F(s)/s^{p_c+1} < \infty$.

Then there exists $\alpha_1 = \alpha_1(N, f) > 0$ *satisfying the following property: for each* $\alpha \in (0, \alpha_1)$ *we may find a* $c_\alpha > 0$ *such that* $V(x) \geq -c_\alpha|x|^{-2}$ *for* $|x| > 0$ *implies* $\alpha_0 \geq \alpha > 0$*.*

Remark 1.2 Notice that Theorem B (i) may be used to see a difference between the cases $V(x) \equiv 0$ and $V(x) \not\equiv 0$. Indeed, since (F6) plays a role only for $N \geq 5$, when $N = 1, 2$, if (V1), (F1)–(F5) and (F9) hold, then we obtain $0 = \alpha_0 < \alpha_{0,\infty}$ due to Theorems A, B (i) and [22, Theorem 1.3].

Let us mention other related results. For the homogeneous problem $e_\infty(\alpha)$, Bellazzini et al. [5] showed that there exists $\bar{\alpha} \geq 0$ such that $e_\infty(\alpha)$ is attained for $\alpha > \bar{\alpha}$ if (F5') given in Proposition 2.1, (F8) and the following assumption are satisfied:

$$\text{there exist } C_1,\ C_2 \geq 0,\ 1 < q \leq p < 2^* - 1 \text{ such that } |f(s)| \leq C_1|s|^q + C_2|s|^p. \tag{1.5}$$

Moreover, they proved that $\bar{\alpha} = 0$ if

$$\text{there exists } 1 < p < p_c \text{ such that } F(s) > s^{p+1} \text{ for small } s > 0. \tag{1.6}$$

Note that (F10) in Proposition 2.2 is a generalization of (1.6). In [22] the threshold $\alpha_{0,\infty}$ was found and Proposition 2.1 was obtained. In particular, the nonexistence part (Proposition 2.1 (ii)) was proved. In Garrisi–Georgiev [13] the one-dimensional case was studied and the orbital stability of the minimizers was obtained if (1.5), (F5′) and the following hold:

$$\text{there exist } 1 < p < 5(= p_c) \text{ and } s_0 \geq 0 \text{ such that } \ F(s) \leq C|s|^{p+1} \text{ for } s \geq s_0.$$

See [12] for a quasilinear homogeneous problem and [7] for a Schrödinger-Poisson problem with pure power nonlinearity. For the inhomogeneous problem $e(\alpha)$, in [6,8,18] the attainability was studied. In [6,8], they deal with the rather special type of nonlinearity, that is, $f(u) = |u|^{p-1}u$ in [6] and $f(u) = Q(x)|u|^{p-1}u$ in [8]. In Jeanjean–Squassina [18, 2.4 A Stuart's type problem] the nonlinear term is $F(x, u)$. They showed that $e(\alpha)$ is attained if F satisfies

$$\lim_{|x|\to\infty} F(x, s) = 0 \text{ uniformly in } s \in \mathbb{R}. \tag{1.7}$$

Here, (1.7) leads to the weak lower semicontinuity of $E(u)$ which our problem does not satisfy.

Let us explain technical details for the proof of Theorem A. To prove Theorem A, we try to establish (1.2) in a scheme similar to [22], and a difficulty is to exclude dichotomy since we treat $V \in L^\infty(\mathbb{R}^N)$ and $E(u)$ is not weak lower semicontinuous. Furthermore, since our nonlinearity is general and there is a term $V(x)$, a scaling argument in [10] or the scaled function $u(\lambda x)$ in the homogeneous case may not be useful. Therefore, we need to bring another idea to overcome this difficulty. In this paper, we perform a careful interaction estimate to exclude dichotomy in Lemma 3.3 where (F5) is used. This usage of the interaction estimate is inspired by Hirata [15] where the unconstrained problem is studied and we try to apply this estimate in the L^2-constraint setting. To do so, we modify any minimizing sequence to be an approximated positive solution of the Euler-Lagrange equation and prove the precompactness of the modified minimizing sequence. This reduction is done in Lemmas 2.6 and 2.8, and is also used in [16] for the homogeneous case. In addition to the reduction, to follow the scheme in [22], we also need the nonexistence result of the minimizer for which the condition $1 \le N \le 4$, (F6) or (V2) is used. See Lemma 3.1. Here we also have a difference between the cases $V(x) \equiv 0$ and $V(x) \not\equiv 0$ because the scaled function $u(\lambda x)$ may not be useful.

Finally we make a comment on the usage of the interaction estimate. Our argument is also applied to a minimizing problems with two constraint conditions and potentials. This will be discussed in [17].

This paper consists of five sections. In Sect. 2 we recall fundamental properties of the problems $e(\alpha)$ and $e_\infty(\alpha)$. In Sect. 3 we study the existence and nonexistence of the minimizers of $e(\alpha)$ and prove Theorem A. In Sect. 4 we prove Theorem B. In "Appendix A" we show that $e(\alpha)$ is not attained if $0 \not\equiv V(x) \ge 0$ and $\lim_{|x|\to\infty} V(x) = 0$.

Notations

- For $p \ge 1$, $L^p(\Omega)$ denotes the space of complex-valued measurable functions u on $\Omega \subset \mathbb{R}^N$ satisfying $\int_\Omega |u|^p dx < \infty$ whose norm is defined by $\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u|^p dx\right)^{1/p}$. When $\Omega = \mathbb{R}^N$, write $\|u\|_p := \|u\|_{L^p(\mathbb{R}^N)}$.
- $L^\infty(\Omega)$ denotes the space of complex-valued essentially bounded measurable functions u on $\Omega \subset \mathbb{R}^N$ whose norm is defined by $\|u\|_{L^\infty(\Omega)} := \mathrm{esssup}_{x\in\Omega}|u(x)|$. When $\Omega = \mathbb{R}^N$, write $\|u\|_\infty := \mathrm{esssup}_{x\in\mathbb{R}^N}|u(x)|$.
- We regard $L^2(\mathbb{R}^N)$ as a Hilbert space over $\mathbb{R}$ by the inner product $\langle u, v\rangle_{L^2} := \mathrm{Re}\int_{\mathbb{R}^N} f(x)\overline{g(x)}dx$.
- The set H stands for the space of complex-valued measurable functions u of the Sobolev space of order 1 whose norm is defined by $\|u\|_H := \left(\int_{\mathbb{R}^N} |u|^2 dx + \int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{1/2}$, i.e., $H := H^1(\mathbb{R}^N)$. We denote its inner product by $\langle u, v\rangle_H := \langle \nabla u, \nabla v\rangle_{L^2} + \langle u, v\rangle_{L^2}$ and the dual space of H by H^*.

2 Preliminaries

We first recall known facts about the homogeneous problem $e_\infty(\alpha)$.

Proposition 2.1 ([22, Theorems 1.1 and 1.5]) *Suppose* (F1)–(F4) *and the following* (F5′)*:*

(F5') There exists $s_0 > 0$ *such that* $F(s_0) > 0$.

Then there exists a unique $\alpha_{0,\infty} \in [0, \infty)$ *such that* (1.4) *and the following (i) and (ii) hold:*

(i) *If* $\alpha > \alpha_{0,\infty}$, *then every minimizing sequence for* $e_\infty(\alpha)$ *has a convergent subsequence in* H *up to translations. Therefore,* $e_\infty(\alpha)$ *is attained, the set of all minimizers is precompact in* H *up to translations and* (1.2) *holds. Moreover, in addition, if* (F7) *and* (F8) *hold, then the set of all minimizers is orbitally stable.*

(ii) *If* $0 < \alpha < \alpha_{0,\infty}$, *then* $e_\infty(\alpha)$ *is not attained.*

Note that (F5) implies (F5′). The next proposition concerns when $\alpha_{0,\infty} = 0$ or $\alpha_{0,\infty} > 0$ holds.

Proposition 2.2 ([22, Theorems 1.3]) *Suppose* (F1)–(F4) *and* (F5′). *Then the following* (i) *and* (ii) *hold:*

(i) *If the following* (F10) *holds:*

(F10) $\liminf_{s\downarrow 0} F(s)/s^{p_c+1} = \infty$,

then $\alpha_{0,\infty} = 0$.

(ii) *If* (F9) *holds, then* $\alpha_{0,\infty} > 0$.

Next, we collect some properties about $F(s)$. We begin with a variant of [22, Lemma 2.2 (i)].

Lemma 2.3 *Suppose* (F1)–(F4), $u_0 \in H$ *and that* (u_n) *is bounded in* H. *If* $\|u_n - u_0\|_p \to 0$ *for some* $p \in [2, \infty]$, *then* $\lim_{n\to\infty} \int_{\mathbb{R}^N} F(|u_n|)dx = \int_{\mathbb{R}^N} F(|u_0|)dx$.

Proof We remark that we may assume $u_n \geq 0$ without loss of generality since $\||u_n| - |u_0|\|_p \leq \|u_n - u_0\|_p$ and $\|\nabla |u|\|_2 \leq \|\nabla u\|_2$ (see [19, Theorem 6.17]). By Sobolev's inequality and Hölder's inequality, $\|u_n - u_0\|_q \to 0$ for any $q \in (2, 2^*)$. We also set $M_0 := \sup_{n\geq 1} \|u_n\|_H < \infty$.

Next, by (F3) and (F4), for each $\varepsilon > 0$, one may find a $C_\varepsilon > 0$ such that

$$|f(s)| \leq \varepsilon|s| + C_\varepsilon |s|^{p_c} \quad \text{for all } s \in \mathbb{R}.$$

From

$$\begin{aligned}
|F(u_n) - F(u_0)| &= \left| \int_0^1 \frac{d}{d\theta} F\left(\theta u_n + (1-\theta)u_0\right) d\theta \right| \\
&\leq \int_0^1 |f(\theta u_n + (1-\theta)u_0)|\, d\theta |u_n - u_0| \\
&\leq \int_0^1 \left\{ \varepsilon\left(u_n + u_0\right) + C_\varepsilon \left(u_n + u_0\right)^{p_c} \right\} d\theta |u_n - u_0| \\
&= \left\{ \varepsilon\left(u_n + u_0\right) + C_\varepsilon \left(u_n + u_0\right)^{p_c} \right\} |u_n - u_0|
\end{aligned}$$

and Hölder's inequality, we have

$$\begin{aligned}
\left| \int_{\mathbb{R}^N} \{F(u_n) - F(u_0)\}\, dx \right| &\leq \int_{\mathbb{R}^N} \left\{ \varepsilon\left(u_n + u_0\right) + C_\varepsilon \left(u_n + u_0\right)^{p_c} \right\} |u_n - u_0| dx \\
&\leq \varepsilon \left(\|u_n\|_2 + \|u_0\|_2\right) \|u_n - u_0\|_2 \\
&\quad + C_\varepsilon \|u_n + u_0\|_{p_c+1}^{p_c} \|u_n - u_0\|_{p_c+1}.
\end{aligned}$$

Noting $2 < p_c + 1 < 2^*$, we obtain

$$\limsup_{n\to\infty} \left| \int_{\mathbb{R}^N} \{F(u_n) - F(u_0)\}\, dx \right| \leq 4M_0^2 \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\int_{\mathbb{R}^N} F(u_n)dx \to \int_{\mathbb{R}^N} F(u_0)dx$ as $n \to \infty$. □

Next, we borrow one lemma from [15], which is used for the interaction estimate in the proof of Lemma 3.3. For a proof, see [15].

Lemma 2.4 ([15, Lemma 4.4]) *Assume* (F1) *and* (F5). *Let* $\delta_1 > 0$ *be as in* (F5). *Then the following* (i) *and* (ii) *hold:*

(i) *There exists* $\delta_2 \in (0, \delta_1]$ *such that*

$$F(u_1) + F(u_2) - F(u_1 + u_2) + \frac{1}{2}(f(u_1)u_2 + f(u_2)u_1) \le 0 \ \textit{for} \ u_1, u_2 \in [0, \delta_2].$$

(ii) *For each compact set* $K \subset (0, \infty)$*, there exist* $C_K > 0$ *and* $\delta_K > 0$ *such that*

$$\begin{aligned} &F(u_1) + F(u_2) - F(u_1 + u_2) + \frac{1}{2}(f(u_1)u_2 + f(u_2)u_1) \\ &\le -C_K u_2 \ \textit{for} \ u_1 \in K \ \textit{and} \ u_2 \in [0, \delta_K]. \end{aligned}$$

In the next lemma we state fundamental properties of $e(\alpha)$ and $e_\infty(\alpha)$.

Lemma 2.5 *Assume* (F1)–(F4) *and* (V1). *Then the following hold:*

(i) $e(\alpha) > -\infty$ *for* $\alpha > 0$.
(ii) *For* $\alpha > 0$*, every minimizing sequence for* $e(\alpha)$ *is bounded in* H.
(iii) $e(\alpha) \le e_\infty(\alpha) \le 0$ *for* $\alpha \ge 0$.
(iv) $e(\alpha) \le e(\beta) + e_\infty(\alpha - \beta)$ *for* $0 \le \beta < \alpha$.
(v) $e(\alpha)$ *is nonincreasing in* $\alpha \ge 0$.

Proof (i) The proof is almost the same as [22, Lemma 2.2 (ii)]. By the assumptions (F1)–(F4), for $\varepsilon > 0$, there exists a positive constant $C_\varepsilon > 0$ such that

$$F(|u|) \le C_\varepsilon |u|^2 + \varepsilon |u|^{p_c+1}. \tag{2.1}$$

By the Gagliardo–Nirenberg inequality we have

$$\|u\|_{p_c+1}^{p_c+1} \le C \, \|u\|_2^{4/N} \, \|\nabla u\|_2^2 . \tag{2.2}$$

Thus, (2.1) and (2.2) give

$$\left| \int_{\mathbb{R}^N} F(|u|) dx \right| \le C_\varepsilon \, \|u\|_2^2 + \varepsilon C \alpha^{2/N} \, \|\nabla u\|_2^2 .$$

We choose $\varepsilon > 0$ such that $\varepsilon C \alpha^{2/N} = 1/4$. Then for $u \in M(\alpha)$,

$$\int_{\mathbb{R}^N} F(|u|) dx \le C_\varepsilon \alpha + \frac{1}{4} \, \|\nabla u\|_2^2 ,$$

which implies

$$E(u) \ge \frac{1}{4} \, \|\nabla u\|_2^2 - C_\varepsilon \alpha. \tag{2.3}$$

Hence, (i) holds.

(ii) Since $u \in M(\alpha)$, the conclusion immediately follows from (2.3).

(iii) Because $E(u) \le E_\infty(u)$ for each $u \in H$ due to (V1), we easily see that $e(\alpha) \le e_\infty(\alpha)$. For the inequality $e_\infty(\alpha) \le 0$, see [22, Lemma 2.3 (i)].

(iv) For $\varepsilon > 0$, we can find $\varphi_\varepsilon, \psi_\varepsilon \in C_0^\infty(\mathbb{R}^N)$ such that

$$\varphi_\varepsilon \in M(\beta), \ \psi_\varepsilon \in M(\alpha - \beta), \ E(\varphi_\varepsilon) \le e(\beta) + \varepsilon, \ E_\infty(\psi_\varepsilon) \le e_\infty(\alpha - \beta) + \varepsilon.$$

Let $u_{\varepsilon,n}(x) := \varphi_\varepsilon(x) + \psi_\varepsilon(x - n\mathbf{e}_1)$. Since φ_ε and ψ_ε have compact support, we see that $u_{\varepsilon,n} \in M(\alpha)$ for large n and that $e(\alpha) \le E(u_{\varepsilon,n}) = E(\varphi_\varepsilon) + E(\psi_\varepsilon(\cdot - n\mathbf{e}_1))$. From $E(\psi_\varepsilon(\cdot - n\mathbf{e}_1)) \to E_\infty(\psi_\varepsilon)$ as $n \to \infty$ thanks to (V1), it follows that

$$e(\alpha) \le \lim_{n \to \infty} (E(\varphi_\varepsilon) + E(\psi_\varepsilon(\cdot - n\mathbf{e}_1))) = E(\varphi_\varepsilon) + E_\infty(\psi_\varepsilon) \le e(\beta) + e_\infty(\alpha - \beta) + 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, (iv) holds.

(v) By (iii) and (iv), we have

$$e(\alpha) \leq e(\beta) + e_\infty(\alpha - \beta) \leq e(\beta) \text{ for } 0 \leq \beta < \alpha.$$

Thus, $e(\alpha)$ is nonincreasing in α.

□

In the next two lemmas we collect some properties of a minimizing sequence for $e(\alpha)$.

Lemma 2.6 *Assume* (F1)–(F4) *and* (V1). *The following hold:*

(i) *Let $(u_n) \subset M(\alpha)$ be a minimizing sequence for $e(\alpha)$, and let $|u_n|(x) := |u_n(x)|$. Then $(|u_n|)$ is also a minimizing sequence.*
(ii) *If $u_0 \in H$ and (u_n) is a minimizing sequence for $e(\alpha)$ with $\|u_n - u_0\|_2 \to 0$, then $\|u_n - u_0\|_H \to 0$. Furthermore, if $u_0 \in H$ and (u_n) is a minimizing sequence of for $e(\alpha)$ and $\||u_n| - |u_0|\|_2 \to 0$, then $\|u_n - u_0\|_H \to 0$.*

Proof (i) By $\|\nabla |u_n|\|_2^2 \leq \|\nabla u\|_2^2$ ([19, Theorem 6.17]) and $|u_n| \in M(\alpha)$, we see that $E(|u_n|) \leq E(u_n)$ and $(|u_n|)$ is also a minimizing sequence.

(ii) From $\|u_n - u_0\|_2 \to 0$, it follows that

$$u_0 \in M(\alpha) \text{ and } \lim_{n\to\infty} \int_{\mathbb{R}^N} V(x) u_n^2 dx = \int_{\mathbb{R}^N} V(x) u_0^2 dx. \tag{2.4}$$

Moreover, by Lemma 2.5 (ii), (u_n) is bounded in H. Thanks to $\|u_n - u_0\|_2 \to 0$, we obtain $u_n \rightharpoonup u_0$ weakly in H. Thus, Lemma 2.3 and the weak lower semicontinuity of $\|\nabla \cdot\|_2$ yield

$$e(\alpha) \leq E(u_0) \leq \liminf_{n\to\infty} E(u_n) = \lim_{n\to\infty} E(u_n) = e(\alpha),$$

which implies $\|\nabla u_n\|_2^2 \to \|\nabla u_0\|_2^2$. Combining this fact with $\nabla u_n \rightharpoonup \nabla u_0$ weakly in $L^2(\mathbb{R}^N)$, we observe that $\|\nabla u_n - \nabla u_0\|_2 \to 0$ and $\|u_n - u_0\|_H \to 0$.
Assume that (u_n) is a minimizing sequence for $e(\alpha)$ with $\||u_n| - |u_0|\|_2 \to 0$. By Lemma 2.5 (ii), (u_n) is bounded in H, hence, choosing a subsequence if necessary, we may assume $u_n \to u_0$ in $L^2_{\text{loc}}(\mathbb{R}^N)$ without loss of generality. Since $\||u_n| - |u_0|\|_2 \to 0$ and $u_n \to u_0$ in $L^2_{\text{loc}}(\mathbb{R}^N)$, we may find a $w_0 \in L^2(\mathbb{R}^N)$ and a subsequence (u_{n_k}) such that $|u_{n_k}(x)| \leq w_0(x)$ and $u_{n_k}(x) \to u_0(x)$ a.e. $\mathbb{R}^N$. The dominated convergence theorem gives $\|u_{n_k} - u_0\|_2 \to 0$ and the former assertion gives $\|u_{n_k} - u_0\|_H \to 0$ due to the fact that (u_{n_k}) is a minimizing sequence for $e(\alpha)$. Since the limit is independent of subsequences, we have $\|u_n - u_0\|_H \to 0$ and the proof is completed.

□

Remark 2.7 A similar argument to the proof of Lemma 2.6 shows that if $u_0 \in M(\alpha)$ is a minimizer, then so is $|u_0(x)|$. Hence, when $e(\alpha)$ is attained, we may always find a nonnegative minimizer.

Lemma 2.8 *Let $(u_n) \subset M(\alpha)$ be a minimizing sequence for $e(\alpha)$. Then there exist $(v_n) \subset M(\alpha)$ and $(\lambda_n) \subset \mathbb{R}$ such that (λ_n) is bounded and*

$$\|u_n - v_n\|_H \to 0, \quad E'(v_n) + \lambda_n Q'(v_n) \to 0 \text{ strongly in } H^*, \tag{2.5}$$

where $Q(u) := \|u\|_2^2$. Furthermore, if (u_n) is real-valued, then we may choose v_n as real-valued function.

Remark 2.9 We notice that if (v_n) in Lemma 2.8 has a strongly convergent subsequence in H, then so is (u_n).

Proof of Lemma 2.8 We first remark that Q is smooth and $Q'(u)u = 2Q(u)$. By $M(\alpha) = Q^{-1}(\alpha)$, we notice that $M(\alpha)$ is closed and a Hilbert manifold with codimension 1. Moreover, the tangent space of $M(\alpha)$ at u and the tangent derivative $D_{T_u M(\alpha)}E$ of E at u are given by

$$T_u M(\alpha) = \{v \in H \mid \langle \nabla Q(u), v\rangle_H = 0\},$$
$$D_{T_u M(\alpha)} E(u) = E'(u) - \frac{E'(u)\nabla Q(u)}{\|\nabla Q(u)\|_H^2} Q'(u), \tag{2.6}$$

where $\nabla Q(u) \in H$ is the unique element satisfying $\langle \nabla Q(u), v\rangle_H = Q'(u)v$ for every $v \in H$.

We now apply Ekeland's variational principle for $E(u)$ and (u_n) on $M(\alpha)$ to get $v_n \in M(\alpha)$ satisfying

$$\|u_n - v_n\|_H \le \sqrt{\varepsilon_n}, \quad E(v_n) \le E(w) + \sqrt{\varepsilon_n}\|v_n - w\|_H \quad \text{for each } w \in M(\alpha), \tag{2.7}$$

where $\varepsilon_n := E(u_n) - e(\alpha) \ge 0$. Putting $w = u_n$ in (2.7) and the fact $v_n \in M(\alpha)$ assert that (v_n) is also a minimizing sequence. In addition, (2.6) and (2.7) imply that

$$\left\| D_{T_{v_n} M(\alpha)} E(v_n) \right\|_{(T_{u_n} M(\alpha))^*} := \sup\left\{ D_{T_{v_n} M(\alpha)} E(v_n)\varphi \mid \|\varphi\|_H = 1,\ \varphi \in T_{v_n} M(\alpha) \right\} \to 0. \tag{2.8}$$

Since (v_n) is bounded in H, E' maps bounded sets into bounded sets and $\|\nabla Q(v_n)\|_H \ge 2\alpha/\|v_n\|_H$ for any $n \ge 1$ due to $Q'(v_n)v_n = 2Q(v_n) = 2\alpha$, setting $\lambda_n := -E'(v_n)\nabla Q(v_n)/\|\nabla Q(v_n)\|_H^2$, from (2.6) and (2.8), we see that (2.5) holds.

If (u_n) is real-valued, then we restrict ourselves into $H_{\mathbb{R}} := \{u \in H \mid \text{u is real-valued}\}$ and $M_{\mathbb{R}}(\alpha) := M(\alpha) \cap H_{\mathbb{R}}$. Since $e(\alpha) = \inf_{u \in M_{\mathbb{R}}(\alpha)} E(u)$ holds, we may use the above argument on $M_{\mathbb{R}}(\alpha)$ to obtain real-valued functions (v_n) satisfying (2.5). Thus we complete the proof. □

3 Proof of Theorem A

We first observe the case when $e(\alpha)$ is not attained.

Lemma 3.1 *Assume* (F1)–(F5) *and* (V1) *and assume* (F6) *or* (V2). *If there are* $\alpha > 0$ *and* $\beta > 0$ *such that* $e(\alpha) = e(\beta)$ *and* $\alpha > \beta$, *then* $e(\beta)$ *is not attained.*

Proof We first prove the following:

$$\text{If } e(\,\cdot\,) \text{ is constant in } [\beta, \beta + \varepsilon) \text{ for small } \varepsilon > 0, \text{ then } e(\beta) \text{ is not attained.} \tag{3.1}$$

Remark that (3.1) implies our conclusion. Indeed, we see by Lemma 2.5 (v) that $e(\,\cdot\,)$ is nonincreasing. Since $e(\alpha) = e(\beta)$, we observe that $e(\,\cdot\,)$ is constant in the interval $[\beta, \alpha]$. Then by (3.1), $e(\beta)$ is not attained.

Now we prove (3.1) by contradiction and let $u_0 \in M(\beta)$ be a minimizer for $e(\beta)$. Thanks to Remark 2.7, we may assume $u_0 \ge 0$. Notice that u_0 is a (classical) solution of

$$-\Delta u_0 + V(x)u_0 - f(u_0) = -2\lambda u_0 \ \text{ in } \mathbb{R}^N \tag{3.2}$$

for some $\lambda \in \mathbb{R}$. Next, we show by contradiction that $\lambda \le 0$. If $\lambda > 0$, then

$$\frac{d}{dt}E(tu_0)\bigg|_{t=1} = \int_{\mathbb{R}^N} |\nabla u_0|^2 + V(x)u_0^2 - f(u_0)u_0 dx = -2\lambda \int_{\mathbb{R}^N} |u_0|^2 dx = -2\lambda\beta < 0.$$

Hence, for sufficiently small $\eta > 0$, the monotonicity of $e(\alpha)$ yields

$$e(\beta + \varepsilon) \le e((1+\eta)^2\beta) \le E((1+\eta)u_0) < E(u_0) = e(\beta),$$

which is a contradiction. Thus, $\lambda \le 0$.

We prove (3.1). Since $V(x) \le 0 \le u_0(x)$ and $\lambda \le 0$, by (3.2) and $f(s) \ge 0$ $(s \ge 0)$ due to (F5), we have

$$-\Delta u_0 \ge f(u_0) \ge 0 \text{ in } \mathbb{R}^N \text{ and } u_0 \ge 0 \text{ in } \mathbb{R}^N. \tag{3.3}$$

Hence, the strong maximum principle and $u_0 \in M(\beta)$ give $u_0 > 0$ in $\mathbb{R}^N$.

If $N = 1, 2$, then $-\Delta u_0 \ge 0$ in $\mathbb{R}^N$. Since u_0 is a positive super-harmonic function in $\mathbb{R}$ or $\mathbb{R}^2$, we see that u_0 is constant (see [21, Chapter 2, Theorem 29] for $N = 2$). However, this contradicts $u_0 \in L^2(\mathbb{R}^N)$ and $e(\beta)$ is not attained.

If $N = 3, 4$, then we show that (3.2) has no solution in H. This claim is proved in [16, Lemma A.2], however, we give another simple proof which is similar to [4, Lemma 3.12]. Let $c_1 > 0$ and $w(x) := u_0(x) - c_1|x|^{2-N}$. Here $c_1 > 0$ can be chosen so that $w(x) \ge 0$ for all $|x| = 1$ due to $u_0 > 0$ in $\mathbb{R}^N$. From $-\Delta w = -\Delta u_0 \ge 0$ for $|x| > 1$ and $w(x) \to 0$ as $|x| \to \infty$, the weak maximum principle asserts that $w \ge 0$ in $|x| \ge 1$, which implies $u_0(x) \ge c_1|x|^{2-N}$ for $|x| \ge 1$. However, this contradicts $u_0 \in L^2(\mathbb{R}^N)$ when $N = 3, 4$. Hence, $e(\beta)$ is not attained.

We consider the case $N \ge 5$. In this case we assume (F6) or (V2). If (F6) holds, then it follows from the result of [1] that (3.3) has no solution. Hence, $e(\beta)$ is not attained.

On the other hand, when (V2) holds, we first observe from (3.2) that u_0 satisfies the Pohozaev identity:

$$0 = \frac{N-2}{2}\|\nabla u_0\|_2^2 - N\int_{\mathbb{R}^N} F(u_0) - \lambda u_0^2 - \frac{V(x)}{2}u_0^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} (x \cdot \nabla V(x))u_0^2 dx.$$

Then we have

$$\begin{aligned} 0 &\ge e(\beta) \\ &= E(u_0) \\ &= \frac{1}{2}\|\nabla u_0\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u_0^2 dx - \int_{\mathbb{R}^N} F(u_0)dx \\ &= \frac{1}{N}\|\nabla u_0\|_2^2 - \lambda\|u_0\|_2^2 - \frac{1}{2N}\int_{\mathbb{R}^N} x \cdot \nabla V(x)u_0^2 dx \\ &\ge \frac{1}{N}\left(\|\nabla u_0\|_2^2 - \frac{1}{2}\int_{\mathbb{R}^N} x \cdot \nabla V(x)u_0^2 dx\right), \end{aligned}$$

where we used $\lambda \le 0$. Since $\nabla V(x) \in L^\infty(\mathbb{R}^N)$, the strict inequality in (V2) holds on $A \subset \mathbb{R}^N$, where the Lebesgue measure of A is strictly positive. Since $u_0 > 0$ in $\mathbb{R}^N$, we get

$$\frac{1}{2}\int_{\mathbb{R}^N} x \cdot \nabla V(x)u_0^2 dx < \frac{(N-2)^2}{4}\int_{\mathbb{R}^N} \frac{u_0^2}{|x|^2}dx.$$

From Hardy's inequality, it follows that

$$0 \ge Ne(\beta) \ge \|\nabla u_0\|_2^2 - \frac{1}{2}\int_{\mathbb{R}^N} x \cdot \nabla V(x)u_0^2 dx > \|\nabla u_0\|_2^2 - \frac{(N-2)^2}{4}\int_{\mathbb{R}^N} \frac{u_0^2}{|x|^2}dx \ge 0.$$

This is a contradiction and $e(\beta)$ is not attained. Thus (3.1) holds. □

Next we observe a behavior of minimizing sequence when the compactness does not hold.

Lemma 3.2 *Assume* (F1)–(F5) *and* (V1) *and assume* (F6) *or* (V2). *Let* $(u_n) \subset M(\alpha)$ *be a minimizing sequence for* $e(\alpha)$ *such that* $u_n \rightharpoonup u_0$ *weakly in* H *and let* $\beta := \|u_0\|_2^2$. *If either* $0 < \beta < \alpha$ *or both* $\beta = 0$ *and* $e(\alpha) < 0$, *then there exist* $(y_n) \subset \mathbb{R}^N$ *and* $w_0 \in H\backslash\{0\}$ *such that*

$$|y_n| \to \infty,\ u_n(\cdot + y_n) \rightharpoonup w_0 \text{ weakly in } H, \tag{3.4}$$

$$\lim_{n\to\infty} \|u_n - u_0 - w_0(\cdot - y_n)\|_2 = 0 \text{ and } \alpha = \beta + \gamma, \tag{3.5}$$

where $\gamma := \|w_0\|_2^2$. *Moreover, the following hold:*

$$E(u_0) = e(\beta),\ E_\infty(w_0) = e_\infty(\gamma) \text{ and } e(\alpha) = e(\beta) + e_\infty(\gamma). \tag{3.6}$$

Proof We divide the proof into three steps.

Step 1: *We find* $(y_n) \subset \mathbb{R}^N$ *and* $w_0 \in H\backslash\{0\}$ *such that* (3.4) *holds.*

First, we show by contradiction that

$$\liminf_{n\to\infty} \sup_{z\in\mathbb{Z}^N} \|u_n - u_0\|_{L^2(Q^N+z)} > 0 \quad \text{where } Q^N := [0,1]^N. \tag{3.7}$$

Suppose on the contrary that $\sup_{z\in\mathbb{Z}^N} \|u_n - u_0\|_{L^2(Q^N+z)} \to 0$. Then, $u_n \to u_0$ strongly in $L^q(\mathbb{R}^N)$ for $2 < q < 2^*$ (See [23]). By Lemmas 2.3 and 2.5, we have

$$e(\alpha) \le e(\beta) \le E(u_0) \le \lim_{n\to\infty} E(u_n) = e(\alpha). \tag{3.8}$$

When $\beta = 0$ and $e(\alpha) < 0$, we get a contradiction. Hence (3.7) holds provided $\beta = 0$ and $e(\alpha) < 0$.

Next, let us consider the case $0 < \beta < \alpha$. In this case, (3.8) asserts $e(\alpha) = E(u_0) = e(\beta)$ and u_0 is a minimizer due to $\|u_0\|_2^2 = \beta$. However, this contradicts Lemma 3.1. Therefore, (3.7) holds.

From (3.7) and $u_n \to u_0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, we can find $(y_n) \subset \mathbb{R}^N$ such that $\|u_n\|_{L^2(Q^N+y_n)} \to c_0 > 0$ and $|y_n| \to \infty$. Let

$$u_n(\cdot + y_n) \rightharpoonup w_0 \text{ weekly in } H.$$

Note that $w_0 \not\equiv 0$ because $c_0 > 0$. Therefore, (y_n) and w_0 satisfy (3.4). The proof of Step 1 is complete.

Since $|y_n| \to \infty$ $(n \to \infty)$, we have

$$\begin{aligned}\|u_n - u_0 - w_0(\cdot - y_n)\|_2^2 &= \|u_n\|_2^2 + \|u_0\|_2^2 + \|w_0\|_2^2 \\ &\quad - 2\langle u_n, u_0\rangle_{L^2} - 2\langle u_n(\cdot + y_n), w_0\rangle_{L^2} + o(1) \\ &= \|u_n\|_2^2 - \|u_0\|_2^2 - \|w_0\|_2^2 + o(1).\end{aligned} \tag{3.9}$$

In particular,

$$\gamma := \|w_0\|_2^2 \le \liminf_{n\to\infty}(\|u_n\|_2^2 - \|u_0\|_2^2) = \alpha - \beta.$$

Note that $\gamma > 0$ because $w_0 \neq 0$.

Step 2: *We show that* (y_n) *and* w_0 *satisfy* (3.5).

Let $\delta := \lim_{n\to\infty} \|u_n - u_0 - w_0(\cdot - y_n)\|_2^2$. Then, we see by (3.9) that $\delta = \alpha - \beta - \gamma$. Our aim is to show that $\delta = 0$. Suppose on the contrary that

$$\delta > 0. \tag{3.10}$$

By direct calculation we have

$$\begin{aligned}&\frac{1}{2}\left(\|\nabla u_n\|_2^2 - \|\nabla u_0\|_2^2 - \|\nabla w_0(\cdot - y_n)\|_2^2 - \|\nabla (u_n - u_0 - w_0(\cdot - y_n))\|_2^2\right)\\ &= -\|\nabla u_0\|_2^2 + \langle \nabla u_n, \nabla u_0\rangle_{L^2} - \|\nabla w_0(\cdot - y_n)\|_2^2\\ &\quad - \langle \nabla u_0, \nabla w_0(\cdot - y_n)\rangle_{L^2} + \langle \nabla u_n(\cdot + y_n), \nabla w_0\rangle_{L^2}\\ &= o(1).\end{aligned} \tag{3.11}$$

Similarly,

$$\frac{1}{2}\int_{\mathbb{R}^N} V(x)\left(|u_n|^2 - |u_0|^2 - |w_0(\cdot - y_n)|^2 - |u_n - u_0 - w_0(\cdot - y_n)|^2\right)dx = o(1). \tag{3.12}$$

By the Brezis–Lieb lemma [9, Theorem 2], we have

$$\begin{aligned}&\int_{\mathbb{R}^N} F(|u_n|)dx = \int_{\mathbb{R}^N} F(|u_0|)dx + \int_{\mathbb{R}^N} F(|u_n - u_0|)dx + o(1),\\ &\int_{\mathbb{R}^N} F(|u_n(\cdot + y_n) - u_0(\cdot + y_n)|)dx = \int_{\mathbb{R}^N} F(|w_0|)dx\\ &\quad + \int_{\mathbb{R}^N} F(|u_n(\cdot + y_n) - u_0(\cdot + y_n) - w_0|)dx + o(1).\end{aligned}$$

Thus,

$$\begin{aligned}&\int_{\mathbb{R}^N} F(|u_n|)dx - \int_{\mathbb{R}^N} F(|u_0|)dx\\ &- \int_{\mathbb{R}^N} F(|w_0(\cdot - y_n)|)dx - \int_{\mathbb{R}^N} F(|u_n - u_0 - w_0(\cdot - y_n)|)dx = o(1).\end{aligned} \tag{3.13}$$

Combining (3.11)–(3.13), we have

$$E(u_n) - E(u_0) - E(w_0(\cdot - y_n)) - E(u_n - u_0 - w_0(\cdot - y_n)) = o(1). \tag{3.14}$$

Since $V(x) \to 0$ as $|x| \to \infty$, $u_n \rightharpoonup u_0$ weakly in H and $|y_n| \to \infty$, we have

$$\int_{\mathbb{R}^N} V(x)|u_n(x) - u_0(x) - w_0(x - y_n)|^2 dx \to 0. \tag{3.15}$$

Noting

$$\begin{aligned}E(u_n - u_0 - w_0(\cdot - y_n)) &= E_\infty(u_n - u_0 - w_0(\cdot - y_n))\\ &\quad + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u_n(x) - u_0(x) - w_0(x - y_n)|^2 dx,\end{aligned}$$

we have

$$\liminf_{n\to\infty} E(u_n - u_0 - w_0(\cdot - y_n)) \geq e_\infty(\delta) \quad \text{and} \quad \liminf_{n\to\infty} E(w_0(\cdot - y_n)) \geq e_\infty(\gamma). \tag{3.16}$$

Hence, by (3.14)–(3.16) we have

$$e(\alpha) \geq e(\beta) + e_\infty(\gamma) + e_\infty(\delta). \tag{3.17}$$

By (3.17) and Lemma 2.5 (iv), we have

$$e(\alpha) \geq e(\beta) + e_\infty(\gamma) + e_\infty(\delta) \geq e(\beta+\gamma) + e_\infty(\delta) \geq e(\beta+\gamma+\delta) = e(\alpha). \tag{3.18}$$

Hence, $e(\alpha) = e(\beta) + e_\infty(\gamma) + e_\infty(\delta)$. Since $\delta > 0$, by Proposition 2.1 (i), we see that if $\gamma + \delta > \alpha_{0,\infty}$, then $e_\infty(\gamma) + e_\infty(\delta) > e_\infty(\gamma+\delta)$. This gives a contradiction because

$$e(\alpha) = e(\beta) + e_\infty(\gamma) + e_\infty(\delta) > e(\beta) + e_\infty(\gamma+\delta) \geq e(\beta+\gamma+\delta) = e(\alpha).$$

Thus, $\gamma + \delta \leq \alpha_{0,\infty}$ and $e_\infty(\gamma) = e_\infty(\delta) = 0$ thanks to Proposition 2.1. By (3.18) we have $e(\alpha) = e(\beta)$. Thus, when $\beta = 0$ and $e(\alpha) < 0$, we obtain a contradiction and (3.10) does not hold, which gives $\delta = 0$.

In the case $0 < \beta < \alpha$, by (3.16), $e_\infty(\delta) = 0 = e_\infty(\gamma)$ and (3.14), we have

$$\begin{aligned} e(\beta) &\leq E(u_0) + E(w_0(\cdot - y_n)) + E(u_n - u_0 - w_0(\cdot - y_n)) + o(1) \\ &= E(u_n) + o(1) \to e(\alpha). \end{aligned} \tag{3.19}$$

Since $\|u_0\|_2^2 = \beta$, by (3.19), we see that $e(\beta)$ is attained by u_0 as well as $e(\beta) = e(\alpha)$. However, by Lemma 3.1, $e(\beta)$ is not attained and we obtain a contradiction. Hence, $\delta = 0$ and Step 2 is proved.

Step 3: *We show that* (y_n) *and* w_0 *satisfy* (3.6).

In Step 2 we saw that (3.14)–(3.16) hold when $\delta > 0$ is assumed. However, (3.14)–(3.16) hold even in the case $\delta = 0$, since (3.10) is not used in deriving (3.14)–(3.16). By (3.14)–(3.16) we have

$$\begin{aligned} e(\alpha) &= \liminf_{n\to\infty} E(u_n) \\ &= \liminf_{n\to\infty} \left(E(u_0) + E(w_0(\cdot - y_n)) + E(u_n - u_0 - w_0(\cdot - y_n))\right) \\ &\geq E(u_0) + E_\infty(w_0) + \liminf_{n\to\infty} E\left(u_n - u_0 - w_0(\cdot - y_n)\right) \\ &\geq e(\beta) + e_\infty(\gamma) + e_\infty(\delta), \end{aligned} \tag{3.20}$$

where $\delta = \lim_{n\to\infty} \|u_n - u_0 - w_0(\cdot - y_n)\|_2^2$. In Step 2 we have shown that $\delta = 0$, and hence $\alpha = \beta + \gamma$. Since $\gamma > 0$ and $e_\infty(\delta) = 0$, by Lemma 2.5 (iv), we have

$$e(\beta) + e_\infty(\gamma) + e_\infty(\delta) = e(\beta) + e_\infty(\gamma) \geq e(\alpha). \tag{3.21}$$

By (3.21) and (3.20) we see that $e(\alpha) = e(\beta) + e_\infty(\gamma)$. Hence, by (3.20), $E(u_0) = e(\beta)$ and $E_\infty(w_0) = e_\infty(\gamma)$. Thus, Step 3 is proved, and the proof of Lemma 3.2 is completed. □

Now we prove the precompactness of minimizing sequence.

Lemma 3.3 *Assume* (F1)–(F5) *and* (V1) *and assume* (F6) *or* (V2). *Let* $\alpha > 0$. *If* $e(\alpha) < 0$, *then every minimizing sequence for* $e(\alpha)$ *has a strong convergent subsequence in* H.

Proof Let $(u_n) \subset M(\alpha)$ be a minimizing sequence for $e(\alpha)$. By Lemma 2.6, it suffices to show that $(|u_n|)$ has a strongly convergent subsequence in $L^2(\mathbb{R}^N)$. Moreover, from Lemma 2.8 and Remark 2.9, we may assume that (u_n) satisfies

$$E'(u_n) + \lambda_n Q'(u_n) \ \to 0 \text{ strongly in } H^* \quad \text{and} \quad (u_n)_- := \max\{-u_n(x), 0\} \to 0 \text{ strongly in } L^2(\mathbb{R}^N) \tag{3.22}$$

for some bounded sequence $(\lambda_n) \subset \mathbb{R}$. We may also suppose

$$u_n \rightharpoonup u_0 \text{ weakly in } H \ \text{ and } \ \lambda_n \to \lambda \text{ in } \mathbb{R}.$$

Let $\beta := \|u_0\|_2^2$. Then, $\beta \leq \alpha$.

If $\beta = \alpha$, then $u_n \to u_0$ strongly in $L^2(\mathbb{R}^N)$ and Lemma 2.6 asserts that (u_n) has a strongly convergent subsequence in H. Hence, the conclusion holds.

When $0 \leq \beta < \alpha$, by Lemma 3.2, there exist $(y_n) \subset \mathbb{R}$ and $w_0 \in H\setminus\{0\}$ such that (3.4)–(3.6) hold. From (3.22) and the definition of w_0 in Step 1 of Lemma 3.2, it follows that

$$-\Delta w_0 + 2\lambda w_0 = f(w_0) \quad \text{in } \mathbb{R}^N, \quad w_0 \geq 0 \quad \text{in } \mathbb{R}^N. \tag{3.23}$$

Since $f(s) \geq 0$ for $s \geq 0$ by (F5) and $-\Delta w_0 + (2\lambda)_+ w_0 \geq -\Delta w_0 + 2\lambda w_0 = f(w_0) \geq 0$ in $\mathbb{R}^N$, the strong maximum principle and $\|w_0\|_2^2 = \alpha - \beta > 0$ give

$$w_0 > 0 \quad \text{in } \mathbb{R}^N. \tag{3.24}$$

Now we may exclude the case $\beta = 0$. In this case, we have $e(\alpha) = e_\infty(\alpha) = E_\infty(w_0)$ and w_0 is a minimizer for $e_\infty(\alpha)$. However, (V1) and (3.24) give a contradiction:

$$e(\alpha) \leq E(w_0) < E_\infty(w_0) = e_\infty(\alpha) = e(\alpha).$$

Hence, the case $\beta = 0$ does not occur.

Hereafter we prove that the case

$$0 < \beta < \alpha \tag{3.25}$$

does not occur. Suppose on the contrary that (3.25) holds.

We divide the proof into two steps.

Step 1 *We show that* $\lambda > 0$.

By (3.23), we observe that w_0 satisfies the Pohozaev identity

$$0 = \frac{N-2}{2}\|\nabla w_0\|_2^2 - N\int_{\mathbb{R}^N} F(w_0) - \lambda w_0^2 dx.$$

Therefore, we obtain

$$0 \geq e_\infty(\alpha - \beta) = E_\infty(w_0) = \frac{1}{N}\|\nabla w_0\|_2^2 - \lambda\|w_0\|_2^2.$$

Now we infer from (3.24) that $\lambda \geq \frac{1}{N(\alpha-\beta)}\|\nabla w_0\|_2^2 > 0$.

Step 2 *Conclusion.*

In this step, we borrow the idea from [15]. Set

$$w_n(x) := w_0(x - n\mathbf{e}_1), \quad \tau_n := \frac{\sqrt{\alpha}}{\|u_0 + w_n\|_2} \quad \text{and} \quad \kappa_n := \langle u_0, w_n\rangle_{L^2}.$$

Remark that $\tau_n(u_0 + w_n) \in M(\alpha)$, $\kappa_n \to 0$ as $n \to \infty$ and

$$\tau_n^2 = \frac{\alpha}{\alpha + 2\kappa_n} = 1 - \frac{2\kappa_n}{\alpha} + O(\kappa_n^2) \quad \text{and} \quad \tau_n = 1 - \frac{\kappa_n}{\alpha} + O(\kappa_n^2).$$

Since it follows from (3.22), $\|u_0\|_2^2 = \beta > 0$ and a similar argument to w_0 that

$$- \Delta u_0 + V(x)u_0 + 2\lambda u_0 = f(u_0) \quad \text{in } \mathbb{R}^N, \quad u_0 > 0 \quad \text{in } \mathbb{R}^N, \tag{3.26}$$

combining this fact with (3.23) and (3.26), we have

$$\begin{aligned}
&\frac{\tau_n^2}{2}\int_{\mathbb{R}^N} |\nabla(u_0+w_n)|^2 + V(x)(u_0+w_n)^2 dx \\
&= \frac{1}{2}\left(1 - \frac{2\kappa_n}{\alpha} + O(\kappa_n^2)\right)\int_{\mathbb{R}^N} |\nabla u_0|^2 + V(x)u_0^2 \\
&\quad + |\nabla w_n|^2 + V(x)w_n^2 + 2\nabla u_0\cdot\nabla w_n + 2V(x)u_0w_n dx \\
&= \frac{1}{2}\left(1 - \frac{2\kappa_n}{\alpha}\right)\int_{\mathbb{R}^N} |\nabla u_0|^2 + V(x)u_0^2 + |\nabla w_0|^2 + V(x)w_n^2 dx \\
&\quad + \left(1 - \frac{2\kappa_n}{\alpha}\right)\int_{\mathbb{R}^N} \frac{1}{2}(\nabla u_0\cdot\nabla w_n + V(x)u_0w_n) \\
&\quad + \frac{1}{2}(\nabla u_0\cdot\nabla w_n + V(x)u_0w_n)dx + O(\kappa_n^2) \\
&= \frac{1}{2}\left(1 - \frac{2\kappa_n}{\alpha}\right)\int_{\mathbb{R}^N} |\nabla u_0|^2 + V(x)u_0^2 + |\nabla w_0|^2 + V(x)w_n^2 dx \\
&\quad + \left(1 - \frac{2\kappa_n}{\alpha}\right)\int_{\mathbb{R}^N} \frac{1}{2}(-2\lambda u_0w_n + f(u_0)w_n) \\
&\quad + \frac{1}{2}(-2\lambda u_0w_n + f(w_n)u_0) + \frac{1}{2}V(x)u_0w_n dx + O(\kappa_n^2) \\
&= \frac{1}{2}\left(1 - \frac{2\kappa_n}{\alpha}\right)\int_{\mathbb{R}^N} |\nabla u_0|^2 + V(x)u_0^2 + |\nabla w_0|^2 + V(x)w_n^2 dx \\
&\quad + \left(1 - \frac{2\kappa_n}{\alpha}\right)\left\{-2\lambda\kappa_n + \frac{1}{2}\int_{\mathbb{R}^N} f(u_0)w_n + f(w_n)u_0 dx\right\} \\
&\quad + \frac{1}{2}\left(1 - \frac{2\kappa_n}{\alpha}\right)\int_{\mathbb{R}^N} V(x)u_0w_n dx + O(\kappa_n^2).
\end{aligned}$$

From $u_0, w_0 \in L^\infty(\mathbb{R})$ with $u_0, w_0 \geq 0$, (F3) and (F5), it follows that

$$0 \leq \int_{\mathbb{R}^N} f(u_0)w_n + f(w_n)u_0 dx \leq \int_{\mathbb{R}^N} C_0\,(u_0w_n + w_nu_0)\,dx = 2C_0\kappa_n. \tag{3.27}$$

Since $V(x) \leq 0$ and we may assume $1 - 2\kappa_n/\alpha \geq 0$, we have

$$\begin{aligned}
&E(\tau_n(u_0+w_n)) \\
&\leq \frac{1}{2}\left(1 - \frac{2\kappa_n}{\alpha}\right)\int_{\mathbb{R}^N} |\nabla u_0|^2 + V(x)u_0^2 + |\nabla w_n|^2 + V(x)w_n^2 dx - 2\lambda\kappa_n \\
&\quad + \int_{\mathbb{R}^N} \frac{1}{2}(f(u_0)w_n + f(w_n)u_0)dx - \int_{\mathbb{R}^N} F(\tau_n(u_0+w_n))dx + O(\kappa_n^2) \\
&\leq E(u_0) + E_\infty(w_n) - \frac{\kappa_n}{\alpha}\int_{\mathbb{R}^N} |\nabla u_0|^2 + V(x)u_0^2 + |\nabla w_0|^2 dx - 2\lambda\kappa_n \\
&\quad + \int_{\mathbb{R}^N} \frac{1}{2}(f(u_0)w_n + f(w_n)u_0)dx \\
&\quad + \int_{\mathbb{R}^N} F(u_0) + F(w_n) - F(\tau_n(u_0+w_n))dx + O(\kappa_n^2).
\end{aligned} \tag{3.28}$$

Noting $f \in C^{\nu}_{\mathrm{loc}}(\mathbb{R})$ due to (F5), we have

$$\begin{aligned}\int_{\mathbb{R}^N} F(\tau_n(u_0+w_n))dx &= \int_{\mathbb{R}^N} F\left(\left(1-\frac{\kappa_n}{\alpha}+O(\kappa_n^2)\right)(u_0+w_n)\right)dx\\ &= \int_{\mathbb{R}^N} F(u_0+w_n)+f(u_0+w_n)\left(-\frac{\kappa_n}{\alpha}\right)(u_0+w_n)dx+O(\kappa_n^{1+\nu}).\end{aligned} \tag{3.29}$$

By (3.28), (3.29) and $\alpha = \|u_0\|_2^2 + \|w_0\|_2^2$, we have

$$\begin{aligned}
&E(\tau_n(u_0+w_n))\\
&\le E(u_0)+E_\infty(w_0)-\frac{\kappa_n}{\alpha}\int_{\mathbb{R}^N}|\nabla u_0|^2+V(x)u_0^2+|\nabla w_0|^2dx-2\frac{\lambda}{\alpha}\left(\|u_0\|_2^2+\|w_0\|_2^2\right)\kappa_n\\
&\quad+\int_{\mathbb{R}^N}\frac{1}{2}(f(u_0)w_n+f(w_n)u_0)dx\\
&\quad+\int_{\mathbb{R}^N}F(u_0)+F(w_n)-F(u_0+w_n)+\frac{\kappa_n}{\alpha}f(u_0+w_n)(u_0+w_n)dx+O(\kappa_n^{1+\nu})\\
&=E(u_0)+E_\infty(w_0)-\frac{\kappa_n}{\alpha}\int_{\mathbb{R}^N}f(u_0)u_0+f(w_n)w_ndx\\
&\quad+\int_{\mathbb{R}^N}F(u_0)+F(w_n)-F(u_0+w_n)+\frac{1}{2}(f(u_0)w_n+f(w_n)u_0)dx\\
&\quad+\frac{\kappa_n}{\alpha}\int_{\mathbb{R}^N}f(u_0+w_n)(u_0+w_n)dx+O(\kappa_n^{1+\nu})\\
&=E(u_0)+E_\infty(w_0)+\int_{\mathbb{R}^N}F(u_0)+F(w_n)-F(u_0+w_n)+\frac{1}{2}(f(u_0)w_n+f(w_n)u_0)dx\\
&\quad+\frac{\kappa_n}{\alpha}\int_{\mathbb{R}^N}(f(u_0+w_n)-f(u_0))u_0+(f(u_0+w_n)-f(w_n))w_ndx+O(\kappa_n^{1+\nu}).
\end{aligned} \tag{3.30}$$

From (3.23), (3.26), $V(x) \to 0$ as $|x| \to \infty$ and $\lambda > 0$ due to Step 1, it follows that u_0 and w_0 decay exponentially as $|x| \to \infty$. In fact, we may prove that if $0 < \eta_1 < 2\lambda < \eta_2$, then there exist $C_{\eta_1} > 0$ and $C_{\eta_2} > 0$ such that

$$C_{\eta_2}e^{-\sqrt{\eta_2}|x|} \le u_0(x) \le C_{\eta_1}e^{-\sqrt{\eta_1}|x|} \quad \text{and} \quad C_{\eta_2}e^{-\sqrt{\eta_2}|x|} \le w_0(x) \le C_{\eta_1}e^{-\sqrt{\eta_1}|x|} \tag{3.31}$$

Noting $|f(u_0+w_n)-f(u_0)| \le Cw_n^{\nu}$, we see that

$$\begin{aligned}\int_{\mathbb{R}^N}|f(u_0+w_n)-f(u_0)||u_0|dx &\le C\int_{\mathbb{R}^N}w_n^{\nu}u_0dx = C\int_{\mathbb{R}^N}(w_nu_0)^{\nu}u_0^{1-\nu}dx\\ &\le C\left(\int_{\mathbb{R}^N}w_nu_0dx\right)^{\nu}\left(\int_{\mathbb{R}^N}u_0dx\right)^{1-\nu} = O(\kappa_n^{\nu}).\end{aligned}$$

By a similar argument, we have

$$\begin{aligned}\int_{\mathbb{R}^N}|f(u_0+w_n)-f(w_n)||w_n|dx &\le C\int_{\mathbb{R}^N}(u_0w_n)^{\nu}w_n^{1-\nu}dx\\ &\le C\left(\int_{\mathbb{R}^N}u_0w_ndx\right)^{\nu}\left(\int_{\mathbb{R}^N}w_ndx\right)^{1-\nu} = O(\kappa_n^{\nu}).\end{aligned}$$

Using two inequalities, by (3.30) we have

$$
\begin{aligned}
&E(\tau_n(u_0+w_n)) \\
&\quad \le E(u_0) + E_\infty(w_0) + \int_{\mathbb{R}^N} F(u_0) \\
&\qquad + F(w_n) - F(u_0+w_n) + \frac{1}{2}(f(u_0)w_n + f(w_n)u_0)dx + O(\kappa_n^{1+\nu}). \qquad (3.32)
\end{aligned}
$$

Let $\delta_2 > 0$ be given in Lemma 2.4 (i). We can choose an $R_0 > 0$ such that if $n \ge 2R_0$, then

$$
\max_{x\in\mathbb{R}^N\setminus(B_{R_0}(O)\cup B_{R_0}(n\mathbf{e}_1))} u_0(x) \le \delta_2 \quad \text{and} \quad \max_{x\in\mathbb{R}^N\setminus(B_{R_0}(O)\cup B_{R_0}(n\mathbf{e}_1))} w_n(x) \le \delta_2.
$$

By Lemma 2.4 (i) we see that if $n \ge 2R_0$, then

$$
\int_{\mathbb{R}^N\setminus(B_{R_0}(O)\cup B_{R_0}(n\mathbf{e}_1))} F(u_0) + F(w_n) - F(u_0+w_n) + \frac{1}{2}(f(u_0)w_n + f(w_n)u_0)dx \le 0. \qquad (3.33)
$$

Next, set

$$
K := \left\{u_0(x) \mid x \in \overline{B_{R_0}(O)}\right\} \cup \left\{w_n(x) \mid x \in \overline{B_{R_0}(n\mathbf{e}_1)}\right\}.
$$

Then $K \subset (0,\infty)$ and K is compact. Let δ_K be given in Lemma 2.4. We can choose $n_{R_0} \ge 2R_0$ such that if $n \ge n_{R_0}$, then

$$
\max_{x\in\overline{B_{R_0}(n\mathbf{e}_1)}} u_0(x) \le \delta_K \quad \text{and} \quad \max_{x\in\overline{B_{R_0}(O)}} w_n(x) \le \delta_K.
$$

By Lemma 2.4 (ii) we see that if $n \ge n_{R_0}$, then

$$
\begin{aligned}
&\int_{\overline{B_{R_0}(O)}\cup\overline{B_{R_0}(n\mathbf{e}_1)}} F(u_0) + F(w_n) - F(u_0+w_n) + \frac{1}{2}(f(u_0)w_n + f(w_n)u_0)dx \\
&\quad \le -C_K\left(\int_{\overline{B_{R_0}(O)}} w_n(x)dx + \int_{\overline{B_{R_0}(n\mathbf{e}_1)}} u_0(x)dx\right). \qquad (3.34)
\end{aligned}
$$

Thus, from (3.32)–(3.34), we see that if $n \ge n_{R_0}$, then

$$
\begin{aligned}
E(\tau_n(u_0+w_n)) &\le E(u_0) + E_\infty(w_0) \\
&\quad - C_K\left(\int_{\overline{B_{R_0}(O)}} w_n(x)dx + \int_{\overline{B_{R_0}(n\mathbf{e}_1)}} u_0(x)dx\right) + O(\kappa_n^{1+\nu}). \qquad (3.35)
\end{aligned}
$$

Now recalling (3.31), we obtain

$$
\int_{\overline{B_{R_0}(O)}} w_n(x)dx + \int_{\overline{B_{R_0}(n\mathbf{e}_1)}} u_0(x)dx \ge C_{\eta_2}e^{-\sqrt{\eta_2}n} \quad \text{for } \eta_2 > 2\lambda.
$$

Remark also that for each $\eta_1 \in (0, 2\lambda)$, it is possible to prove

$$
\kappa_n \le C_{\eta_1}e^{-\sqrt{\eta_1}n}.
$$

For instance, see [2, Proposition 1.2], [3, Lemma II.2] and [17].

Put $\eta_1 := (\sqrt{2\lambda}-\varepsilon)^2$ and $\eta_2 := (\sqrt{2\lambda}+\varepsilon)^2$. If $\varepsilon > 0$ is sufficiently small, then

$$
\sqrt{\eta_2} - (1+\nu)\sqrt{\eta_1} = -\nu\sqrt{2\lambda} + (2+\nu)\varepsilon < 0.
$$

Thus,

$$\kappa_n^{1+\nu} e^{\sqrt{\eta_2} n} \le C_{\eta_1}^{1+\nu} e^{(\sqrt{\eta_2}-(1+\nu)\sqrt{\eta_1})n} \to 0 \text{ as } n \to \infty.$$

Therefore, $O(\kappa_n^{1+\nu}) = o(e^{-\sqrt{\eta_2} n})$. By (3.35) we see that if n is large, then

$$\begin{aligned} e(\alpha) \le E(\tau_n(u_0 + w_n)) &\le E(u_0) + E_\infty(w_0) - c_{\eta_2} e^{-\sqrt{\eta_2} n} + o(e^{-\sqrt{\eta_2} n}) \\ &< E(u_0) + E_\infty(w_0) = e(\alpha), \end{aligned}$$

which is a contradiction. Hence, (3.25) does not occur and the proof is completed. □

Proof of Theorem A Let $\alpha_0 := \inf\{\alpha \ge 0 |\ e(\alpha) < 0\}$. It is clear that $\alpha_0 \le \alpha_{0,\infty}$. Since $\alpha_{0,\infty}$ exists and $\alpha_{0,\infty} < \infty$ thanks to Proposition 2.1, we see that α_0 exists and $\alpha_0 < \infty$. By Lemma 2.5 (v), $e(\alpha)$ is nonincreasing. Since $e(0) = 0$, we easily see that $e(\alpha) = 0$ for $0 < \alpha < \alpha_0$ and that $e(\alpha) < 0$ for $\alpha > \alpha_0$. It follows from Lemma 3.3 that if $\alpha > \alpha_0$, then every minimizing sequence has a strong convergent subsequence in H. It is well known that the orbital stability of S_α follows from the precompactness of every minimizing sequence for $e(\alpha)$. Moreover, Lemma 3.1 and the definition of α_0 imply Theorem A (ii). Therefore, Theorem A holds. □

4 Proof of Theorem B

Proof of Theorem B (i) We first prove $\alpha_0 = 0$ when (V3) holds. By (V3), there is a $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $\|\varphi\|_2 = 1$ and

$$\frac{1}{2} \int_{\mathbb{R}^N} |\nabla \varphi|^2 + V(x)\varphi^2 dx < 0.$$

Replacing $|\varphi|$ if necessary, we may suppose $\varphi \ge 0$. Let $\alpha \in (0, s_0^2/\|\varphi\|_\infty^2)$. Since $\sqrt{\alpha}\varphi \in M(\alpha)$ and $F(\sqrt{\alpha}\varphi) \ge 0$, we get

$$e(\alpha) \le E(\sqrt{\alpha}\varphi) \le \frac{\alpha}{2} \int_{\mathbb{R}^N} |\nabla \varphi|^2 + V(x)\varphi^2 dx < 0.$$

By the monotonicity of $e(\alpha)$ in Lemma 2.5, we see that $\alpha_0 = 0$ holds.

Next, we show that $N = 1, 2$ and (V1) imply (V3). Let $V(x)$ satisfy (V1) and $\varphi \in C_0^\infty(\mathbb{R}^N)$. Put $\varphi_t(x) := t^{N/2}\varphi(tx)$ for $t > 0$. Choose also an $R_0 > 0$ so that $\int_{|x| \le R_0} V(x) dx < 0$. Then we have

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla \varphi_t|^2 + V(x)|\varphi_t|^2 dx &= t^2 \|\nabla \varphi\|_2^2 + t^N \int_{\mathbb{R}^N} V(x)|\varphi(tx)|^2 dx \\ &\le t^2 \left(\|\nabla \varphi\|_2^2 + t^{N-2} \int_{|x| \le R_0} V(x)|\varphi(tx)|^2 dx \right). \end{aligned} \tag{4.1}$$

Remark that

$$\lim_{t \to 0} \int_{|x| \le R_0} V(x)|\varphi(tx)|^2 dx = |\varphi(0)|^2 \int_{|x| \le R_0} V(x) dx.$$

Hence, when $N = 1$, by selecting $\varphi \in C_0^\infty(\mathbb{R})$ so that $\varphi(0) \ne 0$, if $t > 0$ is sufficiently small, then (4.1) and the choice of R_0 imply $\int_{\mathbb{R}} |\nabla \varphi_t|^2 + V(x)|\varphi_t|^2 dx < 0$.
When $N = 2$, from $(-\log |x|)_+^\alpha \in H^1(\mathbb{R}^2)$ for $0 < \alpha < 1/2$, we may find a $\psi_k \in C_0^\infty(\mathbb{R}^2)$ so that $\|\nabla \psi_k\|_2 = 1$, $\psi_k \ge 0$ and $\psi_k(0) \to \infty$ as $k \to \infty$. Setting $\varphi = \psi_k$ and selecting a sufficiently large k_0, we obtain

$$\|\nabla \psi_{k_0}\|_2^2 + |\psi_{k_0}(0)|^2 \int_{|x|\le R_0} V(x)dx < 0.$$

Thus, if $t > 0$ is sufficiently small, then (4.1) gives $\int_{\mathbb{R}^2} |\nabla(\psi_{k_0})_t|^2 + V(x)|(\psi_{k_0})_t|^2 dx < 0$. Therefore, when $N = 2$, (V3) holds.

(ii) We show that there exists $V(x)$ such that $\alpha_0 > 0$. Let $b := \sup_{s>0} F(s)/s^{p_c+1}$. By (F4) and (F9) we see that $b < \infty$. Let C_0 denote the best constant of the inequality $\|u\|_{p_c+1}^{p_c+1} \le C_0 \|u\|_2^{4/N} \|\nabla u\|_2^2$ and define $\alpha_1 = \alpha_1(N, f) > 0$ by $\alpha_1 := (2bC_0)^{-N/2}$. For $\alpha \in (0, \alpha_1)$, we also set $c_\alpha := (N-2)^2(1 - 2bC_0\alpha^{2/N})/4 > 0$ and suppose that $V(x) \ge -c_\alpha |x|^{-2}$ for $|x| > 0$. Then by Hardy's inequality and the definition of b, C_0 and c_α, we obtain

$$\begin{aligned} E(u) &\ge \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{c_\alpha}{2}\int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx - b\|u\|_{p_c+1}^{p_c+1} \\ &\ge \left(\frac{1}{2} - \frac{1}{2} + bC_0\alpha^{2/N} - bC_0\alpha^{2/N}\right)\int_{\mathbb{R}^N} |\nabla u|^2 dx = 0. \end{aligned}$$

This inequality indicates that $e(\alpha) = 0$ and $\alpha_0 \ge \alpha > 0$ follows from from the monotonicity of $e(\alpha)$.

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Appendix A: Nonexistence of minimizer

We consider the following case:

(V4) $0 \not\equiv V(x) \ge 0$ and $\lim_{|x|\to\infty} V(x) = 0$.

Theorem A.1 *Suppose* (V4) *and the following* (F11):

(F11) $f(s) \le f(|s|)$ *for* $s \in \mathbb{R}$, $f(s) \ge 0$ *for* $s \ge 0$, $|f(s)| \le C(|s| + |s|^{p_c})$, $\lim_{s\to\infty} f(s)/s^{p_c} = 0$.

Then $e(\alpha) = e_\infty(\alpha)$ *for* $\alpha \ge 0$ *and* $e(\alpha)$ *is not attained for* $\alpha > 0$.

The assumption (F11) is weaker than (F1)–(F5).

Proof First, we show that $e(\alpha) = e_\infty(\alpha)$. Since $V(x) \ge 0$, we see that $e(\alpha) \ge e_\infty(\alpha)$. On the other hand, for any $u \in M(\alpha)$ and $n \in \mathbb{N}$, we obtain

$$e(\alpha) \le E(u(\cdot - n\mathbf{e}_1)) = E_\infty(u) + \frac{1}{2}\int_{\mathbb{R}^N} V(x + n\mathbf{e}_1)|u|^2 dx.$$

Letting $n \to \infty$, we obtain $e(\alpha) \le E_\infty(u)$. Since u is arbitrary, we see that $e(\alpha) \le e_\infty(\alpha)$. Thus, $e(\alpha) = e_\infty(\alpha)$.

Second, we show by contradiction that $e(\alpha)$ is not attained. Suppose on the contrary that $e(\alpha)$ is attained by $u_0 \in H \cap M(\alpha)$. By Remark 2.7, we may assume $u_0 \ge 0$. Since $E \in C^1(H_{\mathbb{R}}, \mathbb{R})$ due to (F11), there exists a $\lambda \in \mathbb{R}$ such that $-\Delta u_0 + (V(x) + 2\lambda)_+ u_0 \ge -\Delta u_0 + (V(x) + 2\lambda)u_0 = f(u_0) \ge 0$ in $\mathbb{R}^N$. Thus, the weak Harnack inequality [14, Theorem 8.18] yields $u_0 > 0$ in $\mathbb{R}^N$. Using this fact and $0 \not\equiv V(x) \ge 0$, we obtain

$$e(\alpha) = E(u_0) = E_\infty(u_0) + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u_0^2 dx > E_\infty(u_0) \ge e_\infty(\alpha).$$

This is a contradiction, because $e(\alpha) = e_\infty(\alpha)$. Therefore, $e(\alpha)$ has no minimizer. □

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