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Calculus of Variations

Higher integrability for variational integrals with non-standard growth

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Abstract
We consider autonomous integral functionals of the form

$$\mathcal{F}[u] := \int_\Omega f(Du)\,dx \quad \text{where } u : \Omega \to \mathbb{R}^N,\, N \geq 1,$$

where the convex integrand f satisfies controlled (p, q)-growth conditions. We establish higher gradient integrability and partial regularity for minimizers of $\mathcal{F}$ assuming $\frac{q}{p} < 1+\frac{2}{n-1}$, $n \geq 3$. This improves earlier results valid under the more restrictive assumption $\frac{q}{p} < 1 + \frac{2}{n}$.

Mathematics Subject Classification 49N60 · 35J70

1 Introduction

In this note, we study regularity properties of local minimizers of integral functionals

$$\mathcal{F}[u] := \int_\Omega f(Du)\,dx, \tag{1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is a bounded domain, $u : \Omega \to \mathbb{R}^N$, $N \geq 1$ and $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ is a sufficiently smooth integrand satisfying (p, q)-growth of the form

Assumption 1 There exist $0 < \nu \leq L < \infty$ such that $f \in C^2(\mathbb{R}^{N\times n})$ satisfies for all $z, \xi \in \mathbb{R}^{N\times n}$

$$\begin{cases} \nu|z|^p \leq f(z) \leq L(1 + |z|^q), \\ \nu|z|^{p-2}|\xi|^2 \leq \langle \partial^2 f(z)\xi, \xi\rangle \leq L(1 + |z|^2)^{\frac{q-2}{2}}|\xi|^2. \end{cases} \tag{2}$$

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Regularity properties of local minimizers of (1) in the case $p = q$ are classical, see, e.g., [24]. A systematic regularity theory in the case $p < q$ was initiated by Marcellini in [27,28], see [31] for an overview (for a more up-to-date overview see the introduction in [30]). In particular, Marcellini [29] proves (among other things):

(A) If $N = 1$, $2 \le p < q$ and $\frac{q}{p} < 1 + \frac{2}{n}$, then every local minimizer $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ of (1) satisfies $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$.

Local boundedness of the gradient implies that the non-standard growth of f and $\partial^2 f$ in (1) becomes irrelevant and higher regularity (depending on the smoothness of f) follows by standard arguments, see e.g. [27, Chapter 7].

Only very recently, Bella and the author improved in [6] the result (A) in the sense that 'n' in the assumption on the ratio $\frac{q}{p}$ can be replaced by '$n-1$' for $n \ge 3$ (to be precise, [6] considers the non-degenerate version (4) of (2)). The argument in [6] relies on scalar techniques, e.g., Moser-iteration type arguments, and thus cannot be extended to the vectorial case $N > 1$.

For the vectorial case $N > 1$, Esposito, Leonetti and Mingione showed in [18] that

(B) If $2 \le p < q$ and $\frac{q}{p} < 1 + \frac{2}{n}$, then every local minimizer $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ of (1) satisfies $u \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$.

To the best of our knowledge, there is no improvement of (B) with respect to the relation between the exponents p, q and the dimension n available in the literature. Here we provide such an improvement for $n \ge 3$.

Before we state the results, we recall a standard notion of local minimizer in the context of functionals with (p, q)-growth

Definition 1 We call $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ a local minimizer of $\mathcal{F}$ given in (1) iff

$$f(Du) \in L^1_{\mathrm{loc}}(\Omega)$$

and

$$\int_{\operatorname{supp}\varphi} f(Du)\,dx \le \int_{\operatorname{supp}\varphi} f(Du + D\varphi)\,dx$$

for any $\varphi \in W^{1,1}(\Omega, \mathbb{R}^N)$ satisfying $\operatorname{supp}\varphi \Subset \Omega$.

The main result of the present paper is

Theorem 2 *Let $\Omega \subset \mathbb{R}^n$, $n \ge 3$, and suppose Assumption 1 is satisfied with $2 \le p < q < \infty$ such that*

$$\frac{q}{p} < 1 + \frac{2}{n-1}. \tag{3}$$

Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}$ given in (1). Then, $u \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$.

Higher gradient integrability is a first step in the regularity theory for integral functionals with (p, q)-growth, see [7,11,19,20] for further higher integrability results under (p, q)-conditions. Clearly, we cannot expect to improve from $W^{1,q}_{\mathrm{loc}}$ to $W^{1,\infty}_{\mathrm{loc}}$ for $N > 1$, since this even fails in the classic setting $p = q$, see [34]. Direct consequences of Theorem 2 are higher differentiability and a further improvement in gradient integrability in the form:

(i) (Higher differentiability). In the situation of Theorem 2 it holds $|\nabla u|^{\frac{p-2}{2}}\nabla u \in W^{1,2}_{\mathrm{loc}}(\Omega)$, see Theorem 5.
(ii) (Higher integrability). Sobolev inequality and (i) imply $\nabla u \in L^{\kappa p}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n})$ with $\kappa = \frac{n}{n-2}$. Note that $\kappa p > q$ provided $\frac{q}{p} < 1 + \frac{2}{n-2}$.

A further, on first glance less direct, consequence of Theorem 2 is partial regularity of minimizers of (1), see, e.g., [1,7,10,32], for partial regularity results under (p, q)-conditons. For this, we slightly strengthen the assumptions on the integrand and suppose

Assumption 3 There exist $0 < \nu \le L < \infty$ such that $f \in C^2(\mathbb{R}^{N\times n})$ satisfies for all $z, \xi \in \mathbb{R}^{N\times n}$

$$\begin{cases} \nu|z|^p \le f(z) \le L(1+|z|^q), \\ \nu(1+|z|^2)^{\frac{p-2}{2}}|\xi|^2 \le \langle \partial^2 f(z)\xi, \xi\rangle \le L(1+|z|^2)^{\frac{q-2}{2}}|\xi|^2. \end{cases} \tag{4}$$

In [7], Bildhauer and Fuchs prove partial regularity under Assumption 3 with $\frac{q}{p} < 1 + \frac{2}{n}$ ([7] contains also more general conditions including, e.g., the subquadratic case). Here we show

Theorem 4 *Let $\Omega \subset \mathbb{R}^n$, $n \ge 3$, and suppose Assumption 3 is satisfied with $2 \le p < q < \infty$ such that (3). Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}$ given in (1). Then, there exists an open set $\Omega_0 \subset \Omega$ with $|\Omega \setminus \Omega_0| = 0$ such that $\nabla u \in C^{0,\alpha}(\Omega_0, \mathbb{R}^{N\times n})$ for each $0 < \alpha < 1$.*

We do not know if (3) in Theorems 2 and 4 is optimal. Classic counterexamples in the scalar case $N = 1$, see, e.g., [23,28], show that local boundedness of minimizers can fail if $\frac{q}{p}$ is to large depending on the dimension n. In fact, [28, Theorem 6.1] and the recent boundedness result [26] show that $\frac{1}{p} - \frac{1}{q} \le \frac{1}{n-1}$ is the sharp condition ensuring local boundedness in the scalar case $N = 1$ (for sharp results under additional structure assumptions, see, e.g., [14,22]).

For non-autonomous functionals, i.e., $\int_\Omega f(x, Du)\,dx$, rather precise sufficiently & necessary conditions are established in [20], where the conditions on p, q and n has to be balanced with the (Hölder)-regularity in space of the integrand. However, if the integrand is sufficiently smooth in space, the regularity theory in the non-autonomous case essentially coincides with the autonomous case, see [10]. Currently, regularity theory for non-autonomous integrands with non-standard growth, e.g. $p(x)$-Laplacian or double phase functionals are a very active field of research, see, e.g., [2,12,13,15–17,25,33].

Coming back to autonomous integral functionals: In [11] higher gradient integrability is proven assuming so-called 'natural' growth conditions, i.e., no upper bound assumption on $\partial^2 f$, under the relation $\frac{q}{p} < 1 + \frac{1}{n-1}$. Moreover, in two dimensions we cannot improve the previous results on higher differentiability and partial regularity of, e.g., [7,18], see [8] for a full regularity result under Assumption 3 with $n = 2$ and $\frac{q}{p} < 2$. Finally, we mention the recent paper [3] where optimal Lipschitz-estimates with respect to a right-hand side are proven for functionals with (p, q)-growth.

Let us briefly describe the main idea in the proof of Theorem 2 and from where our improvement compared to earlier results comes from. The main point is to obtain suitable a priori estimates for minimizers that may already be in $W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$. The claim then follows by a known regularization and approximation procedure, see, e.g., [18]. For minimizers $v \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ a Caccioppoli-type inequality

$$\int \eta^2 |D(|Dv|^{\frac{p-2}{2}} Dv)|^2 \lesssim \int |\nabla \eta|^2 (1 + |Dv|^q) \tag{5}$$

is valid for all sufficiently smooth cut-off functions η, see Lemma 1. Very formally, the Caccioppoli inequality (5) can be combined with Sobolev inequality and a simple interpolation inequality to obtain

$$\|Dv\|_{L^{\kappa p}}^p \lesssim \|D(|Dv|^{\frac{p-2}{2}}Dv)\|_{L^2}^2 \lesssim \|Dv\|_{L^q}^q \lesssim \|Dv\|_{L^{\kappa p}}^{q\theta}\|Dv\|_{L^p}^{(1-\theta)q},$$

where $\theta = \frac{\frac{1}{p}-\frac{1}{q}}{\frac{1}{p}-\frac{1}{\kappa p}} \in (0,1)$ and $\kappa = \frac{n}{n-2}$. The $\|Dv\|_{L^{\kappa p}}$-factor on the right-hand side can be absorbed provided we have $\frac{q\theta}{p} < 1$, but this is precisely the 'old' (p,q)-condition $\frac{q}{p} < 1+\frac{2}{n}$, this type of argument was previously rigorously implemented in, e.g., [7,19]. Our improvement comes from choosing a cut-of function η in (5) that is optimized with respect to v, which enables us to use Sobolev inequality on $n-1$-dimensional spheres wich gives the desired improvement, see Sect. 3. This idea has its origin in joint works with Bella [4,5] on linear non-uniformly elliptic equations.

With Theorem 2 at hand, we can follows the arguments of [7] almost verbatim to prove Theorem 4. In Sect. 4, we sketch (following [7]) a corresponding ε-regularity result from which Theorem 4 follows by standard methods.

2 Preliminary results

In this section, we gather some known facts. We begin with a well-known higher differentiability result for minimizers of (1) under the assumption that $u \in W^{1,q}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$:

Lemma 1 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and suppose Assumption 1 is satisfied with $2 \leq p < q < \infty$. Let $v \in W^{1,q}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}$ given in (1). Then, $|Dv|^{\frac{p-2}{2}}Dv \in W^{1,2}_{\mathrm{loc}}(\Omega,\mathbb{R}^{N\times n})$ and there exists $c = c(\frac{L}{\nu}, n, N, p, q) \in [1,\infty)$ such that for every $Q \in \mathbb{R}^{N\times n}$ and every $\eta \in C^1_c(\Omega)$*

$$\int_\Omega \eta^2|D(|Dv|^{\frac{p-2}{2}}Dv)|^2\,dx \leq c\int_\Omega (1+|Dv|^2)^{\frac{q-2}{2}}|Dv-Q|^2|\nabla\eta|^2\,dx. \tag{6}$$

The Lemma 1 is known, see e.g. [7,18,28]. Since we did not find a precise reference for estimate (6), we included a prove here following essentially the argument of [18].

Proof of Lemma 1 Without loss of generality, we suppose $\nu = 1$ the general case $\nu > 0$ follows by replacing f with f/ν (and thus L with L/ν). Throughout the proof, we write $\lesssim$ if $\leq$ holds up to a multiplicative constant depending only on n, N, p and q.

Thanks to the assumption $v \in W^{1,q}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$, the minimizer v satisfies the Euler-Largrange equation

$$\int_\Omega \langle \partial f(Dv), D\varphi\rangle\,dx = 0 \qquad \text{for all } \varphi \in W^{1,q}_0(\Omega,\mathbb{R}^N) \tag{7}$$

(for this we use that the convexity and growth conditions of f imply $|\partial f(z)| \leq c(1+|z|^{q-1})$ for some $c = c(L,n,N,q,) < \infty$). Next, we use the difference quotient method, to differentiate the above equation: For $s \in \{1,\ldots,n\}$, we consider the difference quotient operator

$$\tau_{s,h}v := \tfrac{1}{h}(v(\cdot+he_s)-v) \qquad \text{where } v \in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N).$$

Fix $\eta \in C_c^1(\Omega)$. Testing (7) with $\varphi := \tau_{s,-h}(\eta^2(\tau_{s,h}(v-\ell_Q))) \in W_0^{1,q}(\Omega)$, where $\ell_Q(x) = Qx$, we obtain

$$(I) := \int_\Omega \eta^2 \langle \tau_{s,h}\partial f(Dv), \tau_{s,h}Dv\rangle\, dx$$
$$= -2\int_\Omega \eta\langle \tau_{s,h}\partial f(Dv), \tau_{s,h}(v-\ell_Q)\otimes\nabla\eta\rangle\, dx =: (II).$$

Writing $\tau_{s,h}\partial f(Dv) = \frac{1}{h}\partial f(Dv + th\tau_{s,h}Dv)\big|_{t=0}^{t=1}$, the fundamental theorem of calculus yields

$$\int_\Omega\int_0^1 \eta^2\langle \partial^2 f(Dv+th\tau_{s,h}Dv))\tau_{s,h}Dv, \tau_{s,h}Dv\rangle\, dt\, dx = (I)$$
$$=(II) = -2\int_\Omega\int_0^1 \eta\langle\partial^2 f(Dv+th\tau_{s,h}Dv)\tau_{s,h}Dv, (\tau_{s,h}v - Qe_s)\otimes\nabla\eta\rangle\, dt\, dx, \tag{8}$$

where we use $\tau_{h,s}\ell_Q = Qe_s$. Youngs inequality yields

$$|(II)| \le \tfrac{1}{2}(I) + 2(III), \tag{9}$$

where

$$(III) := \int_\Omega\int_0^1 \langle\partial^2 f(Du+th\tau_{s,h}Du)(\tau_{s,h}v-Qe_s)\otimes\nabla\eta, (\tau_{s,h}v-Qe_s)\otimes\nabla\eta\rangle\, dt\, dx.$$

Combining (8), (9) with the assumptions on $\partial^2 f$, see (2), with the elementary estimate

$$|\tau_{s,h}(|Dv|^{\frac{p-2}{2}}Dv)|^2 \lesssim \int_0^1 |Dv+th\tau_{s,h}Dv|^{\frac{p-2}{2}}|\tau_{s,h}Dv|^2\, dt$$

for $h > 0$ sufficiently small (see e.g. [18, Lemma 3.4]), we obtain

$$\int_\Omega \eta^2|\tau_{s,h}(|Dv|^{\frac{p-2}{2}}Dv)|^2\, dx$$
$$\lesssim \int_\Omega\int_0^1 \eta^2|Dv+th\tau_{s,h}Dv|^{\frac{p-2}{2}}|\tau_{s,h}Dv|^2\, dt\, dx \le (I)$$
$$\le 4(III) \le 4L\int_\Omega\int_0^1 (1+|Dv+th\tau_{s,h}Dv|^{q-2})|\nabla\eta|^2|\tau_{s,h}v - Qe_s|^2\, dt\, dx. \tag{10}$$

Estimate (10), the fact $v \in W^{1,q}_{\mathrm{loc}}(\Omega)$ and the arbitrariness of $\eta \in C_c^1(\Omega)$ and $s \in \{1,\dots,n\}$ yield $|Dv|^{\frac{p-2}{2}}Dv \in W^{1,2}_{\mathrm{loc}}(\Omega)$. Sending h to zero in (10), we obtain

$$\int_\Omega \eta^2|\partial_s(|Dv|^{\frac{p-2}{2}}Dv)|^2\, dx \lesssim L\int_\Omega (1+|Dv|^{q-2})|\nabla\eta|^2|\partial_s v - Qe_s|^2\, dx$$

the desired estimate (6) follows by summing over s. □

Next, we state a higher differentiability result under the more restrictive Assumption 3 which will be used in the proof of Theorem 4.

Lemma 2 *Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, and suppose Assumption 3 is satisfied with $2 \le p < q < \infty$. Let $v \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}$ given in (1). Then, $h :=$*

$(1+|Dv|^2)^{\frac{p}{4}} \in W^{1,2}_{\rm loc}(\Omega)$ *and there exists* $c = c(\frac{L}{\nu}, n, N, p, q) \in [1,\infty)$ *such that for every* $Q \in \mathbb{R}^{N\times n}$

$$\int_\Omega \eta^2 |\nabla h|^2\,dx \leq c\int_\Omega (1+|Dv|^2)^{\frac{q-2}{2}}|Dv-Q|^2|\nabla\eta|^2\,dx \quad \textit{for all } \eta\in C^1_c(\Omega). \tag{11}$$

A variation of Lemma 2 can be found in [7] and we only sketch the proof.

Proof of Lemma 2 With the same argument as in the proof of Lemma 1 but using (4) instead of (2), we obtain $v \in W^{2,2}_{\rm loc}(\Omega, \mathbb{R}^N)$ and the Caccioppoli inequality

$$\int_\Omega \eta^2 (1+|Dv|^2)^{\frac{p-2}{2}}|D^2v|^2\,dx \leq c\int_\Omega (1+|Dv|^2)^{\frac{q-2}{2}}|Dv-Q|^2|\nabla\eta|^2\,dx \tag{12}$$

for all $\eta \in C^1_c(\Omega)$, where $c = c(\frac{L}{\nu}, n, N, p, q) < \infty$. Formally, the chain-rule implies

$$|\nabla h|^2 \leq c(1+|Dv|^2)^{\frac{p-2}{2}}|D^2v|^2, \tag{13}$$

where $c = c(n,p) < \infty$, and the claimed estimate (11) follows from (12) and (13). In general, we are not allowed to use the chain rule, but the above reasoning can be made rigorous: Consider a truncated version h_m of h, where $h_m := \Theta_m(|Dv|)$ with

$$\Theta_m(t) := \begin{cases} (1+t^2)^{\frac{p}{4}} & \text{if } 0\leq t\leq m \\ (1+m^2)^{\frac{p}{4}} & \text{if } t\geq m \end{cases}.$$

For h_m we are allowed to use the chain-rule and (12) together with (13) with h replaced by h_m imply (11) with h replaced by h_m. The claimed estimate follows by taking the limit $m\to\infty$, see [7, Proposition 3.2] for details. □

The following technical lemma is contained in [6] (see also [4, proof of Lemma 2.1, Step 1]) and plays a key role in the proof of Theorem 2

Lemma 3 *([6],* **Lemma 3***) Fix* $n \geq 2$. *For given* $0<\rho<\sigma<\infty$ *and* $v \in L^1(B_\sigma)$, *consider*

$$J(\rho,\sigma,v) := \inf\left\{\int_{B_\sigma} |v||\nabla\eta|^2\,dx \mid \eta\in C^1_0(B_\sigma),\ \eta\geq 0,\ \eta = 1 \textit{ in } B_\rho\right\}.$$

Then for every $\delta \in (0,1]$

$$J(\rho,\sigma,v) \leq (\sigma-\rho)^{-(1+\frac{1}{\delta})}\left(\int_\rho^\sigma \left(\int_{\partial B_r} |v|\,d\mathcal{H}^{n-1}\right)^\delta dr\right)^{\frac{1}{\delta}}. \tag{14}$$

For convenience of the reader we include a short proof of Lemma 3

Proof of Lemma 3 Estimate (14) follows directly by minimizing among radial symmetric cut-off functions. Indeed, we obviously have for every $\varepsilon \geq 0$

$$\begin{aligned} &J(\rho,\sigma,v)\\ &\leq \inf\left\{\int_\rho^\sigma \eta'(r)^2\left(\int_{\partial B_r}|v|\,d\mathcal{H}^{n-1}+\varepsilon\right)dr \mid \eta\in C^1(\rho,\sigma),\ \eta(\rho)=1,\ \eta(\sigma)=0\right\}\\ &=: J_{1{\rm d},\varepsilon}. \end{aligned}$$

For $\varepsilon > 0$, the one-dimensional minimization problem $J_{1d,\varepsilon}$ can be solved explicitly and we obtain

$$J_{1d,\varepsilon} = \left(\int_\rho^\sigma \left(\int_{\partial B_r} |v|\, d\mathcal{H}^{n-1} + \varepsilon\right)^{-1} dr\right)^{-1}. \tag{15}$$

To see (15), we observe that using the assumption $v \in L^1(B_\sigma)$ and a simple approximation argument we can replace $\eta \in C^1(\rho, \sigma)$ with $\eta \in W^{1,\infty}(\rho, \sigma)$ in the definition of $J_{1d,\varepsilon}$. Let $\widetilde{\eta} : [\rho, \sigma] \to [0, \infty)$ be given by

$$\widetilde{\eta}(r) := 1 - \left(\int_\rho^\sigma b(r)^{-1}\, dr\right)^{-1} \int_\rho^r b(r)^{-1}\, dr, \quad \text{where } b(r) := \int_{\partial B_r} |v| + \varepsilon.$$

Clearly, $\widetilde{\eta} \in W^{1,\infty}(\rho, \sigma)$ (since $b \geq \varepsilon > 0$), $\widetilde{\eta}(\rho) = 1$, $\widetilde{\eta}(\sigma) = 0$, and thus

$$J_{1d,\varepsilon} \leq \int_\rho^\sigma \widetilde{\eta}'(r)^2 b(r)\, dr = \left(\int_\rho^\sigma b(r)^{-1}\, dr\right)^{-1}.$$

The reverse inequality follows by Hölder's inequality. Next, we deduce (14) from (15): For every $s > 1$, we obtain by Hölder inequality $\sigma - \rho = \int_\rho^\sigma (\frac{b}{b})^{\frac{s-1}{s}} \leq \left(\int_\rho^\sigma b^{s-1}\right)^{\frac{1}{s}} \left(\int_\rho^\sigma \frac{1}{b}\right)^{\frac{s-1}{s}}$ with b as above, and by (15) that

$$J_{1d,\varepsilon} \leq (\sigma - \rho)^{-\frac{s}{s-1}} \left(\int_\rho^\sigma \left(\int_{\partial B_r} |v| + \varepsilon\right)^{s-1} dr\right)^{\frac{1}{s-1}}.$$

Sending ε to zero, we obtain (14) with $\delta = s - 1 > 0$. □

3 Higher integrability - Proof of Theorem 2

In this section, we prove the following higher integrability and differentiability result which clearly contains Theorem 2

Theorem 5 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and suppose Assumption 1 is satisfied with $2 \leq p < q < \infty$ such that $\frac{q}{p} < 1 + \min\{\frac{2}{n-1}, 1\}$. Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}$ given in (1). Then, $u \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ and $|Du|^{\frac{p-2}{2}} Du \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^{N\times n})$. Moreover, for*

$$\chi = \frac{n-1}{n-3} \quad \text{if } n \geq 4 \quad \chi \in \left(\frac{1}{2 - \frac{q}{p}}, \infty\right) \quad \text{if } n = 3 \text{ and} \quad \chi := \infty \quad \text{if } n = 2. \tag{16}$$

there exists $c = c(\frac{L}{\nu}, n, N, p, q, \chi) \in [1, \infty)$ such that for every $B_R(x_0) \Subset \Omega$

$$\fint_{B_{\frac{R}{2}}(x_0)} |Du|^q\, dx + R^2 \fint_{B_{\frac{R}{2}}(x_0)} |D(|Du|^{\frac{p-2}{2}} Du)|^2\, dx \leq c\left(\fint_{B_R(x_0)} 1 + f(Du)\, dx\right)^{\frac{\alpha q}{p}} \tag{17}$$

where

$$\alpha := \frac{1 - \frac{q}{\chi p}}{2 - \frac{q}{p} - \frac{1}{\chi}}. \tag{18}$$

Proof of Theorem 5 Without loss of generality, we suppose $\nu = 1$ the general case $\nu > 0$ follows by replacing f with f/ν. Throughout the proof, we write $\lesssim$ if $\leq$ holds up to a multiplicative constant depending only on L, n, N, p and q.

Following, e.g., [7,18,19], we consider the perturbed integral functionals

$$\mathcal{F}_\lambda(w) := \int_\Omega f_\lambda(Dw)\,dx, \qquad \text{where} \quad f_\lambda(z) := f(z) + \lambda|z|^q \quad \text{with } \lambda \in (0,1). \tag{19}$$

We then derive suitable a priori higher differentiability and integrability estimates for local minimizers of $\mathcal{F}_\lambda$ that are independent of $\lambda \in (0,1)$. The claim then follows with help of a by now standard double approximation procedure in spirit of [18].

Step 1. One-step improvement.

Let $v \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}_\lambda$ defined in (19), $B_1 \Subset \Omega$, and let $\chi > 1$ be defined in (16). We claim that there exists $c = c(L, n, N, p, q, \chi) \in [1, \infty)$ such that for all $\frac{1}{2} \leq \rho < \sigma \leq 1$ and every $\lambda \in (0, 1]$

$$\begin{aligned}&\int_{B_1} 1 + f_\lambda(Dv) + \int_{B_\rho} |D(|Dv|^{\frac{p-2}{2}}Dv)|^2\,dx\\ &\leq \frac{c\Big(\int_{B_1} 1 + f_\lambda(Dv)\Big)^{\frac{\chi}{\chi-1}(1-\frac{q}{\chi p})}}{(\sigma-\rho)^{1+\frac{q}{p}}}\\ &\quad\times\left(\int_{B_1} 1 + f_\lambda(Dv) + \int_{B_\sigma} |D(|Dv|^{\frac{p-2}{2}}Dv)|^2\,dx\right)^{\frac{\chi}{\chi-1}(\frac{q}{p}-1)}\end{aligned} \tag{20}$$

with the understanding $\frac{\infty}{\infty-1} = 1$ and

$$\int_{B_\rho} |D(|Dv|^{\frac{p-2}{2}}Dv)|^2\,dx \lesssim \frac{1}{(\sigma-\rho)^2}\frac{1}{\lambda}\int_{B_\sigma} 1 + f_\lambda(Dv)\,dx. \tag{21}$$

The growth conditions of f_λ and the minimality of v imply $v \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ and thus by Lemma 1

$$\int_\Omega |D(|Dv|^{\frac{p-2}{2}}Dv)|^2\eta^2\,dx \lesssim \int_\Omega (1+|Dv|^2)^{\frac{q-2}{2}}|Dv|^2|\nabla\eta|^2\,dx \tag{22}$$

for all $\eta \in C^1_c(\Omega)$. Estimate (21) follows directly from (22) for $\eta \in C^1_c(B_\sigma)$ with $0 \leq \eta \leq 1$, $\eta \equiv 1$ on B_ρ and $|\nabla\eta| \leq \frac{2}{\sigma-\rho}$, combined with $|z|^q \leq \frac{1}{\lambda}f_\lambda(z)$ and $\lambda \in (0, 1]$.

Hence, it is left to show (20). For this, we use a technical estimate which follows from Lemma 3 and Hölders inequality: For given $0 < \rho < \sigma < \infty$ and $w \in L^q(B_\sigma)$ it holds

$$J(\rho, \sigma, |w|^q) \leq \frac{\Big(\int_{B_\sigma\setminus B_\rho} |w|^p\Big)^{\frac{\chi}{\chi-1}(1-\frac{q}{\chi p})}}{(\sigma-\rho)^{1+\frac{q}{p}}}\left(\int_\rho^\sigma \|w\|^p_{L^{\chi p}(\partial B_r)}\,dr\right)^{\frac{\chi}{\chi-1}(\frac{q}{p}-1)}, \tag{23}$$

where J is defined as in Lemma 3. We postpone the derivation of (23) to the end of this step.

Combining (22) with $(1+|Dv|^2)^{\frac{q-2}{2}}|Dv|^2 \leq (1+|Dv|)^q$ and estimate (23) with $w = 1 + |Dv|$, we obtain

$$\int_{B_\rho} |D(|Dv|^{\frac{p-2}{2}} Dv)|^2 \, dx$$
$$\lesssim \frac{\left(\int_{B_\sigma \setminus B_\rho} (1+|Dv|)^p \, dx\right)^{\frac{\chi}{\chi-1}(1-\frac{q}{\chi p})}}{(\sigma-\rho)^{1+\frac{q}{p}}} \left(\int_\rho^\sigma \|1+|Dv|\|^p_{L^{\chi p}(\partial B_r)} \, dr\right)^{\frac{\chi}{\chi-1}(\frac{q}{p}-1)}. \quad (24)$$

Next, we use the Sobolev inequality on spheres to estimate the second factor on the right-hand side in (24): For $n \geq 2$ there exists $c = c(n, N, \chi) \in [1, \infty)$ such that for all $r > 0$

$$\|Dv\|^p_{L^{\chi p}(\partial B_r)}$$
$$\leq c r^{(n-1)(\frac{1}{\chi}-1)} \left(\int_{\partial B_r} |Dv|^p \, d\mathcal{H}^{n-1} + r^2 \int_{\partial B_r} |D(|Dv|^{\frac{p-2}{2}} Dv)|^2 \, d\mathcal{H}^{n-1}\right). \quad (25)$$

Combining (25) with elementary estimates and assumption $\frac{1}{2} \leq \rho < \sigma \leq 1$, we obtain

$$\begin{aligned}\int_\rho^\sigma \|1+|Dv|\|^p_{L^{\chi p}(\partial B_r)} \, dr &\lesssim \int_\rho^\sigma 1 + \|Dv\|^p_{L^{\chi p}(\partial B_r)} \, dr \\ &\lesssim \int_\rho^\sigma 1 + \left(\int_{\partial B_r} |Dv|^p + |D(|Dv|^{\frac{p-2}{2}} Dv)|^2 \, d\mathcal{H}^{n-1}\right) dr \\ &\lesssim \int_{B_\sigma \setminus B_\rho} 1 + |Dv|^p + |D(|Dv|^{\frac{p-2}{2}} Dv)|^2 \, dx. \quad (26)\end{aligned}$$

Combining (24) and estimate (26), we obtain

$$\int_{B_\rho} |D(|Dv|^{\frac{p-2}{2}} Dv)|^2 \, dx$$
$$\leq \frac{c\left(\int_{B_1} (1+|Dv|)^p \, dx\right)^{\frac{\chi}{\chi-1}(1-\frac{q}{\chi p})}}{(\sigma-\rho)^{1+\frac{q}{p}}} \left(\int_{B_\sigma} 1 + |Dv|^p + |D(|Dv|^{\frac{p-2}{2}} Dv)|^2 \, dx\right)^{\frac{\chi}{\chi-1}(\frac{q}{p}-1)},$$

The claimed estimate (20) now follows since $|z|^p \leq f(z) \leq f_\lambda(z)$, $\frac{\chi}{\chi-1}(1 - \frac{q}{\chi p} + \frac{q}{p} - 1) = \frac{q}{p} \geq 1$ and $\int_{B_1} 1 + f_\lambda(Dv) \, dx \geq |B_1|$.

Finally, we present the computations regarding (23): Lemma 3 yields

$$J(\sigma, \rho, |w|^q) \leq \frac{\left(\int_\rho^\sigma \|w\|^{q\delta}_{L^q(\partial B_r)} \, dr\right)^{\frac{1}{\delta}}}{(\sigma-\rho)^{1+\frac{1}{\delta}}} \qquad \text{for every } \delta > 0.$$

Using two times the Hölder inequality, we estimate

$$\begin{aligned}\left(\int_\rho^\sigma \|w\|^{q\delta}_{L^q(\partial B_r)} \, dr\right)^{\frac{1}{\delta}} &\leq \left(\int_\rho^\sigma \|w\|^{\theta q\delta}_{L^p(\partial B_r)} \|w\|^{(1-\theta)q\delta}_{L^{\chi p}(\partial B_r)} \, dr\right)^{\frac{1}{\delta}} \quad \text{where } \frac{\theta}{p} + \frac{1-\theta}{\chi p} = \frac{1}{q} \\ &\leq \left(\int_\rho^\sigma \|w\|^{\theta q\delta \frac{s}{s-1}}_{L^p(\partial B_r)} \, dr\right)^{\frac{s-1}{s\delta}} \left(\int_\rho^\sigma \|w\|^{(1-\theta)q\delta s}_{L^{\chi p}(\partial B_r)} \, dr\right)^{\frac{1}{\delta s}} \quad \text{for every } s > 1.\end{aligned}$$

Inequality (23) follows with the admissible choice

$$\delta = \frac{p}{q} \quad \text{and} \quad s = \frac{1}{1-\theta} \qquad \left(\text{recall } 1-\theta = \frac{\frac{1}{p}-\frac{1}{q}}{\frac{1}{p}-\frac{1}{\chi p}} \text{ and } p < q\right)$$

which ensures $\theta q\delta\frac{s}{s-1}=(1-\theta)q\delta s=p$.

Step 2. Iteration.

We claim that there exists $c=c(L,n,N,p,q,\chi)\in[1,\infty)$ such that

$$\int_{B_{\frac12}}|Dv|^p+|D(|Dv|^{\frac{p-2}{2}}Dv)|^2\,dx\leq c\Bigl(\int_{B_1}1+f_\lambda(Dv)\,dx\Bigr)^\alpha, \tag{27}$$

where α is defined in (18). For $k\in\mathbb{N}\cup\{0\}$, we set

$$\rho_k=\frac34-\frac{1}{4^{1+k}}\quad\text{and}\quad J_k:=\int_{B_1}1+f_\lambda(Dv)+\int_{B_{\rho_k}}|D(|Dv|^{\frac{p-2}{2}}Dv)|^2\,dx.$$

Estimate (21) and the choice of ρ_k imply for $\lambda\in(0,1]$

$$\sup_{k\in\mathbb{N}}J_k\leq\int_{B_1}1+f_\lambda(Dv)+\int_{B_{\frac34}}|D(|Dv|^{\frac{p-2}{2}}Dv)|^2\,dx\lesssim\frac1\lambda\int_{B_1}1+f_\lambda(Dv)\,dx<\infty. \tag{28}$$

From (20) we deduce the existence of $c=c(L,n,N,p,q,\chi)\in[1,\infty)$ such that for every $k\in\mathbb{N}$

$$J_{k-1}\leq c4^{(1+\frac qp)k}\Bigl(\int_{B_1}1+f_\lambda(Dv)\Bigr)^{\frac{\chi}{\chi-1}(1-\frac{q}{\chi p})}J_k^{\frac{\chi}{\chi-1}\frac{q-p}{p}}. \tag{29}$$

Assumption $\frac qp<1+\min\{1,\frac{2}{n-1}\}$ and the choice of χ yield

$$\frac{\chi}{\chi-1}\frac{q-p}{p}\overset{(16)}{=}\begin{cases}\frac qp-1 & \text{if } n=2\\ \frac{\chi}{\chi-1}\frac{q-p}{p} & \text{if } n=3\\ \frac{n-1}{2}(\frac qp-1) & \text{if } n\geq4\end{cases}<1,$$

where we use for $n=3$ that $\chi\overset{(16)}{>}\frac{1}{2-\frac qp}>0$ and

$$\frac{\chi}{\chi-1}\frac{q-p}{p}<1\quad\Leftrightarrow\quad\frac{q-p}{p}<1-\frac1\chi\quad\Leftrightarrow\quad\frac1\chi<2-\frac qp.$$

Hence, iterating (29) we obtain (using the uniform bound (28) on J_k and $\frac{\chi}{\chi-1}\frac{q-p}{p}<1$)

$$\int_{B_{\frac12}}|Dv|^p+|D(|Dv|^{\frac{p-2}{2}}Dv)|^2\,dx\leq J_0\lesssim\Bigl(\int_{B_1}1+f_\lambda(Dv)\Bigr)^{\frac{\chi}{\chi-1}(1-\frac{q}{\chi p})\sum_{k=0}^\infty(\frac{\chi}{\chi-1}\frac{q-p}{p})^k} \tag{30}$$

and the claimed estimate (27) follow from

$$\alpha=\frac{\chi}{\chi-1}(1-\frac{q}{\chi p})\sum_{k=0}^\infty(\frac{\chi}{\chi-1}\frac{q-p}{p})^k.$$

Step 3. Conclusion.

We assume $B_1\Subset\Omega$ and show that there exists $c=c(L,n,N,p,q,\chi)\in[1,\infty)$

$$\int_{B_{\frac18}}|Du|^q\,dx\leq c\Bigl(\int_{B_1}1+f(Du)\,dx\Bigr)^{\frac{\alpha q}{p}}, \tag{31}$$

where α is given as in (18) above. Clearly, standard scaling, translation and covering arguments yield

$$\fint_{B_{\frac{R}{2}}(x_0)} |Du|^q \, dx \le c\left(\fint_{B_R(x_0)} 1 + f(Du)\, dx\right)^{\frac{\alpha q}{p}}$$

for all $B_R(x_0) \Subset \Omega$ and $c = c(L, n, N, p, q, \chi) \in [1, \infty)$. The claimed estimate (17) then follows from Lemma 1.

Following [18], we introduce in addition to $\lambda \in (0, 1)$ a second small parameter $\varepsilon > 0$ which is related to a suitable regularization of u. For $\varepsilon \in (0, \varepsilon_0)$, where $0 < \varepsilon_0 \le 1$ is such that $B_{1+\varepsilon_0} \Subset \Omega$, we set $u_\varepsilon := u * \varphi_\varepsilon$ with $\varphi_\varepsilon := \varepsilon^{-n}\varphi(\frac{\cdot}{\varepsilon})$ and φ being a non-negative, radially symmetric mollifier, i.e. it satisfies

$$\varphi \ge 0, \quad \operatorname{supp} \varphi \subset B_1, \quad \int_{\mathbb{R}^n} \varphi(x)\, dx = 1, \quad \varphi(\cdot) = \widetilde{\varphi}(|\cdot|) \quad \text{for some } \widetilde{\varphi} \in C^\infty(\mathbb{R}).$$

Given $\varepsilon, \lambda \in (0, \varepsilon_0)$, we denote by $v_{\varepsilon,\lambda} \in u_\varepsilon + W_0^{1,q}(B_1)$ the unique function satisfying

$$\int_{B_1} f_\lambda(Dv_{\varepsilon,\lambda})\, dx \le \int_{B_1} f_\lambda(Dv)\, dx \qquad \text{for all } v \in u_\varepsilon + W_0^{1,q}(B_1). \tag{32}$$

Combining Sobolev inequality with the assumption $\frac{q}{p} < 1 + \frac{2}{n-2}$ and estimate (27), we have

$$\begin{aligned}
\left(\int_{B_{\frac{1}{8}}} |Dv_{\varepsilon,\lambda}|^q \, dx\right)^{\frac{p}{q}} &\lesssim \int_{B_{\frac{1}{8}}} |Dv_{\varepsilon,\lambda}|^p + |D(|Dv_{\varepsilon,\lambda}|^{\frac{p-2}{2}} Dv_{\varepsilon,\lambda})|^2 \, dx \\
&\overset{(27)}{\lesssim} \left(\int_{B_1} 1 + f_\lambda(Dv_{\varepsilon,\lambda})\, dx\right)^{\alpha} \\
&\overset{(19),(32)}{\le} \left(\int_{B_1} 1 + f(Du_\varepsilon) + \lambda |Du_\varepsilon|^q \, dx\right)^{\alpha} \\
&\le \left(|B_1| + \int_{B_{1+\varepsilon}} f(Du)\, dx + \lambda \int_{B_1} |Du_\varepsilon|^q \, dx\right)^{\alpha},
\end{aligned} \tag{33}$$

where we used Jensen's inequality and the convexity of f in the last step. Similarly,

$$\begin{aligned}
\int_{B_1} |Dv_{\varepsilon,\lambda}|^p \, dx &\overset{(2)}{\le} \int_{B_1} f(Dv_{\varepsilon,\lambda})\, dx \overset{(19)(32)}{\le} \int_{B_1} f(Du_\varepsilon) + \lambda |Du_\varepsilon|^q \, dx \\
&\le \int_{B_{1+\varepsilon}} f(Du)\, dx + \lambda \int_{B_1} |Du_\varepsilon|^q \, dx.
\end{aligned} \tag{34}$$

Fix $\varepsilon \in (0, \varepsilon_0)$. In view of (33) and (34), we find $w_\varepsilon \in u_\varepsilon + W_0^{1,p}(B_1)$ such that as $\lambda \to 0$, up to subsequence,

$$\begin{aligned}
v_{\varepsilon,\lambda} &\rightharpoonup w_\varepsilon \qquad \text{weakly in } W^{1,p}(B_1), \\
Dv_{\varepsilon,\lambda} &\rightharpoonup Dw_\varepsilon \qquad \text{weakly in } L^q(B_{\frac{1}{8}}).
\end{aligned}$$

Hence, a combination of (33), (34) with the weak lower-semicontinuity of convex functionals yield

$$\|Dw_\varepsilon\|_{L^q(B_{\frac{1}{8}})} \le \liminf_{\lambda \to 0} \|Dv_{\varepsilon,\lambda}\|_{L^{\kappa p}(B_{\frac{1}{8}})} \lesssim \left(\int_{B_{1+\varepsilon}} f(Du)\, dx + 1\right)^{\frac{\alpha}{p}} \tag{35}$$

$$\int_{B_1} |Dw_\varepsilon|^p \, dx \le \int_{B_1} f(Dw_\varepsilon) \, dx \le \int_{B_{1+\varepsilon}} f(Du) \, dx. \tag{36}$$

Since $w_\varepsilon \in u_\varepsilon + W_0^{1,q}(B_1)$ and $u_\varepsilon \to u$ in $W^{1,p}(B_1)$, we find by (36) a function $w \in u + W_0^{1,p}(B_1)$ such that, up to subsequence,

$$Dw_\varepsilon \rightharpoonup Dw \quad \text{weakly in } L^p(B_1).$$

Appealing to the bounds (35), (36) and lower semicontinuity, we obtain

$$\|Dw\|_{L^q(B_{\frac{1}{8}})} \lesssim \left(\int_{B_1} f(Du) \, dx + 1 \right)^{\frac{\alpha}{p}} \tag{37}$$

$$\int_{B_1} f(Dw) \, dx \le \int_{B_1} f(Du) \, dx. \tag{38}$$

Inequality (38), strict convexity of f and the fact $w \in u + W_0^{1,p}(B_1)$ imply $w = u$ and thus the claimed estimate (31) is a consequence of (37). □

4 Partial regularity - Proof of Theorem 4

Theorem 4 follows from, the higher integrability statement Theorem 2, the ε-regularity statement of Lemma 4 below and a well-known iteration argument.

Lemma 4 *Let $\Omega \subset \mathbb{R}^n$, $n \ge 3$, and suppose Assumption 3 is satisfied with $2 \le p < q < \infty$ such that $\frac{q}{p} < 1 + \frac{2}{n-1}$. Fix $M > 0$. There exists $C^* = C^*(n, N, p, q, \frac{L}{\nu}, M) \in [1, \infty)$ such that for every $\tau \in (0, \frac{1}{4})$ there exists $\varepsilon = \varepsilon(M, \tau) > 0$ such that the following is true: Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}$ given in (1). Suppose for some ball $B_r(x) \Subset \Omega$*

$$|(Du)_{x,r}| \le M,$$

where we use the shorthand $(w)_{x,r} := \fint_{B_r(x)} w \, dy$, and

$$E(x,r) := \fint_{B_r(x)} |Du - (Du)_{x,r}|^2 \, dy + \fint_{B_r(x)} |Du - (Du)_{x,r}|^q \, dy \le \varepsilon,$$

then

$$E(x, \tau r) \le C^* \tau^2 E(x, r).$$

With the higher integrability of Theorem 5 and the Caccioppoli inequality of Lemma 2 at hand, we can prove Lemma 4 following almost verbatim the proof of the corresponding result [7, Lemma 4.1], which contain the statement of Lemma 4 under the assumption $\frac{q}{p} < 1 + \frac{2}{n}$ (note that in [7] somewhat more general growth conditions including also the case $1 < p < q$ are considered). Thus, we only sketch the argument.

Proof of Lemma 4 Fix $M > 0$. Suppose that Lemma 4 is wrong. Then there exists $\tau \in (0, \frac{1}{4})$, a local minimizer $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, which in view of Theorem 2 satisfies $u \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, and a sequence of balls $B_{r_m}(x_m) \Subset B_R$ satisfying

$$|(Du)_{x_m, r_m}| \le M, \quad E(x_m, r_m) =: \lambda_m \quad \text{with} \quad \lim_{m \to \infty} \lambda_m = 0, \tag{39}$$

$$E(x_m, \tau r_m) > C^* \tau^2 \lambda_m^2, \tag{40}$$

where C^* is chosen below. We consider the sequence of rescaled functions given by

$$v_m(z) := \frac{1}{\lambda_m r_m}(u(x_m + r_m z) - a_m - r_m A_m z),$$

where $a_m := (u)_{x_m,r_m}$ and $A_m := (Du)_{x_m,r_m}$. Assumption (39) implies $\sup_m |A_m| \le M$ and thus, up to subsequence,

$$A_m \to A \in \mathbb{R}^{N\times n}.$$

The definition of v_m yields

$$Dv_m(z) = \lambda_m^{-1}(Du(x_m + r_m z) - A_m), \quad (v_m)_{0,1} = 0, \quad (Dv_m)_{0,1} = 0 \tag{41}$$

Assumptions (39) and (40) imply

$$\fint_{B_1} |Dv_m|^2 \, dz + \lambda_m^{q-2} \fint_{B_1} |Dv_m|^q \, dz = \lambda_m^{-1} E(x_m, r_m) = 1, \tag{42}$$

$$\fint_{B_\tau} |Dv_m - (Dv_m)_{0,\tau}|^2 \, dz + \lambda_m^{q-2} \fint_{B_\tau} |Dv_m - (Dv_m)_{0,\tau}|^q \, dz > C^* \tau^2. \tag{43}$$

The bound (42) together with (41) imply the existence of $v \in W^{1,2}(B_1, \mathbb{R}^N)$ such that, up to extracting a further subsequence,

$$\begin{aligned} v_m &\rightharpoonup v && \text{in } W^{1,2}(B_1, \mathbb{R}^N), \\ \lambda_m Dv_m &\to 0 && \text{in } L^2(B_1, \mathbb{R}^{N\times n}) \text{ and almost everywhere} \\ \lambda_m^{1-\frac{2}{q}} v_m &\rightharpoonup 0 && \text{in } W^{1,q}(B_1, \mathbb{R}^N). \end{aligned}$$

The function v satisfies the linear equation with constant coefficients

$$\int_{B_1} \langle \partial^2 f(A) Dv, D\varphi \rangle \, dz = 0 \qquad \text{for all } \varphi \in C_0^1(B_1),$$

see, e.g., [21] or [7, Proposition 4.2]. Standard estimates for linear elliptic systems with constant coefficients imply $v \in C^\infty_{\mathrm{loc}}(B_1, \mathbb{R}^N)$ and existence of $C^{**} < \infty$ depending only on n, N and the ellipticity contrast of $\partial^2 f(A)$ (and thus on $\frac{L}{\nu}$, p, q, and M) such that

$$\fint_{B_\tau} |Dv - (Dv)_{0,\tau}|^2 \le C^{**} \tau^2. \tag{44}$$

Choosing $C^* = 2C^{**}$ we obtain a contradiction between (43) and (44) provided we have as $m \to \infty$

$$Dv_m \to Dv \qquad \text{in } L^2_{\mathrm{loc}}(B_1), \tag{45}$$

$$\lambda_m^{1-\frac{2}{q}} Dv_m \to 0 \quad \text{in } L^q_{\mathrm{loc}}(B_1). \tag{46}$$

Exanctly as in [7, Proposition 4.3] (with $\mu = 2 - p$, see also [9, Section 3.4.3.2] for a more detailed presentation of the proof), we have for all $\rho \in (0, 1)$,

$$\lim_{m\to\infty} \int_{B_\rho} \int_0^1 (1 - s) \Big(1 + |A_m + \lambda_m (Dv + s Dw_m)|^2 \Big)^{\frac{p-2}{2}} |Dw_m|^2 \, dz = 0, \tag{47}$$

where $w := v_m - v$, and thus the local L^2-convergence (45) follows. It is left to prove (46). For this, we introduce for $\rho \in (0,1)$ and $T > 0$ the sequence of subsets

$$U_m := U_m(\rho, T) := \{ z \in B_\rho \,:\, \lambda_m |Dv_m| \le T \}.$$

The local Lipschitz regularity of v, $q > 2$ and (45) imply for all $\rho \in (0,1)$ and $T > 0$

$$\begin{aligned}\limsup_{m\to\infty} \int_{U_m(\rho,T)} \lambda_m^{q-2} |Dv_m|^q \, dz \lesssim & \limsup_{m\to\infty} \int_{U_m(\rho,T)} \lambda_m^{q-2} |Dw_m|^q \, dz \\ \lesssim & \limsup_{m\to\infty} \int_{B_\rho} (M^{q-2} + \lambda_m^{q-2} |Dv|^{q-2}) |Dw_m|^2 \, dz \\ =& 0,\end{aligned}$$

where here and for the rest of the proof $\lesssim$ means $\le$ up to a multiplicative constant depending only on L, n, N, p and q. Hence, it is left to show that there exists $T > 0$ such that

$$\limsup_{m\to\infty} \int_{B_\rho \setminus U_m(\rho,T)} \lambda_m^{q-2} |Dv_m|^q \, dz \le 0 \quad \text{for all } \rho \in (0,1).$$

As in [7], we introduce a sequence of auxiliary functions

$$\psi_m := \lambda_m^{-1} \Big[(1 + |A_m + \lambda_m Dv_m|^2)^{\frac{p}{4}} - (1 + |A_m|^2)^{\frac{p}{4}} \Big],$$

which satisfy

$$\limsup_{m\to\infty} \|\psi_m\|_{W^{1,2}(B_\rho)} \lesssim c(\rho) \in [1,\infty) \qquad \text{for all } \rho \in (0,1). \tag{48}$$

Indeed, by Theorem 2 and Lemma 2, we have for every $\rho \in (0,1)$ and every $Q \in \mathbb{R}^{N\times n}$

$$\int_{B_{\rho r_m}(x_m)} |\nabla (1 + |Du(x)|^2)^{\frac{p}{4}}|^2 \, dx \lesssim r_m^{-2} c(\rho) \int_{B_{r_m}(x_m)} (1 + |\nabla u(x)|)^{q-2} |Du(x) - Q|^2 \, dx$$

and thus by rescaling and setting $Q = A_m$

$$\int_{B_\rho} |\nabla \psi_m|^2 \, dz \lesssim c(\rho) \int_{B_1} (1 + |A|^{q-2} + |\lambda_m Dv_m|^{q-2})) |Dv_m|^2 \, dz \overset{(42)}{\lesssim} c(\rho)(1 + M^{q-2}).$$

The identity $\psi_m = \lambda_m^{-1} \int_0^1 \frac{d}{dt} \Theta(A_m + t\lambda_m v_m) \, dt$ with $\Theta(F) := (1 + |F|^2)^{\frac{p}{4}}$ implies

$$|\psi_m| \le c(|Dv_m| + \lambda_m^{\frac{p-2}{2}} |Dv_m|^{\frac{p}{2}})$$

(see [7, p. 555] for details) and thus with help of (47), we obtain

$$\limsup_{m\to\infty} \int_{B_\rho} |\psi_m|^2 \, dz \lesssim c(\rho).$$

For T sufficiently large (depending on M) there exists $c > 0$ such that for all $z \in B_\rho \setminus U_m(\rho, T)$

$$\psi_m(z) \ge c \lambda_m^{-1} \lambda_m^{\frac{p}{2}} |Dv_m(z)|^{\frac{p}{2}} \quad \text{and thus} \quad \lambda_m^{2(1+\frac{q}{p})} \psi_m^{\frac{2q}{p}}(z) \ge c^{\frac{2q}{p}} \lambda_m^{q-2} |Dv_m(z)|^q$$

Estimate (48) and Sobolev embedding imply $\limsup_{m\to\infty} \|\psi_m\|_{L^{\frac{2n}{n-2}}(B_\rho)} \lesssim c(\rho) \in [1,\infty)$. Hence, using assumption $\frac{q}{p} < 1 + \frac{2}{n-1}$ (and thus $\frac{2q}{p} < \frac{2n}{n-2}$), we obtain for every $\rho \in (0,1)$

$$\limsup_{m\to\infty} \int_{B_\rho\setminus U_m(\rho,T)} \lambda_m^{q-2}|Dv_m|^q\,dz \lesssim \lambda_m^{2(1+\frac{q}{p})} \int_{B_\rho} \psi_m^{\frac{2q}{p}}(z)\,dz \lesssim c(\rho) \limsup_{m\to\infty} \lambda_m^{2(1+\frac{q}{p})} = 0,$$

which finishes the proof. □

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