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Calculus of Variations

Heat kernel on Ricci shrinkers

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Abstract
In this paper, we systematically study the heat kernel of the Ricci flows induced by Ricci shrinkers. We develop several estimates which are much sharper than their counterparts in general closed Ricci flows. Many classical results, including the optimal Logarithmic Sobolev constant estimate, the Sobolev constant estimate, the no-local-collapsing theorem, the pseudo-locality theorem and the strong maximum principle for curvature tensors, are essentially improved for Ricci flows induced by Ricci shrinkers. Our results provide many necessary tools to analyze short time singularities of the Ricci flows of general dimension.

Mathematics Subject Classification 53C25 · 53E20

Contents

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1 Introduction

A Ricci shrinker is a triple (M^n, g, f) of smooth manifold M^n, Riemannian metric g and a smooth function f satisfying

$$Rc + \text{Hess}\, f = \frac{1}{2} g. \tag{1}$$

By a normalization of f, we can assume that

$$R + |\nabla f|^2 = f, \tag{2}$$

$$\int e^{-f} (4\pi)^{-\frac{n}{2}} dV = e^{\boldsymbol{\mu}}, \tag{3}$$

where $\boldsymbol{\mu}$ is the functional of Perelman. As usual, we define

$$\mathcal{M}_n(A) := \left\{ (M^n, g, f) \mid \boldsymbol{\mu} \geq -A \right\}. \tag{4}$$

Lying on the intersection of critical metrics and geometric flows, the study of Ricci shrinkers has already become a very important topic in geometric analysis. Up to dimension 3, all Ricci shrinkers are classified. In dimension 2, the only Ricci shrinkers are $\mathbb{R}^2$, S^2 and $\mathbb{R}P^2$ with standard metrics, due to the classification of Hamilton [25]. In dimension 3, we know that $\mathbb{R}^3$, $S^2 \times \mathbb{R}$, S^3 and their quotients are all possible Ricci shrinkers, based on the work of Perelman [46], Petersen–Wylie [47], Naber [43], Ni–Wallach [45] and Cao–Chen–Zhu [8]. If we assume the curvature operator to be nonnegative, then the Ricci shrinkers are also classified, see Munteanu–Wang [42]. However, an important motivation for the study of the Ricci shrinkers is that the Ricci shrinkers are models for short time singularities of the Ricci flows. In dimension 3, by the Hamilton–Ivey pinch [25,26,30], one may naturally assume that the Ricci shrinker has nonnegative curvature operator. If the dimension is strictly greater than 3, the loss of pinch estimate makes the nonnegativity of curvature operator an unsatisfactory condition and should be dropped. Also, it is well known (cf. Haslhofer–Müller [27]) that most interesting singularity models are non-compact. Therefore, to prepare for the singularity analysis of high dimensional Ricci flow, we shall focus only on the study of *non-compact Ricci shrinkers without any curvature assumption*. Since M is non-compact, the inequality

$$\sup_M |Rm| < \infty \tag{5}$$

may fail. The failure of Riemannian curvature bound causes serious consequences. Many fundamental analysis tools, e.g., maximum principle and integration by parts, cannot be applied directly without estimates of the manifold at infinity.

In this paper, we shall provide a solid foundation for many fundamental analysis tools in the Ricci shrinkers. We shall mostly take the point of view that Ricci shrinkers are time slices of self-similar Ricci flow solutions. After a delicate choice of cutoff functions and calculations, we show that most of the fundamental tools, including maximum principle, existence of (conjugate) heat solutions, uniqueness and stochastic completeness, integration by parts, etc., work well on the Ricci shrinker spacetime. Then we use these fundamental tools to study the geometric properties of the Ricci flows induced by the Ricci shrinkers. Therefore, we are able to check that most known important properties of the compact Ricci flows, including monotonicity of Perelman's functional, no-local-collapsing and pseudo-locality theorem of Perelman, curvature tensor strong maximum principle of Hamilton, do apply on noncompact Ricci shrinkers. Furthermore, since the Ricci flows induced by the Ricci shrinkers are self-similar, we obtain many special properties of the Ricci shrinkers. The first property is the estimate of sharp Logarithmic Sobolev constant, which can be regarded as an improvement of the fact that Perelman's functional is monotone along each Ricci flow.

Theorem 1 (Optimal Logarithmic Sobolev constant) *Let (M^n, p, g, f) be a Ricci shrinker. Then $\boldsymbol{\mu}(g, \tau)$ is a continuous function for $\tau > 0$ such that $\boldsymbol{\mu}(g, \tau)$ is decreasing for $\tau \le 1$ and increasing for $\tau \ge 1$. In particular, we have*

$$\boldsymbol{\nu}(g) := \inf_{\tau > 0} \boldsymbol{\mu}(g, \tau) = \boldsymbol{\mu}(g). \tag{6}$$

Consequently, the following properties hold.

- *Logarithmic Sobolev inequality. In other words, for each compactly supported locally Lipschitz function u and each $\tau > 0$, we have*

$$\int u^2 \log u^2 dV - \left(\int u^2 dV\right) \log\left(\int u^2 dV\right) + \left(\boldsymbol{\mu} + n + \frac{n}{2}\log(4\pi\tau)\right) \int u^2 dV$$
$$\le \tau \int \left\{4|\nabla u|^2 + R u^2\right\} dV. \tag{7}$$

- *Sobolev inequality. Namely, for each compactly supported locally Lipschitz function u, we have*

$$\left(\int u^{\frac{2n}{n-2}} dV\right)^{\frac{n-2}{n}} \le C e^{-\frac{2\boldsymbol{\mu}}{n}} \int \left\{4|\nabla u|^2 + R u^2\right\} dV \tag{8}$$

for some dimensional constant $C = C(n)$.

In geometric analysis, it is a fundamental problem to estimate uniform Sobolev constant. When the underlying manifold is noncompact, the uniform Sobolev constant in general does not exist. However, (8) says that there is a uniform (Scalar-)Sobolev constant, depending only on n and $\boldsymbol{\mu}$. In particular, if the scalar curvature is bounded, i.e., $\sup_M R < \infty$, then there exists a classical Sobolev constant. Namely, for each $u \in C_c^\infty(M)$, we have

$$\left(\int u^{\frac{2n}{n-2}} dV\right)^{\frac{n-2}{n}} \le C e^{-\frac{2\boldsymbol{\mu}}{n}} \int \left\{|\nabla u|^2 + u^2\right\} dV$$

for some $C = C(n, \sup_M R)$. Note that the term $e^{-\frac{2\boldsymbol{\mu}}{n}}$ is almost $|B(p,1)|^{-\frac{2}{n}}$ by Lemma 2.5 of [34].

The proof of Theorem 1 follows a similar route as done in Proposition 9.5 of [34], by using the monotonicity of Perelman's functional along Ricci flow and the invariance of Perelman's functional under diffeomorphism actions.

Secondly, we can improve the no-local-collapsing theorem of Perelman on the Ricci shrinker Ricci flow. By the fundamental work of Perelman [46], the Ricci flow spacetime can be regarded as a "Ricci-flat" spacetime in terms of reduced volume and reduced distance. Now we can regard Ricci shrinker as a special time slice of the induced Ricci flow. On a Ricci flat manifold, an elementary comparison argument shows that $\frac{|B(x,r)|}{|B(x,1)|}$ grows at most Euclideanly and at least linearly (cf. [59,64], and Theorem 2.5 of [35]). This comparison geometry picture has a spacetime version which is used to illustrate the no-local-collapsing (cf. [46,53]). Although the comparison argument (even the space-time version) does not apply directly in the Ricci shrinker case, we can still show that similar phenomena hold for Ricci shrinkers.

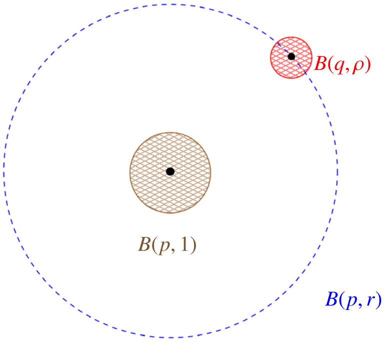

Fig. 1 Propagation of non-collapsing on Ricci-shrinkers

Theorem 2 (Improved no-local-collapsing theorem) *Suppose* (M^n, p, g, f) *is a Ricci shrinker,* $r > 1$. *Then*

$$\begin{cases} \dfrac{1}{C} r \le \dfrac{|B(p,r)|}{|B(p,1)|} \le C r^n, & \text{(9a)} \\ \inf_{\rho \in (0, r^{-1})} \rho^{-n} |B(q,\rho)| \ge \dfrac{1}{C} |B(p,1)|. & \text{(9b)} \end{cases}$$

Here q *is any point on* $\partial B(p,r)$, *and* C *is a dimensional constant.*

Although the volume estimate (9a) behaves like the Ricci-flat case, its proof is totally different and much more involved. The proof builds on the the Sobolev inequality (8) and an improvement (cf. Remark 8) of the induction argument due to Munteanu and Wang [41]. The non-collapsing estimate (9b) in general does not hold for Ricci-flat manifold. This indicates that Ricci shrinkers are more rigid than Ricci-flat manifold. See Figure 1 for intuition.

The proof of (9b) relies on (6) and an effective volume estimate in [53]. The scale $\rho \in (0, r^{-1})$ is chosen such that $R\rho^2 \le C(n)$ inside $B(q,r)$. If we further assume scalar curvature is uniformly bounded on M, then we shall obtain that every unit ball on the Ricci shrinker M is uniformly non-collapsed. Theorem 2 can be regard as a special case of Theorem 23 and Theorem 23, which are more general versions of the no-local-collapsing. In particular, it indicates that any Ricci shrinker must be κ-noncollpased for some constant $\kappa > 0$, see Remark 7. The proof of Theorems 2, 22 and 23 can be found in Sect. 9. Note that Theorem 2 indicates that the Ricci shrinkers are similar to the Ricci-flat manifolds. Actually, there exist many other similarities between the Ricci-flat manifolds and the Ricci Shrinkers. For example, in [29,34], it is proved that each sequence of non-collapsed Ricci shrinkers sub-converges to a limit Riemannian conifold Ricci shrinker. Such results are analogue of the weak compactness theorem of non-collapsed Ricci-flat manifolds, by the deep work of Cheeger, Colding and Naber (cf. [12,14,20]).

Thirdly, the pseudo-locality theorem of Perelman has an elegant version on the Ricci shrinker Ricci flow. The pseudo-locality theorem of Perelman [46] is a fundamental tool in the study of Ricci flow. It claims that the Ricci flow cannot turn an almost Euclidean domain to a very curved region in a short time period. In the literature, it is known that the pseudo-locality theorem hold for Ricci flow with bounded Riemannian curvature, which condition is clearly not available in the current setting. However, using the existence of special cutoff function, we can show maximum principle and stochastic completeness for conjugate heat kernel. By carefully checking the integration by parts, we obtain that the traditional

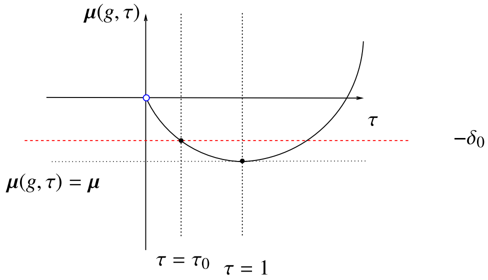

Fig. 2 $\boldsymbol{\mu}(g,\tau)$ of a Ricci shrinker with bounded geometry

pseudo-locality theorem holds on the Ricci flow spacetime induced by the Ricci shrinker. Furthermore, the pseudo-locality has the following special version for Ricci shrinkers.

Theorem 3 (Improved pseudo-locality theorem) *Suppose that (M^n, p, g, f) is a non-flat Ricci shrinker. Then we have*

$$\boldsymbol{\mu} < -\delta_0 \tag{10}$$

for some small positive constant $\delta_0 = \delta_0(n)$. Furthermore, the following properties are equivalent.

(a) M has bounded geometry. Namely, the norm of Riemannian curvature tensor is bounded from above and the injectivity radius is bounded from below.
(b) The infinitesimal functional satisfies

$$\lim_{\tau\to 0^+} \boldsymbol{\mu}(g,\tau) = 0. \tag{11}$$

(c) The infinitesimal functional satisfies the gap

$$\lim_{\tau\to 0^+} \boldsymbol{\mu}(g,\tau) > -\delta_0. \tag{12}$$

If one of the above conditions hold, we can define

$$\tau_0 := \sup\left\{\tau \,|\, \boldsymbol{\mu}(g,s) \geq -\delta_0, \quad \forall\, s\in(0,\tau)\right\}. \tag{13}$$

Then for some positive constant $C = C(n)$, we have the following explicit estimates

$$\begin{cases} \sup_{x\in M} |Rm|(x) \leq C\tau_0^{-1}, & \text{(14a)} \\ \inf_{x\in M} inj(x) \geq \dfrac{1}{C}\sqrt{\tau_0}. & \text{(14b)} \end{cases}$$

We remark that the gap inequality (10) is not new. It was first proved by Yokota in [57,58]. However, our proof of (10) is completely different and is the base for the proof of (11), (12) and (14). Theorem 3 also indicates that the bounded geometry for Ricci shrinkers is equivalent to the gap inequality (12). This criterion has divided all Ricci shrinkers into two categories characterized by their graphs of entropies, which are illustrated by Figure 2 and Figure 3. Note that Figure 2 represents the functional behavior of a typical Ricci shrinker, for example, the cylinder $S^k \times \mathbb{R}^{n-k}$ for $k \geq 2$. Figure 3 represents the functional behavior of a Ricci shrinker with unbounded geometry. However, it is not clear whether such Ricci shrinker exists. For Ricci shrinkers with bounded geometry, it follows from (13) and (14) that the number $\sqrt{\tau_0}$ can be understood as the regularity scale. Actually, under the scale $\sqrt{\tau_0}$,

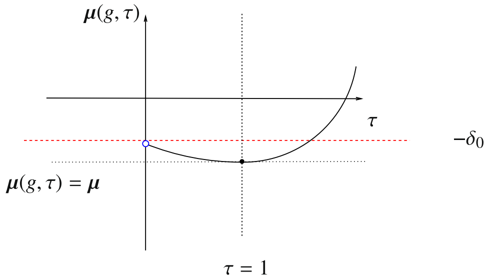

Fig. 3 $\mu(g,\tau)$ of a Ricci shrinker with unbounded geometry

all the higher curvature derivatives norm $|\nabla^k Rm|$ are bounded by $C(n,k)\tau_0^{-1-\frac{k}{2}}$, in light of the estimates of Shi [49].

There exist several other special versions and consequences of the pseudo-locality theorems. The proof of all of them, including the proof of Theorem 3, can be found in Sect. 10.

Fourthly, the curvature tensor strong maximum principle, developed by R. Hamilton, works on Ricci shrinker Ricci flows and also has an improved version. Using the curvature tensor maximum principle, Hamilton shows that the nonnegativity of curvature operator is preserved under the Ricci flow and the kernel space is parallel. Therefore, the manifold splits as product when kernel space is nontrivial. Since different time slices of a Ricci shrinker Ricci flow are the same up to scaling and diffeomorphism, the preservation of curvature conditions is automatic. The interesting problem on Ricci shrinker is to show the strong maximum principle, i.e., the splitting of the manifold when eigenvalues of curvature operator satisfy some nonnegativity condition. On this perspective, we can improve the traditional strong maximum principle of curvature operator to the following format.

Theorem 4 (Improved strong maximum principle of curvature tensor) *Suppose* (M^n, g, f) *is a Ricci shrinker and* $\lambda_1 \leq \lambda_2 \leq \cdots$ *are the eigenvalue functions of the curvature operator* Rm. *Then the following properties hold.*

– *If* $\lambda_2 \geq 0$ *as a function, then there is a* $k \in \{0, 1, 2, \ldots, n\}$ *and a closed symmetric space* N^k *such that* (M^n, g) *is isometric to a quotient of* $N^k \times \mathbb{R}^{n-k}$.
– *If* $\lambda_2 \geq 0$ *as a function and* $\lambda_2 > 0$ *at one point, then* (M^n, g) *is isometric to a quotient of round sphere* S^n.

The statement in Theorem 4 should be well known to experts in Ricci flow if we replace λ_2 by λ_1. In fact, by the work of Munteanu–Wang [42] and Petersen–Wylie [47], we know that the same geometry conclusion hold if we replace λ_2 in Theorem 4 by $\lambda_1+\lambda_2$. Their proof builds on the celebrated work of Böhm–Wilking [5] on the closed Ricci flow satisfying $\lambda_1+\lambda_2>0$ and also relies on a weighted Riemannian curvature integral estimate $\int |Rm|^2 e^{-f} dV < \infty$. If $\lambda_1+\lambda_2 \geq 0$, the Riemannian curvature integral estimate can be deduced from the Ricci curvature integral bound $\int |Rc|^2 e^{-f} dV < \infty$, which follows from a clever integration-by-parts. In Theorem 4, with only condition $\lambda_2 \geq 0$, Riemannian curvature integral estimate $\int |Rm|^2 e^{-f} dV < \infty$ becomes nontrivial. As done in [34], we apply local conformal transformations and the classical Cheeger–Colding theory to study the local structure of Ricci shrinkers. Combining the L^2-curvature estimate of Jiang–Naber [31] with the improved no-local-collapsing Theorem 2, we are able to show that $\int |Rm|^2 e^{-f} dV < \infty$ always holds true (i.e., Theorem 26). Consequently, the work of Petersen–Wylie [47] applies and the curvature tensor strong maximum principle holds for Ricci shrinkers. Then we are able to obtain $\lambda_1 \geq 0$ from the condition $\lambda_2 \geq 0$. Clearly, the condition $\lambda_2 \geq 0$ is weaker than $\lambda_1+\lambda_2 \geq 0$

and Theorem 4 is an improvement of the results of Munteanu–Wang [42] and Petersen–Wylie [47]. Note that $\lambda_2 \geq 0$ is a novel condition in the Ricci flow literature. It is not clear whether $\lambda_2 \geq 0$ is preserved by the Ricci flow on a closed manifold. Actually, in Theorem 4, the same conclusion holds if one replace the condition $\lambda_2 \geq 0$ by an even weak condition

$$\lambda_2 \geq -\epsilon \frac{\lambda_1^2}{|R - 2\lambda_1|}$$

for some $\epsilon = \epsilon(n)$. The details can be found in Theorem 27. The proof of Theorems 4 and 27 appear in Sect. 11.

The proof of the previous four theorems requires some elementary, but delicate, geometric and analytic facts on Ricci shrinkers.

- The level sets of f are comparable with geodesic balls.
- A special cutoff function.
- Special heat solution and conjugate heat solution on the Ricci shrinker Ricci flow.
- The existence of heat kernel and stochastic completeness of the backward heat solution.
- The existence and uniqueness of bounded (conjugate) heat solutions.

After the above estimates are developed, we check that the entropy of Perelman is monotone along the Ricci flow induced by the Ricci shrinker, whose proof needs more delicate integration by parts. Then the proof of Theorem 1 follows a similar route as the one in Proposition 9.5 of [34], with more involved technique. From Theorem 1, we can obtain Theorem 2 by repeatedly choosing proper test function u. When integration by parts are assumed, one can formally follows the routine of Perelman to obtain the differential Harnack inequality (i.e., Theorem 21), and then the traditional pseudo-locality theorem. Combining with a standard localization technique, one can deduce Theorem 3. However, as the functional derivatives contain quadratic Ricci curvature term, many terms concerning high order derivatives need to be carefully handled to verify the integration by parts. This causes many technical difficulties. One key difficulty is the delicate heat kernel estimate to derive the differential Harnack inequality. Therefore, the following heat kernel estimate is in the central position for developing fundamental analytic estimates on Ricci shrinker.

Theorem 5 (Heat Kernel estimate) *Let (M^n, g, f) be a Ricci shrinker in $\mathcal{M}_n(A)$. Then the following properties hold.*

(i) (Heat kernel upper bound)

$$H(x,t,y,s) \leq \frac{e^{-\mu}}{(4\pi(t-s))^{\frac{n}{2}}}.$$

(ii) (Heat kernel lower bound) For any $0 < \delta < 1$, $D > 1$ and $0 < \epsilon < 4$, there exists a constant $C = C(n,\delta,D) > 0$ such that

$$H(x,t,y,s) \geq \frac{C^{\frac{4}{\epsilon}} e^{\mu(\frac{4}{\epsilon}-1)}}{(4\pi(t-s))^{n/2}} \exp\left(-\frac{d_t^2(x,y)}{(4-\epsilon)(t-s)}\right)$$

for any $t \in [-\delta^{-1}, 1-\delta]$ and $d_t(p,y) + \sqrt{t-s} \leq D$.

(iii) (Heat kernel integral bound) For any $0 < \delta < 1$, $D > 1$ and $\epsilon > 0$, there exists a constant $C = C(n, A, \delta, D, \epsilon) > 1$ such that

$$\int_{M \setminus B_s(x, r\sqrt{t-s})} H(x,t,y,s)\, dV_s(y) \leq C \exp\left(-\frac{(r-1)^2}{4(1+\epsilon)}\right)$$

for any $t \in [-\delta^{-1}, 1-\delta]$, $d_t(p,x) + \sqrt{t-s} \leq D$ and $r \geq 1$.

We briefly discuss the proof of Theorem 5. Notice that the Logarithmic Sobolev inequality for all scales implies the ultracontractivity of the heat kernel by Davies' methods (see Chapter 2 of [21]). We prove that the same result (i) holds for Ricci shrinkers. The lower bound of the heat kernel can be estimated by considering the reduced distance (i.e., Theorem 16). We first obtain an on-diagonal lower bound of the heat kernel, in which case the estimate of the reduced distance is straightforward. Then we derive the general off-diagonal lower bound by exploiting a Harnack property (i.e., (200)). To prove the integral upper bound, we consider the probability measure $\upsilon_s(y) := H(x, t, y, s)\, dV_s(y)$. Following the work of Hein–Naber [28], we show that υ_s satisfies a type of Logarithmic Sobolev inequality (i.e., Theorem 13). The equivalence of the Logarithmic Sobolev inequality and the Gaussian concentration (i.e., Theorem 14) then shows that we can estimate the integral upper bound of the heat kernel by its pointwise lower bound.

Organization of the paper In Sect. 2, we review the definition of the Ricci flows induced by the Ricci shrinkers. We also present the estimates of the potential function and volume upper bound. In Sect. 3, we introduce a family of cutoff functions and prove a maximum principle (i.e., Theorem 6) on Ricci shrinker spacetime. Moreover, we prove the existence and other basic properties of the heat kernel on spacetime. In Sect. 4, we prove the monotonicity of Perelman's entropy (i.e., Theorem 10). In Sect. 5, we prove Theorem 1. In Sect. 6, we prove the logarithmic Sobolev inequality (i.e., Theorem 13) and the Gaussian concentration (i.e., Theorem 14) of the probability measure induced by the heat kernel. In Sect. 7, Theorem 5 is proved. In Sect. 8, we prove the differential Harnack inequality (i.e., Theorem 21) by using the heat kernel estimates. In Sect. 9, we provide the proof of Theorem 2. In Sect. 10, we prove the pseudo-locality theorem (i.e., Theorems 24) and 3. In the last section, we obtain an L^2-integral bound of the Riemannian curvature (i.e., Theorem 26). As a consequence, we prove Theorem 4.

2 Preliminaries

For any Ricci shrinker (M^n, g, f), let $\psi^t : M \to M$ be a 1-parameter family of diffeomorphisms generated by $X(t) = \frac{1}{1-t}\nabla_g f$. That is

$$\frac{\partial}{\partial t}\psi^t(x) = \frac{1}{1-t}\nabla_g f\left(\psi^t(x)\right). \tag{15}$$

By a direct calculation, see [18, Chapter 4], the rescaled pull-back metric $g(t) := (1-t)(\psi^t)^* g$ and the pull-back function $f(t) := (\psi^t)^* f$ satisfy the equation

$$Rc(g(t)) + \mathrm{Hess}_{g(t)} f(t) = \frac{1}{2(1-t)} g(t), \tag{16}$$

where $\{(M, g(t)), -\infty < t < 1\}$ is a Ricci flow solution with $g(0) = g$, that is,

$$\partial_t g = -2Rc(g(t)). \tag{17}$$

For notational simplicity, we will omit the subscript $g(t)$ if there is no confusion. From (16) and (17), it is easy to show that

$$\partial_t f = |\nabla f|^2, \tag{18}$$

$$R + \Delta f = \frac{n}{2(1-t)}, \tag{19}$$

$$R + |\nabla f|^2 = \frac{f}{1-t}. \tag{20}$$

Now we define

$$\bar{\tau} = 1 - t, \quad F(x,t) = \bar{\tau} f(x,t) \quad \text{and} \quad \bar{v}(x,t) = (4\pi\bar{\tau})^{-n/2} e^{-f(x,t)}. \tag{21}$$

It follows from (18), (19) and (20) that

$$\partial_t F = \bar{\tau}|\nabla f|^2 - f = -\bar{\tau} R, \tag{22}$$

$$\bar{\tau} R + \Delta F = \frac{n}{2}, \tag{23}$$

$$\bar{\tau}^2 R + |\nabla F|^2 = F. \tag{24}$$

Now we define

$$\square := \partial_t - \Delta_t, \tag{25}$$

$$\square^* := -\partial_t - \Delta_t + R. \tag{26}$$

We have special heat solution and conjugate heat solution:

$$\square\left(F + \frac{n}{2}t\right) = 0, \tag{27}$$

$$\square^* \bar{v} = 0. \tag{28}$$

Note that (27) is equivalent to

$$\square F = -\frac{n}{2}. \tag{29}$$

Now we have the following estimate of F by using the same method as [9,27].

Lemma 1 *There exists a point $p \in M$ where F attains its infimum and F satisfies the quadratic growth estimate*

$$\frac{1}{4}\left(d_t(x,p) - 5n\bar{\tau} - 4\right)_+^2 \le F(x,t) \le \frac{1}{4}\left(d_t(x,p) + \sqrt{2n\bar{\tau}}\right)^2 \tag{30}$$

for all $x \in M$ and $t < 1$, where $a_+ := \max\{0, a\}$.

Proof This originates from the work of Cao–Zhou [9, Theorem 1.1]. We follow the argument of Haslhofer–Müller [27]. It follows from [15] that for any Ricci shrinker $R \ge 0$ since its corresponding Ricci flow solution is ancient. So from (24), we have

$$|\nabla F|^2 \le F. \tag{31}$$

It implies that $\sqrt{F}$ is $\frac{1}{2}$-Lipschitz, since

$$|\nabla\sqrt{F}| = \frac{1}{2}\left|\frac{\nabla F}{\sqrt{F}}\right| \le \frac{1}{2}.$$

On the other hand, for any $x, y \in M$, we choose a minimizing geodesic $\gamma(s), 0 \le s \le d = d_t(x,y)$ joining $x = \gamma(0)$ and $y = \gamma(d)$. Assume that $d > 2$, we construct a function

$$\phi(s) = \begin{cases} s, & s \le 1 \\ 1, & 1 \le x \le d-1 \\ d - s. & d-1 \le x \le d \end{cases}$$

The second variation formula for shortest geodesic implies that

$$\int_0^d \phi^2 Rc(\gamma', \gamma')\, ds \le (n-1)\int_0^d \phi'^2\, ds = 2(n-1). \tag{32}$$

Note that from the Eq. (16),

$$\bar{\tau}\mathrm{Rc}(\gamma', \gamma') = \frac{1}{2} - \mathrm{Hess}F(\gamma', \gamma'). \tag{33}$$

Therefore from (24) we have

$$\begin{aligned} \frac{d}{2} - \frac{2}{3} - 2\bar{\tau}(n-1) &\le \int_0^d \phi^2 \mathrm{Hess}F(\gamma', \gamma')\, ds \\ &\le -2\int_0^1 \phi \nabla_{\gamma'} F\, ds + 2\int_{d-1}^d \phi \nabla_{\gamma'} F\, ds \\ &\le \sup_{s\in[0,1]} |\nabla_{\gamma'} F| + \sup_{s\in[d,d-1]} |\nabla_{\gamma'} F| \\ &\le \sqrt{F(x)} + \sqrt{F(y)} + 1, \end{aligned} \tag{34}$$

where we used (31) in the last inequality. It is now immediate from (34) that F has a minimum point p. It is clear that $|\nabla F| = 0$ and $\Delta F \ge 0$ at the point p by the minimum principle. Hence from (23) and (24) we have

$$F(p) = \bar{\tau}^2 R \le \bar{\tau}(\bar{\tau} R + \Delta F) = \frac{\bar{\tau} n}{2}.$$

For any $q \in M$ such that $d_t(p, q) = d$, it is straightforward from (31) and (34) that

$$\frac{1}{4}(d - 5n\bar{\tau} - 4)_+^2 \le \frac{1}{4}\left(d - \frac{10}{3} - 4\bar{\tau}(n-1) - \sqrt{2n\bar{\tau}}\right)_+^2 \le F(q) \le \frac{1}{4}\left(d + \sqrt{2n\bar{\tau}}\right)^2.$$

□

Note that $F(\cdot, t)$ is a pull-back function of $f(\cdot, 0)$ up to the scale $\bar{\tau}$, we can choose a base point $p \in M$ such that p is a minimum point for all $F(\cdot, t)$. Now from Lemma 1, $F(\cdot, t)$ can be regarded as an approximation of $\frac{d_t^2}{4}$.

With Lemma 1, we have the following volume estimate whose proof follows from [9, Theorem 1.2].

Lemma 2 *There exists a constant $C = C(n) > 0$ such that for any Ricci shrinker (M^n, g, f) with $p \in M$ a minimum point of f,*

$$|B_t(p, r)|_t \le \begin{cases} Ce^{\mu} r^n & \text{if } r \ge 2\sqrt{\bar{\tau} n}; \\ Cr^n & \text{if } r < 2\sqrt{\bar{\tau} n}. \end{cases}$$

Proof We set $\rho = 2\sqrt{F}$ and $D(r) = \{x \in M \mid \rho \le r\}$. Moreover we define $V(r) = \int_{D(r)} dV_t$ and $\chi(r) = \int_{D(r)} R(t)\, dV_t$. It follows from a similar computation as [9, (3.5)], by using (23) and (24), that

$$nV - rV' = 2\bar{\tau}\chi - \frac{4\bar{\tau}^2}{r}\chi'. \tag{35}$$

If we set $r_0 = \sqrt{2\bar{\tau}(n+2)}$, by integrating (35) we obtain, see [9, (3.6)] for details, that

$$V(r) \le 2r^n r_0^{-n} V(r_0)$$

for any $r \geq 2\sqrt{\bar{\tau} n}$. Then it follows from Lemma 1 that for any $r \geq 2\sqrt{\bar{\tau} n}$,

$$|B_t(p,r)|_t \leq V\left(r + \sqrt{2n\bar{\tau}}\right) \leq V(2r) \leq 2^{n+1} r^n r_0^{-n} V(r_0).$$

By definition, we have

$$\begin{aligned} D(r_0) &= \left\{ x \in M \,\middle|\, F \leq \frac{\bar{\tau}(n+2)}{2} \right\} \\ &= \left\{ x \in M \,\middle|\, f(x,t) \leq \frac{n+2}{2} \right\} = \left\{ x \in M \,\middle|\, f(\psi^t(x)) \leq \frac{n+2}{2} \right\}. \end{aligned}$$

Moreover, since $g(t) = \bar{\tau}(\psi^t)^* g$,

$$V(r_0) \leq \bar{\tau}^{\frac{n}{2}} \int_{f(x) \leq \frac{n+2}{2}} dV \leq \bar{\tau}^{\frac{n}{2}} |\{x \mid f(x) \leq (n+2)/2\}|.$$

For any x such that $f(x) \leq (n+2)/2$. it follows from Lemma 1 that $d(p,x) \leq c_0(n)$. Therefore for any $r \geq 2\sqrt{\bar{\tau} n}$,

$$|B_t(p,r)|_t \leq C(n)|B(p,c_0)| r^n \leq C(n) e^{\boldsymbol{\mu}} r^n,$$

where the last inequality follows from [34, Lemma 2.3].

Finally, the case $r \leq 2\sqrt{\bar{\tau} n}$ follows from the comparison theorem [55, Theorem 1.2] by using (16). Indeed, for any x with $d_t(p,x) \leq 2\sqrt{\bar{\tau} n}$, it follows from Lemma 1 that $f(x,t) = \bar{\tau}^{-1} F(x,t) \leq C$. Therefore, from (20) we obtain $|\nabla f|(x,t) \leq C\bar{\tau}^{-1/2}$. Now it follows from [55, (1.5) of Theorem 1.2] that for any $s \leq r$,

$$\int_{B_t(p,r)} e^{-f(x,t)} \, dV_t \leq e^{Cr\bar{\tau}^{-1/2}} \frac{r^n}{s^n} \int_{B_t(p,s)} e^{-f(x,t)} \, dV_t \leq C \frac{r^n}{s^n} \int_{B_t(p,s)} e^{-f(x,t)} \, dV_t.$$

Then the conclusion follows if we let $s \to 0$. □

3 Cutoff functions, maximum principle and heat kernel

Now we construct a family of cutoff functions which is important when we perform integration by parts throughout the paper.

Fix a function $\eta \in C^\infty([0,\infty))$ such that $0 \leq \eta \leq 1$, $\eta = 1$ on $[0,1]$ and $\eta = 0$ on $[2,\infty)$. Furthermore, $-C \leq \eta'/\eta^{\frac{1}{2}} \leq 0$ and $|\eta''| + |\eta'''| \leq C$ for a universal constant $C > 0$. For each $r \geq 1$, we define

$$\phi^r := \eta\left(\frac{F}{r}\right). \tag{36}$$

Then ϕ^r is a smooth function on $M \times (-\infty, 1)$. The following estimates of ϕ^r will be repeatedly used in this paper.

Lemma 3 *There exists a constant $C = C(n)$ such that*

$$(\phi^r)^{-1} |\nabla \phi^r|^2 \leq C r^{-1}, \tag{37}$$

$$|\phi^r_t| \leq C \bar{\tau}^{-1}, \tag{38}$$

$$|\Delta \phi^r| \leq C(\bar{\tau}^{-1} + r^{-1}), \tag{39}$$

$$|\Box \phi^r| \leq C r^{-1}, \tag{40}$$

$$|\Box^* \phi^r| \le C \left(r^{-1} + \bar{\tau}^{-1} + \bar{\tau}^{-2} r \right). \tag{41}$$

Proof Note that $F \le 2r$ on the support of ϕ^r, it follows from the assumption of η and (31) that

$$\frac{|\nabla \phi^r|^2}{\phi^r} = r^{-2} \eta'^2 \eta^{-1} |\nabla F|^2 \le C r^{-2} F \le C r^{-1}.$$

This finishes the proof of (37). Similarly, by using (22), (24), (29) and (31), we can prove

$$|\phi^r_t| = r^{-1} |\eta' F_t| \le C r^{-1} \bar{\tau} R \le C r^{-1} \bar{\tau}^{-1} F \le C \bar{\tau}^{-1},$$
$$|\Box \phi^r| = |(\partial_t - \Delta) \phi^r| = |r^{-1} \eta' \Box F - r^{-2} \eta'' |\nabla F|^2| = | - n r^{-1} \eta'/2 - r^{-2} \eta'' |\nabla F|^2 | \le C r^{-1}.$$

So (38) and (40) are proved. Then we have

$$|\Delta \phi^r| = | - \Box \phi^r + \partial_t \phi^r| \le |\Box \phi^r| + |\phi^r_t| \le C (\bar{\tau}^{-1} + r^{-1}).$$

Hence we obtain (39). Finally, using (24) again, we have

$$\begin{aligned} |\Box^* \phi^r| &= |\left(-\partial_t - \Delta + R\right) \phi^r| = |\left(\Box - 2\partial_t + R\right) \phi^r| \\ &\le |\Box \phi^r| + 2|\phi^r_t| + R\phi^r \le |\Box \phi^r| + 2|\phi^r_t| + \bar{\tau}^{-2} F \phi^r \\ &\le C \left(r^{-1} + \bar{\tau}^{-1} + \bar{\tau}^{-2} r \right), \end{aligned}$$

which proves (41). □

Now we move on to show the maximum principle on general Ricci shrinkers. On a closed manifold, maximum principle holds automatically. If the underlying manifold is noncompact, then some additional assumptions are needed in order the maximum principle to hold. For example, in [35, Theorem 15.2], a condition

$$\int_a^b \int u_+^2(x,t) e^{-cd^2(x)} \, dV \, dt < \infty \tag{42}$$

is needed for the maximum principle of the static heat equation subsolution u. In our current setting of Ricci shrinker spacetime, the metrics are evolving under Ricci curvature. Then the distance distortion of different time slices is not easy to estimate directly without Ricci curvature bound. Fortunately, we can replace d^2 by f and obtain a maximum principle under a condition similar to (42).

Theorem 6 (Maximum principle on Ricci shrinkers) *Let (M^n, g, f) be a Ricci shrinker. Given any closed interval $[a, b] \subset (-\infty, 1)$ and a function u which satisfies $\Box u \le 0$ on $M \times [a, b]$, suppose that*

$$\int_a^b \int u_+^2(x,t) e^{-2f(x,t)} \, dV_t(x) \, dt < \infty. \tag{43}$$

If $u(\cdot, a) \le c$, then $u(\cdot, b) \le c$.

Proof From Lemmas 1 and 2, it is easy to see

$$\int_a^b \int e^{-2f(x,t)} \, dV_t(x) \, dt < \infty.$$

Therefore, we only need to prove the special case when $c = 0$, by considering $u - c$.

Multiplying both sides of $\square u \le 0$ by $u_+(\phi^r)^2 e^{-2f}$ and integrating on the spacetime $M \times [a, b]$, then we obtain

$$\int_a^b \int \left(\frac{u_+^2}{2}\right)_t (\phi^r)^2 e^{-2f}\, dV_t\, dt \le \int_a^b \int \Delta u u_+ (\phi^r)^2 e^{-2f}\, dV_t\, dt. \tag{44}$$

For the left side of (44), we have

$$\begin{aligned}
&\int_a^b \int \left(\frac{u_+^2}{2}\right)_t (\phi^r)^2 e^{-2f}\, dV_t\, dt \\
&= \int \frac{u_+^2}{2} (\phi^r)^2 e^{-2f}\, dV_b - \int_a^b \int u_+^2 \phi^r \phi^r_t e^{-2f}\, dV_t\, dt \\
&\quad + \int_a^b \int u_+^2 (\phi^r)^2 f_t e^{-2f}\, dV_t\, dt + \int_a^b \int \frac{u_+^2}{2} (\phi^r)^2 R e^{-2f}\, dV_t\, dt \\
&\ge \int \frac{u_+^2}{2} (\phi^r)^2 e^{-2f}\, dV_b - \int_a^b \int u_+^2 \phi^r \phi^r_t e^{-2f}\, dV_t\, dt \\
&\quad + \int_a^b \int u_+^2 (\phi^r)^2 |\nabla f|^2 e^{-2f}\, dV_t\, dt,
\end{aligned} \tag{45}$$

where we have used $R \ge 0$, $f_t = |\nabla f|^2$ and $u_+(\cdot, a) = 0$. For the right side of (44), we have

$$\begin{aligned}
&\int_a^b \int \Delta u u_+ (\phi^r)^2 e^{-2f}\, dV_t\, dt \\
&= \int_a^b \int -|\nabla (u_+ \phi^r)|^2 e^{-2f}\, dV_t\, dt + \int_a^b \int |\nabla \phi^r|^2 u_+^2 e^{-2f}\, dV_t\, dt \\
&\quad + \int_a^b \int 2\langle \nabla u_+, \nabla f\rangle u_+ (\phi^r)^2 e^{-2f}\, dV_t\, dt \\
&= \int_a^b \int -|\nabla (u_+ \phi^r)|^2 e^{-2f}\, dV_t\, dt + \int_a^b \int |\nabla \phi^r|^2 u_+^2 e^{-2f}\, dV_t\, dt \\
&\quad + \int_a^b \int 2\langle \nabla (u_+ \phi^r), \nabla f\rangle u_+ \phi^r e^{-2f}\, dV_t\, dt \\
&\quad - \int_a^b \int 2\langle \nabla \phi^r, \nabla f\rangle u_+^2 \phi^r e^{-2f}\, dV_t\, dt.
\end{aligned} \tag{46}$$

Combining (45) and (46), we obtain

$$\int \frac{u_+^2}{2} (\phi^r)^2 e^{-2f}\, dV_b \le I + II, \tag{47}$$

where

$$\begin{aligned}
I = &- \int_a^b \int u_+^2 (\phi^r)^2 |\nabla f|^2 e^{-2f}\, dV_t\, dt - \int_a^b \int |\nabla (u_+ \phi^r)|^2 e^{-2f}\, dV_t\, dt \\
&+ \int_a^b \int 2\langle \nabla (u_+ \phi^r), \nabla f\rangle u_+ \phi^r e^{-2f}\, dV_t\, dt \le 0,
\end{aligned} \tag{48}$$

and

$$II = \int_a^b \int u_+^2 \phi^r \phi_t^r e^{-2f} \, dV_t \, dt + \int_a^b \int |\nabla \phi^r|^2 u_+^2 e^{-2f} \, dV_t \, dt \\ - \int_a^b \int 2\langle \nabla \phi^r, \nabla f \rangle u_+^2 \phi^r e^{-2f} \, dV_t \, dt. \tag{49}$$

From our construction of ϕ^r, it is easy to see that all functions involved in last three integrals are supported in the spacetime set

$$K_r := \{r \leq F(x,t) \leq 2r, \ a \leq t \leq b\}. \tag{50}$$

Moreover, all the cutoff function terms can be estimated by (37) and (38). For example, we have

$$|\langle \nabla \phi^r, \nabla f \rangle| \leq \bar{\tau}^{-1} |\nabla \phi^r| |\nabla F| \leq C \bar{\tau} r^{-1/2} \sqrt{F} \leq C(1-b)^{-1} \quad \text{on } K_r.$$

Plugging (37), (38) and the above inequality into (49), we arrive at

$$II \leq C \left((1-b)^{-1} + r^{-1} \right) \iint_{K_r} u_+^2 e^{-2f} \, dV_t \, dt. \tag{51}$$

It follows from (47), (48) and (51) that

$$\int \frac{u_+^2}{2} (\phi^r)^2 e^{-2f} \, dV_b \leq C \left((1-b)^{-1} + r^{-1} \right) \iint_{K_r} u_+^2 e^{-2f} \, dV_t \, dt.$$

Note that the left hand side of the above inequality is independent of r. Letting $r \to +\infty$, the finite integral assumption (43) implies that

$$\int \frac{u_+^2}{2} e^{-2f} \, dV_b \leq 0.$$

Therefore, $u(\cdot, b) \leq 0$ by the continuity of u and positivity of $e^{-2f(\cdot,b)}$. □

The condition (43) is satisfied in many cases. For example, if u is a bounded heat solution. The technique used in the proof of Theorem 6 will be repeatedly used in this paper.

Now we control the spacetime integral of $|\mathrm{Hess}\, F|^2$.

Lemma 4 *For any $\lambda > 0$, $a < b < 1$, there exists a constant $C = C(a, b, \lambda)$ such that*

$$\int_a^b \int |Hess\, F|^2 e^{-\lambda F} \, dV_t \, dt \leq C.$$

Proof From (29) and direct computations,

$$\Box |\nabla F|^2 = -2 |\mathrm{Hess}\, F|^2.$$

Multiplying both sides of the above equation by $\phi^r e^{-\lambda F}$ and integrating on the spacetime $M \times [a, b]$, we obtain

$$2 \int_a^b \int |\mathrm{Hess}\, F|^2 \phi^r e^{-\lambda F} \, dV_t \, dt \\ = - \left(\int |\nabla F|^2 \phi^r e^{-\lambda F} \, dV_t \right) \Big|_a^b + \int_a^b \int \Box^* \phi^r |\nabla F|^2 e^{-\lambda F} \, dV_t \, dt$$

$$+\lambda\int_a^b\int\left(\phi^r(\lambda|\nabla F|^2-F_t-\Delta F)-2\langle\nabla\phi^r,\nabla F\rangle\right)|\nabla F|^2\phi^r\,dV_t\,dt$$
$$\leq-\left(\int|\nabla F|^2\phi^r e^{-\lambda F}\,dV_t\right)\Big|_a^b+\int_a^b\int|\Box^*\phi^r|Fe^{-\lambda F}\,dV_t\,dt$$
$$+\lambda\int_a^b\int\left((\lambda+2\bar{\tau}^{-1})F+2r^{-\frac{1}{2}}F^{\frac{1}{2}}\right)Fe^{-\lambda F}\,dV_t\,dt.$$

Now we let $r\to\infty$ and the conclusion follows from Lemmas 1 and 2. □

Theorem 7 *On the Ricci flow spacetime $M\times(-\infty,1)$ induced by a Ricci shrinker (M,g,f), there exists a positive heat kernel function $H(x,t,y,s)$ for all $x,y\in M$ and $s,t\in(-\infty,1)$ with $x\neq y$ and $s<t$. It satisfies*

$$\Box_{x,t}H(x,t,y,s):=(\partial_t-\Delta_x)\,H(x,t,y,s)=0, \tag{52}$$
$$\Box^*_{y,s}H(x,t,y,s):=\left(-\partial_s-\Delta_y+R(y,s)\right)H(x,t,y,s)=0, \tag{53}$$
$$\lim_{t\to s^+}H(x,t,y,s)=\delta_y, \tag{54}$$
$$\lim_{s\to t^-}H(x,t,y,s)=\delta_x. \tag{55}$$

Furthermore, the heat kernel H satisfies the semigroup property

$$H(x,t,y,s)=\int H(x,t,z,\rho)H(z,\rho,y,s)\,dV_\rho(z),\quad\forall\,x,y\in M,\ \rho\in(s,t)\subset(-\infty,1), \tag{56}$$

and the following integral relationships

$$\int H(x,t,y,s)\,dV_t(x)\leq 1, \tag{57}$$
$$\int H(x,t,y,s)\,dV_s(y)=1. \tag{58}$$

Proof We shall divide the proof of Theorem 7 into four steps.
Step 1. Existence of a heat kernel function H solving heat equation and conjugate heat equation.

Fix a compact interval $I=[a,b]\subset(-\infty,1)$ and a compact set $\Omega\subset M$ with smooth boundary, there exists a Dirichlet heat kernel. The proof can be found in in [19, Chapter 24, Sect. 5]. Regarding $(-\infty,1)$ as the union of $[-2^k,1-2^{-k}]$, it is easy to see that the Dirichlet heat kernel actually exists on $\Omega\times(-\infty,1)$. Now we let $\{\Omega_i\}$ be an exhaustion of M by relatively compact domains with smooth boundary such that $\overline{\Omega_i}\subset\Omega_{i+1}$. Let $H_i(x,t,y,s)$ be the Dirichlet heat kernel of $(\overline{\Omega_i},g)$. Then the following properties hold.

$$\partial_t H_i(x,t,y,s)=\Delta_{x,t}H_i(x,t,y,s), \tag{59}$$
$$\partial_s H_i(x,t,y,s)=-\Delta_{y,s}H_i(x,t,y,s)+R(y,s)H_i(x,t,y,s); \tag{60}$$
$$\lim_{t\searrow s}H_i(x,t,y,s)=\delta_y, \tag{61}$$
$$\lim_{s\nearrow t}H_i(x,t,y,s)=\delta_x. \tag{62}$$

Let $\mathbf{n}$ be the outward normal vector of $\partial\Omega_i$, then the positivity of H_i implies that $\frac{\partial H_i}{\partial \mathbf{n}} \le 0$. Since $R \ge 0$ on Ricci shrinkers, direct computation shows that

$$\partial_t \int_{\Omega_i} H_i(x,t,y,s)\,dV_t(x) = \int_{\Omega_i} \left(\Delta_{x,t} - R\right) H_i(x,t,y,s)\,dV_t(x) \le \int_{\partial\Omega_i} \frac{\partial H_i}{\partial \mathbf{n}}\,d\sigma_t(x) \le 0. \tag{63}$$

Hence from (61), we have

$$\int_{\Omega_i} H_i(x,t,y,s)\,dV_t(x) \le 1. \tag{64}$$

Similarly, we have

$$\partial_s \int_{\Omega_i} H_i(x,t,y,s)\,dV_s(y) = -\int_{\Omega_i} \Delta_{y,s} H_i(x,t,y,s)\,dV_s(y) = -\int_{\partial\Omega_i} \frac{\partial H_i}{\partial \mathbf{n}}\,d\sigma_s(y) \ge 0, \tag{65}$$

which implies that

$$\int_{\Omega_i} H_i(x,t,y,s)\,dV_s(y) \le 1. \tag{66}$$

As $H_i > 0$ on $\Omega_i \times (-\infty, 1)$, it follows from the classical maximum principle that

$$0 \le H_i \le H_{i+1} \tag{67}$$

on $\Omega_i \times \Omega_i \times (-\infty, 1)$. Now we define the heat kernel on $M \times (-\infty, 1)$ by

$$H(x,t,y,s) := \lim_{i\to\infty} H_i(x,t,y,s). \tag{68}$$

From the well-known mean value theorem (cf. Theorem 25.2 in [19]) the interior regularity estimates for the heat equation and conjugate heat equation, it follows from (64) and (66) that H_i is uniformly bounded when s, t are fixed. Threfore, H exists as a smooth function. Its positivity is guranteed by (67). The regularity estimates also imply that the convergence from H_i to H is locally smooth. In particular, we can take limit of (59) and (60) to obtain that H solves heat equation and conjugate heat equation on $M \times (-\infty, 1)$. In other words, (52) and (53) are satisfied.

Step 2 The heat kernel is a fundamental solution of heat equation and conjugate heat equation.

Let ϕ be a smooth function on M with compact support K. For fixed y and s, we have

$$\begin{aligned}
&\left|\partial_t \int_{\Omega_i} H_i(x,t,y,s)\phi(x)\,dV_t(x)\right| \\
&= \left|\int_{\Omega_i} (\Delta_{x,t} - R) H_i(x,t,y,s)\phi(x)\,dV_t(x)\right| \\
&\le \left|\int_{\Omega_i} H_i(x,t,y,s)\Delta\phi(x)\,dV_t(x)\right| + \left|\int_{\Omega_i} R H_i(x,t,y,s)\phi(x)\,dV_t(x)\right| \\
&\le C \left|\int_{\Omega_i} H_i(x,t,y,s)\,dV_t(x)\right| \le C,
\end{aligned} \tag{69}$$

where C is independent of H_i. Notice that the last two inequalities hold since we just need to restrict the integral on K, and for a fixed s, when t is close to s, the metrics are uniformly equivalent on $K \times [s, t]$. Combining (61) with (69), we obtain

$$\left| \int_{\Omega_i} H_i(x, t, y, s)\phi(x)\, dV_t(x) - \phi(y) \right| \le C(t - s). \tag{70}$$

Since ϕ has compact support, it is clear that

$$\lim_{i \to \infty} \int_{\Omega_i} H_i(x, t, y, s)\phi(x)\, dV_t(x) = \int H(x, t, y, s)\phi(x) dV_t(x).$$

Plugging the above equation into (70) yields that

$$\left| \int H(x, t, y, s)\phi(x)\, dV_t(x) - \phi(y) \right| \le C(t - s),$$

which means that

$$\lim_{t \to s^+} \int H(x, t, y, s)\phi(x)\, dV_t(x) = \phi(y).$$

By the arbitrary choice of ϕ, we obtain (54). Therefore, H is a fundamental solution of the heat equation. Similary, we can use the limit argument to derive (55) and claim that H is a fundamental solution of the conjugate heat equation.
Step 3 The heat kernel satisfies the semigroup property.

From its construction, H_i satisfies the semigroup property:

$$H_i(x, t, y, s) = \int_{\Omega_i} H_i(x, t, z, \rho)H_i(z, \rho, y, s)\, dV_\rho(z), \quad \forall\, x, y \in \Omega_i,\ \rho \in (s, t) \subset (-\infty, 1). \tag{71}$$

For each compact set $K \subset M$, it is clear that $K \subset \Omega_i$ for large i. By the positivity of each H_i, we have

$$\begin{aligned} H(x, t, y, s) &= \lim_{i \to \infty} H_i(x, t, y, s) = \lim_{i \to \infty} \int_{\Omega_i} H_i(x, t, z, \rho)H_i(z, \rho, y, s)\, dV_\rho(z) \\ &\ge \lim_{i \to \infty} \int_K H_i(x, t, z, \rho)H_i(z, \rho, y, s)\, dV_\rho(z) = \int_K H(x, t, z, \rho)H(z, \rho, y, s)\, dV_\rho(z). \end{aligned}$$

By the arbitrary choice of $K \subset M$, the above inequality implies that

$$H(x, t, y, s) \ge \int H(x, t, z, \rho)H(z, \rho, y, s)\, dV_\rho(z). \tag{72}$$

By (67), (68) and the positivity of H, we have

$$\begin{aligned} H_i(x, t, y, s) &= \int_{\Omega_i} H_i(x, t, z, \rho)H_i(z, \rho, y, s)\, dV_\rho(z) \le \int_{\Omega_i} H(x, t, z, \rho)H(z, \rho, y, s) dV_\rho(z) \\ &< \int H(x, t, z, \rho)H(z, \rho, y, s) dV_\rho(z), \end{aligned}$$

whose limit form is

$$H(x, t, y, s) \le \int H(x, t, z, \rho)H(z, \rho, y, s) dV_\rho(z). \tag{73}$$

Therefore, the semigroup property (56) follows from the combination of (72) and (73).
Step 4 The integral relationships (57) and (58) are satisfied.

On each compact set $K \subset M$, since $K \subset \Omega_i$ for large i and each H_i is positive on Ω_i, we have

$$\int_K H(x,t,y,s)dV_t(x) = \lim_{i\to\infty}\int_K H_i(x,t,y,s)dV_t(x) \le \lim_{i\to\infty}\int_{\Omega_i} H_i(x,t,y,s)dV_t(x) \le 1,$$

where (64) is applied in the last step. The arbitrary choice of K then yields that

$$\int H(x,t,y,s)dV_t(x) \le 1,$$

which is nothing but (57). Similar reasoning can pass (66) to obtain

$$\int_K H(x,t,y,s)dV_s(y) \le 1, \tag{74}$$

where the inequality will be improved to equality (58) in the following argument. In fact, let ϕ^r be the cutoff function defined in (36). For any fixed x and t, it follows from the cutoff function estimate (40) that

$$\left|\partial_s \int H(x,t,y,s)\phi^r(y,s)\,dV_s(y)\right|$$
$$= \left|\int H(x,t,y,s)\Box_{y,s}\phi^r(y,s)\,dV_s(y)\right| \le Cr^{-1}\int H(x,t,y,s)\,dV_s(y).$$

Plugging (74) into the above inequality, we obtain

$$\left|\partial_s \int H(x,t,y,s)\phi^r(y,s)\,dV_s(y)\right| \le Cr^{-1}.$$

When r is large, x is covered by the support of ϕ^r at the time t. Using (55), the above inequality implies that

$$\left|\int H(x,t,y,s)\phi^r(y,s)\,dV_s(y) - 1\right| \le Cr^{-1}(t-s).$$

Since r could be arbitrarily large in the above inequality, we obtain (58) by letting $r \to \infty$. □

Lemma 5 *Suppose* $[a,b] \subset (-\infty,1)$ *and* u_a *is a bounded function on the time slice* $(M, g(a))$. *Then*

$$u(x,t) := \int H(x,t,y,a)u_a(y)dV_a(y), \quad \forall\, t \in [a,b] \tag{75}$$

is the unique bounded heat solution with initial value u_a.

Proof Clearly, u is a well-defined heat solution with the initial value u_a. Suppose $\tilde{u}$ is another heat solution with initial value u_a. Then $u - \tilde{u}$ is a bounded heat solution with initial value 0. Therefore, we can apply maximum principle Theorem 6 on $\pm(u-\tilde{u})$ to obtain that

$$u - \tilde{u} \equiv 0 \quad \text{on } M \times [a,b].$$

In other words, $\tilde{u} \equiv u$ and the uniqueness is proved. □

Corollary 1 *Suppose* u_a *is a smooth, bounded, integrable function on* $(M, g(a))$. *Let* u *be the unique bounded heat solution on* $M \times [a,b]$ *starting from* u_a. *Then we have*

$$\sup_M |\nabla u(\cdot,b)| \le \sup_M |\nabla u(\cdot,a)|. \tag{76}$$

Proof Fix $r >> 1$ and multiply both sides of $\square u = 0$ by $u(\phi^r)^2$ and integrating on $M \times [a, b]$, we obtain

$$\left(\frac{1}{2}\int u^2(\phi^r)^2\, dV_t\right)\Big|_a^b - \int_a^b \int u^2 \phi^r \phi_t^r\, dV_t\, dt = \int_a^b \int \left\{-|\nabla(u\phi^r)|^2 + |\nabla \phi^r|^2 u^2\right\} dV_t\, dt. \tag{77}$$

By Lemma 5 we know

$$u = \int H(x, t, y, a) u_a(y) dV_a.$$

Then it follows from (57) that u is bounded and integrable. Consequently, u^2 is integrable. It follows from (37) and (38) that by letting $r \to \infty$, we obtain from (77)

$$\int_a^b \int |\nabla u|^2\, dV_t\, dt \le -\left(\frac{1}{2}\int u^2 dV_t\right)\Big|_a^b + C\bar{\tau}^{-1}\int_a^b \int u^2\, dV_t\, dt < \infty.$$

Therefore, the assumption of Theorem 6 is satisfied. Since $\square |\nabla u|^2 = -2|\mathrm{Hess}\, u|^2 \le 0$, following the maximum principle, we arrive at (76). □

Proposition 1 *Suppose u is a heat solution and w is a conjugate heat solution on $M \times [a, b]$ for $[a, b] \subset (-\infty, 1)$ such that*

$$\sup_{t \in [a,b]} \int |w|\, dV_t + \sup_{M \times [a,b]} |u| \le C < \infty.$$

Then we have

$$\int uw\, dV_b = \int uw\, dV_a. \tag{78}$$

Proof Fix $r >> 1$. We calculate

$$\begin{aligned}\partial_t \int wu\phi^r\, dV_t &= \int \left\{w\square(u\phi^r) - (u\phi^r)\square^* w\right\} dV_t = \int w\square\left(u\phi^r\right) dV_t \\ &= \int w\left\{u\square\phi^r + \phi^r\square u - 2\langle\nabla u, \nabla\phi^r\rangle\right\} dV_t \\ &= \int w\left\{u\square\phi^r - 2\langle\nabla u, \nabla\phi^r\rangle\right\} dV_t.\end{aligned} \tag{79}$$

Note that $|\nabla u| \le C$ by Corollary 1. Plugging the cutoff function estimates (37) and (40) into the above inequality, we obtain

$$\left|\left(\int wu\phi^r\, dV_t\right)\Big|_a^b\right| \le C(r^{-1} + r^{-\frac{1}{2}}).$$

Taking $r \to \infty$, the right hand side of the above inequality tends to zero, the left hand side converges to

$$\int wu\, dV_b - \int wu\, dV_a,$$

since u is bounded and w is integrable. Consequently, we arrive at (78). □

Lemma 6 *Suppose* $[a, b] \subset (-\infty, 1)$ *and* w_b *is an integrable function on the time slice* $(M, g(b))$. *Then*

$$w(y, s) := \int H(x, b, y, s) w_b(x) \, dV_b(x) \tag{80}$$

is the unique conjugate heat solution with initial value w_b *such that*

$$\sup_{t \in [a,b]} \int |w| \, dV_t < \infty. \tag{81}$$

Proof Fix a time $a_0 \in [a, b]$ and let h be an arbitrary smooth function with compact support. Then we solve the heat equation starting from h to obtain a unique bounded function u as

$$u(x, t) = \int H(x, t, y, a) h(y) \, dV_a(y).$$

Since w is given by (80), it follows from (58) that w satisfies (81). Suppose $\tilde{w}$ is another conjugate heat solution starting from w_b satisfying (81). Then we can apply Lemma 5 to the couple of u and $\tilde{w} - w$ to obtain that for any $t \in [a_0, b]$,

$$\int (\tilde{w}(x, t) - w(x, t)) \, u(x, t) \, dV_t(x) = 0.$$

In particular,

$$\int (\tilde{w}(x, a_0) - w(x, a_0)) \, h(x) \, dV_{a_0}(x) = 0.$$

By the arbitrary choice of h, we obtain $\tilde{w}(\cdot, a_0) - w(\cdot, a_0) \equiv 0$. Then by the arbitrary choice of a_0, we see that

$$\tilde{w}(\cdot, t) \equiv w(\cdot, t), \quad \forall \, t \in [a, b].$$

Therefore, the uniqueness is proved. □

Lemma 7 *Suppose* w *is a bounded function on* $M \times [a, b]$ *satisfying* $\square^* w \leq 0$ *and* (81). *Then we have*

$$\sup_M w(\cdot, a) \leq \sup_M w(\cdot, b). \tag{82}$$

Proof Without loss of generality, by adding a constant, we may assume that $\sup_M w(\cdot, b) = 0$. Then it suffices to show that

$$\sup_M w(\cdot, a) \leq 0. \tag{83}$$

At the time slice $t = a$, we choose an arbitrary nonnegative smooth function h with compact support. Then we solve the forward heat solution starting from h and denote the function by u. It is clear that $u \geq 0$. Similar to the proof of Proposition 1, we obtain that

$$\int w(x, a) h(x) \, dV_a(x) \leq \int w(x, b) u(x, b) \, dV_b(x) \leq 0,$$

since at time $t = b$ we have $u \geq 0$ and $w \leq 0$. Therefore, the inequality (83) follows from the arbitrary choice of h. □

Theorem 8 (Bounded heat solution) *Suppose $t_0 \in (-\infty, 1)$ and h is a bounded function on the time-slice $(M, g(t_0))$. On $M \times (t_0, 1)$, starting from h, there is a unique heat solution u which is bounded on each compact time-interval of $[t_0, 1)$. The solution is*

$$u(x,t) = \int H(x,t,y,t_0)h(y)\,dV_{t_0}(y), \quad \forall\, x \in M,\ t \in (t_0, 1). \tag{84}$$

Similarly, for any bouned integrable function h, starting from h there is a unique conjugate heat solution w which is bounded and integrable uniformly on each compact time interval of $(-\infty, t_0]$. The solution is

$$w(x,t) = \int H(y,t_0,x,t)h(y)\,dV_{t_0}(y), \quad \forall\, x \in M,\ t \in (-\infty, t_0). \tag{85}$$

Theorem 9 (Maximum principle of bounded functions) *Suppose u is a bounded super-heat-solution, i.e., $\square u \le 0$ on $M \times [a,b]$. Then*

$$\sup_M u(\cdot, b) \le \sup_M u(\cdot, a). \tag{86}$$

Similarly, if w is a bounded super-conjugate-heat-solution, i.e., $\square^* w \le 0$ on $M \times [a,b]$ satisfying (81). Then

$$\sup_M w(\cdot, b) \ge \sup_M w(\cdot, a). \tag{87}$$

From (27) and (28) from previous section, on the space-time $M \times (-\infty, 1)$, there are standard heat solution and conjugate heat solutions $F + \frac{n}{2}t$ and $\bar{v} = (4\pi(1-t))^{-\frac{n}{2}}e^{-f}$. We can apply Theorems 6 and 9 to compare other supersolutions or subsolutions with $F + \frac{n}{2}t$ and $\bar{v} = (4\pi(1-t))^{-\frac{n}{2}}e^{-f}$. In particular, we have the following Lemma.

Lemma 8 *Given a smooth function ϕ with compact support on a Ricci shrinker (M^n, g, f). For any $b < 1$, let $w(x,t) = \int H(y,b,x,t)\phi(y)\,dV_b(y)$ be the bounded solution of conjugate heat equation with $w(\cdot, b) = \phi$. Then there exists a constant $C > 0$ such that for $t \le b$*

$$w(x,t) \le C\bar{v}(x,t) = C\frac{e^{-f(x,t)}}{(4\pi\bar{\tau})^{n/2}}. \tag{88}$$

Lemma 8 tells us that starting from a compact supported function, the solution of the conjugate heat equation is at least exponentially decaying.

4 Monotonicity of Perelman's entropy

Recall that on any compact Riemannian manifold (M^n, g), Perelman's $\mathcal{W}$ entropy [46] is defined as

$$\mathcal{W}(g, \phi, \tau) = \int \left(\tau(|\nabla\phi|^2 + R) + \phi - n\right) \frac{e^{-\phi}}{(4\pi\tau)^{n/2}}\,dV \tag{89}$$

for ϕ a smooth function and $\tau > 0$. Let $u^2 = \frac{e^{-\phi}}{(4\pi\tau)^{n/2}}$, we can rewrite above functional as

$$\overline{\mathcal{W}}(g, u, \tau) = \int \tau(4|\nabla u|^2 + Ru^2) - u^2 \log u^2\,dV - \left(n + \frac{n}{2}\log(4\pi\tau)\right)\int u^2\,dV. \tag{90}$$

For a general Ricci shrinker (M^n, g, f), we define the $\boldsymbol{\mu}$-functional as

$$\boldsymbol{\mu}(g,\tau) = \inf\left\{\overline{\mathcal{W}}(g,u,\tau)\,\middle|\, u \in W_*^{1,2}(M)\right\}, \tag{91}$$

where

$$W_*^{1,2}(M) = \left\{u \,\middle|\, \int |\nabla u|^2\, dV < \infty,\ \int u^2\, dV = 1 \text{ and } \int d^2(p,\cdot)u^2\, dV < \infty\right\}. \tag{92}$$

The last integral condition $\int d^2(p,\cdot)u^2\,dV < \infty$ is imposed for two reasons. First, it follows from Lemma 1 and (20) that

$$\int Ru^2\, dV < \infty. \tag{93}$$

Second, the term $\int u^2 \log u^2\, dV$ in the definition of $\overline{\mathcal{W}}(g,u,\tau)$ is well defined. Indeed, if we consider the rescaled measure $d\tilde{V} := e^{-d^2(p,x)}V$, then it follows from the volume estimate Lemma 2 that $\tilde{V}(M)$ is finite. Given a $u \in W_*^{1,2}$, we set $A := \{x \in M \mid u(x) < 1\}$ and $\tilde{u} := \chi_A u$, where χ_A is the characteristic function of the set A. Then it is clear that $\int d^2(p,x)\tilde{u}^2(x)\, dV < \infty$. By a direct calculation,

$$\int \tilde{u}^2 \log \tilde{u}^2\, dV = \int \hat{u}^2 \log \hat{u}^2\, d\tilde{V} - \int d^2(p,\cdot)\tilde{u}^2\, dV, \tag{94}$$

where $\hat{u}^2 = \tilde{u}^2 e^{d^2(p,\cdot)}$. By Jensen's inequality, we obtain

$$\int \hat{u}^2 \log \hat{u}^2\, d\tilde{V} \ge \left(\int \hat{u}^2\, d\tilde{V}\right) \log\left(\frac{1}{\tilde{V}(M)} \int \hat{u}^2\, d\tilde{V}\right) > -\infty$$

since $\int \hat{u}^2\, d\tilde{V} = \int \tilde{u}^2\, dV \in [0,1]$. Therefore it follows from (94) that

$$\int \tilde{u}^2 \log \tilde{u}^2\, dV > -\infty.$$

In other words, it implies that for any $u \in W_*^{1,2}(M)$, the negative part of $u^2 \log u^2$ is integrable and $\overline{\mathcal{W}}(g,u,\tau) \in [-\infty, +\infty)$. In fact, it will be proved later, see Proposition 15 that $\overline{\mathcal{W}}(g,u,\tau)$ cannot be $-\infty$.

Remark 1 The space $W_*^{1,2}(M)$ can be regarded as a collection of probability measure v such that

(i) $v = \rho V$, that is, v is absolutely continuous with respect to the volume form V.
(ii) v has finite moment of second order ($v \in P_2(M)$), that is, for any point $q \in M$,

$$\int d^2(q,\cdot)\, dv < \infty.$$

(iii) The Fisher information

$$F(\rho) := 4\int |\nabla\sqrt{\rho}|^2\, dV < \infty.$$

Now we show that for any Ricci shrinker, we can always restrict the infimum on all smooth functions with compact support.

Proposition 2 *For any Ricci shrinker* (M^n, g, f),

$$\boldsymbol{\mu}(g,\tau) = \inf\left\{\overline{\mathcal{W}}(g,u,\tau)\,\middle|\, u \in C_0^\infty(M) \text{ and } \int u^2\, dV = 1\right\}. \tag{95}$$

Proof For any function $u \in W^{1,2}_*(M)$ such that $\overline{\mathcal{W}}(g,u,\tau)$ is finite, we define a positive constant

$$c_r^2 = \int u^2 (\phi^r)^2 \, dV.$$

It is clear from the definition that $c_r \le 1$ and $\lim_{r\to\infty} c_r = 1$. From direct computations,

$$\begin{aligned}
&\overline{\mathcal{W}}(g, c_r^{-1} u\phi^r, \tau)\\
&= \int c_r^{-2}\tau(4|\nabla(u\phi^r)|^2 + R(u\phi^r)^2) - (c_r^{-1}u\phi^r)^2 \log(c_r^{-1}u\phi^r)^2 \, dV - n - \frac{n}{2}\log(4\pi\tau)\\
&= \int 4\tau c_r^{-2}\left((\phi^r)^2|\nabla u|^2 + |\nabla\phi^r|^2 u^2 + 2u\phi^r\langle\nabla u, \nabla\phi^r\rangle\right) + c_r^{-2}\tau R(u\phi^r)^2 \, dV\\
&\quad - \int (c_r^{-1}\phi^r)^2 u^2 \log u^2 + (c_r^{-1}\phi^r)^2 \log(\phi^r)^2 u^2 \, dV + \log c_r^2 - n - \frac{n}{2}\log(4\pi\tau).
\end{aligned}$$

Now by the definition of $W^{1,2}_*$ and the dominated convergence theorem,

$$\begin{aligned}
&\lim_{r\to\infty} \overline{\mathcal{W}}(g, c_r^{-1}u\phi^r, \tau) - \overline{\mathcal{W}}(g,u,\tau)\\
&= -\lim_{r\to\infty} \int \left(1 - (c_r^{-1}\phi^r)^2\right) u^2 \log u^2 \, dV.
\end{aligned}$$

Since $u^2 \log u^2$ is absolutely integrable, by the dominated convergence theorem,

$$\lim_{r\to\infty} \int \left(1 - (c_r^{-1}\phi^r)^2\right) u^2 \log u^2 \, dV = 0$$

and hence

$$\lim_{r\to\infty} \overline{\mathcal{W}}(g, c_r^{-1}u\phi^r, \tau) = \overline{\mathcal{W}}(g,u,\tau).$$

Similarly, if $\overline{\mathcal{W}}(g,u,\tau) = -\infty$, then

$$\lim_{r\to\infty} \overline{\mathcal{W}}(g, c_r^{-1}u\phi^r, \tau) = -\infty.$$

For a fixed r, it is not hard to choose a sequence of smooth functions u_s with compact support by the usual smoothing process such that

$$\lim_{s\to\infty} \overline{\mathcal{W}}(g, u_s, \tau) = \overline{\mathcal{W}}(g, c_r^{-1}u\phi^r, \tau).$$

□

Now we prove the celebrated monotonicity theorem of Perelman on Ricci shrinkers.

Theorem 10 *For any Ricci shrinker (M^n, g, f) and $\tau > 0$,*

$$\boldsymbol{\mu}(g(t), \tau - t) \tag{96}$$

is increasing for $t < \min\{1, \tau\}$.

Proof We fix a time $t_1 < \min\{1,\tau\}$ and an nonnegative smooth function $\sqrt{\bar{w}}$ with compact support such that $\int \bar{w}\, dV_{t_1} = 1$. By defining

$$w(x,t) = \int H(y, t_1, x, t)\bar{w}(y)\, dV_{t_1}(y), \tag{97}$$

it is straightforward to check that

$$\int w(x,t)\,dV_t(x) = \iint H(y,t_1,x,t)\bar{w}(y)\,dV_t(x)\,dV_{t_1}(y) = \int \bar{w}(y)dV_{t_1}(y) = 1,$$

where we have used stochastic completeness (58) for the last equality.

Lemma 9 *For any time $t_0 < t_1$,*

$$4\int_{t_0}^{t_1}\int |\nabla\sqrt{w}|^2\,dV_t\,dt = \int_{t_0}^{t_1}\int \frac{|\nabla w|^2}{w}\,dV_t\,dt < \infty. \tag{98}$$

Proof of Lemma 9: By direct computations,

$$\begin{aligned}
&\int_{t_0}^{t_1}\int \frac{|\nabla w|^2}{w}\phi^r\,dV_t\,dt\\
&=\int_{t_0}^{t_1}\int \langle\nabla(\log w),\nabla w\rangle\phi^r\,dV_t\,dt\\
&=-\int_{t_0}^{t_1}\int(\log w)\Delta w\phi^r\,dV_t\,dt-\int_{t_0}^{t_1}\int\log w\langle\nabla w,\nabla\phi^r\rangle\,dV_t\,dt\\
&=I+II.
\end{aligned}\tag{99}$$

We estimate I first.

$$\begin{aligned}
I:=&-\int_{t_0}^{t_1}\int(\log w)\Delta w\phi^r\,dV_t\,dt\\
=&\int_{t_0}^{t_1}\int(\log w)w_t\phi^r\,dV_t\,dt-\int_{t_0}^{t_1}\int(\log w)Rw\phi^r\,dV_t\,dt\\
=&\left(\int(\log w)w\phi^r\,dV_t\right)\Big|_{t_0}^{t_1}-\int_{t_0}^{t_1}\int(\log w)_t w\phi^r\,dV_t\,dt-\int_{t_0}^{t_1}\int(\log w)w\phi^r_t\,dV_t\,dt\\
&+\int_{t_0}^{t_1}\int(\log w)Rw\phi^r\,dV_t\,dt-\int_{t_0}^{t_1}\int(\log w)Rw\phi^r\,dV_t\,dt\\
=&\left(\int(\log w)w\phi^r\,dV_t\right)\Big|_{t_0}^{t_1}-\int_{t_0}^{t_1}\int w_t\phi^r\,dV_t\,dt-\int_{t_0}^{t_1}\int(\log w)w\phi^r_t\,dV_t\,dt\\
=&\left(\int(\log w)w\phi^r\,dV_t\right)\Big|_{t_0}^{t_1}-\left(\int w\phi^r\,dV_t\right)\Big|_{t_0}^{t_1}\\
&+\int_{t_0}^{t_1}\int w\phi^r_t\,dV_t\,dt-\int_{t_0}^{t_1}\int wR\phi^r\,dV_t\,dt-\int_{t_0}^{t_1}\int(\log w)w\phi^r_t\,dV_t\,dt
\end{aligned}$$

Now it is easy to show that all integrals in I are bounded. Indeed, from Lemma 8, there exists a constant C such that

$$w(x,t)\le Ce^{-f(x,t)}$$

on $M\times[t_0,t_1]$, where C depends only on t_1, t_2 and the upper bound of $w(\cdot,t_1)$.

Therefore for $t\in[t_0,t_1]$

$$\int|w(\log w)|\,dV_t\le C\int w^{1/2}+w^2\,dV_t\le C\int e^{-f/2}+e^{-f}\,dV_t\le C.$$

Moreover, by using (20),

$$\int_{t_0}^{t_1} \int wR\,dV_t\,dt \le C \int_{t_0}^{t_1} \int fe^{-f}\,dV_t\,dt \le C.$$

Now we estimate II in (99).

$$\begin{aligned}
|II| &\le \left| \int_{t_0}^{t_1} \int \log w \langle \nabla w, \nabla \phi^r \rangle \,dV_t\,dt \right| \\
&\le \int_{t_0}^{t_1} \int |\log w||\nabla w||\nabla \phi^r|\,dV_t\,dt \\
&= \int_{t_0}^{t_1} \int |\log w| \frac{|\nabla w|}{\sqrt{w}} \frac{|\nabla \phi^r|}{\sqrt{\phi^r}} \sqrt{w}\sqrt{\phi^r}\,dV_t\,dt \\
&\le \frac{1}{2} \int_{t_0}^{t_1} \int \frac{|\nabla w|^2}{w} \phi^r\,dV_t\,dt + \frac{1}{2} \int_{t_0}^{t_1} \int w|\log w|^2 \frac{|\nabla \phi^r|^2}{\phi^r}\,dV_t\,dt. \qquad (100)
\end{aligned}$$

By our construction of ϕ^r, $\frac{|\nabla \phi^r|^2}{\phi^r}$ is uniformly bounded. Reasoning as before,

$$\int_{t_0}^{t_1} \int w|\log w|^2 \frac{|\nabla \phi^r|^2}{\phi^r}\,dV_t\,dt \le C \iint w^{1/2} + w^2\,dV_t\,dt \le C \iint e^{-f/2} + e^{-f}\,dV_t\,dt \le C.$$

Now it is easy to see from (99) and (100) that

$$\int_{t_0}^{t_1} \int \frac{|\nabla w|^2}{w} \phi^r\,dV_t\,dt \le C,$$

where C depends only on t_0, t_1 and the upper bound of $w(\cdot, t_1)$. By taking $r \to \infty$, we have proved Lemma 9.

Now we define the function ϕ as

$$w(x,t) = \frac{e^{-\phi}}{(4\pi(\tau - t))^{n/2}}.$$

By direct computations, see Theorem 9.1 of [46], that if we set

$$v = \big((\tau - t)(2\Delta\phi - |\nabla\phi|^2 + R) + \phi - n\big)\,w,$$

then for $t < \tau$,

$$\Box^* v = -2(\tau - t)\left| Rc + \operatorname{Hess}\phi - \frac{g}{2(\tau - t)} \right|^2 w \le 0, \qquad (101)$$

that is, v is a subsolution of the conjugate heat equation.

We set $\tau_1 = \tau_1(t) = \tau - t$ for simplicity. By the definition,

$$v = \tau_1 \left(-2\Delta w + \frac{|\nabla w|^2}{w} + Rw \right) - w\log w - \left(n + \frac{n}{2}\log(4\pi\tau_1) \right) w. \qquad (102)$$

Now we multiply both sides of (101) by ϕ^r so that

$$\int_{t_0}^{t_1} \int v_t \phi^r\,dV_t\,dt \ge -\int_{t_0}^{t_1} \int \Delta v \phi^r\,dV_t\,dt + \int_{t_0}^{t_1} \int Rv\phi^r\,dV_t\,dt. \qquad (103)$$

The left side of (103) is

$$\int_{t_0}^{t_1}\int v_t\phi^r\,dV_t\,dt = -\int_{t_0}^{t_1}\int v\phi_t^r\,dV_t\,dt + \int_{t_0}^{t_1}\int Rv\phi^r\,dV_t\,dt + \left(\int v\phi^r\,dV_t\right)\Big|_{t_0}^{t_1}. \tag{104}$$

The right side of (103) is

$$-\int_{t_0}^{t_1}\int \Delta v\phi^r\,dV_t\,dt + \int_{t_0}^{t_1}\int Rv\phi^r\,dV_t\,dt = -\int_{t_0}^{t_1}\int v\Delta\phi^r\,dV_t\,dt + \int_{t_0}^{t_1}\int R\phi^r\,dV_t\,dt \tag{105}$$

Therefore, we have

$$\left(\int v\phi^r\,dV_t\right)\Big|_{t_0}^{t_1} \geq \int_{t_0}^{t_1}\int v\Box\phi^r\,dV_t\,dt. \tag{106}$$

Now it is important to use the exact expression of $\Box\phi^r$, that is,

$$\Box\phi^r = -nr^{-1}\eta'/2 - r^{-2}\eta''|\nabla F|^2. \tag{107}$$

We consider the first term of v and prove the following lemma. □

Lemma 10

$$\lim_{r\to\infty}\int_{t_0}^{t_1}\int \Delta w\Box\phi^r\,dV_t\,dt = 0. \tag{108}$$

Proof of Lemma 10: From (107), we have

$$\begin{aligned}\int_{t_0}^{t_1}\int \Delta w\Box\phi^r\,dV_t\,dt &= -\frac{n}{2r}\int_{t_0}^{t_1}\int \Delta w\eta'\,dV_t\,dt - r^{-2}\int_{t_0}^{t_1}\int \Delta w\eta''|\nabla F|^2\,dV_t\,dt\\ &= I + II.\end{aligned}$$

Now

$$\begin{aligned}|I| &= \left|-\frac{n}{2r}\int_{t_0}^{t_1}\int \Delta w\eta'\,dV_t\,dt\right| = \left|\frac{n}{2r}\int_{t_0}^{t_1}\int \langle\nabla w, \nabla\eta'\rangle\,dV_t\,dt\right|\\ &= \left|\frac{n}{2r^2}\int_{t_0}^{t_1}\int \langle\nabla w, \nabla F\rangle\eta''\,dV_t\,dt\right| \leq \frac{n}{2r^2}\int_{t_0}^{t_1}\int |\nabla w||\nabla F||\eta''|\,dV_t\,dt\\ &= \frac{n}{2r^2}\int_{t_0}^{t_1}\int \frac{|\nabla w|}{\sqrt{w}}|\nabla F||\eta''|\sqrt{w}\,dV_t\,dt\\ &\leq \frac{n}{2r^2}\left(\int_{t_0}^{t_1}\int \frac{|\nabla w|^2}{w}\,dV_t\,dt\right)^{1/2}\left(\int_{t_0}^{t_1}\int |\nabla F|^2|\eta''|^2 w\,dV_t\,dt\right)^{1/2}.\end{aligned} \tag{109}$$

Now the first integral of (109) is bounded by (98) while the second

$$\int_{t_0}^{t_1}\int |\nabla F|^2|\eta''|^2 w\,dV_t\,dt \leq C\int_{t_0}^{t_1}\int Fw\,dV_t\,dt \leq C\int_{t_0}^{t_1}\int Fe^{-f}\,dV_t\,dt \leq C \tag{110}$$

where the last constant C depends only on t_0, t_1 and the upper bound of $w(\cdot, t_1)$.

It is immediate that from (109) by taking $r\to\infty$ that $\lim_{r\to\infty} I = 0$.

We continue to estimate II.

$$
\begin{aligned}
|II| &= \left| -r^{-2} \int_{t_0}^{t_1} \int \Delta w \eta'' |\nabla F|^2 \, dV_t \, dt \right| \\
&\le \left| r^{-2} \int_{t_0}^{t_1} \int \langle \nabla w, \nabla \eta'' \rangle |\nabla F|^2 \, dV_t \, dt \right| + \left| r^{-2} \int_{t_0}^{t_1} \int \langle \nabla w, \nabla |\nabla F|^2 \rangle \eta'' \, dV_t \, dt \right| \\
&= III + IV.
\end{aligned}
$$

Now we have

$$
\begin{aligned}
III &= \left| r^{-2} \int_{t_0}^{t_1} \int \langle \nabla w, \nabla \eta'' \rangle |\nabla F|^2 \, dV_t \, dt \right| \le r^{-3} \int_{t_0}^{t_1} \int |\nabla w| |\nabla F|^3 |\eta'''| \, dV_t \, dt \\
&\le C r^{-3} \int_{t_0}^{t_1} \int \frac{|\nabla w|}{\sqrt{w}} |\nabla F|^3 \sqrt{w} \, dV_t \, dt \\
&\le C r^{-3} \left(\int_{t_0}^{t_1} \int \frac{|\nabla w|^2}{w} \, dV_t \, dt \right)^{1/2} \left(\int_{t_0}^{t_1} \int |\nabla F|^6 w \, dV_t \, dt \right)^{1/2} \le C
\end{aligned}
$$

since

$$
\int_{t_0}^{t_1} \int |\nabla F|^6 w \, dV_t \, dt \le \int_{t_0}^{t_1} \int F^3 e^{-f} \, dV_t \, dt \le C.
$$

Therefore $\lim_{r \to \infty} III = 0$.

Similarly,

$$
\begin{aligned}
IV &= \left| r^{-2} \int_{t_0}^{t_1} \int \langle \nabla w, \nabla |\nabla F|^2 \rangle \eta'' \, dV_t \, dt \right| \le C r^{-2} \int_{t_0}^{t_1} \int |\nabla w| |\nabla F| |\mathrm{Hess} F| |\eta''| \, dV_t \, dt \\
&\le C r^{-3/2} \int_{t_0}^{t_1} \int |\nabla w| |\mathrm{Hess} F| \, dV_t \, dt = C r^{-3/2} \int_{t_0}^{t_1} \int \frac{|\nabla w|}{\sqrt{w}} |\mathrm{Hess} F| \sqrt{w} \, dV_t \, dt \\
&\le C r^{-3/2} \left(\int_{t_0}^{t_1} \int \frac{|\nabla w|^2}{w} \, dV_t \, dt \right)^{1/2} \left(\int_{t_0}^{t_1} \int |\mathrm{Hess} F|^2 w \, dV_t \, dt \right)^{1/2}.
\end{aligned}
$$

Now from Lemma 4 the last integral is bounded since $w \le C e^{-f}$, so $\lim_{r \to \infty} IV = 0$. Therefore, Lemma 10 is proved.

We can estimate the integral of $v \Box \phi^r$.

From the expression of v in (102), we have

$$
\begin{aligned}
&\int_{t_0}^{t_1} \int v \Box \phi^r \, dV_t \, dt \\
&= \int_{t_0}^{t_1} \int \left(\tau_1 (-2\Delta w + \frac{|\nabla w|^2}{w} + R w) - w \log w - \left(n + \frac{n}{2} \log(4\pi \tau_1) \right) w \right) \Box \phi^r \, dV_t \, dt.
\end{aligned}
$$

Since we have $|\Box \phi^r| \le C r^{-1}$ from (40) and all terms except the first above have bounded integral on spacetime, it is easy to show, by taking into account of the claim, that

$$
\lim_{r \to \infty} \int_{t_0}^{t_1} \int v \Box \phi^r \, dV_t \, dt = 0. \tag{111}
$$

Now from (106),

$$
\lim_{r \to \infty} \int v \phi^r \, dV_{t_1} \ge \lim_{r \to \infty} \int v \phi^r \, dV_{t_0}. \tag{112}
$$

Since we choose $\sqrt{w}(\cdot, t_1)$ to be a smooth function with compact support, it is immediate that

$$\lim_{r\to\infty} \int v\phi^r \, dV_{t_1} = \overline{\mathcal{W}}(g(t_1), \sqrt{w}(\cdot, t_1), \tau - t_1). \tag{113}$$

□

Lemma 11

$$\sqrt{w}(\cdot, t_0) \in W^{1,2}_*$$

and

$$\lim_{r\to\infty} \int \Delta w \phi^r \, dV_{t_0} = 0. \tag{114}$$

Proof of Lemma 11: From the definition of v,

$$\begin{aligned}
&\lim_{r\to\infty} \int v\phi^r \, dV_{t_0} \\
&= \lim_{r\to\infty} \int \left((\tau_1(-2\Delta w + \frac{|\nabla w|^2}{w} + Rw) - w\log w - (n + \frac{n}{2}\log(4\pi\tau_1))w \right) \phi^r \, dV_{t_0}.
\end{aligned}$$

All terms except for the first two in the above integral are absolutely integrable, due to $w \le Ce^{-f}$ and $R \le \tau^{-2}F$.

Combining with (112), we conclude that

$$\lim_{r\to\infty} \int \left(-2\Delta w + \frac{|\nabla w|^2}{w} \right) \phi^r \, dV_{t_0}$$

is bounded above.

Then we have

$$\begin{aligned}
&\lim_{r\to\infty} \int \left(-2\Delta w + \frac{|\nabla w|^2}{w} \right) \phi^r \, dV_{t_0} \\
&= \lim_{r\to\infty} \int 2\langle \nabla w, \nabla \phi^r \rangle + \frac{|\nabla w|^2}{w} \phi^r \, dV_{t_0} \\
&\ge \lim_{r\to\infty} \int -\frac{|\nabla w|^2}{2w}\phi^r - 2\frac{|\nabla \phi^r|^2}{\phi^r} w + \frac{|\nabla w|^2}{w}\phi^r \, dV_{t_0} \\
&= \frac{1}{2} \lim_{r\to\infty} \int \frac{|\nabla w|^2}{w} \phi^r \, dV_{t_0},
\end{aligned} \tag{115}$$

where we have used

$$\lim_{r\to\infty} \int \frac{|\nabla \phi^r|^2}{\phi^r} w \, dV_{t_0} = 0.$$

To prove (114), for any $\epsilon > 0$,

$$\begin{aligned}
\lim_{r\to\infty} \left| \int \Delta w \phi^r \, dV_{t_0} \right| &= \lim_{r\to\infty} \left| \int \langle \nabla w, \nabla \phi^r \rangle \, dV_{t_0} \right| \\
&\le \lim_{r\to\infty} \int \epsilon \frac{|\nabla w|^2}{w} \phi^r + \epsilon^{-1} \frac{|\nabla \phi^r|^2}{4\phi^r} w \, dV_{t_0}
\end{aligned}$$

$$= \epsilon \int \frac{|\nabla w|^2}{w} \, dV_{t_0}$$

By taking $\epsilon \to 0$, we conclude that (114) holds. Therefore, the proof of Lemma 11 is complete.

Therefore,

$$\lim_{r \to \infty} \int v\phi^r \, dV_{t_0} = \overline{\mathcal{W}}(g(t_0), \sqrt{w}(\cdot, t_0), \tau - t_0).$$

In summary, we have shown from (112) that

$$\overline{\mathcal{W}}(g(t_1), \sqrt{w}(\cdot, t_1), \tau - t_1) \geq \overline{\mathcal{W}}(g(t_0), \sqrt{w}(\cdot, t_0), \tau - t_0) \geq \boldsymbol{\mu}(g(t_0), \tau - t_0).$$

Since τ, t_0, t_1 and $\sqrt{w}(\cdot, t_1)$ are arbitrary, the proof of Theorem 10 is complete. □

Corollary 2 *On a Ricci shrinker (M^n, g, f), the functional $\boldsymbol{\mu}(g, \tau)$ is decreasing for $0 < \tau < 1$ and increasing for $\tau > 1$.*

Proof The same argument appeared in Step 1, Proposition 9.5 of [34]. We repeat the argument here for the convenience of the readers.

For a fixed constant $\tau_0 > 1$, from Theorem 10,

$$\boldsymbol{\mu}(g(t), \tau_0 - t) = \boldsymbol{\mu}((1-t)(\psi^t)^* g, \tau_0 - t) = \boldsymbol{\mu}\left(g, \frac{\tau_0 - t}{1 - t}\right)$$

is increasing for $t < 1$. Now as t goes from 0 to 1, $\frac{\tau_0 - t}{1-t}$ goes from τ_0 to ∞. As $\tau_0 > 1$ is arbitrary, we have proved that $\boldsymbol{\mu}(g, \tau)$ is increasing for all $\tau > 1$. Similarly, for any $\tau_0 < 1$, as t goes from 0 to τ_0, $\frac{\tau_0 - t}{1-t}$ goes from τ_0 to 0. Therefore, $\boldsymbol{\mu}(g, \tau)$ is decreasing for all $\tau < 1$. □

5 Optimal logarithmic Sobolev constant—part I

For any Ricci shrinker (M^n, g, f) with the normalization (2), we define

$$\boldsymbol{\mu} = \boldsymbol{\mu}(g) := \log \int \frac{e^{-f}}{(4\pi)^{n/2}} \, dV. \tag{116}$$

It follows from a direct calculation that $e^{\boldsymbol{\mu}}$ is comparable to the volume of the unit ball $B(p, 1)$.

Lemma 12 (cf. Lemma 2.5 of [34]) *For any Ricci shrinker (M^n, g, f), there exists a constant $C = C(n) > 1$ such that*

$$C^{-1} e^{\boldsymbol{\mu}} \leq |B(p, 1)| \leq C e^{\boldsymbol{\mu}}.$$

Next we recall from [1] some standard definitions and properties of the space which satisfies the curvature-dimension estimate.

Definition 1 A Riemannian manifold (M, g, v), equipped with a reference measure $v = e^{-W} V$ where $W \in C^2$ and V is the standard volume form, satisfies the $CD(K, \infty)$ condition if the generalized Ricci tensor

$$Ric_W := Ric + \text{Hess}\, W \geq K g.$$

In particular, on a Ricci shrinker (M^n, g, f), if we define

$$f_0 = f + \boldsymbol{\mu} + \frac{n}{2}\log(4\pi), \tag{117}$$

$$v_0 = e^{-f_0}V \tag{118}$$

then v_0 is a probability measure and $(M, g, v_0) \in CD(\frac{1}{2}, \infty)$. Then the following celebrated theorem of Bakry–Émery can be applied on Ricci shrinkers.

Theorem 11 (Bakry–Émery theorem [2]) *For any Riemannian manifold (M, g, v) satisfying the $CD(K, \infty)$ condition for some $K > 0$, the following logarithmic Sobolev inequality holds*

$$\int \rho \log \rho \, dv \le \frac{1}{2K}\int \frac{|\nabla \rho|^2}{\rho}\, dv, \tag{119}$$

where v and $\rho\, v$ are probability measures which have finite moments of second order and ρ is locally Lipschitz.

The original proof by Bakry and Émery is complete for compact manifolds. A proof using the optimal transport by Lott and Villani for the general case can be found in [39, Corollary 6.12], see also [52, Theorem 21.2]. For the self-containedness, we give a proof of the Bakry–Émery theorem for Ricci shrinkers.

Theorem 12 *For any Ricci shrinker (M^n, g, f) and any nonnegative function ρ such that $\sqrt{\rho} \in W^{1,2}(M, v_0)$ and $\int d^2(p, \cdot)\rho \, dv_0 < \infty$,*

$$\int \rho \log \rho \, dv_0 - \left(\int \rho \, dv_0\right) \log \left(\int \rho \, dv_0\right) \le \int \frac{|\nabla \rho|^2}{\rho}\, dv_0.$$

If the equality holds, then either ρ is a constant or (M^n, g) splits off a $\mathbb{R}$ factor.

Before we prove Theorem 12, we prove the following two lemmas.

Lemma 13 *For any smooth function $u(t, x)$ on $M \times [0, T]$ such that*

$$\square_f u := (\partial_t - \Delta_f)u \le 0,$$

and for some constant $a > 0$,

$$\int_0^T \int u^2(t, x)e^{-ad^2(p,x)}\, dv_0 dt < \infty,$$

if $u(\cdot, 0) \le c$, then $u \le c$ on $M \times [0, T]$.

Proof The proof follows from [35, Theorem 15.2] verbatim by using Δ_f and the measure v_0 instead of Δ and the volume form V. □

We define a new familiy of cutoff functions by setting

$$\overline{\phi}^r := \eta\left(\frac{f}{r}\right),$$

where η is the same function as in (36) and f is the potential function at time 0. A direct calculation shows that

$$\Delta_f \overline{\phi}^r = r^{-2}\eta''|\nabla f|^2 + r^{-1}\eta'\Delta_f f = r^{-2}\eta''|\nabla f|^2 + r^{-1}\eta'(\frac{n}{2} - f).$$

Then it is clear that $\Delta_f \overline{\phi}^r$ is supported on $\{f \geq r\}$ and there exists a constant $C = C(n)$ such that

$$|\Delta_f \overline{\phi}^r| \leq C. \tag{120}$$

Lemma 14 *For any smooth bounded function u on M,*

$$\lim_{r\to\infty} \int (\Delta_f u)\overline{\phi}^r \, dv_0 = 0.$$

Proof From the integration by parts,

$$\lim_{r\to\infty} \int (\Delta_f u)\overline{\phi}^r \, dv_0 = \lim_{r\to\infty} \int u(\Delta_f \overline{\phi}^r) \, dv_0 = 0,$$

where the last equality holds since u is bounded and v_0 is a probability measure. □

Proof of Theorem 12: We only prove the inequality for ρ_0 such that $\sqrt{\rho_0}$ is a compactly supported smooth function and the general case follows from approximations as in Proposition 95. In addition, we assume that $\int \rho_0 \, dv_0 = 1$.

Given such ρ_0, we consider the heat flow with respect to the measure v_0, that is,

$$\begin{cases} \partial_t \rho = \Delta_f \rho, \\ \rho(0, \cdot) = \rho_0. \end{cases}$$

It is clear that there exists a constant C such that $\rho \leq C$ on $M \times [0, \infty)$. Now we set

$$E(t) := \left(\int \rho \log \rho \, dv_0 \right)(t).$$

By direct computations

$$\begin{aligned} \partial_t \int \rho(\log \rho)\overline{\phi}^r \, dv_0 &= \int \rho_t(\log \rho + 1)\overline{\phi}^r \, dv_0 \\ &= \int \Delta_f \rho(\log \rho + 1)\overline{\phi}^r \, dv_0 \\ &= \int -\frac{|\nabla \rho|^2}{\rho}\overline{\phi}^r + \Delta_f(\rho \log \rho)\overline{\phi}^r \, dv_0. \end{aligned}$$

Therefore, for any $T > 0$,

$$\begin{aligned} &\left(\int \rho(\log \rho)\overline{\phi}^r \, dv_0 \right)(T) - \left(\int \rho(\log \rho)\overline{\phi}^r \, dv_0 \right)(0) \\ &= \int_0^T \int -\frac{|\nabla \rho|^2}{\rho}\overline{\phi}^r + \Delta_f(\rho \log \rho)\overline{\phi}^r \, dv_0 dt. \end{aligned}$$

It follows from Lemma 14 that

$$\int_0^T \int \frac{|\nabla \rho|^2}{\rho} \, dv_0 dt < \infty \tag{121}$$

and

$$E(T) - E(0) = -\int_0^T \int \frac{|\nabla \rho|^2}{\rho} \, dv_0 dt. \tag{122}$$

We compute

$$\partial_t \int \frac{|\nabla\rho|^2}{\rho}\overline{\phi}^r \, dv_0 = \int \Box_f\left(\frac{|\nabla\rho|^2}{\rho}\right)\overline{\phi}^r + \Delta_f\left(\frac{|\nabla\rho|^2}{\rho}\right)\overline{\phi}^r \, dv_0. \tag{123}$$

From Bochner's formula,

$$\begin{aligned}\partial_t|\nabla\rho|^2 &= 2\langle\nabla\Delta_f\rho, \nabla\rho\rangle\\ &= \Delta_f|\nabla\rho|^2 - 2|\text{Hess}\,\rho|^2 - 2(Rc + \text{Hess}\, f)(\nabla\rho, \nabla\rho)\\ &= \Delta_f|\nabla\rho|^2 - 2|\text{Hess}\,\rho|^2 - |\nabla\rho|^2,\end{aligned}$$

where we have used the Ricci shrinker equation for the last equality.

Therefore,

$$\Box_f|\nabla\rho|^2 = -2|\text{Hess}\,\rho|^2 - |\nabla\rho|^2. \tag{124}$$

A direct calculation shows that

$$\Box_f\frac{|\nabla\rho|^2}{\rho} = -\frac{2}{\rho}\left|\text{Hess}\,\rho - \frac{d\rho\otimes d\rho}{\rho}\right|^2 - \frac{|\nabla\rho|^2}{\rho}. \tag{125}$$

It follows from (124) and Lemma 13 that there exists a constant $C > 0$ such that

$$\frac{|\nabla\rho|^2}{\rho} \le C. \tag{126}$$

Therefore, by (123) and Lemma 14, for any $T > S > 0$,

$$\begin{aligned}&\left(\int \frac{|\nabla\rho|^2}{\rho}\, dv_0\right)(T) - \left(\int \frac{|\nabla\rho|^2}{\rho}\, dv_0\right)(S)\\ &= \int_S^T \int -\frac{2}{\rho}\left|\text{Hess}\,\rho - \frac{d\rho\otimes d\rho}{\rho}\right|^2 - \frac{|\nabla\rho|^2}{\rho}\, dv_0 dt.\end{aligned} \tag{127}$$

It follows from (122) that for any $t \ge 0$,

$$E'(t) = -\left(\int \frac{|\nabla\rho|^2}{\rho}\, dv_0\right)(t) \le 0 \tag{128}$$

Moreover, for any $t > s \ge 0$, it follows from (127) that

$$-E'(t) + E'(s) \le \int_s^t E'(z)\, dz \le 0. \tag{129}$$

Then it is easy to see from (129) that

$$E'(t) \ge E'(0)e^{-t}. \tag{130}$$

Now we claim that $E(t) \to 0$ if $t \to \infty$. Since $E(t)$ is decreasing by (128), we only need to prove the claim by considering a sequence $t_i \to \infty$. We define $u_i = \sqrt{\rho(t_i, \cdot)}$, then

$$\int u_i^2\, dv_0 = 1 \tag{131}$$

and by (130),

$$\int |\nabla u_i|^2\, dv_0 \to 0. \tag{132}$$

Then by taking a subsequence, we claim that u_i converges to u_∞ weakly in $W^{1,2}(M, v_0)$. It is clear from (131) and (132) that $u_\infty \equiv 1$. Since we can assume that u_i converges to 1 almost everywhere,

$$\lim_{i\to\infty} \int u_i^2 \log u_i^2 \, dv_0 = 0 \tag{133}$$

by the dominated convergence theorem. Therefore, $E(t) \to 0$ if $t \to \infty$.

It follows from (122) and (130) that

$$\int \rho_0 \log \rho_0 \, dv_0 = E(0) = -\int_0^\infty E'(t)\, dt \le -E'(0) \int_0^\infty e^{-t}\, dt = \int \frac{|\nabla \rho_0|^2}{\rho_0}\, dv_0.$$

If the equality holds and ρ_0 is not a constant, it follows from (127) that

$$\mathrm{Hess}(\log \rho) = \frac{1}{\rho}\left(\mathrm{Hess}\,\rho - \frac{d\rho \otimes d\rho}{\rho}\right) = 0.$$

Therefore, (M^n, g) splits off a $\mathbb{R}$ factor.

In summary, the proof of Theorem 12 is complete.

Using the Bakry–Émery theorem, Carrillo and Ni have proved in [10] the following result. □

Proposition 3 (Carrillo–Ni [10]) *For any Ricci shrinker (M^n, g, f), we have*

$$\overline{\mathcal{W}}(g, e^{-\frac{f_0}{2}}, 1) = \boldsymbol{\mu}(g, 1) = \boldsymbol{\mu}, \tag{134}$$

where f_0 is the normalization of f defined in (117).

Proof We shall follow the argument of Carrillo and Ni. The proof is given for the self-containedness.

For any Ricci shrinker (M^n, g, f) and any smooth function u on M with compact support such that $\int u^2\, dV = 1$, we define $w = u^2 e^{f_0}$. Then it is clear that both v_0 and wv_0 belong to $P_2(M)$ from the estimates of f and dV.

It follows from Theorem 12 that

$$\int w \log w \, dv_0 \le \int \frac{|\nabla w|^2}{w}\, dv_0. \tag{135}$$

By rewriting (135) in terms of u, we have

$$\int u^2 \log u^2 \, dV + \int f_0 u^2 \, dV \le \int 4|\nabla u|^2 + |\nabla f_0|^2 u^2 + 4\langle \nabla u, \nabla f_0\rangle u \, dV. \tag{136}$$

It follows from the integration by parts for the last term that (136) becomes

$$\int u^2 \log u^2 \, dV + \boldsymbol{\mu} + \frac{n}{2}\log(4\pi) \le \int 4|\nabla u|^2 + u^2(|\nabla f|^2 - 2\Delta f - f)\, dV. \tag{137}$$

It follows from the $|\nabla f|^2 + R = f$ and $\Delta f + R = \frac{n}{2}$ that $|\nabla f|^2 - 2\Delta f - f = R - f$. Therefore, by (137) that

$$\overline{\mathcal{W}}(g, u, 1) = \int \left\{4|\nabla u|^2 + Ru^2 - u^2 \log u^2\right\} dV - n - \frac{n}{2}\log(4\pi) \ge \boldsymbol{\mu}.$$

By the arbitrary choice of u, the above inequality means that

$$\boldsymbol{\mu}(g, 1) \ge \boldsymbol{\mu}. \tag{138}$$

On the other hand, if we set $u_1 = e^{-\frac{f_0}{2}}$, it follows from direct calculation that

$$\overline{\mathcal{W}}(g, u_1, 1) = \int \left(|\nabla f|^2 + R + f - n\right) e^{-f_0} dV + \boldsymbol{\mu}.$$

Recall that $R + |\nabla f|^2 = f$ and $R + \Delta f = \frac{n}{2}$ on a Ricci shrinker. So the above equation can be simplified as

$$\overline{\mathcal{W}}(g, u_1, 1) - \boldsymbol{\mu} = \int (2f - n)\, e^{-f_0} dV = -2\int (\Delta_f f) e^{-f_0} dV = -2\int (\Delta_{f_0} f) e^{-f_0} dV = 0.$$

Then it follows from definition that

$$\boldsymbol{\mu}(g, 1) \le \overline{\mathcal{W}}(g, u_1, 1) = \boldsymbol{\mu}. \tag{139}$$

Therefore, (134) follows from the combination of (138) and (139). □

Corollary 3 *For any Ricci shrinker (M^n, g, f), if there exist more than one minimizer $u \in W_*^{1,2}$ for $\overline{\mathcal{W}}(g, u, 1)$, then (M, g) must split off a $\mathbb{R}$ factor.*

Proof If u is a minimizer other than $e^{-\frac{f_0}{2}}$, then the same proof as Proposition 3 shows that

$$\int w \log w \, dv_0 = \int \frac{|\nabla w|^2}{w} dv_0,$$

where $w = u^2 e^{f_0}$. Then the conclusion follows from the equality case of Theorem 12. □

Proposition 3 indicates that $\boldsymbol{\mu}$ is the optimal log-Sobolev constant for (M^n, g, f) on scale 1. We shall improve (134) by showing that $\boldsymbol{\mu}$ is in fact the optimal log-Sobolev constant for all scales. Note that the same result has already been proved for compact Ricci shrinkers in Proposition 9.5 of [34].

Proposition 4 *For any Ricci shrinker (M^n, g, f), we have*

$$\boldsymbol{\nu}(g) := \inf_{\tau > 0} \boldsymbol{\mu}(g, \tau) = \boldsymbol{\mu}. \tag{140}$$

We first show two important intermediate steps before we prove Proposition 4.

Lemma 15 *For each $\tau \in (0, 1)$, we have*

$$\boldsymbol{\mu}(g, \tau) \ge \boldsymbol{\mu} = \boldsymbol{\mu}(g, 1). \tag{141}$$

Proof Fix $\eta_0 \in (0, 1)$. Let w be a nonnegative, compactly supported smooth function satisfying the normalization condition $\int w\, dV = 1$. We now regard w as a smooth function on the time slice $t = 0$ and solve the conjugate heat equation $\square^* w = 0$. Then w is a smooth function on the space-time $M \times (-\infty, 0)$. It follows from Lemma 8 that there exists a constant $C > 0$ such that

$$w(x, t) \le C(4\pi(1 - t))^{-\frac{n}{2}} e^{-f(x,t)}, \quad \forall\, x \in M,\ t \in (-\infty, 0]. \tag{142}$$

By the diffeomorphism invariance of the $\overline{\mathcal{W}}$-functional, it is easy to see that

$$\overline{\mathcal{W}}\left(g(t), \sqrt{w(\cdot, t)}, \eta_0 - t\right) = \overline{\mathcal{W}}\left((1 - t)(\psi^t)^* g, \sqrt{w(\cdot, t)}, \eta_0 - t\right) = \overline{\mathcal{W}}\left(g, u(\cdot, t), \theta(t)\right) \tag{143}$$

where we have used the notation

$$u(\cdot,t):=(1-t)^{\frac{n}{4}}\sqrt{((\psi^t)^{-1})^* w(\cdot,t)}, \tag{144}$$

$$\theta(t):=\frac{\eta_0-t}{1-t}. \tag{145}$$

Notice that $\int u^2\,dV \equiv 1$ according to our construction. It follows from definition and direct calculations that

$$\begin{aligned}
&\overline{\mathcal{W}}(g,u(\cdot,t),\theta(t))\\
&=\int\left\{\theta\left(4|\nabla u|^2+Ru^2\right)-u^2\log u^2\right\}dV-n-\frac{n}{2}\log(4\pi\theta)\\
&=\theta\left\{\int\left\{\left(4|\nabla u|^2+Ru^2\right)-u^2\log u^2\right\}dV-n-\frac{n}{2}\log(4\pi)\right\}\\
&\quad+(\theta-1)\left\{\int u^2\log u^2 dV+n+\frac{n}{2}\log(4\pi)\right\}-\frac{n}{2}\log\theta\\
&\geq \theta\boldsymbol{\mu}(g,1)+(\theta-1)\left\{\int u^2\log u^2 dV+n+\frac{n}{2}\log(4\pi)\right\}-\frac{n}{2}\log\theta.
\end{aligned} \tag{146}$$

By (144), the inequality (142) can be understood as

$$u^2(x,t)\leq Ce^{-f(x,0)}$$

for some constant C indepenent of t. Consequently, as $f\geq 0$, we obtain

$$\int u^2\log u^2 dV\leq\int\{-f+\log C\}u^2 dV\leq\log C-\int f\cdot u^2 dV\leq\log C.$$

Note that $\theta(t)<1$ when $t<0$. Plugging the above inequality into (146), and noting that

$$\overline{\mathcal{W}}\left(g(0),\sqrt{w(\cdot,0)},\eta_0\right)\geq\overline{\mathcal{W}}\left(g(t),\sqrt{w(\cdot,t)},\eta_0-t\right),\quad\forall\, t\in(-\infty,0),$$

we can use (143) to obtain

$$\overline{\mathcal{W}}\left(g(0),\sqrt{w(\cdot,0)},\eta_0\right)\geq\theta\boldsymbol{\mu}(g,1)+(\theta-1)\left\{\log C+n+\frac{n}{2}\log(4\pi)\right\}-\frac{n}{2}\log\theta.$$

From (145), it is clear that $\lim_{t\to-\infty}\theta(t)=1$. On the right hand side of the above inequlaity, letting $t\to-\infty$, we arrive at

$$\overline{\mathcal{W}}\left(g(0),\sqrt{w(\cdot,0)},\eta_0\right)\geq\boldsymbol{\mu}(g,1).$$

Since $w(\cdot,0)$ could be arbitrary smooth nonnegative function satisfying the normalization condition, and $g=g(0)$, in light of (95), it is clear that (141) follows from the above inequality. □

Lemma 16 *For each* $\tau\in(1,\infty)$, *we have*

$$\boldsymbol{\mu}(g,\tau)\geq\boldsymbol{\mu}=\boldsymbol{\mu}(g,1). \tag{147}$$

Proof For any $u\in W_*^{1,2}$ and $\tau>1$,

$$\overline{\mathcal{W}}(g,u,\tau)=\int\left\{\tau(4|\nabla u|^2+Ru^2)-u^2\log u^2\right\}dV-n-\frac{n}{2}\log(4\pi\tau)$$

$$\geq \int \left\{(4|\nabla u|^2 + Ru^2) - u^2 \log u^2\right\} dV - n - \frac{n}{2}\log(4\pi\tau)$$
$$\geq \boldsymbol{\mu}(g,1) - \frac{n}{2}\log\tau = \boldsymbol{\mu} - \frac{n}{2}\log\tau.$$

By the arbitrary choice of $u \in W_*^{1,2}$, it follows that

$$\boldsymbol{\mu}(g,\tau) \geq \boldsymbol{\mu} - \frac{n}{2}\log\tau.$$

Let $\tau \to 1^+$, we obtain that

$$\liminf_{\tau\to 1^+} \boldsymbol{\mu}(g,\tau) \geq \boldsymbol{\mu}.$$

By Corollary 2, we know that $\boldsymbol{\mu}(g,\tau)$ is an increasing function of τ for $\tau \in (1,\infty)$. Then it is clear that (147) follows directly from the above inequality. □

Proof of Proposition 4: It follows from the combination of Lemmas 15 and 16. □

Lemma 17 *Suppose (M, g) is a complete Riemannian manifold with Sobolev constant C_{RS}. Namely, for each smooth function u with compact support, we have*

$$\left(\int u^{\frac{2n}{n-2}} dV\right)^{\frac{n-2}{n}} \leq C_{RS}\int \left\{4|\nabla u|^2 + Ru^2\right\} dV. \tag{148}$$

Then for each positive τ, the following estimates hold for any $u \in W_*^{1,2}$,

$$e^{-\frac{2E}{n}} \leq \tau \int \left\{4|\nabla u|^2 + Ru^2\right\} dV \leq \max\left\{n^2, 2E\right\}, \tag{149}$$

where

$$E = \overline{\mathcal{W}}(g,u,\tau) + \frac{n}{2}\log(4\pi e^2 C_{RS}). \tag{150}$$

Proof By Jensen's inequality, we know that

$$\int u^2 \log u^2\, dV = \frac{n-2}{2}\int u^2 \log u^{\frac{4}{n-2}}\, dV \leq \frac{n-2}{2}\log\left(\int u^{\frac{2n}{n-2}}\, dV\right).$$

Plugging the Sobolev inequality (148) into the above inequality yields that

$$\int u^2 \log u^2\, dV \leq \frac{n}{2}\log C_{RS} + \frac{n}{2}\log\int \left\{4|\nabla u|^2 + Ru^2\right\} dV. \tag{151}$$

It follows that

$$\overline{\mathcal{W}}(g,u,\tau) \geq \int \left\{\tau(4|\nabla u|^2 + Ru^2) - u^2\log u^2\right\} dV - n - \frac{n}{2}\log(4\pi\tau)$$
$$\geq \int \tau\left\{4|\nabla u|^2 + Ru^2\right\} dV - \frac{n}{2}\log\int \tau\left\{4|\nabla u|^2 + Ru^2\right\} dV - n - \frac{n}{2}\log(4\pi C_{RS}).$$

Let $x = \int \left(4|\nabla u|^2 + Ru^2\right) dV$. The above inequality can be rewritten as

$$\tau x - \frac{n}{2}\log(\tau x) \leq \overline{\mathcal{W}}(g,u,\tau) + n + \frac{n}{2}\log(4\pi C_{RS}) = E. \tag{152}$$

Since $\tau x > 0$, it follows from (152) that

$$\tau x \geq e^{-\frac{2}{n}E}. \tag{153}$$

On the other hand, it is clear that

$$s-\frac{n}{2}\log s \ge \frac{s}{2} \quad \text{on } [n^2,\infty). \tag{154}$$

Suppose $\tau x \ge n^2$, then the combination of (152) and (154) implies that $\tau x \le 2E$. Consequently, we always have

$$\tau x \le \max\left\{n^2, 2E\right\}. \tag{155}$$

Clearly, (149) follows from the combination of (153) and (155). □

Corollary 4 (Sobolev inequality) *Let* $\left\{(M^n, g(t)),\ t\in(-\infty,1)\right\}$ *be the Ricci flow solution of a Ricci shrinker* (M^n, p, g, f)*, there exists a constant* $C=C(n)$ *such that at any time* $t<1$*,*

$$\left(\int u^{\frac{2n}{n-2}}\,dV_t\right)^{\frac{n-2}{n}} \le Ce^{-\frac{2\boldsymbol{\mu}}{n}}\int \left\{4|\nabla u|^2+Ru^2\right\}dV_t \tag{156}$$

for any smooth function u with compact support.

Proof We consider the Schrödinger operator $H=-2\Delta+\frac{R}{2}$ and the quadratic forms $Q(u):=\int (Hu)u\,dV_t$ with its corresponding Markov semigroup $\{e^{-Hs},\ s\ge 0\}$. Since $\boldsymbol{\mu}(g(t),\tau)=\boldsymbol{\mu}(g,\frac{\tau}{1-t})\ge\boldsymbol{\mu}$, we have

$$\int u^2\log u\,dV_t \le \tau Q(u)+\beta(\tau)$$

for any $\int u^2\,dV_t=1$, where $\beta(\tau)=-\frac{n}{2}-\frac{n}{4}\log(4\pi\tau)-\boldsymbol{\mu}$. Then it follows from [21, Corollary 2.2.8] that for any $s>0$,

$$\|e^{-Hs}\|_{\infty,2}\le e^{M(s)}\le Cs^{-\frac{n}{4}}e^{-\frac{\boldsymbol{\mu}}{2}}, \tag{157}$$

where $M(s):=\frac{1}{s}\int_0^s \beta(\tau)\,d\tau$. Now we use the same argument as in [21, Theorem 2.4.2] to derive the Sobolev inequaltiy. It follows from (157) that for any $u\in L^2$,

$$\|e^{-Hs}u\|_\infty \le Cs^{-\frac{n}{4}}e^{-\frac{\boldsymbol{\mu}}{2}}\|u\|_2. \tag{158}$$

Since e^{-Hs} is self-adjoint, by taking the conjugation of (158) we obtain

$$\|e^{-Hs}u\|_2 \le Cs^{-\frac{n}{4}}e^{-\frac{\boldsymbol{\mu}}{2}}\|u\|_1. \tag{159}$$

Therefore, for any $s>0$,

$$\|e^{-Hs}u\|_\infty \le Cs^{-\frac{n}{4}}e^{-\frac{\boldsymbol{\mu}}{2}}\|e^{-\frac{Hs}{2}}u\|_2 \le Cs^{-\frac{n}{2}}e^{-\boldsymbol{\mu}}\|u\|_1. \tag{160}$$

Combining (160) with the fact that e^{-Hs} is a contraction on L^∞, it follows from the Riesz-Thorin interpolation that for any $q\in[1,\infty]$.

$$\|e^{-Hs}u\|_\infty \le Cs^{-\frac{n}{2q}}e^{-\frac{\boldsymbol{\mu}}{q}}\|u\|_q. \tag{161}$$

We now write

$$H^{-\frac{1}{2}}u=a+b$$

where

$$a = \Gamma^{-1}(1/2) \int_0^T s^{-\frac{1}{2}} e^{-Hs} u \, ds,$$
$$b = \Gamma^{-1}(1/2) \int_T^\infty s^{-\frac{1}{2}} e^{-Hs} u \, ds.$$

It follows from (161) that

$$\|b\|_\infty \le C\Gamma^{-1}(1/2) \int_T^\infty s^{-\frac{1}{2}-\frac{n}{2q}} e^{-\frac{\boldsymbol{\mu}}{q}} \|u\|_q \, ds = ce^{-\frac{\boldsymbol{\mu}}{q}} \|u\|_q T^{\frac{1}{2}-\frac{n}{2q}}$$

for some constant $c = c(n)$. Given $\lambda > 0$, we define $T > 0$ by $\frac{\lambda}{2} = ce^{-\frac{\boldsymbol{\mu}}{q}} \|u\|_q T^{\frac{1}{2}-\frac{n}{2q}}$. It is clear that

$$|\{x : |H^{-\frac{1}{2}} u(x)| \ge \lambda\}| \le |\{x : |a(x)| \ge \lambda/2\}| \le 2^q \lambda^{-q} \|a\|_q^q \le C\lambda^{-q} T^{\frac{q}{2}} \|u\|_q^q,$$

since e^{-Hs} is a contraction on L^q. For any $1 < q < n$, we set $\frac{1}{r} = \frac{1}{q} - \frac{1}{n}$, then it follows from our choice of λ that

$$|\{x : |H^{-\frac{1}{2}} u(x)| \ge \lambda\}| \le Ce^{-\frac{\boldsymbol{\mu} q}{n-q}} \lambda^{-r} \|u\|_q^r.$$

In other words,

$$\|H^{-\frac{1}{2}} u\|_{r,w} \le Ce^{-\frac{\boldsymbol{\mu} q}{r(n-q)}} \|u\|_q \tag{162}$$

where $\|\cdot\|_{r,w}$ denotes the weak L^r space. Therefore, it follows from the Marcinkiewicz interpolation theorem that

$$\|H^{-\frac{1}{2}} u\|_p \le Ce^{-\frac{2\boldsymbol{\mu}}{p(n-2)}} \|u\|_2 = Ce^{-\frac{\boldsymbol{\mu}}{n}} \|u\|_2, \tag{163}$$

where $\frac{1}{p} = \frac{1}{2} - \frac{1}{n}$. Therefore, (156) is a direct consequence. □

Remark 2 It follows from the above corollary that the Yamabe invariant of (M^n, g, f)

$$Y([g]) := \inf_{u \in C_0^\infty(M)} \frac{\int \frac{4(n-1)}{n-2} |\nabla u|^2 + Ru^2 \, dV}{\left(\int u^{\frac{2n}{n-2}} \, dV\right)^{\frac{n-2}{n}}} > 0. \tag{164}$$

Here Y depends only on the conformal class of g. Hence it implies some connections between a Ricci shrinker and its conformal class. Note that it is shown in [63] that each Ricci shrinker has a conformal metric such that its Ricci curvature has local bound depending only on the dimension. This fact plays a key role in [34].

Proposition 5 *On a Ricci shrinker (M^n, g, f), the functional $\boldsymbol{\mu}(g, \tau)$ is a continuous function of $\tau \in (0, \infty)$.*

Proof Fix $\tau_0 \in (0, \infty)$. We need to show both the upper semi-continuity and the lower semi-continuity as τ_0.

The upper-semicontinuity is more or less standard. Fix $u \in W_*^{1,2}$, we have

$$\begin{aligned}\limsup_{\tau \to \tau_0} \boldsymbol{\mu}(g, \tau) &\le \limsup_{\tau \to \tau_0} \overline{\mathcal{W}}(g, u, \tau) \\ &= \limsup_{\tau \to \tau_0} \int \tau(4|\nabla u|^2 + Ru^2) - u^2 \log u^2 \, dV - n - \frac{n}{2} \log(4\pi\tau)\end{aligned}$$

$$= \int \tau_0(4|\nabla u|^2 + Ru^2) - u^2 \log u^2 \, dV - n - \frac{n}{2}\log(4\pi\tau_0)$$
$$= \overline{\mathcal{W}}(g, u, \tau_0).$$

By taking the infimum among all qualified u's, we have

$$\limsup_{\tau \to \tau_0} \boldsymbol{\mu}(g, \tau) \le \boldsymbol{\mu}(g, \tau_0). \tag{165}$$

Hence $\boldsymbol{\mu}(g, \tau)$ is upper semicontinuous.

The lower semicontinuity relies on the estimate (149) in Lemma 17. Actually, for arbitrary $u \in W_*^{1,2}$ satisfying the normalization condition, direct calculation shows that

$$\begin{aligned} \overline{\mathcal{W}}(g, u, \tau) &= \overline{\mathcal{W}}(g, u, \tau_0) + (\tau - \tau_0) \int \left\{4|\nabla u|^2 + Ru^2\right\} dV - \frac{n}{2}\log\left(\frac{\tau}{\tau_0}\right) \\ &\ge \boldsymbol{\mu}(g, \tau_0) - |\tau - \tau_0| \int \left\{4|\nabla u|^2 + Ru^2\right\} dV - \frac{n}{2}\log\left(\frac{\tau}{\tau_0}\right). \end{aligned} \tag{166}$$

For any $\tau_i \to \tau_0$, we choose $u_i \in W_*^{1,2}$ such that

$$\overline{\mathcal{W}}(g, u_i, \tau_i) - \boldsymbol{\mu}(g, \tau_i) < i^{-1}. \tag{167}$$

Together with (165), this implies that

$$\limsup_{i \to \infty} \overline{\mathcal{W}}(g, u_i, \tau_i) = \limsup_{i \to \infty} \boldsymbol{\mu}(g, \tau_i) \le \boldsymbol{\mu}(g, \tau_0). \tag{168}$$

By Corollary 4, the Sobolev constant on each Ricci shrinker is finite. It follows from (149) and (150) that $\int (4|\nabla u_i|^2 + Ru_i^2)\, dV$ is uniformly bounded. In (166), replacing u by u_i and letting $i \to \infty$, we obtain

$$\liminf_{i \to \infty} \overline{\mathcal{W}}(g, u_i, \tau_i) \ge \boldsymbol{\mu}(g, \tau_0).$$

Combining the above inequality with (167), we obtain that

$$\liminf_{i \to \infty} \boldsymbol{\mu}(g, \tau_i) \ge \boldsymbol{\mu}(g, \tau_0),$$

which is the lower semi-continuity at τ_0. The continuity of $\boldsymbol{\mu}(g, \tau)$ with respect to τ at τ_0 follows from the combination of the above inequality and (165). □

6 Optimal logarithmic Sobolev constant—part II

We first prove the log-Sobolev inequality for the conjugate heat kernel following [28]. The proof in [28] is for spacetime with bounded geometry. Since we do not impose any curvature restriction here, more should be done due to the integration by parts.

Theorem 13 *For any Ricci shrinker (M^n, g, f) with its heat kernel $H(x, t, y, s)$,*

$$\int \rho \log \rho \, dv_s - \left(\int \rho \, dv_s\right) \log \left(\int \rho \, dv_s\right) \le (t - s) \int \frac{|\nabla \rho|^2}{\rho} \, dv_s.$$

Here $dv_s(y) = H(x, t, y, s) dV_s(y)$ for any $x \in M$ and $s < t < 1$ and ρ is any nonnegative function such that $\sqrt{\rho} \in W^{1,2}(M, v_s)$ and $\int d^2(p, \cdot)\rho \, dv_s < \infty$. If the equality holds, then either ρ is a constant or (M^n, g) splits off a $\mathbb{R}$ factor.

Proof By a similar approximation process as in Sect. 4, we only need to prove the inequality for any ρ such that $\sqrt{\rho}$ is a compactly supported smooth function. Without loss of generality, we assume $s = 0$ and fix $T > 0$ and $q \in M$. Moreover, we set $w(x,t) = H(q,T,x,t)$, $dv = w(y,0)\,dV_0(y)$ and $\rho(x,t)$ is the bounded solution of the heat equation starting from $\rho(x)$. In the proof, we denote $\rho(x,t)$ by ρ with the time t implicitly understood. We also assume that ρ is uniformly bounded by 1 on $M \times [0,T]$.

It is clear from the definition of w that

$$\lim_{t\nearrow T} \int \rho(\log\rho) w\phi^r\,dV_t = \rho(q,T)\log\rho(q,T) = \left(\int \rho\,dv\right)\log\left(\int \rho\,dv\right)$$

and

$$\int \rho(\log\rho) w\phi^r\,dV_0 = \int \rho\log\rho\phi^r\,dv.$$

Therefore, we have

$$\int \rho\log\rho\phi^r\,dv - \left(\int \rho\,dv\right)\log\left(\int \rho\,dv\right) = \int_0^T -\partial_t \int \rho(\log\rho) w\phi^r\,dV_t\,dt.$$

By direct computations

$$\begin{aligned}
&-\partial_t \int \rho(\log\rho) w\phi^r\,dV_t \\
&= -\int \rho_t(\log\rho+1) w\phi^r + \rho(\log\rho) w_t\phi^r + \rho(\log\rho) w\phi^r_t - R\rho(\log\rho) w\phi^r\,dV_t \\
&= -\int \Delta\rho(\log\rho+1) w\phi^r - \rho(\log\rho)\Delta w\phi^r + \rho(\log\rho) w\phi^r_t\,dV_t \\
&= \int \frac{|\nabla\rho|^2}{\rho} w\phi^r + 2w\langle\nabla(\rho\log\rho), \nabla\phi^r\rangle - \rho(\log\rho) w\Box\phi^r\,dV_t. \qquad (169)
\end{aligned}$$

Similarly,

$$\partial_t \int \frac{|\nabla\rho|^2}{\rho} w\phi^r\,dV_t = \int \Box\left(\frac{|\nabla\rho|^2}{\rho}\right) w\phi^r - 2w\left\langle\nabla\left(\frac{|\nabla\rho|^2}{\rho}\right), \nabla\phi^r\right\rangle + \frac{|\nabla\rho|^2}{\rho} w\Box\phi^r\,dV_t.$$

Since

$$\Box\frac{|\nabla\rho|^2}{\rho} = -\frac{2}{\rho}\left|\mathrm{Hess}\,\rho - \frac{d\rho\otimes d\rho}{\rho}\right|^2 \qquad (170)$$

we have for any $s \in [0,T]$,

$$\begin{aligned}
&\int \frac{|\nabla\rho|^2}{\rho} w\phi^r\,dV_s \\
&= \int \frac{|\nabla\rho|^2}{\rho}\phi^r\,dv - \int_0^s \int 2w\langle\nabla\left(\frac{|\nabla\rho|^2}{\rho}\right), \nabla\phi^r\rangle\,dV_t\,dt \\
&\quad + \int_0^s \int \frac{|\nabla\rho|^2}{\rho} w\Box\phi^r\,dV_t\,dt - \int_0^s \int \frac{2}{\rho}\left|\mathrm{Hess}\,\rho - \frac{d\rho\otimes d\rho}{\rho}\right|^2 w\phi^r\,dV_t\,dt. \qquad (171)
\end{aligned}$$

With (169) and (171), we have proved so far that if r is sufficiently large,

$$\int \rho \log \rho \, dv - \left(\int \rho \, dv \right) \log \left(\int \rho \, dv \right) - T \int \frac{|\nabla \rho|^2}{\rho} \, dv$$
$$= \int_0^T \int 2w \langle \nabla(\rho \log \rho), \nabla \phi^r \rangle \, dV_t \, dt - \int_0^T \int \rho(\log \rho) w \Box \phi^r \, dV_t \, dt$$
$$+ \int_0^T \int_0^s \int \frac{|\nabla \rho|^2}{\rho} w \Box \phi^r \, dV_t dt ds - \int_0^T \int_0^s \int 2w \langle \nabla \left(\frac{|\nabla \rho|^2}{\rho} \right), \nabla \phi^r \rangle \, dV_t dt ds$$
$$- \int_0^T \int_0^s \int \frac{2}{\rho} \left| \mathrm{Hess}\, \rho - \frac{d\rho \otimes d\rho}{\rho} \right|^2 w \phi^r \, dV_t \, dt \, ds$$
$$= I + II + III + IV + V,$$

where

$$I = \int_0^T \int 2w \langle \nabla(\rho \log \rho), \nabla \phi^r \rangle \, dV_t \, dt,$$
$$II = \int_0^T \int -\rho(\log \rho) w \Box \phi^r \, dV_t \, dt,$$
$$III = \int_0^T \int_0^s \int \frac{|\nabla \rho|^2}{\rho} w \Box \phi^r \, dV_t dt ds,$$
$$IV = \int_0^T \int_0^s \int -2w \langle \nabla \left(\frac{|\nabla \rho|^2}{\rho} \right), \nabla \phi^r \rangle \, dV_t dt ds,$$
$$V = \int_0^T \int_0^s \int -\frac{2}{\rho} \left| \mathrm{Hess}\, \rho - \frac{d\rho \otimes d\rho}{\rho} \right|^2 w \phi^r \, dV_t \, dt ds.$$

It remains to show that when $r \to \infty$ the sum is less or equal to 0.

We first notice that as ρ is smooth with compact support, by using (170) and the maximum principle,

$$\frac{|\nabla \rho|^2}{\rho} \le C.$$

Here the assumption in Theorem 6 can be checked as (98).

Now we have for the first term I

$$\lim_{r \to \infty} |I| \le \lim_{r \to \infty} 2 \int_0^T \int w |\nabla \rho| (1 + |\log \rho|) |\nabla \phi^r| \, dV_t \, dt \le \lim_{r \to \infty} C r^{-1/2} = 0.$$

For the second term II,

$$\lim_{r \to \infty} |II| \le \lim_{r \to \infty} \int_0^T \int w \rho |\log \rho| |\Box \phi^r| \, dV_t \, dt \le \lim_{r \to \infty} C r^{-1} = 0.$$

Similarly for the third term III,

$$\lim_{r \to \infty} |III| \le \lim_{r \to \infty} \int_0^T \int_0^s \int \frac{|\nabla \rho|^2}{\rho} w |\Box \phi^r| \, dV_t dt ds \le \lim_{r \to \infty} C r^{-1} = 0.$$

The fourth term IV is more involved, by computation we have

$$\nabla \frac{|\nabla \rho|^2}{\rho} = 2 \frac{\langle \mathrm{Hess}\, \rho, \nabla \rho \rangle}{\rho} - \frac{|\nabla \rho|^2}{\rho^2} \nabla \rho = 2 \frac{\langle \mathrm{Hess}\, \rho - \rho^{-1} d\rho \otimes d\rho, \nabla \rho \rangle}{\rho} + \frac{|\nabla \rho|^2}{\rho^2} \nabla \rho. \tag{172}$$

From (172), we have

$$\begin{aligned}|IV| &\le \int_0^T \int_0^s \int 2w|\nabla\phi^r| \left(2\frac{|h||\nabla\rho|}{\rho} + \frac{|\nabla\rho|^3}{\rho^2}\right) dV_t dt ds \\ &\le \int_0^T \int_0^s \int 2\epsilon \frac{|h|^2}{\rho} w\phi^r + 2\epsilon^{-1} \frac{|\nabla\rho|^2}{\rho} \frac{|\nabla\phi^r|^2}{\phi^r} w + 2w|\nabla\phi^r| \frac{|\nabla\rho|^3}{\rho^2} dV_t dt ds \\ &\le -\epsilon V + C\epsilon^{-1} r^{-1} + 2r^{-1/2} \int_0^T \int_0^s \int w \frac{|\nabla\rho|^3}{\rho^2} dV_t dt ds\end{aligned}$$

where we denote $\text{Hess}\,\rho - \rho^{-1} d\rho \otimes d\rho$ by h and $\epsilon \in (0, 1)$.

To deal with the last integral, we notice from Lemma 18 that

$$\frac{|\nabla\rho|^3}{\rho^2} = \frac{|\nabla\rho|^{3/2}}{\rho^{3/2}} \frac{|\nabla\rho|^{3/2}}{\rho^{3/4}} \rho^{1/4} \le \frac{C}{t^{3/4}} \left(\rho^{1/6} \log \frac{M}{\rho}\right)^{3/2} \le \frac{C}{t^{3/4}}$$

and hence

$$\int_0^s \int w \frac{|\nabla\rho|^3}{\rho^2} dV_t dt \le C \int_0^s t^{-3/4} dt \le C.$$

Therefore, $\lim_{r\to\infty} V$ is finite and $\lim_{r\to\infty} |IV| \le -\epsilon\,(\lim_{r\to\infty} V)$. By taking $\epsilon \to 0$, we obtain that $\lim_{r\to\infty} |IV| = 0$ and hence

$$\int \rho \log \rho \, dv - \left(\int \rho \, dv\right) \log \left(\int \rho \, dv\right) - T \int \frac{|\nabla\rho|^2}{\rho} dv \tag{173}$$

$$= -\int_0^T \int_0^s \int \frac{2}{\rho} \left|\text{Hess}\,\rho - \frac{d\rho \otimes d\rho}{\rho}\right|^2 w \, dV_t \, dt \, ds \le 0. \tag{174}$$

If the equality holds and ρ is not a constant, it follows from (173) that

$$\text{Hess}(\log \rho) = \frac{1}{\rho} \left(\text{Hess}\,\rho - \frac{d\rho \otimes d\rho}{\rho}\right) = 0.$$

Therefore, (M^n, g) splits off a $\mathbb{R}$ factor. □

For fixed x, t and s, Theorem 13 implies that the probability measure $dv_s(y) = H(x, t, y, s) dV_s(y)$ satisfies the log-Sobolev inequality with the constant $\frac{1}{2(t-s)}$. It is a standard fact that log-Sobolev condition implies the Talagrand's inequality and equivalently, the Gaussian concentration, see [52, Theorems 22.17, 22.10]. In particular we have the following theorem, see also [28, Theorem 1.13].

Theorem 14 (Gaussian concentration) *For any Ricci shrinker (M^n, g, f) with its heat kernel $H(x, t, y, s)$ and reference measure $dv_s(y) = H(x, t, y, s) dV_s(y)$ and any $\sigma > 0$*

$$v_s(A) v_s^{\frac{1}{\sigma}}(B) \le \exp\left(-\frac{r^2}{4(1+\sigma)(t-s)}\right)$$

where A and B are two sets on M such that $d_s(A, B) \ge r > 0$.

Proof From Theorem (13), we have for any probability measure ρdv_s,

$$\int \rho \log \rho \, dv_s \le (t-s) \int \frac{|\nabla\rho|^2}{\rho} dv_s. \tag{175}$$

By a further approximation, we can assume (175) holds for any locally Lipschitz ρ. Now it follows from [52, Theorem 22.17] that dv_s satisfies the T_2 Talagrand inequality, that is,

$$W_2(\eta, v_s) \le \sqrt{4(t-s)} \left(\int \rho \log \rho \, dv_s \right)^{1/2} \tag{176}$$

for any measure $\eta \in P_2(M)$, where W_2 is the Wasserstein distance of second order. For any two sets A and B on M such that $d_s(A, B) \ge r > 0$. We set $\eta = \frac{1_A}{v_s(A)} v_s$ and $v = \frac{1_B}{v_s(B)} v_s$. Then on the one hand,

$$\begin{aligned} W_2(\eta, v) &\le W_2(\eta, v_s) + W_2(v, v_s) \\ &\le \sqrt{4(t-s)} \left(\left(\int \frac{1_A}{v_s(A)} \log \frac{1_A}{v_s(A)} \, dv_s \right)^{1/2} + \left(\int \frac{1_B}{v_s(B)} \log \frac{1_B}{v_s(B)} \, dv_s \right)^{1/2} \right) \\ &= \sqrt{4(t-s)} \left(\left(\log \frac{1}{v_s(A)} \right)^{1/2} + \left(\log \frac{1}{v_s(B)} \right)^{1/2} \right) \end{aligned}$$

and hence

$$\begin{aligned} W_2^2(\eta, v) &\le 4(t-s) \left(\left(\log \frac{1}{v_s(A)} \right)^{1/2} + \left(\log \frac{1}{v_s(B)} \right)^{1/2} \right)^2 \\ &\le 4(t-s) \left((1+\sigma) \log \frac{1}{v_s(A)} + (1+\sigma^{-1}) \log \frac{1}{v_s(B)} \right). \end{aligned}$$

On the other hand, it follows from the definition of W_2 that

$$W_2^2(\eta, v) = \int d_s^2(x, y) \, d\pi(x, y) \ge r^2,$$

where π is the optimal transport between η and v.

Therefore by computation

$$v_s(A) v_s^{\frac{1}{\sigma}}(B) \le \exp\left(-\frac{r^2}{4(1+\sigma)(t-s)} \right).$$

□

In fact, with the Gaussian concentration, we can prove that v_s has finite square-exponential moment.

Corollary 5 *For any Ricci shrinker (M^n, g, f) with its heat kernel $H(x, t, y, s)$ and reference measure $dv_s(y) = H(x, t, y, s) dV_s(y)$, if $a < \frac{1}{4(t-s)}$, then*

$$\int e^{a d_s^2(p, x)} \, dv_s < \infty.$$

Proof We choose a constant $\sigma > 0$ such that $a < \frac{1}{4(1+\sigma)(t-s)}$. It follows from Theorem 14 that for any integer $k \ge 2$,

$$v_s(M \backslash B_s(p, k)) \le \exp\left(-\frac{(k-1)^2}{4(1+\sigma)(t-s)} \right).$$

Hence

$$\int e^{ad_s^2(p,x)}\,dv_s \le C\left(1+\sum_{k=2}^{\infty}\int_{B_s(p,k+1)\setminus B_s(p,k)} e^{ad_s^2(p,x)}\,dv_s\right)$$
$$\le C + C\sum_{k=2}^{\infty}(k+1)^n \exp\left(a(k+1)^2 - \frac{(k-1)^2}{4(1+\sigma)(t-s)}\right)$$

where we have used Lemma 2. Since $a < \frac{1}{4(1+\sigma)(t-s)}$, it is easy to show that the last sum is finite. □

7 Heat kernel estimates

We first prove a pointwise upper bound for the heat kernel H. The idea of the proof is from [21, Chapter 2], see also [61].

Theorem 15 (Ultracontractivity) *For any Ricci shrinker* (M^n, g, f)*,*

$$H(x,t,y,s) \le \frac{e^{-\mu}}{(4\pi(t-s))^{\frac{n}{2}}}.$$

Proof We fix $x \in M$ and two constants $s < T < 1$. For notational simplicity, we assume that $\tau = T - t$ and $\partial_\tau = -\partial_t$. We also fix a function $p(\tau) = \frac{T-s}{T-s-\tau}$ for $\tau \in [0, T-s)$. For any nonnegative smooth function h with compact support we define

$$w(y,\tau) = \int H(x,T,y,T-\tau)h(x)\,dV_T(x), \tag{177}$$

then $\square^* w = 0$.

Now we compute,

$$\begin{aligned}
\partial_\tau ||w\phi^r||_{p(\tau)} &= \partial_\tau\left(\int (w\phi^r)^{p(\tau)}\,dV_{T-\tau}\right)^{\frac{1}{p(\tau)}}\\
&= -\frac{p'(\tau)}{p^2(\tau)}||w\phi^r||_{p(\tau)}\log\left(\int (w\phi^r)^{p(\tau)}\,dV_{T-\tau}\right)\\
&\quad+\frac{1}{p(\tau)}\left(\int (w\phi^r)^{p(\tau)}dV_{T-\tau}\right)^{\frac{1}{p(\tau)}-1}\left(\int (w\phi^r)^{p(\tau)}(\log w\phi^r)p'(\tau)dV_{T-\tau}\right)\\
&\quad+\frac{1}{p(\tau)}\left(\int (w\phi^r)^{p(\tau)}dV_{T-\tau}\right)^{\frac{1}{p(\tau)}-1}\left(\int p(\tau)(w\phi^r)^{p(\tau)-1}(w\phi^r)_\tau + R(w\phi^r)^{p(\tau)}dV_{T-\tau}\right).
\end{aligned}$$

If we multiply both sides above by $p^2(\tau)||w\phi^r||_{p(\tau)}^{p(\tau)}$ and use the fact

$$(w\phi^r)_\tau = \Delta w\phi^r - Rw\phi^r + w\phi^r_\tau = \Delta(w\phi^r) - Rw\phi^r - (\square_\tau\phi^r)w - 2\langle\nabla w, \nabla\phi^r\rangle, \tag{178}$$

then we have

$$\begin{aligned}
&p^2(\tau)||w\phi^r||_{p(\tau)}^{p(\tau)}\partial_\tau||w\phi^r||_{p(\tau)}\\
&= -p'(\tau)||w\phi^r||_{p(\tau)}^{p(\tau)+1}\log\left(\int (w\phi^r)^{p(\tau)}\,dV_{T-\tau}\right)\\
&\quad+p(\tau)p'(\tau)||w\phi^r||_{p(\tau)}\int (w\phi^r)^{p(\tau)}\log(w\phi^r)\,dV_{T-\tau}
\end{aligned}$$

$$- p^2(\tau)(p(\tau)-1)||w\phi^r||_{p(\tau)} \int (w\phi^r)^{p(\tau)-2}|\nabla(w\phi^r)|^2 \, dV_{T-\tau}$$
$$- (p(\tau)-1)||w\phi^r||_{p(\tau)} \int R(w\phi^r)^{p(\tau)} \, dV_{T-\tau} + X, \tag{179}$$

where

$$X = p^2(\tau)||w\phi^r||_{p(\tau)} \int (w\phi^r)^{p(\tau)-1} \left(-(\Box_\tau \phi^r)w - 2\langle \nabla w, \nabla \phi^r \rangle\right) dV_{T-\tau}.$$

Now we divide both sides of (179) by $||w\phi^r||_{p(\tau)}$, then

$$\begin{aligned} &p^2(\tau)||w\phi^r||_{p(\tau)}^{p(\tau)} \partial_\tau \log ||w\phi^r||_{p(\tau)} \\ &= -p'(\tau)||w\phi^r||_{p(\tau)}^{p(\tau)} \log \left(\int (w\phi^r)^{p(\tau)} \, dV_{T-\tau} \right) \\ &\quad + p(\tau)p'(\tau) \int (w\phi^r)^{p(\tau)} \log(w\phi^r) \, dV_{T-\tau} \\ &\quad - 4(p(\tau)-1) \int |\nabla(w\phi^r)^{\frac{p(\tau)}{2}}|^2 \, dV_{T-\tau} \\ &\quad - (p(\tau)-1) \int R(w\phi^r)^{p(\tau)} \, dV_{T-\tau} + Y, \end{aligned} \tag{180}$$

where

$$Y = p^2(\tau) \int (w\phi^r)^{p(\tau)-1} \left(-(\Box_\tau \phi^r)w - 2\langle \nabla w, \nabla \phi^r \rangle\right) dV_{T-\tau}.$$

We denote $v = (w\phi^r)^{\frac{p(\tau)}{2}} / ||(w\phi^r)^{\frac{p(\tau)}{2}}||_2$ so that $||v||_2 = 1$. Now by direct computations,

$$v^2 \log v^2 = p(\tau)v^2 \log(w\phi^r) - 2v^2 \log ||(w\phi^r)^{\frac{p(\tau)}{2}}||_2.$$

So (180) becomes

$$\begin{aligned} &p^2(\tau)\partial_\tau \log ||w\phi^r||_{p(\tau)} \\ &= p'(\tau) \int v^2 \log v^2 \, dV_{T-\tau} - 4(p(\tau)-1) \int |\nabla v|^2 \, dV_{T-\tau} - (p(\tau)-1) \int Rv^2 \, dV_{T-\tau} + Z \end{aligned}$$

where

$$Z = \frac{p^2(\tau)}{||w\phi^r||_{p(\tau)}^{p(\tau)}} \int (w\phi^r)^{p(\tau)-1} \left(-(\Box_\tau \phi^r)w - 2\langle \nabla w, \nabla \phi^r \rangle\right) dV_{T-\tau}.$$

Now we obtain

$$p^2(\tau)\partial_\tau \log ||w\phi^r||_{p(\tau)} = p'(\tau) \left(\int v^2 \log v^2 \, dV_{T-\tau} - \frac{p(\tau)-1}{p'(\tau)} \int 4|\nabla v|^2 + Rv^2 \, dV_{T-\tau} \right) + Z. \tag{181}$$

Since $\frac{p(\tau)-1}{p'(\tau)} = \frac{\tau(T-s-\tau)}{T-s} > 0$, we have from (181)

$$p^2(\tau)\partial_\tau \log ||w\phi^r||_{p(\tau)} \le p'(\tau) \left(-\boldsymbol{\mu} - n - \frac{n}{2} \log(4\pi(p(\tau)-1)/p'(\tau)) \right) + Z. \tag{182}$$

Now we divide both sides by $p^2(\tau)$, we have

$$\partial_\tau \log ||w\phi^r||_{p(\tau)} \le \frac{p'(\tau)}{p^2(\tau)}\left(-\boldsymbol{\mu} - n - \frac{n}{2}\log(4\pi) - \frac{n}{2}\log\left(\frac{p(\tau)-1)}{p'(\tau)}\right)\right) + U(\tau), \tag{183}$$

where

$$U(\tau) = \frac{1}{||w\phi^r||_{p(\tau)}^{p(\tau)}}\int (w\phi^r)^{p(\tau)-1}\left(-(\Box_\tau\phi^r)w - 2\langle\nabla w, \nabla\phi^r\rangle\right) dV_{T-\tau}.$$

Now we integrate both sides of (183) and estimate the two terms of right side separately. For a number $L < T - s$, we integrate (183) from 0 to L so that

$$\begin{aligned}
&\log ||w\phi^r||_{p(L)} - \log ||w\phi^r||_1 \\
&\le \int_0^L \frac{p'(\tau)}{p^2(\tau)}\left(-\boldsymbol{\mu} - n - \frac{n}{2}\log(4\pi) - \frac{n}{2}\log\left(\frac{p(\tau)-1)}{p'(\tau)}\right)\right) d\tau + \int_0^L U(\tau)\,d\tau \\
&= I(L) + II(L).
\end{aligned} \tag{184}$$

By direct computations,

$$\begin{aligned}
I(T-s) &= \int_0^{T-s} \frac{p'(\tau)}{p^2(\tau)}\left(-\boldsymbol{\mu} - n - \frac{n}{2}\log(4\pi) - \frac{n}{2}\log\left(\frac{p(\tau)-1)}{p'(\tau)}\right)\right) d\tau \\
&= -\frac{n}{2}\log(T-s) - \boldsymbol{\mu} - \frac{n}{2}\log(4\pi).
\end{aligned} \tag{185}$$

Now we consider the term $U(\tau)$.

$$|U(\tau)| \le \frac{1}{||w\phi^r||_{p(\tau)}^{p(\tau)}}\int w^{p(\tau)}|\Box_\tau\phi^r| + 2|\nabla w||\nabla\phi^r|\,dV_{T-\tau}. \tag{186}$$

Since we construct w through a smooth function with compact support,

$$w \le Ce^{-f}$$

for a constant C uniformly on $M \times [T-s-L, T-s]$. On the other hand, by Lemma 11 $\sqrt{w} \in W_*^{1,2}$ for any $\tau > 0$, in particular

$$\int \frac{|\nabla w|^2}{w}\,dV_{T-\tau} < \infty.$$

Now the second term in (186) can be estimated as

$$\int |\nabla w||\nabla\phi^r|\,dV_{T-\tau} = \int \frac{|\nabla w|}{\sqrt{w}}|\nabla\phi^r|\sqrt{w}\,dV_{T-\tau} \le \left(\int \frac{|\nabla w|^2}{w}\,dV_{T-\tau}\right)^{\frac{1}{2}}\left(\int |\nabla\phi^r|^2 w\,dV_{T-\tau}\right)^{\frac{1}{2}}.$$

For any fixed L, it is easy to say $U(\tau)$ is uniformly bounded for any $\tau \in [T-s-L, T-s]$ and $r \ge 1$. By taking $r \to \infty$ in (184), from the dominated convergence theorem,

$$\log ||w||_{p(L)} - \log ||w||_1 \le I(L).$$

Now by taking $L \to T - s$ we have

$$\log ||w||_\infty - \log ||w||_1 \le -\frac{n}{2}\log(T-s) - \boldsymbol{\mu} - \frac{n}{2}\log(4\pi).$$

Therefore,

$$\int H(x,T,y,s)h(x)\,dV_T(x) \le \frac{e^{-\mu}}{(4\pi(T-s))^{n/2}} \iint H(x,T,y,s)h(x)\,dV_T(x)\,dV_s(y)$$
$$= \frac{e^{-\mu}}{(4\pi(T-s))^{n/2}} \int h(x)\,dV_T(x).$$

Since $h(x)$ can be any smooth function with compact support, we derive that

$$H(x,T,y,s) \le \frac{e^{-\mu}}{(4\pi(T-s))^{n/2}}.$$

□

Now we derive the lower bound of H. Recall that the reduced distance between (x,t) and (y,s) are defined as

$$l_{(x,t)}(y,s) = \frac{1}{2\sqrt{t-s}} \inf\left\{\mathcal{L}(\gamma) : \gamma : [s,t] \to M \text{ between } (x,t) \text{ and } (y,s)\right\}, \tag{187}$$

where

$$\mathcal{L}(\gamma) = \int_s^t \sqrt{t-z}\left(|\gamma'(z)|_z^2 + R(\gamma(z),z)\right)\,dz. \tag{188}$$

Now we have the following important estimate, see Corollary 9.5 of [46]. The proof is motivated by [16, Proposition 1].

Theorem 16 *For any Ricci shrinker* (M^n,g,f),

$$H(x,t,y,s) \ge \frac{e^{-l_{(x,t)}(y,s)}}{(4\pi(t-s))^{\frac{n}{2}}}.$$

Proof We set

$$L(x,t,y,s) = \frac{e^{-l_{(x,t)}(y,s)}}{(4\pi(t-s))^{\frac{n}{2}}}. \tag{189}$$

It follows from the definition of $l_{(x,t)}(y,s)$, see [46] and [56], that

$$-\partial_s L(x,t,y,s) \le \Delta_{y,s} L(x,t,y,s) - R(y,s)L(x,t,y,s) \tag{190}$$

and

$$\lim_{s\nearrow t} L(x,t,y,s) = \delta_x. \tag{191}$$

For any $x,y\in M$, $s<T$ and small $\epsilon>0$ we have

$$\begin{aligned}
&\int L(x,T,z,T-\epsilon)H(z,T-\epsilon,y,s)\phi^r(z,T-\epsilon)\,dV_{T-\epsilon}(z)\\
&\quad - \int L(x,T,z,s+\epsilon)H(z,s+\epsilon,y,s)\phi^r(z,s+\epsilon)\,dV_{s+\epsilon}(z)\\
&= \int_{s+\epsilon}^{T-\epsilon} \partial_t\left(\int L(x,T,z,t)H(z,t,y,s)\phi^r(z,t)\,dV_t\right)\,dt\\
&= \int_{s+\epsilon}^{T-\epsilon}\int L_t H\phi^r\,dV_t\,dt + \int_{s+\epsilon}^{T-\epsilon}\int LH_t\phi^r\,dV_t\,dt
\end{aligned}$$

$$
\begin{aligned}
&+\int_{s+\epsilon}^{T-\epsilon}\int LH\phi_t^r\,dV_t\,dt-\int_{s+\epsilon}^{T-\epsilon}\int LH\phi^r R\,dV_t\,dt\\
&\geq -\int_{s+\epsilon}^{T-\epsilon}\int \Delta LH\phi^r dV_t dt+\int_{s+\epsilon}^{T-\epsilon}\int L\Delta H\phi^r dV_t dt+\int_{s+\epsilon}^{T-\epsilon}\int LH\phi_t^r dV_t dt.
\end{aligned}
\tag{192}
$$

Here and after we omit all z, t for notational simplicity.

By the integration by parts, we have

$$
-\int_{s+\epsilon}^{T-\epsilon}\int \Delta LH\phi^r\,dV_t\,dt=-\int_{s+\epsilon}^{T-\epsilon}\int L\left(\Delta H\phi^r+H\Delta\phi^r+2\langle\nabla H,\nabla\phi^r\rangle\right)\,dV_t\,dt
\tag{193}
$$

Therefore,

$$
\begin{aligned}
&\int L(x,T,z,T-\epsilon)H(z,T-\epsilon,y,s)\phi^r(z,T-\epsilon)\,dV_{T-\epsilon}(z)\\
&\quad-\int L(x,T,z,s+\epsilon)H(z,s+\epsilon,y,s)\phi^r(z,s+\epsilon)\,dV_{s+\epsilon}(z)\\
&\geq\int_{s+\epsilon}^{T-\epsilon}\int LH\Box\phi^r-2L\langle\nabla H,\nabla\phi^r\rangle\,dV_t\,dt.
\end{aligned}
\tag{194}
$$

Now we multiply both sides of $\Box H=0$ by $(\phi^r)^2H$ and do the integration.

$$
\begin{aligned}
\int_{s+\epsilon}^{T-\epsilon}\int|\nabla(\phi^r H)|^2\,dV_t\,dt&\leq\int_{s+\epsilon}^{T-\epsilon}\int|\nabla\phi^r|^2H^2\,dV_t\,dt+\int_{s+\epsilon}^{T-\epsilon}\int\frac{H^2}{2}(\phi^r)_t^2\,dV_t\,dt\\
&\quad-\left(\int\frac{H^2}{2}(\phi^r)^2\,dV_t\right)\Bigg|_{s+\epsilon}^{T-\epsilon}.
\end{aligned}
$$

It is immediate by taking $r\to\infty$ that

$$
\int_{s+\epsilon}^{T-\epsilon}\int|\nabla H|^2\,dV_t\,dt<\infty.
\tag{195}
$$

For fixed ϵ, we have

$$
\begin{aligned}
&\left|\int_{s+\epsilon}^{T-\epsilon}\int LH\Box\phi^r-2L\langle\nabla H,\nabla\phi^r\rangle\,dV_t\,dt\right|\\
&\quad\leq\int_{s+\epsilon}^{T-\epsilon}\int LH|\Box\phi^r|+2L|\nabla H||\nabla\phi^r|\,dV_t\,dt=I+II.
\end{aligned}
\tag{196}
$$

For the first term,

$$
\lim_{r\to\infty}I=\lim_{r\to\infty}\int_{s+\epsilon}^{T-\epsilon}\int LH|\Box\phi^r|\,dV_t\,dt=0
$$

since L is uniformly bounded on $M\times[s+\epsilon,T-\epsilon]$ and H is integrable.

For the second term,

$$
II=\int_{s+\epsilon}^{T-\epsilon}\int L|\nabla H||\nabla\phi^r|\,dV_t\,dt\leq 2\left(\int_{s+\epsilon}^{T-\epsilon}\int L^2|\nabla\phi^r|^2\,dV_t\,dt\right)^{\frac{1}{2}}\left(\int_{s+\epsilon}^{T-\epsilon}\int|\nabla H|^2\,dV_t\,dt\right)^{\frac{1}{2}}.
$$

Now we claim

$$\int_{s+\epsilon}^{T-\epsilon} \int L^2 \, dV_t \, dt < \infty. \tag{197}$$

Indeed, it follows from [58, Eq. (3.3)] that for any $t \in [s+\epsilon, T-\epsilon]$,

$$l_{(x,T)}(z,t) \ge \sqrt{\frac{1-t}{T-t}} f(z,t) - \sqrt{\frac{1-T}{T-t}} f(x,T)$$

and hence

$$L(x,T,z,t) \le \eta_1 e^{-\eta_2 F(z,t)},$$

where

$$\eta_1 = \frac{\exp\left(\sqrt{\frac{1-T}{\epsilon}} f(x,T)\right)}{(4\pi\epsilon)^{\frac{n}{2}}} \quad \text{and} \quad \eta_2 = \frac{1}{\sqrt{(T-s-\epsilon)(1-s-\epsilon)}}.$$

Therefore, it is clear from Lemmas 1 and 2 that the claim (197) holds.

It is immediate from (195) that

$$\lim_{r\to\infty} II \le \lim_{r\to\infty} 2\left(\int_{s+\epsilon}^{T-\epsilon}\int L^2|\nabla\phi^r|^2 \, dV_t \, dt\right)^{\frac{1}{2}} \left(\int_{s+\epsilon}^{T-\epsilon}\int |\nabla H|^2 \, dV_t \, dt\right)^{\frac{1}{2}} = 0. \tag{198}$$

Now it follows from (194) that by taking $r \to \infty$,

$$\int L(x,T,z,T-\epsilon)H(z,T-\epsilon,y,s)\, dV_{T-\epsilon}(z) \ge \int L(x,T,z,s+\epsilon)H(z,s+\epsilon,y,s)\, dV_{s+\epsilon}(z). \tag{199}$$

As $\epsilon \to 0$, both $H(z,T-\epsilon,y,s)$ and $L(x,T,z,s+\epsilon)$ are uniformly bounded (in terms of z). We conclude from the definition of δ function that by taking $\epsilon \to 0$

$$H(x,T,y,s) \ge L(x,T,y,s).$$

□

We also need the following gradient estimate from [60].

Lemma 18 *For any Ricci shrinker (M^n, g, f), suppose u is a positive bounded solution of the heat equation $\square u = 0$ on $M \times [0,T]$, then*

$$\frac{|\nabla u|}{u} \le \sqrt{\frac{1}{t}}\sqrt{\log\frac{\Lambda}{u}}$$

where $\Lambda = \max_{M\times[0,T]} u$.

Proof From a direction computation

$$\square\left(t\frac{|\nabla u|^2}{u} - u\log\frac{\Lambda}{u}\right) = -\frac{2}{u}\left|\mathrm{Hess}\, u - \frac{du\otimes du}{u}\right|^2 \le 0.$$

Now the theorem follows from Theorem 6 if

$$\int_0^T \int \frac{|\nabla u|^2}{u} e^{-2f} \, dV_t \, dt < \infty.$$

Notice that this follows the same proof as Lemma 9. □

Now we have the following corollary of Lemma 18, see [60, Eq. (3.44)].

Corollary 6 *With the same conditions as Lemma 18, for any* $\sigma > 0$,

$$u(y,t) \le \Lambda^{\frac{\sigma}{1+\sigma}} u(x,t)^{\frac{1}{1+\sigma}} \exp\left(\frac{d_t^2(x,y)}{4\sigma t}\right). \tag{200}$$

Proof We rewrite Lemma 18 as

$$\left|\nabla\sqrt{\log\frac{\Lambda}{u}}\right| \le \frac{1}{2\sqrt{t}},$$

and hence

$$\sqrt{\log\frac{\Lambda}{u(x,t)}} \le \sqrt{\log\frac{\Lambda}{u(y,t)}} + \frac{d_t(x,y)}{2\sqrt{t}}.$$

By squaring both sides above, we have

$$\begin{aligned}\log\frac{\Lambda}{u(x,t)} &\le \left(\sqrt{\log\frac{\Lambda}{u(y,t)}} + \frac{d_t(x,y)}{2\sqrt{t}}\right)^2 \\ &\le (1+\sigma)\log\frac{\Lambda}{u(y,t)} + \frac{1+\sigma}{\sigma}\frac{d_t^2(x,y)}{4t}.\end{aligned}$$

Then the conclusion follows immediately. □

We now prove the pointwise lower bound of the heat kernel H.

Theorem 17 *For any Ricci shrinker* (M^n, g, f), $0 < \delta < 1$, $D > 1$ *and* $0 < \epsilon < 4$, *there exists a constant* $C = C(n, \delta, D) > 0$ *such that*

$$H(x,t,y,s) \ge \frac{C^{\frac{4}{\epsilon}} e^{\boldsymbol{\mu}(\frac{4}{\epsilon}-1)}}{(4\pi(t-s))^{n/2}} \exp\left(-\frac{d_t^2(x,y)}{(4-\epsilon)(t-s)}\right)$$

for any $t \in [-\delta^{-1}, 1-\delta]$ *and* $d_t(p,y) + \sqrt{t-s} \le D$.

Proof From Theorem 16,

$$H(y,t,y,s) \ge \frac{e^{-l_{(y,t)}(y,s)}}{(4\pi(t-s))^{\frac{n}{2}}}. \tag{201}$$

By the definition of l and $\partial_z f(y,z) = |\nabla f|^2 \ge 0$,

$$\begin{aligned}l_{(y,t)}(y,s) &\le \frac{1}{2\sqrt{t-s}}\int_s^t \sqrt{t-z}R(y,z)\,dz \\ &\le \frac{1}{2\sqrt{t-s}}\int_s^t \frac{\sqrt{t-z}}{1-z} f(y,z)\,dz \\ &\le \frac{f(y,t)}{2\sqrt{t-s}}\int_s^t \frac{\sqrt{t-z}}{1-z}\,dz \le \frac{(t-s)}{3(1-t)^2}F(y,t)\end{aligned} \tag{202}$$

and hence

$$H(y,t,y,s) \ge \frac{C}{(4\pi(t-s))^{n/2}}$$

for some constant $C = C(n, \delta, D) > 0$.

By using (200) for the heat kernel on $M \times [\frac{t+s}{2}, t]$, we obtain

$$H(y,t,y,s) \le e^{-\mu\frac{\sigma}{1+\sigma}}(4\pi(t-s))^{-\frac{n}{2}\frac{\sigma}{1+\sigma}} H^{\frac{1}{1+\sigma}}(x,t,y,s)\exp\left(\frac{d_t^2(x,y)}{4\sigma(t-s)}\right)$$

where we have used the result in Theorem 15 for the upper bound.

Therefore,

$$H(x,t,y,s) \ge \frac{C^{1+\sigma}e^{\mu\sigma}}{(4\pi(t-s))^{n/2}}\exp\left(-\frac{1+\sigma}{\sigma}\frac{d_t^2(x,y)}{4(t-s)}\right).$$

The conclusion follows by choosing $\sigma = 4/\epsilon - 1$. □

Remark 3 From the proof a more precise bound is, for any $0 < \epsilon < 4$,

$$H(x,t,y,s) \ge \frac{e^{\mu(\frac{4}{\epsilon}-1)}}{(4\pi(t-s))^{n/2}}\exp\left(-\frac{d_t^2(x,y)}{(4-\epsilon)(t-s)} - \frac{4(t-s)}{3(1-t)^2\epsilon}F(y,t)\right). \tag{203}$$

In order to further estimate the upper bound of H, it is crucial to compare distance functions from different time slices. We first prove the second order estimate of the heat equation soluton on Ricci shrinkers, see [3, Lemma 3.1].

Lemma 19 *Let $(M^n, g(t))$, $t \in [0, 1)$ be the Ricci flow solution of a Ricci shrinker and let u be a postive solution to the heat equation $\Box u = 0$ and $u \le \Lambda$ on $M \times [0, T]$. Then there exists a constant $C = C(n)$ such that*

$$|\Delta u| + \frac{|\nabla u|^2}{u} - \Lambda R \le \frac{C\Lambda}{t}. \tag{204}$$

Proof By rescaling, we assume that $\Lambda = 1$. Let $L_1 = -\Delta u + \frac{|\nabla u|^2}{u} - R$, then it follows from [3, Eqs. (3.3), (3.4)] that

$$\Box L_1 \le -\frac{1}{n}L_1^2 + \frac{1}{e^2t^2}. \tag{205}$$

From (205) we have

$$\begin{aligned}\Box(L_1\phi^r) &= \phi^r\Box L_1 + L_1\Box\phi^r - 2\langle\nabla\phi^r, \nabla L_1\rangle \\ &\le \phi^r(-\frac{1}{n}L_1^2 + \frac{1}{e^2t^2}) + L_1\Box\phi^r - 2\langle\nabla\phi^r, \nabla L_1\rangle \\ &= \phi^r(-\frac{1}{n}L_1^2 + \frac{1}{e^2t^2}) + L_1\Box\phi^r - 2\frac{\langle\nabla(L_1\phi^r), \nabla\phi^r\rangle}{\phi^r} + 2\frac{L_1|\nabla\phi^r|^2}{\phi^r}.\end{aligned} \tag{206}$$

Now at the maximum point of $L_1\phi^r$, we have

$$-\frac{1}{n}(L_1\phi^r)^2 + (\phi^r e^{-1}t^{-1})^2 + (L_1\phi^r)\left(\Box\phi^r + 2\frac{|\nabla\phi^r|^2}{\phi^r}\right) \ge 0, \tag{207}$$

so we obtain

$$L_1\phi^r \le n\left(\Box\phi^r + 2\frac{|\nabla\phi^r|^2}{\phi^r}\right) + \sqrt{n}\phi^r e^{-1}t^{-1} \le C(n)(r^{-1} + t^{-1}). \tag{208}$$

By taking $r \to \infty$, we have $L_1 \le C(n)t^{-1}$. Now if we set $L_2 = \Delta u + \frac{|\nabla u|^2}{u} - R$, then similarly

$$\Box L_2 \le -\frac{1}{2n}L_2^2 + \frac{1+\frac{4}{n}}{e^2t^2}. \tag{209}$$

Therefore by the same method, we prove that $L_2 \le C(n)t^{-1}$.

Now the proof is complete. □

By applying the above lemma to the heat kernel, we immediately have from Theorem 15 that

Lemma 20 *For any Ricci shrinker (M^n, g, f), there exists a constant $C = C(n)$ such that*

$$|\partial_t H(x,t,y,s)| = |\Delta_x H(x,t,y,s)| \le C\frac{e^{-\mu}}{(t-s)^{\frac{n}{2}}}\left(R(x,t) + \frac{1}{t-s}\right) \tag{210}$$

for any $s < t < 1$.

Now we can prove the local distance distorsion on Ricci shrinkers. Notice that a similar estimate has been obtained on compact manifolds, see [3, Theorem 1.1].

Theorem 18 (Local distance distorsion estimate) *For any Ricci shrinker $(M^n, p, g, f) \in \mathcal{M}_n(A)$, $0 < \delta < 1$ and $D > 1$, there exists a constant $Y = Y(n, A, \delta, D) > 1$ such that for any two points q and z in M with $d_t(p,q) \le D$ and $d_t(q,z) = r \le D$,*

$$Y^{-1}d_s(q,z) \le d_t(q,z) \le Y d_s(q,z)$$

for any $t \in [-\delta^{-1}, 1-\delta-r^2]$ and $s \in [t - Y^{-1}r^2, t + Y^{-1}r^2]$.

Proof In the proof, all constants C_i and c_i depend on n, A, δ and D. Fix a time $T \in [-\delta^{-1}, 1-\delta-r^2]$, a point q with $d_T(p,q) \le D$ and $r \le D$, we set $w(x,t) = H(x,t,q,T-r^2)$. It follows from Theorem 17 that $w(y,T) \ge C_1 r^{-n}$ for any y with $d_T(q,y) \le r$. For any $y \in B_T(q,r)$, we have from Lemma 20 that

$$|\partial_t w(y,t)| \le C_2 r^{-n}(R(y,t) + r^{-2}) \tag{211}$$

for $t \in [T - r^2/2, T + r^2]$. Since $d_T(p,y) \le d_T(p,q) + d_T(q,y) \le 2D$, it is clear from Lemma 1 that $F(y,T) \le c_1$. Moreover, it follows from (22) and (24) that

$$|\partial_t F(y,t)| = |(1-t)R(y,t)| \le \frac{F(y,t)}{1-t} \le c_2 F(y,t).$$

Therefore, it is clear that for any $t \in [T - r^2/2, T + r^2]$, $F(y,t) \le c_3$ and hence $R(y,t) \le c_4$ from (24). Since $r \le D$, we have from (211)

$$|\partial_t w(y,t)| \le C_3 r^{-n-2}. \tag{212}$$

Now we set $c_5 = C_1(2C_3)^{-1}$, it follows from $w(q,T) \ge C_1 r^{-n}$ and (212) that $w(y,t) \ge \frac{C_1}{2}r^{-n}$ on $B_T(q,r) \times [T - c_5 r^2, T + c_5 r^2]$. On the one hand, it follows from Corollary 6 that $w \ge C_4 r^{-n}$ on $B_t(y,r) \times \{t\}$. On the other hand, by Lemma 1, F and hence R is bounded on $B_t(y,r) \times \{t\}$, we conclude from Theorem 23 that

$$|B_t(y,r)|_t \ge C_5 r^n.$$

For any point z with $d_T(q,z)=r$, we consider a geodesic γ connecting q and z. We claim that for any $t\in[T-c_5r^2, T+c_5r^2]$, $d_t(q,z)\le C_6 r$, where $C_6=8(C_4C_5)^{-1}$. Otherwise, we take a maximal set $\{y_i\}_{i=1}^N\subset\gamma$ such that $B_t(y_i,r)$ are mutually disjoint. In particular, it implies that $\{B_t(y_i,2r)\}$ covers γ. Then it is easy to see $C_6 r\le 4Nr$ and hence $N\ge\frac{C_6}{4}$. However, it follows from (57) that

$$1\ge\int w\,dV_t\ge\sum_{i=1}^N\int_{B_t(y_i,r)} w\,dV_t\ge\sum_{i=1}^N C_4 r^{-n}|B_t(y_i,r)|_t\ge NC_4C_5\ge 2,$$

which is a contradiction. Now we set $c_6=c_5(2C_6)^{-2}$ and claim that $d_t(y,z)\ge(2C_6)^{-1}r$ for any $t\in[T-c_6r^2,T+c_6r^2]$. Otherwise, we can find a time $t\in[T-c_6r^2,T+c_6r^2]$ such that $d_t(y,z)=(2C_6)^{-1}r$. Since $c_6r^2=c_5(2C_6)^{-2}r^2$, the argument before shows that $r=d_T(q,z)\le C_6 d_t(q,z)=r/2$ and this is impossible.

Therefore, by choosing $Y=\max\{c_6^{-1},2C_6\}$, the conclusion follows. □

Now we prove that H has the exponential decay in the integral sense.

Theorem 19 *For any Ricci shrinker $(M^n,p,g,f)\in\mathcal{M}_n(A)$, $0<\delta<1$, $D>1$ and $\epsilon>0$, there exists a constant $C=C(n,A,\delta,D,\epsilon)>1$ such that*

$$\int_{M\backslash B_s(x,r\sqrt{t-s})} H(x,t,y,s)\,dV_s(y)\le C\exp\left(-\frac{(r-1)^2}{4(1+\epsilon)}\right)$$

for any point $x\in M$, $t\in[-\delta^{-1},1-\delta]$, $d_t(p,x)+\sqrt{t-s}\le D$ and $r\ge1$.

Proof It follows from Theorem (14) with $\sigma=\epsilon$ that

$$\begin{aligned}&\left(\int_{B_s(x,\sqrt{t-s})} H(x,t,y,s)\,dV_s(y)\right)^{\frac{1}{\epsilon}}\left(\int_{M\backslash B_s(x,r\sqrt{t-s})} H(x,t,y,s)\,dV_s(y)\right)\\&\quad\le\exp\left(-\frac{(r-1)^2}{4(1+\epsilon)}\right)\end{aligned} \tag{213}$$

for any $r\ge1$. So we only need to prove the first integral to be bounded below.

Theorem 18 shows that there exists a constant $Y=Y(n,A,\delta,D)>1$ such that for any y with $d_s(x,y)\le\sqrt{t-s}$, we have $d_t(x,y)\le Y\sqrt{t-s}$. Therefore, it follows from Theorem 17 that

$$H(x,t,y,s)\ge C(t-s)^{-n/2}$$

for any y with $d_s(x,y)\le\sqrt{t-s}$.

It implies that

$$\int_{B_s(x,\sqrt{t-s})} H(x,t,y,s)\,dV_s(y)\ge C(t-s)^{-n/2}|B_s(x,\sqrt{t-s})|_s\ge C$$

where we have used the fact that R is locally bounded. □

As we have proved that all distance functions to the base point p are comparable, we prove the following weaker upper bound.

Theorem 20 *For any Ricci shrinker $(M^n,p,g,f)\in\mathcal{M}_n(A)$, $x\in M$ and $s<t<1$, there exist constants $C=C(n,A,x,t,s)>1$ and $c=c(n,A,x,t,s)>0$ such that*

$$H(x,t,y,s)\le Ce^{-cd_0^2(p,y)}.$$

Proof Fix $s < t < 1$ and x and we require that all constants in the proof depend on n, x, s, t and A. Notice that since s and t are fixed, f is comparable to $d_0^2(p,\cdot)$ by Lemma 1.

For an $\epsilon > 0$ to be chosen later, we have from the semigroup property

$$\begin{aligned} H(x,t,y,s) &= \int H(x,t,z,l)H(z,l,y,s)\,dV_l(z) \\ &= \int_{d_0(p,z)\geq \epsilon d_0(p,y)} H(x,t,z,l)H(z,l,y,s)\,dV_l(z) \\ &\quad + \int_{d_0(p,z)\leq \epsilon d_0(p,y)} H(x,t,z,l)H(z,l,y,s)\,dV_l(z) = I + II \end{aligned}$$

where $l = \frac{s+t}{2}$.

Now from Theorem 19

$$\begin{aligned} I &= \int_{d_0(p,z)\geq \epsilon d_0(p,y)} H(x,t,z,l)H(z,l,y,s)\,dV_l(z) \\ &\leq C_1 \int_{d_0(p,z)\geq \epsilon d_0(p,y)} H(x,t,z,l)\,dV_l(z) \leq C_2 e^{-c_1\epsilon^2 d_0^2(p,y)}. \end{aligned} \tag{214}$$

Note that here we can always assume that $\epsilon d_0(p,y)$ is large.

We choose ϕ which is identical 1 on $B_l(p, c_2\epsilon d_0(p,y))$ and supported on $B_l(p, 2c_2\epsilon d_0(p,y))$ where we choose c_2 that $B_0(p,\epsilon d_0(p,y)) \subset B_l(p, c_2\epsilon d_0(p,y))$.

If we set $w = \frac{e^{-f}}{(4\pi\tau)^{n/2}}$, there are c_3 and c_4 that for any $z \in M$

$$c_3 e^{c_4\epsilon^2 d_0^2(p,y)} w(z,l) \geq \phi(z). \tag{215}$$

Now, we have

$$\begin{aligned} II &= \int_{B_0(p,\epsilon d_0(p,y))} H(x,t,z,l)H(z,l,y,s)\,dV_l(z) \leq c_5 \int_{B_l(p,c_2\epsilon d_0(p,y))} H(z,l,y,s)\,dV_l(z) \\ &\leq c_5 \int H(z,l,y,s)\phi(z)\,dV_l(z) \leq c_6 e^{c_4\epsilon^2 d_0^2(p,y)} w(y,s), \end{aligned}$$

where $c_6 = c_5c_3$ and the last inequality follows from Lemma 8. Indeed, if we consider $m(u,s) := \int H(z,l,u,s)\phi(z)\,dV_l(z)$, then it follows from (215) and Lemma 8 that

$$m(u,s) \leq c_3 e^{c_4\epsilon^2 d_0^2(p,y)} w(u,s) \tag{216}$$

for any $u \in M$ and $s \leq l$. In particular, (216) holds if $u = y$.

By the definition of w,

$$w(y,s) \leq c_7 e^{-c_8 d_0^2(p,y)}.$$

Hence,

$$II \leq c_9 e^{-(c_8 - c_4\epsilon^2)d_0^2(p,y)}. \tag{217}$$

If we choose $\epsilon = \sqrt{\frac{c_8}{2c_4}}$, it follows from (214) and (217) that

$$H(x,t,y,s) \leq Ce^{-cd_0^2(p,y)}.$$

□

8 Differential Harnack inequality on Ricci shrinkers

In this subsection, we prove that Perelman's differential Harnack inequality holds on Ricci shrinkers.

For any Ricci shrinker (M^n, g, f), we fix a point $q \in M$ and a time $T < 1$. Moreover, we set

$$w(x,t) = H(q,T,x,t) = \frac{e^{-b(x,t)}}{(4\pi(T-t))^{n/2}} \tag{218}$$

and $\tau = T - t$.

We first prove

Lemma 21 *For any r such that $\phi^r = 1$ on an open neighborhood of (q, T),*

$$\lim_{t \nearrow T} \int bw\phi^r \, dV_t = \frac{n}{2}. \tag{219}$$

Proof We set $K_r = \operatorname{supp}\phi^r \bigcap M \times [T-1, T]$. Since we only care about the integral on the compact set K_r when t is sufficiently close to T, we can assume that the distances on different time slices from t to T are uniformly comparable. Now all constants C's in the rest of the proof depend on $q, T, \boldsymbol{\mu}$ and the geometry on K_r. In particular, they are independent of τ.

Now we have from Theorem 19 that

$$\int_{d_t(q,x) \ge 2A\sqrt{\tau}} w(x,t) \, dV_t \le Ce^{-A^2/2}. \tag{220}$$

Moreover, from Theorem 17,

$$b(x,t) \le C\left(1 + \frac{d_t^2(q,x)}{\tau}\right) \tag{221}$$

for $(x, t) \in K_r$.

Now we set $d_t = d_t(q, x)$, then for any $A \ge 1$, we have

$$\begin{aligned}\int_{K_r \cap \{d_t \ge 2A\sqrt{\tau}\}} bw \, dV_t \le C \int_{K_r \cap \{d_t \ge 2A\sqrt{\tau}\}} w + \tau^{-1} d_t^2 w \, dV_t \le Ce^{-A^2/2} \\ + C\tau^{-1} \int_{K_r \cap \{d_t \ge 2A\sqrt{\tau}\}} d_t^2 w \, dV_t.\end{aligned}$$

Now we have

$$\begin{aligned}\int_{K_r \cap \{d_t \ge 2A\sqrt{\tau}\}} d_t^2 w \, dV_t &= \sum_{k=1}^{\infty} \int_{K_r \cap \{2^k A\sqrt{\tau} \le d_t \le 2^{k+1} A\sqrt{\tau}\}} d_t^2 w \, dV_t \\ &\le \sum_{k=1}^{\infty} 2^{2k+2} A^2 \tau \int_{K_r \cap \{2^k A\sqrt{\tau} \le d_t \le 2^{k+1} A\sqrt{\tau}\}} w \, dV_t \\ &\le \sum_{k=1}^{\infty} 2^{2k+2} A^2 e^{-2^{2k-3} A^2} \tau.\end{aligned}$$

Therefore, we conclude that

$$\int_{K_r\cap\{d_t\geq 2A\sqrt{\tau}\}} bw\phi^r\, dV_t \leq \int_{K_r\cap\{d_t\geq 2A\sqrt{\tau}\}} bw\, dV_t \leq \eta(A) \tag{222}$$

where $\eta(A)\to 0$ if $A\to+\infty$.

In addition, it follows from Theorem 15 that $b(x,t)\geq \boldsymbol{\mu}$ and hence

$$\int_{K_r\cap\{d_t\geq 2A\sqrt{\tau}\}} bw\phi^r\, dV_t \geq \boldsymbol{\mu}\int_{K_r\cap\{d_t\geq 2A\sqrt{\tau}\}} w\, dV_t \geq -Ce^{-A^2/2} \tag{223}$$

where the last inequality is from (220).

The inequalities (222) and (223) indicates that the integral $\int bw\phi^r\, dV_t$ is concentrated on the scale $\sqrt{\tau}$.

We take a sequence $\tau_i\to 0$ and set $g_i(t)=\tau_i^{-1}g(T-\tau_i t)$ and $w_i(\cdot,t)=\tau_i^{n/2}w(\cdot,T-\tau_i t)$. Then we have

$$\partial_t w_i=\Delta_i w_i - R_i w_i,$$

where Δ_i and R_i are with respect to g_i.

Since g_i is a blow-up sequence for the metric g and K_r has bounded geometry, it is easy to show that (M,g_i,q) subconverges to $(\mathbb{R}^n,g_E,0)$ and w_i converges a positive smooth function w_∞ on $\mathbb{R}^n\times(0,\infty)$ such that

$$\partial_t w_\infty=\Delta_{g_E}w_\infty.$$

Now we can show as (55) that w_∞ is in fact a fundamental solution of the heat equation on the Euclidean space. Moreover it is easy to see by Fatou's inequality that

$$\int w_\infty\, dx\leq 1$$

for any time $t>1$. Now it follows from [22, Corollary 9.6] that w_∞ is the heat kernel based at 0, that is,

$$w_\infty(x,t)=\frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}}.$$

From the smooth convergence,

$$\lim_{i\to\infty}\int_{K_r\cap\{d_{T-\tau_i}\leq 2A\sqrt{\tau_i}\}} bw\, dV_{T-\tau_i}=\int_{|x|\leq 2A}\frac{|x|^2}{4}\frac{e^{-\frac{|x|^2}{4}}}{(4\pi)^{n/2}}\, dx. \tag{224}$$

By direct computations,

$$\int\frac{|x|^2}{4}\frac{e^{-\frac{|x|^2}{4}}}{(4\pi)^{n/2}}\, dx=\frac{n}{2}.$$

Therefore, it is straightforward from (222), (223) and the fact that ϕ^r is equal to 1 on a neighborhood of (q,T) that

$$\lim_{t\nearrow T}\int bw\phi^r\, dV_t=\frac{n}{2}.$$

□

Remark 4 The same proof of Lemma 21 shows that if u is a bounded smooth function on $M \times [T-1, T]$, then

$$\lim_{t \nearrow T} \int bwu\phi^r \, dV_t = \frac{n}{2} u(q, T). \tag{225}$$

Now we set $d = d_T(q, \cdot)$, it follows from (203) that

$$H(q, T, x, t) \geq C \frac{e^{-c_1 \frac{d^2}{\tau} - c_2 \tau F}}{\tau^{\frac{n}{2}}}. \tag{226}$$

In terms of b, we have

$$b(x, t) \leq c_1 \frac{d^2}{\tau} + c_2 \tau F(x, T) + c_3 \tag{227}$$

We denote $K_t^r = \{r \leq F(\cdot, t) \leq 2r\}$, then we have

Lemma 22 *There exist constants C_0 and C_1 which depend only on $\boldsymbol{\mu}$, q and T such that*

$$\int_{T-1}^{T} \int_{K_t^r} |b| w \, dV_t \, dt \leq C_0 \tag{228}$$

for any $r \geq C_1$.

Proof From Lemma 1, there exists $C_1 > 0$ such that for any $x \in K_t^r$ where $t \in [T-1, T]$,

$$\frac{1}{5} d_t^2(p, x) \leq F(x, t) \leq d_t^2(p, x)$$

if $r \geq C_1$.

It follows from (227) that $|b| \leq -\boldsymbol{\mu} + c_1 \frac{d^2}{\tau} + c_2$. So we only need to estimate

$$\int_{T-1}^{T} \int_{K_t^r} d^2 w \, dV_t \, dt. \tag{229}$$

Now it follows from the definition of ϕ^r that $K_t^r \subset \{c_4 r \leq d^2 \leq c_5 r\}$ if C_1 is sufficiently large, therefore

$$\int_{T-1}^{T} \int_{K_t^r} d^2 w \, dV_t \, dt \leq C \int_{T-1}^{T} r \int_{K_t^r} w \, dV_t \, dt \leq C \int_{T-1}^{T} r e^{-\frac{c_3 r}{\tau}} \, dt \leq C_0. \tag{230}$$

Note that here we have used (220). □

Now we have the following spacetime integral estimate.

Lemma 23

$$\int_{T-1}^{T-\epsilon} \int (|\nabla b|^2 + R) w \, dV_t \, dt \leq C \log \epsilon^{-1}, \tag{231}$$

where C depends only on $\boldsymbol{\mu}$, n, q and T.

Proof From the evolution equation

$$\partial_t w = -\Delta w + Rw, \tag{232}$$

we immediately have

$$\partial_t b = -\Delta b + |\nabla b|^2 - R + \frac{n}{2\tau}. \tag{233}$$

From an elementary computation,

$$\begin{aligned}
&\partial_t \int wb\phi^r \, dV_t \\
&\quad = \int \left(b_t w\phi^r + bw_t\phi^r + bw\phi^r_t - bw\phi^r R\right) dV_t \\
&\quad = \int \left(-\Delta b + |\nabla b|^2 - R + \frac{n}{2\tau}\right) w\phi^r - b\Delta w\phi^r + bw\phi^r_t \, dV_t \\
&\quad = \int \langle \nabla b, \nabla(w\phi^r)\rangle + \langle \nabla w, \nabla(b\phi^r)\rangle + |\nabla b|^2 w\phi^r - Rw\phi^r + bw\phi^r_t + \frac{n}{2\tau} w\phi^r \, dV_t \\
&\quad = \int \langle \nabla b, \nabla\phi^r\rangle w + \langle \nabla w, \nabla\phi^r\rangle b - (|\nabla b|^2 + R)w\phi^r + bw\phi^r_t + \frac{n}{2\tau} w\phi^r \, dV_t, \qquad (234)
\end{aligned}$$

where we have used $\nabla w = -w\nabla b$.

On the one hand we have,

$$\begin{aligned}
\int \langle \nabla b, \nabla \phi^r\rangle w \, dV_t &\le \int |\nabla b||\nabla \phi^r| w \, dV_t \\
&\le \frac{1}{4}\int |\nabla b|^2 w\phi^r \, dV_t + \int \frac{|\nabla \phi^r|^2}{\phi^r} w \, dV_t. \qquad (235)
\end{aligned}$$

On the other hand

$$\begin{aligned}
\int \langle \nabla w, \nabla \phi^r\rangle b \, dV_t &\le \int |\nabla w||\nabla \phi^r| b \, dV_t = \int |\nabla b||\nabla \phi^r| wb \, dV_t \\
&\le \frac{1}{4}\int |\nabla b|^2 w\phi^r \, dV_t + \int \frac{|\nabla \phi^r|^2}{\phi^r} b^2 w \, dV_t. \qquad (236)
\end{aligned}$$

Now (234) becomes

$$\partial_t \int wb\phi^r \, dV_t \le -\frac{1}{2}\int (|\nabla b|^2 + R)w\phi^r \, dV_t + X + \frac{n}{2\tau} \tag{237}$$

where

$$X = \int bw\phi^r_t - \frac{|\nabla \phi^r|^2}{\phi^r} w - \frac{|\nabla \phi^r|^2}{\phi^r} b^2 w \, dV_t.$$

Integrate (237) from $T-1$ to $T-\epsilon$, we have

$$\frac{1}{2}\int_{T-1}^{T-\epsilon}\int (|\nabla b|^2 + R)w\phi^r \, dV_t \, dt \le \left(\int wb\phi^r \, dV_t\right)\Big|_{T-\epsilon}^{T-1} + \frac{n}{2}\log \epsilon^{-1} + Y \tag{238}$$

where

$$Y = \int_{T-1}^{T-\epsilon}\int bw\phi^r_t - \frac{|\nabla \phi^r|^2}{\phi^r} w - \frac{|\nabla \phi^r|^2}{\phi^r} b^2 w \, dV_t \, dt.$$

At the time $T-1$, since $b = -\log w - \frac{n}{2}\log 4\pi$, we have

$$\int wb\phi^r \, dV_{T-1} = \int w\left(-\log w - \frac{n}{2}\log 4\pi\right)\phi^r \, dV_{T-1} \le C \tag{239}$$

where the last inequality can be seen from Theorem 20.

Moverover,

$$\int wb\phi^r \, dV_{T-\epsilon} \geq \boldsymbol{\mu} \int w\phi^r \, dV_{T-1} \geq \boldsymbol{\mu}. \tag{240}$$

Now it follows from Theorem 20 and Lemma 2 that

$$\lim_{r\to\infty} |Y| = 0. \tag{241}$$

So if we let $r \to \infty$ in (238), the proof is complete. □

From Lemma 23, we have

Lemma 24 *There exist a sequence $\tau_i \to 0$ and a constant $C > 0$ such that*

$$\tau_i \int (|\nabla b|^2 + R) w \, dV_{T-\tau_i} \leq C. \tag{242}$$

Proof If the conclusion does not hold, we can find a function $C(\tau)$ such that $\lim_{\tau\to 0} C(\tau) = +\infty$ and

$$\int (|\nabla b|^2 + R) w \, dV_{T-\tau} \geq \frac{C(\tau)}{\tau}. \tag{243}$$

But it obviously contradicts Lemma 23 if ϵ is sufficiently small. □

Lemma 25 *For any $\theta > 0$,*

$$\int_{T-1}^{T} \int \tau^\theta (|\nabla b|^2 + R) w \, dV_t \, dt < \infty. \tag{244}$$

Proof It follows from Lemma 23 that

$$\begin{aligned}
&\int_{T-1}^{T} \int \tau^\theta (|\nabla b|^2 + R) w \, dV_t \, dt \\
&= \sum_{k=0}^{\infty} \int_{T-2^{-k}}^{T-2^{-k-1}} \int \tau^\theta (|\nabla b|^2 + R) w \, dV_t \, dt \\
&\leq \sum_{k=0}^{\infty} 2^{-\theta k} \int_{T-1}^{T-2^{-k-1}} \int (|\nabla b|^2 + R) w \, dV_t \, dt \\
&\leq \sum_{k=0}^{\infty} 2^{-\theta k} \log 2^{-k-1} < \infty.
\end{aligned}$$

□

Now we fix a nonnegative function u on the time slice $T-1$ such that $\sqrt{u}$ smooth and compactly supported. We denote by the same u as its heat equation solution.

Then we have

Lemma 26 *There exists a constant $C > 0$ such that*

$$\frac{|\nabla u|^2}{u} \leq C$$

on $M \times [T-1, T]$.

Proof The conclusion follows directly from

$$\square \frac{|\nabla u|^2}{u} = -\frac{2}{u}\left|\text{Hess}\, u - \frac{du \otimes du}{u}\right|^2$$

and Theorem 6. Note that the assumption in Theorem 6 can be checked similarly as Lemma 9 □

We also need the following lemma, whose proof is similar to Lemma 4.

Lemma 27 *There exists a constant $C > 0$ such that*

$$\int_{T-1}^{T}\int |Hess\ F|^2 w\, dV_t\, dt \le C. \tag{245}$$

Proof From the evolution equation $\square|\nabla F|^2 = -2|\text{Hess}\, F|^2$, we have

$$\partial_t \int |\nabla F|^2 w\phi^r\, dV_t = \int (\Delta|\nabla F|^2 - 2|\text{Hess}\, F|^2) w\phi^r - |\nabla F|^2 \Delta w\phi^r + |\nabla F|^2 w\phi^r_t\, dV_t.$$

Integrate above from $T-1$ to T, we get

$$\begin{aligned}
&\int_{T-1}^{T}\int 2|\text{Hess}\, F|^2 w\phi^r\, dV_t\, dt \\
&\le \int_{T-1}^{T}\int -2\langle\nabla|\nabla F|^2, \nabla\phi^r\rangle w + |\nabla F|^2 w\square\phi^r\, dV_t\, dt - \left(\int |\nabla F|^2 w\phi^r\, dV_t\right)\Big|_{T-1}^{T} \\
&\le \int_{T-1}^{T}\int |\text{Hess}\, F|^2 w\phi^r + 4|\nabla F|^2 \frac{|\nabla\phi^r|^2}{\phi^r} w + |\nabla F|^2 w\square\phi^r\, dV_t\, dt - \left(\int |\nabla F|^2 w\phi^r\, dV_t\right)\Big|_{T-1}^{T}.
\end{aligned}$$

From (37) and (40), there exists a constant C independent of r such that

$$|\nabla F|^2 \frac{|\nabla\phi^r|^2}{\phi^r} + |\nabla F|^2 |\square\phi^r| \le C.$$

Therefore,

$$\int_{T-1}^{T}\int |\text{Hess}\, F|^2 w\phi^r\, dV_t\, dt \le C.$$

Now the lemma follows by taking $r \to \infty$. □

With the same proof, we have

Lemma 28 *There exists a constant $C > 0$ such that*

$$\int_{T-1}^{T}\int |Hess\, u|^2 w\, dV_t\, dt \le C.$$

As before, we set

$$v = \left(\tau(2\Delta b - |\nabla b|^2 + R) + b - n\right) w,$$

and therefore

$$\partial_t v = -\Delta v + Rv + 2\tau\left|Rc + \text{Hess}\, b - \frac{g}{2\tau}\right|^2 w. \tag{246}$$

Now we prove

Lemma 29 *There exist a sequence $\tau_i \to 0$ and a constant $C > 0$ independent of r and i such that*

$$\int v u \phi^r \, dV_{T-\tau_i} \le C.$$

Proof From integration by parts, we have

$$\begin{aligned}\int v u \phi^r \, dV_t &= \int \big(\tau(2\Delta b - |\nabla b|^2 + R) + b - n\big)\, w u \phi^r \, dV_t \\ &= \int -2\tau\langle \nabla b, \nabla w\rangle u\phi^r - 2\tau\langle \nabla b, \nabla u\rangle w\phi^r - 2\tau\langle \nabla b, \nabla \phi^r\rangle w u \, dV_t \\ &\quad + \int \big(\tau(R - |\nabla b|^2) + b - n\big)\, w u \phi^r \, dV_t \\ &= \int \big(\tau(|\nabla b|^2 + R) + b - n\big)\, w u \phi^r - 2\tau\langle \nabla b, \nabla u\rangle w\phi^r - 2\tau\langle \nabla b, \nabla \phi^r\rangle w u \, dV_t.\end{aligned}$$

In addition,

$$\begin{aligned}&\int -2\tau\langle \nabla b, \nabla u\rangle w\phi^r - 2\tau\langle \nabla b, \nabla \phi^r\rangle w u \, dV_t \\ &\le \int 2\tau|\nabla b||\nabla u| w\phi^r + 2\tau|\nabla b||\nabla \phi^r| w u \, dV_t \\ &\le \tau \int 2|\nabla b|^2 w u \phi^r + \frac{|\nabla u|^2}{u} w\phi^r + \frac{|\nabla \phi^r|^2}{\phi^r} w u \, dV_t.\end{aligned}$$

Now the conclusion follows immediately from Lemmas 21, 24 and 26. □

We are now ready to estimate the squared term in (246).

Lemma 30

$$\int_{T-1}^{T} \int \tau \left| Rc + Hess\, b - \frac{g}{2\tau} \right|^2 w u \, dV_t \, dt < \infty.$$

Proof We denote $A = 2\tau \left| Rc + \mathrm{Hess}\, b - \frac{g}{2\tau} \right|^2 w$. By computations,

$$\begin{aligned}\partial_t \int v u \phi^r \, dV_t &= \int v_t u \phi^r + v u_t \phi^r + u v \phi^r_t - R u v \phi^r \, dV_t \\ &= \int -\Delta v u \phi^r + A u \phi^r + \Delta u v \phi^r + u v \phi^r_t \, dV_t \\ &= \int u v \Box \phi^r - 2v\langle \nabla \phi^r, \nabla u\rangle + A u \phi^r \, dV_t.\end{aligned}$$

Now we have

$$\begin{aligned}&\int u v \Box \phi^r \, dV_t \\ &= \int \big(\tau(2\Delta b - |\nabla b|^2 + R) + b - n\big)\, w u \Box \phi^r \, dV_t \\ &= \int -2\tau\langle \nabla b, \nabla w\rangle u \Box\phi^r - 2\tau\langle \nabla b, \nabla u\rangle w \Box\phi^r - 2\tau\langle \nabla b, \nabla \Box\phi^r\rangle u w \, dV_t \\ &\quad + \int \big(\tau(R - |\nabla b|^2) + b - n\big)\, w u \Box\phi^r \, dV_t\end{aligned}$$

$$
\begin{aligned}
&= \int \big(\tau(|\nabla b|^2 + R) + b - n\big)\, wu\Box\phi^r \, dV_t \\
&\quad - 2\tau \int \langle \nabla b, \nabla u\rangle w\Box\phi^r + \langle \nabla b, \nabla \Box\phi^r\rangle uw \, dV_t . \qquad (247)
\end{aligned}
$$

For the last integral,

$$
2\int \langle \nabla b, \nabla u\rangle w\Box\phi^r \, dV_t \le 2\int |\nabla b||\nabla u|w|\Box\phi^r| \, dV_t \le \int |\nabla b|^2 wu|\Box\phi^r| + \frac{|\nabla u|^2}{u} w|\Box\phi^r| \, dV_t
$$

and

$$
2\int \langle \nabla b, \nabla\Box\phi^r\rangle uw \, dV_t \le 2\int |\nabla b||\nabla\Box\phi^r|uw \, dV_t \le \int_{K_t^r} |\nabla b|^2 wu + |\nabla\Box\phi^r|^2 uw \, dV_t .
$$

By the explicit expression $\Box\phi^r = -nr^{-1}\eta'/2 - r^{-2}\eta''|\nabla F|^2$, we have

$$
\begin{aligned}
|\nabla\Box\phi^r| &= \left| -nr^{-2}\nabla F\eta''/2 - r^{-3}\eta'''\nabla F|\nabla F|^2 - 2r^{-2}\eta''\operatorname{Hess} F(\nabla F)\right| \\
&\le Cr^{-2}|\nabla F|\left(1 + |\operatorname{Hess} F| + r^{-1}|\nabla F|^2\right).
\end{aligned}
$$

In addition,

$$
\begin{aligned}
&\int v\langle \nabla\phi^r, \nabla u\rangle \, dV_t \\
&= \int \big(\tau(2\Delta b - |\nabla b|^2 + R) + b - n\big)\, w\langle \nabla\phi^r, \nabla u\rangle \, dV_t \\
&= \int -2\tau\langle \nabla b, \nabla w\rangle\langle \nabla\phi^r, \nabla u\rangle - 2\tau \operatorname{Hess}\phi^r(\nabla b, \nabla u)w - 2\tau \operatorname{Hess} u(\nabla b, \nabla\phi^r)w \, dV_t \\
&\quad + \int \big(\tau(R - |\nabla b|^2) + b - n\big)\, w\langle \nabla\phi^r, \nabla u\rangle \, dV_t \\
&= \int \big(\tau(|\nabla b|^2 + R) + b - n\big)\, w\langle \nabla\phi^r, \nabla u\rangle \, dV_t - 2\tau \operatorname{Hess}\phi^r(\nabla b, \nabla u)w \\
&\quad - 2\tau \operatorname{Hess} u(\nabla b, \nabla\phi^r)w \, dV_t .
\end{aligned}
$$

To estimate the last two terms, since $|\nabla u|$ is uniformly bounded,

$$
\int \operatorname{Hess}\phi^r(\nabla b, \nabla u)w \, dV_t \le \int |\operatorname{Hess}\phi^r||\nabla b||\nabla u|w \, dV_t \le C\int_{K_t^r} |\nabla b|^2 w + |\operatorname{Hess}\phi^r|^2 w \, dV_t .
$$

Note that we have

$$
|\operatorname{Hess}\phi^r| = \left| r^{-2}\eta'' F_i F_j + r^{-1}\eta' \operatorname{Hess} F\right| \le Cr^{-1}\left(|\operatorname{Hess} F| + r^{-1}|\nabla F|^2\right)
$$

and

$$
\int \operatorname{Hess} u(\nabla b, \nabla\phi^r)w \, dV_t \le \int |\operatorname{Hess} u||\nabla b||\nabla\phi^r|w \, dV_t \le Cr^{-\frac{1}{2}}\int_{K_t^r} |\nabla b|^2 w + |\operatorname{Hess} u|^2 w \, dV_t .
$$

Now we integrate (247) from $T-1$ to $T-\tau_i$,

$$
\begin{aligned}
&\int_{T-1}^{T-\tau_i}\int Au\phi^r \, dV_t \\
&\le \left(\int vu\phi^r \, dV_t\right)\Big|_{T-1}^{T-\tau_i} + \int_{T-1}^{T-\tau_i}\int \big(\tau(|\nabla b|^2 + R) + b - n\big)\, w(2\langle \nabla\phi^r, \nabla u\rangle - u\Box\phi^r) \, dV_t \, dt \\
&\quad + \int_{T-1}^{T-\tau_i} \tau \int_{K_t^r} |\nabla b|^2 wu|\Box\phi^r| + \frac{|\nabla u|^2}{u} w|\Box\phi^r| + |\nabla b|^2 wu \, dV_t \, dt
\end{aligned}
$$

$$+ Cr^{-4}\int_{T-1}^{T-\tau_i} \tau \int_{K_t^r} |\nabla F|^2 \left(1+|\text{Hess}\, F|^2 + r^{-2}|\nabla F|^4\right) uw\, dV_t\, dt$$
$$+ C\int_{T-1}^{T-\tau_i} \tau \int_{K_t^r} |\nabla b|^2 w + |\text{Hess}\, u|^2 w\, dV_t\, dt$$
$$+ Cr^{-2}\int_{T-1}^{T-\tau_i} \tau \int_{K_t^r} \left(|\text{Hess}\, F|^2 + r^{-2}|\nabla F|^4\right) w\, dV_t\, dt.$$

Therefore,

$$\begin{aligned}
&\int_{T-1}^{T-\tau_i}\int Au\phi^r\, dV_t \\
&\le \left(\int vu\phi^r\, dV_t\right)\Bigg|_{T-1}^{T-\tau_i} + Cr^{-\frac{1}{2}}\int_{T-1}^{T}\int_{K_t^r}\left(\tau(|\nabla b|^2+R)+|b|+n\right) w\, dV_t\, dt \\
&\quad + \int_{T-1}^{T}\tau\int_{K_t^r} Cr^{-1}(|\nabla b|^2 w + w) + |\nabla b|^2 w\, dV_t\, dt \\
&\quad + Cr^{-2}\int_{T-1}^{T}\tau\int_{K_t^r}\left(1+|\text{Hess}\, F|^2\right) uw\, dV_t\, dt \\
&\quad + C\int_{T-1}^{T}\tau\int_{K_t^r}|\nabla b|^2 w + |\text{Hess}\, u|^2 w\, dV_t\, dt
\end{aligned} \tag{248}$$

For a fixed i, from Theorem 20, Lemmas 4, 25, 26 and 27, we have by taking $r \to \infty$ that

$$\int_{T-1}^{T-\tau_i}\int Au\, dV_t = \lim_{r\to\infty}\int_{T-1}^{T-\tau_i}\int Au\phi^r\, dV_t = \lim_{r\to\infty}\left(\int vu\phi^r\, dV_t\right)\Bigg|_{T-1}^{T-\tau_i} \le C, \tag{249}$$

where the last inequality follows from Lemma 29.

Now the lemma follows from (249) by taking $i \to \infty$. □

A consequence of Lemma 30 is

Lemma 31 *There exists a sequence $\tau_j \to 0$ such that*

$$\lim_{j\to\infty}\int \tau_j^2\left|Rc + Hess\, b - \frac{g}{2\tau}\right|^2 wu + \tau_j^{\frac{3}{2}}|\nabla b|^2 w\, dV_t = 0.$$

Proof It follows from Lemmas 30 and 25 that

$$\int_{T-1}^{T}\int \tau\left|Rc + \text{Hess}\, b - \frac{g}{2\tau}\right|^2 wu + \tau^{\frac{1}{2}}|\nabla b|^2 w\, dV_t < \infty.$$

Now the conclusion is obvious. □

Note that the sequence τ_j may not be the same sequence τ_i in Lemma 24.

Finally, we can prove Perelman's differential Harnack inequality.

Theorem 21

$$\tau(2\Delta b - |\nabla b|^2 + R) + b - n \le 0.$$

Proof As $T-1$ can be any time $S<T$, we just need to prove $v \le 0$ on $T-1$.

For the chosen τ_j obtained in Lemma 31, we have

$$\begin{aligned}
&\int vu\phi^r\, dV_{T-\tau_j}\\
&= \int \big(\tau_j(2\Delta b - |\nabla b|^2 + R) + b - n\big)\, wu\phi^r\, dV_{T-\tau_j}\\
&= \int \tau_j(\Delta b + R - \frac{n}{2\tau_j})wu\phi^r - \tau_j\langle \nabla b, \nabla u\rangle w\phi^r\, dV_{T-\tau_j}\\
&\quad + \int -\tau_j\langle \nabla b, \nabla \phi^r\rangle uw + (b-\frac{n}{2})wu\phi^r\, dV_{T-\tau_j}.
\end{aligned} \tag{250}$$

On the one hand,

$$\begin{aligned}
&\int \tau_j(\Delta b + R - \frac{n}{2\tau_j})wu\phi^r\, dV_{T-\tau_j}\\
&\le \tau_j\left(\int \left|Rc + \mathrm{Hess}\, b - \frac{g}{2\tau}\right|^2 wu\, dV_{T-\tau_j}\right)^{\frac{1}{2}}\left(\int wu\, dV_{T-\tau_j}\right)^{\frac{1}{2}}\\
&\le C\left(\int \tau_j^2\left|Rc + \mathrm{Hess}\, b - \frac{g}{2\tau}\right|^2 wu\, dV_{T-\tau_j}\right)^{\frac{1}{2}}.
\end{aligned} \tag{251}$$

On the other hand,

$$\begin{aligned}
&\int -\tau_j\langle \nabla b, \nabla u\rangle w\phi^r - \tau_j\langle \nabla b, \nabla \phi^r\rangle uw\, dV_{T-\tau_j}\\
&\le C\tau_j\int |\nabla b| w\, dV_{T-\tau_j} \le C\left(\int \tau_j^{\frac{3}{2}}|\nabla b|^2\, dV_{T-\tau_j}\right)^{\frac{1}{2}}\left(\int \tau_j^{\frac{1}{2}} w\, dV_{T-\tau_j}\right)^{\frac{1}{2}}\\
&= C\tau_j^{\frac{1}{4}}\left(\int \tau_j^{\frac{3}{2}}|\nabla b|^2\, dV_{T-\tau_j}\right)^{\frac{1}{2}}.
\end{aligned} \tag{252}$$

In addition, it follows from Lemma 21 and Remark 4 that

$$\int (b-\frac{n}{2})wu\phi^r\, dV_{T-\tau_j} = 0. \tag{253}$$

Combining (251), (252) and (253), it follows immediately from Lemmas 31 and 21 that

$$\lim_{j\to\infty}\int vu\phi^r\, dV_{T-\tau_j} = 0.$$

Now we consider (248), with τ_i replaced by τ_j, and let $j \to \infty$.

$$\begin{aligned}
&\int vu\phi^r\, dV_{T-1}\\
&\le Cr^{-\frac{1}{2}}\int_{T-1}^{T}\int_{K_t^r}\big(\tau(|\nabla b|^2 + R) + |b| + n\big)\, w\, dV_t\, dt\\
&\quad + \int_{T-1}^{T}\tau\int_{K_t^r} Cr^{-1}(|\nabla b|^2 w + w) + |\nabla b|^2 w\, dV_t\, dt\\
&\quad + Cr^{-2}\int_{T-1}^{T}\tau\int_{K_t^r}\big(1 + |\mathrm{Hess}\, F|^2\big)\, uw\, dV_t\, dt
\end{aligned}$$

$$+ C \int_{T-1}^{T} \tau \int_{K_t^r} |\nabla b|^2 w + |\mathrm{Hess}\, u|^2 w \, dV_t \, dt.$$

It is easy to see all integrals above converge to zero if $r \to \infty$, by Lemmas 22, 26, 27 and 28. Therefore,

$$\int vu \, dV_{T-1} \le 0.$$

By the arbitrary choice of u at $T-1$, we have proved that $v \le 0$. □

Remark 5 Note that as in Perelman's paper [46], Theorem 16 is a corollary of Theorem 21. Our proof of Theorem 21 is different from most literature, for instance [11,44], in that we do not need a pointwise gradient estimate of the conjugate heat kernel, see [44, Lemma 2.2].

Remark 6 The proof of Theorem 21 shows the following identity. For any $S < T < 1$,

$$\int vu \, dV_S = -\int_S^T \int 2\tau \left| Rc + \mathrm{Hess}\, b - \frac{g}{2\tau} \right|^2 wu \, dV_t \, dt.$$

9 The no-local-collapsing theorems

We need to use the local entropy in [53]. Let us first recall some notations. Let Ω be a domain in M. Then we define (cf. (91) and (92) and Sect. 2 of [53]):

$$\boldsymbol{\mu}(\Omega, g, \tau) := \inf \left\{ \overline{\mathcal{W}}(g, u, \tau) \,\middle|\, u \in \mathcal{W}_*^{1,2}(M), \ u \text{ is supported on } \Omega \right\}, \tag{254}$$

$$\boldsymbol{\nu}(\Omega, g, \tau) := \inf_{s \in (0,\tau)} \boldsymbol{\mu}(\Omega, g, s). \tag{255}$$

When the meaning is clear in the context, the metric g may be dropped. Note that if Ω does not appear, it means the default set is M. We shall exploit the argument in [53] to obtain volume ratio estimate.

Theorem 22 *Suppose (M^n, g, f) is a Ricci shrinker and $B = B(x, r) \subset M$ is a geodesic ball with $R \le \Lambda$, then we have*

$$r^{-n}|B| \ge c \cdot e^{\boldsymbol{\mu} - \Lambda r^2}, \tag{256}$$

for some $c = c(n) > 0$. If $r \in (0, 1)$, then (256) can be improved to

$$r^{-n}|B| \ge c \cdot e^{\boldsymbol{\mu}(g, r^2) - \Lambda r^2}. \tag{257}$$

Proof We first show (256). By Theorem 3.3 of [53], we know that

$$r^{-n}|B| \ge c(n) e^{\boldsymbol{\nu}(B, r^2)} e^{-\Lambda r^2}, \tag{258}$$

where $\boldsymbol{\nu}(B, r^2)$ is the local $\boldsymbol{\nu}$-functional of B on the scale r^2. Since (M, g) is a Ricci shrinker, it follows from (6) in Theorem 1 that

$$\boldsymbol{\nu}(B, r^2) \ge \boldsymbol{\nu}(M, r^2) = \inf_{\tau \in (0, r^2)} \boldsymbol{\mu}(M, g, \tau) \ge \boldsymbol{\mu}. \tag{259}$$

If $r \in (0, 1)$, then $r^2 \in (0, 1)$. By the monotonicity in Theorem 1, the above inequality can be written as

$$\boldsymbol{\nu}(B, r^2) \ge \boldsymbol{\mu}(g, r^2). \tag{260}$$

Therefore, we obtain (256) and (257), after we plugging (259) and (260) into (258) respectively. □

Theorem 23 *Suppose (M^n, g, f) is a Ricci shrinker and $B = B(q, r) \subset M$ is a geodesic ball with $R \leq \Lambda$, then we have*

$$r^{-n}|B| \geq c \cdot e^{\boldsymbol{\mu}}(1 + \Lambda r^2)^{-\frac{n}{2}}. \tag{261}$$

Proof Choose $\rho_0 \in [0, r]$ such that $\inf_{s \in [0,r]} s^{-n}|B(q, s)|$ is achieved at ρ_0. There are two cases $\rho_0 = 0$ and $\rho_0 > 0$, which we shall discuss separately.
Case 1 $\rho_0 = 0$.

In this case, we have

$$|B(q, r)| \geq \omega_n r^n, \tag{262}$$

where ω_n is the volume of the unit Euclidean ball. Actually, it is not hard to observe that

$$\boldsymbol{\mu} \leq 0. \tag{263}$$

Let $\tau \to 0^+$, it is clear that $(M^n, p, \tau^{-1}g)$ converges to $(\mathbb{R}^n, 0, g_E)$ in the Cheeger–Gromov sense. By Lemma 3.2 of [36], we have

$$\limsup_{\tau \to 0^+} \boldsymbol{\mu}(g, \tau) = \limsup_{\tau \to 0^+} \boldsymbol{\mu}(\tau^{-1}g, 1) \leq \boldsymbol{\mu}(g_E, 1) = 0. \tag{264}$$

As $\boldsymbol{\mu}(g, \tau)$ is decreasing on $(0, 1)$ by Lemma 15, then (263) follows from the above inequality. Consequently, (261) follows from the combination of (262) and (263).
Case 2 $\rho_0 > 0$.

We choose a nonincreasing smooth function η on $\mathbb{R}$ such that $\eta = 1$ on $(-\infty, 1/2]$ and 0 on $[1, \infty)$. We also define $u(x) = \eta(\frac{d(q,x)}{\rho_0})$. From (156) in Corollary 4, we obtain

$$\begin{aligned}
|B(q, \rho_0/2)|^{\frac{n-2}{n}} &\leq Ce^{-\frac{2\boldsymbol{\mu}}{n}} \int \left\{4|\nabla u|^2 + Ru^2\right\} dV \\
&\leq Ce^{-\frac{2\boldsymbol{\mu}}{n}} \left(\rho_0^{-2}|B(q, r)| + \int Ru^2\, dV\right) \\
&\leq Ce^{-\frac{2\boldsymbol{\mu}}{n}} \rho_0^{-2}(1 + \Lambda r^2)|B(q, \rho_0)|
\end{aligned}$$

where the last inequality follows from $R \leq \Lambda \leq \Lambda r^2 \rho_0^{-2}$. According to the choice of ρ_0, we obtain

$$|B(q, \rho_0/2)| \geq 2^{-n}|B(q, \rho_0)|.$$

Combining the previous two steps yields that

$$|B(q, \rho_0)| \geq 2^n|B(q, \rho_0/2)| \geq Ce^{\boldsymbol{\mu}}(1 + \Lambda r^2)^{-\frac{n}{2}} \rho_0^n.$$

Recall that $r^{-n}|B(q, r)| \geq \rho_0^{-n}|B(q, \rho_0)|$ by our choice of ρ_0. Therefore, (261) follows directly from the above inequality. □

Remark 7 Theorem 23 indicates that any Ricci shrinker is κ-noncollapsed for some positive constant κ which depends only on the dimension n and the lower bound of $\boldsymbol{\mu}$.

Note that Theorem 22 is based on the Logarithmic Sobolev inequality, and Theorem 23 relies on the Sobolev inequality. Each of Theorems 22 and 23 has its own advantage and will be used in the remainder of the section. Bascially, Theorem 22 is sharper when r is very small and Theorem 23 is more accurate in the situation when Λr^2 is large.

Using the Sobolev constant estimate in Corollary 4, we can further improve Theorem 6.1 of [41] stating that for any noncompact Ricci shrinker, the volume increases at least linearly.

Proposition 6 *For any noncompact Ricci shrinker (M^n, p, g, f), there exist big positive constant $r_0 = r_0(n)$ and small positive constant $\epsilon_0 = \epsilon_0(n)$ such that*

$$|B(p,r)| \geq \epsilon_0 e^{\boldsymbol{\mu}} r, \quad \forall\, r \geq r_0. \tag{265}$$

Proof Similar to the proof of Lemma 2, we follow the notation of [41] to denote

$$\begin{aligned} &\rho{:=}2\sqrt{f}, \quad D(r){:=}\{x \in M \mid \rho \leq r\}, \quad A(s,r){:=}D(r)\backslash D(s); \\ &V(r){:=}|D(r)|, \quad \chi(r){:=}\int_{D(r)} R\, dV. \end{aligned}$$

From Lemma 1, $V(r)$ is almost the volume of geodesic ball $B(p,r)$, with the advantage that the estimate of $V(r)$ is relatively easier than the estimate of $|B(p,r)|$. Actually, by Eqs. (6.24) and (6.25) of [41], we know that

$$V(t+1) \leq 2V(t), \tag{266}$$

$$V(t+1) - V(t) \leq C_1 \frac{V(t)}{t}, \tag{267}$$

whenever $t \geq C_1$ for some dimensional constant $C_1 = C_1(n)$. Now we define

$$r_0{:=}\max\{100n, 10C_1\}. \tag{268}$$

Therefore, in order to prove (265), it suffices to show that

$$V(r) \geq \epsilon_0 e^{\boldsymbol{\mu}} r, \quad \forall\, r \geq r_0, \tag{269}$$

where $\epsilon_0 = \epsilon_0(n)$ will be determined later.

We shall prove (269) by a contradiction argument. If (269) were wrong, then there exists an $r \geq 2r_0$ such that $V(r) \leq \epsilon_0 e^{\boldsymbol{\mu}} r$ for ϵ_0 to be determined later, we claim that

$$V(t_m) \leq 2\epsilon_0 e^{\boldsymbol{\mu}} t_m, \quad t_m = r + m, \quad \forall m \in \mathbb{N}. \tag{270}$$

Indeed, by our assumption the case $m = 0$ is true. We assume that the conclusion is true for all $m = 0, 1, 2, \ldots, k$ and proceed to show it holds for $m = k + 1$.

For any $t \geq r_0$, we define

$$u(x){:=}\begin{cases} 1 & \text{on} \quad A(t, t+1), \\ t + 2 - \rho(x) & \text{on} \quad A(t+1, t+2), \\ \rho(x) - (t-1) & \text{on} \quad A(t-1, t), \\ 0 & \text{otherwise}. \end{cases}$$

Let $t = t_m$ and plug the above u into the Sobolev inequality (156). We obtain

$$|A(t_m, t_{m+1})|^{\frac{n-2}{n}} \leq C_3 e^{-\frac{2\boldsymbol{\mu}}{n}} \left(|A(t_{m-1}, t_m)| + |A(t_{m+1}, t_{m+2})| + \chi(t_{m+2}) - \chi(t_{m-1})\right) \tag{271}$$

for some $C_3 = C_3(n)$. For $0 \leq m \leq k$, it follows from our induction assumption and (267) that

$$|A(t_m, t_{m+1})| = V(t_{m+1}) - V(t_m) \leq C_1 \frac{V(t_m)}{t_m} \leq 2C_1\epsilon_0 e^{\boldsymbol{\mu}}. \tag{272}$$

Summing (271) from $m = 0$ to $m = k$, we have

$$\begin{aligned}\sum_{m=0}^{k} |A(t_m, t_{m+1})|^{\frac{n-2}{n}} &\leq C_3 e^{-\frac{2\boldsymbol{\mu}}{n}} \sum_{m=0}^{k} \left(|A(t_{m-1}, t_m)| + |A(t_{m+1}, t_{m+2})| + \chi(t_{m+2}) - \chi(t_{m-1})\right)\\ &\leq 3C_3 e^{-\frac{2\boldsymbol{\mu}}{n}} \left(|A(t_{-1}, t_{k+2})| + \chi(t_{k+2})\right).\end{aligned}$$

Recall that $\chi(t) \leq \frac{n}{2} V(t)$ by (3.4) of [9]. Plugging this fact into the above inequality yields that

$$\sum_{m=0}^{k} |A(t_m, t_{m+1})|^{\frac{n-2}{n}} \leq 3C_3 e^{-\frac{2\boldsymbol{\mu}}{n}} \left(V(t_{k+2}) + \frac{n}{2} V(t_{k+2})\right) \leq C_4 e^{-\frac{2\boldsymbol{\mu}}{n}} V(t_{k+1}), \tag{273}$$

where $C_4 = (6 + 3n)C_3 = C_4(n)$. Now we choose

$$\epsilon_0 := (2C_1)^{-1}(2C_4)^{-\frac{n}{2}}. \tag{274}$$

Clearly, $\epsilon_0 = \epsilon_0(n)$. Then it follows from (272) that

$$2C_4 e^{-\frac{2\boldsymbol{\mu}}{n}} |A(t_m, t_{m+1})| \leq |A(t_m, t_{m+1})|^{\frac{n-2}{n}}, \quad \forall m \in \{1, 2, \ldots, k\}.$$

It is clear from (273) that

$$2C_4 e^{-\frac{2\boldsymbol{\mu}}{n}} (V(t_{k+1}) - V(r)) \leq C_4 e^{-\frac{2\boldsymbol{\mu}}{n}} V(t_{k+1})$$

and hence

$$V(t_{k+1}) \leq 2V(r) \leq 2\epsilon_0 e^{\boldsymbol{\mu}} r \leq 2\epsilon_0 e^{\boldsymbol{\mu}} t_{k+1}.$$

Therefore, the induction is complete and (270) is proved. By the arbitrary choice of m, the total volume of the Ricci shrinker is finite, which contradicts Lemma 6.2 of [41](See also Theorem 3.1 of [7] by Cao–Zhu). Therefore, the proof of (269) is established by this contradiction. Consequently, (265) holds by Lemma 1. Note that r_0 and ϵ_0 are defined in (268) and (274). Both of them can be calculated explicitly. □

Remark 8 In [41, Theorem 6.1], the authors have obtained a weaker lower bound

$$|B(p, r)| \geq C e^{c\boldsymbol{\mu}} r$$

for two constants $C > 0$ and $c > 1$ depending only on n.

We are now ready to prove the improved no-local-collapsing, i.e., Theorem 2.

Proof of Theorem 2 It follows from Lemma 2 and Proposition 6 that

$$\epsilon_0 r \leq |B(p, r)| e^{-\boldsymbol{\mu}} \leq C r^n.$$

By Lemma 12, we know $e^{\boldsymbol{\mu}}|B(p, 1)|^{-1}$ is uniformly bounded from above and from below. Multiplying each term of the above inequality by $e^{\boldsymbol{\mu}}|B(p, 1)|^{-1}$ and adjusting C if necessary, we arrive at

$$\frac{1}{C} r \leq \frac{|B(p, r)|}{|B(p, q)|} \leq C r^n,$$

which is nothing but (9a). We proceed to prove (9b). Recall that $q \in \partial B(p, r)$ and $\rho \in (0, r^{-1})$ for some $r > 1$. Triangle inequality implies that

$$B(q, \rho) \subset B(p, 2r).$$

It follows from Lemma 1 that $f \le Cr^2$ for some $C = C(n)$ on $B(p, 2r)$. Since $R + |\nabla f|^2 = f$ and $R \ge 0$, it follows that $R \le Cr^2$ on $B(p, 2r)$. In particular, we have $R\rho^2 \le Rr^{-2} \le C(n)$ on $B(q, \rho)$. Consequently, we can apply Theorem 23 on the ball $B(q, \rho)$ to obtain (9b). □

10 The pseudolocality theorems

In this section, we prove the pseudo-locality theorems on Ricci shrinker and discuss their applications.

Based on the Harnack estimate, following a classical point-picking, or maximum principle argument, we are able to obtain the following pseudo-locality theorem.

Theorem 24 *There exist positive numbers $\epsilon_0 = \epsilon_0(n)$ and $\delta_0 = \delta_0(n)$ with the following properties.*

Let $\{(M^n, g(t)), -\infty < t < 1\}$ be the Ricci flow induced from a Ricci shrinker (M^n, p, g). Suppose $t_0 \in (-\infty, 1)$ and $B_{g(t_0)}(x, r) \subset M$ is a geodesic ball satisfying

$$\boldsymbol{\nu}(B_{g(t_0)}(x, r), g(t_0), r^2) > -\delta_0. \tag{275}$$

Then for each $t \in (t_0, \min\{t_0 + \epsilon_0^2 r^2, 1\})$ and $y \in B_{g(t)}(x, 0.5r)$, we have

$$\begin{cases} |Rm|(y, t) \le (t - t_0)^{-1}; & (276a)\\ \inf\limits_{0<\rho<\sqrt{t-t_0}} |B_{g(t)}(y, \rho)|\rho^{-n} \ge \dfrac{1}{2}\omega_n. & (276b) \end{cases}$$

The statement in Theorem 24 is a slight improvement of Theorem 10.1 of [46]. The basic idea of the proof is already contained in Propositions 3.1 and 3.2 of Tian–Wang [51]. Note that the isoperimetric constant estimate in Peleman's statement is only used to (cf. Lemma 3.5 of [53]) estimate the local entropy (i.e., (254) and (255)) $\boldsymbol{\nu}(B_{g(t_0)}(x, r), g(t_0), r^2)$. The statement (275) seems to be more straightforward. The conclusion (276) follows from a standard point-picking argument, whenever the differential Harnack estimate, i.e., Theorem 21 holds. More details can be found in [32, Sect. 30], [11, Sect. 8], [19, Chapter 21], or [54].

As Ricci shrinker Ricci flows are self-similar, we can improve the estimate (276) by the following property.

Theorem 25 *Suppose (M^n, p, g, f) is a Ricci shrinker, $B = B(q, r) \subset M$ is a geodesic ball satisfying*

$$\boldsymbol{\nu}(B, g, r^2) > -\delta_0. \tag{275}$$

Then we have

$$\sup_{x\in B(q, 0.5\epsilon_0 r)} |Rm|(x) \le \max\{1, \epsilon_0 D r\} \cdot (\epsilon_0 r)^{-2}, \tag{276}$$

where $D = d(p, q) + \sqrt{2n}$.

Proof of Theorem 25 We fix $\xi \le \epsilon_0$ a small positive number, whose value will be determined later (i.e., (281)). We set

$$t:=-(\xi r)^2, \quad \tilde{q}=(\psi^t)^{-1}(q), \quad D:=d(p,q)+\sqrt{2n}, \tag{277}$$

where ψ^s is the diffeormorphism (i.e., (15)) generated by $\frac{\nabla f}{1-s}$.

Claim By choosing ξ properly, we have

$$d(q,\tilde{q}) \le \frac{\epsilon_0 r}{2}. \tag{278}$$

By (15) and (2), along the flow line $\psi^s(\tilde{q})$ where s goes from t to 0, we compute

$$d(q,\tilde{q}) \le \int_t^0 \frac{|\nabla f|(\psi^s(\tilde{q}))}{1-s}\,ds \le \int_t^0 \frac{\sqrt{f(\psi^s(\tilde{q}))}}{1-s}\,ds. \tag{279}$$

From the definition of ψ^s, we have

$$\frac{d}{ds} f(\psi^s(\tilde{q})) = \frac{|\nabla f|^2(\psi^s(\tilde{q}))}{1-s} \le \frac{f(\psi^s(\tilde{q}))}{1-s}.$$

For each $s \ge t = -(\xi r)^2$, the integration of the above inequality yields that

$$f(\psi^s(\tilde{q})) \le \frac{1-t}{1-s} f(\psi^t(\tilde{q})) = \frac{1-t}{1-s} f(q) \le \frac{1-t}{1-s}\cdot\frac{D^2}{4},$$

where we applied (30) in the last step. Therefore, it follows from (279) that

$$d(q,\tilde{q}) \le D\sqrt{1-t}\int_t^0 \frac{1}{2}(1-s)^{-3/2}\,ds = D\left(\sqrt{1-t}-1\right).$$

Plugging the fact that $t=-(\xi r)^2$ into the above inequality, we arrive at

$$d(q,\tilde{q}) \le D\left(\sqrt{1+(\xi r)^2}-1\right). \tag{280}$$

Now we define ξ as follows.

$$\xi := \begin{cases} \epsilon_0, & \text{if } Dr \le \epsilon_0^{-1}; \\ \sqrt{\frac{\epsilon_0}{Dr}} & \text{if } Dr > \epsilon_0^{-1}. \end{cases} \tag{281}$$

Therefore, if $Dr \le \epsilon_0^{-1}$, it follows from (280) that

$$d(q,\tilde{q}) \le D\left(\sqrt{1+(\epsilon_0 r)^2}-1\right) \le \frac{D(\epsilon_0 r)^2}{2} \le \frac{\epsilon_0 r}{2}.$$

If $Dr > \epsilon_0^{-1}$, it also follows from (280) that

$$d(q,\tilde{q}) \le D\left(\sqrt{1+(\xi r)^2}-1\right) \le \frac{D}{2}\cdot(\xi r)^2 = \frac{\epsilon_0 r}{2}.$$

Therefore, no matter what the value of r is, we always have (278). The proof of the Claim is complete.

We proceed to prove (276). Since $g(t)=(1-t)(\psi^t)^* g$, it is clear that

$$\psi^t\left(B_t\left(\tilde{q},\sqrt{1-t}\,r\right)\right) = B(q,r).$$

It follows from the scaling property of $\boldsymbol{\nu}$ that

$$\boldsymbol{\nu}\left(B_{g(t)}\left(\tilde{q},\sqrt{1-t}\,r\right),g(t),(1-t)r^2\right)=\boldsymbol{\nu}(B,g,r^2)>-\delta_0.$$

Therefore, we can apply Theorem 24. For each $s\in(t,\min\{t+(\epsilon_0 r)^2,1\}]$ and $x\in B_{g(s)}(\tilde{q},0.5r)$, we have

$$\begin{cases}|Rm|(x,s)\le(s-t)^{-1}; & \text{(284a)}\\ \inf\limits_{0<\rho<\sqrt{s-t}}|B_{g(s)}(x,\rho)|\rho^{-n}\ge\dfrac{1}{2}\omega_n. & \text{(284b)}\end{cases}$$

In particular, we can choose $s=0$. Since $g=g(0)$, for each $x\in B(\tilde{q},0.5r)$, we obtain

$$\begin{cases}|Rm|(x)\le(\xi r)^{-2}; & \text{(285a)}\\ \inf\limits_{0<\rho<\xi r}|B_g(x,\rho)|\rho^{-n}\ge\dfrac{1}{2}\omega_n. & \text{(285b)}\end{cases}$$

Note that $B(q,0.5\epsilon_0 r)\in B(\tilde{q},\epsilon_0 r)\subset B(\tilde{q},0.5r)$ by (278). Plugging (281) into (285a), we obtain (276). □

Now we apply Theorems 24 and 25 to study the geometric properties of (M,g) in terms of $\boldsymbol{\mu}$. In particular, we are ready to finish the proof of Theorem 3.

Proof of Theorem 3 We divide the proof into several steps.

Step 1 The gap property (10) holds.

It suffices to show that $\boldsymbol{\mu}\ge-\delta_0$ implies that (M,g) is isometric to the Euclidean space.

Following directly from its definition, as $B(x,r)\subset M$, it is clear that

$$\boldsymbol{\nu}(B(x,r),g,r^2)\ge\boldsymbol{\nu}(M,g,r^2)=\boldsymbol{\nu}(g,r^2).$$

Combining the above inequality with the optimal Logarithmic Sobolev inequality, we obtain

$$\boldsymbol{\nu}(B(x,r),g,r^2)\ge\boldsymbol{\mu}. \tag{286}$$

Therefore, if $\boldsymbol{\mu}\ge-\delta_0$, then each ball $B(x,r)$ will satisfy the condition (275). By choosing $r>>D$, we can apply (276) to obtain that

$$|Rm|(x)\le\epsilon_0 Dr\cdot(\epsilon_0 r)^{-2}=D\epsilon_0^{-1}r^{-1}.$$

Let $r\to\infty$, we obtain that $|Rm|(x)\equiv 0$. By the arbitrary choice of x, we obtain that $|Rm|\equiv 0$. In particular, $Rc\equiv 0$. Then the Ricci shrinker equation implies that $f_{ij}=\frac{g_{ij}}{2}$. Therefore, (M,g) is isometric to a metric cone which is also a smooth manifold. This forces that (M,g) is isometric to the standard Euclidean space $(\mathbb{R}^n,g_E)$. Thus, the proof of (10) is complete.

Step 2 The inequality (12) and (13) imply the curvature and injectivity radius bound (14).

Recall that (10) means $\boldsymbol{\mu}(g,1)<-\delta_0$. If (12) holds, by continuity and monotonicity of $\boldsymbol{\mu}(g,\tau)$, it is clear that there exists some $\tau\in(0,1)$ such that

$$\boldsymbol{\mu}(g,\tau)=-\delta_0.$$

Then the τ_0 in (14) is well defined. Namely, τ_0 is the largest $\tau \in (0, 1)$ such that the above equality holds. It follows from the definition of τ_0 and $\boldsymbol{\nu}$ that

$$\boldsymbol{\nu}(g, \tau_0) = \boldsymbol{\mu}(g, \tau_0) = -\delta_0. \tag{287}$$

For each ball $B_{g(0)}(x, r) \subset M$, we know $\boldsymbol{\nu}(B_{g(0)}(x, r), g, \tau_0) \geq -\delta_0$. In particular, we can choose $r = \sqrt{\tau_0}$. Now we apply Theorem 24 on the time slice $t_0 = 0$, with scale $\sqrt{\tau_0}$, to obtain that

$$|Rm|(x, t) \leq t^{-1}, \quad \forall\, x \in M, \quad \forall\, t \in (0, \epsilon_0^2 r^2].$$

In particular, we have

$$\sup_{x \in M} |Rm|(x, \epsilon_0^2 \tau_0) \leq \epsilon_0^{-2} \tau_0^{-1}.$$

Up to rescaling, since $g(0) = g$, we arrive atl

$$\sup_{x \in M} |Rm|_g(x) \leq \epsilon_0^{-2} \tau_0^{-1}(1 - \epsilon_0^2 \tau_0) = \epsilon_0^{-2} \tau_0^{-1} - 1 < C(n) \tau_0^{-1},$$

which is nothing but (14a). Plugging (287) into (257) of Theorem 22, we obtain that each geodesic ball $B(\cdot, \sqrt{\tau_0})$ has volume bounded below by $c(n)\tau_0^{\frac{n}{2}}$. Therefore, the injectivity radius estimate of Cheeger–Gromov–Taylor [13] applies and we arrive at (14b). The proof of (14) is complete.

Step 3 The bounded geometry estimate (14) implies the equality (11), i.e., $\lim_{\tau \to 0^+} \boldsymbol{\mu}(g, \tau) = 0$.

We shall argue in the way similar to that in Theorem 1.1 of [62], with more details on the regularity estimate.

Assume otherwise that there exists a sequence $\tau_i \to 0^+$ such that

$$\lim_{i \to \infty} \boldsymbol{\mu}(g, \tau_i) = \boldsymbol{\mu}_\infty < 0. \tag{288}$$

If we set $g_i = \tau_i^{-1} g$, then all metrics g_i have uniformly bounded geometry. More precisely, there exist positive constants K and v_0 such that

$$\begin{cases} |Rm_i| \leq K\tau_i, & \text{(289a)} \\ |B(q, r)|_{g_i} \geq v_0 r^n (1 + \tau_i K r^2)^{-\frac{n}{2}}. & \text{(289b)} \end{cases}$$

Notice that for any i, there exists a large domain

$$B_i := \left\{ x \,\middle|\, 2\sqrt{f} \leq r_i \right\} \tag{290}$$

for some large $r_i >> 1$ such that

$$\boldsymbol{\mu}(B_i, g_i, 1) - \boldsymbol{\mu}(g_i, 1) = \boldsymbol{\mu}(B_i, g_i, 1) - \boldsymbol{\mu}(g, \tau_i) < i^{-1}. \tag{291}$$

The geometry bound (289) actually implies higher order derivatives of curvatures and $\sqrt{f}$ are also uniformly bounded (cf. Sect. 4 of [34]). Therefore, it is not hard to see that ∂B_i is smooth. All the covariant derivatives of second fundamental forms of ∂B_i are bounded independent of i.

It follows from [48] that a minimizer u_i of $\boldsymbol{\mu}(B_i, g_i, 1)$ exists. More precisely, $u_i \in W_0^{1,2}(B_i)$ is a positive smooth function on B_i satisfying the normalization condition

$$\int_{B_i} u_i^2 \, dV_i = 1 \tag{292}$$

and solve the Dirichlet problem

$$\begin{cases} -4\Delta_i u_i + R_i u_i - 2u_i \log u_i - \lambda_i u_i = 0, & \text{in} \quad B_i; \quad (293a) \\ u_i = 0, \quad \text{on} \quad \partial B_i. & (293b) \end{cases}$$

Here dV_i, Δ_i and R_i denote the volume form, Laplacian operator and scalar curvature with respect to g_i respectively. The number λ_i is defined by

$$\lambda_i := n + \frac{n}{2}\log(4\pi) + \boldsymbol{\mu}(B_i, g_i, 1).$$

Recall that $\lim_{\tau\to 0^+} \boldsymbol{\mu}(g,\tau) \le 0$ by (264). Then it follows from (291) that λ_i is uniformly bounded. Since curvature is uniformly bounded, the classical L^2-Sobolev constant of (B_i, g_i) is uniformly bounded. In light of (293), the Moser iteration then implies $\|u_i\|_{C^0}$ is uniformly bounded, see [62, Lemma 2.1(a)] or the proof of Proposition 3.1 of [51]. Then it follows from [23, Corollary 8.36] that $\|u_i\|_{C^{1,\frac{1}{2}}(\bar{B}_i)}$ are uniformly bounded. Since all ∂B_i have uniformly higher regularities, the bootstrapping, see [23, Theorem 6.19], shows that $\|u_i\|_{C^{k,\frac{1}{2}}(\bar{B}_i)}$ are uniformly bounded for any $k \ge 2$.

Let q_i be a point where u_i achieves maximum value in B_i. By (293), at q_i we have

$$R_i u_i - 2u_i \log u_i - \lambda_i u_i \le 0,$$

whence we derive

$$u_i(q_i) \ge \exp\left(\frac{R_i - \lambda_i}{2}\right) \ge c_0 \tag{294}$$

for some uniform constant c_0.

In light of (289) and the discussion below (291), we know that (M^n, q_i, g_i) subconverges to Euclidean space $(\mathbb{R}^n, 0, g_E)$ in C^∞-Cheeger–Gromov topology. The set B_i converges to a limit set B_∞. If $d(q_i, \partial B_i) \to \infty$, then $B_\infty = \mathbb{R}^n$. Otherwise, by the estimate of second fundamental form and its covariant derivatives, ∂B_i converge to a smooth $(n-1)$-dimensional set ∂B_∞. In light of the uniform bound of $\|u_i\|_{C^{k,\frac{1}{2}}}$ and the uniform regularity of ∂B_i, by taking subsequence if necessary, we can assume that u_i converges in smooth topology to a smooth function $u_\infty \in C^\infty(\bar{B}_\infty)$. Furthermore, $u_\infty \equiv 0$ on ∂B_∞.

In view of (294), the convergence process implies that

$$0 < c^2 = \int_{B_\infty} u_\infty^2 dV_\infty \le 1. \tag{295}$$

Furthermore, we have on B_∞ that

$$-4\Delta_{g_E} u_\infty - 2u_\infty \log u_\infty - \lambda_\infty u_\infty = 0, \tag{296}$$

where $\lambda_\infty = n + \frac{n}{2}\log(4\pi) + \boldsymbol{\mu}_\infty$. Let $\tilde{u} = c^{-1}u_\infty$. Then $\int_{B_\infty} \tilde{u}^2 dV_\infty = 1$. The above equation becomes

$$-4\Delta_{g_E}\tilde{u} - 2\tilde{u}\log\tilde{u} - \left(n + \frac{n}{2}\log(4\pi) + \boldsymbol{\mu}_\infty + 2\log c\right)\tilde{u} = 0.$$

Since $c \in (0, 1)$ by (295) and $\boldsymbol{\mu}_\infty < 0$ by (288), then an integration by parts shows that

$$\boldsymbol{\mu}(g_E, 1) \le \overline{\mathcal{W}}(g_E, \tilde{u}, 1) = \boldsymbol{\mu}_\infty + 2\log c < 0,$$

which is a contradiction. So we finish the proof of Step 3.

Step 4 The three properties are equivalent.

By Step 2, it is clear that $(c) \Rightarrow (a)$. Then Step 3 means that $(a) \Rightarrow (b)$. It is obvious that $(b) \Rightarrow (c)$. Therefore, we obtained the equivalence of properties (a), (b) and (c) in Theorem 3. The proof of the Theorem is complete. □

Corollary 7 *There exists a small positive number $\epsilon = \epsilon(n) > 0$ such that for any nonflat Ricci shrinker (M^n, p, g, f), we have*

$$d_{PGH}\left\{(M^n, p, g), (\mathbb{R}^n, 0, g_E)\right\} > \epsilon. \tag{297}$$

Proof We argue by contradiction.

If (297) were wrong, then we can have a sequence of nonflat Ricci shrinkers (M_i, p_i, g_i) such that

$$d_{PGH}\left((M_i, p_i, g_i), \left(\mathbb{R}^n, 0, g_E\right)\right) \to 0.$$

By Proposition 5.8 of [34], it is clear that $\boldsymbol{\mu}_i = \boldsymbol{\mu}(M_i, p_i, g_i)$ is uniformly bounded from below. Using Theorem 1.1 of [34], the above convergence can be improved to be in the C^∞-Cheeger–Gromov sense

$$(M_i, p_i, g_i) \longrightarrow \left(\mathbb{R}^n, 0, g_E\right).$$

It is not hard to see that $\boldsymbol{\mu}$ is continuous with respect to the above convergence (cf. Theorem 1.2(c) of [34]). Therefore, we have

$$\boldsymbol{\mu}_i = \boldsymbol{\mu}(M_i, p_i, g_i) \to \boldsymbol{\mu}(\mathbb{R}^n, 0, g_E) = 0.$$

It follows that $\boldsymbol{\mu}_i > -\delta_0$ for large i. Therefore, each (M_i, g_i) is isometric to Euclidean space by Theorem 3. This contradicts our choice of (M_i, g_i). The proof of (297) is established by this contradiction. □

Corollary 8 *Let (M^n, g, f) be a Ricci shrinker and let $q \in M$ be a point such that*

$$\nu(B(q, \epsilon_0^{-1}), g, \epsilon_0^{-2}) > -\delta_0.$$

Then there exist a positive constant $C = C(n)$ such that

$$|Rm|(\psi^t(x)) \le CD(1-t) \le CD\frac{f(x)}{f(\psi^t(x))}$$

for any $x \in B(q, \frac{1}{2}e^{-CD}D^{-\frac{1}{2}})$ and $t \in [0, 1)$, where $D = d(p, q) + \sqrt{2n}$.

Proof By the assumption, it follows from Theorem 24 by choosing $r = \epsilon_0^{-1}$ that

$$|Rm|(x, t) \le \frac{1}{t} \tag{298}$$

for any $t \in (0, 1)$ and $d_{g(t)}(q, x) \le \frac{1}{2}\epsilon_0^{-1}$. In addition, from Theorem 25 we have

$$|Rm|(x) \le D \tag{299}$$

for any $x \in B(q, \frac{1}{2})$. From (298), (299) and [15, Theorem 3.1] that there exist a positive constant $C = C(n)$ such that for any $x \in B_t(q, \frac{1}{2}D^{-\frac{1}{2}})$,

$$|Rm|(x, t) \le CD. \tag{300}$$

From (300), it is easy to see by comparing the distances that

$$B\left(q, \frac{1}{2}e^{-CD}D^{-\frac{1}{2}}\right) \subset B_t\left(q, \frac{1}{2}D^{-\frac{1}{2}}\right) \tag{301}$$

for any $t \in [0, 1)$.

Therefore, for any $x \in B(q, \frac{1}{2}e^{-CD}D^{-\frac{1}{2}})$,

$$|Rm|(\psi^t(x)) = (1-t)|Rm|(x, t) \le CD(1-t). \tag{302}$$

Along the flow line of $\psi^t(x)$,

$$\frac{d}{dt}f(\psi^t(x)) = \frac{|\nabla f|^2(\psi^t(x))}{1-t} \le \frac{f(\psi^t(x))}{1-t}, \tag{303}$$

and hence by solving the corresponding ODE,

$$f(\psi^t(x)) \le \frac{f(x)}{1-t}. \tag{304}$$

Combining (302) and (304), the conclusion follows. □

Since f is almost $\frac{d^2}{4}$ by Lemma 1, Corollary 8 shows that the curvature is quadratically decaying along the flow line. Next we prove that if there exists a tubular neighborhood of some level set of f whose isoperimetric constant is almost Euclidean, then globally the curvature is quadratically decaying.

Corollary 9 *For any Ricc shrinker (M^n, g, f), if there exists an $a > 0$ such that for any $x \in f^{-1}(a)$,*

$$\nu(B(x, \epsilon_0^{-1}), g, \epsilon_0^{-2}) > -\delta_0,$$

then the curvature is quadratically decaying and each end has a unique smooth tangent cone at infinity.

Proof We can assume that (M, f) is nonflat, otherwise there is nothing to prove. Now we reparametrize ψ^t by defining for any $s \in (-\infty, \infty)$

$$\tilde{\psi}^s = \psi^{1-e^{-s}}.$$

It is clear from the definition of ψ^t that

$$\frac{d}{ds}\tilde{\psi}^s(x) = \nabla f(\tilde{\psi}^s(x)).$$

In other words, $\tilde{\psi}^s$ is the one-parameter group of diffeomorphisms generated by ∇f. Now we set

$$\epsilon_1 = \epsilon_1(a, n) = \frac{1}{2}e^{-CD_1}D_1^{-\frac{1}{2}},$$

where $D_1 = 2\sqrt{a} + 5n + \sqrt{2n} + 4$.

We claim that any $x \in T_{\epsilon_1}(f^{-1}(a)) := \bigcup_{q \in f^{-1}(a)} B(q, \epsilon_1)$ is not a stationary point of $\tilde{\psi}^s$. Otherwise, it follows from Corollary 8 that

$$|Rm|(x) = |Rm|(\tilde{\psi}^s(x)) \le CD_1 e^{-s}$$

for any $s \geq 0$. However, when $s \to \infty$, $|Rm|(x) = 0$ and this contradicts our nonflatness assumption.

Now we choose $c < a < d$ such that for any $x \in \partial T_{\frac{\epsilon_1}{2}}(f^{-1}(a))$, either $f(x) \leq c$ or $f(x) \geq d$. By continuity, there exists a positive constant $\epsilon \ll \epsilon_1$ such that for any $x \in T_\epsilon(f^{-1}(a))$, $f(x) \in (c+\epsilon, d-\epsilon)$. We set $U := T_\epsilon(f^{-1}(a))$ and claim that for any $y \in U$, there exists an $x \in f^{-1}(a)$ such that $\tilde{\psi}^s(x) = y$ for some s. If $f(y) = a$, then the claim is obvious. If $f(y) < a$, we consider the flow line $\tilde{\psi}^s(y)$ for $s \geq 0$. Notice that by the definition of $\tilde{\psi}^s$,

$$\frac{d}{ds} f(\tilde{\psi}^s(y)) = |\nabla f|^2(\tilde{\psi}^s) \geq 0.$$

Therefore, by the local compactness and our previous no stationary argument, the flow will continue and along the flow f is strictly increasing as long as $\tilde{\psi}^s(y)$ stays in $T_{\epsilon_1}(f^{-1}(a))$. We set s_0 to be the first time such that $\tilde{\psi}^s(y)$ reaches $\partial T_{\frac{\epsilon_1}{2}}(f^{-1}(a))$. In particular, $f(\tilde{\psi}^s(y)) \leq c$ or $f(\tilde{\psi}^s(y)) \geq d$. Since $f(y) \in (c+\epsilon, d-\epsilon)$, it must be $f(\tilde{\psi}^s(y)) \geq d$. As $f(y) < a < f(\tilde{\psi}^{s_0}(y))$, there exists an $s \in (0, s_0)$ such that $f(\tilde{\psi}^s(y)) = a$ by continuity. Therefore, if we set $x = \tilde{\psi}^s(y) \in f^{-1}(a)$, then $\tilde{\psi}^{-s}(x) = y$ and the claim follows. Similarly, for the case $f(y) > a$, the claim is also true.

Next we prove that for any y such that $f(y) > a$, there exists an $x \in U$ such that $\tilde{\psi}^s(x) = y$ for some s. Fix such y, we choose any curve $\{\gamma(z) : z \in [0, 1]\}$ such that $\gamma(0) = p$ and $\gamma(1) = y$. In particular, since p is the minimum point of f, there exists a $z_0 \in [0, 1)$ such that $\gamma(z_0) \in f^{-1}(a)$ and for all $z \in (z_0, 1]$, $f(\gamma(z)) > a$. Now we define $I \subset [z_0, 1]$ such that $z \in I$ if and only if there exists an $x \in U$ such that $\tilde{\psi}^s(x) = \gamma(z)$ for some s. In particular, I is not empty as $z_0 \in I$. It is clear that I is open, since U is open. Now we prove the closedness of I. For a sequence $z_i \in I$ such that $z_i \to z_\infty \in [z_0, 1]$, $f(z_i) > a$ if i is sufficiently large. By our definition of I and the claim with its proof, there exists $x_i \in f^{-1}(a)$ and $s_i > 0$ such that $\tilde{\psi}^{s_i}(x_i) = \gamma(z_i)$. Note that s_i must be bounded. Indeed, by Corollary 8,

$$|Rm|(\gamma(z_i)) = |Rm|(\tilde{\psi}^{s_i}(x_i)) \leq C D_1 e^{-s_i}.$$

If $s_i \to \infty$, then it forces $|Rm|(\gamma(z_\infty)) = 0$ and this is a contradiction. By compactness and taking the subsequence, there exist $x_\infty \in f^{-1}(a)$ and $s_\infty \geq 0$ such that $x_i \to x_\infty$ and $s_i \to s_\infty$. By continuity, $\tilde{\psi}^{s_\infty}(x_i) = \gamma(z_\infty)$. To summarize, $I = [z_0, 1]$ and in particular, $\tilde{\psi}^s(x) = \gamma(1) = y$ for some $x \in U$ and $s \in \mathbb{R}$. By the claim again, we have proved that for any y with $f(y) \geq a$, there exists an $x \in f^{-1}(a)$ such that $\psi^s(x) = y$ for some $s \geq 0$.

Therefore, for any point y outside the compact set $\{f \leq a\}$, it follows from Corollary 8 that

$$|Rm|(y) \leq \frac{C D_1 a}{f(y)} \leq C \frac{\max\{1, a^{\frac{3}{2}}\}}{f(y)}. \tag{305}$$

See Figure 4 for intuition in the case $a > 1$.

In other words, the curvature is quadratically decaying. Since a Ricci shrinker can be regarded as an ancient Ricci flow, it follows from Shi's local estimates [50] that

$$|\nabla^k Rm|(y) \leq \frac{C_k}{d^{k+2}(p, y)}$$

for all $k = 1, 2, \ldots$. It follows immediately that any tangent cone at infinity must be smooth. Finally, the uniqueness follows from [17, Theorem 2], see also [33, Lemma A.3]. □

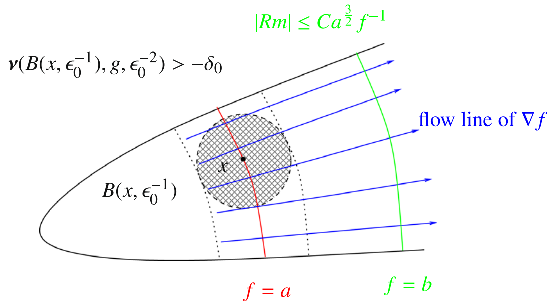

Fig. 4 The quadratic decay of curvature

Remark 9 The proof of Corollary 9 shows that the manifold $\{x \in M \mid f(x) \ge a\}$ is diffeomorphic to $f^{-1}(a) \times [0, 1)$.

11 Strong maximum principle for curvature operator

The purpose of this section is to prove Theorem 4. We remind the readers that all constants C's in this section depend only on the dimension n.

We first show an L^2-integral estimate of Riemannian curvature.

Theorem 26 *Suppose (M^n, p, g, f) is a Ricci shrinker satisfying $\boldsymbol{\mu} \ge -A$, and λ is a positive number. Then we have*

$$\int |Rm|^2 e^{-\lambda f}\, dV \le I \tag{306}$$

for some $I = I(n, A, \lambda) < \infty$.

Theorem 26 is the consequence of the improved no-local-collapsing theorem (i.e., Theorem 2), the local conformal transformation technique (cf. Sect. 3 of [34]), and the curvature estimate of Jiang–Naber (i.e., [31]).

Lemma 32 *For any Ricci shrinker (M^n, p, g, f) and any constant $D > 100n$, we have*

$$\int_{A(D,2D)} |Rc|^2 e^{-f}\, dV \le C e^{\boldsymbol{\mu}} D^{n+2} e^{-D^2/5} \tag{307}$$

where $A(D, 2D)$ is the annulus $B(p, 2D) \backslash B(p, D)$.

Proof Fix a cutoff function ψ on $\mathbb{R}$ such that $\psi = 1$ on $[1, 2]$ and $\psi = 0$ outside $[\frac{1}{2}, 3]$. By defining $\eta(x) = \psi(\frac{d(p,x)}{D})$, we compute

$$\begin{aligned}\int \eta^2 |Rc|^2 e^{-f}\, dV &= \int \eta^2 \langle \frac{g}{2} - \operatorname{Hess} f, Rc\rangle e^{-f}\, dV \\ &= \int \left(\frac{1}{2}\eta^2 R + 2\eta Rc(\nabla \eta, \nabla f)\right) e^{-f}\, dV \\ &\le \int \left(\frac{1}{2}\eta^2 R + \frac{1}{2}\eta^2 |Rc|^2 + 2|\nabla \eta|^2 |\nabla f|^2\right) e^{-f}\, dV\end{aligned}$$

where for the second line we have used $\mathrm{div}(Rc\, e^{-f})=0$. Consequently, by Lemmas 1 and 2, we have

$$\int \eta^2|Rc|^2 e^{-f}\, dV \le \int \left(\eta^2 R + 4|\nabla\eta|^2|\nabla f|^2\right) e^{-f}\, dV \le C\int_{A(D/2,3D)} f e^{-f}\, dV.$$

Plugging the estimates in Lemmas 1 and 2 into the above inequality, we arrive at (307). □

In the proof of Lemma 32, if we choose ψ such that $\psi=1$ on $(-\infty,1]$ and $\psi=0$ on $[2,\infty)$, then a similar argument shows the following Lemma.

Lemma 33 *For any Ricci shrinker (M^n,p,g,f), we have*

$$\int |Rc|^2 e^{-f}\, dV \le Ce^{\boldsymbol{\mu}}. \tag{308}$$

The details of the proof of Lemma 33 is almost identical to that of Lemma 32. So we leave it to interested readers. Note that Lemma 33 provides an explicit upper bound of [40, Theorem 1.1]. Starting from Lemmas 32 and 33, we are ready to prove Theorem 26.

Proof of Theorem 26: We only prove the case when $\lambda=1$. The general case is similar and is left to interested readers.

For any point $q\in M$ such that $d(p,q)=D>100n$, we set $r=\frac{1}{D}$, $\bar f=f-f(q)$, then under the conformal transformation $\bar g := e^{-\frac{2\bar f}{n-2}}g$, we have

$$\overline{Rc}=\frac{1}{n-2}\left\{df\otimes df+(n-1-f)e^{\frac{2\bar f}{n-2}}\bar g\right\}, \tag{309}$$

$$\overline{Rm}=e^{-\frac{2\bar f}{n-2}}\left[Rm+\frac{1}{n-2}\left(\frac{df\otimes df}{n-2}+\frac{g}{2}\left(1-\frac{|\nabla f|^2}{n-2}\right)-\frac{Rc}{n-2}\right)\owedge g\right], \tag{310}$$

where the proof and the definition of the Kulkarni–Nomizu product $\owedge$ can be found in [4, Theorem 1.165]. It follows from [34, Lemma 3.5] that

$$B_{\bar g}\left(q,e^{-\frac{1}{n-2}}r\right)\subset B(q,r)\subset B_{\bar g}\left(q,e^{\frac{1}{n-2}}r\right). \tag{311}$$

Therefore, by the same proof as in [34, Lemma 3.7], we have

$$|\bar f|\le C \quad\text{and}\quad \left|\overline{Rc}\right|_{\bar g}\le CD^2 \quad\text{on}\quad B_{\bar g}\left(q,e^{\frac{1}{n-2}}r\right). \tag{312}$$

Since $R\le CD^2$ on $B(q,r)$, it follows from Theorem 23 that $|B(q,r)|\ge Ce^{\boldsymbol{\mu}}r^n$ and hence

$$\left|B_{\bar g}\left(q,e^{\frac{1}{n-2}}r\right)\right|_{\bar g}\ge Ce^{\boldsymbol{\mu}}r^n. \tag{313}$$

One can also use Theorem 2 to obtain the above estimate directly.

By defining $\tilde g := r^{-2}\bar g$, we have $|\widetilde{Rc}|_{\tilde g}\le C$ on $B_{\tilde g}(q,e^{\frac{1}{n-2}})$ and $|B_{\tilde g}(q,e^{\frac{1}{n-2}})|_{\tilde g}\ge Ce^{\boldsymbol{\mu}}$. By shrinking balls to its half size if necessary, it follows from [31, Theorem 1.6] that

$$r^{4-n}\int_{B_{\bar g}(q,e^{\frac{1}{n-2}}r)}|\overline{Rm}|^2\, dV_{\bar g}=\int_{B_{\tilde g}(q,e^{\frac{1}{n-2}})}|\widetilde{Rm}|^2\, dV_{\tilde g}\le I_0 \tag{314}$$

for some constant $I_0=I_0(n,A)$.

From (310), we have on $B(q,r)$,

$$|Rm|^2\le C\left(|\overline{Rm}|^2+|\nabla f|^4+|Rc|^2\right)\le C\left(|\overline{Rm}|^2+f^2+|Rc|^2\right).$$

Therefore, we have

$$\begin{aligned}
&\int_{B(q,r)} |Rm|^2 e^{-f}\, dV \\
&\le C\left(\int_{B_{\bar g}(q, e^{\frac{1}{n-2}} r)} |\overline{Rm}|^2 e^{-f}\, dV_{\bar g} + \int_{B(q,r)} f^2 e^{-f}\, dV + \int_{B(q,r)} |Rc|^2 e^{-f}\, dV\right) \\
&\le C e^{-\frac{D^2}{5}} \left(D^{4-n} I_0 + D^{n+2} e^{\mu}\right)
\end{aligned}$$

where we have used Lemma 32 and (314). Consequently, there exists $I_1 = I_1(n, A)$ such that

$$\int_{B(q,r)} |Rm|^2 e^{-f}\, dV \le I_1 D^{n+2} e^{-\frac{D^2}{5}}. \tag{315}$$

For any constant $D > 100n$, we apply Vitali's lemma for the covering $\{B(q, \frac{1}{4D})\}_{q \in A(D,2D)}$. If we assume that $\{B(q_i, \frac{1}{4D})\}_{1 \le i \le k}$ is a maximal collection of mutually disjoint sets, then $\{B(q_i, \frac{1}{2D})\}_{1 \le i \le k}$ cover $A(D, 2D)$. It is clear from definition that

$$\sum_{i=1}^{k} \left|B\left(q_i, \frac{1}{4D}\right)\right| \le |A(D, 2D)| \le |B(p, 2D)|.$$

By Lemma 2 and (313), we obtain $k \le C D^{2n}$. Combining (315) with the above inequality implies that

$$\int_{A(D,2D)} |Rm|^2 e^{-f}\, dV \le \sum_{i=1}^{k} \int_{B(q_i, \frac{1}{2D})} |Rm|^2 e^{-f}\, dV \le k I_1 D^{n+2} e^{-\frac{D^2}{5}} \le C I_1 D^{3n+2} e^{-\frac{D^2}{5}}. \tag{316}$$

Similarly, by exploiting Lemma 33, we have

$$\int_{B(p,D_0)} |Rm|^2 e^{-f}\, dV \le I_2 \tag{317}$$

where $D_0 = 100n$ and $I_2 = I_2(n, A)$.

Now we set $D_i = 2^i D_0$ and decompose the integral as

$$\int |Rm|^2 e^{-f}\, dV = \int_{B(p,D_0)} |Rm|^2 e^{-f}\, dV + \sum_{i \ge 0} \int_{A(D_i, 2D_i)} |Rm|^2 e^{-f}\, dV.$$

Plugging (316) and (317) into the above equation, we arrive at

$$\int |Rm|^2 e^{-f}\, dV \le I_2 + \sum_{i \ge 0} C I_1 D_i^{3n+2} e^{-\frac{D_i^2}{5}} = I_2 + C I_1 \sum_{i \ge 0} 2^{i(3n+2)} D_0^{3n+2} e^{-\frac{4^i D_0^2}{5}} := I.$$

Since both I_1 and I_2 depend only on n and A, it is clear that I relies only on n and A and we arrive at (306). The proof of Theorem 26 is complete. □

From (306) and [40, Theorem 1.2], a direct corollary of Theorem 26 is the following estimate.

Corollary 10 *For any Ricci shrinker* $(M^n, g, f) \in \mathcal{M}_n(A)$*, there exists a constant* $I = I(n, A) < \infty$ *such that*

$$\int |\nabla Rc|^2 e^{-f}\, dV = \int |div(Rm)|^2 e^{-f}\, dV \leq I.$$

Theorem 26 is an important step for verifying maximum principle on curvature operators. The curvature operator on two-forms are defined as $\mathcal{R} : \Lambda^2 \to \Lambda^2 : \mathcal{R}(e^i \wedge e^j, e^k \wedge e^l) = R_{ijkl}$. The two-form $e^i \wedge e^j := e^i \otimes e^j - e^j \otimes e^i$ and the inner product on Λ^2 is defined as $\langle A, B\rangle := -\frac{1}{2} tr(AB)$ for $A, B \in \Lambda^2 = \mathfrak{so}(n)$. In other words, for $w = \frac{1}{2}\sum_{i,j} w_{ij} e^i \wedge e^j$, we have

$$\mathcal{R}(w)_{ij} = \frac{1}{2} R_{ijkl} w_{kl}.$$

In the setting of Ricci shrinker (M^n, g, f), the following equation (see [24]) holds:

$$\Delta_f \mathcal{R} = \mathcal{R} - 2Q(\mathcal{R}).$$

Here $Q(\mathcal{R}) := \mathcal{R}^2 + \mathcal{R}^\#$ and $\mathcal{R}^\#$ is defined as

$$\mathcal{R}^\#(u, v) = -\frac{1}{2} tr(ad_u\, \mathcal{R}\, ad_v\, \mathcal{R})$$

for any $u, v \in \Lambda^2$. If we choose an orthonormal basis $\{\phi_i\}$ of Λ^2, then

$$\mathcal{R}^\#(u, v) = -\frac{1}{2} \sum_{i,j} \langle [\mathcal{R}(\phi_i), \phi_j], u\rangle \langle [\mathcal{R}(\phi_j), \phi_i], v\rangle.$$

If we assume $\lambda_1 \leq \lambda_2 \leq \cdots$ are all eigenvalues of $\mathcal{R}$ on Λ^2, then we have the following rigidity theorem.

Theorem 27 *There exists a constant* $\epsilon = \epsilon(n) > 0$ *such that for any Ricci shrinkers* (M^n, g, f)*, if* $\lambda_2 \geq -\epsilon \dfrac{\lambda_1^2}{|R - 2\lambda_1|}$*, then* $\lambda_1 \geq 0$*. Consequently,* (M^n, g) *is isometric to a quotient of* $N^k \times \mathbb{R}^{n-k}$ *for some* $0 \leq k \leq n$*, where* N^k *is a closed symmetric space.*

Proof It suffices to prove $\lambda_1 \geq 0$. Namely, (M^n, g) has nonnegative curvature operator. The further conclusion follows from [42, Corollary 4].

We fix a point q and assume that ϕ_1 is an eigenvector of λ_1. Extending ϕ_1 by parallel transport on a small neighborhood of q, we have

$$\Delta_f \mathcal{R}(\phi_1, \phi_1) = \mathcal{R}(\phi_1, \phi_1) - 2Q(\mathcal{R})(\phi_1, \phi_1).$$

Therefore if we assume that ϕ_i are eigenvectors of λ_i, then in the barrier sense,

$$\begin{aligned} \Delta_f \lambda_1 &\leq \lambda_1 - \left(2\lambda_1^2 - \sum_{i,j} \langle [\mathcal{R}(\phi_i), \phi_j], \phi_1\rangle \langle [\mathcal{R}(\phi_j), \phi_i], \phi_1\rangle\right) \\ &= \lambda_1 - \left(2\lambda_1^2 + \sum_{i,j} C_{ij}^2 \lambda_i \lambda_j\right) \end{aligned} \tag{318}$$

where $C_{i,j} = \langle [\phi_i, \phi_j], \phi_1\rangle$. Notice that $C_{i,j} = 0$ if $i = 1$ or $j = 1$.

We claim that $|C_{i,j}| \leq 2$. Indeed, if we assume that ϕ_i, ϕ_j and ϕ_1 are represented by the antisymmetric matrices A, B and C respectively, then $C_{i,j} = -\frac{1}{2} tr((AB - BA)C) =$

$-tr(ABC)$. By choosing a basis such that $A_{2k-1,2k} = a_k = -A_{2k,2k-1}$ for $k \le [n/2]$ and 0 otherwise, we have

$$\begin{aligned}|tr(ABC)| \le& \sum_{k,l} |a_k||B_{2k,l}C_{l,2k-1} - B_{2k-1,l}C_{l,2k}| \\ \le& \frac{1}{2}\sum_{k,l}(B_{2k,l}^2 + C_{l,2k-1}^2 + B_{2k-1,l}^2 + C_{l,2k}^2) \\ \le& \frac{1}{2}(|B|^2 + |C|^2) = 2.\end{aligned}$$

Here we have used the fact that $|A|^2 = |B|^2 = |C|^2 = 2$.

Next we prove that if ϵ is properly chosen, then we have

$$P{:=}2\lambda_1^2 + \sum_{i,j} C_{ij}^2 \lambda_i \lambda_j \ge 0.$$

From the definition of λ_i, we notice that $\sum \lambda_i = R/2$. Therefore, we fix λ_1 and λ_2 and minimize P under the restriction $\sum \lambda_i = R/2$. We can assume that $\lambda_2 < 0$, otherwise $P \ge 0$ from its definition. We also set $c_n = n(n-1)/2$ and assume that $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_{s+1}$ are all eigenvalues smaller than 0. Therefore,

$$P \ge P_1{:=}2\lambda_1^2 + 2\sum_{\substack{2\le i\le s+1\\ s+2\le j\le c_n}} C_{ij}^2 \lambda_i \lambda_j.$$

It is easy to show that P_1 is minimized when $\lambda_2 = \lambda_3 = \cdots = \lambda_{s+1}$ and $\lambda_{s+2} = \cdots = \lambda_{c_n}$. It follows that

$$\begin{aligned}\frac{P_1}{2} \ge& \lambda_1^2 + \sum_{\substack{2\le i\le s+1\\ s+2\le j\le c_n}} \frac{1}{c_n - s - 1} C_{i,j}^2 \lambda_2 (R/2 - \lambda_1 - s\lambda_2) \\ \ge& \lambda_1^2 + 4s\lambda_2(R/2 - \lambda_1 - s\lambda_2).\end{aligned}$$

By solving the above quadratic inequality, we obtain that P_1 and hence P are nonnegative if

$$\lambda_2 \ge \frac{\frac{R}{2} - \lambda_1 - \sqrt{(\frac{R}{2} - \lambda_1)^2 + \lambda_1^2}}{2s}.$$

If we choose $\epsilon = \frac{1}{(1+\sqrt{2})(c_n-2)}$, then it is clear that for any $1 \le s \le c_n - 2$,

$$\lambda_2 \ge -\epsilon \frac{\lambda_1^2}{R - 2\lambda_1} \ge \frac{\frac{R}{2} - \lambda_1 - \sqrt{(\frac{R}{2} - \lambda_1)^2 + \lambda_1^2}}{2(c_n - 2)} \ge \frac{\frac{R}{2} - \lambda_1 - \sqrt{(\frac{R}{2} - \lambda_1)^2 + \lambda_1^2}}{2s}.$$

Therefore, from (318) we obtain $\Delta_f \lambda_1 \le \lambda_1$. Since $\lambda_1 \in L^2(e^{-f}\, dV)$ by (306), then it follows from [47, Theorem 4.4] that $\lambda_1 \ge 0$. □

We conclude this section by the proof of Theorem 4.

Proof of Theorem 4: Since $\lambda_2 \ge 0$, we can apply Theorem 27 to obtain $\lambda_1 \ge 0$. Therefore, M^n is a finite quotient of $N^k \times \mathbb{R}^{n-k}$. Note that only the case $k = n$ is possible. For otherwise the second smallest eigenvalue must be 0. Since N^n is a compact Einstein manifold such that the curvature operator is 2-positive, it follows from [5] that its universal covering must be S^n. □

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