Calc. Var. (2021) 60:22
https://doi.org/10.1007/s00526-020-01891-6

Calculus of Variations

On the Brezis–Nirenberg problem for a Kirchhoff type equation in high dimension

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Received: 10 June 2020 / Accepted: 16 December 2020 / Published online: 18 January 2021

Abstract
The present paper deals with a parametrized Kirchhoff type problem involving a critical nonlinearity in high dimension. Existence, non existence and multiplicity of solutions are obtained under the effect of a subcritical perturbation by combining variational properties with a careful analysis of the fiber maps of the energy functional associated to the problem. The particular case of a pure power perturbation is also addressed. Through the study of the Nehari manifolds we extend the general case to a wider range of the parameters.

Contents

Mathematics Subject Classification 35J20 · 35B33

Communicated by P. H. Rabinowitz.

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1 Introduction and main results

Nonlocal boundary value problems of the type

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2dx\right)\Delta u = f(x,u), & \text{in } \Omega \\ u=0, & \text{on } \partial\Omega \end{cases}$$

are related to the stationary version of the equation

$$\frac{\partial^2 u}{\partial t^2} - \left(a+b\int_\Omega |\nabla u|^2dx\right)\Delta u = f(t,x,u),$$

proposed by Kirchhoff [11] as a generalization of the D'Alembert's wave equation to describe the transversal oscillations of a stretched string. Here u denotes the displacement, f is the external force, b is the initial tension and a is related to the intrinsic properties of the string. The importance of these kind of problems and its mathematical developments were made very clear after the recent short survey [17].

Recently, the existence and multiplicity of solutions of Kirchhoff problems under the effect of a critical nonlinearity f have received considerable attention. Indeed, the challenging feature of such problems is due to the presence of a nonlocal term together with the lack of compactness of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{2^\star}(\Omega)$ which prevent the application of standard variational methods.

The existence and multiplicity of solutions of Kirchhoff type equations with critical exponents have been investigated by using different techniques as truncation and variational methods, the Nehari manifold approach, the Ljusternik–Schnirelmann category theory, genus theory (see for instance [3,4,7] and the references therein).

In the recent works [1,6,8,9,12,13,20], an application of the Lions' Concentration Compactness principle allows to prove the Palais Smale condition of the energy functional, a key property for the application of the well known Mountain Pass Theorem. Notice that according to the space dimension N, the geometry of the energy functional changes and when $N \geq 4$ (coercive case) the property holds when a and b satisfy a suitable constraint (see [8,9,13,20]).

Indeed, when $N \geq 4$, in [5] it is shown that the interaction between the Kirchhoff operator and the critical term leads to some useful variational properties of the energy functional such as the weak lower semicontinuity and the Palais Smale property when $a^{\frac{N-4}{2}}b \geq C_1(N)$ or $a^{\frac{N-4}{2}}b > C_2(N)$ respectively, for suitable constants $C_1(N) < C_2(N)$.

In this paper we study the following critical Kirchhoff problem

$$(\mathcal{P}_\lambda) \quad \begin{cases} -\left(a+b\int_\Omega |\nabla u|^2dx\right)\Delta u = |u|^{2^*-2}u+\lambda f(x,u), & \text{in } \Omega \\ u=0, & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subseteq \mathbb{R}^N$ $(N > 4)$ is a bounded domain, a, b are positive fixed numbers, 2^* is the Sobolev critical exponent, λ is a positive parameter, f a subcritical Carathéodory function.

In the present paper, through a careful analysis of the fiber maps associated to the energy functional, we will study the existence, non existence and the multiplicity of solutions of $(\mathcal{P}_\lambda)$. Indeed, by using the fibration method introduced in [16] and the notion of extremal values of [10], we will describe the topological changes of the energy functional, when the parameters a, b, λ vary. As it will become clear throughout our study, from the very geometry of the fibers, we will be able to deduce a precise, and in some cases complete picture on existence, non-existence and multiplicity results.

When the nonlinearity f is a pure power term, i.e. $f(x,u) = |u|^{p-2}u$ for some $p \in (2, 2^*)$, we will go further in our study and through a detailed analysis of the Nehari set associated to problem $(\mathcal{P}_\lambda)$ (see [14,15]), we will show the existence of two critical hyperbolas on the plane (a, b), that separates the plane into regions where the energy functional exhibits distinct topological properties. Some of the ideas used here come from [18,19], where the subcritical case was studied and a complete bifurcation diagram was provided. Our work contains new results in the framework of Kirchhoff type equations with critical nonlinearity and extends the results of [13] (for a detailed comparison see below).

To give a better description of our results, let us endow the Sobolev space $H_0^1(\Omega)$ with the classical norm $\|u\| = \left(\int_\Omega |\nabla u|^2\, dx\right)^{\frac{1}{2}}$ and denote by $\|u\|_q$ the Lebesgue norm in $L^q(\Omega)$ for $1 \le q \le 2^*$, i.e. $\|u\|_q = \left(\int_\Omega |u|^q\, dx\right)^{\frac{1}{q}}$. Let S_N be the embedding constant of $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$, i.e.

$$S_N = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\|u\|^2}{\|u\|_{2^*}^2}. \tag{1}$$

Let us recall that

$$S_N = \frac{N(N-2)}{4}\omega_N^{\frac{2}{N}}$$

(where ω_N is the volume of the unit ball in $\mathbb{R}^N$) is sharp, but is never achieved unless $\Omega = \mathbb{R}^N$.

For $N > 4$ let us introduce the following constants which will have a crucial role in the sequel:

$$C_1(N) = \frac{4(N-4)^{\frac{N-4}{2}}}{N^{\frac{N-2}{2}} S_N^{\frac{N}{2}}} \qquad \text{and} \qquad C_2(N) = \frac{2(N-4)^{\frac{N-4}{2}}}{(N-2)^{\frac{N-2}{2}} S_N^{\frac{N}{2}}},$$

and notice that $C_1(N) < C_2(N)$.

On the nonlinearity f we will assume the following:

- $(\mathcal{F}_1)$ $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying $f(x, 0) = 0$ for a.a. $x \in \Omega$;
- $(\mathcal{F}_2)$ $f(x, v) > 0$ for every $v > 0$ and a.a. $x \in \Omega$, $f(x, v) < 0$ for every $v < 0$ and a.a. $x \in \Omega$. Moreover there exists $\mu > 0$ such that $f(x, v) \ge \mu > 0$ for a.a. $x \in \Omega$ and every $v \in I$, being I an open interval of $(0, +\infty)$;
- $(\mathcal{F}_3)$ there exist $c > 0$, $p \in (2, 2^*)$ such that $|f(x, v)| \le c(1 + |v|^{p-1})$ for every $v \in \mathbb{R}$ and a.a. $x \in \Omega$;
- $(\mathcal{F}_4)$ $f(x, v) = o(|v|)$ for $v \to 0$ and uniformly in $x \in \Omega$.

Denote by $\Phi_\lambda : H_0^1(\Omega) \to \mathbb{R}$ the energy functional associated to $(\mathcal{P}_\lambda)$,

$$\Phi_\lambda(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{2^*}\|u\|_{2^*}^{2^*} - \lambda \int_\Omega F(x, u)dx \qquad \text{for every } u \in H_0^1(\Omega),$$

where

$$F(x, v) = \int_0^v f(x, t)dt.$$

Note that from $(\mathcal{F}_1)$ and $(\mathcal{F}_3)$, Φ_λ is well defined and $\Phi_\lambda \in C^1(H_0^1(\Omega))$.

Our first result establishes the existence of global minimizers when $a^{\frac{N-4}{2}} b \ge C_1(N)$.

Theorem 1.1 *Assume $(\mathcal{F}_1)$–$(\mathcal{F}_4)$. If $a^{\frac{N-4}{2}} b > C_1(N)$, then there exists $\lambda_0^* := \lambda_0^*(a, b) > 0$ such that:*

(i) *For each $\lambda > \lambda_0^*$, problem $(\mathcal{P}_\lambda)$ has a non-zero solution u_λ, which is a global minimizer to Φ_λ with negative energy.*
(ii) *Problem $(\mathcal{P}_{\lambda_0^*})$ has a non-zero solution $u_{\lambda_0^*}$, which is a global minimizer to $\Phi_{\lambda_0^*}$ with zero energy.*
(iii) *If $0 < \lambda < \lambda_0^*$, then $\Phi_\lambda(u) > 0$ for all $u \in H_0^1(\Omega)\backslash\{0\}$ and 0 is a global minimizer of Φ_λ.*

If $a^{\frac{N-4}{2}} b = C_1(N)$, then for each $\lambda > 0$, problem $(\mathcal{P}_\lambda)$ has a non-zero solution u_λ, which is a global minimizer to Φ_λ with negative energy. Furthermore, if $(a_k)_k$, $(b_k)_k$ are sequences satisfying $a_k^{\frac{N-4}{2}} b_k \downarrow C_1(N)$, $a_k \to a > 0$ and $b_k \to b > 0$, then $\lambda_0^*(a_k, b_k) \to 0$.

In the sequel, λ_0^* is as in Theorem 1.1. For $\lambda < \lambda_0^*$ but close to λ_0^* we can still prove the existence of a non trivial local minimizer as it is shown in the next result.

Theorem 1.2 *Assume $(\mathcal{F}_1)$–$(\mathcal{F}_4)$. If $a^{\frac{N-4}{2}} b > C_1(N)$, then there exists $\varepsilon > 0$ such that for each $\lambda_0^* - \varepsilon < \lambda < \lambda_0^*$, problem $(\mathcal{P}_\lambda)$ has a non-zero solution u_λ, which is a local minimizer to Φ_λ with positive energy. Moreover $\Phi_\lambda(u) > 0$ for all $u \in H_0^1(\Omega)\backslash\{0\}$.*

A second solution of (P_λ) of mountain pass type is ensured by the next theorem provided $a^{\frac{N-4}{2}} b \geq C_2(N)$.

Theorem 1.3 *Assume $(\mathcal{F}_1)$–$(\mathcal{F}_4)$. If $a^{\frac{N-4}{2}} b > C_2(N)$, then there exists $\varepsilon > 0$ such that for each $\lambda > \lambda_0^* - \varepsilon$, problem $(\mathcal{P}_\lambda)$ has a non-zero solution v_λ, which is of a mountain pass type to Φ_λ, with positive energy. If $a^{\frac{N-4}{2}} b = C_2(N)$, then the same result holds for λ sufficiently large.*

For the next result, we need the additional hypothesis:

$(\mathcal{F}_5)$ For each $u \in H_0^1(\Omega)\backslash\{0\}$, the function $(0, \infty) \ni t \mapsto \int_\Omega f(x, tu(x))dx$ is C^1.

Theorem 1.4 *Assume $(\mathcal{F}_1)$–$(\mathcal{F}_5)$. If $a^{\frac{N-4}{2}} b > C_2(N)$, then there exists $\lambda^* := \lambda^*(a, b) \in (0, \lambda_0^*)$, such that if $\lambda \in (0, \lambda^*)$, then $(\mathcal{P}_\lambda)$ has no non-zero solution. Moreover, there exists $u \in H_0^1(\Omega)\backslash\{0\}$ such that $\Phi_\lambda'(u)u = 0$ if, and only if $\lambda \geq \lambda^*$.*

Now we focus on the power case $f(x, u) = |u|^{p-2}u$ with $p \in (2, 2^*)$. In this case, some conclusions of Theorems 1.1, 1.3 and 1.4 had already been established in [13]. Indeed, a comparison between the constants α_2 (defined in [13]) and $C_2(N)$ shows (after some obvious modifications with respect to $a > 0$) that $\alpha_2 = C_2(N)$. Therefore [13, Theorem B.8] corresponds to our Theorem 1.1 with the following observations:

(1) In [13, Theorem B.8] the existence of a global minimum of the energy functional u_λ is only proved for $\alpha_2 = C_2(N) \leq a^{\frac{N-4}{2}} b$ and λ sufficiently large in order to make the infimum negative, while in our case, we find u_λ for all $C_1(N) \leq a^{\frac{N-4}{2}} b$ (remember that $C_1(N) < C_2(N)$) and there is a threshold λ_0^* for the sign of the energy of Φ_λ. Moreover, we proved the existence of a local minimizer with positive energy in case $\Phi_\lambda(u) > 0$ for $u \neq 0$ and $a^{\frac{N-4}{2}} b > C_1(N)$ (see Theorem 1.2).
(2) The arguments used in [13, Theorem B.8], to prove a mountain pass geometry to Φ_λ require λ to be sufficiently large in order to make the infimum negative. We show that this geometry is preserved even in the case where $\Phi_\lambda(u) > 0$ for all $u \in H_0^1(\Omega)\backslash\{0\}$ (see Theorems 1.2, 1.3).

(3) Theorem 1.4 was proved in [13, Theorem B.8] for λ sufficiently small. We also show that there exists $u \in H_0^1(\Omega)\backslash\{0\}$ such that $\Phi'_\lambda(u)u = 0$ if, and only if $\lambda \geq \lambda^*$. However, when $f(x,u) = |u|^{p-2}u$ this result can be improved (see Theorem 4.3).

Concerning item (1), in fact, we have now a fairly complete result. Combining Theorem 1.1 with [13, Proposition 4.2] we conclude that the curve $a^{\frac{N-4}{2}}b = C_1(N)$ is a threshold in the following sense:

Theorem 1.5 *Suppose that $f(x,u) = |u|^{p-2}u$. If $0 < a^{\frac{N-4}{2}}b \leq C_1(N)$, then Φ_λ has a global minimizer with negative energy for all $\lambda > 0$. If $a^{\frac{N-4}{2}}b > C_1(N)$, then Φ_λ has a global minimizer with negative energy if, and only if, $\lambda > \lambda_0^*(a,b) > 0$, it has two global minimizers with zero energy for $\lambda = \lambda_0^*(a,b)$, and has zero as unique minimizer if $\lambda < \lambda_0^*(a,b)$. Moreover, if $(a_k)_k, (b_k)_k$ are sequences satisfying $a_k^{\frac{N-4}{2}} b_k \downarrow C_1(N)$, $a_k \to a > 0$ and $b_k \to b > 0$, then $\lambda_0^*(a_k,b_k) \to 0$. In all cases the global minimizer is a solution to problem $(\mathcal{P}_\lambda)$.*

Theorem 1.5 settles down the existence of global minimizers with negative energy for all ranges of $a^{\frac{N-4}{2}}b$. It complements [13, Theorem 1.2 and Theorem B.8].

Concerning the second solution, we complement [13, Theorem 1.1] with the following results.

Theorem 1.6 *Suppose that $f(x,u) = |u|^{p-2}u$ and $0 < a^{\frac{N-4}{2}}b < C_2(N)$. Then there exists $p_0(a,b) \in (2,2^*)$ such that if $p \in (p_0(a,b),2^*)$, then for all $\lambda > 0$, problem $(\mathcal{P}_\lambda)$ has a non-zero solution v_λ with positive energy.*

Theorem 1.7 *Suppose that $f(x,u) = |u|^{p-2}u$. For each $a,b > 0$ there exists $\tilde{\lambda} := \tilde{\lambda}(a,b,p) > 0$ such that for all $\lambda > \tilde{\lambda}$, problem $(\mathcal{P}_\lambda)$ has a non-zero solution v_λ with positive energy.*

We note here that in [13, Theorem 1.1], it was proved that for each fixed p, the conclusion of Theorem 1.6 holds true for sufficiently small b. We refer the reader to Theorem 4.4 and Remark 4.1, in particular to item (ii), where we show that the technique used to prove [13, Theorem 1.1] (which we also used) can not hold for all values of a, b, p. However, the above theorem ensures that for each p problem $(\mathcal{P}_\lambda)$ still has a second solution provided λ is big enough.

We conclude this work with an existence result à la Brezis Nirenberg [2] which is a consequence of our study in the limit case ($b \downarrow 0$).

Theorem 1.8 *For each $\lambda > 0$ and $p \in (2,2^*)$, the problem*

$$(\mathcal{Q}_\lambda) \qquad \begin{cases} -\Delta u = |u|^{2^*-2}u + \lambda|u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

has a nontrivial solution.

The last remark of this Section explains the reason why we focus on positive parameters λ:

Remark 1.1 If $\lambda \leq 0$, problem $(\mathcal{P}_\lambda)$ might have only the zero solution. Indeed, assume that Ω is a star shaped domain and $f(v) = |v|^{p-2}v$ with $p \in (2,2^*)$. Then, if u is a solution of $(\mathcal{P}_\lambda)$ then $w = (a + b\|u\|^2)^{-\frac{1}{2^*-2}}u$ satisfies the equation $-\Delta w = |w|^{2^*-2}w + \mu|w|^{p-2}w$ for some $\mu \leq 0$. Applying the Pohozaev identity we deduce that $w = 0$.

The work is organized as follows:

- in Sect. 2 we collect some preliminaries results that will be used throughout the work;
- in Sect. 3 we prove Theorems 1.1, 1.2, 1.3 and 1.4;
- in Sect. 4 we prove Theorems 1.5, 1.6, 1.7 and 1.8,
- in "Appendices A and B" we present some technical results concerning the Nehari set associated to problem $(\mathcal{P}_\lambda)$ and $(\mathcal{P}_0)$ respectively.

2 Preliminaries results

In this Section we provide some auxiliary results which will be used throughout the work. Here only hypotheses $(\mathcal{F}_1)$–$(\mathcal{F}_4)$ are used. For each $a, b > 0$, define $g, h : (0, \infty) \to \mathbb{R}$ by

$$g(t) = \frac{a}{2} + \frac{b}{4}t^2 - S_N^{\frac{-2^*}{2}} \frac{t^{2^*-2}}{2^*},$$
$$h(t) = a + bt^2 - S_N^{\frac{-2^*}{2}} t^{2^*-2}.$$

A simple calculation shows that

Lemma 2.1 *There holds:*

(i) *g has a unique local minimizer at*

$$t_0 = \left(\frac{2^* b}{2(2^* - 2)} S_N^{\frac{2^*}{2}} \right)^{\frac{1}{2^*-4}}. \tag{2}$$

Moreover, $g(t_0) > 0$ if and only if $a^{\frac{N-4}{2}} b > C_1(N)$, while if $a^{\frac{N-4}{2}} b = C_1(N)$, then $g(t_0) = 0$.

(ii) *h has a unique local minimizer at*

$$t_0 = \left(\frac{2b}{2^* - 2} S_N^{\frac{2^*}{2}} \right)^{\frac{1}{2^*-4}}. \tag{3}$$

Moreover, $h(t_0) > 0$ if and only if $a^{\frac{N-4}{2}} b > C_2(N)$, while if $a^{\frac{N-4}{2}} b = C_2(N)$, then $h(t_0) = 0$.

Remark 2.1 Lemma 2.1 gives the same conclusion if instead of g, h we use $t^2 g(t)$ and $t^2 h(t)$. Indeed, note for example that $t^2 g(t) = 0$ and $(t^2 g(t))' = 0$ if, and only if, $g(t) = g'(t) = 0$.

As a consequence of Lemma 2.1 and Remark 2.1 we have

Corollary 2.1 *Suppose that $a^{\frac{N-4}{2}} b < C_2(N)$, then the function $\overline{g}(t) = t^2 g(t)$ has only two critical points, $0 < t_{a,b}^- < t_{a,b}^+$. Moreover, $t_{a,b}^-$ is a local maximum and $t_{a,b}^+$ is a local minimum with $\overline{g}''(t_{a,b}^-) < 0 < \overline{g}''(t_{a,b}^+)$. Furthermore if $a^{\frac{N-4}{2}} b = C_2(N)$, then the function $g(t)t^2$ is increasing and has a unique critical point at $t_{a,b}$ satisfying $\overline{g}''(t_{a,b}) = 0$ and*

$$g(t_{a,b})t_{a,b}^2 = \frac{(2^* - 2)^2 a^2}{4 \cdot 2^* (4 - 2^*) b}.$$

Proposition 2.1 *Suppose that $u \in H_0^1(\Omega) \setminus \{0\}$, then*

(i) *for all* $t > 0$ *we have*

$$\frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 t^2 - \|u\|_{2^*}^{2^*}\frac{t^{2^*-2}}{2^*} > g(\|u\|t)\|u\|^2;$$

(ii) *for all* $t > 0$ *we have*

$$a\|u\|^2 + b\|u\|^4 t^2 - \|u\|_{2^*}^{2^*} t^{2^*-2} > h(\|u\|t)\|u\|^2.$$

Proof (i) Indeed note that

$$t^2\left[\frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 t^2 - \|u\|_{2^*}^{2^*}\frac{t^{2^*-2}}{2^*}\right] = \frac{a}{2}(\|u\|t)^2 + \frac{b}{4}(\|u\|t)^4 - \frac{\|u\|_{2^*}^{2^*}}{\|u\|^{2^*}}\frac{(\|u\|t)^{2^*}}{2^*}$$
$$> \frac{a}{2}(\|u\|t)^2 + \frac{b}{4}(\|u\|t)^4 - S_N^{-\frac{2^*}{2}}\frac{(\|u\|t)^{2^*}}{2^*}, \qquad t > 0.$$

The conclusion follows from Lemma 2.1. The strict inequality above is a consequence of the non existence of minimizers for (1). The proof of (ii) is similar. □

The next Lemma gives some important variational properties of the energy functional Φ_λ.

Lemma 2.2 *The following holds true.*

(1) *Let* a, b *be positive numbers such that* $a^{\frac{N-4}{2}} b \geq C_1(N)$. *Suppose that* $\lambda_k \to \lambda \geq 0$ *and* $u_k \rightharpoonup u$. *Then,* $\Phi_\lambda(u) \leq \liminf_k \Phi_{\lambda_k}(u_k)$. *Moreover, if* $a^{\frac{N-4}{2}} b > C_1(N)$ *and* $\Phi_\lambda(u) = \lim_k \Phi_{\lambda_k}(u_k)$, *then* $u_k \to u$.

(2) *Let* a, b *be positive numbers such that* $a^{\frac{N-4}{2}} b \geq C_2(N)$. *Suppose that* $\lambda_k \to \lambda \geq 0$, $\Phi_{\lambda_k}(u_k) \to c \in \mathbb{R}$ *and* $\Phi'_{\lambda_k}(u_k) \to 0$. *If* $a^{\frac{N-4}{2}} b = C_2(N)$ *assume also that*

$$c \neq \frac{(2^*-2)^2 a^2}{4 \cdot 2^*(4-2^*)b}.$$

Then, u_k *has a convergent subsequence.*

(3) *Let* a, b *be positive numbers such that* $a^{\frac{N-4}{2}} b \geq C_2(N)$. *Suppose that* $\lambda_k \to \lambda \geq 0$ *and* $u_k \rightharpoonup u$. *Then,* $\Phi'_\lambda(u)(u) \leq \liminf_k \Phi'_{\lambda_k}(u_k)(u_k)$.

Proof Item (1) can be found, after some mild modifications, in [5, Lemma 2.1]. In a similar way (3) can be proved. Item (2) follows easily from [5, Lemma 2.2] when $a^{\frac{N-4}{2}} b > C_2(N)$ (see also [13, Proposition B.1]). The case $a^{\frac{N-4}{2}} b = C_2(N)$ can be deduced from [13, Proposition B.4]. Note from Corollary 2.1 that

$$\frac{(2^*-2)^2 a^2}{4 \cdot 2^*(4-2^*)b} = g(t_{a,b})t_{a,b}^2,$$

and one can immediately see, after introducing the parameter a, that $g(t_{a,b})t_{a,b}^2 = g(\tau_b^0)$, where $g(\tau_b^0)$ was defined in [13, Lemma B3]. □

For each $\lambda \geq 0$ and $u \in H_0^1(\Omega)\setminus\{0\}$, define the fiber maps associated to Φ_λ, $\psi_{\lambda,u} : (0, +\infty) \to \mathbb{R}$ by

$$\psi_{\lambda,u}(t) := \Phi_\lambda(tu) = \frac{a}{2}\|u\|^2 t^2 + \frac{b}{4}\|u\|^4 t^4 - \frac{1}{2^*}\|u\|_{2^*}^{2^*} t^{2^*} - \lambda \int_\Omega F(x, tu)dx.$$

Proposition 2.2 *Suppose $\lambda \geq 0$ and $u \in H_0^1(\Omega)\setminus\{0\}$, then*

(i) *there exists a neighborhood V of the origin such that $\psi_{\lambda,u}(t) > 0$ for all $t \in V \cap (0, +\infty)$. Moreover $\psi_{\lambda,u}(t) \to \infty$ as $t \to \infty$ and $\psi_{\lambda,u}$ is bounded from below;*

(ii) *there exists a neighborhood V of the origin such that $\psi'_{\lambda,u}(t) > 0$ for all $t \in V \cap (0, +\infty)$. Moreover $\psi'_{\lambda,u}(t) \to \infty$ as $t \to \infty$ and $\psi'_{\lambda,u}$ is bounded from below.*

Proof (i) Note that

$$\psi_{\lambda,u}(t) = t^2\left(\frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 t^2 - \frac{1}{2^*}\|u\|_{2^*}^{2^*} t^{2^*-2} - \lambda\int_\Omega \frac{F(x,tu)}{t^2}dx\right).$$

From $(\mathcal{F}_4)$ we deduce the existence of V. On the other hand we have

$$\psi_{\lambda,u}(t) = t^4\left(\frac{a}{2}\|u\|^2 t^{-2} + \frac{b}{4}\|u\|^4 - \frac{1}{2^*}\|u\|_{2^*}^{2^*} t^{2^*-4} - \lambda\int_\Omega \frac{F(x,tu)}{t^4}dx\right).$$

Since $2 < p < 2^* < 4$, we conclude from $(\mathcal{F}_3)$ that $\psi_{\lambda,u}(t) \to \infty$ as $t \to \infty$. The last part is obvious.

(ii) Note that

$$\psi'_{\lambda,u}(t) = t\left(a\|u\|^2 + b\|u\|^4 t^2 - \|u\|_{2^*}^{2^*} t^{2^*-2} - \lambda\int_\Omega \frac{f(x,tu)u}{t}dx\right).$$

From $(\mathcal{F}_4)$ again we deduce the existence of V. On the other hand we have

$$\psi'_{\lambda,u}(t) = t^3\left(a\|u\|^2 t^{-2} + b\|u\|^4 - \|u\|_{2^*}^{2^*} t^{2^*-4} - \lambda\int_\Omega \frac{f(x,tu)}{t^3}dx\right).$$

Since $2 < p < 2^* < 4$, we conclude from $(\mathcal{F}_4)$ that $\psi'_{\lambda,u}(t) \to \infty$ as $t \to \infty$. The last part is obvious. ◻

The remaining part of this Section is devoted to define a suitable extremal parameter λ_0^* which will be crucial in our arguments. Consider the system

$$\begin{cases} \psi_{\lambda,u}(t) = 0, \\ \psi'_{\lambda,u}(t) = 0, \\ \psi_{\lambda,u}(t) = \inf_{s>0}\psi_{\lambda,u}(s). \end{cases} \tag{4}$$

Proposition 2.3 *Assume that $a^{\frac{N-4}{2}} b \geq C_1(N)$ and take $u \in H_0^1(\Omega)\setminus\{0\}$. Then there exists a unique positive $\lambda_0(u)$ satisfying (4).*

Proof Note that

$$\psi_{\lambda,u}(t) - \psi_{\lambda',u}(t) = (\lambda' - \lambda)\int_\Omega F(x,tu)dx. \tag{5}$$

Since $F(x,v) \geq 0$ for all $v \in \mathbb{R}$ (see $(\mathcal{F}_2)$), we conclude from (5) that $\psi_{\lambda,u}(t) - \psi_{\lambda',u}(t) \geq 0$ for all $t \in \mathbb{R}$ and $0 \leq \lambda < \lambda'$. Moreover, on compact sets of the form $[c,d]$, with $0 < c < d$, we deduce that $\psi_{\lambda,u} \to \psi_{\lambda',u}$ uniformly as $\lambda \to \lambda'$. From Proposition 2.2, there exists a neighborhood of the origin $V_{\lambda'}$ such that $\psi_{\lambda',u}(t) > 0$ if $t \in V_{\lambda'} \cap (0,+\infty)$, therefore $\psi_{\lambda,u}(t) > 0$ for all $0 \leq \lambda < \lambda'$. Once $\psi_{0,u}$ is positive on $(0,\infty)$ (see Proposition 2.1) and tends to ∞ as $t \to \infty$ we conclude that for λ sufficiently small, the fiber map $\psi_{\lambda,u}$ is positive in $(0,\infty)$. On the other hand, fixed $t > 0$ one can easily see that $\psi_{\lambda,u}(t) \to -\infty$ as $\lambda \to \infty$. Therefore, there exists a unique $\lambda_0(u)$ solving system (4).

Now we claim that $\lambda_0(u) > 0$. Indeed, from Lemma 2.1 and Proposition 2.1 we have that

$$\psi_{0,u}(t) > g(\|u\|t)(\|u\|t)^2 \geq 0, \quad \forall t > 0.$$

From (5) we conclude that $\lambda_0(u) > 0$.

□

Remark 2.2 The proof of Proposition 2.3 also shows that if $a^{\frac{N-4}{2}}b < C_1(N)$, then there exists $u \in H_0^1(\Omega)\backslash\{0\}$ such that $\lambda_0(u) < 0$.

Proposition 2.4 *For each $u \in H_0^1(\Omega)\backslash\{0\}$ one has: $\lambda_0(u)$ is the unique parameter $\lambda > 0$ for which the fiber map $\psi_{\lambda,u}$ has a critical point with zero energy and satisfies* $\inf_{t>0} \psi_{\lambda,u}(t) = \inf_{t>0} \psi_{\lambda_0(u),u}(t) = 0$. *Moreover, if $\lambda > \lambda_0(u)$, then* $\inf_{t>0} \psi_{\lambda,u}(t) < 0$ *while if* $0 < \lambda \leq \lambda_0(u)$, *then* $\inf_{t>0} \psi_{\lambda,u}(t) = 0$.

Proof Choose any $t > 0$ that solves (4). If $\lambda > \lambda_0(u)$, then $\psi_{\lambda,u}(t) < \psi_{\lambda_0(u),u}(t) = 0$ and the claim follows. If $\lambda \leq \lambda_0(u)$, then $\psi_{\lambda,u}(t) \geq \psi_{\lambda_0(u),u}(t) \geq 0$ for all $t \geq 0$ and the conclusion follows at once.

□

We introduce the following extremal parameter (see [10])

$$\lambda_0^* = \inf_{u \in H_0^1(\Omega)\backslash\{0\}} \lambda_0(u).$$

Proposition 2.5 *The following holds true.*

(i) *If $a^{\frac{N-4}{2}}b > C_1(N)$, then $\lambda_0^* > 0$.*
(ii) *If $a^{\frac{N-4}{2}}b = C_1(N)$, then $\lambda_0^* = 0$. Moreover if $u_k \in H_0^1(\Omega)\backslash\{0\}$ satisfies $\lambda_0(u_k) \to \lambda_0^* = 0$, then $u_k \rightharpoonup 0$ and $\frac{\|u_k\|^2}{\|u_k\|_{2^*}^2} \to S_N$.*

Proof (i) Let us prove that $\lambda_0^* > 0$. Notice first that the function $u \to \lambda_0(u)$ is zero homogeneous. Indeed, if $(t, \lambda_0(u))$ solves system (4) and $\mu > 0$, then

$$\begin{cases} \psi_{\lambda,\mu u}(t) = \psi_{\lambda,u}(\mu t) = 0, \\ \psi'_{\lambda,\mu u}(t) = \psi'_{\lambda,u}(\mu t) = 0, \end{cases}$$

by uniqueness, $\lambda(\mu u) = \lambda(u)$. We argue by contradiction assuming that $\lambda_0^* = 0$. Then, there exists $\{u_k\} \subseteq H_0^1(\Omega)\backslash\{0\}$ such that $\lambda_k := \lambda_0(u_k) \to 0$. By homogeneity we can assume that $\|u_k\| = 1$. Then for each k, there exists $t_k > 0$ such that $\Phi_{\lambda_k}(t_k u_k) = \psi_{\lambda_k,u_k}(t_k) = 0$ or equivalently

$$\frac{a}{2} + \frac{b}{4}t_k^2 - \frac{1}{2^*}\|u_k\|_{2^*}^{2^*} t_k^{2^*-2} - \lambda_k \int_\Omega \frac{F(x, t_k u_k)}{t_k^2} dx = 0.$$

Thus, by Proposition 2.1, we obtain for each $k \in \mathbb{N}$

$$g(t_k) < \frac{a}{2} + \frac{b}{4}t_k^2 - \frac{1}{2^*}\|u_k\|_{2^*}^{2^*} t_k^{2^*-2} \leq \lambda_k \int_\Omega \frac{F(x, t_k u_k)}{t_k^2} dx. \tag{6}$$

Notice that from $(\mathcal{F}_3)$ and $(\mathcal{F}_4)$, one has that for each $\varepsilon > 0$ there exists $c > 0$ such that $|f(x, v)| \leq \varepsilon|v| + c|v|^{p-1}$ for all $x \in \Omega$, $v \in \mathbb{R}$. Thus, $|F(x, v)| \leq \frac{\varepsilon}{2}v^2 + \frac{c}{p}|v|^p$ for all

$x \in \Omega$, $v \in \mathbb{R}$. Hence, we deduce that $\{t_k\}$ is bounded in $(0, +\infty)$ and converge to some $\bar{t} > 0$. Thus, from (6) and Lemma 2.1 we deduce that

$$0 < g(\bar{t}) \leq \lim_{k\to\infty} \lambda_k \int_\Omega \frac{F(x, t_k u_k)}{t_k^2} dx = 0,$$

which is a contradiction.

(ii) Without loss of generality we assume that $0 \in \Omega$. Fix $\varphi \in C_0^\infty(\Omega)$ such that $\varphi \geq 0$ and $\varphi(x) = 1$ in the open ball centered at 0 of radius R for some $R > 0$. For each $\varepsilon > 0$, define

$$v_\varepsilon(x) = \frac{\varphi(x)}{(\varepsilon + |x|^2)^{\frac{N-2}{2}}}.$$

Let $u_\varepsilon = v_\varepsilon / \|v_\varepsilon\|$ and note that $u_\varepsilon \in H_0^1(\Omega)$ and (see [2])

$$\|u_\varepsilon\| = 1, \quad \|u_\varepsilon\|_{2^*}^{2^*} = S_N^{\frac{-2^*}{2}} + O(\varepsilon^{\frac{2^* N}{4}}), \quad \|v_\varepsilon\| = \frac{c}{\varepsilon^{\frac{N-2}{4}}} + k(\varepsilon), \tag{7}$$

where $c > 0$ does not depend on ε, $k(\varepsilon) > c_1 > 0$ for small $\varepsilon > 0$, where c_1 is a constant. Now given any $\lambda > 0$ and fixed $t > 0$, note that

$$\begin{aligned}\psi_{\lambda,u_\varepsilon}(t) &= \frac{a}{2}t^2 + \frac{b}{4}t^4 - \frac{1}{2^*}\|u_\varepsilon\|_{2^*}^{2^*} t^{2^*} - \lambda \int_\Omega F(x, tu_\varepsilon)dx \\ &= t^2 g(t) - \frac{1}{2^*} O(\varepsilon^{\frac{2^* N}{4}}) t^{2^*} - \lambda \int_\Omega F(x, tu_\varepsilon)dx.\end{aligned}$$

Take $t = t_0$ where t_0 is given by Lemma 2.1 and notice that, since $a^{\frac{N-4}{2}} b = C_1(N)$, then $g(t_0) = 0$. We have that

$$\psi_{\lambda,u_\varepsilon}(t_0) = -\frac{1}{2^*} O(\varepsilon^{\frac{2^* N}{4}}) t_0^{2^*} - \lambda \int_\Omega F(x, t_0 u_\varepsilon)dx.$$

Let us estimate $\int_\Omega F(x, t_0 u_\varepsilon)dx$ from below. By assumption $(\mathcal{F}_2)$, one has that $f(x, v) \geq \mu \chi_I(v)$ (being χ_I the characteristic function of the interval I), so there exist $\alpha, \beta > 0$ such that $F(x, v) \geq \tilde{F}(v) := \mu \int_0^v \chi_I(t)dt \geq \beta$ for every $v \geq \alpha$. Following Corollary 2.1 of [2] and using the positivity and monotonicity of F,

$$\begin{aligned}\int_\Omega F(x, t_0 u_\varepsilon)dx &\geq \int_{|x|\leq R} F(x, t_0 u_\varepsilon)dx \geq \int_{|x|\leq R} F\left(x, \frac{t_0}{\|v_\varepsilon\|(\varepsilon + |x|^2)^{\frac{N-2}{2}}}\right) dx \\ &\geq \int_{|x|\leq R} \tilde{F}\left(\frac{t_0}{\|v_\varepsilon\|(\varepsilon + |x|^2)^{\frac{N-2}{2}}}\right) dx = c_1 \varepsilon^{\frac{N}{2}} \int_0^{R\varepsilon^{-\frac{1}{2}}} \tilde{F}\left(\frac{t_0}{\|v_\varepsilon\|}\left(\frac{\varepsilon^{-1}}{1+s^2}\right)^{\frac{N-2}{2}}\right) s^{N-1} ds\end{aligned}$$

Notice that

$$\tilde{F}\left(\frac{t_0}{\|v_\varepsilon\|}\left(\frac{\varepsilon^{-1}}{1+s^2}\right)^{\frac{N-2}{2}}\right) \geq \beta \text{ if } s \text{ is such that } \frac{t_0}{\|v_\varepsilon\|}\left(\frac{\varepsilon^{-1}}{1+s^2}\right)^{\frac{N-2}{2}} \geq \alpha. \tag{8}$$

The second inequality of (8) is equivalent to

$$\frac{t_0 \varepsilon^{\frac{2-N}{4}}}{(c + \varepsilon^{\frac{N-2}{4}} k(\varepsilon))(1+s^2)^{\frac{N-2}{2}}} \geq \alpha,$$

which is true if $s \le c_2\varepsilon^{-\frac{1}{4}}$ for some constant c_2 and small ε. Therefore, by taking a smaller R if necessary, we deduce from (8) that

$$\int_\Omega F(x, t_0u_\varepsilon)dx \ge c_3\varepsilon^{\frac{N}{2}} \int_0^{R\varepsilon^{-\frac{1}{4}}} \beta s^{N-1}ds = c_3\varepsilon^{\frac{N}{4}},$$

for some positive constant c_3. Thus,

$$\psi_{\lambda,u_\varepsilon}(t_0) \le \varepsilon^{\frac{N}{4}}\left[-\frac{1}{2^*}\frac{O(\varepsilon^{\frac{2^*N}{4}})}{\varepsilon^{\frac{N}{4}}}t_0^{2^*} - \lambda c_3\right] < 0,$$

for small ε and hence $\lambda_0(u_\varepsilon) < \lambda$. Once λ was arbitrary we deduce that $\lambda_0^* = 0$.

Now suppose that $u_k \in H_0^1(\Omega)\backslash\{0\}$ satisfies $\lambda_k := \lambda_0(u_k) \to \lambda_0^* = 0$. As in i) we may assume that $\|u_k\| = 1$ and $u_k \rightharpoonup u$. Moreover there exists $t_k > 0$ such that

$$\frac{a}{2} + \frac{b}{4}t_k^2 - \frac{1}{2^*}\|u_k\|_{2^*}^{2^*}t_k^{2^*-2} - \lambda_k\int_\Omega \frac{F(x, t_ku_k)}{t_k^2}dx = 0 \qquad \text{for each } k \in \mathbb{N}.$$

From $(\mathcal{F}_3)$ and $(\mathcal{F}_4)$ we conclude that $t_k \to t > 0$ and $\|u_k\|_{2^*}^{2^*} \to s > 0$ and hence

$$\frac{a}{2} + \frac{b}{4}t^2 - \frac{1}{2^*}st^{2^*-2} = 0.$$

From the assumption on a and b we conclude that $s = S_N^{\frac{-2^*}{2}}$ and hence u_k is a minimizing sequence to S_N. Moreover, if $u \ne 0$, then (the first inequality is a consequence of Lemma 2.1 and the fact that $\|u\| \le 1$)

$$\begin{aligned} 0 \le \frac{a}{2} + \frac{b}{4}t^2 - \frac{S_N^{\frac{-2^*}{2}}}{2^*}\|u\|^{2^*}t^{2^*-2} &\le \frac{a}{2} + \frac{b}{4}t^2 - \frac{1}{2^*}\|u\|_{2^*}^{2^*}t^{2^*-2} \\ &\le \liminf_{k\to\infty}\left(\frac{a}{2} + \frac{b}{4}t_k^2 - \frac{1}{2^*}\|u_k\|_{2^*}^{2^*}t_k^{2^*-2} - \lambda_k\int_\Omega \frac{F(x, t_ku_k)}{t_k^2}dx\right) \\ &= 0, \end{aligned}$$

and consequently u is a minimizer to S_N, which is an absurd, therefore $u = 0$.

□

Proposition 2.6 *For each $\lambda \le \lambda_0^*$ and each $u \in H_0^1(\Omega)\backslash\{0\}$, $\inf_{t>0}\psi_{\lambda,u}(t) = 0$; for each $\lambda > \lambda_0^*$ there exists $u \in H_0^1(\Omega)\backslash\{0\}$ such that $\Phi_\lambda(u) < 0$.*

Proof From Proposition 2.4, if $\lambda \le \lambda_0^* \le \lambda_0(u)$, $\inf_{t>0}\psi_{\lambda,u}(t) = 0$ for each $u \in H_0^1(\Omega)\backslash\{0\}$; while if $\lambda > \lambda_0^*$, there exists $u \in H_0^1(\Omega)\backslash\{0\}$ such that $\inf_{t>0}\psi_{\lambda,u}(t) < 0$ which implies at once the claim.

□

3 Existence and non-existence results: general case

In this Section we study the existence of global/local minimizers and mountain pass type solutions to Φ_λ. At the end of the Section we show a non-existence result for small $\lambda > 0$. We note here that in the first three subsections, only hypotheses $(\mathcal{F}_1)$–$(\mathcal{F}_4)$ are needed, while in the fourth subsection we need to add hypothesis $(\mathcal{F}_5)$.

3.1 Global minimizers for $\lambda \geq \lambda_0^*$

For each $\lambda > 0$ define

$$I_\lambda = \inf\{\Phi_\lambda(u) : u \in H_0^1(\Omega)\}.$$

Theorem 3.1 *Suppose that* $a^{\frac{N-4}{2}} b \geq C_1(N)$ *and* $\lambda > \lambda_0^*$. *Then, there exists* $u_\lambda \in H_0^1(\Omega)\backslash\{0\}$ *such that* $I_\lambda = \Phi_\lambda(u_\lambda) < 0$.

Proof In fact, one can easily see by using $(\mathcal{F}_3)$, $(\mathcal{F}_4)$ and the Sobolev embeddings that Φ_λ is coercive. From Lemma 2.2 Φ_λ is also sequentially weakly lower semi-continuous and therefore by direct minimization arguments, there exists $u_\lambda \in H_0^1(\Omega)$ such that $I_\lambda = \Phi_\lambda(u_\lambda)$. Moreover, from Proposition 2.6 there exists $w \in H_0^1(\Omega)$ such that $\Phi_\lambda(w) < 0$, hence $I_\lambda < 0$ and $u_\lambda \neq 0$. □

Theorem 3.2 *Suppose that* $a^{\frac{N-4}{2}} b \geq C_1(N)$ *and* $\lambda = \lambda_0^*$. *The following holds true.*

(i) *If* $a^{\frac{N-4}{2}} b > C_1(N)$, *there exists* $u_{\lambda_0^*} \in H_0^1(\Omega)\backslash\{0\}$ *such that* $I_{\lambda_0^*} = \Phi_{\lambda_0^*}(u_{\lambda_0^*})$. *Moreover,* $I_{\lambda_0^*} = 0$.

(ii) *If* $a^{\frac{N-4}{2}} b = C_1(N)$, $u = 0$ *is the only minimizer for* $I_{\lambda_0^*}$.

Proof (i) In fact, take a sequence $\lambda_k \downarrow \lambda_0^*$. From Theorem 3.1, for each k, we can find $u_k \in H_0^1(\Omega)\backslash\{0\}$ such that $I_{\lambda_k} = \Phi_{\lambda_k}(u_k) < 0$. Since $\lambda_k \downarrow \lambda_0^*$ it follows (as in the proof of Theorem 3.1) that $\{u_k\}$ is bounded and therefore we may assume that $u_k \rightharpoonup u$ in $H_0^1(\Omega)$. From Lemma 2.2 we obtain

$$\Phi_{\lambda_0^*}(u) \leq \liminf_{k\to\infty} \Phi_{\lambda_k}(u_k) \leq 0.$$

Proposition 2.6 ensures that $\Phi_{\lambda_0^*}(w) \geq 0$ for each $w \in H_0^1(\Omega)$ and thus $\lim_{k\to\infty} \Phi_{\lambda_k}(u_k) = \Phi_{\lambda_0^*}(u) = 0$, or $I_{\lambda_0^*} = \Phi_{\lambda_0^*}(u) = 0$.

To conclude the proof, we have to show that $u \neq 0$. In fact

$$\frac{a}{2}\|u_k\|^2 + \frac{b}{4}\|u_k\|^4 - \frac{S_N^{-\frac{2^*}{2}}}{2^*}\|u_k\|^{2^*} \leq \frac{a}{2}\|u_k\|^2 + \frac{b}{4}\|u_k\|^4 - \frac{1}{2^*}\|u_k\|^{2^*} \leq \lambda_k \int_\Omega F(x, u_k)dx.$$

Thus,

$$g(\|u_k\|) = \frac{a}{2} + \frac{b}{4}\|u_k\|^2 - \frac{S_N^{-\frac{2^*}{2}}}{2^*}\|u_k\|^{2^*-2} \leq \lambda_k \int_\Omega \frac{F(x, u_k)}{\|u_k\|^2}dx.$$

If $u = 0$, from $(\mathcal{F}_3)$ and $(\mathcal{F}_4)$, the right hand side in the above inequality would tend to zero against the fact that $g(\|u_k\|) \geq \min_{[0,+\infty[} g > 0$ (see Lemma 2.1).

(ii) From Proposition 2.5 we know that $\lambda_0^* = 0$ and hence

$$\Phi_{\lambda_0^*}(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{2^*}\|u\|_{2^*}^{2^*}.$$

The hypothesis $a^{\frac{N-4}{2}} b = C_1(N)$ implies that $u = 0$ is the only minimizer for this functional. Indeed, from Proposition 2.1 and Lemma 2.1 we have that

$$\Phi_{\lambda_0^*}(u) > g(\|u\|)\|u\|^2 \geq 0, \forall u \in H_0^1(\Omega)\backslash\{0\}.$$

□

Proposition 3.1 *Suppose that* $a^{\frac{N-4}{2}}b > C_1(N)$. *If* $u \in H_0^1(\Omega)\backslash\{0\}$ *satisfies* $I_{\lambda_0^*} = \Phi_{\lambda_0^*}(u)$, *then* $\lambda_0^* = \lambda_0(u)$.

Proof The equality $\lambda_0^* = \lambda_0(u)$ is a consequence of the definition of λ_0^*. □

Theorem 3.3 *If* $a_k^{\frac{N-4}{2}} b_k \downarrow C_1(N)$, $a_k \to a > 0$ *and* $b_k \to b > 0$, *then* $\lambda_k := \lambda_0^* \to 0$. *Moreover, if* $u_k \in H_0^1(\Omega)\backslash\{0\}$ *satisfies* $\lambda_k = \lambda_0(u_k)$, *then* $u_k \rightharpoonup 0$ *and* $\frac{\|u_k\|_2^2}{\|u_k\|_{2^*}^2} \to S_N$.

Proof For each $\varepsilon > 0$, define u_ε as in the proof of Proposition 2.5. Given any $\lambda > 0$ and fixed $t > 0$, note from (7) that

$$\begin{aligned}\psi_{\lambda,u_\varepsilon}(t) &= \frac{a_k}{2}t^2 + \frac{b_k}{4}t^4 - \frac{1}{2^*}\|u_\varepsilon\|_{2^*}^{2^*}t^{2^*} - \lambda\int_\Omega F(x,tu_\varepsilon)dx\\ &= t^2 g_k(t) - \frac{1}{2^*}O(\varepsilon^{\frac{2^*N}{4}})t^{2^*} - \lambda\int_\Omega F(x,tu_\varepsilon)dx,\end{aligned}$$

where g_k is the analogous of g with a_k and b_k instead of a and b. By taking $t = t_{0,k}$ where $t_{0,k}$ is given in (2) (with a_k and b_k instead of a and b) we have that $t_{0,k} \to t_0 > 0$ (t_0 as in (2)) and

$$\lim_k \psi_{\lambda,u_\varepsilon}(t_{0,k}) = \varepsilon^{\frac{N}{4}}\left[-\frac{1}{2^*}\frac{O(\varepsilon^{\frac{2^*N}{4}})}{\varepsilon^{\frac{N}{4}}}t_0^{2^*} - \lambda\int_\Omega \frac{F(x,t_0u_\varepsilon)}{\varepsilon^{\frac{N}{4}}}dx\right],$$

Since

$$\int_\Omega F(x,t_0u_\varepsilon)dx \geq c\varepsilon^{\frac{N}{4}},$$

for some positive constant c, we get that $\psi_{\lambda,u_\varepsilon}(t_{0,k}) < 0$ for small ε and big k. Then $\lambda_k \leq \lambda_0(u_\varepsilon) < \lambda$. Once λ was arbitrary we deduce that $\lambda_0^* = 0$.

Now suppose that $u_k \in H_0^1(\Omega)\backslash\{0\}$ satisfies $\lambda_k := \lambda_0(u_k) \to \lambda_0^* = 0$. We may assume that $\|u_k\| = 1$ and $u_k \rightharpoonup u$. Moreover there exists $t_k > 0$ such that

$$\frac{a_k}{2} + \frac{b_k}{4}t_k^2 - \frac{1}{2^*}\|u_k\|_{2^*}^{2^*}t_k^{2^*-2} - \lambda_k\int_\Omega \frac{F(x,t_ku_k)}{t_k^2} = 0.$$

From $(\mathcal{F}_3)$ and $(\mathcal{F}_4)$ we conclude that $t_k \to t > 0$ and $\|u_k\|_{2^*}^{2^*} \to s > 0$ and hence

$$\frac{a}{2} + \frac{b}{4}t^2 - \frac{1}{2^*}st^{2^*-2} = 0.$$

From the fact that $a^{\frac{N-4}{2}}b = C_1(N)$ we infer that $s = S_N^{\frac{-2^*}{2}}$ and hence $(u_k)_k$ is a minimizing sequence to S_N. Moreover, if $u \neq 0$, then (the first inequality is a consequence of Lemma 2.1 and the fact that $\|u\| \leq 1$)

$$\begin{aligned}0 \leq \frac{a}{2} + \frac{b}{4}t^2 - \frac{S_N^{\frac{-2^*}{2}}}{2^*}\|u\|^{2^*}t^{2^*-2} &\leq \frac{a}{2} + \frac{b}{4}t^2 - \frac{1}{2^*}\|u\|_{2^*}^{2^*}t^{2^*-2}\\ &\leq \liminf_{k\to\infty}\left(\frac{a_k}{2} + \frac{b_k}{4}t_k^2 - \frac{1}{2^*}\|u_k\|_{2^*}^{2^*}t_k^{2^*-2} - \lambda_k\int_\Omega \frac{F(x,t_ku_k)}{t_k^2}dx\right)\\ &= 0,\end{aligned}$$

and consequently u is a minimizer to S_N, which is an absurd, therefore $u = 0$. □

3.2 Mountain pass type solution for $\lambda \geq \lambda_0^*$

Proposition 3.2 *For each $\lambda > 0$, there exists $R_\lambda > 0$ such that*

$$\inf\{\Phi_\lambda(u) : \|u\| = R_\lambda\} > 0.$$

Proof Indeed, given $\varepsilon > 0$, from $(\mathcal{F}_3)$, $(\mathcal{F}_4)$ and Sobolev embeddings, there exists a positive constant c such that

$$\begin{aligned}\Phi_\lambda(u) &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{c}{2^*}\|u\|^{2^*} - \lambda c(\varepsilon\|u\|^2 + \|u\|^p) \\ &= \left(\frac{a}{2} - \lambda c\varepsilon\right)\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{2^*}\|u\|^{2^*} - \lambda c\|u\|^p, \forall u \in H_0^1(\Omega).\end{aligned}$$

By choosing $\varepsilon > 0$ conveniently the proof is complete. □

For each $\lambda \geq \lambda_0^*$ define

$$\Gamma_\lambda = \{\gamma \in C([0, 1], H_0^1(\Omega)) : \gamma(0) = 0,\ \gamma(1) = u_{\lambda_0^*}\},$$

where $u_{\lambda_0^*}$ is as in Theorem 3.2. and

$$c_\lambda = \inf_{\gamma\in\Gamma_\lambda} \max_{t\in[0,1]} \Phi_\lambda(\gamma(t)).$$

Theorem 3.4 *There holds:*

(i) *If $a^{\frac{N-4}{2}} b > C_2(N)$, then for each $\lambda \geq \lambda_0^*$, there exist $w_\lambda \in H_0^1(\Omega)\backslash\{0\}$ such that $\Phi_\lambda(w_\lambda) = c_\lambda$ and $\Phi'_\lambda(w_\lambda) = 0$.*

(ii) *If $a^{\frac{N-4}{2}} b = C_2(N)$, then the above conclusion holds for λ sufficiently large.*

Proof The proof is standard and we write only the main steps. Note that $\Phi_\lambda(0) = 0$ and $\Phi_\lambda(u_{\lambda_0^*}) \leq 0$. In fact, from Theorem 3.2 we know that $\Phi_{\lambda_0^*}(u_{\lambda_0^*}) = 0$ and if $\lambda > \lambda^*$, from Proposition 2.4 and Proposition 3.1, we must conclude that $\Phi_\lambda(u_{\lambda_0^*}) < 0$. These together with Proposition 3.2 implies a mountain pass geometry to Φ_λ.

(i) If $a^{\frac{N-4}{2}} b > C_2(N)$, from Lemma 2.2, Φ_λ satisfies the Palais-Smale condition at any level and the proof is complete.

(ii) If $a^{\frac{N-4}{2}} b = C_2(N)$, it is enough to prove that (see Lemma 2.2)

$$c_\lambda \neq \frac{(2^* - 2)^2 a^2}{4 \cdot 2^*(4 - 2^*)b}.$$

We will actually show that $c_\lambda \to 0$ as $\lambda \to \infty$. Indeed, given $\varepsilon > 0$, fix any $\lambda' > 0$. From $(\mathcal{F}_1)$ and $(\mathcal{F}_4)$, there exists $\delta > 0$ such that $0 < \psi_{\lambda',u_{\lambda_0^*}}(t) \leq \varepsilon$ for all $t \in (0, \delta]$. Since the function $(\lambda', \infty) \ni \lambda \mapsto \psi_{\lambda,u_{\lambda_0^*}}(\delta)$ is continuous, decreasing and tends to $-\infty$ as $\lambda \to \infty$ (see the proof of Proposition 2.3), it follows that there exists a unique parameter $\mu > \lambda'$ such that $\psi_{\mu,u_{\lambda_0^*}}(\delta) = 0$. Now observe that on compact sets $[t_0, t_1] \subset (0, \infty)$, we can always choose λ so large that $\psi_{\lambda,u_{\lambda_0^*}}(t) < 0$ for all $t \in [t_0, t_1]$. By taking δ even smaller if necessary, we can suppose that

$$c_\lambda \leq \max_{t\in[0,1]} \Phi_\lambda(tu_{\lambda_0^*}) = \max_{t\in[0,1]} \psi_{\mu,u_{\lambda_0^*}}(t) = \max_{t\in(0,\delta)} \psi_{\mu,u_{\lambda_0^*}}(t) = \psi_{\mu,u_{\lambda_0^*}}(t_{max}),$$

where $t_{max} \in (0, \delta)$. Since $\psi_{\mu,u_{\lambda_0^*}}(t_{max}) \leq \psi_{\lambda',u_{\lambda_0^*}}(t_{max}) \leq \varepsilon$, it follows that $c_\lambda \to 0$ as $\lambda \to \infty$. Choosing λ sufficiently large there holds

$$c_\lambda < \frac{(2^* - 2)^2 a^2}{4 \cdot 2^*(4 - 2^*)b},$$

and Lemma 2.2 applies. □

3.3 Local minimizers and mountain pass type solutions for $\lambda < \lambda_0^*$

From Proposition 2.6, $I_\lambda = \inf_{H_0^1(\Omega)} \Phi_\lambda \geq 0$ for $\lambda \leq \lambda_0^*$, and consequently $u = 0$ is a global minimizer of Φ_λ. It is the unique global minimizer if $\lambda < \lambda_0^*$, while when $\lambda = \lambda_0^*$ (see Theorem 3.2) there exists a second global minimizer $u_{\lambda_0^*} \neq 0$. We will prove that for $\lambda < \lambda_0^*$, close to λ_0^*, Φ_λ has a local minimizer with positive energy.

First we prove a refined version of Proposition 3.2: fix $u_{\lambda_0^*} \in H_0^1(\Omega)\backslash\{0\}$ such that $\lambda_0^* = \lambda_0(u_{\lambda_0^*})$ (see Theorem 3.2 and Proposition 3.1). Denote $R = \|u_{\lambda_0^*}\|$.

Proposition 3.3 *Suppose that $\lambda \leq \lambda_0^*$, then there exists $0 < r < R$ and $M > 0$ such that*

$$\inf\{\Phi_\lambda(u) : u \in H_0^1(\Omega),\ \|u\| = r\} \geq M.$$

Proof Indeed, as in the proof of Proposition 3.2, given $\varepsilon > 0$, there exists a positive constant c, depending only on N and p, such that

$$\Phi_\lambda(u) \geq \left(\frac{a}{2} - \lambda c\varepsilon\right)\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{c}{2^*}\|u\|^{2^*} - \lambda c\|u\|^p,\ \forall u \in H_0^1(\Omega),$$

therefore

$$\Phi_\lambda(u) \geq \left(\frac{a}{2} - \lambda_0^* c\varepsilon\right)\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{c}{2^*}\|u\|^{2^*} - \lambda_0^* c\|u\|^p,\ \forall u \in H_0^1(\Omega).$$

If we choose ε in such a way that $\frac{a}{2} - \lambda_0^* c\varepsilon > 0$ the proof is complete. □

Now consider the set

$$K = \{u \in H_0^1(\Omega)\backslash\{0\} : \Phi_{\lambda_0^*}(u) = 0\}.$$

Note by Theorem 3.2 that $K \neq \emptyset$. In the next corollary we denote $B(0, r) = \{u \in H_0^1(\Omega) : \|u\| < r\}$.

Corollary 3.1 *Suppose that $a^{\frac{N-4}{2}} b > C_1(N)$. There holds:*

(1) *There exists $r > 0$ such that $K \cap B(0, r) = \emptyset$.*
(2) *K is compact.*

Proof The proof of (1) follows from Proposition 3.3. For the proof of (2), take $u_n \in K$. Since $\Phi_{\lambda_0^*}$ is coercive, we can suppose that u_n is bounded and, up to a subsequence, $u_n \rightharpoonup u$. Note that $0 \leq \Phi_{\lambda_0^*}(u) \leq \lim_n \Phi_{\lambda_0^*}(u_n) = 0$ and thus, from Lemma 2.2, 1), we conclude that $u_n \to u$ and from (1) it follows that $u \neq 0$, which implies that K is compact and the proof is complete. □

Given $\delta > 0$ define

$$K_\delta = \{u \in H_0^1(\Omega) : \operatorname{dist}(u, K) \leq \delta\}.$$

For the next result, r is given as in Corollary 3.1.

Corollary 3.2 *Suppose that* $a^{\frac{N-4}{2}}b > C_1(N)$*, then*

(1) K_δ *is sequentially weakly closed.*
(2) *There exists* $\delta > 0$ *such that* $K_\delta \cap B(0, r) = \emptyset$.

Proof (1) It is enough to prove that the distance function $H_0^1(\Omega) \ni u \mapsto \operatorname{dist}(u, K)$ is sequentially weakly lower semi-continuous. Suppose that $u_n \rightharpoonup u$ in $H_0^1(\Omega)$ and assume, on the contrary, that there exists $c > 0$ such that

$$\liminf_{n\to\infty} \operatorname{dist}(u_n, K) < c < \operatorname{dist}(u, K). \tag{9}$$

Since K is compact, for each n we can find $v_n \in K$ such that $\operatorname{dist}(u_n, K) = \|u_n - v_n\|$. Up to a subsequence, we can suppose that $v_n \to v \in K$. Also by (9), up to a subsequence, we can assume that for large n there holds $\operatorname{dist}(u_n, K) < c$, therefore from (9) and for large n, we conclude that

$$\begin{aligned} \|u_n - v\| &\le \|u_n - v_n\| + \|v_n - v\| \\ &< c + \|v_n - v\| \\ &< \operatorname{dist}(u, K) + \|v_n - v\| \\ &\le \|u - v\| + \|v_n - v\| \\ &\le \liminf_{k\to\infty} \|u_k - v\| + \|v_n - v\|, \end{aligned}$$

which implies a contradiction and thus the proof is complete.

(2) is a consequence of Corollary 3.1. □

For the next proposition we choose δ as in Corollary 3.2.

Proposition 3.4 *Suppose that* $a^{\frac{N-4}{2}}b > C_1(N)$*, then there exits* $\overline{\varepsilon} > 0$ *such that*

$$\inf\{\Phi_{\lambda_0^*}(u) : u \in \partial K_\delta\} > 2\overline{\varepsilon}.$$

Proof On the contrary, we can find a sequence $u_n \in \partial K_\delta$ such that $\Phi_{\lambda_0^*}(u_n) \to 0$ as $n \to \infty$. Since $\Phi_{\lambda_0^*}$ is coercive, we can assume that, up to a subsequence, $u_n \rightharpoonup u$ in $H_0^1(\Omega)$. Once $0 \le \Phi_{\lambda_0^*}(u) \le \lim_n \Phi_{\lambda_0^*}(u_n) = 0$, it follows from Lemma 2.2, 1) that $u_n \to u$ and thus $\Phi_{\lambda_0^*}(u) = 0$ with $u \in \partial K_\delta$. Since $u \notin K$, we have that $u = 0$, which contradicts item (2) of Corollary 3.2. □

Proposition 3.5 *Suppose that* $a^{\frac{N-4}{2}}b > C_1(N)$*, then* $\inf\{\Phi_\lambda(u) : u \in H_0^1(\Omega),\ u \in K\} \to 0$ *as* $\lambda \uparrow \lambda_0^*$.

Proof In fact, let $u \in H_0^1(\Omega)$ be such that $\lambda_0^* = \lambda_0(u)$ (see Theorem 3.2 and Proposition 3.1). Note that

$$0 \le \inf\{\Phi_\lambda(u) : u \in H_0^1(\Omega),\ u \in K\} \le \Phi_\lambda(u) \to 0, \text{ as } \lambda \uparrow \lambda_0^*.$$

□

For each $\lambda \le \lambda_0^*$ and $\delta > 0$, define

$$\hat{I}_\lambda = \inf\{\Phi_\lambda(u) : u \in H_0^1(\Omega),\ u \in K_\delta\}.$$

Theorem 3.5 *Assume that* $a^{\frac{N-4}{2}}b > C_1(N)$. *There exists* $\delta > 0$ *and* $\varepsilon > 0$ *such that if* $\lambda \in (\lambda_0^* - \varepsilon, \lambda_0^*)$, *then the infimum* $\hat{I}_\lambda$ *is achieved by some* $u_\lambda \in K_\delta$ *satisfying* $\text{dist}(u_\lambda, K) < \delta$. *Moreover* u_λ *is a local minimizer and a critical point to* Φ_λ *and* $\hat{I}_\lambda > 0$.

Proof Indeed, choose $\delta > 0$ as in Proposition 3.4. By Proposition 3.5 we can find $\varepsilon > 0$ such that for all $\lambda \in (\lambda_0^* - \varepsilon, \lambda_0^*)$ there holds $\inf\{\Phi_\lambda(u) : u \in H_0^1(\Omega),\ u \in K\} < \overline{\varepsilon}$, where $\overline{\varepsilon}$ is given by Proposition 3.4. Moreover, we can also assume by Proposition 3.4 that $\inf\{\Phi_{\lambda_0^*}(u) : u \in \partial K_\delta\} > \overline{\varepsilon}$ for all $\lambda \in (\lambda_0^* - \varepsilon, \lambda_0^*)$.

Now let u_n be a minimizing sequence to $\hat{I}_\lambda$. Once Φ_λ is coercive, we can suppose that $u_n \rightharpoonup u$ in $H_0^1(\Omega)$. By Corollary 3.2 we have that $u \in K_\delta$. Since $\Phi_\lambda(u) \le \liminf_{n\to\infty} \Phi_\lambda(u_n) = \hat{I}_\lambda$, it follows that $\Phi_\lambda(u) = \hat{I}_\lambda$. By the previous paragraph we conclude that $u_\lambda \notin \partial K_\delta$ and hence the proof is complete. □

Now we show the existence of a mountain pass type solution: let $\varepsilon > 0$ be given as in Theorem 3.5 and for each $\lambda \in (\lambda_0^* - \varepsilon, \lambda_0^*)$, choose $u_\lambda \in H_0^1(\Omega)\backslash\{0\}$ such that $\hat{I}_\lambda = \Phi_\lambda(u_\lambda)$. Define

$$\Gamma_\lambda = \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0,\ \gamma(1) = u_\lambda\},$$

and

$$c_\lambda = \inf_{\gamma\in\Gamma_\lambda} \max_{t\in[0,1]} \Phi_\lambda(\gamma(t)).$$

Theorem 3.6 *Assume that* $a^{\frac{N-4}{2}}b > C_2(N)$, *then for each* $\lambda \in (\lambda_0^* - \varepsilon, \lambda_0^*)$, *there exists* $w_\lambda \in H_0^1(\Omega)\backslash\{0\}$ *such that* $\Phi_\lambda(w_\lambda) = c_\lambda$ *and* $\Phi_\lambda'(w_\lambda) = 0$.

Proof Note that $\min\{\Phi_\lambda(0), \Phi_\lambda(u_\lambda)\} < M$, where M is given by Proposition 3.2. Therefore Φ_λ has a mountain pass geometry. From Lemma 2.2 we know that Φ_λ satisfies the Palais-Smale condition and thus the proof is complete. □

Now we are in position to prove Theorems 1.1, 1.2, 1.3:

Proof of Theorem 1.1 It follows from Theorems 3.1, 3.2, 3.3 and the definition of λ_0^*. □

Proof of Theorem 1.2 It follows from Theorem 3.5. □

Proof of Theorem 1.3 It follows from Theorems 3.4 and 3.6. □

3.4 Non-existence result

Suppose $(\mathcal{F}_5)$. Therefore the following system is well defined:

$$\begin{cases} \psi_{\lambda,u}'(t) = 0, \\ \psi_{\lambda,u}''(t) = 0, \\ \psi_{\lambda,u}'(t) = \inf_{s>0} \psi_{\lambda,u}'(s). \end{cases} \tag{10}$$

The next Proposition can be proved in the same way as Proposition 2.3

Proposition 3.6 *Assume that* $u \in H_0^1(\Omega)\backslash\{0\}$, *then there exists a unique* $\lambda(u) > 0$ *satisfying* (10).

Proposition 3.7 *For each $u \in H_0^1(\Omega)\backslash\{0\}$ one has: $\lambda(u)$ is the unique parameter $\lambda > 0$ for which the fiber map $\psi_{\lambda,u}$ has a critical point with second derivative zero and satisfies $\inf_{t>0} \psi'_{\lambda,u}(t) = 0$. Moreover, if $0 < \lambda < \lambda(u)$, then $\psi_{\lambda,u}$ has no critical points.*

Proof If $0 < \lambda < \lambda(u)$, then $\psi'_{\lambda,u}(s) > \psi'_{\lambda(u),u}(s) \geq 0$ for each $t > 0$. □

Corollary 3.3 *For each $u \in H_0^1(\Omega)\backslash\{0\}$ one has that $\lambda(u) < \lambda_0(u)$.*

Proof Indeed, assume on the contrary that $\lambda_0(u) \leq \lambda(u)$, then from Proposition 3.7, the definition of $\lambda_0(u)$ and Proposition 2.2, we deduce that $\psi_{\lambda_0(u),u}$ is increasing, which contradicts the definition of $\lambda_0(u)$, therefore, $\lambda(u) < \lambda_0(u)$. □

Define the extremal value (see [10])

$$\lambda^* = \inf_{u \in H_0^1(\Omega)\backslash\{0\}} \lambda(u).$$

Proposition 3.8 *There holds:*

(i) *If $a^{\frac{N-4}{2}} b > C_2(N)$, then $0 < \lambda^* < \lambda_0^*$.*
(ii) *If $a^{\frac{N-4}{2}} b = C_2(N)$, then $\lambda^* = 0$. Moreover if $u_k \in H_0^1(\Omega)\backslash\{0\}$ satisfies $\lambda(u_k) \to \lambda^* = 0$, then $u_k \rightharpoonup 0$ and $\frac{\|u_k\|_2^2}{\|u_k\|_{2^*}^2} \to S_N$.*

Proof We only prove that $\lambda^* < \lambda_0^*$ (the rest of the proof is similar to the proof of Proposition 2.5). Indeed, from Theorem 3.2 and Proposition 3.1, there exists $u \in H_0^1(\Omega)\backslash\{0\}$ such that $\lambda_0^* = \lambda_0(u)$, therefore from Corollary 3.3 we obtain $\lambda^* \leq \lambda(u) < \lambda_0(u) = \lambda_0^*$. □

Proposition 3.9 *For each $\lambda < \lambda^*$, the fiber map $\psi_{\lambda,u}$ is increasing and has no critical points.*

Proof This follows form the fact that $\lambda < \lambda^* \leq \lambda(u)$ for every $u \in H_0^1(\Omega)\backslash\{0\}$ and Proposition 3.7. □

Theorem 3.7 *If $a^{\frac{N-4}{2}} b > C_2(N)$ and $\lambda \in (0, \lambda^*)$, then $(\mathcal{P}_\lambda)$ has no non-zero solution.*

Proof In fact, from Proposition 3.9 we have that $\psi'_{\lambda,u}(t) > 0$ for all $t > 0$ and $u \in H_0^1(\Omega)\backslash\{0\}$, therefore Φ_λ has no critical points other than $u = 0$. □

The next result provides the existence of $u \in H_0^1(\Omega)\backslash\{0\}$ such that $\Phi'_{\lambda^*}(u)u = 0$.

Proposition 3.10 *Suppose that $a^{\frac{N-4}{2}} b > C_2(N)$. Then, there exists $u \in H_0^1(\Omega)\backslash\{0\}$ such that $\lambda^* = \lambda(u)$.*

Proof Let λ_k be a sequence of positive numbers converging to λ^*. Thus, there exists $u_k \in H_0^1(\Omega)\backslash\{0\}$ with $\|u_k\| = 1$ (by the homogeneity of the map $u \to \lambda(u)$) such that $\lambda_k = \lambda(u_k)$. We deduce then, the existence of $u \in H_0^1(\Omega)$ such that $u_k \rightharpoonup u$. We claim that $u \neq 0$. By the definition of λ_k, there exists $t_k = t(u_k) > 0$ such that

$$\psi'_{\lambda_k,u_k}(t_k) = \Phi'_{\lambda_k}(t_k u_k)(u_k) = 0$$

that is

$$a + bt_k^2 - \|u_k\|_{2^*}^{2^*} t_k^{2^*-2} - \lambda_k \int_\Omega \frac{f(x, t_k u_k)u_k}{t_k} dx = 0.$$

Thus, we obtain

$$0 < h(t_k) \le a + bt_k^2 - S_N^{-\frac{2^*}{2}} t_k^{2^*-2} \le \lambda_k \int_\Omega \frac{f(x, t_k u_k)u_k}{t_k} dx. \tag{11}$$

From the above inequality, $(\mathcal{F}_3)$ and $(\mathcal{F}_4)$ we deduce that $\{t_k\}$ is bounded in $(0, +\infty)$ and it admits a subsequence still denoted by $\{t_k\}$ converging to some $\bar{t} > 0$. Also, from (11) and Lemma 2.1 we deduce that $u \neq 0$. By Proposition 3.7, $\psi'_{\lambda^*,u}(t) > 0$ for every $t > 0$. But since $t_k u_k \rightharpoonup \bar{t}u$, by 3) Lemma 2.2 it follows

$$\psi'_{\lambda^*,u}(\bar{t}) = \Phi'_{\lambda^*}(\bar{t}u)(\bar{t}u) \le \liminf_k \Phi'_{\lambda_k}(t_k u_k)(t_k u_k) = \liminf_k \psi'_{\lambda_k,u_k}(t_k) = 0,$$

which leads to a contradiction. □

As a consequence we have:

Proof of Theorem 1.4 It follows from Theorem 3.7, Proposition 3.7 and Proposition 3.10. □

4 A particular case: $f(x, u) = |u|^{p-2}u$

In this Section we consider the particular case where $f(x, u) = |u|^{p-2}u$, that is

$$\begin{cases} -\left(a + b\displaystyle\int_\Omega |\nabla u|^2 dx\right) \Delta u = |u|^{2^*-2}u + \lambda |u|^{p-2}u, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{12}$$

and $p \in (2, 2^*)$. We will compare the results obtained here with the literature. In fact we will extend and complement some results of [13]. For some values of p in fact, we have a fairly complete picture. One can easily see that $f(x, u) = |u|^{p-2}u$ satisfies all hypothesis $(\mathcal{F}_1)$-$(\mathcal{F}_5)$ and therefore, with respect to problem (12) we have, as a consequence of Theorems 1.1, 1.2, 1.3 and 1.4, the following:

Theorem 4.1 *There exists a function* $\lambda_0^* : (0, \infty)^2 \to [0, \infty)$ *satisfying the following.*

(i) *If* $a^{\frac{N-4}{2}} b > C_1(N)$, *then* $\lambda_0^*(a, b) > 0$ *and:*

  (1) *For each* $\lambda > \lambda_0^*(a, b)$, *problem* (12) *admits a positive solution, which is a global minimizer to* Φ_λ *with negative energy.*
  (2) *If* $\lambda = \lambda_0^*(a, b)$, *then problem* (12) *admits a positive solution, which is a global minimizer to* $\Phi_{\lambda_0^*(a,b)}$ *with zero energy.*
  (3) *For* $\lambda \in (0, \lambda_0^*(a, b))$, *then only global minimizer to* Φ_λ *is* $u = 0$.

(ii) *If* $a^{\frac{N-4}{2}} b = C_1(N)$, *then* $\lambda_0^*(a, b) = 0$ *and for each* $\lambda > 0$, *problem* (12) *admits a positive solution, which is a global minimizer to* Φ_λ *with negative energy.*

(iii) *Moreover*

$$\lambda_0^*(a_k, b_k) \to 0, \text{ if } a_k \to a > 0, b_k \to b > 0, \ a_k^{\frac{N-4}{2}} b_k \downarrow C_1(N).$$

(iv) *If* $a^{\frac{N-4}{2}} b > C_1(N)$, *then there exists* $\varepsilon := \varepsilon(a, b) > 0$ *such that: for each* $\lambda \in (\lambda_0^*(a, b) - \varepsilon, \lambda_0^*(a, b))$, *problem* (12) *admits a positive solution, which is a local minimizer to* Φ_λ *with positive energy.*

Recall that $C_1(N) < C_2(N)$.

Theorem 4.2 *There exists a function* $\lambda^* : (0,\infty)^2 \to [0,\infty)$ *satisfying the following.*

(i) *If* $a^{\frac{N-4}{2}} b > C_2(N)$, *then* $0 < \lambda^*(a,b) < \lambda_0^*(a,b)$.
(ii) *If* $a^{\frac{N-4}{2}} b = C_2(N)$, *then* $0 = \lambda^*(a,b) < \lambda_0^*(a,b)$.
(iii) *If* $a^{\frac{N-4}{2}} b > C_2(N)$, *then there exists* $\varepsilon := \varepsilon(a,b) > 0$ *such that for each* $\lambda > \lambda_0^*(a,b) - \varepsilon$, *problem* (12) *admits a positive mountain pass type solution with positive energy.*
(iv) *If* $a^{\frac{N-4}{2}} b = C_2(N)$, *then there exists* $\tilde{\lambda} > 0$ *such that for each* $\lambda > \tilde{\lambda}$, *problem* (12) *admits a positive mountain pass type solution with positive energy.*
(v) *If* $a^{\frac{N-4}{2}} b > C_2(N)$ *and* $\lambda \in (0, \lambda^*(a,b))$, *then problem* (12) *has no non-zero solutions.*

We note that items i) and ii) of Theorem 4.2 follow from Proposition 3.8. Combining Theorem 4.1 with [13, Proposition 4.2] we conclude that the curve $a^{\frac{N-4}{2}} b = C_1(N)$ is a threshold in the sense stated in Theorem 1.5:

Proof ofTheorem 1.5 By inspection, one can easily see that the constant α_2 defined in [13] corresponds to our $C_2(N)$ with obvious modifications with respect to $a > 0$. Since $C_1(N) < C_2(N)$ and for each a, b satisfying $0 < a^{\frac{N-4}{2}} b \le C_1(N)$, there exists $u \in H_0^1(\Omega)\backslash\{0\}$ such that $\Phi_\lambda(u) < 0$ for all $\lambda > 0$, it follows that [13, Proposition 4.2] can be applied and then Φ_λ has a global minimizer with negative energy for all $\lambda > 0$. The rest of the proof is a consequence of Theorem 4.1. □

In order to get more results concerning our problem $(\mathcal{P}_\lambda)$, let us introduce and study the Nehari sets associated to Φ_λ: for each $a, b, \lambda \in \mathbb{R}$ let

$$\mathcal{N} := \mathcal{N}_{a,b,\lambda} = \{u \in H_0^1(\Omega)\backslash\{0\} : \Phi_\lambda'(u)u = 0\} = \{u \in H_0^1(\Omega)\backslash\{0\} : \psi_{\lambda,u}'(1) = 0\}.$$

We split the above set in three disjoint sets

$$\mathcal{N}^0 := \mathcal{N}^0_{a,b,\lambda} = \{u \in H_0^1(\Omega)\backslash\{0\} : \psi_{\lambda,u}'(1) = 0, \psi_{\lambda,u}''(1) = 0\},$$
$$\mathcal{N}^+ := \mathcal{N}^+_{a,b,\lambda} = \{u \in H_0^1(\Omega)\backslash\{0\} : \psi_{\lambda,u}'(1) = 0, \psi_{\lambda,u}''(1) > 0\},$$
$$\mathcal{N}^- := \mathcal{N}^-_{a,b,\lambda} = \{u \in H_0^1(\Omega)\backslash\{0\} : \psi_{\lambda,u}'(1) = 0, \psi_{\lambda,u}''(1) < 0\}.$$

By using the implicit function theorem and the Lagrange's multiplier rule we have that:

Proposition 4.1 *Suppose that* $a, b > 0$ *and* $\lambda \ge 0$. *Then, whenever* $\mathcal{N}^-, \mathcal{N}^+$ *are not empty, they are* C^1 *manifolds of co-dimension* 1 *in* $H_0^1(\Omega)$. *Moreover, every critical point of* Φ_λ *restricted to* $\mathcal{N}^- \cup \mathcal{N}^+$ *is a critical point to* Φ_λ. *Moreover, if* $u \in \mathcal{N}^+$ *is a local minimizer of* $\Phi_{\lambda|_{\mathcal{N}^+}}$, *then it is a local minimizer of* Φ_λ *over* $H_0^1(\Omega)$.

To understand the Nehari sets we prove:

Proposition 4.2 *For each* $a, b > 0$ *and* $\lambda \ge 0$ *and* $u \in H_0^1(\Omega)$, *only one of the next* $i) - iii)$ *occurs.*

(i) *The function* $\psi_{\lambda,u}$ *is increasing and has no critical points.*
(ii) *The function* $\psi_{\lambda,u}$ *has only one critical point in* $(0,+\infty)$ *at the value* $t_\lambda(u)$. *Moreover,* $\psi_{\lambda,u}''(t_\lambda(u)) = 0$ *and* $\psi_{\lambda,u}$ *is increasing.*
(iii) *The function* $\psi_{\lambda,u}$ *has only two critical points,* $0 < t_\lambda^-(u) < t_\lambda^+(u)$. *Moreover,* $t_\lambda^-(u)$ *is a local maximum and* $t_\lambda^+(u)$ *is a local minimum with* $\psi_{\lambda,u}''(t_\lambda^-(u)) < 0 < \psi_{\lambda,u}''(t_\lambda^+(u))$.

Proof We have $\psi'_{\lambda,u}(t) = 0$ if and only if

$$a\|u\|^2 = -b\|u\|^4 t^2 + \|u\|_{2^*}^{2^*} t^{2^*-2} + \frac{\lambda}{p}\|u\|_p^p t^{p-2}.$$

Let $\varphi(t) = -b\|u\|^4 t^2 + \|u\|_{2^*}^{2^*} t^{2^*-2} + \frac{\lambda}{p}\|u\|_p^p t^{p-2}$ for each $t > 0$. Then, it is easy to see that there exists a unique maximum point t^* of φ such that $\varphi(t^*) > 0$. Thus, the following cases occur. If $a\|u\|^2 > \varphi(t^*)$, then, $\psi'_{\lambda,u}(t) > 0$ for every $t > 0$ and i) holds. If $a\|u\|^2 = \varphi(t^*)$, then, $\psi'_{\lambda,u}(t) > 0$ for every $t \neq t^*$ and $\psi''_{\lambda,u}(t^*) = a\|u\|^2 - \varphi(t^*) - t^*\varphi'(t^*) = 0$, so that ii) is verified. Finally, if $a\|u\|^2 < \varphi(t^*)$, then, there exist $t_1 < t^* < t_2$ such that $a\|u\|^2 = \varphi(t_1) = \varphi(t_2)$ and $a\|u\|^2 > \varphi(t)$ for $t < t_1$ and $t > t_2$, $a\|u\|^2 < \varphi(t)$ for $t_1 < t < t_2$ so that (iii) is satisfied with $t_\lambda^-(u) = t_1$ and $t_\lambda^+(u) = t_2$. □

4.1 A refined non-existence result

Recall from Theorem 3.7 that if $a^{\frac{N-4}{2}} b > C_2(N)$ and $\lambda \in (0, \lambda^*)$, then $(\mathcal{P}_\lambda)$ has no non-zero solution. This is clear, since for that range of parameters, the Nehari set is empty. We show how to improve the non-existence result. First we need some preliminaries results:

Corollary 4.1 *Assume that $a^{\frac{N-4}{2}} b > C_2(N)$, then for each $u \in H_0^1(\Omega)\backslash\{0\}$ satisfying $\lambda^* = \lambda(u)$ we have that*

$$-(2a + 4b\|u\|^2)\Delta u - 2^*|u|^{2^*-2}u - \lambda^* p|u|^{p-2}u = 0.$$

Proof Define $J_{\lambda^*} : H_0^1(\Omega) \to \mathbb{R}$ by $J_{\lambda^*}(w) = \Phi'_{\lambda^*}(w)w$. From Lemma 2.2 item 3), J_{λ^*} attains its infimum. Moreover, by the definition of λ^*,

$$\inf\{J_{\lambda^*}(w) : w \in H_0^1(\Omega)\} = J_{\lambda^*}(u).$$

(see also Proposition 3.1). We conclude that $J'_{\lambda^*}(u) = 0$, which is the desired equation. □

Theorem 4.3 *If $a^{\frac{N-4}{2}} b > C_2(N)$ and Ω is star-shaped, then there exists $\varepsilon > 0$ such that $(\mathcal{P}_\lambda)$ has no non-zero solution for each $\lambda \in (0, \lambda^* + \varepsilon)$.*

Proof The case $\lambda \in (0, \lambda^*)$ is given by Theorem 3.7. Suppose on the contrary that $(\mathcal{P}_{\lambda^*})$ has a non-zero solution u. From Proposition 3.7 and the definition of λ^*, we have that $u \in \mathcal{N}_{\lambda^*}^0 = \mathcal{N}_{\lambda^*}$ (note from Proposition 3.10 that $\mathcal{N}_{\lambda^*}^0 \neq \emptyset$) and hence $\lambda^* = \lambda(u)$. From Corollary 4.1 we deduce that

$$\begin{cases} -(a + b\|u\|^2)\Delta u - |u|^{2^*-2}u - \lambda^*|u|^{p-2}u = 0, \\ -(2a + 4b\|u\|^2)\Delta u - 2^*|u|^{2^*-2}u - \lambda^* p|u|^{p-2}u = 0, \end{cases}$$

which implies that

$$-[(2-p)a + (4-p)b\|u\|^2]\Delta u = (2^* - p)|u|^{2^*-2}u,$$

which leads, from Pohozaev identity, to $u = 0$, a contradiction. Now suppose that there exists a sequence $\lambda_k \downarrow \lambda^*$ and a corresponding sequence of non-zero solutions u_k of $(\mathcal{P}_{\lambda_k})$. Then

$$a + b\|u_k\|^2 - \|v_k\|_{2^*}^{2^*}\|u_k\|^{2^*-2} - \lambda_k\|v_k\|_p^p\|u_k\|^{p-2} = 0,$$

where $v_k = u_k/\|u_k\|$. Therefore $(u_k)_k$ is bounded and does not converge to 0. From Lemma 2.2 item 2), we conclude that $u_k \to u \in H_0^1(\Omega)\setminus\{0\}$ and

$$-(a+b\|u\|^2)\Delta u - |u|^{2^*-2}u - \lambda^*|u|^{p-2}u = 0,$$

that is u is a non zero solution of $(\mathcal{P}_{\lambda^*})$, a contradiction. □

4.2 Existence of the second solution when $a^{\frac{N-4}{2}} b < C_2(N)$

For each $a, b, \lambda > 0$, define (whenever $\mathcal{N}^0, \mathcal{N}^-$ are not empty)

$$c^0 := c^0(a,b,\lambda) = \inf\{\Phi_\lambda(u) : u \in \mathcal{N}^0\},$$
$$c^- := c^-(a,b,\lambda) = \inf\{\Phi_\lambda(u) : u \in \mathcal{N}^-\}.$$

and

$$\sigma := \inf\{\liminf_{n\to\infty} \Phi_\lambda(u_k) : u_k \in \mathcal{M}\},$$

where

$$\mathcal{M} = \{u_k \in \mathcal{N} : \lim_{n\to\infty} \psi''_{u_k}(1) = 0\}.$$

With a simple modification of [13, Lemma 3.4] we can prove:

Lemma 4.1 *There holds*

$$\frac{(p-2)^2a^2}{4p(4-p)b} \le \sigma \le c^0.$$

Now we prove a result which complements [13, Theorem 1.1].

Theorem 4.4 *Assume $a^{\frac{N-4}{2}} b < C_2(N)$. Then, there exists $p_0(a,b) \in (2, 2^*)$ such that if $p \in (p_0(a,b), 2^*)$, for all $\lambda > 0$, there exists $v_\lambda \in \mathcal{N}^-$ for which $c^-(a,b,\lambda) = \Phi_\lambda(v_\lambda)$.*

Proof From Proposition B.2 in the Appendix we know that

$$c^-(a,b,0) < \frac{(2^*-2)^2a^2}{4\cdot 2^*(4-2^*)b}. \tag{13}$$

Note that the function $[2, 2^*) \ni p \mapsto \frac{(p-2)^2a^2}{4p(4-p)b}$ is increasing and is zero for $p = 2$, therefore from (13), there exists a unique $p_0 := p_0(a,b) \in (2, 2^*)$ such that

$$c^-(a,b,0) = \frac{(p_0-2)^2a^2}{4p_0(4-p_0)b}.$$

As a consequence

$$c^-(a,b,0) < \frac{(p-2)^2a^2}{4p(4-p)b},$$

for all $p \in (p_0(a,b), 2^*)$. From Proposition A.1 and Corollary B.1 in the Appendix and Lemma 4.1 we deduce that

$$c^-(a,b,\lambda) \le c^-(a,b,0) < \frac{(p-2)^2a^2}{4p(4-p)b} \le \sigma, \ \forall\lambda > 0$$

and from [13, Corollary 3.3 and Proposition 4.1], the proof is complete. □

Remark 4.1 Note that:

(i) Our method to prove Theorem 4.4 also proves [13, Theorem 1.1]. Indeed, fix $p \in (2, 2^*)$. By one hand we know from Proposition A.1 in the Appendix that $c^-(a, b, 0)$ is non-decreasing in b. On the other hand

$$\lim_{b \downarrow 0} \frac{(p-2)^2 a^2}{4p(4-p)b} = \infty,$$

therefore by choosing b sufficiently small we conclude that

$$c^-(a, b, \lambda) \leq c^-(a, b, 0) < \frac{(p-2)^2 a^2}{4p(4-p)b}, \ \forall \lambda > 0.$$

which is [13, Corollary 3.3] and consequently implies [13, Theorem 1.1].

(ii) Observe that the method employed in [13, Corollary 3.3], which was used to prove [13, Theorem 1.1], does not work for all values of p and $a, b > 0$ with $a^{\frac{N-4}{2}} b < C_2(N)$. Indeed, fix $a, b > 0$ with $a^{\frac{N-4}{2}} b < C_2(N)$. Choose $p \in (2, 2^*)$ such that

$$\frac{(p-2)^2 a^2}{4p(4-p)b} < c^-(a, b, 0).$$

Therefore from Proposition A.2 in Appendix we deduce that for small λ,

$$\frac{(p-2)^2 a^2}{4p(4-p)b} < c^-(a, b, \lambda),$$

which contradicts the inequality in [13, Proposition 3.1] that was used to prove [13, Corollary 3.3].

Proof of Theorem 1.6 From Theorem 4.4, there exists $v_\lambda \in \mathcal{N}^-$ such that $\Phi_\lambda(v_\lambda) = c^-(a, b, \lambda)$. From Proposition 4.1 the proof is complete. □

However, without any restriction on p or a, b, we can prove the following:

Theorem 4.5 *For each $a, b > 0$ there exists $\tilde{\lambda} := \tilde{\lambda}(a, b, p) > 0$ such that for all $\lambda > \tilde{\lambda}$, there exists $v_\lambda \in \mathcal{N}^-$ satisfying $c^-(a, b, \lambda) = \Phi_\lambda(v_\lambda)$.*

Proof We claim that $c^-(a, b, \lambda) \to 0$ as $\lambda \to \infty$. To prove it, fix $u \in H_0^1(\Omega) \setminus \{0\}$. Given $\varepsilon > 0$, fix any $\lambda' > 0$. Then there exists $\delta > 0$ such that $0 < \psi_{\lambda',u}(t) \leq \varepsilon$ for all $t \in (0, \delta]$. Since the function $(\lambda', \infty) \ni \lambda \mapsto \psi_{\lambda,u}(\delta)$ is continuous, decreasing and tends to $-\infty$ as $\lambda \to \infty$, it follows that there exists a unique parameter $\mu > \lambda'$ such that $\psi_{\mu,u}(\delta) = 0$. Therefore $0 < t_\mu^- < \delta$ and $\psi_{\mu,u}(t_\mu^-) \leq \psi_{\lambda',u}(t_\mu^-) \leq \varepsilon$. By the arbitrariness of ε, the claim is proved.

Now choose $\tilde{\lambda}$ such that

$$c^-(a, b, \tilde{\lambda}) < \frac{(p-2)^2 a}{4p(4-p)b},$$

then from Proposition A.1 in the Appendix we deduce that

$$c^-(a, b, \lambda) < \frac{(p-2)^2 a}{4p(4-p)b}, \ \forall \lambda > \tilde{\lambda}. \tag{14}$$

Now we divide the proof in two cases: if $a^{\frac{N-4}{2}}b < C_2(N)$, then we can apply [13, Corollary 3.3 and Proposition 4.1] and the proof is complete. Now assume that $a^{\frac{N-4}{2}}b \geq C_2(N)$. Let $(u_k)_k \in \mathcal{N}^-$ be a minimizing sequence to $c^-(a,b,\lambda)$. Since

$$a\|u_k\|^2 + b\|u_k\|^4 - \|u_k\|_{2^*}^{2^*} - \lambda\|u_k\|_p^p = 0, \forall k, \tag{15}$$

we deduce that there exist positive constants d_1, d_2 such that $d_1 \leq \|u_k\| \leq d_2$ for all $k \in \mathbb{N}$. Without loss of generality we can assume that $u_k \rightharpoonup u$ in $H_0^1(\Omega)$ and $\|u_k\| \to t > 0$. We claim that $u \neq 0$. Indeed, from (15) and the Sobolev embedding we also have that

$$h(\|u_k\|) = a + b\|u_k\|^2 - S_N^{-\frac{2^*}{2}}\|u_k\|^{2^*-2} \leq C\lambda\|u_k\|^{p-2},$$

where C is some positive constant. Then, if $u = 0$, we would reach the contradiction $0 < h(t) \leq 0$ (see Proposition 2.1 and Lemma 2.1). From Lemma 2.2 we have that

$$\begin{aligned}\psi'_{\lambda,u}(1) &= a\|u\|^2 + b\|u\|^4 - \|u\|_{2^*}^{2^*} - \lambda\|u\|_p^p \\ &\leq \liminf_{k\to\infty}(a\|u_k\|^2 + b\|u_k\|^4 - \|u_k\|_{2^*}^{2^*} - \lambda\|u_k\|_p^p) = 0,\end{aligned}$$

which implies that the fiber map $\psi_{\lambda,u}$ satisfies (ii) or (iii) of Proposition 4.2. We claim that it satisfies (iii). Indeed, if it satisfies (ii), then $u \in \mathcal{N}^0$ and from Lemma 2.2 and (14) we obtain that

$$\Phi_\lambda(u) \leq \liminf_{k\to\infty}\Phi_\lambda(u_k) = c^-(a,b,\lambda) < \frac{(p-2)^2a}{4p(4-p)b},$$

which contradicts Lemma 4.1. Therefore $\psi_{\lambda,u}$ satisfies iii) and there exists $t_\lambda^-(u) \leq 1$ such that $t_\lambda^-(u)u \in \mathcal{N}^-$. From Lemma 2.2

$$\Phi_\lambda(t_\lambda^-(u)u) \leq \liminf_{k\to\infty}\Phi_\lambda(t_\lambda^-(u)u_k) \leq \liminf_{k\to\infty}\Phi_\lambda(u_k) = c^-(a,b,\lambda),$$

and the proof is complete. □

Remark 4.2 Note that

(i) Theorem 4.5 complements the results of [13], globally in a, b and locally in λ.
(ii) Recall from Theorems 3.4 and 3.6 that if $a^{\frac{N-4}{2}}b > C_2(N)$ and $\lambda > \lambda_0^* - \varepsilon$, then Φ_λ has a mountain pass type solution. One may ask if the solutions found in Theorem 4.5 and in those theorems are the same? Or at least, is it true that $c^-(a,b,\lambda) = c_\lambda$?

Proof of Theorem 1.7 From Theorem 4.5, there exists $v_\lambda \in \mathcal{N}^-$ such that $\Phi_\lambda(v_\lambda) = c^-(a,b,\lambda)$. From Proposition 4.1 the proof is complete. □

4.3 Brezis–Nirenberg problem: the limit case $b \to 0$

In this Section we show how to recover a well known result from Brezis and Nirenberg [2] as a byproduct of our study. To emphasize the more important role of the parameter b, we use the notation $\psi_{b,\lambda,u} = \psi_{\lambda,u}$, $t_{b,\lambda}^-(u) = t_\lambda^-(u)$, $\Phi_{b,\lambda} = \Phi_\lambda$ and so on.

Lemma 4.2 *Fix $a > 0$, then*

$$c^-(a,0,0) = \frac{a^{\frac{N}{2}}}{N}S_N^{\frac{N}{2}}.$$

Proof Indeed, first observe that

$$\Phi_{0,0}(u) = \frac{1}{N}a\|u\|^2, \ \forall u \in \mathcal{N}_{0,0}^{-},$$

which implies from the definition of S_N that

$$\Phi_{0,0}(u) \geq \frac{a^{\frac{N}{2}}}{N} S_N^{\frac{N}{2}}, \ \forall u \in \mathcal{N}_{0,0}^{-}.$$

Now suppose that $(u_k)_k$ is a minimizing sequence to S_N satisfying $\|u_k\|_{2^*} = 1$ for all $k \in \mathbb{N}$. From Lemma A.1 and Remark A.1 in Appendix, for each k, there exists $t_k := t_{0,0}(u_k)$ such that $t_k u_k \in \mathcal{N}_{0,0}^{-}$. From

$$a t_k^2 \|u_k\|^2 - t_k^{2^*} \|u_k\|_{2^*}^{2^*} = 0,$$

we have that

$$t_k \to (a S_N)^{\frac{1}{2^*-2}}, \ k \to \infty.$$

Therefore

$$\Phi_{0,0}(t_k u_k) = \frac{1}{N} a t_k^2 \|u_k\|^2 \to \frac{1}{N} a \, (a S_N)^{\frac{2}{2^*-2}} \, S_N = \frac{a^{\frac{N}{2}}}{N} S_N^{\frac{N}{2}},$$

and the proof is complete. □

Proposition 4.3 *Fix $a > 0$, then for each $\lambda > 0$ we have that*

$$c^{-}(a, 0, \lambda) < c^{-}(a, 0, 0) = \frac{a^{\frac{N}{2}}}{N} S_N^{\frac{N}{2}}.$$

Proof For each $\varepsilon > 0$, choose $u_\varepsilon \in H_0^1(\Omega)$ such that (see [2])

$$\int_\Omega |\nabla u_\varepsilon|^2 = 1, \ \int_\Omega |u_\varepsilon|^{2^*} = S_N^{\frac{-2^*}{2}} + O(\varepsilon^{\frac{2^* N}{4}}), \ \int_\Omega |u_\varepsilon|^p = \frac{\varepsilon^{\frac{2p-N(p-2)}{4}}}{(c + O(1)\varepsilon^{\frac{N-2}{2}})^{\frac{p}{2}}},$$

where c is a positive constant independent on ε. From Lemma A.1 and Remark A.1 in Appendix, for each $\varepsilon > 0$, there exists $t_{\varepsilon,\lambda} := t_{0,\lambda}^{-}(u_\varepsilon)$ such that $t_{\varepsilon,\lambda} u_\varepsilon \in \mathcal{N}^{-}$. Denote $f_\varepsilon(\lambda) = \psi_{0,\lambda}(t_{\varepsilon,\lambda} u_\varepsilon) = \Phi_{0,\lambda}(t_{\varepsilon,\lambda} u_\varepsilon)$. From Lemma A.2 (and its proof) we know that

$$f_\varepsilon(\lambda) - f_\varepsilon(0) = f_\varepsilon'(\theta)\lambda = -\frac{t_{\varepsilon,\theta}^p}{p} \lambda \|u_\varepsilon\|_p^p,$$

and hence

$$f_\varepsilon(\lambda) = f_\varepsilon(0) - \frac{t_{\varepsilon,\theta}^p}{p} \lambda \|u_\varepsilon\|_p^p, \ \forall \varepsilon, \tag{16}$$

where $\theta := \theta_\varepsilon \in (0, \lambda)$. Now some calculations are in order: note from

$$a t_{\varepsilon,\theta}^2 = t_{\varepsilon,\theta}^{2^*} \|u_\varepsilon\|_{2^*}^{2^*} + \lambda t_{\varepsilon,\theta}^p \|u_\varepsilon\|_p^p, \ \forall \varepsilon,$$

that there exists a positive constant c_1 such that

$$t_{\varepsilon,\theta} \geq c_1, \ \forall \varepsilon. \tag{17}$$

Moreover, since

$$a t_{\varepsilon,0}^2 - t_{\varepsilon,0}^{2^*}\|u_\varepsilon\|_{2^*}^{2^*} = 0, \ \forall \varepsilon,$$

we conclude that

$$t_{\varepsilon,0} = \left(\frac{a}{S_N^{\frac{-2^*}{2}} + O(\varepsilon^{\frac{2^*N}{4}})}\right)^{\frac{1}{2^*-2}} = \left(\frac{a}{S_N^{\frac{-2^*}{2}}}\right)^{\frac{1}{2^*-2}} + O(\varepsilon^{\frac{2^*N}{4(2^*-2)}}), \ \forall \varepsilon$$

and hence

$$\begin{aligned} f_\varepsilon(0) &= \frac{a}{2} t_{\varepsilon,0}^2 - \frac{t_{\varepsilon,0}^{2^*}}{2^*}\|u_\varepsilon\|_{2^*}^{2^*} \\ &= \frac{a}{2}\left[\left(\frac{a}{S_N^{\frac{-2^*}{2}}}\right)^{\frac{2}{2^*-2}} + O(\varepsilon^{\frac{2^*N}{2(2^*-2)}})\right] \\ &\quad -\frac{1}{2^*}\left[\left(\frac{a}{S_N^{\frac{-2^*}{2}}}\right)^{\frac{2^*}{2^*-2}} + O(\varepsilon^{\frac{2^*2^*N}{4(2^*-2)}})\right]\left(S_N^{\frac{-2^*}{2}} + O(\varepsilon^{\frac{2^*N}{4}})\right), \\ &= \frac{a^{\frac{N}{2}}}{N} S_N^{\frac{N}{2}} + O(\varepsilon^{\frac{2^*N}{4}}). \end{aligned} \tag{18}$$

We combine (16) and (18) to obtain that

$$\begin{aligned} f_\varepsilon(\lambda) &= \frac{a^{\frac{N}{2}}}{N} S_N^{\frac{N}{2}} + O(\varepsilon^{\frac{2^*N}{4}}) - \frac{t_{\varepsilon,\theta}^p}{p}\lambda \frac{\varepsilon^{\frac{2p-N(p-2)}{4}}}{(c+O(1)\varepsilon^{\frac{N-2}{2}})^{\frac{p}{2}}}, \\ &= \frac{a^{\frac{N}{2}}}{N} S_N^{\frac{N}{2}} + \varepsilon^{\frac{2p-N(p-2)}{4}}\left[\frac{O(\varepsilon^{\frac{2^*N}{4}})}{\varepsilon^{\frac{2p-N(p-2)}{4}}} - \frac{t_{\varepsilon,\theta}^p}{p}\lambda\frac{1}{(c+O(1)\varepsilon^{\frac{N-2}{2}})^{\frac{p}{2}}}\right]. \end{aligned}$$

Since

$$\frac{2^*N}{4} > 1 > \frac{2p-N(p-2)}{4},$$

we conclude from (17) that for sufficiently small ε, we must have that $f_\varepsilon(\lambda) < \frac{a^{\frac{N}{2}}}{N} S_N^{\frac{N}{2}}$ which concludes the proof. □

Remark 4.3 Fix $a > 0$ and $\lambda \geq 0$:

(i) By using a continuity argument, one can easily see that the Nehari manifold $\mathcal{N}_{b,\lambda}^-$ is not empty for b on a neighborhood of 0.

(ii) However, it is possible to adapt the calculations made in Theorem 3.7, to prove the existence of $b^* > 0$ such that if $b \in [0, b^*)$, then $\mathcal{N}_{b,\lambda}^- \neq \emptyset$, while if $b > b^*$, then $\mathcal{N}_{b,\lambda} = \emptyset$ (see "Appendix B").

As a corollary of Theorem 4.5 we obtain the following result à la Brezis Nirenberg [2]:

Theorem 4.6 *Let $a = 1$ and $b_k \downarrow 0$. Then, there exists a sequence $(v_k)_k$ of solutions of $(\mathcal{P}_\lambda)$ such that $v_k \to v$ where v is a nontrivial solution of*

$$(\mathcal{Q}_\lambda) \quad \begin{cases} -\Delta u = |u|^{2^*-2}u + \lambda |u|^{p-2}u, & in\ \Omega \\ u = 0, & on\ \partial\Omega. \end{cases}$$

Proof Fix $\lambda > 0$. From Remark 4.3 we can assume that $c^-(1, b_k, \lambda)$ is well defined for all k. Let also $\varepsilon > 0$ such that $c^-(1, 0, \lambda) + \varepsilon < \frac{1}{N} S_N^{\frac{N}{2}}$ (see Proposition 4.3). Thus, by Proposition A.2 in Appendix, for k big enough, one has

$$c^-(1, b_k, \lambda) < c^-(1, 0, \lambda) + \varepsilon < \frac{1}{N} S_N^{\frac{N}{2}}.$$

We claim that $(v_k)_k$ is bounded in $H_0^1(\Omega)$. Indeed, we know that

$$0 = \Phi'_{b_k,\lambda}(v_k)(v_k) = \|v_k\|^2 + b_k\|v_k\|^4 - \|v_k\|_{2^*}^{2^*} - \lambda\|v_k\|_p^p \tag{19}$$

$$c^-(1, b_k, \lambda) = \Phi_{b_k,\lambda}(v_k) = \frac{1}{2}\|v_k\|^2 + \frac{b_k}{4}\|v_k\|^4 - \frac{1}{2^*}\|v_k\|_{2^*}^{2^*} - \frac{\lambda}{p}\|v_k\|_p^p. \tag{20}$$

Denote $t_k = t_{0,\lambda}^-(v_k)$ and note from Lemma A.2 in the Appendix that $0 < t_k \le 1$ for all k. This property combined with Proposition A.2 implies that

$$\begin{aligned} 0 &\le \lim_{k\to\infty} \Phi_{0,\lambda}(t_k v_k) - c^-(1, 0, \lambda), \\ &\le \lim_{k\to\infty} \Phi_{b_k,\lambda}(t_k v_k) - c^-(1, 0, \lambda), \\ &\le \lim_{k\to\infty} \Phi_{b_k,\lambda}(v_k) - c^-(1, 0, \lambda), \\ &= \lim_{k\to\infty} \left(\Phi_{b_k,\lambda}(v_k) - c^-(1, b_k, \lambda)\right) = 0, \end{aligned}$$

and hence $(t_k v_k)_k$ is a minimizing sequence to $c^-(1, 0, \lambda)$. We claim that $(t_k)_k$ is bounded away from 0. Suppose on the contrary that $t_k \to 0$ as $k \to \infty$. Since $t_k v_k \in \mathcal{N}_{0,\lambda}^-$ we know that

$$2t_k^2\|v_k\|^2 - 2^* t_k^{2^*}\|v_k\|_{2^*}^{2^*} - p\lambda t_k^p\|v_k\|_p^p < 0, \quad \forall k.$$

Thus

$$2\frac{\|v_k\|^2}{\|v_k\|_{2^*}^{2^*}} - 2^* t_k^{2^*-2} < p\lambda t_k^{p-2}\frac{\|v_k\|_p^p}{\|v_k\|_{2^*}^{2^*}}, \quad \forall k,$$

and hence

$$\frac{\|v_k\|^2}{\|v_k\|_{2^*}^{2^*}} = o(1). \tag{21}$$

From

$$\|v_k\|^2 + b_k\|v_k\|^4 - \|v_k\|_{2^*}^{2^*} - \lambda\|v_k\|_p^p = 0, \quad \forall k,$$

and (21) we deduce that

$$\frac{b_k\|v_k\|^4}{\|v_k\|_{2^*}^{2^*}} = 1 + \lambda\frac{\|v_k\|_p^p}{\|v_k\|_{2^*}^{2^*}} + o(1), \quad \forall k. \tag{22}$$

Since

$$\Phi_{b_k,\lambda}(v_k) = \|v_k\|_{2^*}^{2^*}\left(\frac{1}{2}\frac{\|v_k\|^2}{\|v_k\|_{2^*}^{2^*}} + \frac{1}{4}\frac{b_k\|v_k\|^4}{\|v_k\|_{2^*}^{2^*}} - \frac{1}{2^*} - \frac{\lambda}{p}\frac{\|v_k\|_p^p}{\|v_k\|_{2^*}^{2^*}}\right), \quad \forall k,$$

it follows from (22) that

$$\Phi_{b_k,\lambda}(v_k) = \|v_k\|_{2^*}^{2^*}\left[\frac{1}{4}\left(1+\lambda\frac{\|v_k\|_p^p}{\|v_k\|_{2^*}^{2^*}}\right)-\frac{1}{2^*}-\frac{\lambda}{p}\frac{\|v_k\|_p^p}{\|v_k\|_{2^*}^{2^*}}+o(1)\right]$$
$$= \|v_k\|_{2^*}^{2^*}\left[\frac{2^*-4}{2^*4}+\left(\frac{p-4}{2^*4}\right)\lambda\frac{\|v_k\|_p^p}{\|v_k\|_{2^*}^{2^*}}+o(1)\right],$$

which is a contradiction since $\Phi_{b_k,\lambda}(v_k) = c^-(1, b_k, \lambda) > 0$ for all k and therefore t_k is bounded away from 0. Once $(t_k v_k)_k$ is a minimizing sequence to $c^-(1, 0, \lambda)$, it has to be bounded, that is, there exists $d > 0$ such that

$$t_k^2\int |\nabla v_k|^2 \le d, \quad \forall k,$$

and as a consequence $(v_k)_k$ is bounded in $H_0^1(\Omega)$.

Eventually passing to a subsequence, there exists $v \in H_0^1(\Omega)$ such that $v_k \rightharpoonup v$ weakly in $H_0^1(\Omega)$, $v_k \to v$ strongly in $L^q(\Omega)$ for $q < 2^*$, $|v_k|^{2^*-2}v_k \rightharpoonup |v|^{2^*-2}v$ weakly in $(L^{2^*})'$. Thus, since v_k is a critical point of $\Phi_{k,\lambda}$, for every $\varphi \in H_0^1(\Omega)$,

$$(1+b_k\|v_k\|^2)\int_\Omega \nabla v_k\nabla\varphi - \int_\Omega |v_k|^{2^*-2}v_k\varphi - \lambda\int_\Omega |v_k|^{p-2}v_k\varphi = 0,$$

passing to the limit as $k \to \infty$ we deduce that

$$\int_\Omega \nabla v\nabla\varphi - \int_\Omega |v|^{2^*-2}v\varphi - \lambda\int_\Omega |v|^{p-2}v\varphi = 0,$$

which implies that v is a solution of $(\mathcal{Q}_\lambda)$. Let us show that $v \neq 0$. Assume by contradiction that $v = 0$. By (19), dividing by $\|v_k\|^2$ we get

$$1+b_k\|v_k\|^2 - S_N^{-\frac{2}{2^*}}\|v_k\|^{2^*-2} \le 1+b_k\|v_k\|^2 - \|v_k\|_{2^*}^{2^*-2} = \lambda\|v_k\|_p^{p-2} \le c_1\lambda\|v_k\|^{p-2}$$

and $(\|v_k\|)_k$ is bounded away from zero. Passing to a subsequence we can assume that $\|v_k\| \to l > 0$. From (19) and (20) (recall that $0 = v = \lim_k v_k$ in L^p), we obtain that

$$l^2 = \lim_k \|v_k\|_{2^*}^{2^*}$$

and

$$\lim_k c^-(1, b_k, \lambda) = \frac{1}{2}l^2 - \frac{1}{2^*}\lim_k \|v_k\|_{2^*}^{2^*} = \frac{1}{N}l^2.$$

Since $\|v_k\|^2 \ge S_N\|v_k\|_{2^*}^2$ we obtain that $l^2 \ge S_N^{\frac{N}{2}}$ which implies

$$\lim_k c^-(1, b_k, \lambda) \ge \frac{1}{N}S_N^{\frac{N}{2}},$$

against the initial assumptions. Thus, $v \neq 0$. Let us prove now that $v_k \to v$ in $H_0^1(\Omega)$ and $\Phi_{0,\lambda}(v) = c^-(1, 0, \lambda)$. Indeed, since $v_k \in \mathcal{N}_{b_k,\lambda}^-$ for all k, we have that

$$\Phi_{b_k,\lambda}(v_k) = \frac{2^*-2}{22^*}\|v_k\|^2 + \frac{2^*-4}{42^*}b_k\|v_k\|^4 - \frac{2^*-p}{2^*p}\|v_k\|_p^p, \ \forall k.$$

Since v solves $(\mathcal{Q}_\lambda)$ we conclude from Remark A.1 in the Appendix that $v \in \mathcal{N}^-_{0,\lambda}$ and hence

$$c^-(1,0,\lambda) \le \Phi_{0,\lambda}(v) = \frac{2^*-2}{22^*}\|v\|^2 - \frac{2^*-p}{2^*p}\|v\|_p^p \le \liminf_{k\to\infty} \Phi_{b_k,\lambda}(v_k) = c^-(1,0,\lambda),$$

and therefore $\|v_k\| \to \|v\|$ as $k \to \infty$, which implies that $v_k \to v$ in $H_0^1(\Omega)$ and $\Phi_{0,\lambda}(v) = c^-(1,0,\lambda)$. □

Proof of Theorem 1.8 See Theorem 4.6. □

Acknowledgements The authors would like to thank the anonymous Referee for his/her remarkable suggestions which helped improving the quality of the manuscript. F. Faraci has been supported by the Università degli Studi di Catania, "Piano della Ricerca 2016/2018 Linea di intervento 2". She is member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). K. Silva has been supported by CNPq-Grant 408604/2018-2.

Appendix A: Some topological properties of the Nehari manifolds

We collect some topological properties concerning the Nehari manifold $\mathcal{N}^-$. Since the dependency on each parameter will be considered, we will write the full notation $\Phi_{a,b,\lambda}$, $t^-_{a,b,\lambda}(u)$, $\mathcal{N}^-_{a,b,\lambda}$ and so on.

Similarly to Proposition 4.2 we can prove:

Lemma A.1 *For each $a > 0$, $b \in \mathbb{R}$, $\lambda \in \mathbb{R}$ and $u \in H_0^1(\Omega)\setminus\{0\}$, only one of the next $i) - iv)$ occurs.*

(i) *The function $\psi_{a,b,\lambda,u}$ is increasing and has no critical points.*
(ii) *The function $\psi_{a,b,\lambda,u}$ has only one critical point in $]0,+\infty[$ at the value $t_{a,b,\lambda}(u)$. Moreover, $\psi''_{a,b,\lambda,u}(t_{a,b,\lambda}(u)) = 0$ and $\psi_{a,b,\lambda,u}$ is increasing.*
(iii) *The function $\psi_{a,b,\lambda,u}$ has only two critical points, $0 < t^-_{a,b,\lambda}(u) < t^+_{a,b,\lambda}(u)$. Moreover, $t^-_{a,b,\lambda}(u)$ is a local maximum and $t^+_{a,b,\lambda}(u)$ is a local minimum with $\psi''_{a,b,\lambda,u}(t^-_{a,b,\lambda}(u)) < 0 < \psi''_{a,b,\lambda,u}(t^+_{a,b,\lambda}(u))$.*
(iv) *The function $\psi_{a,b,\lambda,u}$ has only one critical point in $]0,+\infty[$ at the value $t^-_{a,b,\lambda}(u)$. Moreover, $t^-_{a,b,\lambda}(u)$ is a local maximum and $\psi''_{a,b,\lambda,u}(t^-_{a,b,\lambda}(u)) < 0$.*

Remark A.1 If $b \le 0$ and $\lambda \ge 0$, then only item iv) of Lemma A.1 occurs. Moreover, if $b > 0$, then only one of the items $i) - iii)$ occurs.

Lemma A.2 *Fix $u \in H_0^1(\Omega)\setminus\{0\}$ and $a > 0$. Let $V \subset \mathbb{R}^2$ be an open set and assume that $t^-_{a,b,\lambda}(u)$ is defined for all $(b,\lambda) \in V$. Then the function $V \ni (b,\lambda) \mapsto t^-_{a,b,\lambda}(u)$ is C^1. Moreover the following holds true.*

(i) *The functions $t^-_{a,b,\lambda}(u)$ and $\psi_{a,b,\lambda,u}(t^-_{a,b,\lambda}(u))$ are increasing with respect to b;*
(ii) *The functions $t^-_{a,b,\lambda}(u)$ and $\psi_{a,b,\lambda,u}(t^-_{a,b,\lambda}(u))$ are decreasing with respect to λ.*

Proof Denote $t_{b,\lambda} = t^-_{a,b,\lambda}(u)$ and note from the implicit function theorem that $\psi'_{a,b,\lambda,u}(t_{b,\lambda}) = 0$ and $\psi''_{a,b,\lambda,u}(t_{b,\lambda}) < 0$ implies that $t_{b,\lambda}$ is C^1 as a function of $(b,\mu,\lambda) \in V$. Since

$$at^2_{b,\lambda}\|u\|^2 + bt^4_{b,\lambda}\|u\|^4 - t^{2^*}_{b,\lambda}\mu\|u\|^{2^*}_{2^*} - \lambda t^p_{b,\lambda}\|u\|^p_p = 0,$$

we conclude by differentiating both sides, with respect to b, that

$$\frac{\partial t_{b,\lambda}}{\partial b} = -\frac{t_{b,\lambda}^4 \|u\|^4}{\psi''_{a,b,\lambda,u}(t_{b,\lambda})} > 0,$$

and hence $t_{b,\lambda}$ is increasing in b. Now let $f(b) = \psi_{a,b,\lambda,u}(t^-_{a,b,\lambda}(u))$ and observe that

$$f'(b) = \frac{\partial t_{b,\lambda}}{\partial b}\psi'_{a,b,\lambda,u}(t_{b,\lambda}) + \frac{t_{b,\lambda}^4 \|u\|^4}{4} > 0,$$

which implies that f is increasing and hence (i) is proved. The proof of (ii) is similar. □

Remark A.2 Note that a similar result can also be proved with respect to the functions $t^+_{a,b,\lambda}(u)$ and $\psi_{a,b,\lambda,u}(t^+_{a,b,\lambda}(u))$.

Denote

$$\mathcal{M}_{a,b,\lambda} = \left\{\frac{u}{\|u\|} : u \in \mathcal{N}^-_{a,b,\lambda}\right\}.$$

Lemma A.3 *There holds:*

(i) *If $b_1 < b_2$, then $\mathcal{M}_{b_2} \subset \mathcal{M}_{b_1}$.*
(ii) *If $\lambda_1 < \lambda_2$, then $\mathcal{M}_{\lambda_1} \subset \mathcal{M}_{\lambda_2}$.*

Proof (i) Take $u \in \mathcal{M}_{a,b_2,\lambda}$. Once $\psi'_{a,b_1,\lambda}(t) \leq \psi'_{a,b_2,\lambda}(t)$ for all $t > 0$, it follows that $\psi'_{a,b_1,\lambda}(t^-_{a,b_2,\lambda}(u)) < \psi'_{a,b_2,\lambda}(t^-_{a,b_2,\lambda}(u)) = 0$ and hence, from Lemma A.1 we conclude that $u \in \mathcal{M}_{a,b_1,\lambda}$.

(ii) Take $u \in \mathcal{M}_{a,b,\lambda_1}$. Once $\psi'_{a,b,\lambda_2}(t) \leq \psi'_{a,b,\lambda_1}(t)$ for all $t > 0$, it follows that $\psi'_{a,b,\lambda_2}(t^-_{a,b,\lambda_1}(u)) < \psi'_{a,b,\lambda_1}(t^-_{a,b,\lambda_1}(u)) = 0$ and hence, from Proposition A.1 we conclude that $u \in \mathcal{M}_{a,b,\lambda_1}$. □

Proposition A.1 *Fix $a > 0$ and let I be an interval. Then, the following holds true.*

(i) *Fix $b \in \mathbb{R}$. If $c^-(a, b, \lambda)$ is defined for all $\lambda \in I$, then it is non-increasing as a function of λ.*
(ii) *Fix $\lambda \in \mathbb{R}$. If $c^-(a, b, \lambda)$ is defined for all $b \in I$, then it is non-decreasing as a function of b.*

Proof i) Indeed, fix $\lambda_1 < \lambda_2$ and $u \in \mathcal{M}_{a,b,\lambda_1}$. Since from Lemma A.3 we have that $u \in \mathcal{M}_{a,b,\lambda_2}$, it follows from Lemma A.2 that

$$c^-(a, b, \lambda_2) \leq \psi_{a,b,\lambda_2}(t^-_{a,b,\lambda_2}(u)) < \psi_{a,b,\lambda_1}(t^-_{a,b,\lambda_1}(u)), \forall u \in \mathcal{M}_{a,b,\lambda_1}. \tag{23}$$

and hence $c^-(a, b, \lambda_2) \leq c^-(a, b, \lambda_1)$. The proof of ii) is similar. □

Proposition A.2 *Fix $a > 0$ and let I be an interval. Then, the following holds true.*

(i) *Fix $\lambda \in \mathbb{R}$. If $c^-(a, b, \lambda)$ is defined for all $b \in I$, then it is right continuous as a function of b.*
(ii) *Fix $b \in \mathbb{R}$. If $c^-(a, b, \lambda)$ is defined for all $\lambda \in I$, then it is right continuous as a function of λ.*

Proof (i) Fix $b_0 \in I$. We claim that $\lim_{b\downarrow b_0} c^-(a,b,\lambda) = c^-(a,b_0,\lambda)$. Indeed, once $I \ni b \mapsto c^-(a,b,\lambda)$ is non-decreasing, we can assume that $\lim_{b\downarrow b_0} c^-(a,b,\lambda) = c \geq c^-(a,b_0,\lambda)$. Suppose on the contrary that $c > c^-(a,b_0,\lambda)$. Given $\varepsilon > 0$ choose $u \in \mathcal{M}_{a,b_0,\lambda}$ such that $\Phi_{a,b_0,\lambda}(t^-_{a,b_0,\lambda}(u)u) \in [c^-(a,b_0,\lambda), c^-(a,b_0,\lambda)+\varepsilon)$ and $c^-(a,b_0,\lambda)+\varepsilon < c$. From Lemma A.2 we conclude that for small $\delta > 0$

$$c^-(a,b_0+\delta,\lambda) \leq \Phi_{a,b_0+\delta,\lambda}(t^-_{a,b_0+\delta,\lambda}(u)u) < c^-(a,b_0,\lambda)+\varepsilon < c,$$

which is a contradiction and thus $I \ni b \mapsto c^-(a,b,\lambda)$ is right continuous. The proof of (ii) is similar. □

Appendix B: The case $\lambda = 0$

We collect some results concerning the fiber maps ψ when $\lambda = 0$. The parameter now is $b > 0$, while $a > 0$ is fixed. For this reason, we write $\psi_{b,u}$ and Φ_b instead of $\psi_{0,u}$ and Φ_0 and so on. As we already know, for each $u \in H_0^1(\Omega)\backslash\{0\}$ the fiber map $\psi_{b,u}$ has satisfies Proposition 4.2. One can see now that the systems $\psi_{b,u}(t) = \psi'_{b,u}(t) = 0$ and $\psi'_{b,u}(t) = \psi''_{b,u}(t) = 0$ admits a unique solution, with respect to t, b, which are given respectively by (see [5] and [18])

$$t_0(u) = \left(\frac{2^*a}{4-2^*}\frac{\|u\|^2}{\|u\|_{2^*}^{2^*}}\right)^{\frac{1}{2^*-2}},$$

$$b_0(u) = a^{\frac{4-N}{2}} S_N^{\frac{N}{2}} C_1(N)\left(\frac{\|u\|_{2^*}}{\|u\|}\right)^N,$$

and

$$t(u) = \left(\frac{2a}{4-2^*}\frac{\|u\|^2}{\|u\|_{2^*}^{2^*}}\right)^{\frac{1}{2^*-2}},$$

$$b(u) = a^{\frac{4-N}{2}} S_N^{\frac{N}{2}} C_2(N)\left(\frac{\|u\|_{2^*}}{\|u\|}\right)^N.$$

As a conclusion of this analysis and similar to Propositions 2.4 and 3.7 we have

Proposition B.1 *There holds*

(i) *For each $b \geq b_0(u)$ and each $u \in H_0^1(\Omega)\backslash\{0\}$, $\inf_{t>0}\psi_{b,u}(t) = 0$; for each $b < b_0(u)$ there exists $u \in H_0^1(\Omega)\backslash\{0\}$ such that $\Phi_b(u) < 0$.*
(ii) *For each $b \geq b(u)$, the set $\mathcal{N}_b = \emptyset$; for each $b < b(u)$, the sets $\mathcal{N}_b^+$, $\mathcal{N}_b^-$ and $\mathcal{N}_b^0$ are non empty.*

Therefore:

Lemma B.1 *The following holds true.*

(i) *If $a^{\frac{N-4}{2}} b < C_1(N)$, then there exists $u \in H_0^1(\Omega)\backslash\{0\}$ such that $\Phi_b(u) < 0$.*
(ii) *If $a^{\frac{N-4}{2}} b \geq C_1(N)$, then $\psi_{b,u}(t) > 0$ for all $u \in H_0^1(\Omega)\backslash\{0\}$ and $t > 0$.*
(iii) *If $a^{\frac{N-4}{2}} b < C_2(N)$, then $\mathcal{N}_b^0, \mathcal{N}_b^-, \mathcal{N}_b^+$ are non-empty.*
(iv) *If $a^{\frac{N-4}{2}} b \geq C_2(N)$, then $\mathcal{N}_b = \emptyset$.*

Remark B.1 Comparing Lemmas B.1 and 2.2 we see that

(i) Φ_b is weak lower semi-continuous if, and only, $\Phi_b(u) \geq 0$ for all $u \in H_0^1(\Omega)$.
(ii) If $\Phi_b'(u)u > 0$ for all $u \in H_0^1(\Omega)\setminus\{0\}$, then Φ_b satisfies the Palais-Smale condition. Equivalently $\mathcal{N}_b = \emptyset$.

Corollary B.1 *If $a^{\frac{N-4}{2}} b < C_2(N)$, then for all $\lambda > 0$ we have $\mathcal{N}_b^- \neq \emptyset$.*

Proof Indeed, this is a consequence of Lemmas B.1 and A.3. We also refer the reader to [13, Lemma 2.6]. □

The next lemma is an application of Lemma A.2 and Remark A.2:

Lemma B.2 *Fix $u \in H_0^1(\Omega)\setminus\{0\}$. The following holds true.*

(i) *The function $(0, b(u)) \ni \mapsto t_b^-(u)$ is continuous and increasing.*
(ii) *The function $(0, b(u)) \ni \mapsto t_b^+(u)$ is continuous and decreasing.*
(iii)

$$\lim_{b \uparrow b(u)} t_b^-(u) = t(u) = \lim_{b \uparrow b(u)} t_b^+(u).$$

The following proposition can be found in [18,19] (with some adaptations). We give an outline of the proof (recall from Lemma B.1 that $\mathcal{N}_b^0, \mathcal{N}_b^-$ are not empty for all $a, b > 0$ satisfying $a^{\frac{N-4}{2}} b < C_2(N)$):

Proposition B.2 *Suppose that $a^{\frac{N-4}{2}} b < C_2(N)$, then*

$$\Phi_b(u) = \frac{(2^* - 2)^2 a^2}{4 \cdot 2^*(4 - 2^*) b}, \ \forall u \in \mathcal{N}_b^0.$$

Moreover,

$$c^-(a, b, 0) < \frac{(2^* - 2)^2 a^2}{4 \cdot 2^*(4 - 2^*) b} = c^0(a, b, 0).$$

Proof The first part is trivial. Now suppose on the contrary that there exists $u \in \mathcal{N}_b^-$ such that

$$\Phi_b(u) \geq \frac{(2^* - 2)^2 a^2}{4 \cdot 2^*(4 - 2^*) b}.$$

From Lemma B.2 we have that $t_b^-(u) = 1 < t_{b'}^-(u) < t_{b'}^+(u) < t_b^+(u)$ for each $0 < b < b' < b(u)$ and hence

$$\begin{aligned} \Phi_{b'}(t_{b'}^-(u)u) &> \Phi_{b'}(u) \\ &> \Phi_b(u) \\ &\geq \frac{(2^* - 2)^2 a^2}{4 \cdot 2^*(4 - 2^*) b}, \end{aligned}$$

which implies that

$$\frac{(2^* - 2)^2 a^2}{4 \cdot 2^*(4 - 2^*) b} < \lim_{b' \uparrow b(u)} \Phi_{b'}(t_{b'}^-(u)u) = \Phi_{b(u)}(t_b(u)u) = \frac{(2^* - 2)^2 a^2}{4 \cdot 2^*(4 - 2^*) b(u)},$$

a contradiction since $b < b(u)$. □

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