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Calculus of Variations

G-invariant Szegő kernel asymptotics and CR reduction

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Abstract
Let $(X, T^{1,0}X)$ be a compact connected orientable CR manifold of dimension $2n+1$ with non-degenerate Levi curvature. Assume that X admits a connected compact Lie group G action. Under certain natural assumptions about the group G action, we show that the G-invariant Szegő kernel for $(0, q)$ forms is a complex Fourier integral operator, smoothing away $\mu^{-1}(0)$ and there is a precise description of the singularity near $\mu^{-1}(0)$, where μ denotes the CR moment map. We apply our result to the case when X admits a transversal CR S^1 action and deduce an asymptotic expansion for the mth Fourier component of the G-invariant Szegő kernel for $(0, q)$ forms as $m \to +\infty$ and when $q = 0$, we recover Xiaonan Ma and Weiping Zhang's result about the existence of the G-invariant Bergman kernel for ample line bundles. As an application, we show that if m large enough, quantization commutes with reduction.

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1 Introduction and statement of the main results

Let $(X, T^{1,0}X)$ be a CR manifold of dimension $2n+1$, $n \geq 1$. Let $\Box_b^{(q)}$ be the Kohn Lalpacian acting on $(0, q)$ forms. The orthogonal projection $S^{(q)} : L^2_{(0,q)}(X) \to \operatorname{Ker} \Box_b^{(q)}$

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onto Ker $\Box_b^{(q)}$ is called the Szegő projection, while its distribution kernel $S^{(q)}(x, y)$ is called the Szegő kernel. The study of the Szegő projection and kernel is a classical subject in several complex variables and CR geometry. A very important case is when X is a compact strictly pseudoconvex CR manifold. Assume first that X is the boundary of a strictly pseudoconvex domain. Boutet de Monvel-Sjöstrand [2] showed that $S^{(0)}(x, y)$ is a complex Fourier integral operator.

The Boutet de Monvel-Sjöstrand description of the Szegő kernel had a profound impact in many research areas, especially through [4]: several complex variables, symplectic and contact geometry, geometric quantization, Kähler geometry, semiclassical analysis, quantum chaos, etc. cf. [6,8,11,22,29,32], to quote just a few. These ideas also partly motivated the introduction of the recent direct approaches and their various extensions, see [18,19,21,22].

Now, we consider a connected compact Lie group G acting on X. The study of G-invariant Szegő kernel is closely related to Mathematical physics and geometric quantization of CR manifolds. It is a fundamental problem to establish G-invariant Boutet de Monvel-Sjöstrand type theorems for G-invariant Szegő kernels and study the consequence of the G-invariant Szegő kernel. This is the motivation of this work. In this paper, we consider G-invariant Szegő kernel for $(0, q)$ forms and we show that the G-invariant Szegő kernel for $(0, q)$ forms is a complex Fourier integral operator. In particular, $S^{(q)}(x, y)$ is smoothing outside $\mu^{-1}(0)$ and there is a precise description of the singularity near $\mu^{-1}(0)$, where μ denotes the CR moment map. We apply our result to the case when X admits a transversal CR S^1 action and deduce an asymptotic expansion for the mth Fourier component of the Szegő kernel for $(0, q)$ forms as $m \to +\infty$. As an application, we show that, if m large enough, quantization commutes with reduction.

In [20], Ma and Zhang have studied the asymptotic expansion of the invariant Bergman kernel of the spin^c Dirac operator associated with high tensor powers of a positive line bundle on a symplectic manifold admitting a Hamiltonian action of a compact connected Lie group and its relation to the asymptotic expansion of Bergman kernel on symplectic reduced space, also the Toeplitz operator aspect in [20, Section 4.5]. Their approach is inspired by the analytic localization techniques developed by Bismut and Lebeau [3]. About the quantization commutes with reduction problem in symplectic geometry, we refer the readers to [22]. In the second part of [22], Ma described how the G-invariant Bergman kernel concentrates on the Bergman kernel of the reduced space. Note that the "quantization commutes with reduction" in the situations in symplectic case is a very active subject. When the action connected Lie group is compact and the symplectic manifold is also compact, this question was solved finally by Meinrenken [24] and Tian-Zhang [31]. When the action connected Lie group is compact and the symplectic manifold is noncompact, this is a famous conjecture of Vergne and was solved by Ma-Zhang in [23].

It should be mentioned that in [7], Charles relates the Toeplitz operators on a compact complex manifold M with the Toeplitz operators on the "reduced" space for torus action, and in [26], Paoletti studied equivariant Szegő kernels on complex manifolds (cf. [20, Remark 0.5]).

We now formulate the main results. We refer to Sect. 2 for some notations and terminology used here. Let $(X, T^{1,0}X)$ be a compact connected orientable CR manifold of dimension $2n + 1$, $n \geq 1$, where $T^{1,0}X$ denotes the CR structure of X. Fix a global non-vanishing real 1-form $\omega_0 \in C^\infty(X, T^*X)$ such that $\langle\, \omega_0\, ,\, u\,\rangle = 0$, for every $u \in T^{1,0}X \oplus T^{0,1}X$. The Levi form of X at $x \in X$ is the Hermitian quadratic form on $T^{1,0}_xX$ given by $\mathcal{L}_x(U, \overline{V}) = -\frac{1}{2i}\langle\, d\omega_0(x)\, ,\, U \wedge \overline{V}\,\rangle$, for all $U, V \in T^{1,0}_xX$. In this work, we assume that

Assumption 1.1 *The Levi form is non-degenerate of constant signature (n_-, n_+) on X. That is, the Levi form has exactly n_- negative and n_+ positive eigenvalues at each point of X, where $n_- + n_+ = n$.*

Let $HX = \left\{\mathrm{Re}\, u;\ u \in T^{1,0}X\right\}$ and let $J : HX \to HX$ be the complex structure map given by $J(u + \overline{u}) = iu - i\overline{u}$, for every $u \in T^{1,0}X$. In this work, we assume that X admits a d-dimensional connected compact Lie group G action. We assume throughout that

Assumption 1.2 *The G action preserves ω_0 and J. That is, $g^*\omega_0 = \omega_0$ on X and $g_*J = Jg_*$ on HX, for every $g \in G$, where g^* and g_* denote the pull-back map and push-forward map of G, respectively.*

Let $\mathfrak{g}$ denote the Lie algebra of G. For any $\xi \in \mathfrak{g}$, we write ξ_X to denote the vector field on X induced by ξ. That is, $(\xi_X u)(x) = \frac{\partial}{\partial t}\left(u(\exp(t\xi)\circ x)\right)|_{t=0}$, for any $u \in C^\infty(X)$.

Definition 1.3 The moment map associated to the form ω_0 is the map $\mu : X \to \mathfrak{g}^*$ such that, for all $x \in X$ and $\xi \in \mathfrak{g}$, we have

$$\langle \mu(x), \xi\rangle = \omega_0(\xi_X(x)). \tag{1.1}$$

In this work, we assume that

Assumption 1.4 *0 is a regular value of μ, the action G on $\mu^{-1}(0)$ is freely and*

$$\underline{\mathfrak{g}}_x \bigcap \underline{\mathfrak{g}}_x^{\perp_b} = \{0\} \text{ at every point } x \in Y, \tag{1.2}$$

where $\underline{\mathfrak{g}} = \mathrm{Span}\,(\xi_X;\ \xi \in \mathfrak{g})$, $\underline{\mathfrak{g}}^{\perp_b} = \left\{v \in HX;\ b(\xi_X, v) = 0,\ \ \forall \xi_X \in \underline{\mathfrak{g}}\right\}$, b is the nondegenerate bilinear form on HX given by (2.4).

By Assumption 1.4, $\mu^{-1}(0)$ is a d-codimensional submanifold of X. Let $Y := \mu^{-1}(0)$ and let $HY := HX \bigcap TY$. Note that if the Levi form is positive at Y, then (1.2) holds. Under the condition (1.2), in Sect. 2.5, we will show that $\dim\,(HY \bigcap JHY) = 2n - 2d$ at every point of Y, $\mu^{-1}(0)/G =: Y_G$ is a CR manifold with natural CR structure induced by $T^{1,0}X$ of dimension $2n - 2d + 1$ and we can identify HY_G with $HY \bigcap JHY$.

Fix a G-invariant smooth Hermitian metric $\langle \cdot \,|\, \cdot \rangle$ on $\mathbb{C}TX$ so that $T^{1,0}X$ is orthogonal to $T^{0,1}X$, $\underline{\mathfrak{g}}$ is orthogonal to $HY \bigcap JHY$ at every point of Y, $\langle u \,|\, v \rangle$ is real if u, v are real tangent vectors, $\langle T \,|\, T \rangle = 1$ and T is orthogonal to $T^{1,0}X \oplus T^{0,1}X$, where T is given by (2.2). The Hermitian metric $\langle \cdot \,|\, \cdot \rangle$ on $\mathbb{C}TX$ induces, by duality, a Hermitian metric on $\mathbb{C}T^*X$ and also on the bundles of $(0, q)$ forms $T^{*0,q}X$, $q = 0, 1, \cdots, n$. We shall also denote all these induced metrics by $\langle \cdot \,|\, \cdot \rangle$. Fix $g \in G$. Let $g^* : \Lambda^r_x(\mathbb{C}T^*X) \to \Lambda^r_{g^{-1}\circ x}(\mathbb{C}T^*X)$ be the pull-back map. Since G preserves J, we have $g^* : T^{*0,q}_x X \to T^{*0,q}_{g^{-1}\circ x}X$, for all $x \in X$. Thus, for $u \in \Omega^{0,q}(X)$, we have $g^*u \in \Omega^{0,q}(X)$. Put $\Omega^{0,q}(X)^G := \left\{u \in \Omega^{0,q}(X);\ g^*u = u,\ \ \forall g \in G\right\}$. Since the Hermitian metric $\langle \cdot \,|\, \cdot \rangle$ on $\mathbb{C}TX$ is G-invariant, the L^2 inner product $(\cdot \,|\, \cdot)$ on $\Omega^{0,q}(X)$ induced by $\langle \cdot \,|\, \cdot \rangle$ is G-invariant. Let $u \in L^2_{(0,q)}(X)$ and $g \in G$, we can also define g^*u in the standard way (see the discussion in the beginning of Sect. 3.2). Put $L^2_{(0,q)}(X)^G := \left\{u \in L^2_{(0,q)}(X);\ g^*u = u,\ \ \forall g \in G\right\}$. Let $\Box^{(q)}_b : \mathrm{Dom}\,\Box^{(q)}_b \to L^2_{(0,q)}(X)$ be the Gaffney extension of Kohn Laplacian (see (3.1)). Put $(\mathrm{Ker}\,\Box^{(q)}_b)^G := \mathrm{Ker}\,\Box^{(q)}_b \bigcap L^2_{(0,q)}(X)^G$. The G-invariant Szegő projection is the orthogonal projection $S^{(q)}_G : L^2_{(0,q)}(X) \to (\mathrm{Ker}\,\Box^{(q)}_b)^G$ with respect to $(\cdot \,|\, \cdot)$. Let $S^{(q)}_G(x, y) \in D'(X \times X, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ be the distribution kernel of $S^{(q)}_G$. The first main result of this work is the following

Theorem 1.5 *With the assumptions and notations above, suppose that* $\Box_b^{(q)}: \mathrm{Dom}\,\Box_b^{(q)} \to L^2_{(0,q)}(X)$ *has closed range. If* $q \notin \{n_-, n_+\}$, *then* $S_G^{(q)} \equiv 0$ *on* X.

Suppose $q \in \{n_-, n_+\}$. *Let* D *be an open set of* X *with* $D \bigcap \mu^{-1}(0) = \emptyset$. *Then,* $S_G^{(q)} \equiv 0$ *on* D.

Let $p \in \mu^{-1}(0)$ *and let* U *be an open set of* p *and let* $x = (x_1, \ldots, x_{2n+1})$ *be local coordinates defined in* U. *Then, there exist continuous operators* $S_-^G, S_+^G : \Omega_0^{0,q}(U) \to \Omega^{0,q}(U)$ *such that*

$$S_G^{(q)} \equiv S_-^G + S_+^G \ \text{ on } U,$$

and $S_-^G(x,y)$, $S_+^G(x,y)$ *satisfy*

$$S_\mp^G(x,y) \equiv \int_0^\infty e^{i\Phi_\mp(x,y)t} a_\mp(x,y,t)dt \ \text{ on } U,$$

with

$$\begin{aligned}
&a_+(x,y,t), a_-(x,y,t) \in S_{\mathrm{cl}}^{n-\frac{d}{2}}(U \times U \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*),\\
&a_-(x,y,t) = 0 \ \text{ if } q \neq n_-, \quad a_+(x,y,t) = 0 \ \text{ if } q \neq n_+,\\
&a_-^0(x,x) \neq 0, \quad \forall x \in U, \quad \text{if } q = n_-, \quad a_+^0(x,x) \neq 0, \quad \forall x \in U, \quad \text{if } q = n_+,
\end{aligned} \tag{1.3}$$

$a_-^0(x,x)$, $a_+^0(x,x)$, $x \in \mu^{-1}(0) \bigcap U$, *are given by* (1.8) *below,* $\Phi_-(x,y) \in C^\infty(U \times U)$,

$$\begin{aligned}
&\mathrm{Im}\,\Phi_-(x,y) \geq 0,\\
&d_x\Phi_-(x,x) = -d_y\Phi_-(x,x) = -\omega_0(x), \quad \forall x \in U \bigcap \mu^{-1}(0),
\end{aligned} \tag{1.4}$$

there is a constant $C \geq 1$ *such that, for all* $(x,y) \in U \times U$,

$$\begin{aligned}
&|\Phi_-(x,y)| + \mathrm{Im}\,\Phi_-(x,y) \leq C\left(\inf\left\{d^2(g \circ x, y);\ g \in G\right\} + d^2(x, \mu^{-1}(0)) + d^2(y, \mu^{-1}(0))\right),\\
&|\Phi_-(x,y)| + \mathrm{Im}\,\Phi_-(x,y) \geq \frac{1}{C}\left(\inf\left\{d^2(g \circ x, y);\ g \in G\right\} + d^2(x, \mu^{-1}(0)) + d^2(y, \mu^{-1}(0))\right),\\
&Cd^2(x, \mu^{-1}(0)) \geq \mathrm{Im}\,\Phi_-(x,x) \geq \frac{1}{C}d^2(x, \mu^{-1}(0)), \quad \forall x \in U,
\end{aligned} \tag{1.5}$$

and $\Phi_-(x,y)$ *satisfies* (1.18) *below and* (1.19) *below, and* $\Phi_+(x,y) \in C^\infty(U \times U)$, $-\overline{\Phi}_+(x,y)$ *satisfies* (1.4), (1.5), (1.18) *below and* (1.19) *below.*

We refer the reader to the discussion before (2.1) and Definition 3.1 for the precise meanings of $A \equiv B$ and the symbol space $S_{\mathrm{cl}}^{n-\frac{d}{2}}$, respectively.

Let $\Phi \in C^\infty(U \times U)$. We assume that Φ satisfies (1.4), (1.5), (1.18), (1.19). We will show in Theorem 5.2 that the functions Φ and Φ_- are equivalent on U in the sense of Definition 5.1 if and only if there is a function $f \in C^\infty(U \times U)$ with $f(x,x) = 1$, for every $x \in \mu^{-1}(0)$, such that $\Phi(x,y) - f(x,y)\Phi_-(x,y)$ vanishes to infinite order at $\mathrm{diag}\left((\mu^{-1}(0) \bigcap U) \times (\mu^{-1}(0) \bigcap U)\right)$. From this observation, we see that the leading term $a_-^0(x,x)$, $x \in \mu^{-1}(0)$, is well-defined. To state the formula for $a_-^0(x,x)$, we introduce some notations. For a given point $x_0 \in X$, let $\{W_j\}_{j=1}^n$ be an orthonormal frame of $(T^{1,0}X, \langle\cdot\,|\,\cdot\rangle)$ near x_0, for which the Levi form is diagonal at x_0. Put

$$\mathcal{L}_{x_0}(W_j, \overline{W}_\ell) = \mu_j(x_0)\delta_{j\ell}, \quad j, \ell = 1, \ldots, n.$$

We will denote by

$$\det \mathcal{L}_{x_0} = \prod_{j=1}^n \mu_j(x_0). \tag{1.6}$$

Let $\{T_j\}_{j=1}^n$ denote the basis of $T^{*0,1}X$, dual to $\{\overline{W}_j\}_{j=1}^n$. We assume that $\mu_j(x_0) < 0$ if $1 \leq j \leq n_-$ and $\mu_j(x_0) > 0$ if $n_- + 1 \leq j \leq n$. Put

$$\begin{aligned} \mathcal{N}(x_0, n_-) &:= \left\{cT_1(x_0) \wedge \ldots \wedge T_{n_-}(x_0);\ c \in \mathbb{C}\right\}, \\ \mathcal{N}(x_0, n_+) &:= \left\{cT_{n_-+1}(x_0) \wedge \ldots \wedge T_n(x_0);\ c \in \mathbb{C}\right\} \end{aligned}$$

and let

$$\tau_{n_-} = \tau_{x_0,n_-} : T^{*0,q}_{x_0}X \to \mathcal{N}(x_0, n_-), \quad \tau_{n_+} = \tau_{x_0,n_+} : T^{*0,q}_{x_0}X \to \mathcal{N}(x_0, n_+), \tag{1.7}$$

be the orthogonal projections onto $\mathcal{N}(x_0, n_-)$ and $\mathcal{N}(x_0, n_+)$ with respect to $\langle \cdot | \cdot \rangle$, respectively.

Fix $x \in \mu^{-1}(0)$, consider the linear map

$$\begin{aligned} R_x : \underline{\mathfrak{g}}_x &\to \underline{\mathfrak{g}}_x, \\ u &\to R_x u, \quad \langle R_x u \,|\, v \rangle = \langle d\omega_0(x), Ju \wedge v \rangle. \end{aligned}$$

Let $\det R_x = \lambda_1(x) \cdots \lambda_d(x)$, where $\lambda_j(x)$, $j = 1, 2, \ldots, d$, are the eigenvalues of R_x.

Fix $x \in \mu^{-1}(0)$, put $Y_x = \{g \circ x;\ g \in G\}$. Y_x is a d-dimensional submanifold of X. The G-invariant Hermitian metric $\langle \cdot | \cdot \rangle$ induces a volume form dv_{Y_x} on Y_x. Put

$$V_{\text{eff}}(x) := \int_{Y_x} dv_{Y_x}.$$

Note that the function $V_{\text{eff}}(x)$ was already appeared in Ma-Zhang [23, (0,10)] as exactly the role in the expansion, cf. [23, (0.14)].

Theorem 1.6 *With the notations used above, for $a^0_-(x, y)$ and $a^0_+(x, y)$ in (1.3), we have*

$$a^0_{\mp}(x, x) = 2^{d-1} \frac{1}{V_{\text{eff}}(x)} \pi^{-n-1+\frac{d}{2}} |\det R_x|^{-\frac{1}{2}} |\det \mathcal{L}_x| \tau_{x,n_{\mp}}, \quad \forall x \in \mu^{-1}(0). \tag{1.8}$$

We now assume that X admits an S^1 action: $S^1 \times X \to X$. We write $e^{i\theta}$ to denote the S^1 action. Let $T \in C^\infty(X, TX)$ be the global real vector field induced by the S^1 action given by $(Tu)(x) = \frac{\partial}{\partial\theta}\left(u(e^{i\theta} \circ x)\right)|_{\theta=0}$, $u \in C^\infty(X)$. We assume that the S^1 action $e^{i\theta}$ is CR and transversal (see Definition 4.1). We take $\omega_0 \in C^\infty(X, T^*X)$ to be the global real one form determined by $\langle \omega_0, u \rangle = 0$, for every $u \in T^{1,0}X \oplus T^{0,1}X$ and $\langle \omega_0, T \rangle = -1$. In this paper, we assume that

Assumption 1.7

$$\begin{aligned} &T \text{ is transversal to the space } \underline{\mathfrak{g}} \text{ at every point } p \in \mu^{-1}(0), \\ &e^{i\theta} \circ g \circ x = g \circ e^{i\theta} \circ x, \quad \forall x \in X, \quad \forall \theta \in [0, 2\pi[, \quad \forall g \in G, \end{aligned} \tag{1.9}$$

and

$$G \times S^1 \text{ acts freely near } \mu^{-1}(0).$$

Let $u \in \Omega^{0,q}(X)$ be arbitrary. Define

$$Tu := \frac{\partial}{\partial\theta}\left((e^{i\theta})^* u\right)|_{\theta=0} \in \Omega^{0,q}(X).$$

For every $m \in \mathbb{Z}$, let

$$\begin{aligned} \Omega^{0,q}_m(X) &:= \left\{u \in \Omega^{0,q}(X);\ Tu = imu\right\}, \quad q = 0, 1, 2, \ldots, n, \\ \Omega^{0,q}_m(X)^G &= \left\{u \in \Omega^{0,q}(X)^G;\ Tu = imu\right\}, \quad q = 0, 1, 2, \ldots, n. \end{aligned}$$

We denote $C^\infty_m(X) := \Omega^{0,0}_m(X)$, $C^\infty_m(X)^G := \Omega^{0,0}_m(X)^G$. From the CR property of the S^1 action and (1.9), it is not difficult to see that $Tg^*\overline{\partial}_b = g^*T\overline{\partial}_b = \overline{\partial}_b g^*T = \overline{\partial}_b Tg^*$ on $\Omega^{0,q}(X)$, for all $g \in G$. Hence,

$$\overline{\partial}_b : \Omega^{0,q}_m(X)^G \to \Omega^{0,q+1}_m(X)^G, \quad \forall m \in \mathbb{Z}.$$

We now assume that the Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathbb{C}TX$ is $G \times S^1$ invariant. Then the L^2 inner product $(\cdot | \cdot)$ on $\Omega^{0,q}(X)$ induced by $\langle \cdot | \cdot \rangle$ is $G \times S^1$-invariant. We then have

$$\begin{aligned} &Tg^*\overline{\partial}^*_b = g^*T\overline{\partial}^*_b = \overline{\partial}^*_b g^*T = \overline{\partial}^*_b Tg^* \text{ on } \Omega^{0,q}(X), \quad \forall g \in G,\\ &Tg^*\Box^{(q)}_b = g^*T\Box^{(q)}_b = \Box^{(q)}_b g^*T = \Box^{(q)}_b Tg^* \text{ on } \Omega^{0,q}(X), \quad \forall g \in G, \end{aligned}$$

where $\overline{\partial}^*_b$ is the L^2 adjoint of $\overline{\partial}_b$ with respect to $(\cdot | \cdot)$.

Let $L^2_{(0,q),m}(X)^G$ be the completion of $\Omega^{0,q}_m(X)^G$ with respect to $(\cdot | \cdot)$. We write $L^2_m(X)^G := L^2_{(0,0),m}(X)^G$. Put $H^q_{b,m}(X)^G := (\mathrm{Ker}\,\Box^{(q)}_b)^G_m := (\mathrm{Ker}\,\Box^{(q)}_b)^G \bigcap L^2_{(0,q),m}(X)^G$. It is not difficult to see that, for every $m \in \mathbb{Z}$, $(\mathrm{Ker}\,\Box^{(q)}_b)^G_m \subset \Omega^{0,q}_m(X)^G$ and $\dim\,(\mathrm{Ker}\,\Box^{(q)}_b)^G_m < \infty$. The mth G-invariant Szegő projection is the orthogonal projection $S^{(q)}_{G,m} : L^2_{(0,q)}(X) \to (\mathrm{Ker}\,\Box^{(q)}_b)^G_m$ with respect to $(\cdot | \cdot)$. Let $S^{(q)}_{G,m}(x, y) \in C^\infty(X \times X, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ be the distribution kernel of $S^{(q)}_{G,m}$. The second main result of this work is the following

Theorem 1.8 *With the assumptions and notations used above, if $q \notin n_-$, then, as $m \to +\infty$, $S^{(q)}_{G,m} = O(m^{-\infty})$ on X.*

Suppose $q = n_-$. Let D be an open set of X with $D \bigcap \mu^{-1}(0) = \emptyset$. Then, as $m \to +\infty$, $S^{(q)}_{G,m} = O(m^{-\infty})$ on D.

Let $p \in \mu^{-1}(0)$ and let U be an open set of p and let $x = (x_1, \ldots, x_{2n+1})$ be local coordinates defined in U. Then, as $m \to +\infty$,

$$\begin{aligned} &S^{(q)}_{G,m}(x, y) = e^{im\Psi(x,y)} b(x, y, m) + O(m^{-\infty}),\\ &b(x, y, m) \in S^{n-\frac{d}{2}}_{\mathrm{loc}}(1; U \times U, T^{*0,q}X \boxtimes (T^{*0,q}X)^*),\\ &b(x, y, m) \sim \textstyle\sum_{j=0}^\infty m^{n-\frac{d}{2}-j} b_j(x, y) \text{ in } S^{n-\frac{d}{2}}_{\mathrm{loc}}(1; U \times U, T^{*0,q}X \boxtimes (T^{*0,q}X)^*),\\ &b_j(x, y) \in C^\infty(U \times U, T^{*0,q}X \boxtimes (T^{*0,q}X)^*), \quad j = 0, 1, 2, \ldots, \end{aligned}$$

and

$$b_0(x, x) = 2^{d-1} \frac{1}{V_{\mathrm{eff}}(x)} \pi^{-n-1+\frac{d}{2}} |\det R_x|^{-\frac{1}{2}} |\det \mathcal{L}_x| \tau_{x,n_-}, \quad \forall x \in \mu^{-1}(0), \tag{1.10}$$

where τ_{x,n_-} is given by (1.7), and $\Psi(x, y) \in C^\infty(U \times U)$, $d_x\Psi(x, x) = -d_y\Psi(x, x) = -\omega_0(x)$, for every $x \in \mu^{-1}(0)$, $\Psi(x, y) = 0$ if and only if $x = y \in \mu^{-1}(0)$ and there is a constant $C \geq 1$ such that, for all $(x, y) \in U \times U$,

$$\begin{aligned} &\mathrm{Im}\,\Psi(x, y) \geq \frac{1}{C}\Big(d(x, \mu^{-1}(0))^2 + d(y, \mu^{-1}(0))^2 + \inf_{g\in G,\theta\in S^1} d(e^{i\theta} \circ g \circ x, y)^2\Big),\\ &\mathrm{Im}\,\Psi(x, y) \leq C\Big(d(x, \mu^{-1}(0))^2 + d(y, \mu^{-1}(0))^2 + \inf_{g\in G,\theta\in S^1} d(e^{i\theta} \circ g \circ x, y)^2\Big). \end{aligned}$$

(We refer the reader to Theorem 1.12 for more properties of the phase $\Psi(x, y)$.)

We refer the reader to the discussion in the beginning of Sect. 2.2 and Definition 2.1 for the precise meanings of $A = B + O(m^{-\infty})$ and the symbol space $S^{n-\frac{d}{2}}_{\mathrm{loc}}$, respectively.

It is was proved in Theorem 1.12 in [15]) that when X admits a transversal and CR S^1 action and the Levi form is non-degenerate of constant signature on X, then $\Box_b^{(q)}$ has L^2 closed range.

Let $Y_G := \mu^{-1}(0)/G$. In Theorem 2.5, we will show that Y_G is a CR manifold with natural CR structure induced by $T^{1,0}X$ of dimension $2n-2d+1$. Let $\mathcal{L}_{Y_G,x}$ be the Levi form on Y_G at $x \in Y_G$ induced naturally from $\mathcal{L}$. Note that the bilinear form b is non-degenerate on $\mu^{-1}(0)$, where b is given by (2.4). Hence, on $(\underline{\mathfrak{g}}, \underline{\mathfrak{g}})$, b has constant signature on $\mu^{-1}(0)$. Assume that on $(\underline{\mathfrak{g}}, \underline{\mathfrak{g}})$, b has r negative eigenvalues and $d-r$ positive eigenvalues on $\mu^{-1}(0)$. Hence $\mathcal{L}_{Y_G}$ has $q-r$ negative and $n-d-q+r$ positive eigenvalues at each point of Y_G. Let $\Box_{b,Y_G}^{(q-r)}$ be the Kohn Laplacian for $(0, q-r)$ forms on Y_G. Fix $m \in \mathbb{N}$. Let $H_{b,m}^{q-r}(Y_G) := \left\{u \in \Omega^{0,q-r}(Y_G);\ \Box_{b,Y_G}^{(q-r)}u = 0,\ \ Tu = imu\right\}$. We will apply Theorem 1.8 to establish an isomorphism between $H_{b,m}^{q}(X)^G$ and $H_{b,m}^{q-r}(Y_G)$ if m large enough. We introduce some notations.

Since $\underline{\mathfrak{g}}_x$ is orthogonal to $H_xY \bigcap JH_xY$ and $H_xY \bigcap JH_xY \subset \underline{\mathfrak{g}}_x^{\perp_b}$ (see Lemma 2.4 and (2.5) for the meaning of $\underline{\mathfrak{g}}_x^{\perp_b}$), for every $x \in Y$, we can find a G-invariant orthonormal basis $\{Z_1, \ldots, Z_n\}$ of $T^{1,0}X$ on Y such that

$$
\begin{aligned}
&\mathcal{L}_x(Z_j(x), \overline{Z}_k(x)) = \delta_{j,k}\lambda_j(x), \quad j,k = 1, \ldots, n,\\
&Z_j(x) \in \underline{\mathfrak{g}}_x + iJ\underline{\mathfrak{g}}_x, \quad j = 1, 2, \ldots, d,\\
&Z_j(x) \in \mathbb{C}H_xY \bigcap J(\mathbb{C}H_xY), \quad j = d+1, \ldots, n.
\end{aligned}
$$

Let $\{e_1, \ldots, e_n\}$ denote the orthonormal basis of $T^{*0,1}X$ on Y, dual to $\left\{\overline{Z}_1, \ldots, \overline{Z}_n\right\}$. Fix $s = 0, 1, 2, \ldots, n-d$. For $x \in Y$, put

$$
B_x^{*0,s}X = \left\{ \sum_{d+1 \le j_1 < \cdots < j_s \le n} a_{j_1,\ldots,j_s} e_{j_1} \wedge \cdots \wedge e_{j_s};\ a_{j_1,\ldots,j_s} \in \mathbb{C},\ \forall d+1 \le j_1 < \cdots < j_s \le n \right\}
$$

and let $B^{*0,s}X$ be the vector bundle of Y with fiber $B_x^{*0,s}X$, $x \in Y$. Let $C^\infty(Y, B^{*0,s}X)^G$ denote the set of all G-invariant sections of Y with values in $B^{*0,s}X$. Let

$$
\iota_G : C^\infty(Y, B^{*0,s}X)^G \to \Omega^{0,s}(Y_G)
$$

be the natural identification.

Assume that $\lambda_1 < 0, \ldots, \lambda_r < 0$, and $\lambda_{d+1} < 0, \ldots, \lambda_{n_- - r + d} < 0$. For $x \in Y$, put

$$
\hat{\mathcal{N}}(x, n_-) = \left\{c e_{d+1} \wedge \cdots \wedge e_{n_- - r + d};\ c \in \mathbb{C}\right\},
$$

and let

$$
\begin{aligned}
\hat{p} = \hat{p}_x : \mathcal{N}(x, n_-) &\to \hat{\mathcal{N}}(x, n_-),\\
u = c e_1 \wedge \cdots \wedge e_r \wedge e_{d+1} \wedge \cdots \wedge e_{n_- - r + d} &\to c e_{d+1} \wedge \cdots \wedge e_{n_- - r + d}.
\end{aligned}
$$

Let $\iota : Y \to X$ be the natural inclusion and let $\iota^* : \Omega^{0,q}(X) \to \Omega^{0,q}(Y)$ be the pull-back of ι. Let $q = n_-$. Let $S_{Y_G,m}^{(q-r)} : L^2_{(0,q-r)}(Y_G) \to H_{b,m}^{q-r}(Y_G)$ be the orthogonal projection and let

$$
f(x) = \sqrt{V_{\rm eff}(x)}|\det R_x|^{-\frac{1}{4}} \in C^\infty(Y)^G.
$$

Let

$$
\begin{aligned}
\sigma_m : H_{b,m}^{q}(X)^G &\to H_{b,m}^{q-r}(Y_G),\\
u &\to m^{-\frac{d}{4}} S_{Y_G,m}^{(q-r)} \circ \iota_G \circ \hat{p} \circ \tau_{x,n_-} \circ f \circ \iota^* \circ u.
\end{aligned}
$$

In Sect. 6.2, we will show that

Theorem 1.9 *With the notations and assumptions above, suppose that $q = n_-$. If m is large, then $\sigma_m : H^q_{b,m}(X)^G \to H^{q-r}_{b,m}(Y_G)$ is an isomorphism.*

In particular, if m large enough, then $\dim H^q_{b,m}(X)^G = \dim H^{q-r}_{b,m}(Y_G)$.

Remark 1.10 Let's sketch the idea of the proof of Theorem 1.9. W can consider σ_m as a map from $\Omega^{0,q}(X) \to H^{q-r}_{b,m}(Y_G)$:

$$\begin{aligned}\sigma_m : \Omega^{0,q}(X) &\to H^{q-r}_{b,m}(Y_G) \subset \Omega^{0,q-r}(Y_G),\\ u &\to m^{-\frac{d}{4}} S^{(q-r)}_{Y_G,m} \circ \iota_G \circ \hat{p} \circ \tau_{x,n_-} \circ f \circ \iota^* \circ S^{(q)}_{G,m} u.\end{aligned}$$

Let $\sigma^*_m : \Omega^{0,q-r}(Y_G) \to \Omega^{0,q}(X)$ be the adjoint of σ_m. From Theorem 1.8 and some calculation of complex Fourier integral operators, we will show in Sect. 6.2 that $F_m = \sigma^*_m \sigma_m : \Omega^{0,q}(X) \to \Omega^{0,q}(X)$ is the same type of operator as $S^{(q)}_{G,m}$ and

$$\frac{1}{C_0} F_m = (I + R_m) S^{(q)}_{G,m}, \tag{1.11}$$

where $C_0 > 0$ is a constant and R_m is also the same type of operator as $S^{(q)}_{G,m}$, but the leading symbol of R_m vanishes at $\mathrm{diag}\,(Y \times Y)$. By using the fact that the leading symbol of R_m vanishes at $\mathrm{diag}\,(Y \times Y)$, we will show in Lemma 6.8 that $\|R_m u\| \le \varepsilon_m \|u\|$, for all $u \in \Omega^{0,q}(X)$, for all $m \in \mathbb{N}$, where ε_m is a sequence with $\lim_{m\to\infty} \varepsilon_m = 0$. In particular, if m is large enough, then the map

$$I + R_m : \Omega^{0,q}(X) \to \Omega^{0,q}(X) \tag{1.12}$$

is injective. From (1.11) and (1.12), we deduce that, if m is large enough, then $F_m : H^q_{b,m}(X)^G \to H^q_{b,m}(X)^G$ is injective. Hence $\sigma_m : H^q_{b,m}(X)^G \to H^{q-r}_{b,m}(Y_G)$ is injective.

Similarly, we can repeat the argument above with minor change and deduce that if m is large enough, then the map $\hat{F}_m = \sigma_m \sigma^*_m : H^{q-r}_{b,m}(Y_G) \to H^{q-r}_{b,m}(Y_G)$ is injective. Hence, if m is large enough, then the map $\sigma^*_m : H^{q-r}_{b,m}(Y_G) \to H^q_{b,m}(X)^G$ is injective. Thus, $\dim H^q_{b,m}(X)^G = \dim H^{q-r}_{b,m}(Y_G)$ and σ_m is an isomorphism if m large enough.

Let's apply Theorem 1.9 to complex case. Let (L, h^L) be a holomorphic line bundle over a connected compact complex manifold (M, J) with $\dim_{\mathbb{C}} M = n$, where J denotes the complex structure map of M and h^L is a Hermitian fiber metric of L. Let R^L be the curvature of L induced by h^L. Assume that R^L is non-degenerate of constant signature (n_-, n_+) on M. Let K be a connected compact Lie group with Lie algebra $\mathfrak{k}$. We assume that $\dim_{\mathbb{R}} K = d$ and K acts holomorphically on (M, J), and that the action lifts to a holomorphic action on L. We assume further that h^L is preserved by the K-action. Then R^L is a K-invariant form. Let $\omega = \frac{i}{2\pi} R^L$ and let $\tilde{\mu} : M \to \mathfrak{k}^*$ be the moment map induced by ω. Assume that $0 \in \mathfrak{k}^*$ is regular and the action of K on $\tilde{\mu}^{-1}(0)$ is freely. The analogue of the Marsden-Weinstein reduction holds (see [10]). More precisely, the complex structure J on M induces a complex structure J_K on $M_0 := \tilde{\mu}^{-1}(0)/K$, for which the line bundle $L_0 := L/K$ is a holomorphic line bundle over M_0.

For any $\xi \in \mathfrak{k}$, we write ξ_M to denote the vector field on M induced by ξ. Let $\underline{\mathfrak{k}} = \mathrm{Span}\,(\xi_M;\ \xi \in \mathfrak{k})$. On $\tilde{\mu}^{-1}(0)$, let b^L be the bilinear form on $\underline{\mathfrak{k}} \times \underline{\mathfrak{k}}$ given by $b^L(\cdot\,,\cdot) = \omega(\cdot\,, J\cdot)$. Assume that b^L has r negative eigenvalues and $d - r$ positive eigenvalues on $\tilde{\mu}^{-1}(0)$. Let $q = n_-$. For $m \in \mathbb{N}$, let $H^q(M, L^m)^K$ denote the K-invariant qth Dolbeault cohomology group with values in L^m and let $H^{q-r}(M_0, L_0^m)$ denote the $(q-r)$th Dolbeault cohomology group with values in L_0^m. Theorem 1.9 implies that, if m is large enough, then there is an isomorphism map: $\widetilde{\sigma}_m : H^q(M, L^m)^K \to H^{q-r}(M_0, L_0^m)$. In particular, if m is large enough,then

$$\dim H^q(M, L^m)^K = \dim H^{q-r}(M_0, L_0^m). \tag{1.13}$$

Note that when $m = 1$ and $q = 0$, the equality (1.13) was first proved in [10, §5]. For $m = 1$, the equality (1.13) was established in [30,33] when L is positive. Zhang [33] combined the methods and results in [31] with Braverman's idea [5] to construct a suitable quasi-isomorphisim to prove the equality (1.13). The proof of the equality (1.13) in [30] is completely algebraic, while the the proof of the equality (1.13) in [33] is purely analytic where different quasi-homomorphisms between Dolbeault complexes under considerations were constructed to prove the equality (1.13). If m is large enough and $q = 0$, an isomorphism map in (1.13) was also constructed in [20, (0.27), Corollary 4.13].

If m large enough and $q = 0$, an isomorphism map in (1.13) was also constructed in [20, (0.27), Corollary 4.13]. The point of [20, (0.27), Corollary 4.13] is to study the isometric aspect of this map, as an consequence of the asymptotic of G-invariant Bergman kernel of Ma-Zhang [20], they gave another proof that it is an isomorphism for m large, and this approaches of the isomorphism for m large is adopted in this paper. It should be mentioned that in this situation, a version of the full asymptotics of $S^{(0)}_{G,m}(x, y)$ including (1.10) was established in [20, Theorem 0.1, 0.2].

1.1 The phase functions $\Phi_-(x, y)$ and $\Psi(x, y)$

In this section, we collect some properties of the phase functions $\Phi_-(x, y)$, $\Psi(x, y)$ in Theorem 1.5 and Theorem 1.8.

Let $v = (v_1, \ldots, v_d)$ be local coordinates of G defined in a neighborhood V of e_0 with $v(e_0) = (0, \ldots, 0)$. From now on, we will identify the element $e \in V$ with $v(e)$. Fix $p \in \mu^{-1}(0)$. In Theorem 3.7, we will show that there exist local coordinates $v = (v_1, \ldots, v_d)$ of G defined in a neighborhood V of e_0 with $v(e_0) = (0, \ldots, 0)$, local coordinates $x = (x_1, \ldots, x_{2n+1})$ of X defined in a neighborhood $U = U_1 \times U_2$ of p with $0 \leftrightarrow p$, where $U_1 \subset \mathbb{R}^d$ is an open set of $0 \in \mathbb{R}^d$, $U_2 \subset \mathbb{R}^{2n+1-d}$ is an open set of $0 \in \mathbb{R}^{2n+1-d}$ and a smooth function $\gamma = (\gamma_1, \ldots, \gamma_d) \in C^\infty(U_2, U_1)$ with $\gamma(0) = 0 \in \mathbb{R}^d$ such that

$$\begin{aligned}&(v_1, \ldots, v_d) \circ (\gamma(x_{d+1}, \ldots, x_{2n+1}), x_{d+1}, \ldots, x_{2n+1})\\ &= (v_1 + \gamma_1(x_{d+1}, \ldots, x_{2n+1}), \ldots, v_d + \gamma_d(x_{d+1}, \ldots, x_{2n+1}), x_{d+1}, \ldots, x_{2n+1}),\\ &\forall (v_1, \ldots, v_d) \in V, \ \ \forall (x_{d+1}, \ldots, x_{2n+1}) \in U_2,\end{aligned} \tag{1.14}$$

$$\begin{aligned}&\underline{\mathfrak{g}} = \operatorname{span}\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}\right\},\\ &\mu^{-1}(0) \bigcap U = \{x_{d+1} = \cdots = x_{2d} = 0\},\\ &\text{On } \mu^{-1}(0) \bigcap U, \text{ we have } J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial x_{d+j}} + a_j(x)\frac{\partial}{\partial x_{2n+1}}, \ j = 1, 2, \ldots, d,\end{aligned} \tag{1.15}$$

where $a_j(x)$ is a smooth function on $\mu^{-1}(0) \bigcap U$, independent of $x_1, \ldots, x_{2d}, x_{2n+1}$ and $a_j(0) = 0$, $j = 1, \ldots, d$,

$$
\begin{aligned}
&T^{1,0}_pX = \text{span}\,\{Z_1,\ldots,Z_n\}\,,\\
&Z_j = \tfrac{1}{2}(\tfrac{\partial}{\partial x_j} - i\tfrac{\partial}{\partial x_{d+j}})(p),\quad j=1,\ldots,d,\\
&Z_j = \tfrac{1}{2}(\tfrac{\partial}{\partial x_{2j-1}} - i\tfrac{\partial}{\partial x_{2j}})(p),\quad j=d+1,\ldots,n,\\
&\langle\, Z_j \,|\, Z_k\,\rangle = \delta_{j,k},\quad j,k=1,2,\ldots,n,\\
&\mathcal{L}_p(Z_j,\overline{Z}_k) = \mu_j\delta_{j,k},\quad j,k=1,2,\ldots,n
\end{aligned}
\tag{1.16}
$$

and

$$
\begin{aligned}
\omega_0(x) = (1+O(|x|))dx_{2n+1} &+ \textstyle\sum_{j=1}^{d} 4\mu_j x_{d+j}dx_j\\
&+ \textstyle\sum_{j=d+1}^{n} 2\mu_j x_{2j}dx_{2j-1} - \sum_{j=d+1}^{n} 2\mu_j x_{2j-1}dx_{2j}\\
&+ \textstyle\sum_{j=d+1}^{2n} b_j x_{2n+1}dx_j + O(|x|^2),
\end{aligned}
\tag{1.17}
$$

where $b_{d+1}\in\mathbb{R},\ldots,b_{2n}\in\mathbb{R}$. Put $x''=(x_{d+1},\ldots,x_{2n+1})$, $\hat{x}''=(x_{d+1},x_{d+2},\ldots,x_{2d})$, $\mathring{x}''=(x_{d+1},\ldots,x_{2n})$. We have the following (see Theorem 3.11 and Theorem 3.12)

Theorem 1.11 *With the notations above, the phase function* $\Phi_-(x,y)\in C^\infty(U\times U)$ *is independent of* $(x_1,\ldots,x_d)$ *and* $(y_1,\ldots,y_d)$. *Hence,* $\Phi_-(x,y)=\Phi_-((0,x''),(0,y'')):=\Phi_-(x'',y'')$. *Moreover, there is a constant* $c>0$ *such that*

$$
\operatorname{Im}\Phi_-(x'',y'')\geq c\Big(|\hat{x}''|^2+|\hat{y}''|^2+|\mathring{x}''-\mathring{y}''|^2\Big),\quad \forall((0,x''),(0,y''))\in U\times U. \tag{1.18}
$$

Furthermore,

$$
\begin{aligned}
\Phi_-(x'',y'') = -x_{2n+1}+y_{2n+1} &+ 2i\textstyle\sum_{j=1}^{d}|\mu_j|y_{d+j}^2 + 2i\sum_{j=1}^{d}|\mu_j|x_{d+j}^2\\
&+i\textstyle\sum_{j=d+1}^{n}|\mu_j||z_j-w_j|^2+\sum_{j=d+1}^{n} i\mu_j(\overline{z}_jw_j-z_j\overline{w}_j)\\
&+\textstyle\sum_{j=1}^{d}(-b_{d+j}x_{d+j}x_{2n+1}+b_{d+j}y_{d+j}y_{2n+1})\\
&+\textstyle\sum_{j=d+1}^{n}\frac{1}{2}(b_{2j-1}-ib_{2j})(-z_jx_{2n+1}+w_jy_{2n+1})\\
&+\textstyle\sum_{j=d+1}^{n}\frac{1}{2}(b_{2j-1}+ib_{2j})(-\overline{z}_jx_{2n+1}+\overline{w}_jy_{2n+1})\\
&+(x_{2n+1}-y_{2n+1})f(x,y)+O(|(x,y)|^3),
\end{aligned}
\tag{1.19}
$$

where $z_j=x_{2j-1}+ix_{2j}$, $w_j=y_{2j-1}+iy_{2j}$, $j=d+1,\ldots,n$, μ_j, $j=1,\ldots,n$, *and* $b_{d+1}\in\mathbb{R},\ldots,b_{2n}\in\mathbb{R}$ *are as in* (1.17) *and* f *is smooth and satisfies* $f(0,0)=0$, $f(x,y)=\overline{f}(y,x)$.

We now assume that X admits an S^1 action: $S^1\times X\to X$. We will use the same notations as in Theorem 1.8. Recall that we work with Assumption 1.7. Let $p\in\mu^{-1}(0)$. We can repeat the proof of Theorem 3.7 with minor change and show that there exist local coordinates $v=(v_1,\ldots,v_d)$ of G defined in a neighborhood V of e_0 with $v(e_0)=(0,\ldots,0)$, local coordinates $x=(x_1,\ldots,x_{2n+1})$ of X defined in a neighborhood $U=U_1\times(\hat{U}_2\times]-2\delta,2\delta[)$ of p with $0\leftrightarrow p$, where $U_1\subset\mathbb{R}^d$ is an open set of $0\in\mathbb{R}^d$, $\hat{U}_2\subset\mathbb{R}^{2n-d}$ is an open set of $0\in\mathbb{R}^{2n-d}$, $\delta>0$, and a smooth function $\gamma=(\gamma_1,\ldots,\gamma_d)\in C^\infty(\hat{U}_2\times]-2\delta,2\delta[,U_1)$ with $\gamma(0)=0\in\mathbb{R}^d$ such that $T=-\frac{\partial}{\partial x_{2n+1}}$ and (1.14), (1.15), (1.16), (1.17) hold. We have the following

Theorem 1.12 *With the notations above, the phase function* Ψ *satisfies* $\Psi(x,y)=-x_{2n+1}+y_{2n+1}+\hat{\Psi}(\mathring{x}'',\mathring{y}'')$, *where* $\hat{\Psi}(\mathring{x}'',\mathring{y}'')\in C^\infty(U\times U)$, $\mathring{x}''=(x_{d+1},\ldots,x_{2n})$, $\mathring{y}''=(y_{d+1},\ldots,y_{2n})$, *and* Ψ *satisfies* (1.18) *and* (1.19).

2 Preliminaries

2.1 Standard notations

Let M be a C^∞ paracompact manifold. We let TM and T^*M denote the tangent bundle of M and the cotangent bundle of M, respectively. The complexified tangent bundle of M and the complexified cotangent bundle of M will be denoted by $\mathbb{C}TM$ and $\mathbb{C}T^*M$, respectively. Write $\langle\,\cdot\,,\cdot\,\rangle$ to denote the pointwise duality between TM and T^*M. We extend $\langle\,\cdot\,,\cdot\,\rangle$ bilinearly to $\mathbb{C}TM \times \mathbb{C}T^*M$. Let B be a C^∞ vector bundle over M. The fiber of B at $x \in M$ will be denoted by B_x. Let E be a vector bundle over a C^∞ paracompact manifold M_1. We write $B \boxtimes E^*$ to denote the vector bundle over $M \times M_1$ with fiber over $(x, y) \in M \times M_1$ consisting of the linear maps from E_y to B_x. Let $Y \subset M$ be an open set. From now on, the spaces of distribution sections of B over Y and smooth sections of B over Y will be denoted by $D'(Y, B)$ and $C^\infty(Y, B)$, respectively. Let $E'(Y, B)$ be the subspace of $D'(Y, B)$ whose elements have compact support in Y.

We recall the Schwartz kernel theorem [12, Theorems 5.2.1, 5.2.6], [19, Thorem B.2.7]. Let B and E be C^∞ vector bundles over paracompact orientable C^∞ manifolds M and M_1, respectively, equipped with smooth densities of integration. If $A : C^\infty_0(M_1, E) \to D'(M, B)$ is continuous, we write $K_A(x, y)$ or $A(x, y)$ to denote the distribution kernel of A. The following two statements are equivalent

(1) A is continuous: $E'(M_1, E) \to C^\infty(M, B)$,
(2) $K_A \in C^\infty(M \times M_1, B \boxtimes E^*)$.

If A satisfies (1) or (2), we say that A is smoothing on $M \times M_1$. Let $A, \hat{A} : C^\infty_0(M_1, E) \to D'(M, B)$ be continuous operators. We write

$$A \equiv \hat{A}(\text{on } M \times M_1) \tag{2.1}$$

if $A - \hat{A}$ is a smoothing operator. If $M = M_1$, we simply write "on M".

Let $H(x, y) \in D'(M\times M_1, B\boxtimes E^*)$. We write H to denote the unique continuous operator $C^\infty_0(M_1, E) \to D'(M, B)$ with distribution kernel $H(x, y)$. In this work, we identify H with $H(x, y)$.

2.2 Some standard notations in semi-classical analysis

Let W_1 be an open set in $\mathbb{R}^{N_1}$ and let W_2 be an open set in $\mathbb{R}^{N_2}$. Let E and F be vector bundles over W_1 and W_2, respectively. An m-dependent continuous operator $A_m : C^\infty_0(W_2, F) \to D'(W_1, E)$ is called m-negligible on $W_1 \times W_2$ if, for m large enough, A_m is smoothing and, for any $K \Subset W_1 \times W_2$, any multi-indices α, β and any $N \in \mathbb{N}$, there exists $C_{K,\alpha,\beta,N} > 0$ such that

$$|\partial^\alpha_x \partial^\beta_y A_m(x, y)| \leq C_{K,\alpha,\beta,N}\, m^{-N} \ \text{ on } K, \quad \forall m \gg 1.$$

In that case we write

$$A_m(x, y) = O(m^{-\infty}) \text{ on } W_1 \times W_2, \quad \text{or} \quad A_m = O(m^{-\infty}) \text{ on } W_1 \times W_2.$$

If $A_m, B_m : C^\infty_0(W_2, F) \to D'(W_1, E)$ are m-dependent continuous operators, we write $A_m = B_m + O(m^{-\infty})$ on $W_1 \times W_2$ or $A_m(x, y) = B_m(x, y) + O(m^{-\infty})$ on $W_1 \times W_2$ if $A_m - B_m = O(m^{-\infty})$ on $W_1 \times W_2$. When $W = W_1 = W_2$, we sometime write "on W".

Let X and M be smooth manifolds and let E and F be vector bundles over X and M, respectively. Let $A_m, B_m : C^\infty(M, F) \to C^\infty(X, E)$ be m-dependent smoothing operators. We write $A_m = B_m + O(m^{-\infty})$ on $X \times M$ if on every local coordinate patch D of X and local coordinate patch D_1 of M, $A_m = B_m + O(m^{-\infty})$ on $D \times D_1$. When $X = M$, we sometime write on X.

We recall the definition of the semi-classical symbol spaces

Definition 2.1 Let W be an open set in $\mathbb{R}^N$. Let

$$S(1; W) := \left\{a \in C^\infty(W) \,|\, \forall \alpha \in \mathbb{N}_0^N : \sup_{x\in W}|\partial^\alpha a(x)| < \infty\right\},$$
$$S^0_{\rm loc}(1; W) := \left\{(a(\cdot, m))_{m\in\mathbb{R}} \,|\, \forall \alpha \in \mathbb{N}_0^N, \forall \chi \in C_0^\infty(W) : \sup_{m\in\mathbb{R}, m\geq 1} \sup_{x\in W}|\partial^\alpha(\chi a(x, m))| < \infty\right\}.$$

For $k \in \mathbb{R}$, let

$$S^k_{\rm loc}(1) := S^k_{\rm loc}(1; W) = \left\{(a(\cdot, m))_{m\in\mathbb{R}} \,|\, (m^{-k}a(\cdot, m)) \in S^0_{\rm loc}(1; W)\right\}.$$

Hence $a(\cdot, m) \in S^k_{\rm loc}(1; W)$ if for every $\alpha \in \mathbb{N}_0^N$ and $\chi \in C_0^\infty(W)$, there exists $C_\alpha > 0$ independent of m, such that $|\partial^\alpha(\chi a(\cdot, m))| \leq C_\alpha m^k$ holds on W.

Consider a sequence $a_j \in S^{k_j}_{\rm loc}(1)$, $j \in \mathbb{N}_0$, where $k_j \searrow -\infty$, and let $a \in S^{k_0}_{\rm loc}(1)$. We say

$$a(\cdot, m) \sim \sum_{j=0}^{\infty} a_j(\cdot, m) \text{ in } S^{k_0}_{\rm loc}(1),$$

if, for every $\ell \in \mathbb{N}_0$, we have $a - \sum_{j=0}^{\ell} a_j \in S^{k_{\ell+1}}_{\rm loc}(1)$. For a given sequence a_j as above, we can always find such an asymptotic sum a, which is unique up to an element in $S^{-\infty}_{\rm loc}(1) = S^{-\infty}_{\rm loc}(1; W) := \cap_k S^k_{\rm loc}(1)$.

Similarly, we can define $S^k_{\rm loc}(1; Y, E)$ in the standard way, where Y is a smooth manifold and E is a vector bundle over Y.

2.3 CR manifolds and bundles

Let $(X, T^{1,0}X)$ be a compact, connected and orientable CR manifold of dimension $2n + 1$, $n \geq 1$, where $T^{1,0}X$ is a CR structure of X, that is, $T^{1,0}X$ is a subbundle of rank n of the complexified tangent bundle $\mathbb{C}TX$, satisfying $T^{1,0}X \cap T^{0,1}X = \{0\}$, where $T^{0,1}X = \overline{T^{1,0}X}$, and $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$, where $\mathcal{V} = C^\infty(X, T^{1,0}X)$. There is a unique subbundle HX of TX such that $\mathbb{C}HX = T^{1,0}X \oplus T^{0,1}X$, i.e. HX is the real part of $T^{1,0}X \oplus T^{0,1}X$. Let $J : HX \to HX$ be the complex structure map given by $J(u + \overline{u}) = iu - i\overline{u}$, for every $u \in T^{1,0}X$. By complex linear extension of J to $\mathbb{C}TX$, the i-eigenspace of J is $T^{1,0}X = \left\{V \in \mathbb{C}HX : JV = \sqrt{-1}V\right\}$. We shall also write (X, HX, J) to denote a compact CR manifold.

We fix a real non-vanishing 1 form $\omega_0 \in C(X, T^*X)$ so that $\langle\, \omega_0(x)\,,\, u\,\rangle = 0$, for every $u \in H_xX$, for every $x \in X$. For each $x \in X$, we define a quadratic form on HX by

$$\mathcal{L}_x(U, V) = \frac{1}{2}d\omega_0(JU, V), \forall\, U, V \in H_xX.$$

We extend $\mathcal{L}$ to $\mathbb{C}HX$ by complex linear extension. Then, for $U, V \in T^{1,0}_xX$,

$$\mathcal{L}_x(U, \overline{V}) = \frac{1}{2}d\omega_0(JU, \overline{V}) = -\frac{1}{2i}d\omega_0(U, \overline{V}).$$

The Hermitian quadratic form $\mathcal{L}_x$ on $T^{1,0}_x X$ is called Levi form at x. We recall that in this paper, we always assume that the Levi form $\mathcal{L}$ on $T^{1,0}X$ is non-degenerate of constant signature (n_-, n_+) on X, where n_- denotes the number of negative eigenvalues of the Levi form and n_+ denotes the number of positive eigenvalues of the Levi form. Let $T \in C^\infty(X, TX)$ be the non-vanishing vector field determined by

$$\omega_0(T) = -1, \quad d\omega_0(T, \cdot) \equiv 0 \text{ on } TX. \tag{2.2}$$

Note that X is a contact manifold with contact form ω_0, contact plane HX and T is the Reeb vector field.

Fix a smooth Hermitian metric $\langle \cdot \mid \cdot \rangle$ on $\mathbb{C}TX$ so that $T^{1,0}X$ is orthogonal to $T^{0,1}X$, $\langle u \mid v \rangle$ is real if u, v are real tangent vectors, $\langle T \mid T \rangle = 1$ and T is orthogonal to $T^{1,0}X \oplus T^{0,1}X$. For $u \in \mathbb{C}TX$, we write $|u|^2 := \langle u \mid u \rangle$. Denote by $T^{*1,0}X$ and $T^{*0,1}X$ the dual bundles $T^{1,0}X$ and $T^{0,1}X$, respectively. They can be identified with subbundles of the complexified cotangent bundle $\mathbb{C}T^*X$. Define the vector bundle of $(0, q)$-forms by $T^{*0,q}X := \wedge^q T^{*0,1}X$. The Hermitian metric $\langle \cdot \mid \cdot \rangle$ on $\mathbb{C}TX$ induces, by duality, a Hermitian metric on $\mathbb{C}T^*X$ and also on the bundles of $(0, q)$ forms $T^{*0,q}X$, $q = 0, 1, \cdots, n$. We shall also denote all these induced metrics by $\langle \cdot \mid \cdot \rangle$. Note that we have the pointwise orthogonal decompositions:

$$\begin{aligned} \mathbb{C}T^*X &= T^{*1,0}X \oplus T^{*0,1}X \oplus \{\lambda\omega_0 : \lambda \in \mathbb{C}\}, \\ \mathbb{C}TX &= T^{1,0}X \oplus T^{0,1}X \oplus \{\lambda T : \lambda \in \mathbb{C}\}. \end{aligned}$$

For $x, y \in X$, let $d(x, y)$ denote the distance between x and y induced by the Hermitian metric $\langle \cdot \mid \cdot \rangle$. Let A be a subset of X. For every $x \in X$, let $d(x, A) := \inf\{d(x, y);\ y \in A\}$.

Let D be an open set of X. Let $\Omega^{0,q}(D)$ denote the space of smooth sections of $T^{*0,q}X$ over D and let $\Omega^{0,q}_0(D)$ be the subspace of $\Omega^{0,q}(D)$ whose elements have compact support in D.

2.4 Contact reduction

Let G be a connected compact Lie group with Lie algebra $\mathfrak{g}$ such that $\dim_{\mathbb{R}} G = d$. We assume that the Lie group G acts on X preserving ω_0, i.e. $g^*\omega_0 = \omega_0$, for any $g \in G$. For any $\xi \in \mathfrak{g}$, there is an induced vector field ξ_X on X given by $(\xi_X u)(x) = \frac{\partial}{\partial t}\left(u(\exp(t\xi) \circ x)\right)|_{t=0}$, for any $u \in C^\infty(X)$.

Definition 2.2 The contact moment map associated to the form ω_0 is the map $\mu : X \to \mathfrak{g}^*$ such that, for all $x \in X$ and $\xi \in \mathfrak{g}$, we have

$$\langle \mu(x), \xi \rangle = \omega_0(\xi_X(x)). \tag{2.3}$$

We now recall the contact reduction from [1,9]. It was shown in [1,9] that the contact moment map is G-equivariant, so G acts on $Y := \mu^{-1}(0)$, where G acts on $\mathfrak{g}^*$ through co-adjoint represent. Since we assume that the action of G on Y is freely, $Y_G := \mu^{-1}(0)/G$ is a smooth manifold. Let $\pi : Y \to Y_G$ and $\iota : Y \hookrightarrow X$ be the natural quotient and inclusion, respectively, then there is a unique induced contact form $\widetilde{\omega}_0$ on Y_G such $\pi^*\widetilde{\omega}_0 = \iota^*\omega_0$. We denote by $HY := \operatorname{Ker}\omega_0 \cap T(\mu^{-1}(0)) = HX \cap TY$, then the induced contact plane on Y_G is $HY_G := \pi_* HY$. In particular, $\dim HY = 2n - d$ and $\dim HY_G = 2n - 2d$.

2.5 CR reduction

In this subsection we study the reduction of CR manifolds with non-degenerate Levi curvature which is a CR analogue of the reduction on complex manifolds considered in [27, §2.1]. For the case of strictly pseudoconvex CR manifolds, the CR reduction was also studied in [17].

Recall that we work with Assumption 1.2. Let b be the nondegenerate bilinear form on HX such that

$$b(\cdot,\cdot) = d\omega_0(\cdot, J\cdot). \tag{2.4}$$

We denote by $\underline{\mathfrak{g}} := \mathrm{Span}(\xi_X, \xi \in \mathfrak{g})$ the tangent bundle of the orbits in X. Let

$$\underline{\mathfrak{g}}^{\perp_b} = \left\{v \in HX;\ b(\xi_X, v) = 0,\ \ \forall \xi_X \in \underline{\mathfrak{g}}\right\}. \tag{2.5}$$

Since we assume that $\underline{\mathfrak{g}}_x \cap \underline{\mathfrak{g}}_x^{\perp_b} = \{0\}$, for every $x \in Y$, we immediately get

Lemma 2.3 *When restricted to $\underline{\mathfrak{g}} \times \underline{\mathfrak{g}}$, the bilinear form b is nondegenerate on Y.*

For $x \in Y$, $V \in H_xX$ and $\xi \in \mathfrak{g}$, by (2.3) and (2.4), we have

$$b_x(\xi_X, JV) = -d\omega_0(x)(\xi_X, V) = -\left(d\mu(x)(V)\right)(\xi).$$

Therefore,

$$JV \in \underline{\mathfrak{g}}^{\perp_b}|_Y \iff d\mu(x)(V) = 0. \tag{2.6}$$

Since $Y = \mu^{-1}(0)$, we have

$$d\mu(x)(V) = 0 \iff V \in T_xY. \tag{2.7}$$

In particular, for $x \in Y$,

$$\dim \underline{\mathfrak{g}}_x^{\perp_b} = \dim(H_xX \cap T_xY) = \dim H_xY = 2n - d.$$

By (2.2), (2.7) and the definition of $\underline{\mathfrak{g}}$, we have $\underline{\mathfrak{g}} \subset HX|_Y$. From Lemma 2.3, we can check that $\underline{\mathfrak{g}} + \underline{\mathfrak{g}}^{\perp_b} = HX|_Y$. Since $\underline{\mathfrak{g}}_x \cap \underline{\mathfrak{g}}_x^{\perp_b} = \{0\}$, for every $x \in Y$, this sum is a direct sum.

Let U be a small open G-invariant neighborhood of Y. Since G acts freely on Y, we can thus also assume that G acts freely on $\overline{U}$. Since $\underline{\mathfrak{g}}_x \cap \underline{\mathfrak{g}}_x^{\perp_b} = \{0\}$, for $x \in Y$, we have, for $x \in Y$,

$$H_xU = \underline{\mathfrak{g}}_x \oplus \underline{\mathfrak{g}}_x^{\perp_b}. \tag{2.8}$$

Then, by (2.8), we can choose the horizontal bundles of the fibrations $U \to U_G := U/G$ and $Y \to Y_G$ to be

$$H^HU = \underline{\mathfrak{g}}^{\perp_b}|_U, \quad H^HY := H^HU|_Y \cap HY. \tag{2.9}$$

Hence

$$HY = \underline{\mathfrak{g}}|_Y \oplus H^HY.$$

Lemma 2.4

$$\underline{\mathfrak{g}}^{\perp_b}|_Y = JHY. \tag{2.10}$$

$$HU|_Y = J\underline{\mathfrak{g}}|_Y \oplus HY = \underline{\mathfrak{g}}|_Y \oplus J\underline{\mathfrak{g}}|_Y \oplus H^HY. \tag{2.11}$$

Proof The identity (2.10) follows from (2.6) and (2.7). For $x \in Y$, $V \in H_xY$ and $\xi \in \mathfrak{g}$,

$$b_x(J\xi_X, V) = d\omega_0(x)(\xi_X, V) = (d\mu(x)(V))(\xi) = 0. \tag{2.12}$$

Using (2.12), $\dim H_xU = \dim H_xY + \dim J\underline{\mathfrak{g}}_x$, and the fact that b is nondegenerate on JHY, we obtain (2.11). □

By (2.9), and (2.10), we have $H^HY = JHY \cap HY$. In particular, H^HY is preserved by J, so we can define the homomorphism J_G on HY_G in the following way: For $V \in HY_G$, we denote by V^H its lift in H^HY, and we define J_G on Y_G by

$$(J_GV)^H = J(V^H). \tag{2.13}$$

Hence, we have $J_G : HY_G \to HY_G$ such that $J_G^2 = -\,\mathrm{id}$, where id denotes the identity map $\mathrm{id} : HY_G \to HY_G$. By complex linear extension of J_G to $\mathbb{C}TY_G$, we can define the i-eigenspace of J_G is given by $T^{1,0}Y_G = \left\{V \in \mathbb{C}HY_G : J_GV = \sqrt{-1}V\right\}$.

Theorem 2.5 *The subbundle $T^{1,0}Y_G$ is a CR structure of Y_G.*

Proof Let $u, v \in C^\infty(Y_G, T^{1,0}Y_G)$, then we can find $U, V \in C^\infty(Y_G, TY_G)$ such that

$$u = U - \sqrt{-1}J_GU, \qquad v = V - \sqrt{-1}J_GV.$$

By (2.13), we have

$$u^H = U^H - \sqrt{-1}JU^H, \quad v = V^H - \sqrt{-1}JV^H \in T^{1,0}X \cap \mathbb{C}HY.$$

Since $T^{1,0}X$ is a CR structure and it is clearly that $[u^H, v^H] \in \mathbb{C}HY$, we have $[u^H, v^H] \in T^{1,0}X \cap \mathbb{C}HY$. Hence, there is a $W \in C^\infty(X, HX)$ such that

$$[u^H, v^H] = W - \sqrt{-1}JW.$$

In particular, $W, JW \in HY$. Thus, $W \in HY \cap JHY = H^HY$. Let $X^H \in H^HY$ be a lift of $X \in TY_G$ such that $X^H = W$. Then we have

$$[u, v] = \pi_*[u^H, v^H] = \pi_*(X^H - \sqrt{-1}JX^H) = X - \sqrt{-1}J_GX \in T^{1,0}Y_G,$$

i.e. we have $[C^\infty(Y_G, T^{1,0}Y_G), C^\infty(Y_G, T^{1,0}Y_G)] \subset C^\infty(Y_G, T^{1,0}Y_G)$. Therefore, $T^{1,0}Y_G$ is a CR structure of Y_G. □

3 *G*-invariant Szegő kernel asymptotics

In this section, we will establish asymptotic expansion for the G-invariant Szegő kernel. We first review some known results for Szegő kernel.

3.1 Szegő kernel asymptotics

In this subsection, we don't assume that our CR manifold admits a compact Lie group action but we still assume that the Levi form is non-degenerate of constant signature (n_-, n_+). The Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathbb{C}TX$ induces, by duality, a Hermitian metric on $\mathbb{C}T^*X$ and also on the bundles of $(0, q)$ forms $T^{*0,q}X$, $q = 0, 1, \ldots, n$. We shall also denote all these induced metrics by $\langle \cdot | \cdot \rangle$. For $u \in T^{*0,q}X$, we write $|u|^2 := \langle u | u \rangle$. Let $D \subset X$ be an open set. Let $\Omega^{0,q}(D)$ denote the space of smooth sections of $T^{*0,q}X$ over D and let $\Omega_0^{0,q}(D)$ be the subspace of $\Omega^{0,q}(D)$ whose elements have compact support in D.

Let

$$\overline{\partial}_b : \Omega^{0,q}(X) \to \Omega^{0,q+1}(X)$$

be the tangential Cauchy-Riemann operator. Let $dv(x)$ be the volume form induced by the Hermitian metric $\langle \cdot | \cdot \rangle$. The natural global L^2 inner product $(\cdot | \cdot)$ on $\Omega^{0,q}(X)$ induced by $dv(x)$ and $\langle \cdot | \cdot \rangle$ is given by

$$(\, u \,|\, v \,) := \int_X \langle \, u(x) \,|\, v(x) \,\rangle \, dv(x) \,, \quad u, v \in \Omega^{0,q}(X) \,.$$

We denote by $L^2_{(0,q)}(X)$ the completion of $\Omega^{0,q}(X)$ with respect to $(\cdot | \cdot)$. We write $L^2(X) := L^2_{(0,0)}(X)$. We extend $(\cdot | \cdot)$ to $L^2_{(0,q)}(X)$ in the standard way. For $f \in L^2_{(0,q)}(X)$, we denote $\|f\|^2 := (\, f \,|\, f \,)$. We extend $\overline{\partial}_b$ to $L^2_{(0,r)}(X)$, $r = 0, 1, \ldots, n$, by

$$\overline{\partial}_b : \mathrm{Dom}\, \overline{\partial}_b \subset L^2_{(0,r)}(X) \to L^2_{(0,r+1)}(X) \,,$$

where $\mathrm{Dom}\, \overline{\partial}_b := \{u \in L^2_{(0,r)}(X);\ \overline{\partial}_b u \in L^2_{(0,r+1)}(X)\}$ and, for any $u \in L^2_{(0,r)}(X)$, $\overline{\partial}_b u$ is defined in the sense of distributions. We also write

$$\overline{\partial}^*_b : \mathrm{Dom}\, \overline{\partial}^*_b \subset L^2_{(0,r+1)}(X) \to L^2_{(0,r)}(X)$$

to denote the Hilbert space adjoint of $\overline{\partial}_b$ in the L^2 space with respect to $(\cdot | \cdot)$. Let $\Box^{(q)}_b$ denote the (Gaffney extension) of the Kohn Laplacian given by

$$\begin{gathered} \mathrm{Dom}\, \Box^{(q)}_b = \left\{ s \in L^2_{(0,q)}(X);\ s \in \mathrm{Dom}\, \overline{\partial}_b \cap \mathrm{Dom}\, \overline{\partial}^*_b,\ \overline{\partial}_b s \in \mathrm{Dom}\, \overline{\partial}^*_b,\ \overline{\partial}^*_b s \in \mathrm{Dom}\, \overline{\partial}_b \right\}, \\ \Box^{(q)}_b s = \overline{\partial}_b \overline{\partial}^*_b s + \overline{\partial}^*_b \overline{\partial}_b s \ \text{ for } s \in \mathrm{Dom}\, \Box^{(q)}_b \,. \end{gathered} \tag{3.1}$$

By a result of Gaffney, for every $q = 0, 1, \ldots, n$, $\Box^{(q)}_b$ is a positive self-adjoint operator (see [19, Proposition 3.1.2]). That is, $\Box^{(q)}_b$ is self-adjoint and the spectrum of $\Box^{(q)}_b$ is contained in $\overline{\mathbb{R}}_+$, $q = 0, 1, \ldots, n$. Let

$$S^{(q)} : L^2_{(0,q)}(X) \to \mathrm{Ker}\, \Box^{(q)}_b \tag{3.2}$$

be the orthogonal projections with respect to the L^2 inner product $(\cdot | \cdot)$ and let

$$S^{(q)}(x, y) \in D'(X \times X, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$$

denote the distribution kernel of $S^{(q)}$.

We recall Hörmander symbol space. Let $D \subset X$ be a local coordinate patch with local coordinates $x = (x_1, \ldots, x_{2n+1})$.

Definition 3.1 For $m \in \mathbb{R}$, $S^m_{1,0}(D \times D \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ is the space of all $a(x, y, t) \in C^\infty(D \times D \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ such that, for all compact $K \Subset D \times D$ and all $\alpha, \beta \in \mathbb{N}_0^{2n+1}$, $\gamma \in \mathbb{N}_0$, there is a constant $C_{\alpha,\beta,\gamma} > 0$ such that

$$|\partial^\alpha_x \partial^\beta_y \partial^\gamma_t a(x, y, t)| \le C_{\alpha,\beta,\gamma}(1 + |t|)^{m-\gamma}, \quad \forall (x, y, t) \in K \times \mathbb{R}_+, \quad t \ge 1.$$

Put

$$S^{-\infty}(D \times D \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*) := \bigcap_{m \in \mathbb{R}} S^m_{1,0}(D \times D \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*).$$

Let $a_j \in S^{m_j}_{1,0}(D \times D \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$, $j = 0, 1, 2, \ldots$ with $m_j \to -\infty$, as $j \to \infty$. Then there exists $a \in S^{m_0}_{1,0}(D \times D \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ unique modulo $S^{-\infty}$, such that $a - \sum_{j=0}^{k-1} a_j \in S^{m_k}_{1,0}(D \times D \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*\big)$ for $k = 0, 1, 2, \ldots$.

If a and a_j have the properties above, we write $a \sim \sum_{j=0}^{\infty} a_j$ in $S^{m_0}_{1,0}\big(D \times D \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*\big)$. We write

$$s(x, y, t) \in S^m_{\mathrm{cl}}\left(D \times D \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*\right)$$

if $s(x, y, t) \in S^m_{1,0}\big(D \times D \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*\big)$ and

$$\begin{aligned} s(x, y, t) \sim \textstyle\sum_{j=0}^{\infty} s^j(x, y)t^{m-j} \text{ in } S^m_{1,0}\big(D \times D \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*\big), \\ s^j(x, y) \in C^\infty\big(D \times D, T^{*0,q}X \boxtimes (T^{*0,q}X)^*\big), \ j \in \mathbb{N}_0. \end{aligned}$$

The following was proved in Theorem 4.8 in [15]

Theorem 3.2 *Given $q = 0, 1, 2, \ldots, n$. Assume that $q \notin \{n_-, n_+\}$. Then, $S^{(q)} \equiv 0$ on X.*

We have the following (see Theorem 1.2 in [13], Theorem 4.7 in [15] and see also [2] for $q = 0$)

Theorem 3.3 *We recall that we work with the assumption that the Levi form is non-degenerate of constant signature (n_-, n_+) on X. Let $q = n_-$ or n_+. Suppose that $\Box^{(q)}_b$ has L^2 closed range. Then, $S^{(q)}(x, y) \in C^\infty(X \times X \setminus \mathrm{diag}\,(X \times X), T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$. Let $D \subset X$ be any local coordinate patch with local coordinates $x = (x_1, \ldots, x_{2n+1})$. Then, there exist continuous operators $S_-, S_+ : \Omega^{0,q}_0(D) \to D'(D, T^{*0,q}X)$ such that*

$$S^{(q)} \equiv S_- + S_+ \ \text{ on } D,$$

and $S_-(x, y)$, $S_+(x, y)$ satisfy

$$S_\mp(x, y) \equiv \int_0^\infty e^{i\varphi_\mp(x,y)t} s_\mp(x, y, t)dt \ \text{ on } D,$$

with

$$\begin{aligned} & s_-(x, y, t), s_+(x, y, t) \in S^n_{\mathrm{cl}}\,(D \times D \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*), \\ & s_-(x, y, t) = 0 \ \text{ if } q \neq n_-, \ \ s_+(x, y, t) = 0 \ \text{ if } q \neq n_+, \\ & s^0_-(x, x) \neq 0, \ \ \forall x \in D, \ \ s^0_+(x, x) \neq 0, \ \ \forall x \in D, \end{aligned} \tag{3.3}$$

and the phase functions φ_-, φ_+ satisfy

$$\begin{aligned} & \varphi_+(x, y), \varphi_- \in C^\infty(D \times D), \ \ \mathrm{Im}\,\varphi_-(x, y) \geq 0, \\ & \varphi_-(x, x) = 0, \ \ \varphi_-(x, y) \neq 0 \ \text{ if } \ x \neq y, \\ & d_x\varphi_-(x, y)\big|_{x=y} = -\omega_0(x), \ \ d_y\varphi_-(x, y)\big|_{x=y} = \omega_0(x), \\ & \varphi_-(x, y) = -\overline{\varphi}_-(y, x), \ \ -\overline{\varphi}_+(x, y) = \varphi_-(x, y). \end{aligned}$$

Remark 3.4 It is well-known that for a strictly pseudoconvec CR manifold of dimension 3, $\Box^{(0)}_b$ does not have L^2 closed range in general (see [28]). Kohn [16] proved that if $q = n_- = n_+$ or $|n_- - n_+| > 1$ then $\Box^{(q)}_b$ has L^2 closed range.

The following result describes the phase function in local coordinates (see chapter 8 of part I in [13])

Theorem 3.5 *For a given point $p \in X$, let $\{W_j\}_{j=1}^n$ be an orthonormal frame of $T^{1,0}X$ in a neighborhood of p such that the Levi form is diagonal at p, i.e. $\mathcal{L}_{x_0}(W_j, \overline{W}_s) = \delta_{j,s}\mu_j$, $j, s = 1, \ldots, n$. We take local coordinates $x = (x_1, \ldots, x_{2n+1})$, $z_j = x_j + ix_{d+j}$, $j = 1, \ldots, d$, $z_j = x_{2j-1} + ix_{2j}$, $j = d+1, \ldots, n$, defined on some neighborhood of p such that $\omega_0(p) = dx_{2n+1}$, $x(p) = 0$, and, for some $c_j \in \mathbb{C}$, $j = 1, \ldots, n$,*

$$W_j = \frac{\partial}{\partial z_j} - i\mu_j\overline{z}_j\frac{\partial}{\partial x_{2n+1}} - c_jx_{2n+1}\frac{\partial}{\partial x_{2n+1}} + \sum_{k=1}^{2n} a_{j,k}(x)\frac{\partial}{\partial x_k} + O(|x|^2),\ \ j = 1, \ldots, n, \tag{3.4}$$

where $a_{j,k}(x) \in C^\infty$, $a_{j,k}(x) = O(|x|)$, for every $j = 1, \ldots, n$, $k = 1, \ldots, 2n$. Set $y = (y_1, \ldots, y_{2n+1})$, $w_j = y_j + iy_{d+j}$, $j = 1, \ldots, d$, $w_j = y_{2j-1} + iy_{2j}$, $j = d+1, \ldots, n$. Then, for φ_- in Theorem 3.3, we have

$$\operatorname{Im}\varphi_-(x, y) \geq c\sum_{j=1}^{2n}|x_j - y_j|^2,\ \ c > 0, \tag{3.5}$$

in some neighbourhood of $(0, 0)$ and

$$\begin{aligned}&\varphi_-(x, y) = -x_{2n+1} + y_{2n+1} + i\textstyle\sum_{j=1}^n|\mu_j||z_j - w_j|^2 + \sum_{j=1}^n\Big(i\mu_j(\overline{z}_jw_j - z_j\overline{w}_j)\\ &\quad +c_j(-z_jx_{2n+1} + w_jy_{2n+1}) + \overline{c}_j(-\overline{z}_jx_{2n+1} + \overline{w}_jy_{2n+1})\Big)\\ &\quad +(x_{2n+1} - y_{2n+1})f(x, y) + O(|(x, y)|^3),\end{aligned} \tag{3.6}$$

where f is smooth and satisfies $f(0, 0) = 0$, $f(x, y) = \overline{f}(y, x)$.

The following formula for the leading term s_-^0 on the diagonal follows from [13, §9]. The formula for the leading term s_+^0 on the diagonal follows similarly.

Theorem 3.6 *We assume that the Levi form is non-degenerate of constant signature (n_-, n_+) at each point of X. Suppose that $\Box_b^{(q)}$ has L^2 closed range. If $q = n_\mp$, then, for the leading term $s_\mp^0(x, y)$ of the expansion (3.3) of $s_\mp(x, y, t)$, we have*

$$s_\mp^0(x_0, x_0) = \frac{1}{2}\pi^{-n-1}|\det\mathcal{L}_{x_0}|\tau_{x_0,n_\mp},\ \ x_0 \in D,$$

where $\det\mathcal{L}_{x_0}$ is given by (1.6) and $\tau_{x_0,n_\mp}$ is given by (1.7).

3.2 *G*-invariant Szegő kernel

Fix $g \in G$. Let $g^* : \Lambda_x^r(\mathbb{C}T^*X) \to \Lambda_{g^{-1}\circ x}^r(\mathbb{C}T^*X)$ be the pull-back map. Since G preserves J, we have $g^* : T_x^{*0,q}X \to T_{g^{-1}\circ x}^{*0,q}X$, $\forall x \in X$. Thus, for $u \in \Omega^{0,q}(X)$, we have $g^*u \in \Omega^{0,q}(X)$ and we write $(g^*u)(x) := u(g \circ x)$. Put $\Omega^{0,q}(X)^G := \{u \in \Omega^{0,q}(X);\ g^*u = u,\ \ \forall g \in G\}$. Now, we assume that the Hermitian metric $\langle\cdot|\cdot\rangle$ on $\mathbb{C}TX$ is G-invariant and $\underline{\mathfrak{g}}$ is orthogonal to $HY \bigcap JHY$ at every point of Y. The Hermitian metric is G-invariant means that, for any G-invariant vector fields U and V, $\langle U | V\rangle$ is G-invariant. Then the L^2 inner product $(\cdot|\cdot)$ on $\Omega^{0,q}(X)$ induced by $\langle\cdot|\cdot\rangle$ is G-invariant, that is, $(u | v) = (g^*u | g^*v)$, for all $u, v \in \Omega^{0,q}(X)$, $g \in G$. Let $u \in L^2_{(0,q)}(X)$ and let $g \in G$. Take $u_j \in \Omega^{0,q}(X)$, $j = 1, 2, \ldots$, with $u_j \to u$ in $L^2_{(0,q)}(X)$ as $j \to \infty$. Since $(\cdot|\cdot)$ is G-invariant, there is a $v \in L^2_{(0,q)}(X)$ such that $v = \lim_{j\to\infty} g^*u_j$. We define $g^*u := v$.

It is clear that the definition is well-defined. We have $g^* : L^2_{(0,q)}(X) \to L^2_{(0,q)}(X)$. Put $L^2_{(0,q)}(X)^G := \left\{u \in L^2_{(0,q)}(X);\ g^*u = u,\ \forall g \in G\right\}$. It is not difficult to see that $L^2_{(0,q)}(X)^G$ is the completion of $\Omega^{0,q}(X)^G$ with respect to $(\,\cdot\,|\,\cdot\,)$. We write $L^2(X)^G := L^2_{(0,0)}(X)^G$. Since G preserves J and $(\,\cdot\,|\,\cdot\,)$ is G-invariant, it is straightforward to see that

$$\begin{aligned}
&g^*\overline{\partial}_b = \overline{\partial}_b g^* \ \text{ on Dom}\,\overline{\partial}_b, \quad g^*\overline{\partial}^*_b = \overline{\partial}^*_b g^* \ \text{ on Dom}\,\overline{\partial}^*_b,\\
&g^*\Box^{(q)}_b = \Box^{(q)}_b g^* \ \text{ on Dom}\,\Box^{(q)}_b.
\end{aligned}$$

Put $(\mathrm{Ker}\,\Box^{(q)}_b)^G := \mathrm{Ker}\,\Box^{(q)}_b \bigcap L^2_{(0,q)}(X)^G$. The G-invariant Szegő projection is the orthogonal projection $S^{(q)}_G : L^2_{(0,q)}(X) \to (\mathrm{Ker}\,\Box^{(q)}_b)^G$ with respect to $(\,\cdot\,|\,\cdot\,)$. Let $S^{(q)}_G(x, y) \in D'(X \times X, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ be the distribution kernel of S^G. Let $d\mu$ be a Haar measure on G so that $|G|_{d\mu} := \int_G d\mu = 1$.Then,

$$S^{(q)}_G(x, y) = \int_G S^{(q)}(x, g \circ y)d\mu(g). \tag{3.7}$$

Note that the integral (3.7) is defined in the sense of distribution.

3.3 *G*-invariant Szegő kernel asymptotics near $\mu^{-1}(0)$

In this section, we will study G-invariant Szegő kernel near $\mu^{-1}(0)$.

Let $e_0 \in G$ be the identity element. Let $v = (v_1, \ldots, v_d)$ be the local coordinates of G defined in a neighborhood V of e_0 with $v(e_0) = (0, \ldots, 0)$. From now on, we will identify the element $e \in V$ with $v(e)$. We first need

Theorem 3.7 *Let $p \in \mu^{-1}(0)$. There exist local coordinates $v = (v_1, \ldots, v_d)$ of G defined in a neighborhood V of e_0 with $v(e_0) = (0, \ldots, 0)$, local coordinates $x = (x_1, \ldots, x_{2n+1})$ of X defined in a neighborhood $U = U_1 \times U_2$ of p with $0 \leftrightarrow p$, where $U_1 \subset \mathbb{R}^d$ is an open set of $0 \in \mathbb{R}^d$, $U_2 \subset \mathbb{R}^{2n+1-d}$ is an open set of $0 \in \mathbb{R}^{2n+1-d}$ and a smooth function $\gamma = (\gamma_1, \ldots, \gamma_d) \in C^\infty(U_2, U_1)$ with $\gamma(0) = 0 \in \mathbb{R}^d$ such that*

$$\begin{aligned}
&(v_1, \ldots, v_d) \circ (\gamma(x_{d+1}, \ldots, x_{2n+1}), x_{d+1}, \ldots, x_{2n+1})\\
&= (v_1 + \gamma_1(x_{d+1}, \ldots, x_{2n+1}), \ldots, v_d + \gamma_d(x_{d+1}, \ldots, x_{2n+1}), x_{d+1}, \ldots, x_{2n+1}),\\
&\forall (v_1, \ldots, v_d) \in V, \quad \forall (x_{d+1}, \ldots, x_{2n+1}) \in U_2,
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
&\underline{\mathfrak{g}} = \mathrm{span}\left\{\tfrac{\partial}{\partial x_1}, \ldots, \tfrac{\partial}{\partial x_d}\right\},\\
&\mu^{-1}(0) \textstyle\bigcap U = \{x_{d+1} = \cdots = x_{2d} = 0\},\\
&\textit{On } \mu^{-1}(0) \textstyle\bigcap U, \textit{ we have } J(\tfrac{\partial}{\partial x_j}) = \tfrac{\partial}{\partial x_{d+j}} + a_j(x)\tfrac{\partial}{\partial x_{2n+1}}, \ j = 1, 2, \ldots, d,
\end{aligned} \tag{3.9}$$

where $a_j(x)$ is a smooth function on $\mu^{-1}(0) \bigcap U$, independent of $x_1, \ldots, x_{2d}, x_{2n+1}$ and $a_j(0) = 0$, $j = 1, \ldots, d$,

$$\begin{aligned}
&T^{1,0}_p X = \mathrm{span}\,\{Z_1, \ldots, Z_n\},\\
&Z_j = \tfrac{1}{2}(\tfrac{\partial}{\partial x_j} - i\tfrac{\partial}{\partial x_{d+j}})(p), \quad j = 1, \ldots, d,\\
&Z_j = \tfrac{1}{2}(\tfrac{\partial}{\partial x_{2j-1}} - i\tfrac{\partial}{\partial x_{2j}})(p), \quad j = d+1, \ldots, n,\\
&\langle\, Z_j \,|\, Z_k \,\rangle = \delta_{j,k}, \quad j, k = 1, 2, \ldots, n,\\
&\mathcal{L}_p(Z_j, \overline{Z}_k) = \mu_j \delta_{j,k}, \quad j, k = 1, 2, \ldots, n
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}\omega_0(x) = (1+O(|x|))dx_{2n+1} + \textstyle\sum_{j=1}^{d} 4\mu_j x_{d+j} dx_j \\ + \textstyle\sum_{j=d+1}^{n} 2\mu_j x_{2j} dx_{2j-1} - \sum_{j=d+1}^{n} 2\mu_j x_{2j-1} dx_{2j} \\ + \textstyle\sum_{j=d+1}^{2n} b_j x_{2n+1} dx_j + O(|x|^2),\end{aligned} \tag{3.11}$$

where $b_{d+1} \in \mathbb{R}, \ldots, b_{2n} \in \mathbb{R}$.

Proof From the standard proof of Frobenius Theorem, it is not difficult to see that there exist local coordinates $v = (v_1, \ldots, v_d)$ of G defined in a neighborhood V of e_0 with $v(e_0) = (0, \ldots, 0)$ and local coordinates $x = (x_1, \ldots, x_{2n+1})$ of X defined in a neighborhood U of p with $x(p) = 0$ such that

$$\begin{aligned}&(v_1, \ldots, v_d) \circ (0, \ldots, 0, x_{d+1}, \ldots, x_{2n+1}) \\ &= (v_1, \ldots, v_d, x_{d+1}, \ldots, x_{2n+1}), \quad \forall (v_1, \ldots, v_d) \in V, \ \ \forall (0, \ldots, 0, x_{d+1}, \ldots, x_{2n+1}) \in U,\end{aligned} \tag{3.12}$$

and

$$\underline{\mathfrak{g}} = \text{span}\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}\right\}. \tag{3.13}$$

Since $p \in \mu^{-1}(0)$, we have $\omega_0(p)(\frac{\partial}{\partial x_j}(p)) = 0$, $j = 1, 2, \ldots, d$, and hence $\frac{\partial}{\partial x_j}(p) \in H_pX$, $j = 1, 2, \ldots, d$. Consider the linear map

$$\begin{aligned}R : \underline{\mathfrak{g}}_p &\to \underline{\mathfrak{g}}_p, \\ u &\to Ru, \ \ \langle\, Ru \,|\, v \,\rangle = \langle\, d\omega_0\,,\ Ju \wedge v \,\rangle.\end{aligned}$$

Since R is self-adjoint, by using linear transformation in $(x_1, \ldots, x_d)$, we can take $(x_1, \ldots, x_d)$ such that, for $j, k = 1, 2, \ldots, d$,

$$\langle\, R\frac{\partial}{\partial x_j}(p) \,|\, \frac{\partial}{\partial x_k}(p) \,\rangle = 4\mu_j \delta_{j,k}, \quad \langle\, \frac{\partial}{\partial x_j}(p) \,|\, \frac{\partial}{\partial x_k}(p) \,\rangle = 2\delta_{j,k}. \tag{3.14}$$

By taking linear transformation in $(v_1, \ldots, v_d)$, (3.12) still hold.

Let $\omega_0(\frac{\partial}{\partial x_j}) = a_j(x) \in C^\infty(U)$, $j = 1, 2, \ldots, d$. Since $a_j(x)$ is G-invariant, we have $\frac{\partial a_j(x)}{\partial x_s} = 0$, $j, s = 1, 2, \ldots, d$. By the definition of the moment map, we have

$$\mu^{-1}(0) \bigcap U = \{x \in U;\ a_1(x) = \cdots = a_d(x) = 0\}.$$

Since p is a regular value of the moment map μ, the matrix $\left(\frac{\partial a_j}{\partial x_s}(p)\right)_{1 \le j \le d, d+1 \le s \le 2n+1}$ is of rank d. We may assume that the matrix $\left(\frac{\partial a_j}{\partial x_s}(p)\right)_{1 \le j \le d, d+1 \le s \le 2d}$ is non-singular. Thus, $(x_1, \ldots, x_d, a_1, \ldots, a_d, x_{2d+1}, \ldots, x_{2n+1})$ are also local coordinates of X. Hence, we can take $v = (v_1, \ldots, v_d)$ and $x = (x_1, \ldots, x_{2n+1})$ such that (3.12), (3.13), (3.14) hold and

$$\mu^{-1}(0) \bigcap U = \{x = (x_1, \ldots, x_{2n+1}) \in U;\ x_{d+1} = \cdots = x_{2d} = 0\}. \tag{3.15}$$

On $\mu^{-1}(0) \bigcap U$, let

$$J(\frac{\partial}{\partial x_j}) = b_{j,1}(x)\frac{\partial}{\partial x_1} + \cdots + b_{j,2n+1}(x)\frac{\partial}{\partial x_{2n+1}}, \quad j = 1, 2, \ldots, d.$$

Since we only work on $\mu^{-1}(0)$, $b_{j,k}(x)$ is independent of $x_{d+1}, \ldots, x_{2d}$, for all $j = 1, \ldots, d$, $k = 1, \ldots, 2n+1$. Moreover, it is easy to see that $b_{j,k}(x)$ is also independent of $x_1, \ldots, x_d$, for all $j = 1, \ldots, d$, $k = 1, \ldots, 2n+1$. Let $\widetilde{x}'' = (x_{2d+1}, \ldots, x_{2n+1})$.

Hence, $b_{j,k}(x) = b_{j,k}(\widetilde{x}'')$, $j = 1, \ldots, d$, $k = 1, \ldots, 2n+1$. We claim that the matrix $\big(b_{j,k}(\widetilde{x}'')\big)_{1\leq j\leq d, d+1\leq k\leq 2d}$ is non-singular near p. If not, it is easy to see that there is a non-zero vector $u \in J\underline{\mathfrak{g}} \bigcap HY$, where $Y = \mu^{-1}(0)$. Let $u = Jv$, $v \in \underline{\mathfrak{g}}$. Then, $v \in \underline{\mathfrak{g}} \bigcap JHY = \underline{\mathfrak{g}} \bigcap \underline{\mathfrak{g}}^{\perp_b}$ (see (2.10)). Since $\underline{\mathfrak{g}} \cap \underline{\mathfrak{g}}^{\perp_b} = \{0\}$ on $\mu^{-1}(0)$, we deduce that $v = 0$ and we get a contradiction. The claim follows. From the claim, we can use linear transformation in $(x_{d+1}, \ldots, x_{2d})$ (the linear transform depends smoothly on $\widetilde{x}''$) and we can take $(x_{d+1}, \ldots, x_{2d})$ such that on $\mu^{-1}(0)$,

$$
\begin{aligned}
J(\frac{\partial}{\partial x_j}) &= b_{j,1}(\widetilde{x}'')\frac{\partial}{\partial x_1} + \cdots + b_{j,d}(\widetilde{x}'')\frac{\partial}{\partial x_d} + \frac{\partial}{\partial x_{d+j}} \\
&+ b_{j,2d+1}(\widetilde{x}'')\frac{\partial}{\partial x_{2d+1}} + \cdots + b_{j,2n+1}(\widetilde{x}'')\frac{\partial}{\partial x_{2n+1}},
\end{aligned}
$$

where $j = 1, 2, \ldots, d$. Consider the coordinates change:

$$
\begin{aligned}
&x = (x_1, \ldots, x_{2n+1}) \to u = (u_1, \ldots, u_{2n+1}), \\
&(x_1, \ldots, x_{2n+1}) \to (x_1 - \textstyle\sum_{j=1}^d b_{j,1}(\widetilde{x}'')x_{d+j}, \ldots, x_d - \sum_{j=1}^d b_{j,d}(\widetilde{x}'')x_{d+j}, x_{d+1}, \ldots, x_{2d}, \\
&\quad x_{2d+1} - \textstyle\sum_{j=1}^d b_{j,2d+1}(\widetilde{x}'')x_{d+j}, \ldots, x_{2n+1} - \sum_{j=1}^d b_{j,2n+1}(\widetilde{x}'')x_{d+j}).
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\tfrac{\partial}{\partial x_j} \to \tfrac{\partial}{\partial u_j}, \quad j = 1, \ldots, d, 2d+1, \ldots, 2n+1, \\
&\tfrac{\partial}{\partial x_{d+j}} \to -b_{j,1}\tfrac{\partial}{\partial u_1} - \cdots - b_{j,d}\tfrac{\partial}{\partial u_d} + \tfrac{\partial}{\partial u_{d+j}} \\
&-b_{j,2d+1}\tfrac{\partial}{\partial u_{2d+1}} - \cdots - b_{j,2n+1}\tfrac{\partial}{\partial u_{2n+1}}, \quad j = 1, \ldots, d.
\end{aligned}
$$

Hence, on $\mu^{-1}(0) \bigcap U$, $J(\frac{\partial}{\partial x_j}) \to \frac{\partial}{\partial u_{d+j}}$, $j = 1, \ldots, d$. Thus, we can take $v = (v_1, \ldots, v_d)$ and $x = (x_1, \ldots, x_{2n+1})$ such that (3.8), (3.13), (3.14), (3.15) hold and on $\mu^{-1}(0) \bigcap U$,

$$
J(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial x_{d+j}}, \quad j = 1, 2, \ldots, d.
$$

Let $Z_j = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial x_{d+j}})(p) \in T_p^{1,0}X$, $j = 1, \ldots, d$. From (3.14), we can check that

$$
\mathcal{L}_p(Z_j, \overline{Z}_k) = \mu_j\delta_{j,k}, \quad \langle\, Z_j \,|\, Z_k \,\rangle = \delta_{j,k}, \quad j, k = 1, \ldots, d.
$$

Since $\underline{\mathfrak{g}}_p$ is orthogonal to $H_pY \bigcap JH_pY$ and $H_pY \bigcap JH_pY \subset \underline{\mathfrak{g}}_p^{\perp_b}$, we can find an orthonormal frame $\{Z_1, \ldots, Z_d, V_1, \ldots, V_{n-d}\}$ for $T_p^{1,0}X$ such that the Levi form $\mathcal{L}_p$ is diagonalized with respect to $Z_1, \ldots, Z_d, V_1, \ldots, V_{n-d}$, where $V_1 \in \mathbb{C}H_pY \bigcap J\mathbb{C}H_pY, \ldots, V_{n-d} \in \mathbb{C}H_pY \bigcap J\mathbb{C}H_pY$. Write

$$
\operatorname{Re} V_j = \sum_{k=1}^{2n+1}\alpha_{j,k}\frac{\partial}{\partial x_k}, \quad \operatorname{Im} V_j = \sum_{k=1}^{2n+1}\beta_{j,k}\frac{\partial}{\partial x_k}, \quad j = 1, \ldots, n-d.
$$

We claim that $\alpha_{j,k} = \beta_{j,k} = 0$, for all $k = d+1, \ldots, 2d$, $j = 1, \ldots, n-d$. Fix $j = 1, \ldots, n-d$. Since $\operatorname{Re} V_j \in \underline{\mathfrak{g}}_p^{\perp_b}$ and span $\left\{\frac{\partial}{\partial x_{d+1}}, \ldots, \frac{\partial}{\partial x_{2d}}\right\} \in \underline{\mathfrak{g}}_p^{\perp_b}$, we conclude that

$$
\sum_{k=1}^{d}\alpha_{j,k}\frac{\partial}{\partial x_k} + \sum_{k=2d+1}^{2n+1}\alpha_{j,k}\frac{\partial}{\partial x_k} \in \underline{\mathfrak{g}}_p^{\perp_b} \bigcap H_pY. \tag{3.16}
$$

From (2.10) and (3.16), we deduce that

$$\sum_{k=1}^{d}\alpha_{j,k}\frac{\partial}{\partial x_k}+\sum_{k=2d+1}^{2n+1}\alpha_{j,k}\frac{\partial}{\partial x_k}\in JH_pY\bigcap H_pY=\underline{\mathfrak{g}}_p^{\perp_b}\bigcap H_pY$$

and hence

$$J\Big(\sum_{k=1}^{d}\alpha_{j,k}\frac{\partial}{\partial x_k}+\sum_{k=2d+1}^{2n+1}\alpha_{j,k}\frac{\partial}{\partial x_k}\Big)\in \underline{\mathfrak{g}}_p^{\perp_b}\bigcap H_pY. \tag{3.17}$$

From (3.17) and notice that $J(\mathrm{Re}\, V_j)\in\underline{\mathfrak{g}}_p^{\perp_b}$, we deduce that

$$J(\sum_{k=d+1}^{2d}\alpha_{j,k}\frac{\partial}{\partial x_k})\in\underline{\mathfrak{g}}_p\bigcap\underline{\mathfrak{g}}_p^{\perp_b}=\{0\}\,.$$

Thus, $\alpha_{j,k}=0$, for all $k=d+1,\ldots,2d$, $j=1,\ldots,n-d$. Similarly, we can repeat the procedure above and deduce that $\beta_{j,k}=0$, for all $k=d+1,\ldots,2d$, $j=1,\ldots,n-d$.

Since span $\{\mathrm{Re}\, V_j, \mathrm{Im}\, V_j;\ j=1,\ldots,n-d\}$ is transversal to $\underline{\mathfrak{g}}_p\oplus J\underline{\mathfrak{g}}_p$, we can take linear transformation in $(x_{2d+1},\ldots,x_{2n+1})$ so that

$$\begin{aligned}\mathrm{Re}\, V_j&=\alpha_{j,1}\tfrac{\partial}{\partial x_1}+\cdots+\alpha_{j,d}\tfrac{\partial}{\partial x_d}+\tfrac{\partial}{\partial x_{2j-1+2d}},\quad j=1,2,\ldots,n-d,\\ \mathrm{Im}\, V_j&=\beta_{j,1}\tfrac{\partial}{\partial x_1}+\cdots+\beta_{j,d}\tfrac{\partial}{\partial x_d}+\tfrac{\partial}{\partial x_{2j+2d}},\quad j=1,2,\ldots,n-d.\end{aligned}$$

Consider the coordinates change:

$$\begin{aligned}x=(x_1,\ldots,x_{2n+1})&\to u=(u_1,\ldots,u_{2n+1}),\\ (x_1,\ldots,x_{2n+1})&\to (x_1-\textstyle\sum_{j=1}^d\alpha_{j,1}x_{2j-1+2d}-\sum_{j=1}^d\beta_{j,1}x_{2j+2d},\ldots,x_d\\ &\qquad-\textstyle\sum_{j=1}^d\alpha_{j,d}x_{2j-1+2d}-\sum_{j=1}^d\beta_{j,d}x_{2j+2d},x_{d+1},\ldots,x_{2n+1})\end{aligned}$$

Then,

$$\begin{aligned}\tfrac{\partial}{\partial x_j}&\to\tfrac{\partial}{\partial u_j},\quad j=1,\ldots,2d,\\ \tfrac{\partial}{\partial x_{2j-1+2d}}&\to-\alpha_{j,1}\tfrac{\partial}{\partial u_1}-\cdots-\alpha_{j,d}\tfrac{\partial}{\partial u_d}+\tfrac{\partial}{\partial u_{2j-1+2d}},\quad j=1,\ldots,n-d,\\ \tfrac{\partial}{\partial x_{2j+2d}}&\to-\beta_{j,1}\tfrac{\partial}{\partial u_1}-\cdots-\beta_{j,d}\tfrac{\partial}{\partial u_d}+\tfrac{\partial}{\partial u_{2j+2d}},\quad j=1,\ldots,n-d.\end{aligned}$$

Thus, we can take $v=(v_1,\ldots,v_d)$ and $x=(x_1,\ldots,x_{2n+1})$ such that (3.8), (3.9) and (3.10) hold.

Now, we can take linear transformation in x_{2n+1} so that $\omega_0(p)=dx_{2n+1}$. Let W_j, $j=1,\ldots,n$ be an orthonormal basis of $T^{1,0}X$ such that $W_j(p)=Z_j$, $j=1,\ldots,n$, where $Z_j\in T_p^{1,0}X$, $j=1,\ldots,n$, are as in (3.10). Let $\widetilde{x}=(\widetilde{x}_1,\ldots,\widetilde{x}_{2n+1})$ be the coordinates as in Theorem 3.5. It is easy to see that

$$\begin{aligned}&\widetilde{x}_j=x_j+a_jx_{2n+1}+h_j(x),\quad h_j(x)=O(|x|^2),\quad a_j\in\mathbb{R},\quad j=1,2,\ldots,2n,\\ &\widetilde{x}_{2n+1}=x_{2n+1}+h_{2n+1}(x),\quad h_{2n+1}(x)=O(|x|^2).\end{aligned} \tag{3.18}$$

We may change x_{2n+1} be $x_{2n+1}+h_{2n+1}(0,\ldots,0,x_{d+1},\ldots,x_{2n},0)$ and we have

$$\frac{\partial^2\widetilde{x}_{2n+1}}{\partial x_j\partial x_k}(p)=0,\quad j,k=\{d+1,\ldots,2n\}\,. \tag{3.19}$$

Note that when we change x_{2n+1} to $x_{2n+1}+h_{2n+1}(0,\ldots,0,x_{d+1},\ldots,x_{2n},0)$, $\frac{\partial}{\partial x_j}$ will change to $\frac{\partial}{\partial x_j}+\alpha_j(x)\frac{\partial}{\partial x_{2n+1}}$, $j=d+1,\ldots,2n$, where $\alpha_j(x)$ is a smooth function on

$\mu^{-1}(0)\bigcap U$, independent of $x_1,\ldots,x_d, x_{2n+1}$ and $\alpha_j(0)=0$, $j=d+1,\ldots,2n$. Hence, on $\mu^{-1}(0)\bigcap U$, we have $J(\frac{\partial}{\partial x_j})=\frac{\partial}{\partial x_{d+j}}+a_j(x)\frac{\partial}{\partial x_{2n+1}}$, $j=1,2,\ldots,d$, where $a_j(x)$ is a smooth function on $\mu^{-1}(0)\bigcap U$, independent of $x_1,\ldots,x_{2d}, x_{2n+1}$ and $a_j(0)=0$, $j=1,\ldots,d$.

From (3.4) and (3.18), it is straightforward to see that

$$\begin{aligned}&\omega_0(\widetilde{x})=(1+O(|\widetilde{x}|))d\widetilde{x}_{2n+1}+\textstyle\sum_{j=1}^d 2\mu_j\widetilde{x}_{d+j}d\widetilde{x}_j+\sum_{j=1}^d(-2\mu_j\widetilde{x}_j)d\widetilde{x}_{d+j}\\&\quad+\textstyle\sum_{j=d+1}^n 2\mu_j x_{2j}dx_{2j-1}+\sum_{j=d+1}^n(-2\mu_j\widetilde{x}_{2j-1})d\widetilde{x}_{2j}+\sum_{j=1}^{2n}\hat{b}_j\widetilde{x}_{2n+1}d\widetilde{x}_j+O(|x|^2)\\&=(1+O(|x|))dx_{2n+1}+\textstyle\sum_{j=1}^d(2\mu_jx_{d+j}+\frac{\partial\widetilde{x}_{2n+1}}{\partial x_j})dx_j+\sum_{j=1}^d(-2\mu_jx_j+\frac{\partial\widetilde{x}_{2n+1}}{\partial x_{d+j}})dx_{d+j}\\&\quad+\textstyle\sum_{j=d+1}^n(2\mu_jx_{2j}+\frac{\partial\widetilde{x}_{2n+1}}{\partial x_{2j-1}})dx_{2j-1}+\sum_{j=d+1}^n(-2\mu_jx_{2j-1}+\frac{\partial\widetilde{x}_{2n+1}}{\partial x_{2j}})dx_{2j}\\&\quad+\textstyle\sum_{j=1}^{2n}\widetilde{b}_jx_{2n+1}dx_j+O(|x|^2),\end{aligned}\tag{3.20}$$

where $\widetilde{b}_j\in\mathbb{R}$, $\hat{b}_j\in\mathbb{R}$, $j=1,\ldots,2n$. Note that ω_0 is G-invariant. From this observation and (3.20), we deduce that

$$\begin{aligned}&\tfrac{\partial^2\widetilde{x}_{2n+1}}{\partial x_j\partial x_k}(p)=0,\quad j\in\{1,\ldots,d\},k\in\{1,\ldots,d\}\bigcup\{2d+1,\ldots,2n\},\\&\tfrac{\partial^2\widetilde{x}_{2n+1}}{\partial x_{d+j}\partial x_k}(p)=2\mu_j\delta_{j,k},\quad j,k\in\{1,\ldots,d\}.\end{aligned}\tag{3.21}$$

From (3.21), (3.20) and (3.19), it is straightforward to see that

$$\begin{aligned}\omega_0(x)&=(1+O(|x|))dx_{2n+1}+\textstyle\sum_{j=1}^d 4\mu_jx_{d+j}dx_j\\&\quad+\textstyle\sum_{j=d+1}^n 2\mu_jx_{2j}dx_{2j-1}-\sum_{j=d+1}^n 2\mu_jx_{2j-1}dx_{2j}+\sum_{j=1}^{2n}b_jx_{2n+1}dx_j+O(|x|^2),\end{aligned}\tag{3.22}$$

where $b_1\in\mathbb{R},\ldots,b_{2n}\in\mathbb{R}$. Since $\omega_0(p)(\frac{\partial}{\partial x_j})=0$ on $x_{d+1}=\cdots=x_{2d}=0$, $j=1,2,\ldots,d$, we deduce that $b_1=\cdots=b_d=0$ and we get (3.11). The theorem follows. □

We need

Theorem 3.8 *Let $p\in\mu^{-1}(0)$ and take local coordinates $x=(x_1,\ldots,x_{2n+1})$ of X defined in an open set U of p with $0\leftrightarrow p$ such that (3.9), (3.10) and (3.11) hold. Let $\varphi_-(x,y)\in C^\infty(U\times U)$ be as in Theorem 3.3. Then,*

$$\begin{aligned}\varphi_-(x,y)&=-x_{2n+1}+y_{2n+1}-2\textstyle\sum_{j=1}^d\mu_jx_jx_{d+j}+2\sum_{j=1}^d\mu_jy_jy_{d+j}+i\sum_{j=1}^n|\mu_j||z_j-w_j|^2\\&\quad+\textstyle\sum_{j=1}^n i\mu_j(\overline{z}_jw_j-z_j\overline{w}_j)+\sum_{j=1}^d(-\frac{i}{2}b_{d+j})(-z_jx_{2n+1}+w_jy_{2n+1})\\&\quad+\textstyle\sum_{j=1}^d(\frac{i}{2}b_{d+j})(-\overline{z}_jx_{2n+1}+\overline{w}_jy_{2n+1})+\sum_{j=d+1}^n\frac{1}{2}(b_{2j-1}-ib_{2j})(-z_jx_{2n+1}+w_jy_{2n+1})\\&\quad+\textstyle\sum_{j=d+1}^n\frac{1}{2}(b_{2j-1}+ib_{2j})(-\overline{z}_jx_{2n+1}+\overline{w}_jy_{2n+1})+(x_{2n+1}-y_{2n+1})f(x,y)+O(|(x,y)|^3),\end{aligned}\tag{3.23}$$

where $z_j=x_j+ix_{d+j}$, $j=1,\ldots,d$, $z_j=x_{2j-1}+ix_{2j}$, $j=2d+1,\ldots,2n$, μ_j, $j=1,\ldots,n$, and $b_{d+1}\in\mathbb{R},\ldots,b_{2n}\in\mathbb{R}$ are as in (3.11) and f is smooth and satisfies $f(0,0)=0$, $f(x,y)=\overline{f}(y,x)$.

Proof Let $\widetilde{x}=(\widetilde{x}_1,\ldots,\widetilde{x}_{2n+1})$ be the coordinates as in Theorem 3.5. It is easy to see that

$$\begin{aligned}&\widetilde{x}_j=x_j+a_jx_{2n+1}+h_j(x),\quad h_j(x)=O(|x|^2),\quad a_j\in\mathbb{R},\quad j=1,2,\ldots,2n,\\&\widetilde{x}_{2n+1}=x_{2n+1}+h_{2n+1}(x),\quad h_{2n+1}(x)=O(|x|^2).\end{aligned}\tag{3.24}$$

From (3.4), it is straightforward to see that

$$\begin{aligned}&\omega_0(\widetilde{x})=(1+O(|\widetilde{x}|))d\widetilde{x}_{2n+1}+\textstyle\sum_{j=1}^d 2\mu_j\widetilde{x}_{d+j}d\widetilde{x}_j+\sum_{j=1}^d(-2\mu_j\widetilde{x}_j)d\widetilde{x}_{d+j}\\&\quad+\textstyle\sum_{j=d+1}^n 2\mu_jx_{2j}dx_{2j-1}+\sum_{j=d+1}^n(-2\mu_j\widetilde{x}_{2j-1})d\widetilde{x}_{2j}+\sum_{j=1}^{2n}\hat{b}_j\widetilde{x}_{2n+1}d\widetilde{x}_j+O(|x|^2),\end{aligned}\tag{3.25}$$

where

$$\hat{b}_j = c_j + \overline{c}_j, \quad j \in \{1, \ldots, d\} \bigcup \{2d+1, 2d+3, \ldots, 2n-1\},$$
$$\hat{b}_j = ic_j - i\overline{c}_j, \quad j \in \{d+1, \ldots, 2d\} \bigcup \{2d+2, \ldots, 2n\}.$$

From (3.25) and (3.11), it is not difficulty to see that (see also (3.20))

$$\begin{aligned} &\frac{\partial^2 \widetilde{x}_{2n+1}}{\partial x_j \partial x_k}(p) = 0, \quad j \in \{1, \ldots, d\}, k \in \{1, \ldots, d\}, \\ &\frac{\partial^2 \widetilde{x}_{2n+1}}{\partial x_j \partial x_k}(p) = 0, \quad j \in \{1, \ldots, 2n\}, k \in \{2d+1, \ldots, 2n\}, \\ &\frac{\partial^2 \widetilde{x}_{2n+1}}{\partial x_{d+j} \partial x_k}(p) = 2\mu_j \delta_{j,k}, \quad j,k \in \{1, \ldots, d\}. \end{aligned} \tag{3.26}$$

From (3.24), (3.26) and (3.6), it is straightforward to check that

$$\begin{aligned} \varphi_-(x,y) = &-x_{2n+1} + y_{2n+1} - 2\sum_{j=1}^d \mu_j x_j x_{d+j} + 2\sum_{j=1}^d \mu_j y_j y_{d+j} + i\sum_{j=1}^n |\mu_j||z_j - w_j|^2 \\ &+ \sum_{j=1}^n i\mu_j(\overline{z}_j w_j - z_j \overline{w}_j) + \sum_{j=1}^n \beta_j(-z_j x_{2n+1} + w_j y_{2n+1}) \\ &+ \sum_{j=1}^n \overline{\beta}_j(-\overline{z}_j x_{2n+1} + \overline{w}_j y_{2n+1}) + (x_{2n+1} - y_{2n+1}) f(x,y) + O(|(x,y)|^3), \end{aligned} \tag{3.27}$$

where $\beta_j \in \mathbb{C}$, $j = 1, \ldots, n$ and f is smooth and satisfies $f(0,0) = 0$, $f(x,y) = \overline{f}(y,x)$. We now determine β_j, $j = 1, \ldots, n$. We can compute that

$$\begin{aligned} &\frac{\partial \varphi_-}{\partial x_j}(x,x) = -4\mu_j x_{d+j} - (\beta_j + \overline{\beta}_j)x_{2n+1} + O(|x|^2), \quad j = 1, \ldots, d, \\ &\frac{\partial \varphi_-}{\partial x_{d+j}}(x,x) = -i(\beta_j - \overline{\beta}_j)x_{2n+1} + O(|x|^2), \quad j = 1, \ldots, d, \\ &\frac{\partial \varphi_-}{\partial x_{2j-1}}(x,x) = -2\mu_j x_{2j} - (\beta_j + \overline{\beta}_j)x_{2n+1} + O(|x|^2), \quad j = d+1, \ldots, n, \\ &\frac{\partial \varphi_-}{\partial x_{2j}}(x,x) = 2\mu_j x_{2j-1} + (-i\beta_j + i\overline{\beta}_j)x_{2n+1} + O(|x|^2), \quad j = d+1, \ldots, n. \end{aligned} \tag{3.28}$$

Note that $d_x\varphi_-(x,x) = -\omega_0(x)$. From this observation and (3.11), we deduce that

$$\begin{aligned} &\frac{\partial \varphi_-}{\partial x_j}(x,x) = -4\mu_j x_{d+j} + O(|x|^2), \quad j = 1, \ldots, d, \\ &\frac{\partial \varphi_-}{\partial x_{d+j}}(x,x) = -b_{d+j}x_{2n+1} + O(|x|^2), \quad j = 1, \ldots, d, \\ &\frac{\partial \varphi_-}{\partial x_{2j-1}}(x,x) = -2\mu_j x_{2j} - b_{2j-1}x_{2n+1} + O(|x|^2), \quad j = d+1, \ldots, n, \\ &\frac{\partial \varphi_-}{\partial x_{2j}}(x,x) = 2\mu_j x_{2j-1} - b_{2j}x_{2n+1} + O(|x|^2), \quad j = d+1, \ldots, n. \end{aligned} \tag{3.29}$$

From (3.28) and (3.29), we deduce that

$$\beta_j = -\frac{i}{2}b_{d+j}, \quad j = 1, \ldots, d, \quad \text{and} \quad \beta_j = \frac{1}{2}(b_{2j-1} - ib_{2j}), \quad j = d+1, \ldots, n. \tag{3.30}$$

From (3.30) and (3.27), we get (3.23). □

We now work with local coordinates as in Theorem 3.7. From (3.23), we see that near $(p,p) \in U \times U$, we have $\frac{\partial \varphi_-}{\partial y_{2n+1}} \neq 0$. Using the Malgrange preparation theorem [12, Th. 7.5.7], we have

$$\varphi_-(x,y) = g(x,y)(y_{2n+1} + \hat{\varphi}_-(x,\mathring{y})) \tag{3.31}$$

in some neighborhood of (p,p), where $\mathring{y} = (y_1, \ldots, y_{2n})$, $g, \hat{\varphi}_- \in C^\infty$. Since $\operatorname{Im} \varphi_- \geq 0$, it is not difficult to see that $\operatorname{Im} \hat{\varphi}_- \geq 0$ in some neighborhood of (p,p). We may take U small enough so that (3.31) holds and $\operatorname{Im} \hat{\varphi}_- \geq 0$ on $U \times U$. From [25, Th. 4.2], we see that since $\varphi_-(x,y)$ and $\hat{\varphi}_-(x,y)$ satisfy (3.31), $\varphi_-(x,y)t$ and $(y_{2n+1} + \hat{\varphi}_-(x,\mathring{y}))t$ are equivalent in the sense of Melin–Sjöstrand. More precisely, for any $k \in \mathbb{R}$ and any $b_1(x,y,t) \in S^k_{\mathrm{cl}}\big(U \times U \times$

$\mathbb{R}_+, T^{*0,q}X\boxtimes(T^{*0,q}X)^*\big)$, we can find $b_2(x,y,t)\in S^k_{\rm cl}\left(U\times U\times\mathbb{R}_+, T^{*0,q}X\boxtimes(T^{*0,q}X)^*\right)$ such that

$$\int_0^\infty e^{i\varphi_-(x,y)t}b_1(x,y,t)dt\equiv\int_0^\infty e^{i\hat{\varphi}_-(x,y)t}b_2(x,y,t)dt \ \text{ on } U$$

and vise versa. We can replace the phase φ_- by $y_{2n+1}+\hat{\varphi}_-(x,\mathring{y})$. From now on, we assume that $\varphi_-(x,y)$ has the form

$$\varphi_-(x,y)=y_{2n+1}+\hat{\varphi}_-(x,\mathring{y}). \tag{3.32}$$

It is easy to check that $\varphi_-(x,y)$ satisfies (3.5) and (3.23) with $f(x,y)=0$.

We now study $S^{(q)}_G(x,y)$. From Theorem 3.2, we get

Theorem 3.9 *Assume that $q\notin\{n_-,n_+\}$.Then, $S^{(q)}_G\equiv 0$ on X.*

Assume that $q=n_-$ and $\Box^{(q)}_b$ has L^2 closed range. Fix $p\in\mu^{-1}(0)$ and let $v=(v_1,\ldots,v_d)$ and $x=(x_1,\ldots,x_{2n+1})$ be the local coordinates of G and X as in Theorem 3.7. Assume that $d\mu=m(v)dv=m(v_1,\ldots,v_d)dv_1\cdots dv_d$ on V, where V is an open neighborhood of $e_0\in G$ as in Theorem 3.7. From (3.7), we have

$$S^{(q)}_G(x,y)=\int_G\chi(g)S^{(q)}(x,g\circ y)d\mu(g)+\int_G(1-\chi(g))S^{(q)}(x,g\circ y)d\mu(g),$$

where $\chi\in C^\infty_0(V)$, $\chi=1$ near e_0. Since G is freely on $\mu^{-1}(0)$, if U and V are small, there is a constant $c>0$ such that

$$d(x,g\circ y)\geq c,\ \ \forall x,y\in U, g\in\text{Supp}\,(1-\chi), \tag{3.33}$$

where U is an open set of $p\in\mu^{-1}(0)$ as in Theorem 3.7. From now on, we take U and V small enough so that (3.33) holds. In view of Theorem 3.3, we see that $S^{(q)}(x,y)$ is smoothing away from diagonal. From this observation and (3.33), we conclude that $\int_G(1-\chi(g))S^{(q)}(x,g\circ y)d\mu(g)\equiv 0$ on U and hence

$$S^{(q)}_G(x,y)\equiv\int_G\chi(g)S^{(q)}(x,g\circ y)d\mu(g) \text{ on } U. \tag{3.34}$$

From Theorem 3.3 and (3.34), we have

$$\begin{aligned}&S^{(q)}_G(x,y)\equiv\hat{S}_-(x,y)+\hat{S}_+(x,y)\text{ on }U,\\&\hat{S}_\mp(x,y)=\textstyle\int_G\chi(g)S_\mp(x,g\circ y)d\mu(g),\end{aligned} \tag{3.35}$$

Write $x=(x',x'')=(x',\hat{x}'',\widetilde{x}'')$, $y=(y',y'')=(y',\hat{y}'',\widetilde{y}'')$, where $\hat{x}''=(x_{d+1},\ldots,x_{2d})$, $\hat{y}''=(y_{d+1},\ldots,y_{2d})$, $\widetilde{x}''=(x_{2d+1},\ldots,x_{2n+1})$, $\widetilde{y}''=(y_{2d+1},\ldots,y_{2n+1})$. Since $S^{(q)}_G(x,y)$ is G-invariant, we have $S^{(q)}_G(x,y)=S^{(q)}_G((0,x''),(\gamma(y''),y''))$, where $\gamma\in C^\infty(U_2,U_1)$ is as in Theorem 3.7. From this observation and (3.35), we have

$$S^{(q)}_G(x,y)\equiv\hat{S}_-((0,x''),(\gamma(y''),y''))+\hat{S}_+((0,x''),(\gamma(y''),y''))\text{ on }U. \tag{3.36}$$

Write $\mathring{x}''=(x_{d+1},\ldots,x_{2n})$, $\mathring{y}''=(y_{d+1},\ldots,y_{2n})$ From (3.32), (3.36), Theorem 3.7 and Theorem 3.3, we have

$$\begin{aligned}&\hat{S}_-((0,x''),(\gamma(y''),y''))\\&\equiv\textstyle\int e^{i(y_{2n+1}+\hat{\varphi}_-((0,x''),(v+\gamma(y''),\mathring{y}'')))t}s_-((0,x''),(v+\gamma(y''),y''),t)m(v)dvdt.\end{aligned} \tag{3.37}$$

From (3.23), it is straightforward to see that

$$\det\left(\left(\frac{\partial^2\hat{\varphi}_-}{\partial v_k\partial v_j}(p,p)\right)_{j,k=1}^{d}\right)=(2i)^d|\mu_1|\cdots|\mu_d|\neq 0. \tag{3.38}$$

We pause and introduce some notations. Let W be an open set of $\mathbb{R}^N$, $N\in\mathbb{N}$. From now on, we write $W^{\mathbb{C}}$ to denote an open set in $\mathbb{C}^N$ with $W^{\mathbb{C}}\bigcap\mathbb{R}^N=W$ and for $f\in C^\infty(W)$, from now on, we write $\widetilde{f}\in C^\infty(W^{\mathbb{C}})$ to denote an almost analytic extension of f (see Section 2 in [25]). Let $h(x'',y'')\in C^\infty(U\times U,\mathbb{C}^d)$ be the solution of the system

$$\frac{\partial\widetilde{\hat{\varphi}_-}}{\partial\widetilde{y}_j}((0,x''),(h(x'',y'')+\gamma(y''),\mathring{y}''))=0,\quad j=1,2,\ldots,d, \tag{3.39}$$

and let

$$\Phi_-(x'',y''):=y_{2n+1}+\widetilde{\hat{\varphi}_-}((0,x''),(h(x'',y'')+\gamma(y''),\mathring{y}'')). \tag{3.40}$$

It is known that (see page 147 in [25]) $\mathrm{Im}\,\Phi_-(x'',y'')\geq 0$. Note that

$$\frac{\partial\hat{\varphi}_-}{\partial v_j}|_{\hat{x}''=\hat{y}''=0,\mathring{x}''=\mathring{y}'',x'=v+\gamma(y'')=0}=-\langle\,\omega_0(x)\,,\ \frac{\partial}{\partial x_j}\,\rangle=0,$$

where $x=(0,(0,\mathring{x}''))$. We deduce that for $\hat{x}''=\hat{y}''=0$, $\mathring{x}''=\mathring{y}''$, $v=-\gamma(y'')$ are real critical points. From this observation, we can calculate that

$$d_x\Phi_-|_{x''=y'',\hat{x}''=0}=-f(x'')\omega_0(x),\quad d_y\Phi_-|_{x''=y'',\hat{x}''=0}=f(x'')\omega_0(x), \tag{3.41}$$

where $x=(0,\mathring{x}'')$ and $f\in C^\infty$ is a positive function with $f(p)=1$. By using stationary phase formula of Melin–Sjöstrand [25], we can carry out the v integral in (3.37) and get

$$\hat{S}_-((0,x''),(\gamma(y''),y''))\equiv\int e^{i\Phi_-(x'',y'')t}a_-(x'',y'',t)dt\quad\text{on } U,$$

where $a_-(x'',y'',t)\sim\sum_{j=0}^{\infty}t^{n-\frac{d}{2}-j}a_-^0(x'',y'')$ in $S^{n-\frac{d}{2}}_{1,0}(U\times U\times\mathbb{R}_+,T^{*0,q}X\boxtimes(T^{*0,q}X)^*)$,

$$\begin{aligned}&a_-^j(x'',y'')\in C^\infty(U\times U,T^{*0,q}X\boxtimes(T^{*0,q}X)^*),\quad j=0,1,\ldots,\\&a_-^0(p,p)=\frac{1}{2}m(0)\pi^{-n-1+\frac{d}{2}}|\mu_1|^{\frac{1}{2}}\cdots|\mu_d|^{\frac{1}{2}}|\mu_{d+1}|\cdots|\mu_n|\tau_{p,n_-}.\end{aligned} \tag{3.42}$$

We now study the property of the phase $\Phi_-(x'',y'')$. We need the following which is known (see Section 2 in [25])

Theorem 3.10 *There exist a constant $c>0$ and an open set $\Omega\in\mathbb{R}^d$ such that*

$$\mathrm{Im}\,\Phi_-(x'',y'')\geq c\inf_{v\in\Omega}\left\{\mathrm{Im}\,\hat{\varphi}_-((0,x''),(v+\gamma(y''),\mathring{y}''))+|d_v\hat{\varphi}_-((0,x''),(v+\gamma(y''),\mathring{y}''))|^2\right\}, \tag{3.43}$$

for all $((0,x''),(0,y''))\in U\times U$.

We can now prove

Theorem 3.11 *If U is small enough, then there is a constant $c>0$ such that*

$$\mathrm{Im}\,\Phi_-(x'',y'')\geq c\Big(|\hat{x}''|^2+|\hat{y}''|^2+|\mathring{x}''-\mathring{y}''|^2\Big),\ \ \forall((0,x''),(0,y''))\in U\times U. \tag{3.44}$$

Proof From (3.5), we see that there is a constant $c_1 > 0$ such that

$$\operatorname{Im}\hat{\varphi}_-((0,x''),(v+\gamma(y''),\mathring{y}'')) \geq c_1(|v+\gamma(y'')|^2+|\mathring{x}''-\mathring{y}''|^2),\quad \forall v\in\Omega, \tag{3.45}$$

where Ω is any open set of $0 \in \mathbb{R}^d$. From (3.45) and (3.43), we conclude that there is a constant $c_2 > 0$ such that

$$\operatorname{Im}\Phi_-(x'',y'') \geq c_2(|\mathring{x}''-\mathring{y}''|^2+|d_{y'}\hat{\varphi}_-((0,x''),(0,\mathring{x}''))|^2). \tag{3.46}$$

From (3.23), we see that the matrix

$$\left(\frac{\partial^2\hat{\varphi}_-}{\partial x_j\partial x_k}(p,p)+\frac{\partial^2\hat{\varphi}_-}{\partial y_j\partial y_k}(p,p)\right)_{1\leq k\leq d,\, d+1\leq j\leq 2d}$$

is non-singular. From this observation and notice that $d_{y'}\hat{\varphi}_-((0,x''),(0,\mathring{x}''))|_{\hat{x}''}=0$, we deduce that if U is small enough then there is a constant $c_3 > 0$ such that

$$|d_{y'}\hat{\varphi}_-((0,x''),(0,x''))| \geq c_3|\hat{x}''|. \tag{3.47}$$

From (3.47) and (3.46), the theorem follows. □

From now on, we assume that U is small enough so that (3.44) holds.

We now determine the Hessian of $\Phi_-(x'',y'')$ at (p,p). Let $\hat{h}(x'',y'') := h(x'',y'')+\gamma(y'')$. From (3.39), we have

$$\frac{\partial^2\hat{\varphi}_-}{\partial x_{d+1}\partial y_1}(p,p)+\sum_{j=1}^{d}\frac{\partial^2\hat{\varphi}_-}{\partial y_1\partial y_j}(p,p)\frac{\partial\hat{h}_j}{\partial x_{d+1}}(p,p)=0. \tag{3.48}$$

From (3.23), we can calculate that

$$\frac{\partial^2\hat{\varphi}_-}{\partial x_{d+1}\partial y_1}(p,p)=2\mu_1,\quad \frac{\partial^2\hat{\varphi}_-}{\partial y_1\partial y_j}(p,p)=2i|\mu_1|\delta_{1,j},\quad j=1,2,\ldots,d. \tag{3.49}$$

From (3.49) and (3.48), we obtain $\frac{\partial\hat{h}_1}{\partial x_{d+1}}(p,p)=i\frac{\mu_1}{|\mu_1|}$. We can repeat the procedure above several times and deduce that

$$\frac{\partial\hat{h}_j}{\partial x_{d+k}}(p,p)=\frac{\partial\hat{h}_j}{\partial y_{d+k}}(p,p)=i\frac{\mu_j}{|\mu_j|}\delta_{j,k},\quad j,k=1,2,\ldots,d. \tag{3.50}$$

From (3.50), (3.23), (3.40) and by some straightforward computation (we omit the details), we get

Theorem 3.12 *With the notations used above, we have*

$$\begin{aligned}\Phi_-(x'',y'') = {} & -x_{2n+1}+y_{2n+1}+2i\textstyle\sum_{j=1}^{d}|\mu_j|y_{d+j}^2+2i\sum_{j=1}^{d}|\mu_j|x_{d+j}^2\\ & +i\textstyle\sum_{j=d+1}^{n}|\mu_j||z_j-w_j|^2+\sum_{j=d+1}^{n}i\mu_j(\overline{z}_jw_j-z_j\overline{w}_j)\\ & +\textstyle\sum_{j=1}^{d}(-b_{d+j}x_{d+j}x_{2n+1}+b_{d+j}y_{d+j}y_{2n+1})\\ & +\textstyle\sum_{j=d+1}^{n}\frac{1}{2}(b_{2j-1}-ib_{2j})(-z_jx_{2n+1}+w_jy_{2n+1})\\ & +\textstyle\sum_{j=d+1}^{n}\frac{1}{2}(b_{2j-1}+ib_{2j})(-\overline{z}_jx_{2n+1}+\overline{w}_jy_{2n+1})\\ & +(x_{2n+1}-y_{2n+1})f(x,y)+O(|(x,y)|^3),\end{aligned} \tag{3.51}$$

where $z_j=x_{2j-1}+ix_{2j}$, $j=2d+1,\ldots,2n$, μ_j, $j=1,\ldots,n$, *and* $b_{d+1}\in\mathbb{R},\ldots,b_{2n}\in\mathbb{R}$ *are as in* (3.11) *and* f *is smooth and satisfies* $f(0,0)=0$, $f(x,y)=\overline{f}(y,x)$.

We can change $\Phi_-(x'', y'')$ be $\Phi_-(x'', y'')\frac{1}{f(x'')}$, where $f(x'')$ is as in (3.41). Thus,

$$d_x\Phi_-|_{x''=y'',\hat{x}''=0} = -\omega_0(x), \quad d_y\Phi_-|_{x''=y'',\hat{x}''=0} = \omega_0(x), \tag{3.52}$$

where $x = (0, \widetilde{x}'')$. It is clear that $\Phi_-(x'', y'')$ still satisfies (3.44) and (3.51).

We now determine the leading term $a_-^0(p, p)$. In view of (3.42), we only need to calculate $m(0)$. Put $Y_p = \{g \circ p;\ g \in G\}$. Y_p is a d-dimensional submanifold of X. The G-invariant Hermitian metric $\langle \cdot | \cdot \rangle$ induces a volume form dv_{Y_p} on Y_p. Put

$$V_{\mathrm{eff}}(p) := \int_{Y_p} dv_{Y_p}.$$

For $f(g) \in C^\infty(G)$, let $\hat{f}(g \circ p) := f(g)$, $\forall g \in G$. Then, $\hat{f} \in C^\infty(Y_p)$. Let $d\hat{\mu}$ be the measure on G given by $\int_G f d\hat{\mu} := \int_{Y_p} \hat{f} dv_{Y_p}$, for all $f \in C^\infty(G)$. It is not difficult to see that $d\hat{\mu}$ is a Haar measure and

$$\int_G d\hat{\mu} = V_{\mathrm{eff}}(p). \tag{3.53}$$

Recall that we work with the local coordinates in Theorem 3.7. In view of (3.10), we see that $\left\{\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_1}, \ldots, \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_d}\right\}$ is an orthonormal basis for $\underline{\mathfrak{g}}_p$. Hence $m(0) = 2^{\frac{d}{2}}\frac{1}{V_{\mathrm{eff}}(p)}$. From this observation, (3.53) and (3.42), we get

$$a_-^0(p, p) = 2^{\frac{d}{2}-1}\frac{1}{V_{\mathrm{eff}}(p)}\pi^{-n-1+\frac{d}{2}}|\mu_1|^{\frac{1}{2}}\cdots|\mu_d|^{\frac{1}{2}}|\mu_{d+1}|\cdots|\mu_n|\tau_{p,n_-}. \tag{3.54}$$

Similarly, we can repeat the procedure above and deduce that

$$\hat{S}_+((0, x''), (\gamma(y''), y'')) \equiv \int e^{i\Phi_+(x'',y'')t}a_+(x'', y'', t)dt \quad \text{on } U,$$

where $a_+(x'', y'', t) \sim \sum_{j=0}^{\infty} t^{n-\frac{d}{2}-j}a_+^j(x'', y'')$ in $S_{1,0}^{n-\frac{d}{2}}(U \times U \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$,

$$\begin{aligned}
&a_+^j(x'', y'') \in C^\infty(U \times U, T^{*0,q}X \boxtimes (T^{*0,q}X)^*), \quad j = 0, 1, \ldots, \\
&a_+^0(p, p) = 2^{\frac{d}{2}-1}\frac{1}{V_{\mathrm{eff}}(p)}\pi^{-n-1+\frac{d}{2}}|\mu_1|^{\frac{1}{2}}\cdots|\mu_d|^{\frac{1}{2}}|\mu_{d+1}|\cdots|\mu_n|\tau_{p,n_+},
\end{aligned} \tag{3.55}$$

and $\Phi_+(x'', y'') \in C^\infty(U \times U)$, $\mathrm{Im}\,\Phi_+(x'', y'') \geq 0$, $-\overline{\Phi}_+(x'', y'')$ satisfies (3.44), (3.51) and (3.52).

Summing up, we get one of the main result of this work

Theorem 3.13 *We recall that we work with the assumption that the Levi form is non-degenerate of constant signature (n_-, n_+) on X. Let $q = n_-$ or n_+. Suppose that $\Box_b^{(q)}$ has L^2 closed range. Let $p \in \mu^{-1}(0)$ and let $x = (x_1, \ldots, x_{2n+1})$ be the local coordinates defined in an open set U of p such that $x(p) = 0$ and (3.8), (3.9), (3.10), (3.11) hold. Write $x'' = (x_{d+1}, \ldots, x_{2n+1})$. Then, there exist continuous operators $S_-^G, S_+^G : \Omega_0^{0,q}(U) \to \Omega^{0,q}(U)$ such that*

$$S_G^{(q)} \equiv S_-^G + S_+^G \quad \text{on } U,$$

and $S_-^G(x, y)$, $S_+^G(x, y)$ satisfy

$$S_\mp^G(x, y) \equiv \int_0^\infty e^{i\Phi_\mp(x'',y'')t}a_\mp(x'', y'', t)dt \quad \text{on } U,$$

with

$$a_-(x,y,t), a_+(x,y,t) \in S^{n-\frac{d}{2}}_{\mathrm{cl}}(U \times U \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*),$$
$$a^0_-(x,x) \neq 0, \ \ \forall x \in U, \ \ a^0_+(x,x) \neq 0, \ \ \forall x \in U,$$

$a^0_-(p,p)$ *and* $a^0_+(p,p)$ *are given by* (3.54) *and* (3.55) *respectively,* $\Phi_-(x'',y'') \in C^\infty(U \times U)$ *satisfies* (3.52), (3.44) *and* (3.51), $\Phi_+(x'',y'') \in C^\infty(U \times U)$, $-\overline{\Phi}_+(x'',y'')$ *satisfies* (3.52), (3.44) *and* (3.51).

3.4 *G*-invariant Szegő kernel asymptotics away $\mu^{-1}(0)$

The goal of this section is to prove the following

Theorem 3.14 *Let* D *be an open set of* X *with* $D \bigcap \mu^{-1}(0) = \emptyset$. *Then,* $S^{(q)}_G \equiv 0$ *on* D.

We first need

Lemma 3.15 *Let* $p \notin \mu^{-1}(0)$. *Then, there are open sets* U *of* p *and* V *of* $e \in G$ *such that for any* $\chi \in C^\infty_0(V)$, *we have*

$$\int_G S^{(q)}(x, g\circ y)\chi(g)d\mu(g) \equiv 0 \ \text{ on } U. \tag{3.56}$$

Proof If $q \notin \{n_-, n_+\}$. By Theorem 3.2, we get (3.56). We may assume that $q = n_-$. Take local coordinates $v = (v_1, \ldots, v_d)$ of G defined in a neighborhood V of e_0 with $v(e_0) = (0, \ldots, 0)$, local coordinates $x = (x_1, \ldots, x_{2n+1})$ of X defined in a neighborhood $U = U_1 \times U_2$ of p with $0 \leftrightarrow p$, where $U_1 \subset \mathbb{R}^d$ is an open set of $0 \in \mathbb{R}^d$, $U_2 \subset \mathbb{R}^{2n+1-d}$ is an open set of $0 \in \mathbb{R}^{2n+1-d}$, such that

$$\begin{aligned}
&(v_1, \ldots, v_d) \circ (\gamma(x_{d+1}, \ldots, x_{2n+1}), x_{d+1}, \ldots, x_{2n+1})\\
&= (v_1 + \gamma_1(x_{d+1}, \ldots, x_{2n+1}), \ldots, v_d + \gamma_d(x_{d+1}, \ldots, x_{2n+1}), x_{d+1}, \ldots, x_{2n+1}),\\
&\forall (v_1, \ldots, v_d) \in V, \ \ \forall (x_{d+1}, \ldots, x_{2n+1}) \in U_2,
\end{aligned}$$

and

$$\underline{\mathfrak{g}} = \operatorname{span}\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}\right\},$$

where $\gamma = (\gamma_1, \ldots, \gamma_d) \in C^\infty(U_2, U_1)$ with $\gamma(0) = 0 \in \mathbb{R}^d$. Note that we don't use the local coordinates in Theorem 3.7. It should be notice that G needs not act locally freely on near p, (3.33) need not be true. We can not use off-diagonal expansion for the Szegő kernel to get this lemma. We will use some properties of the phase φ_- and integrations by parts to obtain this lemma. From Theorem 3.3, we have

$$\int_G S^{(q)}(x, g\circ y)\chi(g)d\mu(g) \equiv \int_G S_-(x, g\circ y)\chi(g)d\mu(g) + \int_G S_+(x, g\circ y)\chi(g)d\mu(g) \text{ on } U. \tag{3.57}$$

From Theorem 3.3, we have

$$\begin{aligned}
&\int_G S_-(x, g\circ y)\chi(g)d\mu(g)\\
&\equiv \int e^{i(\varphi_-(x,(v+\gamma(y''),y''))t} s_-(x, (v+\gamma(y''), y''), t)\chi(v)m(v)dvdt,
\end{aligned}$$

where $y'' = (y_{d+1}, \ldots, y_{2n+1})$, $m(v)dv = d\mu|_V$. Since $p \notin \mu^{-1}(0)$ and notice that $d_y\varphi_-(x, x) = \omega_0(x, x)$, we deduce that if V and U are small then $d_v(\varphi_-(x, (v + \gamma(y''), y''))) \neq 0$, for every $v \in V$, $(x, y) \in U \times U$. Hence, by using integration by parts with respect to v, we get

$$\int_G S_-(x, g \circ y)\chi(g)d\mu(g) \equiv 0 \text{ on } U. \tag{3.58}$$

Similarly, we have

$$\int_G S_+(x, g \circ y)\chi(g)d\mu(g) \equiv 0 \text{ on } U. \tag{3.59}$$

From (3.57), (3.58) and (3.59), the lemma follows. □

Lemma 3.16 *Let $p \notin \mu^{-1}(0)$ and let $h \in G$. We can find open sets U of p and V of h such that for every $\chi \in C_0^\infty(V)$, we have $\int_G S^{(q)}(x, g \circ y)\chi(g)d\mu(g) \equiv 0$ on U.*

Proof Let U and V be open sets as in Lemma 3.15. Let $\hat{V} = hV$. Then, $\hat{V}$ is an open set of G. Let $\hat{\chi} \in C_0^\infty(\hat{V})$. We have

$$\begin{aligned}\int_G S^{(q)}(x, g \circ y)\hat{\chi}(g)d\mu(g) &= \int_G S^{(q)}(x, h \circ g \circ y)\hat{\chi}(h \circ g)d\mu(g)\\ &= \int_G S^{(q)}(x, g \circ y)\chi(g)d\mu(g),\end{aligned} \tag{3.60}$$

where $\chi(g) := \hat{\chi}(h \circ g) \in C_0^\infty(V)$. From (3.60) and Lemma 3.15, we deduce that

$$\int_G S^{(q)}(x, g \circ y)\hat{\chi}(g)d\mu(g) \equiv 0 \text{ on } U.$$

The lemma follows. □

Proof of Theorem 3.14 Fix $p \in D$. We need to show that $S_G^{(q)}$ is smoothing near p. Let $h \in G$. By Lemma 3.16, we can find open sets U_h of p and V_h of h such that for every $\chi \in C_0^\infty(V_h)$, we have

$$\int_G S^{(q)}(x, g \circ y)\chi(g)d\mu(g) \equiv 0 \text{ on } U_h. \tag{3.61}$$

Since G is compact, we can find open sets U_{h_j} and V_{h_j}, $j = 1, \ldots, N$, such that $G = \bigcup_{j=1}^N V_{h_j}$. Let $U = D \bigcap \left(\bigcap_{j=1}^N U_{h_j} \right)$ and let $\chi_j \in C_0^\infty(V_{h_j})$, $j = 1, \ldots, N$, with $\sum_{j=1}^N \chi_j = 1$ on G. From (3.61), we have

$$S_G^{(q)}(x, y) = \int_G S^{(q)}(x, g \circ y)d\mu(g) = \sum_{j=1}^N \int_G S^{(q)}(x, g \circ y)\chi_j(g)d\mu(g) \equiv 0 \text{ on } U.$$

The theorem follows. □

From Theorems 3.9, 3.13 and 3.14, we get Theorem 1.5.

4 G-invariant Szegő kernel asymptotics on CR manifolds wit S^1 action

Let $(X, T^{1,0}X)$ be a compact CR manifold of dimension $2n+1$, $n \geq 1$. We assume that X admits an S^1 action: $S^1 \times X \to X$. We write $e^{i\theta}$ to denote the S^1 action. Let $T \in C^\infty(X, TX)$ be the global real vector field induced by the S^1 action given by $(Tu)(x) = \frac{\partial}{\partial\theta}\left(u(e^{i\theta}\circ x)\right)|_{\theta=0}$, $u \in C^\infty(X)$. We recall

Definition 4.1 We say that the S^1 action $e^{i\theta}$ is CR if $[T, C^\infty(X, T^{1,0}X)] \subset C^\infty(X, T^{1,0}X)$ and the S^1 action is transversal if for each $x \in X$, $\mathbb{C}T(x) \oplus T^{1,0}_x X \oplus T^{0,1}_x X = \mathbb{C}T_x X$. Moreover, we say that the S^1 action is locally free if $T \neq 0$ everywhere. It should be mentioned that transversality implies locally free.

We assume now that $(X, T^{1,0}X)$ is a compact connected CR manifold with a transversal CR locally free S^1 action $e^{i\theta}$ and we let T be the global vector field induced by the S^1 action. Let $\omega_0 \in C^\infty(X, T^*X)$ be the global real one form determined by $\langle\, \omega_0\,,\, u\,\rangle = 0$, for every $u \in T^{1,0}X \oplus T^{0,1}X$, and $\langle\, \omega_0\,,\, T\,\rangle = -1$. Note that ω_0 and T satisfy (2.2). Assume that X admits a compact connected Lie group G action and the Lie group G acts on X preserving ω_0 and J. We recall that we work with Assumption 1.7.

We now assume that the Hermitian metric $\langle\,\cdot\,|\,\cdot\,\rangle$ on $\mathbb{C}TX$ is $G \times S^1$ invariant. Then the L^2 inner product $(\,\cdot\,|\,\cdot\,)$ on $\Omega^{0,q}(X)$ induced by $\langle\,\cdot\,|\,\cdot\,\rangle$ is $G \times S^1$-invariant. We then have

$$\begin{aligned}&Tg^*\overline{\partial}^*_b = g^*T\overline{\partial}^*_b = \overline{\partial}^*_b g^*T = \overline{\partial}^*_b Tg^* \ \text{ on } \Omega^{0,q}(X),\ \ \forall g \in G,\\ &Tg^*\Box^{(q)}_b = g^*T\Box^{(q)}_b = \Box^{(q)}_b g^*T = \Box^{(q)}_b Tg^* \ \text{ on } \Omega^{0,q}(X),\ \ \forall g \in G.\end{aligned}$$

Let $L^2_{(0,q),m}(X)^G$ be the completion of $\Omega^{0,q}_m(X)^G$ with respect to $(\,\cdot\,|\,\cdot\,)$. We write $L^2_m(X)^G := L^2_{(0,0),m}(X)^G$. Put $(\mathrm{Ker}\,\Box^{(q)}_b)^G_m := (\mathrm{Ker}\,\Box^{(q)}_b)^G \bigcap L^2_{(0,q),m}(X)^G$. It is not difficult to see that, for every $m \in \mathbb{Z}$, $(\mathrm{Ker}\,\Box^{(q)}_b)^G_m \subset \Omega^{0,q}_m(X)^G$ and $\dim\,(\mathrm{Ker}\,\Box^{(q)}_b)^G_m < \infty$. The mth G-invariant Szegő projection is the orthogonal projection $S^{(q)}_{G,m} : L^2_{(0,q)}(X) \to (\mathrm{Ker}\,\Box^{(q)}_b)^G_m$ with respect to $(\,\cdot\,|\,\cdot\,)$. Let $S^{(q)}_{G,m}(x, y) \in C^\infty(X \times X, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ be the distribution kernel of $S^{(q)}_{G,m}$. We can check that

$$S^{(q)}_{G,m}(x, y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S^{(q)}_G(x, e^{i\theta}\circ y)e^{im\theta}d\theta. \tag{4.1}$$

The goal of this section is to study the asymptotics of $S^{(q)}_{G,m}$ as $m \to +\infty$.

From Theorem 3.14, (4.1) and by using integration by parts several times, we get

Theorem 4.2 *Let $D \subset X$ be an open set with $D \bigcap \mu^{-1}(0) = \emptyset$. Then, $S^{(q)}_{G,m} = O(m^{-\infty})$ on D.*

We now study $S^{(q)}_{G,m}$ near $\mu^{-1}(0)$. We can repeat the proof of Theorem 3.7 with minor change and get

Theorem 4.3 *Let $p \in \mu^{-1}(0)$. There exist local coordinates $v = (v_1, \ldots, v_d)$ of G defined in a neighborhood V of e_0 with $v(e_0) = (0, \ldots, 0)$, local coordinates $x = (x_1, \ldots, x_{2n+1})$ of X defined in a neighborhood $U = U_1 \times (\hat{U}_2 \times]-2\delta, 2\delta[)$ of p with $0 \leftrightarrow p$, where $U_1 \subset \mathbb{R}^d$ is an open set of $0 \in \mathbb{R}^d$, $\hat{U}_2 \subset \mathbb{R}^{2n-d}$ is an open set of $0 \in \mathbb{R}^{2n-d}$, $\delta > 0$, and a smooth function $\gamma = (\gamma_1, \ldots, \gamma_d) \in C^\infty(\hat{U}_2 \times]-2\delta, 2\delta[, U_1)$ with $\gamma(0) = 0 \in \mathbb{R}^d$ such that*

$$\begin{aligned}&(v_1, \ldots, v_d) \circ (\gamma(x_{d+1}, \ldots, x_{2n+1}), x_{d+1}, \ldots, x_{2n+1})\\ &= (v_1 + \gamma_1(x_{d+1}, \ldots, x_{2n+1}), \ldots, v_d + \gamma_d(x_{d+1}, \ldots, x_{2n+1}), x_{d+1}, \ldots, x_{2n+1}),\\ &\forall (v_1, \ldots, v_d) \in V,\ \ \forall (x_{d+1}, \ldots, x_{2n+1}) \in \hat{U}_2 \times]-2\delta, 2\delta[,\end{aligned}$$

$$\begin{aligned}&T=-\frac{\partial}{\partial x_{2n+1}},\quad \underline{\mathfrak{g}}=\operatorname{span}\left\{\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_d}\right\},\\&\mu^{-1}(0)\bigcap U=\{x_{d+1}=\cdots=x_{2d}=0\},\\&\textit{On } \mu^{-1}(0)\bigcap U, \textit{ we have } J(\tfrac{\partial}{\partial x_j})=\frac{\partial}{\partial x_{d+j}}+a_j(x)\frac{\partial}{\partial x_{2n+1}},\ j=1,2,\ldots,d,\end{aligned}\tag{4.2}$$

where $a_j(x)$ *is a smooth function on* $\mu^{-1}(0)\bigcap U$*, independent of* $x_1,\ldots,x_{2d}$*,* x_{2n+1} *and* $a_j(0)=0$*,* $j=1,\ldots,d$*,*

$$\begin{aligned}&T_p^{1,0}X=\operatorname{span}\{Z_1,\ldots,Z_n\},\\&Z_j=\tfrac{1}{2}(\tfrac{\partial}{\partial x_j}-i\tfrac{\partial}{\partial x_{d+j}})(p),\quad j=1,\ldots,d,\\&Z_j=\tfrac{1}{2}(\tfrac{\partial}{\partial x_{2j-1}}-i\tfrac{\partial}{\partial x_{2j}})(p),\quad j=d+1,\ldots,n,\\&\langle Z_j\,|\,Z_k\rangle=\delta_{j,k},\quad j,k=1,2,\ldots,n,\\&\mathcal{L}_p(Z_j,\overline{Z}_k)=\mu_j\delta_{j,k},\quad j,k=1,2,\ldots,n\end{aligned}$$

and

$$\begin{aligned}\omega_0(x)=&(1+O(|x|))dx_{2n+1}+\textstyle\sum_{j=1}^d 4\mu_jx_{d+j}dx_j\\&+\textstyle\sum_{j=d+1}^n 2\mu_jx_{2j}dx_{2j-1}-\sum_{j=d+1}^n 2\mu_jx_{2j-1}dx_{2j}+O(|x|^2).\end{aligned}$$

Remark 4.4 Let $p\in\mu^{-1}(0)$ and let $x=(x_1,\ldots,x_{2n+1})$ be the local coordinates as in Theorem 4.3. We can change x_{2n+1} be $x_{2n+1}-\sum_{j=1}^d a_j(x)x_{d+j}$, where $a_j(x)$, $j=1,\ldots,d$, are as in (4.2). With this new local coordinates $x=(x_1,\ldots,x_{2n+1})$, on $\mu^{-1}(0)\bigcap U$, we have $J(\frac{\partial}{\partial x_j})=\frac{\partial}{\partial x_{d+j}}$, $j=1,2,\ldots,d$. Moreover, it is clear that $\Phi_-(x,y)+\sum_{j=1}^d a_j(x)x_{d+j}-\sum_{j=1}^{d=1} a_j(y)y_{d+j}$ satisfies (1.19). Note that $a_j(x)$ is a smooth function on $\mu^{-1}(0)\bigcap U$, independent of $x_1,\ldots,x_{2d},x_{2n+1}$ and $a_j(0)=0$, $j=1,\ldots,d$.

We now work with local coordinates as in Theorem 4.3. From (3.51), we see that near $(p,p)\in U\times U$, we have $\frac{\partial\Phi_-}{\partial y_{2n+1}}\neq 0$. Using the Malgrange preparation theorem [12, Th. 7.5.7], we have

$$\Phi_-(x,y)=g(x,y)(y_{2n+1}+\hat{\Phi}_-(x'',\mathring{y}''))\tag{4.3}$$

in some neighborhood of (p,p), where $\mathring{y}''=(y_{d+1},\ldots,y_{2n})$, $g,\hat{\Phi}_-\in C^\infty$. Since $\operatorname{Im}\Phi_-\geq 0$, it is not difficult to see that $\operatorname{Im}\hat{\Phi}_-\geq 0$ in some neighborhood of (p,p). We may take U small enough so that (4.3) holds and $\operatorname{Im}\hat{\Phi}_-\geq 0$ on $U\times U$. From [25, Th. 4.2], we see that since $\Phi_-(x,y)$ and $y_{2n+1}+\hat{\Phi}_-(x'',\mathring{y}'')$ satisfy (4.3), $\Phi_-(x,y)t$ and $(y_{2n+1}+\hat{\Phi}_-(x'',\mathring{y}''))t$ are equivalent in the sense of Melin–Sjöstrand (see the discussion after (3.31), for the meaning of equivalent in the sense of Melin–Sjöstrand). We can replace the phase Φ_- by $y_{2n+1}+\hat{\Phi}_-(x,\mathring{y}'')$. From now on, we assume that

$$\Phi_-(x,y)=y_{2n+1}+\hat{\Phi}_-(x'',\mathring{y}'').\tag{4.4}$$

It is easy to check that $\Phi_-(x,y)$ satisfies (3.44) and (3.51) with $f(x,y)=0$. Similarly, from now on, we assume that

$$\Phi_+(x,y)=-y_{2n+1}+\hat{\Phi}_+(x'',\mathring{y}'').\tag{4.5}$$

We now study $S^{(q)}_{G,m}(x,y)$. From Theorem 3.9, we get

Theorem 4.5 *Assume that* $q\notin\{n_-,n_+\}$*. Then,* $S^{(q)}_{G,m}=O(m^{-\infty})$ *on* X*.*

Assume that $q=n_-$. It was proved in Theorem 1.12 in [15] that when X admits a transversal S^1 action, then $\Box_b^{(q)}$ has L^2 closed range. Fix $p\in\mu^{-1}(0)$ and let $v=(v_1,\ldots,v_d)$

and $x = (x_1, \ldots, x_{2n+1})$ be the local coordinates of G and X as in Theorem 4.3 and let U and V be open sets as in Theorem 4.3. We take U small enough so that there is a constant $c > 0$ such that

$$d(e^{i\theta} \circ g \circ x, y) \geq c, \quad \forall (x, y) \in U \times U, \quad \forall g \in G, \theta \in [-\pi, -\delta] \bigcup [\delta, \pi], \tag{4.6}$$

where $\delta > 0$ is as in Theorem 4.3. We will study $S^{(q)}_{G,m}(x, y)$ in U. From (4.1), we have

$$\begin{aligned} S^{(q)}_{G,m}(x, y) &= \tfrac{1}{2\pi} \int_{-\pi}^{\pi} S^{(q)}_G(x, e^{i\theta} \circ y) e^{im\theta} d\theta \\ &= \tfrac{1}{2\pi} \int_{-\pi}^{\pi} e^{-imx_{2n+1}+imy_{2n+1}} S^{(q)}_G(\mathring{x}, e^{i\theta} \circ \mathring{y}) e^{im\theta} d\theta \\ &= I + II, \\ I &= \tfrac{1}{2\pi} \int_{-\pi}^{\pi} e^{-imx_{2n+1}+imy_{2n+1}} \chi(\theta) S^{(q)}_G(\mathring{x}, e^{i\theta} \circ \mathring{y}) e^{im\theta} d\theta, \\ II &= \tfrac{1}{2\pi} \int_{-\pi}^{\pi} e^{-imx_{2n+1}+imy_{2n+1}} (1 - \chi(\theta)) S^{(q)}_G(\mathring{x}, e^{i\theta} \circ \mathring{y}) e^{im\theta} d\theta, \end{aligned}$$

where $\mathring{x} = (x_1, \ldots, x_{2n}, 0) \in U$, $\mathring{y} = (y_1, \ldots, y_{2n}, 0) \in U$, $\chi \in C^\infty_0(]-2\delta, 2\delta[)$, $\chi = 1$ on $[-\delta, \delta]$. We first study II. We have

$$II = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_G e^{-imx_{2n+1}+imy_{2n+1}} (1 - \chi(\theta)) S^{(q)}(\mathring{x}, e^{i\theta} \circ g \circ \mathring{y}) e^{im\theta} d\mu(g) d\theta. \tag{4.7}$$

From (4.7), (4.6) and notice that $S^{(q)}$ is smoothing away from diagonal, we deduce that

$$II = O(m^{-\infty}).$$

We now study I. From Theorem 3.13, (4.1), (4.4) and (4.5), we have

$$\begin{aligned} I &= I_0 + I_1, \\ I_0 &= \tfrac{1}{2\pi} \int_0^\infty \int_{-\pi}^{\pi} e^{-imx_{2n+1}+imy_{2n+1}} \chi(\theta) e^{i(-\theta + \hat{\Phi}_-(\mathring{x}'', \mathring{y}''))t + im\theta} a_-(\mathring{x}'', (\mathring{y}'', -\theta), t) dt d\theta, \\ I_1 &= \tfrac{1}{2\pi} \int_0^\infty \int_{-\pi}^{\pi} e^{-imx_{2n+1}+imy_{2n+1}} \chi(\theta) e^{i(\theta + \hat{\Phi}_+(\mathring{x}'', \mathring{y}''))t + im\theta} a_+(\mathring{x}'', (\mathring{y}'', -\theta), t) dt d\theta. \end{aligned}$$

We first study I_1. From $\frac{\partial}{\partial\theta}\Big(i(\theta + \hat{\Phi}_+(\mathring{x}'', \mathring{y}''))t + im\theta\Big) \neq 0$, we can integrate by parts with respect to θ several times and deduce that

$$I_1 = O(m^{-\infty}).$$

We now study I_0. We have

$$I_0 = \frac{1}{2\pi} \int_0^\infty \int_{-\pi}^{\pi} e^{-imx_{2n+1}+imy_{2n+1}} \chi(\theta) e^{im(-\theta t + \hat{\Phi}_-(\mathring{x}'', \mathring{y}''))t + \theta)} m a_-(\mathring{x}'', (\mathring{y}'', -\theta), mt) dt d\theta. \tag{4.8}$$

We can use the complex stationary phase formula of Melin–Sjöstrand [25, Theorem 2.3] to carry the $dtd\theta$ integration in (4.8) and get (the calculation is similar as in the proof of Theorem 3.17 in [14], we omit the details)

$$\begin{aligned} & I_0 = e^{im\Psi(x,y)} b(\mathring{x}'', \mathring{y}'', m) + O(m^{-\infty}), \\ & \Psi(x, y) = \hat{\Phi}_-(\mathring{x}'', \mathring{y}'') - x_{2n+1} + y_{2n+1}, \\ & b(\mathring{x}'', \mathring{y}'', m) \in S^{n-\frac{d}{2}}_{\mathrm{loc}}(1; U \times U, T^{*0,q}X \boxtimes (T^{*0,q}X)^*), \\ & b(\mathring{x}'', \mathring{y}'', m) \sim \textstyle\sum_{j=0}^\infty m^{n-\frac{d}{2}-j} b_j(\mathring{x}'', \mathring{y}'') \text{ in } S^{n-\frac{d}{2}}_{\mathrm{loc}}(1; U \times U, T^{*0,q}X \boxtimes (T^{*0,q}X)^*), \\ & b_j(\mathring{x}'', \mathring{y}'') \in C^\infty(U \times U, T^{*0,q}X \boxtimes (T^{*0,q}X)^*), \quad j = 0, 1, 2, \ldots, \\ & b_0(p, p) = a^0_-(p, p) = 2^{\frac{d}{2}-1} \tfrac{1}{V_{\mathrm{eff}}(p)} \pi^{-n-1+\frac{d}{2}} |\mu_1|^{\frac{1}{2}} \cdots |\mu_d|^{\frac{1}{2}} |\mu_{d+1}| \cdots |\mu_n| \tau_{p,n_-}. \end{aligned}$$

Assume that $q = n_+ \neq n_-$. If $m \to -\infty$, then the expansion for $S^{(q)}_{G,m}(x, y)$ as $m \to -\infty$ is similar to $q = n_-$ case. When $m \to +\infty$, we can repeat the method above with minor change and deduce that $S^{(q)}_{G,m}(x, y) = O(m^{-\infty})$ on X. Summing up, we get Theorem 1.8.

5 Equivalent of the phase function $\Phi_-(x, y)$

Let $p \in \mu^{-1}(0)$ and let U be a small open set of p. We need

Definition 5.1 With the assumptions and notations used in Theorem 3.13, let $\Phi_1, \Phi_2 \in C^\infty(U \times U)$. We assume that Φ_1 and Φ_2 satisfy (3.52), (3.51) and (3.44). We say that Φ_1 and Φ_2 are equivalent on U if for any $b_1(x, y, t) \in S^{n-\frac{d}{2}}_{\mathrm{cl}}\left(U \times U \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*\right)$ we can find $b_2(x, y, t) \in S^{n-\frac{d}{2}}_{\mathrm{cl}}\left(U \times U \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*\right)$ such that

$$\int_0^\infty e^{i\Phi_1(x,y)t} b_1(x, y, t)dt \equiv \int_0^\infty e^{i\Phi_2(x,y)t} b_2(x, y, t)dt \ \text{ on } U$$

and vise versa.

We characterize now the phase Φ_-.

Theorem 5.2 *Let $\Phi_-(x, y) \in C^\infty(U \times U)$ be as in Theorem 3.13. Let $\Phi \in C^\infty(U \times U)$. We assume that Φ satisfies (3.52), (3.51) and (3.44). The functions Φ and Φ_- are equivalent on U in the sense of Definition 5.1 if and only if there is a function $f \in C^\infty(U \times U)$ with $f(x, x) = 1$ such that $\Phi(x, y) - f(x, y)\Phi_-(x, y)$ vanishes to infinite order at* $\mathrm{diag}\left((\mu^{-1}(0) \bigcap U) \times (\mu^{-1}(0) \bigcap U)\right)$.

Proof The "$\Leftarrow$" part follows from global theory of complex Fourier integral operator of Melin–Sjöstrand [25]. We only need to prove the "$\Rightarrow$" part. Take local coordinates $x = (x_1, \ldots, x_{2n+1})$ defined in some small neighbourhood of p such that $x(p) = 0$ and $\omega_0(p) = dx_{2n+1}$. Since $d_y\Phi(x, y)|_{x=y\in\mu^{-1}(0)} = d_y\Phi_-(x, y)|_{x=y\in\mu^{-1}(0)} = \omega_0(x)$, we have $\frac{\partial\Phi}{\partial y_{2n+1}}(p, p) = \frac{\partial\Phi_-}{\partial y_{2n+1}}(p, p) = 1$. From this observation and the Malgrange preparation theorem [12, Theorem 7.5.7], we conclude that in some small neighborhood of (p, p), we can find $f(x, y), f_1(x, y) \in C^\infty$ such that

$$\Phi_-(x, y) = f(x, y)(y_{2n+1} + h(x, \mathring{y})), \quad \Phi(x, y) = f_1(x, y)(y_{2n+1} + h_1(x, \mathring{y})) \tag{5.1}$$

in some small neighborhood of (p, p), where $\mathring{y} = (y_1, \ldots, y_{2n})$. For simplicity, we assume that (5.1) hold on $U \times U$. It is clear that $\Phi_-(x, y)$ and $y_{2n-1} + h(x, \mathring{y})$ are equivalent in the sense of Definition 5.1, $\Phi(x, y)$ and $y_{2n+1} + h_1(x, \mathring{y})$ are equivalent in the sense of Definition 5.1, we may assume that $\Phi_-(x, y) = y_{2n+1} + h(x, \mathring{y})$ and $\Phi(x, y) = y_{2n+1} + h_1(x, \mathring{y})$. Fix $x_0 \in \mu^{-1}(0) \bigcap U$. We are going to prove that $h(x, \mathring{y}) - h_1(x, \mathring{y})$ vanishes to infinite order at $(x_0, x_0) \in (\mu^{-1}(0) \bigcap U) \times (\mu^{-1}(0) \bigcap U)$. Take

$$b(x, y, t) \sim \sum_{j=0}^\infty b_j(x, y)t^{n-\frac{d}{2}-j} \in S^{n-\frac{d}{2}}_{\mathrm{cl}}\left(U \times U \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*\right)$$

with $b_0(x, x) \neq 0$ at each $x \in U \bigcap \mu^{-1}(0)$. Since Φ and Φ_- are equivalent on U in the sense of Definition 5.1, we can find $a(x, y, t) \in S^{n-\frac{d}{2}}_{\mathrm{cl}}\left(U \times U \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*\right)$

such that

$$\int_0^\infty e^{i\Phi_-(x,y)t}b(x,y,t)dt \equiv \int_0^\infty e^{i\Phi(x,y)t}a(x,y,t)dt \ \text{ on } U.$$

Put $x_0 = (x_0^1, x_0^2, \ldots, x_0^{2n+1})$ and $\mathring{x}_0 = (x_0^1, \ldots, x_0^{2n})$. Take $\tau \in C_0^\infty(\mathbb{R}^{2n+1})$, $\tau_1 \in C_0^\infty(\mathbb{R}^{2n})$, $\chi \in C_0^\infty(\mathbb{R})$ so that $\tau = 1$ near x_0, $\tau_1 = 1$ near $\mathring{x}_0$, $\chi = 1$ near x_0^{2n+1} and $\operatorname{Supp}\tau \Subset U$, $\operatorname{Supp}\tau_1 \times \operatorname{Supp}\chi \Subset U' \times \operatorname{Supp}\chi \Subset U$, where U' is an open neighborhood of $\mathring{x}_0$ in $\mathbb{R}^{2n}$. For each $k > 0$, we consider the distributions

$$\begin{aligned}
&A_k : u \mapsto \int_0^\infty e^{i(y_{2n-1}+h(x,\mathring{y}))t-iky_{2n+1}}\tau(x)b(x,y,t)\tau_1(\mathring{y})\chi(y_{2n+1})u(\mathring{y})dydt,\\
&B_k : u \mapsto \int_0^\infty e^{i(y_{2n+1}+h_1(x,\mathring{y}))t-iky_{2n+1}}\tau(x)a(x,y,t)\tau_1(\mathring{y})\chi(y_{2n+1})u(\mathring{y})dydt,
\end{aligned}$$

for $u \in C_0^\infty(U', T^{*0,q}X)$. By using the stationary phase formula of Melin–Sjöstrand [25], we can show that (cf. the proof of [14, Theorem 3.12]) A_k and B_k are smoothing operators and

$$\begin{aligned}
&A_k(x,\mathring{y}) \equiv e^{ikh(x,\mathring{y})}g(x,\mathring{y},k) + O(k^{-\infty}),\\
&B_k(x,\mathring{y}) \equiv e^{ikh_1(x,\mathring{y})}p(x,\mathring{y},k) + O(k^{-\infty}),\\
&g(x,\mathring{y},k),\ p(x,\mathring{y},k) \in S^{n-\frac{d}{2}}_{\mathrm{loc}}(1; U\times U', T^{*0,q}X \boxtimes (T^{*0,q}X)^*),\\
&g(x,\mathring{y},k) \sim \sum_{j=0}^\infty g_j(x,\mathring{y})k^{n-\frac{d}{2}-j} \text{ in } S^{n-\frac{d}{2}}_{\mathrm{loc}}(1; U\times U', T^{*0,q}X \boxtimes (T^{*0,q}X)^*),\\
&p(x,\mathring{y},k) \sim \sum_{j=0}^\infty p_j(x,y')k^{n-\frac{d}{2}-j} \text{ in } S^{n-\frac{d}{2}}_{\mathrm{loc}}(1; U\times U', T^{*0,q}X \boxtimes (T^{*0,q}X)^*),\\
&g_j(x,\mathring{y}),\ p_j(x,\mathring{y}) \in C^\infty(U\times U', T^{*0,q}X \boxtimes (T^{*0,q}X)^*),\quad j=0,1,\ldots,\\
&g_0(x_0,\mathring{x}_0) \neq 0.
\end{aligned}$$

Since

$$\int_0^\infty e^{i(y_{2n+1}+h(x,\mathring{y}))t}b(x,y,t)dt - \int_0^\infty e^{i(y_{2n+1}+h_1(x,\mathring{y}))t}a(x,y,t)dt$$

is smoothing, by using integration by parts with respect to y_{2n+1}, it is easy to see that $A_k - B_k = O(k^{-\infty})$ (see [14, Section 3]). Thus,

$$\begin{aligned}
&e^{ikh(x,\mathring{y})}g(x,\mathring{y},k) = e^{ikh_1(x,\mathring{y})}p(x,\mathring{y},k) + F_k(x,\mathring{y}),\\
&F_k(x,\mathring{y}') = O(k^{-\infty}).
\end{aligned} \tag{5.2}$$

Now, we are ready to prove that $h(x,\mathring{y}) - h_1(x,\mathring{y})$ vanishes to infinite order at $(x_0,\mathring{x}_0)$. We assume that there exist $\alpha_0 \in \mathbb{N}_0^{2n+1}$, $\beta_0 \in \mathbb{N}_0^{2n}$, $|\alpha_0| + |\beta_0| \ge 1$ such that

$$|\partial_x^{\alpha_0}\partial_{\mathring{y}}^{\beta_0}(ih(x,\mathring{y}) - ih_1(x,\mathring{y}))|_{(x_0,\mathring{x}_0)} = C_{\alpha_0,\beta_0} \neq 0$$

and

$$|\partial_x^{\alpha}\partial_{\mathring{y}}^{\beta}(ih(x,\mathring{y}) - ih_1(x,\mathring{y}))|_{(x_0,\mathring{x}_0)} = 0 \ \text{ if } |\alpha| + |\beta| < |\alpha_0| + |\beta_0|.$$

From (5.2), we have

$$\begin{aligned}
&|\partial_x^{\alpha_0}\partial_{\mathring{y}}^{\beta_0}\Big(e^{ikh(x,\mathring{y})-ikh_1(x,\mathring{y})}g(x,\mathring{y},k) - p(x,\mathring{y},k)\Big)|_{(x_0,\mathring{x}_0)}\\
&= -|\partial_x^{\alpha_0}\partial_{\mathring{y}}^{\beta_0}\Big(e^{-ikh_1(x,\mathring{y})}F_k(x,\mathring{y})\Big)|_{(x_0,\mathring{x}_0)}.
\end{aligned} \tag{5.3}$$

Since $h_1(x_0,\mathring{x}_0) = -x_0^{2n+1}$ and $F_k(x,\mathring{y}) = O(k^{-\infty})$, we have

$$\lim_{k\to\infty} k^{-n+\frac{d}{2}-1}|\partial_x^{\alpha_0}\partial_{\mathring{y}}^{\beta_0}\Big(e^{-ikh_1(x,\mathring{y})}F_k(x,\mathring{y})\Big)|_{(x_0,\mathring{x}_0)} = 0. \tag{5.4}$$

On the other hand, we can check that

$$\begin{aligned}&\lim_{k\to\infty} k^{-n+\frac{d}{2}-1}|\partial_x^{\alpha_0}\partial_{\mathring{y}}^{\beta_0}\Big(e^{ikh(x,\mathring{y})-ikh_1(x,\mathring{y})}g(x,\mathring{y},k)-p(x,\mathring{y},k)\Big)|_{(x_0,\mathring{x}_0)}\\&=C_{\alpha_0,\beta_0}g_0(x_0,\mathring{x}_0)\neq 0\end{aligned}\tag{5.5}$$

since $g_0(x_0, \mathring{x}_0) \neq 0$. From (5.3), (5.4) and (5.5), we get a contradiction. Thus, $h(x, \mathring{y}) - h_1(x, \mathring{y})$ vanishes to infinite order at $(x_0, \mathring{x}_0)$. Since x_0 is arbitrary, the theorem follows. □

6 The proof of Theorem 1.9

6.1 Preparation

Fix $p \in \mu^{-1}(0)$ and let $x = (x_1, \dots, x_{2n+1})$ be the local coordinates as in Remark 4.4 defined in an open set U of p. We may assume that $U = \Omega_1 \times \Omega_2 \times \Omega_3 \times \Omega_4$, where $\Omega_1 \subset \mathbb{R}^d$, $\Omega_2 \subset \mathbb{R}^d$ are open sets of $0 \in \mathbb{R}^d$, $\Omega_3 \subset \mathbb{R}^{2n-2d}$ is an open set of $0 \in \mathbb{R}^{2n-2d}$ and Ω_4 is an open set of $0 \in \mathbb{R}$. From now on, we identify Ω_2 with

$$\{(0,\dots,0,x_{d+1},\dots,x_{2d},0,\dots,0)\in U;\ (x_{d+1},\dots,x_{2d})\in\Omega_2\},$$

Ω_3 with $\{(0, \dots, 0, x_{2d+1}, \dots, x_{2n}, 0) \in U;\ (x_{d+1}, \dots, x_{2n}) \in \Omega_3\}$, $\Omega_2 \times \Omega_3$ with

$$\{(0,\dots,0,x_{d+1},\dots,x_{2n},0)\in U;\ (x_{d+1},\dots,x_{2n})\in\Omega_2\times\Omega_3\}.$$

For $x = (x_1, \dots, x_{2n+1})$, we write $x'' = (x_{d+1}, \dots, x_{2n+1})$, $\mathring{x}'' = (x_{d+1}, \dots, x_{2n})$, $\hat{x}'' = (x_{d+1}, \dots, x_{2d})$,

$$\widetilde{x}''=(x_{2d+1},\dots,x_{2n+1}),\ \ \widetilde{\mathring{x}}''=(x_{2d+1},\dots,x_{2n}).$$

From now on, we identify x'' with $(0, \dots, 0, x_{d+1}, \dots, x_{2n+1}) \in U$, $\mathring{x}'' = (x_{d+1}, \dots, x_{2n})$ with $(0, \dots, 0, x_{d+1}, \dots, x_{2n}, 0) \in U$, $\hat{x}''$ with $(0, \dots, 0, x_{d+1}, \dots, x_{2d}, 0, \dots, 0) \in U$, $\widetilde{x}''$ with $(0, \dots, 0, x_{2d+1}, \dots, x_{2n+1}) \in U$, $\widetilde{\mathring{x}}''$ with $(0, \dots, 0, x_{2d+1}, \dots, x_{2n}, 0)$. Since $G \times S^1$ acts freely on $\mu^{-1}(0)$, we take Ω_2 and Ω_3 small enough so that if $x, x_1 \in \Omega_2 \times \Omega_3$ and $x \neq x_1$, then

$$g\circ e^{i\theta}\circ x\neq g_1\circ e^{i\theta_1}\circ x_1,\ \ \forall(g,e^{i\theta})\in G\times S^1,\ \ \forall(g_1,e^{i\theta_1})\in G\times S^1.\tag{6.1}$$

We now assume that $q = n_-$ and let $\Psi(x, y) \in C^\infty(U \times U)$ be as in Theorem 1.8. From $S^{(q)}_{G,m} = (S^{(q)}_{G,m})^*$, we get

$$e^{im\Psi(x,y)}b(x,y,m)=e^{-im\overline{\Psi}(y,x)}b^*(x,y,m)+O(m^{-\infty}),\tag{6.2}$$

where $(S^{(q)}_{G,m})^* : L^2_{(0,q)}(X) \to L^2_{(0,q)}(X)$ is the adjoint of $S^{(q)}_{G,m} : L^2_{(0,q)}(X) \to L^2_{(0,q)}(X)$ with respect to $(\cdot\,|\,\cdot)$ and $b^*(x, y, m) : T^{*0,q}_x X \to T^{*0,q}_y X$ is the adjoint of $b(x, y, m) : T^{*0,q}_y X \to T^{*0,q}_x X$ with respect to $\langle\cdot\,|\,\cdot\rangle$. From (6.2), we can repeat the proof of Theorem 5.2 with minor change and deduce that

$$\Psi(x,y)+\overline{\Psi}(y,x)\text{ vanishes to infinite order at diag}\Big((\mu^{-1}(0)\bigcap U)\times(\mu^{-1}(0)\bigcap U)\Big).\tag{6.3}$$

From $\overline{\partial}_b S^{(q)}_{G,m} = 0$, we can check that

$$\overline{\partial}_b\Psi(x,y)\text{ vanishes to infinite order at diag}\Big((\mu^{-1}(0)\bigcap U)\times(\mu^{-1}(0)\bigcap U)\Big).\tag{6.4}$$

From (6.3), (6.4) and notice that $\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial x_{d+j}} \in T^{0,1}_xX$, $j = 1,\ldots,d$, where $x \in \mu^{-1}(0)$ (see Remark 4.4), and $\frac{\partial}{\partial x_j}\Psi(x,y) = \frac{\partial}{\partial y_j}\Psi(x,y) = 0$, $j = 1,\ldots,d$, we conclude that

$\frac{\partial}{\partial x_{d+j}}\Psi(x,y)|_{x_{d+1}=\cdots=x_{2d}=0}$ and $\frac{\partial}{\partial y_{d+j}}\Psi(x,y)|_{y_{d+1}=\cdots=y_{2d}=0}$ vanish to infinite order at $\mathrm{diag}\Big((\mu^{-1}(0)\bigcap U)\times(\mu^{-1}(0)\bigcap U)\Big)$.

Let $G_j(x,y) := \frac{\partial}{\partial y_{d+j}}\Psi(x,y)|_{y_{d+1}=\cdots=y_{2d}=0}$, $H_j(x,y) := \frac{\partial}{\partial x_{d+j}}\Psi(x,y)|_{x_{d+1}=\cdots=x_{2d}=0}$. Put

$$\Psi_1(x,y) := \Psi(x,y) - \sum_{j=1}^{d} y_{d+j}G_j(x,y), \quad \Psi_2(x,y) := \Psi(x,y) - \sum_{j=1}^{d} x_{d+j}H_j(x,y).$$

Then, for $j = 1,2,\ldots,d$,

$$\frac{\partial}{\partial y_{d+j}}\Psi_1(x,y)|_{y_{d+1}=\cdots=y_{2d}=0} = 0 \quad \text{and} \quad \frac{\partial}{\partial x_{d+j}}\Psi_2(x,y)|_{x_{d+1}=\cdots=x_{2d}=0} = 0, \tag{6.5}$$

and, for $j = 1,2$,

$$\Psi(x,y) - \Psi_j(x,y) \text{ vanishes to infinite order at } \mathrm{diag}\Big((\mu^{-1}(0)\bigcap U)\times(\mu^{-1}(0)\bigcap U)\Big). \tag{6.6}$$

We also write $u = (u_1,\ldots,u_{2n+1})$ to denote the local coordinates of U. Recall that for any smooth function $f \in C^\infty(U)$, we write $\widetilde{f} \in C^\infty(U^{\mathbb{C}})$ to denote an almost analytic extension of f (see the discussion after (3.38)). We consider the following two systems

$$\frac{\partial\widetilde{\Psi}_1}{\partial\widetilde{u}_{2d+j}}(\widetilde{x},\widetilde{\widetilde{u}''}) + \frac{\partial\widetilde{\Psi}_2}{\partial\widetilde{x}_{2d+j}}(\widetilde{\widetilde{u}''},\widetilde{y}) = 0, \quad j = 1,2,\ldots,2n-2d, \tag{6.7}$$

and

$$\frac{\partial\widetilde{\Psi}_1}{\partial\widetilde{u}_{d+j}}(\widetilde{x},\widetilde{u''}) + \frac{\partial\widetilde{\Psi}_2}{\partial\widetilde{x}_{d+j}}(\widetilde{u''},\widetilde{y}) = 0, \quad j = 1,2,\ldots,2n-d, \tag{6.8}$$

where $\widetilde{\widetilde{u}''} = (0,\ldots,0,\widetilde{u}_{2d+1},\ldots,\widetilde{u}_{2n+1})$, $\widetilde{u''} = (0,\ldots,0,\widetilde{u}_{d+1},\ldots,\widetilde{u}_{2n+1})$. From (6.5) and Theorem 1.12, we can take $\widetilde{\Psi}_1$ and $\widetilde{\Psi}_2$ so that for every $j = 1,2,\ldots,d$,

$$\frac{\partial\widetilde{\Psi}_1}{\partial\widetilde{u}_{d+j}}(\widetilde{x},\widetilde{u''}) = 0 \quad \text{and} \quad \frac{\partial\widetilde{\Psi}_2}{\partial\widetilde{x}_{d+j}}(\widetilde{u''},\widetilde{y}) = 0, \ \text{ if } \widetilde{u}_{d+1} = \cdots = \widetilde{u}_{2d} = 0, \tag{6.9}$$

and, for $j = 1,2$,

$$\widetilde{\Psi}_j(\widetilde{x},\widetilde{y}) = -\widetilde{x}_{2n+1} + \widetilde{y}_{2n+1} + \widetilde{\hat{\Psi}}_j(\widetilde{\mathring{x}''},\widetilde{\mathring{y}''}), \quad \widetilde{\hat{\Psi}}_j \in C^\infty(U^{\mathbb{C}}\times U^{\mathbb{C}}), \tag{6.10}$$

where $\widetilde{\mathring{x}''} = (0,\ldots,0,\widetilde{x}_{d+1},\ldots,\widetilde{x}_{2n},0)$, $\widetilde{\mathring{y}''} = (0,\ldots,0,\widetilde{y}_{d+1},\ldots,\widetilde{y}_{2n},0)$.

From Theorem 1.12, (1.19) and $d_x\Psi(x,x) = -d_y\Psi(x,x) = -\omega_0(x)$, $\forall x \in \mu^{-1}(0)$, it is not difficult to see that

$$\frac{\partial\widetilde{\Psi}_1}{\partial\widetilde{u}_{d+j}}(\widetilde{x}'',\widetilde{x}'') + \frac{\partial\widetilde{\Psi}_2}{\partial\widetilde{x}_{d+j}}(\widetilde{x}'',\widetilde{x}'') = 0, \quad j = 1,2,\ldots,2n-d,$$

and the matrices

$$\left(\frac{\partial^2\Psi}{\partial u_{2d+j}\partial u_{2d+k}}(p,p) + \frac{\partial^2\Psi}{\partial x_{2d+j}\partial x_{2d+k}}(p,p)\right)_{j,k=1}^{2n-2d},$$

$$\left(\frac{\partial^2\Psi}{\partial u_{d+j}\partial u_{d+k}}(p,p)+\frac{\partial^2\Psi}{\partial x_{d+j}\partial x_{d+k}}(p,p)\right)_{j,k=1}^{2n-d}$$

are non-singular. Moreover,

$$\det\left(\tfrac{\partial^2\Psi}{\partial u_{2d+j}\partial u_{2d+k}}(p,p)+\tfrac{\partial^2\Psi}{\partial x_{2d+j}\partial x_{2d+k}}(p,p)\right)_{j,k=1}^{2n-2d}=(4i|\mu_{d+1}|\cdots 4i|\mu_n|)^2,$$
$$\det\left(\tfrac{\partial^2\Psi}{\partial u_{d+j}\partial u_{d+k}}(p,p)+\tfrac{\partial^2\Psi}{\partial x_{d+j}\partial x_{d+k}}(p,p)\right)_{j,k=1}^{2n-d}=(8i|\mu_1|\cdots 8i|\mu_d|)(4i|\mu_{d+1}|\cdots 4i|\mu_n|)^2.$$

Hence, near (p,p), we can solve (6.7) and (6.8) and the solutions are unique. Let $\alpha(x,y)=(\alpha_{2d+1}(x,y),\ldots,\alpha_{2n}(x,y))\in C^\infty(U\times U,\mathbb{C}^{2n-2d})$ and $\beta(x,y)=(\beta_{d+1}(x,y),\ldots,\beta_{2n}(x,y))\in C^\infty(U\times U,\mathbb{C}^{2n-d})$ be the solutions of (6.7) and (6.8), respectively. From (6.9), it is easy to see that

$$\beta(x,y)=(\beta_{d+1}(x,y),\ldots,\beta_{2n}(x,y))=(0,\ldots,0,\alpha_{2d+1}(x,y),\ldots,\alpha_{2n}(x,y)).\quad(6.11)$$

From (6.11), we see that the value of $\widetilde{\Psi}_1(x,\widetilde{\widetilde{u}}'')+\widetilde{\Psi}_2(\widetilde{\widetilde{u}}'',y)$ at critical point $\widetilde{\widetilde{u}}''=\alpha(x,y)$ is equal to the value of $\widetilde{\Psi}_1(x,\widetilde{u}'')+\widetilde{\Psi}_2(\widetilde{u}'',y)$ at critical point $\widetilde{u}''=\beta(x,y)$. Put

$$\Psi_3(x,y):=\widetilde{\Psi}_1(x,\alpha(x,y))+\widetilde{\Psi}_2(\alpha(x,y),y)=\widetilde{\Psi}_1(x,\beta(x,y))+\widetilde{\Psi}_2(\beta(x,y),y).\quad(6.12)$$

$\Psi_3(x,y)$ is a complex phase function. From (6.10), we have

$$\Psi_3(x,y)=-x_{2n+1}+y_{2n+1}+\hat{\Psi}_3(\mathring{x}'',\mathring{y}''),\quad \hat{\Psi}_3(\mathring{x}'',\mathring{y}'')\in C^\infty(U\times U).$$

Moreover, we have the following

Theorem 6.1 *The function* $\Psi_3(x,y)-\Psi(x,y)$ *vanishes to infinite order at* $\mathrm{diag}\Big((\mu^{-1}(0)\bigcap U)\times(\mu^{-1}(0)\bigcap U)\Big)$.

Proof We consider the kernel $S^{(q)}_{G,m}\circ S^{(q)}_{G,m}$ on U. Let $V\Subset U$ be an open set of p. Let $\chi(\mathring{x}'')\in C^\infty_0(\Omega_2\times\Omega_3)$. From (6.1), we can extend $\chi(\mathring{x}'')$ to $W:=\big\{g\circ e^{i\theta}\circ x;\ (g,e^{i\theta})\in G\times S^1, x\in\Omega_2\times\Omega_3\big\}$ by $\chi(g\circ e^{i\theta}\circ\mathring{x}''):=\chi(\mathring{x}'')$, for every $(g,e^{i\theta})\in G\times S^1$. Assume that $\chi=1$ on some neighborhood of V. Let $\chi_1\in C^\infty_0(U)$ with $\chi_1=1$ on some neighborhood of V and $\mathrm{Supp}\,\chi_1\subset\{x\in X;\ \chi(x)=1\}$. We have

$$\chi_1S^{(q)}_{G,m}\circ S^{(q)}_{G,m}=\chi_1S^{(q)}_{G,m}\chi\circ S^{(q)}_{G,m}+\chi_1S^{(q)}_{G,m}(1-\chi)\circ S^{(q)}_{G,m}.\quad(6.13)$$

Let's first consider $\chi_1S^{(q)}_{G,m}(1-\chi)\circ S^{(q)}_{G,m}$. We have

$$(\chi_1S^{(q)}_{G,m}(1-\chi))(x,u)=\frac{1}{2\pi}\chi_1(x)\int_{-\pi}^{\pi}\int_G S^{(q)}(x,g\circ e^{i\theta}\circ u)(1-\chi(u))e^{im\theta}d\mu(g)d\theta.\quad(6.14)$$

If $u\notin\{x\in X;\ \chi(x)=1\}$. Since $\mathrm{Supp}\,\chi_1\subset\{x\in X;\ \chi(x)=1\}$ and $\chi(x)=\chi(g\circ e^{i\theta}\circ x)$, for every $(g,e^{i\theta})\in G\times S^1$, for every $x\in X$, we conclude that $g\circ e^{i\theta}\circ u\notin\mathrm{Supp}\,\chi_1$, for every $(g,e^{i\theta})\in G\times S^1$. From this observation and notice that $S^{(q)}$ is smoothing away from diagonal, we can integrate by parts with respect to θ in (6.14) and deduce that $\chi_1S^{(q)}_{G,m}\circ(1-\chi)=O(m^{-\infty})$ and hence

$$\chi_1S^{(q)}_{G,m}(1-\chi)\circ S^{(q)}_{G,m}=O(m^{-\infty}).\quad(6.15)$$

From (6.13) and (6.15), we get

$$\chi_1S^{(q)}_{G,m}\circ S^{(q)}_{G,m}=\chi_1S^{(q)}_{G,m}\chi\circ S^{(q)}_{G,m}+O(m^{-\infty}).\quad(6.16)$$

We can check that on U,

$$\begin{aligned}&(\chi_1 S^{(q)}_{G,m}\chi\circ S^{(q)}_{G,m})(x,y)\\&=(2\pi)\int e^{im\Psi(x,u'')+im\Psi(u'',y)}\chi_1(x)b(x,\mathring{u}'',m)\chi(\mathring{u}'')b(\mathring{u}'',y,m)dv(\mathring{u}'')+O(m^{-\infty})\\&=(2\pi)\int e^{im\Psi_1(x,u'')+im\Psi_2(u'',y)}\chi_1(x)b(x,\mathring{u}'',m)\chi(\mathring{u}'')b(\mathring{u}'',y,m)dv(\mathring{u}'')+O(m^{-\infty})\\&\text{(here we use (6.6))},\end{aligned}\tag{6.17}$$

where $d\mu(g)d\theta dv(\mathring{u}'')=dv(x)$ on U. We use complex stationary phase formula of Melin–Sjöstrand [25] to carry out the integral (6.17) and get

$$\begin{aligned}&(\chi_1 S^{(q)}_{G,m}\chi\circ S^{(q)}_{G,m})(x,y)=e^{im\Psi_3(x,y)}a(x,y,m)+O(m^{-\infty})\ \text{ on } U,\\&a(x,y,m)\in S^{n-\frac{d}{2}}_{\mathrm{loc}}(1;U\times U,T^{*0,q}X\boxtimes(T^{*0,q}X)^*),\\&a(x,y,m)\sim\sum_{j=0}^{\infty}m^{n-\frac{d}{2}-j}a_j(x,y)\text{ in } S^{n-\frac{d}{2}}_{\mathrm{loc}}(1;U\times U,T^{*0,q}X\boxtimes(T^{*0,q}X)^*),\\&a_j(x,y)\in C^\infty(U\times U,T^{*0,q}X\boxtimes(T^{*0,q}X)^*),\quad j=0,1,2,\ldots,\\&a_0(p,p)\neq 0.\end{aligned}\tag{6.18}$$

From (6.16), (6.18) and notice that $(\chi_1 S^{(q)}_{G,m}\circ S^{(q)}_{G,m})(x,y)=(\chi_1 S^{(q)}_{G,m})(x,y)$, we deduce that

$$e^{im\Psi_3(x,y)}a(x,y,m)=e^{im\Psi(x,y)}\chi_1(x)b(x,y,m)+O(m^{-\infty})\ \text{ on } U.\tag{6.19}$$

From (6.19), we can repeat the proof of Theorem 5.2 with minor change and deduce that $\Psi_3(x,y)-\Psi(x,y)$ vanishes to infinite order at $\mathrm{diag}\Big((\mu^{-1}(0)\bigcap U)\times(\mu^{-1}(0)\bigcap U)\Big)$. □

The following two theorems follow from (6.6), (6.12), Theorem 6.1, complex stationary phase formula of Melin–Sjöstrand [25] and some straightforward computation. We omit the details.

Theorem 6.2 *With the notations used above, let*

$$\begin{aligned}&A_m(x,y)=e^{im\Psi(x,y)}a(x,y,m),\quad B_m(x,y)=e^{im\Psi(x,y)}b(x,y,m),\\&a(x,y,m)\in S^k_{\mathrm{loc}}(1;U\times U,H\boxtimes F^*),\quad b(x,y,m)\in S^\ell_{\mathrm{loc}}(1;U\times U,F\boxtimes E^*),\\&a(x,y,m)\sim\sum_{j=0}^{\infty}m^{k-j}a_j(x,y)\ \textit{in}\ S^k_{\mathrm{loc}}(1;U\times U,H\boxtimes F^*),\\&b(x,y,m)\sim\sum_{j=0}^{\infty}m^{\ell-j}b_j(x,y)\ \textit{in}\ S^\ell_{\mathrm{loc}}(1;U\times U,F\boxtimes E^*),\\&a_j(x,y)\in C^\infty(U\times U,H\boxtimes F^*),\quad b_j(x,y)\in C^\infty(U\times U,F\boxtimes E^*),\quad j=0,1,2,\ldots,\end{aligned}$$

where E, F and H are vector bundles over X. Let $\chi(\mathring{x}'')\in C^\infty_0(\Omega_2\times\Omega_3)$. Then, we have

$$\begin{aligned}&\int A_m(x,u)\chi(\mathring{u}'')B_m(u,y)dv(\mathring{u}'')=e^{im\Psi(x,y)}c(x,y,m)+O(m^{-\infty}),\\&c(x,y,m)\in S^{k+\ell-(n-\frac{d}{2})}_{\mathrm{loc}}(1;U\times U,H\boxtimes E^*),\\&c(x,y,m)\sim\sum_{j=0}^{\infty}m^{k+\ell-(n-\frac{d}{2})-j}c_j(x,y)\ \textit{in}\ S^{k+\ell-(n-\frac{d}{2})}_{\mathrm{loc}}(1;U\times U,H\boxtimes E^*),\\&c_0(x,x)=2^{-n-\frac{d}{2}}\pi^{n-\frac{d}{2}}|\det\mathcal{L}_x|^{-1}|\det R_x|^{\frac{1}{2}}a_0(x,x)b_0(x,x)\chi(\mathring{x}''),\quad\forall x\in\mu^{-1}(0)\textstyle\bigcap U,\end{aligned}$$

where $|\det R_x|$ is in the discussion before Theorem 1.6.

Moreover, if there are $N_1,N_2\in\mathbb{N}$, such that $|a_0(x,y)|\leq C|(x,y)-(x_0,x_0)|^{N_1}$, $|b_0(x,y)|\leq C|(x,y)-(x_0,x_0)|^{N_2}$, for all $x_0\in\mu^{-1}(0)\bigcap U$, where $C>0$ is a constant, then,

$$|c_0(x,y)|\leq\hat{C}|(x,y)-(x_0,x_0)|^{N_1+N_2},$$

for all $x_0\in\mu^{-1}(0)\bigcap U$, where $\hat{C}>0$ is a constant.

Theorem 6.3 *With the notations used above, let*

$$\begin{aligned}
&\mathcal{A}_m(x,\widetilde{y}'')=e^{im\Psi(x,\widetilde{y}'')}\alpha(x,\widetilde{y}'',m),\quad \mathcal{B}_m(\widetilde{x}'',y)=e^{im\Psi(\widetilde{x}'',y)}\beta(\widetilde{x}'',y,m),\\
&\alpha(x,\widetilde{y}'',m)\in S^k_{\rm loc}(1;U\times(\Omega_3\times\Omega_4),H\boxtimes F^*),\quad \beta(\widetilde{x}'',y,m)\\
&\in S^\ell_{\rm loc}(1;(\Omega_3\times\Omega_4)\times U,F\boxtimes E^*),\\
&\alpha(x,\widetilde{y}'',m)\sim\sum_{j=0}^\infty m^{k-j}\alpha_j(x,\widetilde{y}'')\ \text{in}\ S^k_{\rm loc}(1;U\times(\Omega_3\times\Omega_4),H\boxtimes F^*),\\
&\beta(\widetilde{x}'',y,m)\sim\sum_{j=0}^\infty m^{\ell-j}\beta_j(\widetilde{x}'',y)\ \text{in}\ S^\ell_{\rm loc}(1;(\Omega_3\times\Omega_4)\times U,F\boxtimes E^*),\\
&\alpha_j(x,\widetilde{y}'')\in C^\infty(U\times(\Omega_3\times\Omega_4),H\boxtimes F^*),\ \beta_j(\widetilde{x}'',y)\\
&\in C^\infty((\Omega_3\times\Omega_4)\times U,F\boxtimes E^*),\ j=0,1,2,\ldots,
\end{aligned}$$

where E, F and H are vector bundles over X. Let $\chi_1(\widetilde{\mathring{x}}'')\in C^\infty_0(\Omega_3)$. Then, we have

$$\begin{aligned}
&\int\mathcal{A}_m(x,\widetilde{u}'')\chi_1(\widetilde{\mathring{u}}'')\mathcal{B}_m(\widetilde{u}'',y)dv(\widetilde{\mathring{u}})=e^{im\Psi(x,y)}\gamma(x,y,m)+O(m^{-\infty}),\\
&\gamma(x,y,m)\in S^{k+\ell-(n-d)}_{\rm loc}(1;U\times U,H\boxtimes E^*),\\
&\gamma(x,y,m)\sim\sum_{j=0}^\infty m^{k+\ell-(n-d)-j}\gamma_j(x,y)\ \text{in}\ S^{k+\ell-(n-d)}_{\rm loc}(1;U\times U,H\boxtimes E^*),\\
&\gamma_0(x,x)=2^{-n}\pi^{n-d}|\det\mathcal{L}_x|^{-1}|\det R_x|\alpha_0(x,\widetilde{x}'')\beta_0(\widetilde{x}'',x)\chi_1(\widetilde{\mathring{x}}''),\quad\forall x\in\mu^{-1}(0)\bigcap U,
\end{aligned}$$

where $|\det R_x|$ is in the discussion before Theorem 1.6.

Moreover, if there are $N_1,N_2\in\mathbb{N}$, such that $|\alpha_0(x,\widetilde{y}'')|\leq C|(x,\widetilde{y}'')-(x_0,x_0)|^{N_1}$, $|\beta_0(x,\widetilde{y}'')|\leq C|(x,\widetilde{y}'')-(x_0,x_0)|^{N_2}$, for all $x_0\in\mu^{-1}(0)\bigcap U$, where $C>0$ is a constant, then,

$$|\gamma_0(x,y)|\leq\hat{C}|(x,y)-(x_0,x_0)|^{N_1+N_2},$$

for all $x_0\in\mu^{-1}(0)\bigcap U$, where $\hat{C}>0$ is a constant.

6.2 The proof of Theorem 1.9

Since $\underline{\mathfrak{g}}_x$ is orthogonal to $H_xY\bigcap JH_xY$ and $H_xY\bigcap JH_xY\subset\underline{\mathfrak{g}}_x^{\perp_b}$, for every $x\in Y$, we can find a G-invariant orthonormal basis $\{Z_1,\ldots,Z_n\}$ of $T^{1,0}X$ on Y such that

$$\begin{aligned}
&\mathcal{L}_x(Z_j(x),\overline{Z}_k(x))=\delta_{j,k}\lambda_j(x),\quad j,k=1,\ldots,n,\quad x\in Y,\\
&Z_j(x)\in\underline{\mathfrak{g}}_x+iJ\underline{\mathfrak{g}}_x,\quad\forall x\in Y,\quad j=1,2,\ldots,d,\\
&Z_j(x)\in\mathbb{C}H_xY\bigcap J(\mathbb{C}H_xY),\quad\forall x\in Y,\quad j=d+1,\ldots,n.
\end{aligned}$$

Let $\{e_1,\ldots,e_n\}$ denote the orthonormal basis of $T^{*0,1}X$ on Y, dual to $\{\overline{Z}_1,\ldots,\overline{Z}_n\}$. Fix $s=0,1,2,\ldots,n-d$. For $x\in Y$, put

$$B^{*0,s}_xX=\left\{\sum_{d+1\leq j_1<\cdots<j_s\leq n}a_{j_1,\ldots,j_s}e_{j_1}\wedge\cdots\wedge e_{j_s};\ a_{j_1,\ldots,j_s}\in\mathbb{C},\ \forall d+1\leq j_1<\cdots<j_s\leq n\right\}$$

and let $B^{*0,s}X$ be the vector bundle of Y with fiber $B^{*0,s}_xX$, $x\in Y$. Let $C^\infty(Y,B^{*0,s}X)^G$ denote the set of all G-invariant sections of Y with values in $B^{*0,s}X$. Let

$$\iota_G:C^\infty(Y,B^{*0,s}X)^G\to\Omega^{0,s}(Y_G)$$

be the natural identification.

Assume that $\lambda_1<0,\ldots,\lambda_r<0$, and $\lambda_{d+1}<0,\ldots,\lambda_{n_--r+d}<0$. For $x\in Y$, put

$$\hat{\mathcal{N}}(x,n_-)=\left\{ce_{d+1}\wedge\cdots\wedge e_{n_--r+d};\ c\in\mathbb{C}\right\},$$

and let

$$\begin{aligned}&\hat{p}=\hat{p}_x:\mathcal{N}(x,n_-)\to\hat{\mathcal{N}}(x,n_-),\\&u=ce_1\wedge\cdots\wedge e_r\wedge e_{d+1}\wedge\cdots\wedge e_{n_--r+d}\to ce_{d+1}\wedge\cdots\wedge e_{n_--r+d}.\end{aligned}$$

Let $\iota : Y \to X$ be the natural inclusion and let $\iota^* : \Omega^{0,q}(X) \to \Omega^{0,q}(Y)$ be the pull-back of ι. Recall that we work with the assumption that $q = n_-$. Let $\Box^{(q-r)}_{b,Y_G}$ be the Kohn Laplacian for $(0, q-r)$ forms on Y_G. Fix $m \in \mathbb{N}$. Let $H^{q-r}_{b,m}(Y_G) := \Big\{u \in \Omega^{0,q-r}(Y_G);\ \Box^{(q-r)}_{b,Y_G}u = 0,\ \ Tu = imu\Big\}$. Let $S^{(q-r)}_{Y_G,m} : L^2_{(0,q-r)}(Y_G) \to H^{q-r}_{b,m}(Y_G)$ be the orthogonal projection and let $S^{(q-r)}_{Y_G,m}(x, y)$ be the distribution kernel of $S^{(q-r)}_{Y_G,m}$. Let

$$f(x)=\sqrt{V_{\rm eff}\,(x)}|\det\ R_x|^{-\frac{1}{4}}\in C^\infty(Y)^G.$$

Let

$$\begin{aligned}\sigma_m:\Omega^{0,q}(X)&\to H^{q-r}_{b,m}(Y_G),\\ u\quad&\to m^{-\frac{d}{4}}S^{(q-r)}_{Y_G,m}\circ\iota_G\circ\hat{p}\circ\tau_{x,n_-}\circ f\circ\iota^*\circ S^{(q)}_{G,m}u.\end{aligned}$$

Recall that τ_{x,n_-} is given by (1.7). Let $\sigma^*_m : \Omega^{0,q-r}(Y_G) \to \Omega^{0,q}(X)$ be the adjoints of σ_m. It is easy to see that $\sigma^*_m u \in H^q_{b,m}(X)^G := (\mathrm{Ker}\,\Box^{(q)}_b)^G_m$, $\forall u \in \Omega^{0,q-r}(Y_G)$. Let $\sigma_m(x, y)$ and $\sigma^*_m(x, y)$ denote the distribution kernels of σ_m and σ^*_m, respectively.

Let's pause and recall some results for $S^{(q-r)}_{Y_G,m}$. We first introduce some notations. Let $\mathcal{L}_{Y_G,x}$ be the Levi form on Y_G at $x \in Y_G$ induced naturally from $\mathcal{L}$. The Hermitian metric $\langle\cdot|\cdot\rangle$ on $T^{1,0}X$ induces a Hermitian metric $\langle\cdot|\cdot\rangle$ on $T^{1,0}Y_G$. Let $\det\mathcal{L}_{Y_G,x} = \lambda_1\ldots\lambda_{n-d}$, where λ_j, $j = 1,\ldots,n-d$, are the eigenvalues of $\mathcal{L}_{Y_G,x}$ with respect to the Hermitian metric $\langle\cdot|\cdot\rangle$. For $x \in Y_G$, let $\hat{\tau}_x : T^{*0,q-r}_x Y_G \to \hat{\mathcal{N}}(x, n_-)$ be the orthogonal projection.

Let $\pi : Y \to Y_G$ be the natural quotient. Let $S^{(q-r)}_{Y_G} : L^2_{(0,q-r)}(Y_G) \to \mathrm{Ker}\,\Box^{(q-r)}_{b,Y_G}$ be the Szegő projection as (3.2). Since $S^{(q-r)}_{Y_G}$ is smoothing away from diagonal (see Theorem 3.3), it is easy to see that for any $x, y \in Y$, if $\pi(e^{i\theta}\circ x) \neq \pi(e^{i\theta}\circ y)$, for every $\theta \in [0, 2\pi[$, then there are open sets U of $\pi(x)$ in Y_G and V of $\pi(y)$ in Y_G such that for all $\hat{\chi} \in C^\infty_0(U)$, $\widetilde{\chi} \in C^\infty_0(V)$, we have

$$\hat{\chi}S^{(q-r)}_{Y_G,m}\widetilde{\chi}=O(m^{-\infty})\ \text{ on } Y_G. \tag{6.20}$$

Fix $p \in Y$ and let $x = (x_1,\ldots,x_{2n+1})$ be the local coordinates as in Remark 4.4. We will use the same notations as in the beginning of Sect. 6.1. From now on, we identify $\widetilde{x}''$ as local coordinates of Y_G near $\pi(p) \in Y_G$ and we identify $W := \Omega_3 \times \Omega_4$ with an open set of $\pi(p)$ in Y_G. It was proved in Theorem 4.11 in [14] that as $m \to +\infty$,

$$\begin{aligned}&S^{(q-r)}_{Y_G,m}(\widetilde{x}'',\widetilde{y}'')=e^{im\phi(\widetilde{x}'',\widetilde{y}'')}b(\widetilde{x}'',\widetilde{y}'',m)+O(m^{-\infty})\ \text{ on } W,\\&\beta(\widetilde{x}'',\widetilde{y}'',m)\in S^{n-d}_{\rm loc}(1;W\times W,T^{*0,q-r}Y_G\boxtimes(T^{*0,q-r}Y_G)^*),\\&\beta(\widetilde{x}'',\widetilde{y}'',m)\sim\sum_{j=0}^\infty m^{n-d-j}b_j(\widetilde{x}'',\widetilde{y}'')\text{ in } S^{n-d}_{\rm loc}(1;W\times W,T^{*0,q-r}Y_G\boxtimes(T^{*0,q-r}Y_G)^*),\\&\beta_j(\widetilde{x}'',\widetilde{y}'')\in C^\infty(W\times W,T^{*0,q-r}Y_G\boxtimes(T^{*0,q-r}Y_G)^*),\quad j=0,1,2,\ldots,\\&\beta_0(\widetilde{x}'',\widetilde{x}'')=\frac{1}{2}\pi^{-(n-d)-1}|\det\ \mathcal{L}_{Y_G,\widetilde{x}''}|\hat{\tau}_{\widetilde{x}''},\quad\forall\widetilde{x}''\in W,\end{aligned}$$

and

$$\begin{aligned}
&\phi(\widetilde{x}'', \widetilde{y}'') = -x_{2n+1} + y_{2n+1} + \hat{\phi}(\mathring{\widetilde{x}}'', \mathring{\widetilde{y}}'') \in C^\infty(W \times W),\\
&d_{\widetilde{x}''}\phi(\widetilde{x}'', \widetilde{y}'') = -d_{\widetilde{y}''}\phi(\widetilde{x}'', \widetilde{x}'') = -\omega_0(\widetilde{x}''),\\
&\mathrm{Im}\,\hat{\phi}(\mathring{\widetilde{x}}'', \mathring{\widetilde{y}}'') \geq c|\mathring{\widetilde{x}}'' - \mathring{\widetilde{y}}''|^2, \quad \text{where } c > 0 \text{ is a constant},\\
&p_0(\widetilde{x}'', d_{\widetilde{x}''}\phi(\widetilde{x}'', \widetilde{y}'')) \text{ vanishes to infinite order at } \mathring{\widetilde{x}}'' = \mathring{\widetilde{y}}'',\\
&\phi(\widetilde{x}'', \widetilde{y}'') = -x_{2n+1} + y_{2n+1} + i\sum_{j=d+1}^{n}|\mu_j||z_j - w_j|^2\\
&\quad + \sum_{j=d+1}^{n} i\mu_j(\overline{z}_j w_j - z_j\overline{w}_j) + O(|(\mathring{\widetilde{x}}'', \mathring{\widetilde{y}}'')|^3),
\end{aligned} \tag{6.21}$$

where p_0 denotes the principal symbol of $\Box^{(q-r)}_{b,Y_G}$, $z_j = x_{2j-1} + ix_{2j}$, $j = d+1, \ldots, n$, and $\mu_{d+1}, \ldots, \mu_n$ are the eigenvalues of $\mathcal{L}_{Y_G,p}$.

Note that for any $\phi_1(\widetilde{x}'', \widetilde{y}'') \in C^\infty(W \times W)$, if ϕ_1 satisfies (6.21), then $\phi_1 - \phi$ vanishes to infinite order at $\mathring{\widetilde{x}}'' = \mathring{\widetilde{y}}''$ (see Remark 3.6 in [14]). It is not difficult to see that the phase function $\Psi(\widetilde{x}'', \widetilde{y}'')$ satisfies (6.21). Hence, we can replace the phase $\phi(\widetilde{x}'', \widetilde{y}'')$ by $\Psi(\widetilde{x}'', \widetilde{y}'')$ and we have

$$S^{(q-r)}_{Y_G,m}(\widetilde{x}'', \widetilde{y}'') = e^{im\Psi(\widetilde{x}'', \widetilde{y}'')}\beta(\widetilde{x}'', \widetilde{y}'', m) + O(m^{-\infty}) \ \text{ on } W. \tag{6.22}$$

We can now prove

Theorem 6.4 *With the notations used above, if $y \notin Y$, then for any open set D of y with $\overline{D}\bigcap Y = \emptyset$, we have*

$$\sigma_m = O(m^{-\infty}) \ \text{ on } Y_G \times D. \tag{6.23}$$

Let $x, y \in Y$. If $\pi(x) \neq \pi(e^{i\theta} \circ y)$, for every $\theta \in [0, 2\pi[$, then there are open sets U_G of $\pi(x)$ in Y_G and V of y in X such that

$$\sigma_m = O(m^{-\infty}) \ \text{ on } U_G \times V. \tag{6.24}$$

Let $p \in \mu^{-1}(0)$ and let $x = (x_1, \ldots, x_{2n+1})$ be the local coordinates as in Remark 4.4. Then,

$$\begin{aligned}
&\sigma_m(\widetilde{x}'', y) = e^{im\Psi(\widetilde{x}'', y'')}\alpha(\widetilde{x}'', y'', m) + O(m^{-\infty}) \ \text{ on } W \times U,\\
&\alpha(\widetilde{x}'', y'', m) \in S^{n-\frac{3}{4}d}_{\mathrm{loc}}(1; W \times U, T^{*0,q-r}Y_G \boxtimes (T^{*0,q}X)^*),\\
&\alpha(\widetilde{x}'', y'', m) \sim \sum_{j=0}^{\infty} m^{n-\frac{3}{4}d-j}\alpha_j(\widetilde{x}'', y'') \text{ in } S^{n-\frac{3}{4}d}_{\mathrm{loc}}(1; W \times U, T^{*0,q-r}Y_G \boxtimes (T^{*0,q}X)^*),\\
&\alpha_j(\widetilde{x}'', y'') \in C^\infty(W \times U, T^{*0,q-r}Y_G \boxtimes (T^{*0,q}X)^*), \quad j = 0, 1, 2, \ldots,
\end{aligned} \tag{6.25}$$

$$\alpha_0(\widetilde{x}'', \widetilde{x}'') = 2^{-n+2d-1}\pi^{\frac{d}{2}-n-1}\frac{1}{\sqrt{V_{\mathrm{eff}}(\widetilde{x}'')}}|\det \mathcal{L}_{\widetilde{x}''}||\det R_x|^{-\frac{3}{4}}\hat{\tau}_{\widetilde{x}''}\tau_{\widetilde{x}'',n_-}, \quad \forall \widetilde{x}'' \in W, \tag{6.26}$$

where U is an open set of p, $W = \Omega_3 \times \Omega_4$, Ω_3 and Ω_4 are open sets as in the beginning of Sect. 6.1.

Proof Note that $S^{(q)}_{G,m} = O(m^{-\infty})$ away Y. From this observation, we get (6.23). Let $x, y \in Y$. Assume that $\pi(x) \neq \pi(e^{i\theta} \circ y)$, for every $\theta \in [0, 2\pi[$. Since

$$S^{(q)}_{G,m}(x, y) = \frac{1}{2\pi|G|_{d\mu}}\int_{-\pi}^{\pi}\int_G S^{(q)}(x, e^{i\theta} \circ g \circ y)e^{im\theta}d\mu(g)d\theta$$

and $S^{(q)}$ is smoothing away from diagonal, we can integrate by parts with respect to θ and deduce that there are open sets U_1 of x in X and V_1 of y in X such that

$$S^{(q)}_{G,m} = O(m^{-\infty}) \ \text{ on } U_1 \times V_1. \tag{6.27}$$

From (6.20), we see that there are open sets $\hat{U}_G$ of $\pi(x)$ in Y_G and $\hat{V}_G$ of $\pi(y)$ in Y_G such that

$$S^{(q-r)}_{Y_G,m} = O(m^{-\infty}) \ \text{ on } \hat{U}_G \times \hat{V}_G. \tag{6.28}$$

From (6.27) and (6.28), we get (6.24).

Fix $u = (u_1, \ldots, u_{2n+1}) \in Y \bigcap U$. From (6.23) and (6.24), we only need to show that (6.25) and (6.26) hold near u and we may assume that $u = (0, \ldots, 0, u_{2d+1}, \ldots, u_{2n}, 0) = \widetilde{\mathring{u}}''$. Let V be a small neighborhood of u. Let $\chi(\widetilde{\mathring{x}}'') \in C^\infty_0(\Omega_3)$. From (6.1), we can extend $\chi(\widetilde{\mathring{x}}'')$ to

$$Q = \Big\{g \circ e^{i\theta} \circ x;\ (g, e^{i\theta}) \in G \times S^1, x \in \Omega_3\Big\}$$

by $\chi(g \circ e^{i\theta} \circ \widetilde{\mathring{x}}'') := \chi(\widetilde{\mathring{x}}'')$, for every $(g, e^{i\theta}) \in G \times S^1$. Assume that $\chi = 1$ on some neighborhood of V. Let $V_G = \{\pi(x);\ x \in V\}$. Let $\chi_1 \in C^\infty_0(Y_G)$ with $\chi_1 = 1$ on some neighborhood of V_G and $\operatorname{Supp}\chi_1 \subset \{\pi(x) \in Y_G;\ x \in Y, \chi(x) = 1\}$. We have

$$\begin{aligned}\chi_1\sigma_m &= m^{-\frac{d}{4}}\chi_1 S^{(q-r)}_{Y_G,m} \circ \iota_G \circ \hat{p} \circ \tau_{x,n_-} \circ f \circ \iota^* \circ S^{(q)}_{G,m}\\ &= m^{-\frac{d}{4}}\chi_1 S^{(q-r)}_{Y_G,m} \circ \iota_G \circ \hat{p} \circ \tau_{x,n_-} \circ f \circ \iota^* \circ \chi S^{(q)}_{G,m}\\ &\quad + m^{-\frac{d}{4}}\chi_1 S^{(q-r)}_{Y_G,m} \circ \iota_G \circ \hat{p} \circ \tau_{x,n_-} \circ f \circ \iota^* \circ (1-\chi) S^{(q)}_{G,m}.\end{aligned} \tag{6.29}$$

If $u \in Y$ but $u \notin \{x \in X;\ \chi(x) = 1\}$. Since $\operatorname{Supp}\chi_1 \subset \{\pi(x) \in X;\ x \in Y, \chi(x) = 1\}$ and $\chi(x) = \chi(g \circ e^{i\theta} \circ x)$, for every $(g, e^{i\theta}) \in G \times S^1$, for every $x \in X$, we conclude that $\pi(e^{i\theta} \circ u) \notin \operatorname{Supp}\chi_1$, for every $e^{i\theta} \in S^1$. From this observation and (6.20), we get

$$m^{-\frac{d}{4}}\chi_1 S^{(q-r)}_{Y_G,m} \circ \iota_G \circ \hat{p} \circ \tau_{x,n_-} \circ f \circ \iota^* \circ (1-\chi) S^{(q)}_{G,m} = O(m^{-\infty}) \ \text{ on } Y_G \times X. \tag{6.30}$$

From (6.29) and (6.30), we get

$$\chi_1\sigma_m = m^{-\frac{d}{4}}\chi_1 S^{(q-r)}_{Y_G,m} \circ \iota_G \circ \hat{p} \circ \tau_{x,n_-} \circ f \circ \iota^* \circ \chi S^{(q)}_{G,m} + O(m^{\infty}) \ \text{ on } Y_G \times X.$$

From (6.22) and Theorem 1.8, we can check that on U,

$$\chi_1\sigma_m(\widetilde{x}'', y) = (2\pi)\int e^{im\Psi(\widetilde{x}'',\widetilde{v}'')+im\Psi(v'',y)}\chi_1(\widetilde{x})\beta(\widetilde{x}'', \widetilde{\mathring{v}}'', m)\hat{b}(\widetilde{\mathring{v}}'', y, m)dv(\widetilde{\mathring{v}}'') + O(m^{-\infty}), \tag{6.31}$$

where $\hat{b}(\widetilde{\mathring{v}}'', y, m) = \Big(\iota_G \circ \hat{p} \circ \tau_{x,n_-} \circ f \circ \iota^* \circ \chi(\widetilde{\mathring{v}}'') \circ b\Big)(\widetilde{\mathring{v}}'', y, m)$. From (6.31) and Theorem 6.3, we see that (6.25) and (6.26) hold near u. The theorem follows. □

Let

$$F_m := \sigma^*_m\sigma_m : \Omega^{0,q}(X) \to H^q_{b,m}(X)^G, \quad \hat{F}_m := \sigma_m\sigma^*_m : \Omega^{0,q-r}(Y_G) \to H^{q-r}_{b,m}(Y_G).$$

Let $F_m(x, y)$ and $\hat{F}_m(x, y)$ be the distribution kernels of F_m and $\hat{F}_m$ respectively. From Theorems 6.2 and 6.3, we can repeat the proof of Theorem 6.4 with minor change and deduce the following two theorems

Theorem 6.5 *With the notations used above, if $y \notin Y$, then for any open set D of y with $\overline{D} \bigcap Y = \emptyset$, we have $F_m = O(m^{-\infty})$ on $X \times D$.*

Let $x, y \in Y$. If $\pi(x) \neq \pi(e^{i\theta} \circ y)$, for every $\theta \in [0, 2\pi[$, then there are open sets D_1 of x in X and D_2 of y in X such that $F_m = O(m^{-\infty})$ on $D_1 \times D_2$.

Let $p \in \mu^{-1}(0)$ and let $x = (x_1, \ldots, x_{2n+1})$ be the local coordinates as in Remark 4.4. Then,

$$\begin{aligned}&F_m(x, y) = e^{im\Psi(x'', y'')} a(x'', y'', m) + O(m^{-\infty}) \ \text{ on } U \times U,\\&a(x'', y'', m) \in S^{n-\frac{d}{2}}_{\mathrm{loc}}(1; U \times U, T^{*0,q}X \boxtimes (T^{*0,q}X)^*),\\&a(x'', y'', m) \sim \sum_{j=0}^{\infty} m^{n-\frac{d}{2}-j} a_j(\widetilde{x}'', y'') \text{ in } S^{n-\frac{d}{2}}_{\mathrm{loc}}(1; U \times U, T^{*0,q}X \boxtimes (T^{*0,q}X)^*),\\&a_j(x'', y'') \in C^\infty(U \times U, T^{*0,q}X \boxtimes (T^{*0,q}X)^*), \quad j = 0, 1, 2, \ldots,\end{aligned}$$

and

$$a_0(\widetilde{x}'', \widetilde{x}'') = 2^{-3n+4d-1}\pi^{-n-1}\frac{1}{V_{\mathrm{eff}}(\widetilde{x}'')}|\det \mathcal{L}_{\widetilde{x}''}||\det R_x|^{-\frac{1}{2}}\tau_{\widetilde{x}'',n_-}, \quad \forall \widetilde{x}'' \in U \bigcap Y, \tag{6.32}$$

where U is an open set of p.

Theorem 6.6 *Let $x, y \in Y$. If $\pi(x) \neq \pi(e^{i\theta} \circ y)$, for every $\theta \in [0, 2\pi[$, then there are open sets D_G of $\pi(x)$ in Y_G and V_G of $\pi(y)$ in Y_G such that $\hat{F}_m = O(m^{-\infty})$ on $D_G \times V_G$.*

Let $p \in \mu^{-1}(0)$ and let $x = (x_1, \ldots, x_{2n+1})$ be the local coordinates as in Remark 4.4. Then,

$$\begin{aligned}&\hat{F}_m(x, y) = e^{im\Psi(\widetilde{x}'', \widetilde{y}'')} \hat{a}(\widetilde{x}'', \widetilde{y}'', m) + O(m^{-\infty}) \ \text{ on } W \times W,\\&\hat{a}(\widetilde{x}'', \widetilde{y}'', m) \in S^{n-d}_{\mathrm{loc}}(1; W \times W, T^{*0,q-r}Y_G \boxtimes (T^{*0,q-r}Y_G)^*),\\&\hat{a}(\widetilde{x}'', \widetilde{y}'', m) \sim \sum_{j=0}^{\infty} m^{n-d-j} \hat{a}_j(\widetilde{x}'', \widetilde{y}'') \text{ in } S^{n-d}_{\mathrm{loc}}(1; W \times W, T^{*0,q-r}Y_G \boxtimes (T^{*0,q-r}Y_G)^*),\\&\hat{a}_j(\widetilde{x}'', \widetilde{y}'') \in C^\infty(W \times W, T^{*0,q-r}Y_G \boxtimes (T^{*0,q-r}Y_G)^*), \quad j = 0, 1, 2, \ldots,\\&\hat{a}_0(\widetilde{x}'', \widetilde{x}'') = 2^{-3n+\frac{5}{2}d-1}\pi^{-n+\frac{d}{2}-1}|\det \mathcal{L}_{Y_G,\widetilde{x}''}|\hat{\tau}_{\widetilde{x}''}, \quad \forall \widetilde{x}'' \in W\end{aligned}$$

where $W = \Omega_3 \times \Omega_4$, Ω_3 and Ω_4 are open sets as in the beginning of Sect. 6.1.

Let $R_m := \frac{1}{C_0}F_m - S^{(q)}_{G,m} : \Omega^{0,q}(X) \to H^q_{b,m}(X)^G$, where $C_0 = 2^{-3d+3n}\pi^{\frac{d}{2}}$. Since $F_m = F_m S^{(q)}_{G,m}$, it is clear that

$$\frac{1}{C_0}F_m = S^{(q)}_{G,m} + R_m = S^{(q)}_{G,m} + R_m S^{(q)}_{G,m} = (I + R_m)S^{(q)}_{G,m}. \tag{6.33}$$

Our next goal is to show that for m large, $I + R_m : \Omega^{0,q}(X) \to \Omega^{0,q}(X)$ is injective. From Theorem 6.5 and Theorem 1.8, we see that if $y \notin Y$, then for any open set D of y with $\overline{D} \bigcap Y = \emptyset$, we have

$$R_m = O(m^{-\infty}) \ \text{ on } X \times D. \tag{6.34}$$

Let $x, y \in Y$. If $\pi(x) \neq \pi(e^{i\theta} \circ y)$, for every $\theta \in [0, 2\pi[$, then there are open sets D_1 of x in X and D_2 of y in X such that

$$R_m = O(m^{-\infty}) \ \text{ on } D_1 \times D_2. \tag{6.35}$$

Let $p \in \mu^{-1}(0)$ and let $x = (x_1, \ldots, x_{2n+1})$ be the local coordinates as in Remark 4.4. Then,

$$\begin{aligned}&R_m(x, y) = e^{im\Psi(x'', y'')} r(x'', y'', m) + O(m^{-\infty}) \ \text{ on } U \times U,\\&r(x'', y'', m) \in S^{n-\frac{d}{2}}_{\mathrm{loc}}(1; U \times U, T^{*0,q}X \boxtimes (T^{*0,q}X)^*),\\&r(x'', y'', m) \sim \sum_{j=0}^{\infty} m^{n-\frac{d}{2}-j} r_j(x'', y'') \text{ in } S^{n-\frac{d}{2}}_{\mathrm{loc}}(1; U \times U, T^{*0,q}X \boxtimes (T^{*0,q}X)^*),\\&r_j(x'', y'') \in C^\infty(U \times U, T^{*0,q}X \boxtimes (T^{*0,q}X)^*), \quad j = 0, 1, 2, \ldots.\end{aligned} \tag{6.36}$$

Moreover, from (6.32) and (1.10), it is easy to see

$$|r_0(x,y)| \leq C|(x,y)-(x_0,x_0)|, \tag{6.37}$$

for all $x_0 \in \mu^{-1}(0)\bigcap U$, where $C > 0$ is a constant. We need

Lemma 6.7 *Let $p \in \mu^{-1}(0)$ and let $x = (x_1, \ldots, x_{2n+1})$ be the local coordinates as in Remark 4.4 defined in an open set U of p. Let*

$$\begin{aligned} & H_m(x,y) = e^{im\Psi(x'',y'')}h(x,y,m) \ \text{ on } U\times U, \\ & h(x,y,m) \in S^{n-1-\frac{d}{2}}_{\mathrm{loc}}(1; U\times U, T^{*0,q}X \boxtimes (T^{*0,q}X)^*), \\ & h(x,y,m) \sim \textstyle\sum_{j=0}^{\infty} m^{n-1-\frac{d}{2}-j}h_j(x,y) \text{ in } S^{n-1-\frac{d}{2}}_{\mathrm{loc}}(1; U\times U, T^{*0,q}X \boxtimes (T^{*0,q}X)^*), \\ & h_j(x,y) \in C^\infty_0(U\times U, T^{*0,q}X \boxtimes (T^{*0,q}X)^*), \quad j=0,1,2,\ldots. \end{aligned}$$

Assume that $h(x,y,m) \in C^\infty_0(U\times U, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$. Then, there is a constant $\hat{C} > 0$ independent of m such that

$$\|H_m u\| \leq \delta_m \|u\|, \quad \forall u \in \Omega^{0,q}(X), \quad \forall m \in \mathbb{N}, \tag{6.38}$$

where δ_m is a sequence with $\lim_{m\to\infty}\delta_m = 0$.

Proof Fix $N \in \mathbb{N}$. It is not difficult to see that

$$\|H_m u\| \leq \left\|(H_m^* H_m)^{2^N} u\right\|^{\frac{1}{2^{N+1}}} \|u\|^{1-\frac{1}{2^{N+1}}}, \quad \forall u \in \Omega^{0,q}(X), \tag{6.39}$$

where H_m^* denotes the adjoint of H_m. From Theorem 6.2, we can repeat the proof of Theorem 6.4 with minor change and deduce that

$$\begin{aligned} & (H_m^* H_m)^{2^N}(x,y) = e^{im\Psi(x'',y'')}p(x,y,m) + O(m^{-\infty}) \ \text{ on } U\times U, \\ & p(x,y,m) \in S^{n-2^{N+1}-\frac{d}{2}}_{\mathrm{loc}}(1; U\times U, T^{*0,q}X \boxtimes (T^{*0,q}X)^*), \\ & p(x,y,m) \in C^\infty_0(U\times U, T^{*0,q}X \boxtimes (T^{*0,q}X)^*). \end{aligned}$$

Hence,

$$|(H_m^* H_m)^{2^N}(x,y)| \leq \hat{C} m^{n-2^{N+1}-\frac{d}{2}}, \quad \forall (x,y) \in U\times U, \tag{6.40}$$

where $\hat{C} > 0$ is a constant independent of m. Take N large enough so that $n - 2^{N+1} - \frac{d}{2} < 0$. From (6.40) and (6.39), we get (6.38). □

We also need

Lemma 6.8 *Let $p \in \mu^{-1}(0)$ and let $x = (x_1, \ldots, x_{2n+1})$ be the local coordinates as in Remark 4.4 defined in an open set U of p. Let*

$$\begin{aligned} & B_m(x,y) = e^{im\Psi(x'',y'')}g(x,y,m) \ \text{ on } U\times U, \\ & g(x,y,m) \in S^{n-\frac{d}{2}}_{\mathrm{loc}}(1; U\times U, T^{*0,q}X \boxtimes (T^{*0,q}X)^*), \\ & g(x,y,m) \sim \textstyle\sum_{j=0}^{\infty} m^{n-\frac{d}{2}-j}g_j(x,y) \text{ in } S^{n-\frac{d}{2}}_{\mathrm{loc}}(1; U\times U, T^{*0,q}X \boxtimes (T^{*0,q}X)^*), \\ & g_j(x,y) \in C^\infty_0(U\times U, T^{*0,q}X \boxtimes (T^{*0,q}X)^*), \quad j=0,1,2,\ldots, \\ & g(x,y) \in C^\infty_0(U\times U, T^{*0,q}X \boxtimes (T^{*0,q}X)^*). \end{aligned}$$

Suppose that

$$|g_0(x,y)| \leq C|(x,y)-(x_0,x_0)|,$$

for all $x_0 \in \mu^{-1}(0) \bigcap U$, *where* $C > 0$ *is a constant. Then,*

$$\|B_m u\| \leq \varepsilon_m \|u\|, \quad \forall u \in \Omega^{0,q}(X), \quad \forall m \in \mathbb{N}, \tag{6.41}$$

where ε_m *is a sequence with* $\lim_{m\to\infty} \varepsilon_m = 0$.

Proof Fix $N \in \mathbb{N}$. It is not difficult to see that

$$\|B_m u\| \leq \left\|(B_m^* B_m)^{2^N} u\right\|^{\frac{1}{2^{N+1}}} \|u\|^{1-\frac{1}{2^{N+1}}}, \quad \forall u \in \Omega^{0,q}(X), \tag{6.42}$$

where B_m^* denotes the adjoint of B_m. From Theorem 6.2, we can repeat the proof of Theorem 6.4 with minor change and deduce that

$$\begin{aligned}
&(B_m^* B_m)^{2^N}(x, y) = e^{im\Psi(x'', y'')}\hat{g}(x, y, m) + O(m^{-\infty}) \text{ on } U \times U,\\
&\hat{g}(x, y, m) \in S^{n-\frac{d}{2}}_{\rm loc}(1; U \times U, T^{*0,q}X \boxtimes (T^{*0,q}X)^*),\\
&\hat{g}(x, y, m) \sim \sum_{j=0}^{\infty} m^{n-\frac{d}{2}-j}\hat{g}_j(x, y) \text{ in } S^{n-\frac{d}{2}}_{\rm loc}(1; U \times U, T^{*0,q}X \boxtimes (T^{*0,q}X)^*),\\
&\hat{g}_j(x, y) \in C_0^\infty(U \times U, T^{*0,q}X \boxtimes (T^{*0,q}X)^*), \quad j = 0, 1, 2, \ldots,\\
&\hat{g}(x, y, m) \in C_0^\infty(U \times U, T^{*0,q}X \boxtimes (T^{*0,q}X)^*),
\end{aligned}$$

and

$$|\hat{g}_0(x, y)| \leq C|(x, y) - (x_0, x_0)|^{2^{N+1}}, \tag{6.43}$$

for all $x_0 \in \mu^{-1}(0) \bigcap U$, where $C > 0$ is a constant. Let

$$(B_m^* B_m)_0^{2^N}(x, y) = e^{im\Psi(x'', y'')}\hat{g}_0(x, y, m), \quad (B_m^* B_m)_1^{2^N}(x, y) = e^{im\Psi(x'', y'')}h(x, y, m),$$

where $h(x, y, m) = \hat{g}(x, y, m) - \hat{g}_0(x, y, m)$. It is clear that $h(x, y, m) \in S^{n-1-\frac{d}{2}}_{\rm loc}(1; U \times U, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$. From Lemma 6.7, we see that

$$\left\|(B_m^* B_m)_1^{2^N} u\right\| \leq \delta_m \|u\|, \quad \forall u \in \Omega^{0,q}(X), \quad \forall m \in \mathbb{N}, \tag{6.44}$$

where δ_m is a sequence with $\lim_{m\to\infty} \delta_m = 0$.

From (6.43), we see that

$$|\hat{g}_0(x, y)| \leq C_1\Big(|\hat{x}''| + |\hat{y}''| + |\widetilde{\mathring{x}}'' - \widetilde{\mathring{y}}''|\Big)^{2^{N+1}}, \tag{6.45}$$

where $C_1 > 0$ is a constant. From (3.44), we see that

$$|\mathrm{Im}\,\Psi(x, y)| \geq c\Big(|\hat{x}''|^2 + |\hat{y}''|^2 + |\widetilde{\mathring{x}}'' - \widetilde{\mathring{y}}''|^2\Big), \tag{6.46}$$

where $c > 0$ is a constant. From (6.45) and (6.46), we conclude that

$$|(B_m^* B_m)_0^{2^N}(x, y)| \leq \hat{C}m^{-2^N+n-\frac{d}{2}}, \quad \forall (x, y) \in U \times U, \tag{6.47}$$

where $\hat{C} > 0$ is a constant independent of m. From (6.47), we see that if N large enough, then

$$\left\|(B_m^* B_m)_0^{2^N} u\right\| \leq \hat{\delta}_m \|u\|, \quad \forall u \in \Omega^{0,q}(X), \quad \forall m \in \mathbb{N}, \tag{6.48}$$

where $\hat{\delta}_m$ is a sequence with $\lim_{m\to\infty} \hat{\delta}_m = 0$.

From (6.42), (6.44) and (6.48), we get (6.41). □

From (6.34), (6.35), (6.36), (6.37) and Lemma 6.8, we get

Theorem 6.9 *With the notations above, we have* $\|R_m u\| \leq \varepsilon_m \|u\|, \quad \forall u \in \Omega^{0,q}(X), \ \forall m \in \mathbb{N}$, *where* ε_m *is a sequence with* $\lim_{m\to\infty} \varepsilon_m = 0$.

In particular, if m *is large enough, then the map* $I + R_m : \Omega^{0,q}(X) \to \Omega^{0,q}(X)$ *is injective.*

Proof of Theorem 1.9 From (6.33) and Theorem 6.9, we see that if m is large enough, then the map $F_m = \sigma_m^* \sigma_m : H^q_{b,m}(X)^G \to H^q_{b,m}(X)^G$ is injective. Hence, if m is large enough, then the map $\sigma_m : H^q_{b,m}(X)^G \to H^{q-r}_{b,m}(Y_G)$ is injective and $\dim H^q_{b,m}(X)^G \leq \dim H^{q-r}_{b,m}(Y_G)$.

Similarly, we can repeat the proof of Theorem 6.9 with minor change and deduce that, if m is large enough, then the map $\hat{F}_m = \sigma_m \sigma_m^* : H^{q-r}_{b,m}(Y_G) \to H^{q-r}_{b,m}(Y_G)$ is injective. Hence, if m is large enough, then the map $\sigma_m^* : H^{q-r}_{b,m}(Y_G) \to H^q_{b,m}(X)^G$ is injective. Thus, $\dim H^q_{b,m}(X)^G = \dim H^{q-r}_{b,m}(Y_G)$ and σ_m is an isomorphism if m large enough. □

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