




A nonlocal isoperimetric problem with density perimeter

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Abstract

We consider the minimization of an energy functional given by the sum of a density perimeter and a nonlocal interaction of Riesz type with exponent α , under volume constraint, where the strength of the nonlocal interaction is controlled by a parameter γ . We show that for a wide class of density functions the energy admits a minimizer for any value of γ . Moreover these minimizers are bounded. For monomial densities of the form $|x|^p$ we prove that when γ is sufficiently small the unique minimizer is given by the ball of fixed volume. In contrast with the constant density case, here the $\gamma \rightarrow 0$ limit corresponds, under a suitable rescaling, to a small mass $m = |\Omega| \rightarrow 0$ limit when $p < d - \alpha + 1$, but to a large mass $m \rightarrow \infty$ for powers $p > d - \alpha + 1$.

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1 Introduction

We consider the nonlocal isoperimetric problems

$$e(\gamma) := \inf \left\{ \mathcal{E}_\gamma(\Omega) : |\Omega| = 1 \right\} \tag{1.1}$$

over sets of finite perimeter $\Omega \subset \mathbb{R}^d$ with given volume, where $|\cdot|$ denotes the Lebesgue measure in \mathbb{R}^d , and the energy functional \mathcal{E}_γ is defined as

$$\mathcal{E}_\gamma(\Omega) := \int_{\partial^* \Omega} \mathbf{a}(x) \, d\mathcal{H}^{d-1} + \gamma \int_{\Omega} \int_{\Omega} \frac{1}{|x - y|^\alpha} \, dx \, dy \tag{1.2}$$

for $\gamma > 0$, $\alpha \in (0, d)$. Here ∂^* denotes the reduced boundary of a set. The first term in the energy functional is the perimeter of Ω with density $\mathbf{a} : \mathbb{R}^d \rightarrow [0, \infty)$, whereas the second term is a Riesz-type nonlocal interaction energy.

The minimization problem (1.1) is a variant of the classical liquid drop model introduced by Gamow in [26]. Gamow’s model is simply given by (1.1) with $\mathbf{a} \equiv 1$. The most important feature of this geometric variational problem is that the two terms present in the energy functional are in direct competition. For $\mathbf{a} \equiv 1$, the surface energy is minimized by a ball whereas the repulsive term does not admit a minimizer and prefers minimizing sequences with multiple vanishingly small components diverging infinitely apart in order to disperse the mass. The parameter of the problem, that is γ , sets a length scale between these competing forces and drives the competition between the short- and long-range interactions. This problem has generated considerable interest in the calculus of variations community (see e.g. [3, 7, 14, 20–22, 29, 30, 32–34, 38, 42] as well as [15] for a review) with several papers studying parameter regimes of existence and nonexistence of minimizers. Results of [21, 32, 34], for example, show that for large values of γ , the energy \mathcal{E}_γ with $\mathbf{a} \equiv 1$ does not admit a minimizer. There are also several studies characterizing the minimizing sequences [3, 7, 33] even when minimizers fail to exist. In particular, in [3], the authors use a “regularization” of the energy by adding an attractive external potential which guarantees the existence of minimizers for all values of γ .

Also very recently there has been studies on the extensions of the liquid drop model to the anisotropic setting where the surface energy is replaced by an anisotropic surface tension [8, 13, 36]. In these models the surface energy is given by $\int_{\partial^* \Omega} \psi(\nu_\Omega) \, d\mathcal{H}^{d-1}$ for some convex, one-homogeneous function ψ where ν_Ω denotes the outward unit normal to the reduced boundary $\partial^* \Omega$. Such anisotropic extensions do not annihilate the translation invariance of the liquid drop model and a simple scaling argument heuristically justifies that for large γ values minimizers still fail to exist. In contrast, the inclusion of a translation *variant* density in the perimeter functional “regularizes” the liquid drop model in the sense that the problem admits a minimizer for all values of γ .

Isoperimetric problems defined via weighted perimeters

$$\mathcal{P}_\mathbf{a}(\Omega) := \int_{\partial^* \Omega} \mathbf{a}(x) \, d\mathcal{H}^{d-1}(x) \tag{1.3}$$

have been studied for various choices of densities \mathbf{a} . Problems where the volume constraint is also weighted either by the density \mathbf{a} or by some other function have especially attracted

significant interest (see e.g. [5,6,10–12,17–19,24,37,40,41,44] and references therein). The main questions regarding these problems have been existence, boundedness and regularity of isoperimetric sets. These questions have been studied not only for specific densities (radial, monomial, Gauss-like) but also for rather general densities satisfying some boundedness and continuity conditions. To our knowledge, perturbations of density perimeter (either by long-range interactions or by external potentials) have not yet been considered in the literature.

As pointed above, the inclusion of a confining density in the perimeter functional provides a different type of “regularization” of the problem (1.1). Our first main result establishes the compactness of any minimizing sequence with global convergence to a minimizer for any $\gamma \geq 0$ and for a wide class of densities, satisfying a simple coercivity condition:

A1 $\mathbf{a} \in C^0(\mathbb{R}^d)$, $\mathbf{a}(0) = 0$, $\mathbf{a}(x) > 0$ for all $x \neq 0$, and $\lim_{|x| \rightarrow \infty} \mathbf{a}(x) = +\infty$.

In order to state the existence result, we recall that sets $\Omega_n \rightarrow \Omega$ globally if $|\Omega_n \Delta \Omega| \rightarrow 0$, that is, their characteristic functions $\chi_{\Omega_n} \rightarrow \chi_\Omega$ in the $L^1(\mathbb{R}^d)$ -norm.

Theorem 1.1 (Existence of minimizers) *Let \mathbf{a} be any density satisfying the assumption (A1), and fix any $\gamma \geq 0$. Then any minimizing sequence $\{\Omega_n\}_{n \in \mathbb{N}}$ for the problem (1.1) admits a subsequence which converges globally to a minimizer $\Omega_\gamma \subset \mathbb{R}^d$ with $|\Omega_\gamma| = 1$ and $\chi_{\Omega_\gamma} \in BV_{loc}(\mathbb{R}^d \setminus \{0\})$.*

Remark 1.2 The proof of the existence result in Theorem 1.1 can be extended to a somewhat broader class of densities \mathbf{a} satisfying

$$\mathbf{a} \in C^0(\mathbb{R}^d), \quad \mathbf{a}^{-1}(\{0\}) = Z, \quad \mathbf{a} > 0 \text{ in } \mathbb{R}^d \setminus Z, \quad \text{and} \quad \lim_{|x| \rightarrow \infty} \mathbf{a}(x) = +\infty,$$

for any finite set $Z = \{z_1, \dots, z_m\} \subset \mathbb{R}^d$ with $m \in \mathbb{N}$. We present the proof in the case $Z = \{0\}$. In this general situation, any minimizer $\Omega_Z \subset \mathbb{R}^d$ is such that $\chi_{\Omega_Z} \in BV_{loc}(\mathbb{R}^d \setminus Z)$.

We remark that the coercive nature of \mathbf{a} at infinity ensures the existence of minimizers for (1.1), essentially because the splitting of mass off to infinity (the main reason for non-compactness in nonlocal isoperimetric problems) is rendered too costly. However it does not ensure that minimizers need be connected sets. Indeed, for large γ the nonlocal interactions should become large enough to favor the fragmentation of sets, which will repel but be contained at finite distance. This behavior is also observed in nonlocal isoperimetric problems with a confining term [3].

Our existence result relies on a modified version of the relative isoperimetric inequality on annuli and requires only the minimal assumption (A1) on the densities \mathbf{a} . On the other hand, proving boundedness of minimizers is rather technical and we prove it under one of the following additional structural assumptions:

(A2a) $\mathbf{a} \in C_{loc}^{0,1}(\mathbb{R}^d)$ and there exists constants $C_{\mathbf{a}} \geq 1$ and $R_{\mathbf{a}} \geq 1$ such that for every $R \geq R_{\mathbf{a}}$ it holds

$$0 < \sup_{B_{2R} \setminus B_R} \mathbf{a} \leq C_{\mathbf{a}} \inf_{B_{2R} \setminus B_R} \mathbf{a}. \tag{1.4}$$

(A2b) $\mathbf{a} \in C_{loc}^{0,1}(\mathbb{R}^d)$, $\mathbf{a}(x) = \mathbf{a}(|x|)$ and there exists $R_{\mathbf{a}} \geq 1$ such that \mathbf{a} is non-decreasing for $|x| \geq R_{\mathbf{a}}$.

For densities satisfying either of the additional conditions (A2a) or (A2b), using regularity results of quasi-minimizers, such as density bounds, we obtain the boundedness of minimizers of the problem (1.1).

Theorem 1.3 (Boundedness of minimizers) *For any density \mathbf{a} satisfying (A1) and either (A2a) or (A2b), and for any $\gamma \geq 0$, any minimizer Ω_γ of (1.1) is essentially bounded.*

Remark 1.4 We require the assumption (A1) in Theorem 1.3 only to obtain the existence of a minimizer to (1.1). If the existence of a minimizer could be obtained under some other conditions, either (A2a) or (A2b) would be sufficient to obtain the boundedness of minimizers.

Remark 1.5 (Almost polynomial densities) Let $\mathbf{a} \in C_{loc}^{0,1}(\mathbb{R}^d)$ satisfy the following conditions:

- (i) $\mathbf{a}(0) = 0$ and $\mathbf{a}(x) > 0$ for $x \neq 0$.
- (ii) There exist $p > 0$, $R_0 > 0$ and $C_1, C_2 > 0$ such that $C_1|x|^p \leq \mathbf{a}(x) \leq C_2|x|^p$ for all $|x| > R_0$.

Then \mathbf{a} satisfies the assumptions (A1) and (A2a); hence, Theorems 1.1 and 1.3 hold for such densities.

For homogeneous densities, which satisfy the condition $\mathbf{a}(tx) = t^p \mathbf{a}(x)$ for some $p > 0$, a simple scaling argument shows that the minimization problem (1.1) is equivalent to the problem

$$\inf \left\{ \mathcal{E}_1(\Omega) : |\Omega| = m \right\} \tag{1.5}$$

with the correspondence

$$\gamma = m^{-(p+\alpha-d-1)/d}, \quad p \neq p_* := d - \alpha + 1.$$

It is interesting to observe that for homogeneous weights the large mass/small mass behavior of the minimization problem depends on the specific power p . In particular, when $p > p_*$, the corresponding value of γ varies inversely with mass. Thus, with $p > p_*$ the nonlocal energy is dominated by the perimeter term $\mathcal{P}_\mathbf{a}$ for large mass m , and the nonlocal term dominates for small m , exactly the opposite of the behavior for constant \mathbf{a} . For subcritical $p < p_*$ the opposite is true, and the energy is perimeter-dominated for small m . At the critical value $p = p_*$ the two problems (1.1) and (1.5) are not equivalent, and (1.5) is scale invariant: the minimizers at any mass m are all rescaled copies of the same set. Since Theorem 1.1 guarantees the existence of a minimizer for all values of m (or γ), an interesting question is the characterization of minimizers for a range of values of the parameters. We provide a partial answer to this question in the next theorem.

Theorem 1.6 (Global minimizers in the small γ regime) *Let $\mathbf{a}(x) = |x|^p$ with $p > 0$. For γ sufficiently small the ball $B \subset \mathbb{R}^d$ of volume one, centered at the origin is the unique minimizer of $e(\gamma)$.*

For $p < p_*$, the $\gamma \rightarrow 0$ limit is equivalent to the small m regime in (1.5), and the optimality of the spherical ball for small mass is well-known for the unweighted $\mathbf{a} \equiv 1$ case (see [7,29–32]). With $p > p_*$, the situation is reversed and Theorem 1.6 shows that the ball minimizes for all sufficiently large m . Such \mathbf{a} are very coercive at infinity, and the situation is similar to the case studied by G en eraux and Oudet in [27], where the problem (1.5) with constant density is supplemented with a confining term. In that case they also prove the minimality of the ball for very large m .

The proof of Theorem 1.6 relies on a penalization technique, similar to those used in several geometric variational problems involving the perimeter functional (see [1,7,16,20,

39]). Utilizing results from the regularity theory for density perimeters [18,41] we reduce the minimizers of the nonlocal problem to nearly spherical or isoperimetric sets in the small γ regime. The novelty here is, though, that we cannot directly apply the results from the literature due to the degeneracy of the density \mathbf{a} at the origin and the possibility of small nonsmooth components of $\partial^*\Omega_\gamma$ near the origin. Once we reduce the problem to nearly spherical sets we use a Fuglede-type argument (see [23]) to control the isoperimetric and nonlocal deficits between minimizers and the ball and show that for small $\gamma > 0$ these quantities have to be identically zero.

The regime of large γ is also very interesting, but its analysis requires a very different approach. While the existence of minimizers is guaranteed by Theorem 1.1, in this regime, the nonlocal term is dominant and prefers the minimizer to break into smaller pieces distributed in a compact set whose size is determined by the confining term \mathbf{a} . Hence, the characterization of minimizers (i.e., the shape of the disconnected components as well as their locations) depends on the delicate balance between the preferred shapes dictated by the density perimeter and the inter-component interactions. A similar phenomenon is also observed in models of copolymer/homopolymer blends. While existence of minimizers is obtained for all values of m in [9], only in the small m regime the minimizers are uniquely characterized leaving the question of the precise morphology of minimizing configurations for large m open.

Structure of the paper

The paper is organized as follows. In Sects. 2 and 3 we prove Theorems 1.1 and 1.3, respectively. Section 4 is devoted to the proof of Theorem 1.6.

Notation

Throughout the paper ω_d denotes the volume of the unit ball $B_1(0)$ in \mathbb{R}^d and we write $B_r := B_r(0)$ to denote the ball of radius r centered at zero. Constants, denoted by C , can change from line to line (unless otherwise noted). We will denote the \mathbf{a} -volume measure and the \mathbf{a} -surface area measure, respectively, by

$$|\Omega|_{\mathbf{a}} := \int_{\Omega} \mathbf{a}(y) \, dy, \quad \text{and} \quad \mathcal{H}_{\mathbf{a}}^{d-1}(\Omega) := \int_{\Omega} \mathbf{a}(y) \, d\mathcal{H}^{d-1}(y).$$

The relative weighted perimeter of E in F will be denoted by either $\mathcal{P}_{\mathbf{a}}(E, F)$ or $\int_F \mathbf{a}(x) |\nabla \chi_E|$, where χ_E is the characteristic function of the set E and $|\nabla \chi_E|$ is the total variation of χ_E . The weighted perimeter of E in F (or the \mathbf{a} -perimeter) is defined as

$$\mathcal{P}_{\mathbf{a}}(E, F) := \sup \left\{ \int_E \operatorname{div}(\mathbf{a}(x)X(x)) \, dx : X \in C_c^\infty(F; \mathbb{R}^d), \|X\|_{L^\infty} \leq 1 \right\}.$$

In particular, $\mathcal{P}_{\mathbf{a}}(E, F) = \mathcal{H}_{\mathbf{a}}^{d-1}(\partial^*E \cap F)$. We will say that $\chi_E \in BV_{\mathbf{a}}(F)$ if $\mathcal{P}_{\mathbf{a}}(E, F) < +\infty$. Perimeters of sets in the whole space (i.e., when $F = \mathbb{R}^d$) are denoted by $\mathcal{P}_{\mathbf{a}}(E)$ or $\int_{\mathbb{R}^d} \mathbf{a}(x) |\nabla \chi_E|$. We will denote the Euclidean perimeter (when $\mathbf{a} \equiv 1$) by simply \mathcal{P} . Finally, we will denote the nonlocal term by \mathcal{V} , i.e.,

$$\mathcal{V}(\Omega) := \int_{\Omega} \int_{\Omega} \frac{dx \, dy}{|x - y|^\alpha},$$

for any $\alpha \in (0, d)$.

2 Existence of minimizers

In this section we present the proof of Theorem 1.1. It relies on the following modified version of the relative isoperimetric inequality on annuli.

Lemma 2.1 *Let $A_{r,R} = \{x \in \mathbb{R}^d : r \leq |x| < R\} = B_R \setminus B_r$ denote the half-open annulus of inner radius r and outer radius R . Then, there exists a dimensional constant $c_d > 0$, so that*

$$\min \left\{ |\Omega \cap A_{1,2}|^{(d-1)/d}, |A_{1,2} \setminus \Omega|^{(d-1)/d} \right\} \leq c_d \mathcal{P}(\Omega, A_{1,2}) \tag{2.1}$$

for every set of finite perimeter $\Omega \subset \mathbb{R}^d$.

Remark 2.2 Inequality (2.1) is still valid with the same constant c_d over any annulus $A_{2^j, 2^{j+1}}$ with $j \geq 1$, as the inequality is invariant under scalings.

The relative isoperimetric inequality is typically stated for balls in \mathbb{R}^d (see e.g. [47, Thm 5.4.3],) but in fact the same proof verifies that it holds in any domain for which one can prove the validity of the Poincaré inequality,

$$\left(\int_{A_{1,2}} |u - \bar{u}_{r,R}|^{\frac{d}{d-1}} dx \right)^{\frac{d-1}{d}} \leq C_{1,2} \int_{A_{1,2}} |\nabla u| dx,$$

where $\bar{u}_{r,R} := \int_{A_{r,R}} u dx$ is the average on annuli. The latter can be found in [2].

We now turn to the proof of the existence of minimizers for $e(\gamma)$. For translation-invariant nonlocal isoperimetric problems existence is a delicate issue, as minimizing sequences can split, with pieces diverging to infinity. The increasing weight $\mathbf{a}(x)$ raises the cost of splitting, an effect which is quantified in our proof via the relative isoperimetric inequality, Lemma 2.1.

Proof of Theorem 1.1 Let $\{\Omega_n\} \subset \mathbb{R}^d$ be a minimizing sequence of (1.1):

$$\mathcal{E}_\gamma(\Omega_n) \rightarrow e(\gamma) \text{ as } n \rightarrow \infty, \text{ and } |\Omega_n| = 1 \text{ for any } n \geq 1.$$

We first show that $\{\Omega_n\}$ is uniformly bounded in $BV(\mathbb{R}^d \setminus \bar{B}_\varepsilon)$ for all $\varepsilon > 0$ sufficiently small. To see this, note that $\lim_{|x| \rightarrow \infty} \mathbf{a}(x) = +\infty$ implies that there exists $R_1 > 0$ such that $\mathbf{a} \geq 1$ in $\mathbb{R}^d \setminus B_{R_1}$. Define $\delta_\varepsilon := \min\{\mathbf{a}(x) : \varepsilon \leq |x| \leq R_1\}$ for $0 < \varepsilon < 1$ small enough so that $\delta_\varepsilon \leq 1$ (as $\lim_{|x| \rightarrow 0} \mathbf{a}(x) = 0$). Then, for any such choice of ε , $\mathbf{a}(x) \geq \delta_\varepsilon > 0$ for all $x \in \mathbb{R}^d \setminus \bar{B}_\varepsilon$, and we deduce

$$\delta_\varepsilon \mathcal{P}(\Omega_n, \mathbb{R}^d \setminus \bar{B}_\varepsilon) < \mathcal{P}_\mathbf{a}(\Omega_n, \mathbb{R}^d \setminus \bar{B}_\varepsilon) < \mathcal{E}_\gamma(\Omega_n) = e(\gamma) + o_n(1),$$

which confirms uniform boundedness in $BV(\mathbb{R}^d \setminus \bar{B}_\varepsilon)$.

We next show local convergence of $\{\Omega_n\}$ to a limiting set Ω_γ . Invoking compactness results of sets with uniformly bounded perimeter, there exists $\Omega^\varepsilon \subset \mathbb{R}^d$ so that $\Omega_{n_\ell} \rightarrow \Omega^\varepsilon$ locally in $\mathbb{R}^d \setminus \bar{B}_\varepsilon$, as $\ell \rightarrow \infty$. Running a diagonalization argument over a sequence $\varepsilon_k = 1/k \rightarrow 0^+$, there exists a subsequence $n_\ell \rightarrow \infty$ such that for all $k \geq 1$, $\Omega_{n_\ell} \rightarrow \Omega^{1/k}$ locally in $\mathbb{R}^d \setminus \bar{B}_{1/k}$, as $\ell \rightarrow \infty$. In particular, $\Omega^{1/(k+j)} \setminus \bar{B}_{1/k} = \Omega^{1/k} \setminus \bar{B}_{1/k}$ for any $j \geq 1$. Defining the limit set as $\Omega_\gamma := \bigcup_{k=1}^\infty (\Omega^{1/k} \setminus \bar{B}_{1/k})$, we claim that up to subsequence,

$$\begin{aligned} \Omega_n &\rightarrow \Omega_\gamma \text{ locally in } \mathbb{R}^d \text{ as } n \rightarrow \infty, \text{ and} \\ \chi_{\Omega_n} &\rightarrow \chi_{\Omega_\gamma} \text{ pointwise a.e. in } \mathbb{R}^d \text{ as } n \rightarrow \infty. \end{aligned} \tag{2.2}$$

Assuming the claim, we may conclude that $\mathcal{P}(\Omega_\gamma, \mathbb{R}^d \setminus \bar{B}_\varepsilon) \leq \liminf_{n \rightarrow \infty} \mathcal{P}(\Omega_n, \mathbb{R}^d \setminus \bar{B}_\varepsilon)$ for $0 < \varepsilon \ll 1$, which shows $\chi_{\Omega_\gamma} \in BV_{\text{loc}}(\mathbb{R}^d \setminus \{0\})$.

To verify (2.2), let $K \subset \mathbb{R}^d$ be a compact set and for $\varepsilon > 0$, fix $k \geq 1$ so that $|B_{1/k}| \leq \varepsilon$. Then

$$\begin{aligned} |(\Omega_n \triangle \Omega_\gamma) \cap K| &= |(\Omega_n \triangle \Omega^{1/k}) \cap (K \setminus B_{1/k})| + |(\Omega_n \triangle \Omega_\gamma) \cap K \cap B_{1/k}| \\ &\leq o_n(1) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary small, we conclude (2.2). The aforementioned convergence along with the lower semicontinuity of the \mathbf{a} -perimeter functional, together with Fatou’s Lemma shows that

$$\mathcal{E}_\gamma(\Omega_\gamma) = \mathcal{P}_\mathbf{a}(\Omega_\gamma) + \gamma \mathcal{V}(\Omega_\gamma) \leq \liminf_{n \rightarrow \infty} (\mathcal{P}_\mathbf{a}(\Omega_n) + \gamma \mathcal{V}(\Omega_n)) = e(\gamma).$$

We are only left to show that Ω_γ is admissible in (1.1), from which it will follow that $\mathcal{E}_\gamma(\Omega_\gamma) \geq e(\gamma)$; thus obtaining the existence of a minimizer.

We observe that $|\Omega_\gamma| \leq 1$ in view of Fatou’s Lemma, once again. We claim that in fact $|\Omega_\gamma| = 1$. Suppose, on the contrary, that $|\Omega_\gamma| < \beta$ for some $\beta \in (0, 1)$.

The local convergence (2.2) shows that for all $R > 0$, $|\Omega_n \cap B_R| = |\Omega_\gamma \cap B_R| + o_n(1) < \beta$ for all but finitely many n . Thus, the sets Ω_n have very thick tails, which will introduce huge energy cost via the relative isoperimetric inequality. By running a diagonalization argument over $\{R_k = 2^k\}$, there exists an increasing subsequence $n_k \rightarrow +\infty$ such that for all $k \geq 1$,

$$\inf_{n \geq n_k} |\Omega_n \setminus B_{2^k}| > 1 - \beta. \tag{2.3}$$

On the other hand, as $\mathbf{a}(x) \rightarrow +\infty$ as $|x| \rightarrow \infty$, for $M > 1$ arbitrarily large, for all $n \geq 1$, and every $k \geq k_M$ sufficiently large,

$$\mathcal{E}_\gamma(\Omega_n) \geq \mathcal{P}_\mathbf{a}(\Omega_n, \mathbb{R}^d \setminus B_{2^k}) \geq M \mathcal{P}(\Omega_n, \mathbb{R}^d \setminus B_{2^k}). \tag{2.4}$$

Intuitively, when j is large we expect $|\Omega_n \cap A_{2^j, 2^{j+1}}|$ to be much smaller than its complement $|A_{2^j, 2^{j+1}} \setminus \Omega_n|$ in the annulus. Indeed, we claim that there exists $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$ and for all but finitely many n we have

$$|\Omega_n \cap A_{2^j, 2^{j+1}}| < (2^{j+1})^{-d} |A_{2^j, 2^{j+1}} \setminus \Omega_n|. \tag{2.5}$$

For otherwise, there would exist increasing sequences $j_\ell \rightarrow +\infty$ and $n_\ell \rightarrow +\infty$ for which $|\Omega_{n_\ell} \cap A_{2^{j_\ell}, 2^{j_\ell+1}}| \geq (2^{j_\ell+1})^{-d} |A_{2^{j_\ell}, 2^{j_\ell+1}} \setminus \Omega_{n_\ell}|$, for every $\ell \geq 1$. This would imply that

$$|\Omega_n \cap A_{2^j, 2^{j+1}}| \geq \frac{1}{1 + 2^{(j+1)d}} |A_{2^{j_\ell}, 2^{j_\ell+1}}|,$$

and hence,

$$\begin{aligned} 1 = |\Omega_{n_\ell}| &\geq \sum_{\ell=1}^{\infty} |\Omega_{n_\ell} \cap A_{2^{j_\ell}, 2^{j_\ell+1}}| \\ &\geq \sum_{\ell=1}^{\infty} \frac{1}{1 + (2^{j_\ell+1})^d} |A_{2^{j_\ell}, 2^{j_\ell+1}}| > \sum_{\ell=1}^{\infty} \frac{1}{2} \left(1 - \frac{1}{2^d}\right) \omega_d = +\infty, \end{aligned}$$

establishing the claim.

We now fix any $k \geq \max\{k_M, j_0\}$ and $n \gg 1$ sufficiently large to obtain the validity of (2.4) and of (2.5) for all $j \geq k$. Utilizing the relative isoperimetric inequality on $\{A_{2^j, 2^{j+1}} : j \geq k\}$ (see Remark 2.2), we get the lower bound

$$\begin{aligned}
 c_d \mathcal{P}(\Omega_n, \mathbb{R}^d \setminus B_{2^k}) &= \sum_{j=k}^{\infty} c_d \mathcal{P}(\Omega_n, A_{2^j, 2^{j+1}}) \\
 &\geq \sum_{j=k}^{\infty} |\Omega_n \cap A_{2^j, 2^{j+1}}|^{\frac{d-1}{d}} \\
 &\geq \left(\sum_{j=k}^{\infty} |\Omega_n \cap A_{2^j, 2^{j+1}}| \right)^{\frac{d-1}{d}} = |\Omega_n \cap (\mathbb{R}^d \setminus B_{2^k})|^{\frac{d-1}{d}}
 \end{aligned}$$

Increasing the value of $n \geq n_k$ if necessary, it follows from (2.3) and (2.4) that

$$+\infty > e(\gamma) + o_n(1) \geq M \mathcal{P}(\Omega_n, \mathbb{R}^d \setminus B_{2^k}) \geq \frac{M}{c_d} (1 - \beta)^{\frac{d-1}{d}},$$

with $M > 1$ arbitrarily large. Thus, we reach a contradiction. Hence, $|\Omega_\gamma| = 1$, and we have proven that Ω_γ attains the minimum in the nonlocal isoperimetric problem (1.1).

Finally, by the identity $|\Omega_n \Delta \Omega_\gamma| = 2|\Omega_\gamma \setminus \Omega_n| + |\Omega_n| - |\Omega_\gamma|$ together with $|\Omega_n| = 1 = |\Omega_\gamma|$ and (2.2), we deduce the global convergence of the subsequence $\{\Omega_n\}$, as (by local convergence,)

$$|\Omega_\gamma \setminus \Omega_n| \leq |(\Omega_\gamma \setminus \Omega_n) \cap B_r| + |\Omega_\gamma \setminus B_r| \leq |(\Omega_\gamma \setminus \Omega_n) \cap B_r| + o_r(1).$$

Thus, every minimizing sequence Ω_n for $e(\gamma)$ contains a subsequence which converges globally (in $L^1(\mathbb{R}^d)$) to a minimizer of $e(\gamma)$. □

3 Boundedness of minimizers

In this section we prove Theorem 1.3. Since the assumptions (A2a) and (A2b) characterize different types of densities, the proof of the theorem requires two different approaches. For densities which are polynomial-like and have bounded oscillations (i.e., densities satisfying (A2a)) we make use of a series of technical lemmas establishing uniform density bounds for quasi-minimizers of the weighted perimeter functional where the density is measured with respect to weighted volumes. For radial and monotone densities (i.e., densities satisfying (A2b)), on the other hand, we utilize a regularity result, called $\varepsilon - \varepsilon^{(d-1)/d}$ property, which basically says that a set of finite perimeter can be locally modified where one increases its volume by ε , while the perimeter increases at most by a constant multiple of $\varepsilon^{(d-1)/d}$. We present the proof in two subsections.

3.1 Densities with bounded oscillations

We start with densities satisfying the assumptions (A1) and (A2a). First, we prove that any minimizer Ω_γ of (1.1) has finite \mathbf{a} -volume.

Lemma 3.1 *For any density \mathbf{a} satisfying (A1) and (A2a) any minimizer Ω_γ of (1.1) has finite \mathbf{a} -volume, i.e., $|\Omega_\gamma|_{\mathbf{a}} < +\infty$.*

Proof By passing to the limit $\Omega_n \rightarrow \Omega_\gamma$ in (2.5),

$$|\Omega_\gamma \cap A_{2^j, 2^{j+1}}| \leq (2^{j+1})^{-d} |A_{2^j, 2^{j+1}} \setminus \Omega_\gamma|, \tag{3.1}$$

for all $j \geq j_0$, for some j_0 . Fix $j_{\mathbf{a}} \in \mathbb{N}$ with $j_{\mathbf{a}} \geq j_0$, such that $R_{\mathbf{a}} < 2^{j_{\mathbf{a}}}$ where $R_{\mathbf{a}}$ is given as in (1.4). Using (1.4), (3.1), and the relative isoperimetric inequality (see Remark 2.2) we have:

$$\begin{aligned} \int_{\Omega_\gamma \setminus B_{2^{j_{\mathbf{a}}}}} \mathbf{a}(x) \, dx &= \sum_{j=j_{\mathbf{a}}}^\infty \int_{\Omega_\gamma \cap A_{2^j, 2^{j+1}}} \mathbf{a}(x) \, dx \\ &\leq \sum_{j=j_{\mathbf{a}}}^\infty \left(\sup_{A_{2^j, 2^{j+1}}} \mathbf{a} \right) |\Omega_\gamma \cap A_{2^j, 2^{j+1}}| \\ &\leq \sum_{j=j_{\mathbf{a}}}^\infty C_{\mathbf{a}} \left(\inf_{A_{2^j, 2^{j+1}}} \mathbf{a} \right) |\Omega_\gamma \cap A_{2^j, 2^{j+1}}|^{\frac{d-1}{d}} |\Omega_\gamma \setminus B_{2^j}|^{\frac{1}{d}} \\ &\leq C_{\mathbf{a}} |\Omega_\gamma \setminus B_{2^{j_{\mathbf{a}}}}|^{\frac{1}{d}} \sum_{j=j_{\mathbf{a}}}^\infty \left(\inf_{A_{2^j, 2^{j+1}}} \mathbf{a} \right) c_d \int_{A_{2^j, 2^{j+1}}} |\nabla \chi_{\Omega_\gamma}| \\ &\leq c_d C_{\mathbf{a}} |\Omega_\gamma \setminus B_{2^{j_{\mathbf{a}}}}|^{\frac{1}{d}} \sum_{j=j_{\mathbf{a}}}^\infty \int_{A_{2^j, 2^{j+1}}} \mathbf{a}(x) |\nabla \chi_{\Omega_\gamma}|. \end{aligned}$$

Hence, as $|\Omega_\gamma| \leq 1$, we conclude that

$$\int_{\Omega_\gamma \setminus B_{2^{j_{\mathbf{a}}}}} \mathbf{a}(x) \, dx \leq c_d C_{\mathbf{a}} \int_{\mathbb{R}^d \setminus B_{2^{j_{\mathbf{a}}}}} \mathbf{a}(x) |\nabla \chi_{\Omega_\gamma}|. \tag{3.2}$$

Since Ω_γ has finite \mathbf{a} -perimeter, we obtain that $|\Omega_\gamma|_{\mathbf{a}} < +\infty$. □

At the heart of the proof of Theorem 1.3 lies the regularity of quasi-minimal sets with a volume constraint. In order to establish this we will largely follow the argument carried out by Rigot in [43, Chapter 2], where the author studies the case of standard perimeter functional $\mathbf{a}(x) \equiv 1$. As in [43, Chapter 1] and [35, Chapter 21], given a function $g : (0, +\infty) \rightarrow (0, +\infty)$ with $g(x) = o(x^{(d-1)/d})$ for x close to 0, we will say that Ω_γ is a *volume constrained quasi-minimal set* for \mathbf{a} -perimeter if

$$\mathcal{P}_{\mathbf{a}}(\Omega_\gamma) \leq \mathcal{P}_{\mathbf{a}}(F) + g(|F \Delta \Omega_\gamma|)$$

for any $F \subset \mathbb{R}^d$ with $\chi_F \in BV_{\mathbf{a}}(\mathbb{R}^d)$, $|F| = 1$ and $F \Delta \Omega_\gamma \subset\subset \mathbb{R}^d$.

Minimizers of isoperimetric problems with a Riesz-type nonlocal term are also volume constrained quasi-minimizers for the perimeter functional with the choice $g_\gamma(x) \simeq \gamma x$. Indeed, as argued in [32, Proposition 2.1], we define the potential of a Borel set $E \subset \mathbb{R}^d$ by

$$v_E(x) := \int_E \frac{1}{|x - y|^\alpha} \, dy$$

and consider a minimizer Ω_γ of (1.1) together with a set F with prescribed mass and $F \Delta \Omega_\gamma \subset B_r(0)$ for some $r > 0$. A simple argument shows that the interaction energy \mathcal{V} is Lipschitz with respect to symmetric difference:

$$\begin{aligned} \mathcal{V}(F) - \mathcal{V}(E) &= \int_F \int_F \frac{1}{|x - y|^\alpha} \, dx \, dy - \int_E \int_E \frac{1}{|x - y|^\alpha} \, dx \, dy \\ &\leq \int_{E \Delta F} (v_E + v_F) \, dx \\ &\leq C |E \Delta F| \end{aligned} \tag{3.3}$$

with $C = 2 \int_{B_1} \frac{dy}{|y|^\alpha} + 2$.

Hence, any minimizer Ω_γ of \mathcal{E}_γ must satisfy

$$\mathcal{P}_a(\Omega_\gamma) \leq \mathcal{P}_a(F) + C\gamma |F \Delta \Omega_\gamma|, \tag{3.4}$$

for any suitable competitor F as above.

We will now present some technical lemmas essentially studied in [43, Chapter 2] and adapt these results to the case of weighted perimeters. The next lemma, proven for $\mathbf{a} = 1$ by Giusti (see [28, Lemma 2.1]), shows that any set of positive perimeter can be approximated in L^1 by another set without substantially increasing the weighted perimeter. We denote by $[\cdot]_{1,D}$ the Lipschitz seminorm in D .

Lemma 3.2 *Let $D \subset \mathbb{R}^d$ be a bounded domain, $L \subset \mathbb{R}^d$ with $\chi_L \in BV_a(\mathbb{R}^d)$ for a density $\mathbf{a} \in C^{0,1}(D)$, $\inf_D \mathbf{a} > 0$, and such that*

$$\int_D \mathbf{a}(x) |\nabla \chi_L| > 0.$$

Then there exist $\varepsilon > 0$ and $Q_a > 0$, depending on $L \cap D$ and D only, such that, $Q_a \lesssim (\sup_D \mathbf{a})(1 + [\mathbf{a}]_{1,D})$, and for every $v \in (-\varepsilon, \varepsilon)$, there exists $F \subset \mathbb{R}^d$ with $F = L$ in a neighborhood of $\mathbb{R}^d \setminus D$ satisfying

$$\begin{aligned} |F| &= |L| + v, \\ \int_D \mathbf{a}(x) |\nabla \chi_F| &\leq \int_D \mathbf{a}(x) |\nabla \chi_L| + Q_a |v|, \\ \int_D |\chi_F - \chi_L| &\leq Q_a |v|. \end{aligned}$$

Proof By definition of \mathbf{a} -perimeter, there exists $w \in C^1_c(D; \mathbb{R}^d)$, $|w(x)| \leq \mathbf{a}(x)$ a.e. in D , such that

$$\int_D \chi_L \operatorname{div} w \, dx \geq \frac{1}{2} \int_D \mathbf{a}(x) |\nabla \chi_L| > 0. \tag{3.5}$$

Note that, since $\mathbf{a} > 0$ in D , $BV_a(D) \subset BV(D)$. For $t \in (0, 1)$ we put $\eta_t = x + tw(x)$ and $K := \operatorname{spt} w \subset\subset D$. Then $\eta_t \equiv I$ in $\mathbb{R}^d \setminus K$, and for $|t|$ small enough, $\eta_t : D \rightarrow D$ is a diffeomorphism. Letting $L_t := \eta_t(L)$, we claim:

$$\begin{aligned} |L_t| &= \int_L |\det D\eta_t| \, dx \\ \int_D \mathbf{a}(x) |\nabla \chi_{L_t}| &\leq \int_D \mathbf{a}(x) f_t(x) |\nabla \chi_L| + |t| (\sup_K \mathbf{a}) [\mathbf{a}]_{1,K} \int_K f_t(x) |\nabla \chi_L| \end{aligned} \tag{3.6}$$

where $f_t(x) := |\det D\eta_t(x)| |(D\eta_t)^{-1}(x)|$. Indeed, the first equality is clear, and the inequality below it is obtained by noting,

$$\begin{aligned} \int_D \mathbf{a}(x) |\nabla \chi_{L_t}| &\leq \int_D (\mathbf{a} \circ \eta_t)(x) f_t(x) |\nabla \chi_L| \\ &\leq \int_D \mathbf{a}(x) f_t(x) |\nabla \chi_L| + \int_K f_t(x) |\mathbf{a} \circ \eta_t(x) - \mathbf{a}(x)| |\nabla \chi_L| \end{aligned}$$

and estimating, for $x \in K$,

$$|\mathbf{a} \circ \eta_t(x) - \mathbf{a}(x)| = |\mathbf{a}(x + tw(x)) - \mathbf{a}(x)| \leq [\mathbf{a}]_{1,K} |tw(x)| \leq |t| [\mathbf{a}]_{1,K} (\sup_K \mathbf{a}).$$

Also, $\det D\eta_t = 1 + t \operatorname{div} w + t^2 A(x, t)$ and $(D\eta_t)^{-1} = I - tH(x, t)$, with $|A|$ and $|H|$ bounded uniformly by a constant, which depends exclusively on $L \cap D$ and D . For $|t|$ small enough, $f_t(x) \leq 1 + t(\operatorname{div} w + |H|) + O(t^2)$, and so (3.6) shows

$$|L_t| = |L| + t \int_D \chi_L \operatorname{div} w \, dx + t^2 \int_D \chi_L A(x, t) \, dx, \tag{3.7}$$

and

$$\begin{aligned} & \int_D \mathbf{a}(x) |\nabla \chi_{L_t}| \\ & \leq (1 + |t| \cdot \|\operatorname{div} w + |H|\|_{L^\infty(D)}) \int_D \mathbf{a}(x) |\nabla \chi_L| \\ & \quad + |t| [\mathbf{a}]_{1,D} (\sup_D \mathbf{a}) \int_D |\nabla \chi_L| + O(t^2) \\ & \leq \int_D \mathbf{a}(x) |\nabla \chi_L| + |t| (\sup_D \mathbf{a}) (C + [\mathbf{a}]_{1,D}) \int_D |\nabla \chi_L| + O(t^2). \end{aligned} \tag{3.8}$$

In view of (3.5), there exists $\varepsilon' > 0$ sufficiently small so that for any choice of $v \in (-\varepsilon', \varepsilon')$ the relation $t \int_D \chi_L \operatorname{div} w \, dx + t^2 \int_D \chi_L A(x, t) = |v|$ in (3.7) holds true for some t_v , and moreover $|t_v| \leq C'|v|$, with C' depending on $L \cap D$ and D only. We take $F := L_{t_v}$ and observe that F satisfies the first two statements of the lemma, in light of (3.7)–(3.8), for the value $Q_{\mathbf{a}} = 2(\sup_D \mathbf{a})(C + [\mathbf{a}]_{1,D}) \int_D |\nabla \chi_L|$, by decreasing the value of ε' if necessary, in order that $C'\varepsilon' \leq 2$.

To verify the final statement, for $g \in C^1(D)$ let $g_t := g \circ \eta_t^{-1}$, so $g_t - g = g_t - g_t \circ \eta_t$. Then

$$\begin{aligned} \int_D |g_t - g| \, dx &= - \int_D \int_0^1 t w(x) \cdot \nabla g_t(x + tsw(x)) \, ds \, dx \\ &\leq |t| \int_0^1 \int_D \mathbf{a}(x) |\nabla g_t \circ \eta_{ts}| \, dx \, ds \\ &\leq |t| \int_D \mathbf{a}(x) |\nabla g| \, dx + |t|^2 [\mathbf{a}]_{1,D} \int_D |\nabla g| + O(|t|^3), \end{aligned} \tag{3.9}$$

where the last inequality will be derived below. Observe the third bound in the statement of Lemma 3.2 holds for $Q_{\mathbf{a}} = 2(\sup_D \mathbf{a})(1 + [\mathbf{a}]_{1,D}) \int_D |\nabla \chi_L|$, upon decreasing the value of ε' if necessary. An approximation argument justifies estimate (3.9) for $g \in BV_{\mathbf{a}}(\mathbb{R}^d)$ and so in particular for $g = \chi_L$ and $g_t = \chi_L \circ \eta_t^{-1} = \chi_{L_t}$. First, note that

$$\begin{aligned} \int_D \mathbf{a}(x) |\nabla g_t \circ \eta_{ts}| \, dx &= \int_D (\mathbf{a} \circ \eta_{ts}^{-1})(x) |\nabla g_t| |\det D(\eta_{ts}^{-1})| \, dx \\ &\leq (1 + |t| \|H\|_\infty) \int_D (\mathbf{a} \circ \eta_{ts}^{-1})(x) |\nabla(g \circ \eta_t^{-1})| \, dx \\ &\leq (1 + |t| \|H\|_\infty) \int_D \mathbf{a}(\eta_{ts}^{-1} \circ \eta_t)(x) |(D\eta_t)^{-1}| |\nabla g| |\det D\eta_t| \, dx \\ &\leq (1 + |t| \|H\|_\infty)^2 \left(1 + |t| \|\operatorname{div} w\|_\infty \right. \\ & \quad \left. + |t| \|H\|_\infty + O(t^2) \right) \int_D \mathbf{a}(\eta_{ts}^{-1} \circ \eta_t)(x) |\nabla g| \, dx. \end{aligned}$$

Also, it can be checked that $|\mathbf{a}(\eta_{ts}^{-1} \circ \eta_t)(x) - \mathbf{a}(x)| \lesssim [\mathbf{a}]_{1,K} |t|$ for $x \in K$, since we have that $|(\eta_{ts}^{-1} \circ \eta_t)(x) - x| \lesssim |t|(1 + |s|)\|w\|_{L^\infty(K)}$. Hence,

$$\int_D \mathbf{a}(\eta_{ts}^{-1} \circ \eta_t)(x) |\nabla g| \, dx \leq \int_D \mathbf{a}(x) |\nabla g| \, dx + C |t| [\mathbf{a}]_{1,K} \int_D |\nabla g|,$$

where the constant C depends on D only. Recalling (3.7)–(3.8)–(3.9), and the fact that $|t_v| \leq C'|v|$, we can choose

$$Q_{\mathbf{a}} = 2(\sup_D \mathbf{a})(\max\{1, C\} + [\mathbf{a}]_{1,D}) \int_D |\nabla \chi_L|$$

and this concludes the proof. □

Remark 3.3 We note that in the proof of Lemma 3.2 above we only use the Lipschitzianity of \mathbf{a} .

Let us continue with an adaptation of a classical notion in geometric measure theory, to our setting with weight function \mathbf{a} . Given $x \in \mathbb{R}^d$ and $r > 0$ let us define the *weighted relative density function* of the set Ω_γ as

$$h_{\mathbf{a}}(x, r) := \min \left\{ \frac{|\Omega_\gamma \cap B_r(x)|_{\mathbf{a}}}{|B_r(x)|_{\mathbf{a}}}, \frac{|B_r(x) \setminus \Omega_\gamma|_{\mathbf{a}}}{|B_r(x)|_{\mathbf{a}}} \right\}.$$

The rest of the proof is devoted to establishing a uniform lower bound of the form $h_{\mathbf{a}}(x, r) \geq \varepsilon_0 > 0$ for any point $x \in \partial^* \Omega_\gamma$, as long as r is taken sufficiently small. From here we will conclude the boundedness of the minimizer Ω_γ of \mathcal{E}_γ .

Now, in view of the behavior of the density at infinity, $\lim_{|x| \rightarrow \infty} \mathbf{a}(x) = +\infty$, the constant $R_{\mathbf{a}} \geq 1$ in condition (1.4) can be chosen large enough so that $\mathbb{R}^d \setminus B_{R_{\mathbf{a}}} \subset \{x \in \mathbb{R}^d : \mathbf{a}(x) \geq 1\}$. If $\Omega_\gamma \subset B_{2R_{\mathbf{a}}}$, then the minimizer is essentially bounded and we are done. Therefore, in the following we may assume that the total variation measure $|\nabla \chi_{\Omega_\gamma}| = \mathcal{H}^{d-1} \llcorner \partial^* \Omega_\gamma$ is nonvanishing in $\mathbb{R}^d \setminus B_{R_{\mathbf{a}}}$. We may also fix a constant $t_0 \in (0, 1)$ and balls B^1 and B^2 , each of radius t_0 , for which

$$3B^1 \cap B^2 = \emptyset, \quad 3B^1 \cup B^2 \subset\subset \mathbb{R}^d \setminus B_{R_{\mathbf{a}}} \quad \text{and} \quad \int_{B^i} \mathbf{a}(x) |\nabla \chi_{\Omega_\gamma}| > 0 \quad \text{for } i = 1, 2. \tag{3.10}$$

In what follows, B^1 and B^2 are to be used as reference sets, inside of which we will perform small deformations of our minimizer Ω_γ in order to create a competitor set F with $|F| = |\Omega_\gamma|$ (see Lemma 3.2 above with D being B^1 or B^2 , by analyzing two cases). This will allow us to exploit the volume constrained quasi-minimality of Ω_γ with respect to the \mathbf{a} -perimeter, to derive a delicate growth estimate for the weighted relative density function of Ω_γ as a function of the radius r , which will ultimately justify the uniform lower bound on $h_{\mathbf{a}}$ that was claimed above.

Before we continue, let us remark that for densities \mathbf{a} satisfying the assumptions (A1) and (A2a) (hence, in particular, the condition (1.4)), the \mathbf{a} -volume of any two sets $F_1, F_2 \subset\subset B_{2\bar{R}} \setminus B_{\bar{R}}$ are uniformly comparable:

$$C_{\mathbf{a}}^{-1} \frac{|F_1|}{|F_2|} \leq \frac{\inf_{F_1} \mathbf{a} |F_1|}{\sup_{F_2} \mathbf{a} |F_2|} \leq \frac{|F_1|_{\mathbf{a}}}{|F_2|_{\mathbf{a}}} \leq \frac{\sup_{F_1} \mathbf{a} |F_1|}{\inf_{F_2} \mathbf{a} |F_2|} \leq C_{\mathbf{a}} \frac{|F_1|}{|F_2|} \tag{3.11}$$

for any $\bar{R} > R_{\mathbf{a}}$. In particular, for any set $F \subset\subset \mathbb{R}^d \setminus B_{R_{\mathbf{a}}}$,

$$|F| = \int_F 1 \, dx \leq \int_F \mathbf{a}(x) \, dx = |F|_{\mathbf{a}} \tag{3.12}$$

These facts will be used in the following technical results.

The first lemma establishes a bound on the growth rate of the weighted relative density function, for any minimizer Ω_γ of \mathcal{E}_γ , as a function of the radius r on balls having small \mathbf{a} -volume, provided that the set Ω_γ or its complement $\mathbb{R}^d \setminus \Omega_\gamma$ have small density on that ball.

Lemma 3.4 *For every $\gamma > 0$ there exist $0 < \varepsilon' < 1$ with $\max\{1, \gamma^d\} \varepsilon' \ll 1$, and $0 < t_0 < 1$, such that, for any minimizer Ω_γ of \mathcal{E}_γ and for any ball $B_r(x) \subset\subset \mathbb{R}^d \setminus B_{R_a}$ with $0 < r < t_0$ and R_a as in condition (1.4), there holds: If $h_a(x, r) < \varepsilon'$ and $0 < |B_r(x)|_a \leq 1$, then*

$$h_a\left(x, \frac{r}{2}\right) \leq \frac{1}{2} h_a(x, r).$$

Proof Let $x \in \mathbb{R}^d$ and $0 < r < t_0$. Loosely speaking, the main strategy is the following. If $h_a(x, r) = |\Omega_\gamma \cap B_r(x)|_a / |B_r(x)|_a$ is small, we would like to delete the portion of Ω_γ that is inside of $B_{t_*}(x)$ for some $t_* \in (r/2, r)$ appropriately chosen. In a similar fashion, if $h_a(x, r) = |B_r(x) \setminus \Omega_\gamma|_a / |B_r(x)|_a$ is small, we would like to append the ball $B_{t_*}(x)$ to Ω_γ . In these two cases, the resulting set has an additional portion of its boundary located inside $\overline{B_{t_*}}(x)$, when compared to $\Omega_\gamma \cap B_r(x)$ or $B_r(x) \setminus \Omega_\gamma$. As $h_a(x, r)$ is assumed small, we lose in the volume term in (3.4) less than what we are adding on the boundary (F in (3.4) is our resulting set). We must then analyze the contribution of the boundary term in $\overline{B_{t_*}}(x)$. We choose t_* in such a way that we can control $\mathcal{H}_a^{d-1}(\Omega_\gamma \cap \partial B_{t_*}(x))$ in the first case, and $\mathcal{H}_a^{d-1}(\partial B_{t_*}(x) \setminus \Omega_\gamma)$ on the second case, in terms of $|B_r(x)|_a^{(d-1)/d} h_a(x, r)$.

We distinguish four cases.

Case 1: Assume that $h_a(x, r) = |\Omega_\gamma \cap B_r(x)|_a / |B_r(x)|_a$ and $B_r(x) \cap B^1 = \emptyset$, where B^1 denotes the fixed ball in (3.10). By Fubini and Chebyshev inequalities one can find $t_* \in (r/2, r)$ and $C = C(d) > 0$ such that

$$\mathcal{H}_a^{d-1}(\Omega_\gamma \cap \partial B_{t_*}(x)) \leq C \frac{(\sup_{B_r(x)} \mathbf{a})^{1/d}}{|B_r(x)|_a^{1/d}} |\Omega_\gamma \cap B_r(x)|_a, \tag{3.13}$$

Indeed, using Chebyshev’s inequality with $M_\theta := (\theta |B_r(x)|_a^{1/d})^{-1} |\Omega_\gamma \cap B_r(x)|_a$, for $\theta > 0$, yields

$$\begin{aligned} \left| \{t \in (r/2, r) : \mathcal{H}_a^{d-1}(\Omega_\gamma \cap \partial B_t(x)) > M_\theta\} \right| &\leq \frac{1}{M_\theta} \int_{r/2}^r \mathcal{H}_a^{d-1}(\Omega_\gamma \cap \partial B_t(x)) dt \\ &\leq \frac{1}{M_\theta} |\Omega_\gamma \cap (B_r(x) \setminus B_{r/2}(x))|_a \\ &\leq \theta |B_r(x)|_a^{1/d} \leq \theta \left(\sup_{B_r(x)} \mathbf{a} \right)^{1/d} \omega_d^{1/d} r < \frac{r}{2} \end{aligned}$$

for $\theta := (4^d \omega_d \sup_{B_r(x)} \mathbf{a})^{-1/d}$. On the other hand, note that

$$\int_{\mathbb{R}^d} \mathbf{a}(y) |\nabla \chi_{\Omega_\gamma \setminus B_{t_*}(x)}| = \int_{\mathbb{R}^d \setminus B_{t_*}(x)} \mathbf{a}(y) |\nabla \chi_{\Omega_\gamma}| + \mathcal{H}_a^{d-1}(\Omega_\gamma \cap \partial B_{t_*}(x)).$$

With $D = B^1$, as defined in (3.10), and $L = \Omega_\gamma$, we apply Lemma 3.2 to obtain values ε and Q_a satisfying the conclusions of that lemma. The constants ε and Q_a depend on $\Omega_\gamma \cap B^1$ and B^1 only, so, in particular, they are independent of x and r (and t_*). Fix $\varepsilon' \in (0, \varepsilon)$ for the time being, and choose $r \in (0, t_0)$ sufficiently enough so that $h_a(x, r) < \varepsilon'$ and $|B_r(x)| < \varepsilon$. Later on ε' will be reduced accordingly (independent of r).

Let $\Omega' := \Omega_\gamma \setminus B_{t_*}(x)$ and observe, by hypothesis, that $\Omega' \cap B^1 = \Omega_\gamma \cap B^1$. This means that the same constants ε and $Q_{\mathbf{a}}$ still work for Lemma 3.2 when applying it with $L = \Omega'$ and $D = B^1$. Applying Lemma 3.2 to $L = \Omega'$, $D = B^1$ and $v = |\Omega_\gamma \cap B_{t_*}(x)|$ (note that $|v| < |B_r(x)| < \varepsilon$), we find a set $F \subset \mathbb{R}^d$ such that $F = \Omega'$ in a neighborhood of $\mathbb{R}^d \setminus B^1$, $|F| = |\Omega'| + |\Omega_\gamma \cap B_{t_*}(x)| = 1$ and for which

$$\int_{B^1} \mathbf{a}(y) |\nabla \chi_F| \leq \int_{B^1} \mathbf{a}(y) |\nabla \chi_{\Omega'}| + Q_{\mathbf{a}} |\Omega_\gamma \cap B_{t_*}(x)| \tag{3.14}$$

and $|F \Delta \Omega'| \leq Q_{\mathbf{a}} |\Omega_\gamma \cap B_{t_*}(x)|,$

where $Q_{\mathbf{a}} \lesssim (\sup_{B^1} \mathbf{a})(1 + [\mathbf{a}]_{1, B^1})$ is fixed and independent of x and r .

In light of (3.12), we have $|\Omega_\gamma \cap B_{t_*}(x)| \leq |\Omega_\gamma \cap B_{t_*}(x)|_{\mathbf{a}}$, and in addition, using $h_{\mathbf{a}}(x, r) < \varepsilon'$ and $|B_r(x)|_{\mathbf{a}} \leq 1$, we deduce that

$$\begin{aligned} |\Omega_\gamma \cap B_{t_*}(x)|_{\mathbf{a}} &\leq |\Omega_\gamma \cap B_r(x)|_{\mathbf{a}} \\ &= |B_r(x)|_{\mathbf{a}} h_{\mathbf{a}}(x, r) \leq |B_r(x)|_{\mathbf{a}}^{(d-1)/d} h_{\mathbf{a}}(x, r) < \varepsilon'. \end{aligned} \tag{3.15}$$

Since $F = \Omega'$ in a neighborhood of $\mathbb{R}^d \setminus B^1$, we have

$$\int_{\mathbb{R}^d \setminus B^1} \mathbf{a}(y) |\nabla \chi_F| = \int_{\mathbb{R}^d \setminus B^1} \mathbf{a}(y) |\nabla \chi_{\Omega'}|.$$

Combining these facts, we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} \mathbf{a}(y) |\nabla \chi_F| &\leq \int_{\mathbb{R}^d} \mathbf{a}(y) |\nabla \chi_{\Omega'}| + Q_{\mathbf{a}} |\Omega_\gamma \cap B_{t_*}(x)|_{\mathbf{a}} \\ &\stackrel{(3.15)}{\leq} \int_{\mathbb{R}^d} \mathbf{a}(y) |\nabla \chi_{\Omega'}| + Q_{\mathbf{a}} |B_r(x)|_{\mathbf{a}}^{(d-1)/d} h_{\mathbf{a}}(x, r). \end{aligned}$$

In addition,

$$\begin{aligned} \int_{\mathbb{R}^d} \mathbf{a}(y) |\nabla \chi_{\Omega'}| &= \int_{\mathbb{R}^d \setminus B_{t_*}(x)} \mathbf{a}(y) |\nabla \chi_{\Omega_\gamma}| + \mathcal{H}_{\mathbf{a}}^{d-1}(\Omega_\gamma \cap \partial B_{t_*}(x)) \\ &\stackrel{(3.13)}{\leq} \int_{\mathbb{R}^d \setminus B_{t_*}(x)} \mathbf{a}(y) |\nabla \chi_{\Omega_\gamma}| + C \frac{(\sup_{B_r(x)} \mathbf{a})^{1/d}}{|B_r(x)|_{\mathbf{a}}^{1/d}} |\Omega_\gamma \cap B_r(x)|_{\mathbf{a}} \\ &\stackrel{(3.15)}{\leq} \int_{\mathbb{R}^d \setminus B_{t_*}(x)} \mathbf{a}(y) |\nabla \chi_{\Omega_\gamma}| + C (\sup_{B_r(x)} \mathbf{a})^{1/d} |B_r(x)|_{\mathbf{a}}^{(d-1)/d} h_{\mathbf{a}}(x, r). \end{aligned}$$

Assuming, without loss of generality, that $\sup_{B_r(x)} \mathbf{a} \gg 1$, the above estimate yields

$$\int_{\mathbb{R}^d} \mathbf{a}(y) |\nabla \chi_F| \leq \int_{\mathbb{R}^d \setminus B_{t_*}(x)} \mathbf{a}(y) |\nabla \chi_{\Omega_\gamma}| + C (\sup_{B_r(x)} \mathbf{a})^{1/d} |B_r(x)|_{\mathbf{a}}^{(d-1)/d} h_{\mathbf{a}}(x, r) \tag{3.16}$$

On the other hand, recalling (3.12), (3.14), and (3.15), we have

$$\begin{aligned} |F \Delta \Omega_\gamma| &\leq |F \Delta \Omega'| + |\Omega' \Delta \Omega_\gamma| \leq (1 + Q_{\mathbf{a}}) |\Omega_\gamma \cap B_{t_*}(x)| \\ &\leq (1 + Q_{\mathbf{a}}) |\Omega_\gamma \cap B_{t_*}(x)|_{\mathbf{a}} \leq (1 + Q_{\mathbf{a}}) |B_r(x)|_{\mathbf{a}} h_{\mathbf{a}}(x, r) \leq C \varepsilon'. \end{aligned}$$

Recall now that Ω_γ is a volume constrained quasi-minimal set for \mathbf{a} -perimeter with $g_\gamma(x) \simeq \gamma x = o(x^{(d-1)/d})$; see (3.4). By reducing the value of ε' in such a way that $\max\{1, \gamma\} \varepsilon'^{\frac{1}{d}} \ll 1$, we obtain

$$g_\gamma(|F \Delta \Omega_\gamma|) \leq \eta |B_r(x)|_{\mathbf{a}}^{\frac{d-1}{d}} h_{\mathbf{a}}(x, r)^{\frac{d-1}{d}},$$

where η will be chosen below (independent of ε'). Since Ω_γ is a volume constrained quasi-minimizer for the \mathbf{a} -perimeter, the above inequality and (3.16) yield the following estimate:

$$\int_{B_{r_*}(x)} \mathbf{a}(y) |\nabla \chi_{\Omega_\gamma}| \leq \left(C \left(\sup_{B_r(x)} \mathbf{a} \right)^{1/d} h_{\mathbf{a}}(x, r) + \eta h_{\mathbf{a}}(x, r)^{\frac{d-1}{d}} \right) |B_r(x)|_{\mathbf{a}}^{\frac{d-1}{d}}. \tag{3.17}$$

In view of (3.17), using the standard isoperimetric inequality on balls, we obtain that

$$\begin{aligned} & |B_{r/2}(x)|_{\mathbf{a}} h_{\mathbf{a}}(x, r/2) \\ &= \min \{ |\Omega_\gamma \cap B_{r/2}(x)|_{\mathbf{a}}, |B_{r/2}(x) \setminus \Omega_\gamma|_{\mathbf{a}} \} \\ &\leq \sup_{B_{r/2}(x)} \mathbf{a} \cdot \min \{ |\Omega_\gamma \cap B_{r/2}(x)|, |B_{r/2}(x) \setminus \Omega_\gamma| \} \\ &\stackrel{(1.2)}{\leq} (C_{\mathbf{a}} \inf_{B_{r/2}(x)} \mathbf{a}) \cdot C \left(\int_{B_{r/2}(x)} |\nabla \chi_{\Omega_\gamma}| \right)^{\frac{d}{d-1}} \\ &\leq C C_{\mathbf{a}} \left(\inf_{B_{r/2}(x)} \mathbf{a} \right)^{-\frac{1}{d-1}} \left(\int_{B_{r_*}(x)} \mathbf{a}(y) |\nabla \chi_{\Omega_\gamma}| \right)^{\frac{d}{d-1}} \\ &\leq C C_{\mathbf{a}} \left(\inf_{B_{r/2}(x)} \mathbf{a} \right)^{-\frac{1}{d-1}} \left(C \left(\sup_{B_r(x)} \mathbf{a} \right)^{1/d} h_{\mathbf{a}}(x, r) + \eta h_{\mathbf{a}}(x, r)^{\frac{d-1}{d}} \right)^{\frac{d}{d-1}} |B_r(x)|_{\mathbf{a}} \\ &\leq C C_{\mathbf{a}} \left(\inf_{B_{r/2}(x)} \mathbf{a} \right)^{-\frac{1}{d-1}} \left(\left(\sup_{B_r(x)} \mathbf{a} \right)^{\frac{1}{d-1}} h_{\mathbf{a}}(x, r)^{\frac{1}{d-1}} + \eta^{\frac{d}{d-1}} \right) h_{\mathbf{a}}(x, r) |B_r(x)|_{\mathbf{a}} \\ &\leq C C_{\mathbf{a}} \left(\frac{\sup_{B_r(x)} \mathbf{a}}{\inf_{B_{r/2}(x)} \mathbf{a}} \right)^{\frac{1}{d-1}} \left(\varepsilon'^{\frac{1}{d-1}} + \eta^{\frac{d}{d-1}} \right) h_{\mathbf{a}}(x, r) |B_r(x)|_{\mathbf{a}} \\ &\stackrel{(3.11)}{\leq} C C_{\mathbf{a}}^2 \left(\varepsilon'^{\frac{1}{d-1}} + \eta^{\frac{d}{d-1}} \right) h_{\mathbf{a}}(x, r) (2^d C_{\mathbf{a}} |B_{r/2}(x)|_{\mathbf{a}}). \end{aligned}$$

In other words, we conclude that

$$h_{\mathbf{a}} \left(x, \frac{r}{2} \right) \leq C C_{\mathbf{a}}^3 \left(\varepsilon'^{\frac{1}{d-1}} + \eta^{\frac{d}{d-1}} \right) h_{\mathbf{a}}(x, r) \leq \frac{1}{2} h_{\mathbf{a}}(x, r),$$

provided $0 < \eta < (4CC_{\mathbf{a}}^3)^{-\frac{d-1}{d}}$ is chosen small enough; and possibly reducing the value of $\varepsilon' \lesssim \min\{\varepsilon, (4CC_{\mathbf{a}}^3)^{-(d-1)}\}$, independently of x and r .

Case 2: Suppose $h_{\mathbf{a}}(x, r) = |\Omega_\gamma \cap B_r(x)|_{\mathbf{a}} / |B_r(x)|_{\mathbf{a}}$ and $B_r(x) \cap B^1 \neq \emptyset$. Under the hypotheses, $B_r(x) \subset 3B^1$, and (3.10) ensures that $B_r(x) \cap B^2 = \emptyset$. We proceed exactly as in the argument in the first case, with B^1 replaced by B^2 .

Case 3: Assume that $h_{\mathbf{a}}(x, r) = |B_r(x) \setminus \Omega_\gamma|_{\mathbf{a}} / |B_r(x)|_{\mathbf{a}}$ and $B_r(x) \cap B^1 = \emptyset$. We will proceed analogously as the first case, but with

$$\Omega' := \Omega_\gamma \cup B_{t_*}(x)$$

for a certain t_* to be determined below. Again, by Fubini and Chebyshev inequalities, we can find $t_* \in (r/2, r)$ in such a way that

$$\mathcal{H}_{\mathbf{a}}^{d-1}(\partial B_{t_*}(x) \setminus \Omega_\gamma) \leq C \frac{(\sup_{B_r(x)} \mathbf{a})^{1/d}}{|B_r(x)|_{\mathbf{a}}^{1/d}} |B_r(x) \setminus \Omega_\gamma|_{\mathbf{a}}. \tag{3.18}$$

Also, we have that

$$\int_{\mathbb{R}^d} \mathbf{a}(y) |\nabla \chi_{\Omega_\gamma \cup B_{r_*}(x)}| = \int_{\mathbb{R}^d \setminus B_{r_*}(x)} \mathbf{a}(y) |\nabla \chi_{\Omega_\gamma}| + \mathcal{H}_a^{d-1}(\partial B_{r_*}(x) \setminus \Omega_\gamma). \tag{3.19}$$

Fix now ε' as in the previous cases so that $h_a(x, r) < \varepsilon' < \varepsilon$ and $|B_r(x)| < \varepsilon$. Note that (3.12) yields $|\Omega' \setminus \Omega_\gamma| \leq |\Omega' \setminus \Omega_\gamma|_a$. Also, recalling that $|B_r(x)|_a \leq 1$, we have

$$\begin{aligned} |\Omega' \setminus \Omega_\gamma|_a &\leq |B_{r_*}(x) \setminus \Omega_\gamma|_a \leq |B_r(x) \setminus \Omega_\gamma|_a \\ &\leq |B_r(x)|_a h_a(x, r) \leq |B_r(x)|_a^{(d-1)/d} h_a(x, r) < \varepsilon'. \end{aligned}$$

Because of the choice of ε' , we can apply Lemma 3.2 to $L = \Omega'$, $D = B^1$ and $v = -|\Omega' \setminus \Omega_\gamma|$ (with $|v| \leq |B_r(x)| < \varepsilon$) to obtain the existence of a set $F \subset \mathbb{R}^d$ such that $F = \Omega'$ in a neighborhood of $\mathbb{R}^d \setminus B^1$ and $|F| = |\Omega'| - |\Omega' \setminus \Omega_\gamma| = 1$. Moreover, F satisfies the estimates

$$\begin{aligned} \int_{B^1} \mathbf{a}(y) |\nabla \chi_F| &\leq \int_{B^1} \mathbf{a}(y) |\nabla \chi_{\Omega'}| + Q_a |\Omega' \setminus \Omega_\gamma| \\ &\leq \int_{B^1} \mathbf{a}(y) |\nabla \chi_{\Omega'}| + Q_a |B_r(x)|_a^{(d-1)/d} h_a(x, r), \end{aligned}$$

and

$$|F \Delta \Omega'| \leq Q_a |\Omega' \setminus \Omega_\gamma|,$$

where $Q_a \lesssim (\sup_{B^1} \mathbf{a})(1 + [\mathbf{a}]_{1, B^1})$ is fixed, and independent of x and r .

Then, just like in the first case, we deduce

$$\int_{\mathbb{R}^d} \mathbf{a}(y) |\nabla \chi_F| \leq \int_{\mathbb{R}^d} \mathbf{a}(y) |\nabla \chi_{\Omega'}| + C |B_r(x)|_a^{(d-1)/d} h_a(x, r). \tag{3.20}$$

Thus, utilizing (3.19) and (3.18), we can estimate

$$\begin{aligned} \int_{\mathbb{R}^d} \mathbf{a}(y) |\nabla \chi_{\Omega'}| &\leq \int_{\mathbb{R}^d \setminus B_{r_*}(x)} \mathbf{a}(y) |\nabla \chi_{\Omega_\gamma}| + C \frac{(\sup_{B_r(x)} \mathbf{a})^{1/d}}{|B_r(x)|_a^{1/d}} |B_r(x) \setminus \Omega_\gamma|_a \\ &\leq \int_{\mathbb{R}^d \setminus B_{r_*}(x)} \mathbf{a}(y) |\nabla \chi_{\Omega_\gamma}| + C (\sup_{B_r(x)} \mathbf{a})^{1/d} |B_r(x)|_a^{(d-1)/d} h_a(x, r), \end{aligned}$$

which can be combined with (3.20), in turn, to obtain

$$\int_{\mathbb{R}^d} \mathbf{a}(y) |\nabla \chi_F| \leq \int_{\mathbb{R}^d \setminus B_{r_*}(x)} \mathbf{a}(y) |\nabla \chi_{\Omega_\gamma}| + C (\sup_{B_r(x)} \mathbf{a})^{1/d} |B_r(x)|_a^{(d-1)/d} h_a(x, r).$$

On the other hand,

$$\begin{aligned} |F \Delta \Omega_\gamma| &\leq |F \Delta \Omega'| + |\Omega' \Delta \Omega_\gamma| \\ &\leq (Q_a + 1) |\Omega' \setminus \Omega_\gamma| \leq (Q_a + 1) |\Omega' \setminus \Omega_\gamma|_a \\ &\leq (Q_a + 1) |B_r(x)|_a h_a(x, r) \leq C \varepsilon'. \end{aligned}$$

We reach the conclusion like in the first case, utilizing the volume constrained quasi-minimality of Ω_γ , the relative isoperimetric inequality on balls, and the fact that the weight has uniformly bounded oscillation (condition (1.4)).

Case 4: Suppose $h_a(x, r) = |B_r(x) \setminus \Omega_\gamma|_a / |B_r(x)|_a$ and $B_r(x) \cap B^1 \neq \emptyset$. We employ the same construction as in the third case, with B^1 replaced by B^2 , to conclude the proof of this lemma. □

Lemma 3.5 *Given $\gamma > 0$, there exist $0 < \varepsilon_0 < 1$ with $\max\{1, \gamma^d\}\varepsilon_0 \ll 1$, and $0 < t_0 < 1$, such that for any minimizer Ω_γ of \mathcal{E}_γ and any ball $B_r(x) \subset\subset \mathbb{R}^d \setminus B_{R_a}$, for which $0 < r \leq t_0$, and for R_a as in (1.4), the following holds: If $h_a(x, r) < \varepsilon_0$ and $0 < |B_r(x)|_a \leq 1$, then*

$$\text{either } |\Omega_\gamma \cap B_{r/2}(x)| = 0 \text{ or } |B_{r/2}(x) \setminus \Omega_\gamma| = 0.$$

Remark 3.6 We note that, in particular, for any point $x \in \partial^*\Omega_\gamma$ and any ball $B_r(x)$ as above, one has

$$h_a(x, r) = \frac{|\Omega_\gamma \cap B_r(x)|_a}{|B_r(x)|_a} \geq \varepsilon_0 > 0.$$

Proof Under the hypotheses of this lemma, for any $y \in B_{r/2}(x)$ we write $d_k := h_a(y, 2^{-k}r)$. Choose

$$\varepsilon_0 := (2^d C_a)^{-1} \min\{\varepsilon', 1/2\},$$

with $\varepsilon' > 0$ given as in Lemma 3.4. If $h_a(x, r) = |B_r(x)|_a^{-1} |\Omega_\gamma \cap B_r(x)|_a$, then

$$d_1 := |B_{r/2}(x)|_a^{-1} |\Omega_\gamma \cap B_{r/2}(y)|_a \leq 2^d C_a h_a(x, r) < \min\{\varepsilon', 1/2\}$$

where we used the estimate (3.11). Since $d_1 < \varepsilon'$, by Lemma 3.4 we deduce that

$$d_2 = h_a\left(y, \frac{r}{4}\right) \leq \frac{1}{2} h_a\left(y, \frac{r}{2}\right) = \frac{d_1}{2}.$$

In particular, $d_2 < \min\{\varepsilon', 1/2\}$. Hence, by induction it follows that, for all $k \geq 1$, we have $d_{k+1} \leq d_1/2^k$.

Hence, for any $y \in B_{r/2}(x)$, and by virtue of condition (1.4), for all $k \geq 1$, we have

$$\frac{\varepsilon'}{2^{k-1}} \geq \frac{|\Omega_\gamma \cap B_{2^{-k}r}(y)|_a}{|B_{2^{-k}r}(y)|_a} \geq \frac{\inf_{B_r(y)} \mathbf{a}}{\sup_{B_r(y)} \mathbf{a}} \frac{|\Omega_\gamma \cap B_{2^{-k}r}(y)|}{|B_{2^{-k}r}(y)|} \geq C_a^{-1} \frac{|\Omega_\gamma \cap B_{2^{-k}r}(y)|}{|B_{2^{-k}r}(y)|}$$

Thus, y is not a point of density of Ω_γ , which yields $|\Omega_\gamma \cap B_{r/2}(x)| = 0$.

Otherwise, if $h_a(x, r) = r^{-d} |B_r(x) \setminus \Omega_\gamma|_a$, it can be shown in the same fashion that, for any $k \geq 1$ and any $y \in B_{r/2}(x)$,

$$d_k = |B_{2^{-k}r}(y)|_a^{-1} |B_r(y) \setminus \Omega_\gamma|_a < 2^d C_a \varepsilon < \min\{\varepsilon', 1/2\} \quad \text{and} \quad d_{k+1} \leq \frac{d_k}{2}.$$

Therefore,

$$C_a^{-1} \cdot \frac{|B_{2^{-k}r}(y) \setminus \Omega_\gamma|}{|B_{2^{-k}r}(y)|} \leq \frac{|B_{2^{-k}r}(y) \setminus \Omega_\gamma|_a}{|B_{2^{-k}r}(y)|_a} \leq \frac{\varepsilon'}{2^{k-1}} \xrightarrow{k \rightarrow \infty} 0.$$

and so y is not a density point of $\mathbb{R}^d \setminus \Omega_\gamma$. This shows that $|B_{r/2}(x) \setminus \Omega_\gamma| = 0$. □

Proof of Theorem 1.3 (Part 1) We now finish the argument to establish the essential boundedness of the minimizer Ω_γ in Theorem 1.3 under the hypotheses (A1) and (A2a). Let ε_0 and $t_0 > 0$ be given as in Lemma 3.5, and let $j_a \in \mathbb{N}$ be such that $R_a < 2^{j_a}$.

Let $\{R_1, \dots, R_N\} \subset (2^{j_a}, +\infty)$ be any finite collection for which

$$\partial^*\Omega_\gamma \cap (B_{R_{i+1}} \setminus \overline{B_{R_i}}) \neq \emptyset \quad \text{and} \quad R_{i+1} - R_i \geq 1, \quad \text{for all } i.$$

Then we can select balls $B_{r_i}(x^i) \subset\subset B_{R_{i+1}} \setminus B_{R_i}$ with $x^i \in \partial^*\Omega_\gamma$ and $r_i \leq t_0 < 1$ taken sufficiently small so that

$$\frac{1}{2} \leq |B_{r_i}(x^i)|_a < 1.$$

In view of (3.2), and employing Remark 3.6 over each ball, we derive the following bound

$$\begin{aligned}
 c_d C_{\mathbf{a}} \int_{\mathbb{R}^d \setminus B_{2^j \mathbf{a}}} \mathbf{a}(x) |\nabla \chi_{\Omega_\gamma}| &\geq |\Omega_\gamma \setminus B_{2^j \mathbf{a}}|_{\mathbf{a}} \geq \sum_{i=1}^N |\Omega_\gamma \cap (B_{R_{i+1}} \setminus B_{R_i})|_{\mathbf{a}} \\
 &\geq \sum_{i=1}^N |\Omega_\gamma \cap B_{R_i}(x^i)|_{\mathbf{a}} \geq \sum_{i=1}^N \varepsilon_0 |B_{R_i}(x^i)|_{\mathbf{a}} \\
 &\geq \frac{1}{2} \varepsilon_0 N
 \end{aligned}$$

thus proving that the maximal number of such radii $\{R_i\}$ must be finite. This shows that there exists $R_{\max} \in (R_{\mathbf{a}}, +\infty)$, depending on \mathbf{a} , for which $\partial^* \Omega_\gamma \cap (\mathbb{R}^d \setminus B_{R_{\max}}) = \emptyset$. Since Ω_γ is assumed to be essentially closed, then

$$\Omega_\gamma \subset B_{R_{\max}}.$$

This establishes Theorem 1.3 under the hypotheses (A1) and

□ (A2a).

3.2 Radial and monotone densities

Now we turn to densities \mathbf{a} satisfying the assumptions (A1) and (A2b). A key ingredient in the second part of the proof of Theorem 1.3 is the $\varepsilon - \varepsilon^{(d-1)/d}$ property recently proved by Pratelli and Saracco [41]. We state this result as a lemma here for the convenience of the reader.

Lemma 3.7 (cf. Theorem A in [41]) *For any set of finite perimeter $E \subset \mathbb{R}^d$ and for any ball B such that $\mathcal{H}^{d-1}(B \cap \partial^* E) > 0$ there exists $\bar{\varepsilon} > 0$ and $C_E > 0$ such that for any $|\varepsilon| < \bar{\varepsilon}$ there is a set $F \subset \mathbb{R}^d$ satisfying*

$$F \Delta E \subset\subset B, \quad |F| - |E| = \varepsilon, \quad \mathcal{P}_{\mathbf{a}}(F) - \mathcal{P}_{\mathbf{a}}(E) \leq C_E |\varepsilon|^{(d-1)/d}. \quad (3.21)$$

Moreover the constant C_E can be chosen arbitrarily small up to possibly choosing $\bar{\varepsilon}$ smaller.

Now we will finish the proof of Theorem 1.3 following an argument similar to [37, Theorem 5.9] and [32, Lemma 5.1].

Proof of Theorem 1.3 (Part 2) Assume, for a contradiction, that Ω_γ is not bounded. Then $|\Omega_\gamma \setminus B_r| > 0$ for all $r > 0$. In particular, for $r \geq R_{\mathbf{a}}$,

$$|\Omega_\gamma \setminus B_r|_{\mathbf{a}} \geq \mathbf{a}(R_{\mathbf{a}}) |\Omega_\gamma \setminus B_r| > 0.$$

Now let

$$\Omega(r) := \Omega_\gamma \cap B_r, \quad \Omega_r := \Omega_\gamma \cap \partial B_r, \quad \mathcal{P}_{\mathbf{a}}(r) := \mathcal{H}_{\mathbf{a}}^{d-1}(\partial^* \Omega_\gamma \setminus B_r), \quad \text{and} \quad V_{\mathbf{a}}(r) := |\Omega_\gamma \setminus B_r|_{\mathbf{a}}.$$

By the proof of Theorem 4.3 of [37],

$$\mathcal{P}_{\mathbf{a}}(\Omega(r)) < \mathcal{P}_{\mathbf{a}}(\Omega_\gamma) \quad \text{for } r \geq R_{\mathbf{a}}.$$

This follows from a nifty argument involving projection onto the sphere B_r and for the reader's convenience we include the details here. Consider the projection map $\Pi: \partial^* \Omega_\gamma \setminus B_r \rightarrow \partial B_r$. Clearly, Π is strictly 1-Lipschitz and its image $\text{Im}(\Pi)$ satisfies

$$\partial^* \Omega(r) \setminus \partial^* \Omega_\gamma \subset \text{Im}(\Pi).$$

This inclusion would be trivially true for bounded Ω_γ ; however, since we assumed that Ω_γ is unbounded it contains the whole cone $C = \{\lambda x : \lambda \geq 1, x \in H\}$ where $H = (\partial^* \Omega(r) \setminus \partial^* \Omega_\gamma) \setminus \text{Im}(\Pi)$. But since \mathbf{a} is increasing for large values of $|x|$, the cone C , and hence the set Ω_γ , have infinite volume, unless $\mathcal{H}^{d-1}(H) = 0$. Thus $H = \emptyset$ up to \mathcal{H}^{d-1} -measure zero.

Since \mathbf{a} is eventually increasing, for $r \geq R_{\mathbf{a}}$, by the co-area formula we get that

$$\begin{aligned} \mathcal{P}_{\mathbf{a}}(\Omega(r)) &= \int_{\partial^* \Omega_\gamma \cap B_r} \mathbf{a}(|x|) \, d\mathcal{H}^{d-1}(x) + \int_{\partial^* \Omega(r) \setminus \partial^* \Omega_\gamma} \mathbf{a}(\theta) \, d\mathcal{H}^{d-1}(\theta) \\ &< \int_{\partial^* \Omega_\gamma \cap B_r} \mathbf{a}(|x|) \, d\mathcal{H}^{d-1}(x) + \int_{\partial^* \Omega_\gamma \setminus \partial B_r} \mathbf{a}(\Pi(x)) \, d\mathcal{H}^{d-1}(x) \\ &\leq \int_{\partial^* \Omega_\gamma} \mathbf{a}(|x|) \, d\mathcal{H}^{d-1}(x) = \mathcal{P}_{\mathbf{a}}(\Omega_\gamma). \end{aligned}$$

Also, note that

$$\mathcal{P}_{\mathbf{a}}(\Omega(r)) = \mathcal{P}_{\mathbf{a}}(\Omega_\gamma) - \mathcal{P}_{\mathbf{a}}(r) + \mathcal{H}_{\mathbf{a}}^{d-1}(\Omega_r).$$

This, combined with $\mathcal{P}_{\mathbf{a}}(\Omega(r)) < \mathcal{P}_{\mathbf{a}}(\Omega_\gamma)$, implies that

$$\mathcal{P}_{\mathbf{a}}(r) > \mathcal{H}_{\mathbf{a}}^{d-1}(\Omega_r). \tag{3.22}$$

For $r \geq 1$, the standard isoperimetric inequality in the sphere yields that for any subset E_r of the sphere ∂B_r having area at most half of the sphere, we have $\mathcal{H}^{d-2}(\partial^* E_r) \geq c(\mathcal{H}^{d-1}(E_r))^{(d-2)/(d-1)}$ for some constant $c > 0$ where $\partial^* E_r$ denotes the reduced boundary of E_r inside ∂B_r . In fact, as Ω_γ has finite volume, for r sufficiently large, $\mathcal{H}^{d-1}(\Omega_r) \leq \frac{1}{2} \mathcal{H}^{d-1}(\partial B_r)$. Since the density \mathbf{a} is constant on ∂B_r , we get

$$\mathcal{H}_{\mathbf{a}}^{d-2}(\partial^* \Omega_r) \geq c \left(\mathcal{H}_{\mathbf{a}}^{d-1}(\Omega_r) \right)^{\frac{d-2}{d-1}} (\mathbf{a}(r))^{\frac{1}{d-1}}. \tag{3.23}$$

Combining (3.22) and (3.23), we obtain

$$(\mathcal{P}_{\mathbf{a}}(r))^{\frac{1}{d-1}} \mathcal{H}_{\mathbf{a}}^{d-2}(\partial^* \Omega_r) > (\mathcal{H}_{\mathbf{a}}^{d-1}(\Omega_r))^{\frac{1}{d-1}} \mathcal{H}_{\mathbf{a}}^{d-2}(\partial^* \Omega_r) \geq c \mathcal{H}_{\mathbf{a}}^{d-1}(\Omega_r) (\mathbf{a}(r))^{\frac{1}{d-1}}.$$

Since $\mathbf{a}(r) \geq \mathbf{a}(R_{\mathbf{a}})$ for $r \geq R_{\mathbf{a}}$, the estimate above becomes

$$\mathcal{H}_{\mathbf{a}}^{d-2}(\partial^* \Omega_r) \geq c (\mathcal{P}_{\mathbf{a}}(r))^{-\frac{1}{d-1}} \mathcal{H}_{\mathbf{a}}^{d-1}(\Omega_r). \tag{3.24}$$

Also, observe that

$$-\frac{d}{dr} \mathcal{P}_{\mathbf{a}}(r) = \left| \frac{d}{dr} \mathcal{P}_{\mathbf{a}}(r) \right| \geq \mathcal{H}_{\mathbf{a}}^{d-2}(\partial^* \Omega_r) \quad \text{and} \quad -\frac{d}{dr} V_{\mathbf{a}}(r) = \mathcal{H}_{\mathbf{a}}^{d-1}(\Omega_r). \tag{3.25}$$

Hence, by (3.24), we get

$$-\frac{d}{dr} \left((\mathcal{P}_{\mathbf{a}}(r))^{\frac{d}{d-1}} \right) \geq -c \frac{d}{dr} V_{\mathbf{a}}(r).$$

Since both $\mathcal{P}_{\mathbf{a}}(r), V_{\mathbf{a}}(r) \rightarrow 0$ as $r \rightarrow \infty$, integrating both sides from r to ∞ yields

$$(\mathcal{P}_{\mathbf{a}}(r))^{\frac{d}{d-1}} \geq c V_{\mathbf{a}}(r). \tag{3.26}$$

Now, let $R \in \mathbb{R}$ be such that $\Omega_\gamma \cap B_R \neq \emptyset$. Then by Lemma 3.7, there exists $\bar{\varepsilon}$ and $C_{\Omega_\gamma} > 0$ such that for all $|\varepsilon| < \bar{\varepsilon}$ there exists Ω_ε satisfying

$$\Omega_\varepsilon \Delta \Omega_\gamma \subset\subset B_R, \quad |\Omega_\varepsilon| - |\Omega_\gamma| = \varepsilon, \quad \mathcal{P}_a(\Omega_\varepsilon) - \mathcal{P}_a(\Omega_\gamma) \leq C_{\Omega_\gamma} |\varepsilon|^{(d-1)/d}. \tag{3.27}$$

By choosing ε smaller, if necessary, by Lemma 3.2 and Remark 3.3, we can take Ω_ε such that

$$|\Omega_\varepsilon \Delta \Omega_\gamma| < C\varepsilon. \tag{3.28}$$

Let $r > R$ be such that $\varepsilon := V_a(r) < \bar{\varepsilon}$ and define $\Omega_\varepsilon(r) = \Omega_\varepsilon \cap B_r$. Then $|\Omega_\varepsilon(r)| = |\Omega_\gamma| = 1$ and $\Omega_\varepsilon(r)$ is competitor in (1.1). So,

$$\begin{aligned} \mathcal{P}_a(\Omega_\varepsilon(r)) &= \mathcal{P}_a(\Omega_\varepsilon) - \mathcal{P}_a(r) + \mathcal{H}_a^{d-1}(\Omega_r) \\ &\stackrel{(3.26), (3.27)}{\leq} \mathcal{P}_a(\Omega_\gamma) + C_{\Omega_\gamma} \varepsilon^{\frac{d-1}{d}} - c \varepsilon^{\frac{d-1}{d}} + \mathcal{H}_a^{d-1}(\Omega_r) \\ &\leq \mathcal{P}_a(\Omega_\varepsilon(r)) + \gamma \left(\mathcal{V}(\Omega_\varepsilon(r)) - \mathcal{V}(\Omega_\gamma) \right) + C_{\Omega_\gamma} \varepsilon^{\frac{d-1}{d}} - c \varepsilon^{\frac{d-1}{d}} + \mathcal{H}_a^{d-1}(\Omega_r) \\ &\stackrel{(3.3)}{\leq} \mathcal{P}_a(\Omega_\varepsilon(r)) + c\gamma |\Omega_\varepsilon(r) \Delta \Omega_\gamma| + C_{\Omega_\gamma} \varepsilon^{\frac{d-1}{d}} - c \varepsilon^{\frac{d-1}{d}} + \mathcal{H}_a^{d-1}(\Omega_r). \end{aligned}$$

Since, as noted in Lemma 3.7, we can choose C_{Ω_γ} arbitrarily small, and since (3.27) and (3.28) hold, for ε sufficiently small, we have

$$\mathcal{H}_a^{d-1}(\Omega_r) \geq C\varepsilon^{\frac{d-1}{d}}$$

for some $C > 0$. Then, by (3.25), $-\frac{d}{dr} V_a(r) \geq C\varepsilon^{(d-1)/d} = C(V_a(r))^{(d-1)/d}$, i.e.,

$$\frac{d}{dr} \left((V_a(r))^{1/d} \right) \leq -C.$$

This contradicts the fact that $V_a(r) > 0$ for all $r > R_a$ and we establish Theorem 1.3 under the hypotheses (A1) and (A2b). □

4 Global minimizers in the small γ regime

In this section we will prove Theorem 1.6. The proof of this theorem relies on the regularity of almost-minimizers of the perimeter functional. As in [1, Definition 2.5], we will call a set in $E \subset \mathbb{R}^d$ an ω -minimizer of the Euclidean perimeter functional \mathcal{P} in an open set $O \subset \mathbb{R}^d$ with $\omega > 0$ if for every ball $B_r(x) \subset O$ and for every set of finite perimeter $F \subset \mathbb{R}^d$ such that $E \Delta F \subset\subset B_r(x)$ we have

$$\mathcal{P}(E) \leq \mathcal{P}(F) + \omega r^d.$$

This definition also coincides with the definition of almost-minimizers according to [45, Section 1.5] with $\alpha(r) = \omega r$, although it is weaker than the (ω, r) -minimizers as defined in [35]. Nevertheless, by the regularity theory developed in [45], it will be sufficient to justify passage to a limit as $\gamma \rightarrow 0$.

First we show that any minimizer of the energy \mathcal{E}_γ is an ω -minimizer of the Euclidean perimeter functional \mathcal{P} in any open set that does not contain the origin.

Lemma 4.1 *Let $\Omega_\gamma \subset \mathbb{R}^d$ be a solution of the problem (1.1). Then Ω_γ is ω -minimal for the Euclidean perimeter functional \mathcal{P} in $O = \mathbb{R}^d \setminus \overline{B_\delta}$ for any $\delta > 0$.*

Proof We first note that the constraint $|\Omega_\gamma| = 1$ may be replaced by a penalization term in the energy. Namely, arguing as in Step 1 of the proof of [7, Theorem 2.7] we obtain that there exists a constant $\lambda > 0$ such that

$$\min \mathcal{F}_\gamma^\lambda = \mathcal{F}_\gamma^\lambda(\Omega_\gamma) = \mathcal{E}_\gamma(\Omega_\gamma)$$

where

$$\mathcal{F}_\gamma^\lambda(E) = \mathcal{E}_\gamma(E) + \lambda||E| - 1|$$

for any $E \subset \mathbb{R}^d$.

Fix $r > 0$ such that $B_r(x) \cap B_\delta = \emptyset$ and let $F \subset \mathbb{R}^d$ be any set such that $\Omega_\gamma \Delta F \subset\subset B_r(x)$. Since $\mathcal{E}_\gamma(\Omega_\gamma) = \mathcal{F}_\gamma^\lambda(\Omega_\gamma) \leq \mathcal{F}_\gamma^\lambda(F)$, using (3.3) we obtain that

$$\mathcal{P}_a(\Omega_\gamma) \leq \mathcal{P}_a(F) + \omega|\Omega_\gamma \Delta F|$$

for some constant ω depending only on γ and λ .

Let $m := \max\{\mathbf{a}(y) : y \in B_r(x)\}$. Since \mathbf{a} is Lipschitz there exists a constant $C > 0$ such that $\mathbf{a}(y) \geq m - Cr$ for all $y \in B_r(x)$. Then

$$\begin{aligned} (m - Cr)\mathcal{P}(\Omega_\gamma, B_r(x)) &\leq \mathcal{P}_a(\Omega_\gamma, B_r(x)) \leq \mathcal{P}_a(F, B_r(x)) + \omega|\Omega_\gamma \Delta F| \\ &\leq m\mathcal{P}(F, B_r(x)) + \omega|\Omega_\gamma \Delta F|. \end{aligned}$$

Hence,

$$\mathcal{P}(\Omega_\gamma, B_r(x)) \leq \mathcal{P}(F, B_r(x)) + Cr^d$$

provided $\mathcal{P}(\Omega_\gamma, B_r(x)) \leq Cr^{d-1}$. This establishes that Ω_γ is an ω -minimizer of the Euclidean perimeter in $\mathbb{R}^d \setminus \overline{B_\delta}$.

It now remains to prove that $\mathcal{P}(\Omega_\gamma, B_r(x)) \leq Cr^{d-1}$. Note that Ω_γ satisfies the $\varepsilon - \varepsilon^{(d-1)/d}$ property given by (3.21) in Lemma 3.7. Let B^1 and B^2 be two disjoint balls, intersecting $\partial^* \Omega_\gamma$ in a set of positive \mathcal{H}^{d-1} -measure. Let $C_1, C_2 > 0$ and $0 < \varepsilon_1, \varepsilon_2 < 1$ be the corresponding constants satisfying the conditions in (3.21). Take $C = \max\{C_1, C_2\}$, $\bar{\varepsilon} = \min\{\varepsilon_1, \varepsilon_2\}$, and $\bar{r} = \min\{(\bar{\varepsilon}/\omega_d)^{1/d}, \text{dist}(B^1, B^2)\}$.

If $r < \bar{r}$, then $\varepsilon := |B_r(x) \cap \Omega_\gamma| < \omega_d r^d \leq \bar{\varepsilon}$, and $B_r(x)$ cannot intersect both B^1 and B^2 . Without loss of generality, assume that $B_r(x) \cap B^1 = \emptyset$. For this ε , let $F \subset \mathbb{R}^d$ be the set of finite perimeter satisfying (3.21). Clearly, we also have

$$\mathcal{P}_a(F) \leq \mathcal{P}_a(\Omega_\gamma) + C\varepsilon \leq \mathcal{P}_a(\Omega_\gamma) + C\omega_d r^d.$$

Now, let $G = F \setminus B_r(x)$. Then $|G| = |\Omega_\gamma| = 1$ and G is admissible for (1.1). By minimality of Ω_γ and Lipschitzianity of \mathcal{V} (see (3.3)) we obtain

$$\begin{aligned} \mathcal{P}_a(\Omega_\gamma) &\leq \mathcal{P}_a(G) + \gamma(\mathcal{V}(G) - \mathcal{V}(\Omega_\gamma)) \\ &\leq \mathcal{P}_a(F) - \mathcal{P}_a(\Omega_\gamma, B_r(x)) + d\omega_d R_{\max}^p r^{d-1} + C\gamma|G \Delta \Omega_\gamma| \\ &\leq \mathcal{P}_a(\Omega_\gamma) + C\omega_d r^d - \mathcal{P}_a(\Omega_\gamma, B_r(x)) + d\omega_d R_{\max}^p r^{d-1} + C\gamma r^d \\ &\leq \mathcal{P}_a(\Omega_\gamma) - \mathcal{P}_a(\Omega_\gamma, B_r(x)) + Cr^{d-1} \end{aligned}$$

with a constant depending only on d, R_{\max} , and γ , where R_{\max} is the constant obtained in the proof of Theorem 1.3. Recalling that $\mathbf{a}(x) \geq \delta^p$ in $\mathbb{R}^d \setminus \overline{B_\delta}$ we have that

$$\mathcal{P}(\Omega_\gamma, B_r(x)) \leq \delta^{-p} \mathcal{P}_a(\Omega_\gamma, B_r(x)) \leq Cr^{d-1}.$$

If $r \geq \bar{r}$, on the other hand, then

$$\mathcal{P}(\Omega_\gamma, B_r(x)) \leq \delta^{-p} \mathcal{P}_a(\Omega_\gamma, B_r(x)) \leq \delta^{-p} \mathcal{P}_a(\Omega_\gamma) \leq \frac{\delta^{-p} \mathcal{P}_a(\Omega_\gamma)}{\bar{r}^{d-1}} r^{d-1}.$$

Hence, in either case we obtain that $\mathcal{P}(\Omega_\gamma, B_r(x)) \leq Cr^{d-1}$, completing the proof of the lemma. \square

Since the density $\mathbf{a}(x) = |x|^p$ is vanishing at zero we cannot conclude that Ω_γ is an ω -minimizer of the perimeter in \mathbb{R}^d . Nevertheless, combining [45, Sections 1.9 and 1.10] (see also [4,46]), we obtain the following regularity result for any ω -minimizer of perimeter in open sets.

Lemma 4.2 *Let $O \subset \mathbb{R}^d$ be an open set.*

- (i) *If $E \subset \mathbb{R}^d$ is an ω -minimizer of Euclidean perimeter in O then $\partial^*E \cap O$ is a $C^{1,\alpha}$ hypersurface for any $\alpha \in (0, 1/2)$.*
- (ii) *If $E_n \subset \mathbb{R}^d$ is a sequence of uniformly ω -minimizers of Euclidean perimeter in O and if $E_n \rightarrow E$ in $L^1(O)$, then $\partial E_n \rightarrow \partial E$ in $C^{1,\alpha}$ and ∂E_n is a $C^{1,\alpha}$ graph over ∂E .*

We are now ready to present the proof of global minimality of balls in the small γ regime.

Proof of Theorem 1.6 We fix $r_0 = (1/\omega_d)^{1/d}$ with $|B_{r_0}| = 1$, and recall (see e.g. [5,10]) that B_{r_0} is the unique minimizer of the local weighted isoperimetric problem $e(0)$. By Theorem 1.1, for any $\gamma > 0$, there exists $\Omega_\gamma \subset \mathbb{R}^d$, $|\Omega_\gamma| = 1$, minimizing \mathcal{E}_γ . We first claim that $\Omega_\gamma \rightarrow B_{r_0}$ in $L^1(\mathbb{R}^d)$. Indeed, clearly $e(0) \leq \mathcal{E}_0(\Omega_\gamma) \leq \mathcal{E}_\gamma(\Omega_\gamma)$ for all $\gamma > 0$, while on the other hand,

$$\limsup_{\gamma \rightarrow 0} \mathcal{E}_\gamma(\Omega_\gamma) \leq \limsup_{\gamma \rightarrow 0} \mathcal{E}_\gamma(B_{r_0}) = \mathcal{E}_0(B_{r_0}) = e(0).$$

Thus, $\{\Omega_\gamma\}_{\gamma>0}$ give minimizing sequences for $e(0)$. By Theorem 1.1 it follows that for every sequence $\gamma_n \rightarrow 0$ there is a subsequence along which $\{\Omega_{\gamma_n}\}$ is compact in $L^1(\mathbb{R}^d)$, and converges to the minimizer B_{r_0} of $e(0)$. As the minimizer of the limit problem is unique, we conclude $\Omega_\gamma \rightarrow B_{r_0}$ in $L^1(\mathbb{R}^d)$ as claimed.

Next, by Lemma 4.1 Ω_γ is an ω -minimizer of the Euclidean perimeter in $O = \mathbb{R}^d \setminus \overline{B_\delta}$ for any $\delta > 0$; hence, by Lemma 4.2(ii) $\partial^*\Omega_\gamma \cap O$ is a $C^{1,\alpha}$ -graph over the limit set B_{r_0} . Note in particular that Lemma 4.2(ii) assures that for any $\delta > 0$, there exists $\gamma_\delta > 0$ so that

$$\partial^*\Omega_\gamma \cap (B_{r_0-\delta} \setminus \overline{B_\delta}) = \emptyset \tag{4.1}$$

for all $0 < \gamma < \gamma_\delta$. However, since we cannot conclude quasi-minimality (hence, regularity) of Ω_γ in the whole space we cannot *a priori* assume that $\partial\Omega_\gamma \cap B_\delta$ is empty for all $\delta > 0$. Assume for a contradiction that for some $\delta > 0$, $\partial\Omega_\gamma \cap B_\delta$ is non-empty; from (4.1) we may conclude that $|\Omega_\gamma^c \cap B_\delta| > 0$ for all $0 < \gamma < \gamma_\delta$, and hence Ω_γ is multiply-connected, and its boundary is disconnected into disjoint components. That is, $\Omega_\gamma = \tilde{\Omega}_\gamma \setminus \Omega_\gamma^0$, where $\tilde{\Omega}_\gamma$ is simply connected, with smooth boundary, and $\Omega_\gamma^0 := \Omega_\gamma^c \cap B_\delta$, and thus $\partial^*\Omega_\gamma = \partial\tilde{\Omega}_\gamma \cup \partial^*\Omega_\gamma^0$, a disjoint union. We define $m_\gamma := |\Omega_\gamma^0|$. As $|\Omega_\gamma \Delta B_{r_0}| \rightarrow 0$ it follows that $m_\gamma \rightarrow 0$ and $|\tilde{\Omega}_\gamma| = 1 + m_\gamma$. Moreover, let $R_\gamma = \left(\frac{1+m_\gamma}{\omega_d}\right)^{1/d}$ so that $|B_{R_\gamma}| = 1 + m_\gamma$. In fact, using a Fuglede-type argument (cf. [23]), we will prove that $m_\gamma = 0$ for γ sufficiently small, and therefore $\Omega_\gamma = B_{r_0}$ for all small γ .

Since $\partial\tilde{\Omega}_\gamma$ is a graph over ∂B_{R_0} we can think of it as a graph over ∂B_{R_γ} given by

$$\partial\tilde{\Omega}_\gamma = \{\tilde{x}(1 + \psi_\gamma(\tilde{x})) : \tilde{x} \in \partial B_{R_\gamma}\}$$

for some $\psi_\gamma \in C^{1,\alpha}(\partial B_{R_\gamma})$ such that $\int_{\partial B_{R_\gamma}} \psi_\gamma \, d\mathcal{H}^{d-1} = 0$. Now, we can write both $\partial\tilde{\Omega}_\gamma$ and ∂B_{R_γ} over the unit sphere \mathbb{S}^{d-1} . Namely,

$$\partial\tilde{\Omega}_\gamma = \{R_\gamma x(1 + u_\gamma(x)) : x \in \mathbb{S}^{d-1}\} \quad \text{and} \quad \partial B_{R_\gamma} = \{R_\gamma x : x \in \mathbb{S}^{d-1}\},$$

where $u_\gamma(x) = \psi_\gamma(R_\gamma x)$. Note that $\int_{\mathbb{S}^{d-1}} u_\gamma \, d\mathcal{H}^{d-1} = R_\gamma^{1-d} \int_{\partial B_{R_\gamma}} \psi_\gamma \, d\mathcal{H}^{d-1} = 0$. Computing the perimeter, we have

$$\mathcal{P}_a(\tilde{\Omega}_\gamma) = R_\gamma^{d-1} \int_{\mathbb{S}^{d-1}} \mathbf{a}(R_\gamma x(1 + u_\gamma(x)))(1 + u_\gamma)^{d-1} \sqrt{1 + \frac{|\nabla_\tau u_\gamma|^2}{(1 + u_\gamma)^2}} \, d\mathcal{H}^{d-1},$$

where ∇_τ denotes the gradient with respect to \mathbb{S}^{d-1} . Likewise,

$$\mathcal{P}_a(B_{R_\gamma}) = R_\gamma^{d-1} \int_{\mathbb{S}^{d-1}} \mathbf{a}(R_\gamma x) \, d\mathcal{H}^{d-1}.$$

Putting these two together,

$$\begin{aligned} \mathcal{P}_a(\tilde{\Omega}_\gamma) - \mathcal{P}_a(B_{R_\gamma}) &= R_\gamma^{d-1} \int_{\mathbb{S}^{d-1}} (1 + u_\gamma)^{d-1} \mathbf{a}(R_\gamma x(1 + u_\gamma(x))) \left[\sqrt{1 + \frac{|\nabla_\tau u_\gamma|^2}{(1 + u_\gamma)^2}} - 1 \right] \, d\mathcal{H}^{d-1} \\ &\quad + R_\gamma^{d-1} \int_{\mathbb{S}^{d-1}} \left[(1 + u_\gamma)^{d-1} \mathbf{a}(R_\gamma x(1 + u_\gamma(x))) - \mathbf{a}(R_\gamma x) \right] \, d\mathcal{H}^{d-1} \\ &=: R_\gamma^{d-1} I_1 + R_\gamma^{d-1} I_2. \end{aligned}$$

Now we will estimate I_1 and I_2 . Let $\varepsilon > 0$ be small so that $1 + u_\gamma \geq 1/2$ and $\|u_\gamma\|_{C^1(\mathbb{S}^{d-1})} \leq \varepsilon$ for γ sufficiently small. Using the estimate $\sqrt{1+t} \geq 1 + \frac{t}{2} - \frac{t^2}{8}$, we get that

$$\begin{aligned} I_1 &\geq \int_{\mathbb{S}^{d-1}} R_\gamma^p (1 + u_\gamma)^{p+d-1} \left[\frac{1}{2} \frac{|\nabla_\tau u_\gamma|^2}{(1 + u_\gamma)^2} - \frac{1}{8} \frac{|\nabla_\tau u_\gamma|^4}{(1 + u_\gamma)^4} \right] \, d\mathcal{H}^{d-1} \\ &\geq \left(\frac{1}{2} - C\varepsilon \right) \int_{\mathbb{S}^{d-1}} R_\gamma^p (1 + u_\gamma)^{p+d-3} |\nabla_\tau u_\gamma|^2 \, d\mathcal{H}^{d-1} \\ &\geq C \int_{\mathbb{S}^{d-1}} |\nabla_\tau u_\gamma|^2 \, d\mathcal{H}^{d-1} \end{aligned}$$

for ε sufficiently small.

Since $\int_{\mathbb{S}^{d-1}} u_\gamma \, d\mathcal{H}^{d-1} = 0$ and $(1+t)^q \geq 1+qt$, we have

$$I_2 = \int_{\mathbb{S}^{d-1}} R_\gamma^p ((1 + u_\gamma)^{p+d-1} - 1) \, d\mathcal{H}^{d-1} \geq 0.$$

These two estimates imply that

$$\mathcal{P}_a(\tilde{\Omega}_\gamma) - \mathcal{P}_a(B_{R_\gamma}) \geq C \|u_\gamma\|_{H^1(\mathbb{S}^{d-1})}^2 \tag{4.2}$$

for some constant $C > 0$ independent of γ .

As for the nonlocal term, [20, Lemma 5.3] (see also [25, Remark 3.2]) implies that

$$\mathcal{V}(B_{R_\gamma}) - \mathcal{V}(\tilde{\Omega}_\gamma) \leq C \|u_\gamma\|_{H^1(\mathbb{S}^{d-1})}^2.$$

Furthermore,

$$\mathcal{V}(B_{R_\gamma} \setminus \Omega_\gamma^0) - \mathcal{V}(\Omega_\gamma) \leq \mathcal{V}(B_{R_\gamma}) - \mathcal{V}(\tilde{\Omega}_\gamma) + Cm_\gamma \leq C(\|u_\gamma\|_{H^1(\mathbb{S}^{d-1})}^2 + m_\gamma). \tag{4.3}$$

To see the first estimate above, let $\bar{R} > 0$ be large enough such that $\Omega_\gamma, B_{R_\gamma} \subset B_{\bar{R}}$ for all $\gamma > 0$. For $\mathcal{I}(E, F) = \int_E \int_F |x - y|^{-\alpha} dx dy$, we have

$$\mathcal{V}(B_{R_\gamma} \setminus \Omega_\gamma^0) - \mathcal{V}(\Omega_\gamma) = \mathcal{V}(B_{R_\gamma}) - \mathcal{V}(\tilde{\Omega}_\gamma) + 2\mathcal{I}(\Omega_\gamma^0, \tilde{\Omega}_\gamma) - 2\mathcal{I}(\Omega_\gamma^0, B_{R_\gamma}).$$

Since

$$\begin{aligned} |\mathcal{I}(\Omega_\gamma^0, \tilde{\Omega}_\gamma) - \mathcal{I}(\Omega_\gamma^0, B_{R_\gamma})| &\leq \int_{\Omega_\gamma^0} \int_{\tilde{\Omega}_\gamma \Delta B_{R_\gamma}} \frac{1}{|x - y|^\alpha} dx dy \\ &\leq |\Omega_\gamma^0| \sup_{x \in B_{\bar{R}}} \int_{\tilde{\Omega}_\gamma \Delta B_{R_\gamma}} \frac{1}{|x - y|^\alpha} dx dy \leq Cm_\gamma, \end{aligned}$$

the first estimate in (4.3) follows.

Again, using minimality of Ω_γ , i.e. $\mathcal{E}_\gamma(\Omega_\gamma) \leq \mathcal{E}_\gamma(B_{R_\gamma} \setminus \Omega_\gamma^0)$, and the lower and upper bounds (4.2) and (4.3) we get that

$$\begin{aligned} C\|u_\gamma\|_{H^1(\mathbb{S}^{d-1})}^2 &\leq \mathcal{P}_a(\tilde{\Omega}_\gamma) - \mathcal{P}_a(B_{R_\gamma}) = \mathcal{P}_a(\Omega_\gamma) - \mathcal{P}_a(B_{R_\gamma} \setminus \Omega_\gamma^0) \\ &\leq \gamma (\mathcal{V}(B_{R_\gamma} \setminus \Omega_\gamma^0) - \mathcal{V}(\Omega_\gamma)) \leq C\gamma (\|u_\gamma\|_{H^1(\mathbb{S}^{d-1})}^2 + m_\gamma). \end{aligned}$$

Thus, for γ sufficiently small,

$$\|u_\gamma\|_{H^1(\mathbb{S}^{d-1})}^2 \leq C\gamma m_\gamma. \tag{4.4}$$

This, in turn, implies that

$$\mathcal{E}_\gamma(B_{R_\gamma} \setminus \Omega_\gamma^0) - \mathcal{E}_\gamma(\Omega_\gamma) \leq C\gamma m_\gamma. \tag{4.5}$$

The last step in the argument is a lower bound on the above difference. In fact, we will show that replacing the domain $\Omega_\gamma = \tilde{\Omega}_\gamma \setminus \Omega_\gamma^0$ by the sphere B_{r_0} results in a much larger energy difference, due to the reduction of the radius R_γ to r_0 . We observe this right away, in estimating the difference in perimeter, $\mathcal{P}_a(B_{R_\gamma}) + \mathcal{P}_a(\Omega_\gamma^0) - \mathcal{P}_a(B_{r_0})$. Let $\rho_\gamma = (m_\gamma/\omega_d)^{1/d}$ and note that

$$R_\gamma = (r_0^d + \rho_\gamma^d)^{1/d} = r_0 \left(1 + \left(\frac{\rho_\gamma}{r_0} \right)^d \right)^{1/d}.$$

Now, for m_γ (hence, for ρ_γ/ω_d) small, we have

$$\begin{aligned} \mathcal{P}_a(B_{R_\gamma}) + \mathcal{P}_a(\Omega_\gamma^0) - \mathcal{P}_a(B_{r_0}) &\geq \mathcal{P}_a(B_{R_\gamma}) + \mathcal{P}_a(B_{\rho_\gamma}) - \mathcal{P}_a(B_{r_0}) \\ &= d\omega_d(R_\gamma^{d-1+p} + \rho_\gamma^{d-1+p} - r_0^{d-1+p}) \\ &= d\omega_d r_0^{d-1+p} \left(\left[1 + \left(\frac{\rho_\gamma}{r_0} \right)^d \right]^{(d-1+p)/d} + \left(\frac{\rho_\gamma}{r_0} \right)^{d-1+p} - 1 \right) \\ &= d\omega_d r_0^{p-1} \left(1 + \frac{p-1}{d} \right) \rho_\gamma^d + O(\rho_\gamma^{d-1+p}) \\ &= \bar{C} m_\gamma + o(m_\gamma). \end{aligned}$$

Combining this estimate with (4.3) and (4.4), and recalling that the ball maximizes the Riesz potential, for sufficiently small γ , we get

$$\begin{aligned} \mathcal{E}_\gamma(B_{R_\gamma} \setminus \Omega_\gamma^0) - \mathcal{E}_\gamma(B_{r_0}) &= \mathcal{P}_a(B_{R_\gamma}) + \mathcal{P}_a(\Omega_\gamma^0) - \mathcal{P}_a(B_{r_0}) + \gamma(\mathcal{V}(B_{R_\gamma} \setminus \Omega_\gamma^0) - \mathcal{V}(B_{r_0})) \\ &\geq (\bar{C} - \gamma C)m_\gamma \geq Cm_\gamma. \end{aligned} \quad (4.6)$$

Now (4.5) and (4.6) imply that

$$Cm_\gamma + \mathcal{E}_\gamma(B_{r_0}) \leq \mathcal{E}_\gamma(B_{R_\gamma} \setminus \Omega_\gamma^0) \leq \mathcal{E}_\gamma(\Omega_\gamma) + C\gamma m_\gamma \leq \mathcal{E}_\gamma(B_{r_0}) + C\gamma m_\gamma.$$

Hence, for γ sufficiently small, $Cm_\gamma \leq 0$, i.e., $m_\gamma = 0$. This implies that $\Omega_\gamma^0 = \emptyset$ for γ small and that $\partial\Omega_\gamma$ is a graph over ∂B_{r_0} .

Therefore, running through the Fuglede-type argument one more time, we get

$$C\|u_\gamma\|_{H^1(\mathbb{S}^{d-1})}^2 \leq \mathcal{P}_a(\Omega_\gamma) - \mathcal{P}_a(B_{r_0}) \leq \gamma(\mathcal{V}(B_{r_0}) - \mathcal{V}(\Omega_\gamma)) \leq C\gamma\|u_\gamma\|_{H^1(\mathbb{S}^{d-1})}^2.$$

Hence, for γ sufficiently small $u_\gamma \equiv 0$, i.e., $\Omega_\gamma = B_{r_0}$. \square

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References

1. Acerbi, E., Fusco, N., Morini, M.: Minimality via second variation for a nonlocal isoperimetric problem. *Commun. Math. Phys.* **322**(2), 515–557 (2013). <https://doi.org/10.1007/s00220-013-1733-y>
2. Adams, R.A., Fournier, J.J.F.: *Sobolev Spaces. Pure and Applied Mathematics (Amsterdam)*, vol. 140, 2nd edn. Elsevier, Academic Press, Amsterdam (2003)
3. Alama, S., Bronsard, L., Choksi, R., Topaloglu, I.: Droplet breakup in the liquid drop model with background potential. *Commun. Contemp. Math.* **21**(3), 1850022 (2019). <https://doi.org/10.1142/S0219199718500220>
4. Almgren, F.J.J.: Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints. *Mem. Am. Math. Soc.* **165**(4), 199 (1976)
5. Alvino, A., Brock, F., Chiacchio, F., Mercaldo, A., Posteraro, M.R.: Some isoperimetric inequalities on \mathbb{R}^N with respect to weights $|x|^\alpha$. *J. Math. Anal. Appl.* **451**(1), 280–318 (2017). <https://doi.org/10.1016/j.jmaa.2017.01.085>
6. Barchiesi, M., Brancolini, A., Julin, V.: Sharp dimension free quantitative estimates for the Gaussian isoperimetric inequality. *Ann. Probab.* **45**(2), 668–697 (2017). <https://doi.org/10.1214/15-AOP1072>
7. Bonacini, M., Cristoferi, R.: Local and global minimality results for a nonlocal isoperimetric problem on \mathbb{R}^N . *SIAM J. Math. Anal.* **46**(4), 2310–2349 (2014). <https://doi.org/10.1137/130929898>
8. Bonacini, M., Cristoferi, R., Topaloglu, I.: Minimality of polytopes in a nonlocal anisotropic isoperimetric problem (2020)
9. Bonacini, M., Knüpfer, H.: Ground states of a ternary system including attractive and repulsive Coulomb-type interactions. *Calc. Var. Partial Differ. Equ.* **55**(5), 114 (2016). <https://doi.org/10.1007/s00526-016-1047-y>
10. Brasco, L., De Philippis, G., Ruffini, B.: Spectral optimization for the Stekloff–Laplacian: the stability issue. *J. Funct. Anal.* **262**(11), 4675–4710 (2012). <https://doi.org/10.1016/j.jfa.2012.03.017>
11. Brock, F., Chiacchio, F., Mercaldo, A.: An isoperimetric inequality for Gauss-like product measures. *J. Math. Pures Appl.* (9) **106**(2), 375–391 (2016). <https://doi.org/10.1016/j.matpur.2016.02.014>
12. Cabré, X., Ros-Oton, X., Serra, J.: Euclidean balls solve some isoperimetric problems with nonradial weights. *C. R. Math. Acad. Sci. Paris* **350**(21–22), 945–947 (2012). <https://doi.org/10.1016/j.crma.2012.10.031>
13. Choksi, R., Neumayer, R., Topaloglu, I.: Anisotropic liquid drop models. *Adv. Calc. Var.* <https://doi.org/10.1515/acv-2019-0088> (to appear)

14. Choksi, R., Peletier, M.: Small volume fraction limit of the diblock copolymer problem: I. Sharp-interface functional. *SIAM J. Math. Anal.* **42**(3), 1334–1370 (2010). <https://doi.org/10.1137/090764888>
15. Choksi, R., Muratov, C.B., Topaloglu, I.: An old problem resurfaces nonlocally: Gamow's liquid drops inspire today's research and applications. *Not. Am. Math. Soc.* **64**(11), 1275–1283 (2017). <https://doi.org/10.1090/noti1598>
16. Cicalese, M., Leonardi, G.P.: A selection principle for the sharp quantitative isoperimetric inequality. *Arch. Rat. Mech. Anal.* **206**(2), 617–643 (2012). <https://doi.org/10.1007/s00205-012-0544-1>
17. Cinti, E., Glaudo, F., Pratelli, A., Ros-Oton, X., Serra, J.: Sharp quantitative stability for isoperimetric inequalities with homogeneous weights (2020). (preprint)
18. Cinti, E., Pratelli, A.: The $\varepsilon - \varepsilon^\beta$ property, the boundedness of isoperimetric sets in \mathbb{R}^N with density, and some applications. *J. Reine Angew. Math.* **728**, 65–103 (2017). <https://doi.org/10.1515/crelle-2014-0120>
19. De Philippis, G., Franzina, G., Pratelli, A.: Existence of isoperimetric sets with densities “converging from below” on \mathbb{R}^N . *J. Geom. Anal.* **27**(2), 1086–1105 (2017). <https://doi.org/10.1007/s12220-016-9711-1>
20. Figalli, A., Fusco, N., Maggi, F., Millot, V., Morini, M.: Isoperimetry and stability properties of balls with respect to nonlocal energies. *Commun. Math. Phys.* **336**(1), 441–507 (2015). <https://doi.org/10.1007/s00220-014-2244-1>
21. Frank, R.L., Killip, R., Nam, P.T.: Nonexistence of large nuclei in the liquid drop model. *Lett. Math. Phys.* **106**(8), 1033–1036 (2016). <https://doi.org/10.1007/s11005-016-0860-8>
22. Frank, R.L., Lieb, E.H.: A compactness lemma and its application to the existence of minimizers for the liquid drop model. *SIAM J. Math. Anal.* **47**(6), 4436–4450 (2015). <https://doi.org/10.1137/15M1010658>
23. Fuglede, B.: Stability in the isoperimetric problem for convex or nearly spherical domains in \mathbb{R}^n . *Trans. Am. Math. Soc.* **314**, 619–638 (1989). <https://doi.org/10.1090/S0002-9947-1989-0942426-3>
24. Fusco, N., Maggi, F., Pratelli, A.: On the isoperimetric problem with respect to a mixed Euclidean–Gaussian density. *J. Funct. Anal.* **260**(12), 3678–3717 (2011). <https://doi.org/10.1016/j.jfa.2011.01.007>
25. Fusco, N., Pratelli, A.: Sharp stability for the Riesz potential (2019). (preprint)
26. Gamow, G.: Mass defect curve and nuclear constitution. *Proc. R. Soc. Lond. A* **126**(803), 632–644 (1930). <https://doi.org/10.1098/rspa.1930.0032>
27. Générac, F., Oudet, E.: Large volume minimizers of a nonlocal isoperimetric problem: theoretical and numerical approaches. *SIAM J. Math. Anal.* **50**(3), 3427–3450 (2018). <https://doi.org/10.1137/17M1139400>
28. Giusti, E.: The equilibrium configuration of liquid drops. *J. Reine Angew. Math.* **321**, 53–63 (1981). <https://doi.org/10.1515/crll.1981.321.53>
29. Julin, V.: Isoperimetric problem with a Coulomb repulsive term. *Indiana Univ. Math. J.* **63**(1), 77–89 (2014). <https://doi.org/10.1512/iumj.2014.63.5185>
30. Julin, V.: Remark on a nonlocal isoperimetric problem. *Nonlinear Anal.* **154**, 174–188 (2017). <https://doi.org/10.1016/j.na.2016.10.011>
31. Knüpfer, H., Muratov, C.B.: On an isoperimetric problem with a competing nonlocal term I: the planar case. *Commun. Pure Appl. Math.* **66**(7), 1129–1162 (2013). <https://doi.org/10.1002/cpa.21451>
32. Knüpfer, H., Muratov, C. B.: On an isoperimetric problem with a competing nonlocal term II: the general case. *Commun. Pure Appl. Math.* **67**(12), 1974–1994 (2014). <https://doi.org/10.1002/cpa.21479>
33. Knüpfer, H., Muratov, C.B., Novaga, M.: Low density phases in a uniformly charged liquid. *Commun. Math. Phys.* **345**(1), 141–183 (2016). <https://doi.org/10.1007/s00220-016-2654-3>
34. Lu, J., Otto, F.: Nonexistence of a minimizer for Thomas-Fermi-Dirac-von Weizsäcker model. *Commun. Pure Appl. Math.* **67**(10), 1605–1617 (2014). <https://doi.org/10.1002/cpa.21477>
35. Maggi, F.: *Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory*. Cambridge Studies in Advanced Mathematics, vol. 135. Cambridge University Press, Cambridge (2012). <https://doi.org/10.1017/CBO9781139108133>
36. Misiats, O., Topaloglu, I.: On minimizers of an anisotropic liquid drop model. *ESAIM Control Optim. Calc. Var.* <https://doi.org/10.1051/cocv/2020068> (to appear)
37. Morgan, F., Pratelli, A.: Existence of isoperimetric regions in \mathbb{R}^n with density. *Ann. Global Anal. Geom.* **43**(4), 331–365 (2013). <https://doi.org/10.1007/s10455-012-9348-7>
38. Muratov, C.B., Zaleski, A.: On an isoperimetric problem with a competing non-local term: quantitative results. *Ann. Global Anal. Geom.* **47**, 63–80 (2014). <https://doi.org/10.1007/s10455-014-9435-z>
39. Neumayer, R.: A strong form of the quantitative Wulff inequality. *SIAM J. Math. Anal.* **48**(3), 1727–1772 (2016). <https://doi.org/10.1137/15M1013675>
40. Pratelli, A., Saracco, G.: On the isoperimetric problem with double density. *Nonlinear Anal.* **177**(Part B), 733–752 (2018). <https://doi.org/10.1016/j.na.2018.04.009>

41. Pratelli, A., Saracco, G.: The $\varepsilon\text{-}\varepsilon^\beta$ property in the isoperimetric problem with double density, and the regularity of isoperimetric sets. *Adv. Nonlinear Stud.* **20**(3), 539–555 (2020). <https://doi.org/10.1515/ans-2020-2074>
42. Ren, X., Wei, J.: Double tori solution to an equation of mean curvature and Newtonian potential. *Calc. Var. Partial Differ. Equ.* **49**(3–4), 987–1018 (2014). <https://doi.org/10.1007/s00526-013-0608-6>
43. Rigot, S.: Ensembles quasi-minimaux avec contrainte de volume et rectifiabilité uniforme. *Mém. Soc. Math. Fr. (N.S.)* **82**, 104 (2000). <https://doi.org/10.24033/msmf.395>
44. Rosales, C., Cañete, A., Bayle, V., Morgan, F.: On the isoperimetric problem in Euclidean space with density. *Calc. Var. Partial Differ. Equ.* **31**(1), 27–46 (2008). <https://doi.org/10.1007/s00526-007-0104-y>
45. Tamanini, I.: Regularity results for almost minimal oriented hypersurfaces in \mathbb{R}^N . *Quaderni del Dipartimento di Matematica dell' Università di Lecce*. <http://cvgmt.sns.it/paper/1807/>. (1984)
46. White, B.: A strong minimax property of nondegenerate minimal submanifolds. *J. Reine Angew. Math.* **457**, 203–218 (1994). <https://doi.org/10.1515/crll.1994.457.203>
47. Ziemer, W.P.: *Weakly Differentiable Functions: Sobolev Spaces and Functions of Bounded Variation*. Graduate Texts in Mathematics. Springer, New York (1989). <https://doi.org/10.1007/978-1-4612-1015-3>

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