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Calculus of Variations

On the jerky crack growth in elastoplastic materials

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Abstract
The purpose of this paper is to show that in elastoplastic materials cracks can grow only in an intermittent way. This result is rigorously proved in the framework of a simplified model.

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1 Introduction

In this paper we give a contribution to the mathematical derivation of the properties of the quasistatic crack growth in elastoplastic materials. The study of this subject has a long history (see, e.g., [11,14,15]). Our aim is to obtain a precise mathematical result in a simplified model where perfect plasticity interacts with crack growth. In particular, under suitable assumptions we prove that cracks are piecewise constant in time.

In our simplified model the reference configuration Ω is a bounded connected open subset of $\mathbb{R}^2$ with Lipschitz boundary. We consider only the antiplane case, so that the displacement u is a function from Ω into $\mathbb{R}$. We assume that the cracks and the plastic slips may occur only on a prescribed segment Γ, whose interior is contained in Ω and whose end-points belong to $\partial\Omega$. It is not restrictive to assume that $\Gamma := \{(x, 0) : a \le x \le b\}$ for some $a < b$.

Since there is no plastic part in $\Omega \setminus \Gamma$, the displacement u belongs to $H^1(\Omega \setminus \Gamma)$ and the elastic energy is given by

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u|^2 dxdy.$$

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We assume that at each time the crack has the form $\Gamma_a^s := \{(x, 0) : a \leq x \leq s\}$ for some $a \leq s \leq b$ and that the energy spent to produce it is equal to $s - a$. On $\Gamma_s^b := \{(x, 0) : s \leq x \leq b\}$ the plastic slip is determined by the jump of the displacement:

$$[u] = u^+ - u^-,$$

where u^+ and u^- are the traces of u on Γ from above and from below. The plastic dissipation distance between the current displacement u and a previous displacement u_0 is given by

$$\int_{\Gamma_s^b} |[u] - [u_0]| \, dx.$$

The evolution is driven by a time-dependent Dirichlet boundary condition $u = w(t)$ imposed on a prescribed Borel subset $\partial_D \Omega$ of $\partial \Omega$. We first consider the incremental formulation. Given a subdivision $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$ of the interval $[0, T]$, for $i = 1, \ldots, n$ let (u_i, s_i) be a solution of the incremental minimum problem for the pair (u, s):

$$\min_{\substack{u \in H^1(\Omega \setminus \Gamma) \\ u = w(t_i) \text{ on } \partial_D \Omega \\ s_{i-1} \leq s \leq b}} \Big\{ \frac{1}{2} \int_{\Omega \setminus \Gamma} |\nabla u|^2 dx dy + s + \int_{\Gamma_s^b} |[u] - [u_{i-1}]| dx \Big\}.$$

As in [6] we can prove (Theorem 2.5) that, passing to a subsequence, the piecewise constant interpolation of (u_i, s_i) converges, as the fineness of the subdivision tends to zero, to a quasistatic evolution, i.e., a pair (u, s) which satisfies the following conditions:

(a) (irreversibility) s is nondecreasing on $[0, T]$;
(b) (equilibrium) for every $t \in [0, T]$ we have $u(t) \in H^1(\Omega \setminus \Gamma)$, $u(t) = w(t)$ on $\partial_D \Omega$, and

$$\frac{1}{2} \int_{\Omega \setminus \Gamma} |\nabla u(t)|^2 dx dy + s(t) \leq \frac{1}{2} \int_{\Omega \setminus \Gamma} |\nabla \hat{u}|^2 dx dy + \hat{s} + \int_{\Gamma_{\hat{s}}^b} |[\hat{u}] - [u(t)]| dx,$$

for every $\hat{u} \in H^1(\Omega \setminus \Gamma)$, with $\hat{u} = w(t)$ on $\partial_D \Omega$, and every $\hat{s} \in [s(t), b]$;
(c) (energy-dissipation inequality) for every $t_1, t_2 \in [0, T]$, with $t_1 < t_2$, we have

$$\frac{1}{2} \int_{\Omega \setminus \Gamma} |\nabla u(t_2)|^2 dx dy + s(t_2) - s(t_1) + \int_{\Gamma_{s(t_2)}^b} |[u(t_2)] - [u(t_1)]| dx$$
$$\leq \frac{1}{2} \int_{\Omega \setminus \Gamma} |\nabla u(t_1)|^2 dx dy + \int_{t_1}^{t_2} \Big(\int_{\Omega \setminus \Gamma} \nabla u(\tau) \nabla \dot{w}(\tau) dx dy \Big) d\tau,$$

where $\dot{w}$ is the time-derivative of w.

As in [6] we can obtain (Theorem 2.9) an energy-dissipation balance, using a suitable notion of dissipation (Definition 2.6). Therefore our notion of quasistatic evolution is formulated within the framework of rate-independent processes developed in [12,13]. When no plastic slip is present, i.e., $[u(t)] = 0$ on $\Gamma_{s(t)}^b$, this evolution agrees, in the antiplane case, with the variational solution of the crack growth problem introduced in [9] and studied in [2].

The main result of our paper (Theorem 4.1) is that, if (u, s) satisfies hypotheses (a)-(c), and w satisfies suitable continuity conditions, then s is piecewise constant. In other words, the crack growth is jerky. This behaviour is in agreement with the numerical simulations in [1] and with many experimental results (see, e.g., [7,10]). As a consequence of well-known results on perfect plasticity (Theorem 4.14), from this property of s we deduce (Theorem 4.15)

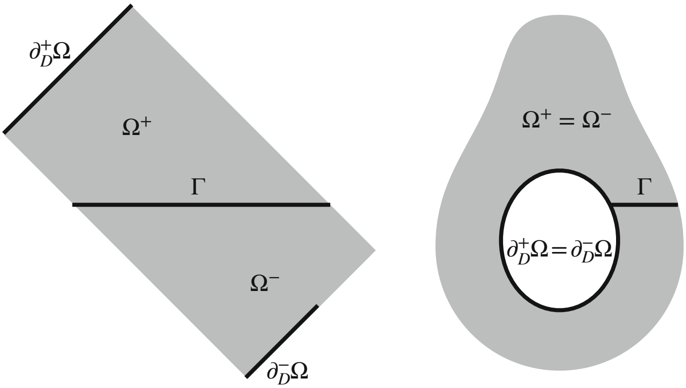

Fig. 1 Examples of sets Ω, $\partial_D\Omega$, and Γ

that u is piecewice absolutely continuous with values in $H^1(\Omega \setminus \Gamma)$. A concluding example (Theorem 5.1) shows that, in general, s is not constant.

A numerical study of the simplified model of the present paper will appear in [5].

2 Formulation of the problem

The reference configuration is a bounded connected open set $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$. On a prescribed Borel subset $\partial_D\Omega$ of $\partial\Omega$ we shall impose a time-dependent Dirichlet boundary condition. On its complement $\partial\Omega \setminus \partial_D\Omega$ we shall consider the homogeneous Neumann boundary condition.

In our simplified model we assume that the cracks and the plastic slips may occur only on a prescribed segment $\Gamma := \{(x, 0) : a \le x \le b\}$ contained in $\overline{\Omega}$, with $(a, 0), (b, 0) \in \partial\Omega$ and $(x, 0) \in \Omega$ for every $a < x < b$. For every $a \le s_1 \le s_2 \le b$ we set $\Gamma_{s_1}^{s_2} := \{(x, 0) : s_1 \le x \le s_2\}$.

We assume that there exists an open neighbourhood U of Γ in $\mathbb{R}^2$ such that $U \cap (\Omega \setminus \Gamma)$ is the union of two disjoint connected open sets U^+ and U^- with Lipschitz boundary. We also assume that for every $a < x < b$ we have $(x, y) \in U^\pm$ whenever $|y|$ is small and $\pm y > 0$. Let $\Omega^\pm$ be the connected component of $\Omega \setminus \Gamma$ containing $U^\pm$. Note that under our hypotheses we have $\Omega \setminus \Gamma = \Omega^+ \cup \Omega^-$ and that it may happen that $\Omega^+ = \Omega^-$, if Ω is not simply connected (see Fig. 1) We set $\partial^\pm\Omega := \partial\Omega^\pm \setminus \Gamma$ and $\partial_D^\pm\Omega := \partial_D\Omega \cap \partial\Omega^\pm$. We assume that

$$\partial_D^+\Omega \quad \text{and} \quad \partial_D^-\Omega \quad \text{have positive one-dimensional measure.} \tag{2.1}$$

Since we are dealing with the antiplane case, the displacement $u = u(x, y)$ is a scalar function belonging to $H^1(\Omega \setminus \Gamma)$. An admissible crack will be a segment of the form Γ_a^s for some $a \le s \le b$. Given a displacement $u \in H^1(\Omega \setminus \Gamma)$, the jump of u across Γ is given by

$$[u] = u^+ - u^-,$$

where u^+ is the trace on the side of Γ corresponding to $y > 0$, and u^- is the trace on the opposite side.

The Dirichlet boundary condition will be prescribed through a function

$$w \in AC([0,T]; H^1(\Omega \setminus \Gamma_a^{s_0})) \tag{2.2}$$

for a suitable $s_0 \in [a, b)$.

Definition 2.1 Let $T > 0$, $s_0 \in [a, b)$, and $w \in AC([0,T]; H^1(\Omega \setminus \Gamma_a^{s_0}))$. A quasistatic evolution with boundary value w on $\partial_D\Omega$ is a pair (u, s), with $u\colon [0,T] \to H^1(\Omega \setminus \Gamma)$ measurable and $s\colon [0,T] \to [s_0, b]$, that satisfies the following conditions:

(a) (irreversibility) s is nondecreasing;
(b) (equilibrium) for every $t \in [0,T]$ we have $u(t) = w(t)$ on $\partial_D\Omega$ and

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t)|^2 dxdy + s(t) \le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla \hat{u}|^2 dxdy + \hat{s} + \int_{\Gamma_{\hat{s}}^b} |[\hat{u}] - [u(t)]| dx,$$

for every $\hat{u} \in H^1(\Omega \setminus \Gamma)$, with $\hat{u} = w(t)$ on $\partial_D\Omega$, and every $\hat{s} \in [s(t), b]$;
(c) (energy-dissipation inequality) for every $t_1, t_2 \in [0,T]$, with $t_1 < t_2$, we have

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t_2)|^2 dxdy + s(t_2) - s(t_1) + \int_{\Gamma_{s(t_2)}^b} |[u(t_2)] - [u(t_1)]| dx$$
$$\le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t_1)|^2 dxdy + \int_{t_1}^{t_2} \Big(\int_{\Omega\setminus\Gamma} \nabla u(\tau)\nabla \dot{w}(\tau) dxdy\Big) d\tau.$$

Remark 2.2 Taking $\hat{u} = w(t)$ and $\hat{s} = b$ in condition (b) above, by (2.2) we obtain that there exists a constant $M_1 > 0$ such that

$$\int_{\Omega\setminus\Gamma} |\nabla u(t)|^2 dxdy \le M_1 \quad \text{for every } t \in [0,T]. \tag{2.3}$$

Together with the measurability of $t \mapsto u(t)$ this implies that the last integral in condition (c) above is well defined. Moreover, since $u(t) = w(t)$ on $\partial_D\Omega$, by (2.1)–(2.3) there exists a constant $M_0 > 0$ such that

$$\int_{\Omega\setminus\Gamma} |u(t)|^2 dxdy \le M_0 \quad \text{for every } t \in [0,T]. \tag{2.4}$$

Remark 2.3 Let us now comment on the term

$$\int_{t_1}^{t_2} \Big(\int_{\Omega\setminus\Gamma} \nabla u(\tau)\nabla \dot{w}(\tau) dxdy\Big) d\tau \tag{2.5}$$

which appears in the energy-dissipation balance. The Euler equation for the equilibrium condition gives that $u(t)$ is harmonic in $\Omega \setminus \Gamma$ for every $t \in [0,T]$. Moreover, if u is sufficiently regular, the equilibrium condition implies that $\frac{\partial u(t)}{\partial \nu} = 0$ on $\partial\Omega \setminus \partial_D\Omega$, where ν is the outward unit normal to $\partial\Omega$, $\big(\frac{\partial u(t)}{\partial y}\big)^+ = \big(\frac{\partial u(t)}{\partial y}\big)^- = 0$ on $\Gamma_a^{s(t)}$, and $\big(\frac{\partial u(t)}{\partial y}\big)^+ = \big(\frac{\partial u(t)}{\partial y}\big)^-$ on $\Gamma_{s(t)}^b$ (the last property follows easily from (3.8) and (3.9), proved below in a more general setting). Therefore, since $(\dot{w})^+(\tau) = (\dot{w})^-(\tau)$ on $\Gamma_{s(t)}^b$ (by our assumption on w and s), integrating by parts we obtain

$$\int_{\Omega\setminus\Gamma} \nabla u(\tau)\nabla \dot{w}(\tau) dxdy = \int_{\partial_D\Omega} \frac{\partial u(\tau)}{\partial \nu} \dot{w}(\tau) dS,$$

where S is the line-measure on $\partial_D\Omega$. Thus (2.5) equals

$$\int_{t_1}^{t_2}\Big(\int_{\partial_D\Omega}\frac{\partial u(\tau)}{\partial\nu}\dot{w}(\tau)dS\Big)d\tau. \tag{2.6}$$

Since $\frac{\partial u(\tau)}{\partial\nu}$ represents the force acting on the boundary, (2.6) represents the work done by this force in the interval $[t_1, t_2]$.

Remark 2.4 The previous remark suggests that Definition 2.1 does not change if w is replaced by another function $w_* \in AC([0,T]; H^1(\Omega\setminus\Gamma_a^{s_0}))$ such that

$$w(t) = w_*(t) \quad \text{on } \partial_D\Omega.$$

This is actually true without any additional regularity assumption. Indeed, if (u, s) is a quasistatic evolution for w, then

$$\int_{\Omega\setminus\Gamma}\nabla u(\tau)\nabla\dot{w}(\tau)dxdy = \int_{\Omega\setminus\Gamma}\nabla u(\tau)\nabla\dot{w}_*(\tau)dxdy \quad \text{for a.e. } \tau\in[0,T].$$

This follows from Lemma 3.1, since $\dot{w}(\tau)-\dot{w}_*(\tau)\in H^1(\Omega\setminus\Gamma)$, $\dot{w}(\tau)-\dot{w}_*(\tau) = 0$ on $\partial_D\Omega$, and $[\dot{w}(\tau)-\dot{w}_*(\tau)] = 0$ on $\Gamma^b_{s(\tau)}$.

The following result shows the existence of a quasistatic evolution with prescribed initial data.

Theorem 2.5 *Let $T > 0$, $s_0\in[a,b)$, $u_0\in H^1(\Omega\setminus\Gamma)$, and let $w\in AC([0,T]; H^1(\Omega\setminus\Gamma_a^{s_0}))$. Assume that $u_0 = w(0)$ on $\partial_D\Omega$ and*

$$\frac{1}{2}\int_{\Omega\setminus\Gamma}|\nabla u_0|^2dxdy + s_0 \le \frac{1}{2}\int_{\Omega\setminus\Gamma}|\nabla\hat{u}|^2dxdy + \hat{s} + \int_{\Gamma^b_{\hat{s}}}|[\hat{u}]-[u_0]|dx,$$

for every $\hat{u}\in H^1(\Omega\setminus\Gamma)$, with $\hat{u} = w(0)$ on $\partial\Omega$, and every $s_0\le\hat{s}\le b$. Then there exists a quasistatic evolution with boundary value w on $\partial_D\Omega$, satisfying the initial conditions $u(0) = u_0$ and $s(0) = s_0$.

To prove the theorem it is convenient to introduce the notion of dissipation, which is a particular case of the one considered in [6, Section 2.3].

Definition 2.6 Let $u\colon[0,T]\to H^1(\Omega\setminus\Gamma)$ and $s\colon[0,T]\to[a,b]$. The dissipation of (u,s) on the interval $[t_1,t_2]\subset[0,T]$ is defined as:

$$\mathrm{Diss}(u(\cdot), s(\cdot); t_1, t_2) := \sup\sum_{i=1}^{k}\Big(s(\tau_i)-s(\tau_{i-1}) + \int_{\Gamma^b_{s(\tau_i)}}|[u(\tau_i)]-[u(\tau_{i-1})]|dx\Big)$$

where the supremum is taken over all finite partitions $t_1 = \tau_0\le\tau_1\le\cdots\le\tau_k = t_2$.

Proof of Theorem 2.5. The proof is a simplified version of the proof of [6, Theorem 2.5]. We fix a sequence of subdivisions $(t_n^i)_{0\le i\le n}$ with

$$0 = t_n^0 < t_n^1 < \cdots < t_n^{n-1} < t_n^n = T, \tag{2.7}$$

$$\lim_{n\to\infty}\max_{1\le i\le n}(t_n^i - t_n^{i-1}) = 0. \tag{2.8}$$

For every n we set $u_n^0 = u_0$, $s_n^0 = s_0$, and for every $i = 1, \dots, n$ we define inductively (u_n^i, s_n^i) as a solution of the incremental minimum problem

$$\min_{\substack{u\in H^1(\Omega\setminus\Gamma)\\ u=w_n^i \text{ on } \partial_D\Omega\\ s_n^{i-1}\le s\le b}} \Big\{\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u|^2 dxdy + s + \int_{\Gamma_s^b} |[u]-[u_n^{i-1}]|dx\Big\}, \tag{2.9}$$

where $w_n^i := w(t_n^i)$.

Note that, by the triangle inequality, from (2.9) we obtain that u_n^i satisfies

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_n^i|^2 dxdy + s_n^i \le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla \hat{u}|^2 dxdy + \hat{s} + \int_{\Gamma_{\hat{s}}^b} |[\hat{u}]-[u_n^i]|dx, \tag{2.10}$$

for every $s_n^i \le \hat{s} \le b$ and every $\hat{u} \in H^1(\Omega\setminus\Gamma)$ with $\hat{u} = w_n^i$ on $\partial_D\Omega$.

To estimate u_n^i we compare (u_n^i, s_n^i) with (w_n^i, s_n^i) in the minimum problem (2.10) and we obtain

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_n^i|^2 dxdy \le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla w_n^i|^2 dxdy + \int_{\Gamma_{s_n^i}^b} |[u_n^i]|dx \le C_1 + \int_{\Gamma_{s_n^i}^b} |[u_n^i]|dx, \tag{2.11}$$

for a suitable constant $C_1 > 0$ independent of i and n . By the Trace Inequality there exists a constant $C_2 > 0$ independent of i and n such that

$$\int_\Gamma |[u_n^i]|dx \le C_2\Big(\int_{\Omega\setminus\Gamma} |\nabla u_n^i|^2 dxdy\Big)^{1/2} + C_2\Big(\int_{\Omega\setminus\Gamma} |u_n^i|^2 dxdy\Big)^{1/2}.$$

Since $u_n^i = w_n^i$ on $\partial_D\Omega$, by (2.1), (2.2), and the Poincaré Inequality there exists a constant $C_3 > 0$ independent of i and n, such that

$$\int_\Gamma |[u_n^i]|dx \le C_3\Big(\int_{\Omega\setminus\Gamma} |\nabla u_n^i|^2 dxdy\Big)^{1/2} + C_3.$$

Therefore (2.11) gives

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_n^i|^2 dxdy \le C_3\Big(\int_{\Omega\setminus\Gamma} |\nabla u_n^i|^2 dxdy\Big)^{1/2} + C_3 + C_1$$

which implies that there exists a constant $C_4 > 0$ independent of i and n such that

$$\int_{\Omega\setminus\Gamma} |\nabla u_n^i|^2 dxdy \le C_4. \tag{2.12}$$

We now compare (u_n^i, s_n^i) with $(u_n^{i-1} + w_n^i - w_n^{i-1}, s_n^{i-1})$ in the minimum problem (2.9) and we obtain

$$\begin{aligned}\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_n^i|^2 dxdy + s_n^i - s_n^{i-1} + \int_{\Gamma_{s_n^i}^b} |[u_n^i]-[u_n^{i-1}]|dx \\ \le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_n^{i-1}|^2 dxdy + \int_{t_n^{i-1}}^{t_n^i}\Big(\int_{\Omega\setminus\Gamma} \nabla u_n^{i-1}\nabla \dot{w}(t) dxdy\Big)dt + R_n^i,\end{aligned}$$

where

$$R_n^i := \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla(w_n^i - w_n^{i-1})|^2 dxdy.$$

Iterating this inequality for every $0 \le i < j \le n$ we obtain

$$\begin{aligned} &\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_n^j|^2 dxdy + \sum_{h=i+1}^{j} \Big(s_n^h - s_n^{h-1} + \int_{\Gamma^b_{s_n^h}} |[u_n^h] - [u_n^{h-1}]| dx \Big) \\ &\le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_n^i|^2 dxdy + \sum_{h=i+1}^{j} \int_{t_n^{h-1}}^{t_n^h} \Big(\int_{\Omega\setminus\Gamma} \nabla u_n^{h-1} \nabla \dot{w}(\tau) dxdy \Big) d\tau + R_n, \end{aligned} \tag{2.13}$$

where $R_n := \sum_{i=1}^n R_n^i$. Since $w \in AC([0,T]; H^1(\Omega \setminus \Gamma_a^{s_0}))$ we have that $R_n \to 0$.

Let $u_n(t)$, $s_n(t)$, and $w_n(t)$ be the piecewise constant interpolations of u_n^i, s_n^i, and w_n^i defined by

$$u_n(t) := u_n^{i-1}, \quad s_n(t) := s_n^{i-1}, \quad w_n(t) := w_n^{i-1} \qquad \text{for } t_n^{i-1} \le t < t_n^i. \tag{2.14}$$

Note that by (2.12) we have

$$\int_{\Omega\setminus\Gamma} |\nabla u_n(t)|^2 dxdy \le C_4 \quad \text{for every } t \in [0,T] \text{ and every } n. \tag{2.15}$$

Inequality (2.13) can be rewritten as

$$\begin{aligned} &\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_n(t_n^j)|^2 dxdy + \mathrm{Diss}(u_n(\cdot), s_n(\cdot); t_n^i, t_n^j) \\ &\le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_n(t_n^i)|^2 dxdy + \int_{t_n^i}^{t_n^j} \Big(\int_{\Omega\setminus\Gamma} \nabla u_n(\tau) \nabla \dot{w}(\tau) dxdy \Big) d\tau + R_n. \end{aligned}$$

Since the function $t \mapsto \big(\int_{\Omega\setminus\Gamma} |\nabla \dot{w}(t)|^2 dxdy \big)^{1/2}$ is integrable, using (2.8) and (2.15) we deduce from the previous inequality that there exists $\tilde{R}_n \to 0$ such that for every $0 \le t_1 < t_2 \le T$ we have

$$\begin{aligned} &\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_n(t_2)|^2 dxdy + \mathrm{Diss}(u_n(\cdot), s_n(\cdot); t_1, t_2) \\ &\le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_n(t_1)|^2 dxdy + \int_{t_1}^{t_2} \Big(\int_{\Omega\setminus\Gamma} \nabla u_n(\tau) \nabla \dot{w}(\tau) dxdy \Big) d\tau + \tilde{R}_n. \end{aligned} \tag{2.16}$$

In particular, by (2.15) this inequality implies that $\mathrm{Diss}(u_n(\cdot), s_n(\cdot); 0, T)$ is bounded uniformly with respect to t and n. To continue the proof we need the following lemmas.

Given a set A, let χ_A be its characteristic function, defined by $\chi_A(x) := 1$ if $x \in A$ and $\chi_A(x) := 0$ if $x \notin A$.

Lemma 2.7 *Assume that $\|u_n(t)\|_{H^1(\Omega\setminus\Gamma)}$ and $\mathrm{Diss}(u_n(\cdot), s_n(\cdot); 0, T)$ are bounded uniformly with respect to t and n. Then there exist a subsequence of (u_n, s_n), not relabelled, a nondecreasing function $s\colon [0,T] \to [a,b]$, and a function $g\colon [0,T] \to L^1(\Gamma)$ such that*

$$s_n(t) \to s(t), \tag{2.17}$$

$$[u_n(t)]\chi_{\Gamma^b_{s_n(t)}} \to g(t)\chi_{\Gamma^b_{s(t)}} \quad \text{strongly in } L^1(\Gamma), \tag{2.18}$$

for every $t \in [0,T]$.

Proof The statement on the convergence of s_n is a consequence of Helly's Theorem. Let D be a countable dense subset of $[0,T]$. By a diagonal argument we can find a subsequence of u_n, not relabelled, and a bounded function $v\colon D \to H^1(\Omega \setminus \Gamma)$ such that $u_n(t) \rightharpoonup v(t)$ weakly in $H^1(\Omega \setminus \Gamma)$ for every $t \in D$. This implies that

$$[u_n(t)] \to [v(t)] \quad \text{strongly in } L^2(\Gamma) \tag{2.19}$$

for every $t \in D$.

To prove (2.18) for every $t \in [0, T]$ we introduce the nondecreasing functions $V_n\colon [0, T] \to \mathbb{R}$ defined by

$$V_n(t) := \text{Diss}(u_n(\cdot), s_n(\cdot); 0, t). \tag{2.20}$$

By Helly's Theorem there exist a subsequence, not relabelled, and a nondecreasing function V such that $V_n(t) \to V(t)$ for every $t \in [0, T]$.

Let $t_0 \in (0, T)$ be a continuity point for both V and s. For every $\varepsilon > 0$ there exists $\delta > 0$ such that $|V(t) - V(t_0)| < \varepsilon$ and $|s(t) - s(t_0)| < \varepsilon$ for every $t \in [0, T]$ with $|t - t_0| < \delta$. Let $t \in D$ with $t_0 < t < t_0 + \delta$. Then $V_n(t) \to V(t) < V(t_0) + \varepsilon$ and $V_n(t_0) \to V(t_0)$. By Definition 2.6 it follows that

$$\int_{\Gamma^b_{s_n(t)}} |[u_n(t)] - [u_n(t_0)]| dx \le \text{Diss}(u_n(\cdot), s_n(\cdot); t_0, t) \le V_n(t) - V_n(t_0) < \varepsilon \tag{2.21}$$

for n large enough. Moreover, since $\|u_n(t)\|_{H^1(\Omega\setminus\Gamma)}$ is uniformly bounded, there exists a constant $C > 0$ such that $\|[u_n(t_0)]\|_{L^2(\Gamma)} \le C$ for every n. This implies that

$$\begin{aligned} &\int_{\Gamma} |[u_n(t_0)]\chi_{\Gamma^b_{s_n(t)}} - [u_n(t_0)]\chi_{\Gamma^b_{s_n(t_0)}}| dx \\ &\le \int_{\Gamma^{s_n(t)}_{s_n(t_0)}} |[u_n(t_0)]| dx \le \|[u_n(t_0)]\|_{L^2(\Gamma)} (s_n(t) - s_n(t_0))^{1/2} < C\varepsilon^{1/2} \end{aligned} \tag{2.22}$$

for sufficiently large n. By the triangle inequality (2.21) and (2.22) give

$$\int_{\Gamma} |[u_n(t)]|\chi_{\Gamma^b_{s_n(t)}} - [u_n(t_0)]\chi_{\Gamma^b_{s_n(t_0)}}| dx < \varepsilon + C\varepsilon^{1/2},$$

and this inequality, together with (2.19), implies that $[u_n(t_0)]\chi_{\Gamma^b_{s_n(t_0)}}$ is a Cauchy sequence in $L^1(\Gamma)$, hence it converges to a function $g(t_0) \in L^1(\Gamma)$. Since $s_n(t_0) \to s(t_0)$ we have $g(t_0) = g(t_0)\chi_{\Gamma^b_{s(t_0)}}$.

Therefore (2.18) holds for all continuity points of both V and s. Since the set of all other points is at most countable, we can apply again the diagonal argument to extract a further subsequence along which (2.18) holds for all t. □

Lemma 2.8 *For every $w \in H^1(\Omega \setminus \Gamma)$, $s \in [a, b]$, and $g \in L^1(\Gamma)$ let $u^w_{s,g}$ be the unique solution of the minimum problem*

$$\min_{\substack{u\in H^1(\Omega\setminus\Gamma)\\ u=w \text{ on } \partial_D\Omega}} \Big\{\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u|^2 dxdy + \int_{\Gamma^b_s} |[u] - g| dx\Big\}. \tag{2.23}$$

Let $w_n, w \in H^1(\Omega \setminus \Gamma)$, $s_n, s \in [a, b]$, and $g_n, g \in L^1(\Gamma)$ be such that

$$w_n \to w \quad \text{strongly in } H^1(\Omega \setminus \Gamma), \tag{2.24}$$

$$s_n \to s, \tag{2.25}$$

$$g_n\chi_{\Gamma^b_{s_n}} \to g\chi_{\Gamma^b_s} \quad \text{strongly in } L^1(\Gamma). \tag{2.26}$$

Then $u^{w_n}_{s_n,g_n} \to u^w_{s,g}$ strongly in $H^1(\Omega \setminus \Gamma)$.

Proof Note that the uniqueness of the solution to (2.23) follows easily from the strict convexity of the functional with respect to ∇u, using (2.1).

We set $u_n := u^{w_n}_{s_n,g_n}$ and $u := u^w_{s,g}$. From the minimality of u_n we have

$$\frac{1}{2}\int_{\Omega\setminus\Gamma}|\nabla u_n|^2dxdy+\int_{\Gamma^b_{s_n}}|[u_n]-g_n|dx\le\frac{1}{2}\int_{\Omega\setminus\Gamma}|\nabla w_n|^2dxdy+\int_{\Gamma^b_{s_n}}|[w_n]-g_n|dx,$$

which gives the boundedness of u_n in $H^1(\Omega\setminus\Gamma)$, thanks to (2.1). Hence there exist a subsequence, not relabelled, and a function $v\in H^1(\Omega\setminus\Gamma)$ with $v=w$ on $\partial_D\Omega$, such that $u_n\rightharpoonup v$ weakly in $H^1(\Omega\setminus\Gamma)$. Using lower semicontinuity it is easy to prove that v solves (2.23), hence $v=u$. By the arbitrariness of the subsequence we conclude that the whole sequence u_n converges to u weakly in $H^1(\Omega\setminus\Gamma)$. To prove the strong convergence we first observe that $[u_n]\to[u]$ strongly in $L^2(\Gamma)$ and

$$\begin{aligned}&\frac{1}{2}\int_{\Omega\setminus\Gamma}|\nabla u_n|^2dxdy+\int_{\Gamma^b_{s_n}}|[u_n]-g_n|dx\\&\quad\le\frac{1}{2}\int_{\Omega\setminus\Gamma}|\nabla(u+w_n-w)|^2dxdy+\int_{\Gamma^b_{s_n}}|[u+w_n-w]-g_n|dx,\end{aligned}$$

by minimality. By (2.24)–(2.26) this implies

$$\limsup_n\int_{\Omega\setminus\Gamma}|\nabla u_n|^2dxdy\le\int_{\Omega\setminus\Gamma}|\nabla u|^2dxdy,$$

which, together with the weak convergence, gives $\nabla u_n\to\nabla u$ strongly in $L^2(\Omega\setminus\Gamma;\mathbb{R}^2)$. Taking (2.1) into account, this implies the strong convergence of u_n to u. □

For every $\alpha,\beta\in\mathbb{R}$ we set $\alpha\vee\beta:=\max\{\alpha,\beta\}$ and $\alpha\wedge\beta:=\min\{\alpha,\beta\}$.

Proof of Theorem 2.5 (continuation) Let s and g be the functions given by Lemma 2.7 and for every $t\in[0,T]$ let $u(t)$ be the solution of the minimum problem

$$\min_{\substack{u\in H^1(\Omega\setminus\Gamma)\\u=w(t)\text{ on }\partial_D\Omega}}\left\{\frac{1}{2}\int_{\Omega\setminus\Gamma}|\nabla u|^2dxdy+\int_{\Gamma^b_{s(t)}}|[u]-g(t)|dx\right\}.$$

Let us prove that $t\mapsto u(t)$ from $[0,T]$ into $H^1(\Omega\setminus\Gamma)$ is measurable. It is enough to show that $t\mapsto u(t)$ is continuous at every continuity point $t_0\in(0,T)$ of both V and s, where V is defined as in the proof of Lemma 2.7. Let us fix such a point t_0 and a sequence $t_k\to t_0$. Taking into account Lemma 2.8, since $u(t_k)=u^{w(t_k)}_{s(t_k),g(t_k)}$, it is sufficient to prove that

$$g(t_k)\chi_{\Gamma^b_{s(t_k)}}\to g(t_0)\chi_{\Gamma^b_{s(t_0)}}\quad\text{strongly in }L^1(\Gamma).\tag{2.27}$$

By Definition 2.6 and (2.20) we have

$$\begin{aligned}&\int_\Gamma|[u_n(t_k)]\chi_{\Gamma^b_{s_n(t_k)}}-[u_n(t_0)]\chi_{\Gamma^b_{s_n(t_0)}}|dx\\&\le\int_{\Gamma^b_{s_n(t_0\vee t_k)}}|[u_n(t_k)]-[u_n(t_0)]|dx+\int_{\Gamma^{s_n(t_0\vee t_k)}_{s_n(t_0\wedge t_k)}}|[u_n(t_0\wedge t_k)]|dx\\&\le V_n(t_0\vee t_k)-V_n(t_0\wedge t_k)+\|[u_n(t_0\wedge t_k)]\|_{L^2(\Gamma)}|s_n(t_k)-s_n(t_0)|^{1/2}.\end{aligned}$$

Since $\|u_n(t)\|_{H^1(\Omega\setminus\Gamma)}$ is bounded uniformly with respect to n and t, there exists a constant $C > 0$ such that

$$\int_\Gamma |[u_n(t_k)]\chi_{\Gamma^b_{s_n(t_k)}} - [u_n(t_0)]\chi_{\Gamma^b_{s_n(t_0)}}|dx \le V_n(t_0 \vee t_k) - V_n(t_0 \wedge t_k) + C|s_n(t_k) - s_n(t_0)|^{1/2}.$$

Passing to the limit as $n \to \infty$ along a suitable subsequence and using Lemma 2.7, we obtain

$$\int_\Gamma |g(t_k)\chi_{\Gamma^b_{s(t_k)}} - [g(t_0)]\chi_{\Gamma^b_{s(t_0)}}|dx \le V(t_0 \vee t_k) - V(t_0 \wedge t_k) + C|s(t_k) - s(t_0)|^{1/2}.$$

Since V and s are continuous in t_0, this gives (2.27) and concludes the proof of the measurability of $t \mapsto u(t)$.

We now prove the equilibrium condition (b) in Definition 2.1. By (2.10) and (2.14), for every t and n we have that

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_n(t)|^2 dxdy + s_n(t) \le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla \hat{u}|^2 dxdy + \hat{s} + \int_{\Gamma^b_{\hat{s}}} |[\hat{u}] - [u_n(t)]|dx \tag{2.28}$$

for every $s_n(t) \le \hat{s} \le b$ and every $\hat{u} \in H^1(\Omega \setminus \Gamma)$ with $\hat{u} = w_n(t)$ on $\partial_D\Omega$. In particular, taking $\hat{s} = s_n(t)$, we see that $u_n(t)$ satisfies the minimum problem (2.23) with $w = w_n(t)$, $s = s_n(t)$, and $g_n = [u_n(t)]$. Since $w_n(t) \to w(t)$ strongly in $H^1(\Omega \setminus \Gamma_a^{s_0})$, $s_n(t) \to s(t)$, and $[u_n(t)]\chi_{\Gamma^b_{s_n(t)}} \to g(t)\chi_{\Gamma^b_{s(t)}}$ strongly in $L^1(\Gamma)$, by Lemma 2.8 we have

$$u_n(t) \to u(t) \quad \text{strongly in } H^1(\Omega \setminus \Gamma) \tag{2.29}$$

for every $t \in [0, T]$.

We now fix $t \in [0, T]$, $s(t) \le \hat{s} \le b$, and $\hat{u} \in H^1(\Omega \setminus \Gamma)$ with $\hat{u} = w(t)$ on $\partial\Omega$. We have to prove that

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t)|^2 dxdy + s(t) \le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla \hat{u}|^2 dxdy + \hat{s} + \int_{\Gamma^b_{\hat{s}}} |[\hat{u}] - [u(t)]|dx. \tag{2.30}$$

Let $\hat{s}_n := \hat{s} \vee s_n(t)$ and $\hat{u}_n := \hat{u} + w_n(t) - w(t)$. Since $\hat{u}_n = w_n(t)$ on $\partial_D\Omega$ and $\hat{s}_n \ge s_n(t)$, by (2.28) we have

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_n(t)|^2 dxdy + s_n(t) \le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla \hat{u}_n|^2 dxdy + \hat{s}_n + \int_{\Gamma^b_{\hat{s}_n}} |[\hat{u}_n] - [u_n(t)]|dx. \tag{2.31}$$

Since $u_n(t) \to u(t)$ and $\hat{u}_n \to \hat{u}$ strongly in $H^1(\Omega \setminus \Gamma)$ by (2.29), while $s_n(t) \to s(t)$ and $\hat{s}_n \to \hat{s}$, we can pass to the limit in (2.31) and we obtain (2.30), which gives the equilibrium condition (b) in Definition 2.1.

We conclude by proving now the energy-dissipation inequality (c) in Definition 2.1. By (2.16) and by Definition 2.6 we have

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_n(t_2)|^2 dxdy + s_n(t_2) - s_n(t_1) + \int_{\Gamma^b_{s_n(t_2)}} |[u_n(t_2)] - [u_n(t_1)]|dx$$
$$\le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_n(t_1)|^2 dxdy + \int_{t_1}^{t_2} \Big(\int_{\Omega\setminus\Gamma} \nabla u_n(\tau)\nabla \dot{w}(\tau) dxdy\Big) d\tau + \tilde{R}_n.$$

for every $0 \le t_1 \le t_2 \le T$ and for every n. By (2.29) we can pass to the limit and obtain condition (c). □

The following theorem shows that the notion of evolution according to Definition 2.1 can be expressed equivalently by using the notion of dissipation introduced in Definition 2.6. This shows the analogy with the definition used in [6].

Theorem 2.9 *Let* $T > 0$*,* $s_0 \in [a, b)$*, and* $w \in AC([0, T]; H^1(\Omega \setminus \Gamma_a^{s_0}))$*. A pair* (u, s) *is a quasistatic evolution with boundary value* w *on* $\partial_D \Omega$ *if and only if* $u\colon [0, T] \to H^1(\Omega \setminus \Gamma)$ *is measurable,* $s\colon [0, T] \to [s_0, b]$*, conditions (a) and (b) of Definition 2.1 are satisfied, and one of the following two conditions holds:*

(c$'$) *(energy-dissipation inequality starting from* 0*) for every* $t \in [0, T]$ *we have*

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t)|^2 dxdy + \mathrm{Diss}(u(\cdot), s(\cdot); 0, t)$$
$$\leq \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(0)|^2 dxdy + \int_0^t \Big(\int_{\Omega\setminus\Gamma} \nabla u(\tau)\nabla \dot{w}(\tau)dxdy\Big)d\tau.$$

(c$''$) *(energy-dissipation balance) for every* $t_1, t_2 \in [0, T]$ *with* $t_1 < t_2$*, we have*

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t_2)|^2 dxdy + \mathrm{Diss}(u(\cdot), s(\cdot); t_1, t_2)$$
$$= \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t_1)|^2 dxdy + \int_{t_1}^{t_2} \Big(\int_{\Omega\setminus\Gamma} \nabla u(t)\nabla \dot{w}(t)dxdy\Big)dt.$$

Proof Let (u, s) be a quasistatic evolution with boundary value w on $\partial_D \Omega$. By (c) and Definition 2.6 we obtain

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t_2)|^2 dxdy + \mathrm{Diss}(u(\cdot), s(\cdot); t_1, t_2)$$
$$\leq \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t_1)|^2 dxdy + \int_{t_1}^{t_2} \Big(\int_{\Omega\setminus\Gamma} \nabla u(\tau)\nabla \dot{w}(\tau)dxdy\Big)d\tau,$$

which clearly implies (c$'$).

To prove that (a)&(b)&(c$'$) $\Longrightarrow$ (c$''$) we argue as in the proof of the energy balance in [6, Section 6] and we obtain

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t)|^2 dxdy + \mathrm{Diss}(u(\cdot), s(\cdot); 0, t)$$
$$\geq \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(0)|^2 dxdy + \int_0^t \Big(\int_{\Omega\setminus\Gamma} \nabla u(\tau)\nabla \dot{w}(\tau)dxdy\Big)d\tau.$$

This inequality, together with (c$'$) gives (c$''$) for $t_1 = 0$. The general case for (c$''$) follows by additivity.

The implication (c$''$) $\Longrightarrow$ (c) is an immediate consequence of Definition 2.6. □

3 Some auxiliary results

In this section we prove a characterization of the solutions of the minimum problems considered in Lemma 2.8, which are connected with the equilibrium condition (b) in Definition 2.1. This is obtained by means of a suitable weak formulation of their boundary conditions on Γ. In the last part of the section we present a technical result that will be crucial in the proof of our main result in Sect. 4.

It is convenient to introduce the notation

$$H^1_{0,D}(\Omega \setminus \Gamma) := \{u \in H^1(\Omega \setminus \Gamma) : u = 0 \text{ on } \partial_D \Omega\}. \tag{3.1}$$

We also set

$$\partial^{\pm} U := \partial U^{\pm} \setminus \Gamma, \tag{3.2}$$

where U and $U^{\pm}$ are the open sets introduced at the beginning of Sect. 2.

Lemma 3.1 *Let $w \in H^1(\Omega \setminus \Gamma)$, $s \in [a, b]$, $g \in L^1(\Gamma)$, and let u be the minimiser of*

$$\min_{\substack{u \in H^1(\Omega\setminus\Gamma) \\ u = w \text{ on } \partial_D\Omega}} \Big(\frac{1}{2} \int_{\Omega\setminus\Gamma} |\nabla u|^2 dxdy + \int_{\Gamma^b_s} |[u] - g| dx \Big). \tag{3.3}$$

Then there exists $\psi \in L^\infty(\Gamma)$, with $\psi = 0$ a.e. on Γ^s_a and $|\psi| \le 1$ a.e. on Γ^b_s, such that

$$\int_{\Omega\setminus\Gamma} \nabla u \nabla \varphi \, dxdy = \int_\Gamma \psi[\varphi]\, dx \quad \text{for every } \varphi \in H^1_{0,D}(\Omega \setminus \Gamma). \tag{3.4}$$

Proof Let $\varphi \in H^1_{0,D}(\Omega \setminus \Gamma)$. Since $u + \varepsilon\varphi = w$ on $\partial_D \Omega$ for every $\varepsilon \in \mathbb{R}$, by minimality

$$\frac{1}{2} \int_{\Omega\setminus\Gamma} |\nabla(u + \varepsilon\varphi)|^2 dxdy + \int_{\Gamma^b_s} |[u] - g + \varepsilon[\varphi]| dx$$
$$- \frac{1}{2} \int_{\Omega\setminus\Gamma} |\nabla u|^2 dxdy - \int_{\Gamma^b_s} |[u] - g| dx \ge 0.$$

Developing the square and using the triangle inequality we get

$$\frac{\varepsilon}{2} \int_{\Omega\setminus\Gamma} |\nabla\varphi|^2 dxdy + \int_{\Omega\setminus\Gamma} \nabla u \nabla\varphi dxdy + \int_{\Gamma^b_s} |[\varphi]| dx \ge 0$$

for every $\varepsilon > 0$. Taking the limit as $\varepsilon \to 0+$ we obtain

$$\int_{\Omega\setminus\Gamma} \nabla u \nabla\varphi dxdy \ge - \int_{\Gamma^b_s} |[\varphi]| dx.$$

Using the same inequality also for $-\varphi$, we deduce that

$$\Big| \int_{\Omega\setminus\Gamma} \nabla u \nabla\varphi \, dxdy \Big| \le \int_{\Gamma^b_s} |[\varphi]| dx \tag{3.5}$$

for every $\varphi \in H^1_{0,D}(\Omega \setminus \Gamma)$. Given $\varphi \in H^1(U^+)$ with $\varphi = 0$ on $\partial^+ U$, we can extend it by 0 and we obtain a function in $H^1_{0,D}(\Omega \setminus \Gamma)$. Therefore (3.5) gives

$$\Big| \int_{U^+} \nabla u \nabla\varphi dxdy \Big| \le \int_{\Gamma^b_s} |\varphi^+| dx \tag{3.6}$$

for every $\varphi \in H^1(U^+)$ with $\varphi = 0$ on $\partial^+ U$, where φ^+ denotes the trace of φ on Γ from above. Moreover, (3.5) gives also

$$\int_{U^+} \nabla u \nabla\varphi \, dxdy + \int_{U^-} \nabla u \nabla\varphi dxdy = 0 \tag{3.7}$$

for every $\varphi \in H^1_0(U \cap \Omega)$.

Let μ be the distribution on $U \cap \Omega$ defined by

$$\langle \mu, \varphi \rangle := \int_{U^+} \nabla u \nabla \varphi \, dx dy = -\int_{U^-} \nabla u \nabla \varphi \, dx dy$$

for every $\varphi \in C_c^\infty(U \cap \Omega)$. By (3.6) it is easy to prove that there exist $\psi \in L^\infty(\Gamma)$, with $\psi = 0$ a.e. on Γ_a^s and $|\psi| \leq 1$ a.e. on Γ_s^b, such that

$$\langle \mu, \varphi \rangle = \int_\Gamma \psi \varphi \, dx \quad \text{for every } \varphi \in C_c^\infty(U \cap \Omega).$$

By density

$$\int_{U^+} \nabla u \nabla \varphi \, dx dy = \int_\Gamma \psi \varphi \, dx \quad \text{and} \quad \int_{U^-} \nabla u \nabla \varphi \, dx dy = -\int_\Gamma \psi \varphi \, dx$$

for every $\varphi \in H_0^1(U \cap \Omega)$.

Given $\varphi \in H^1(U^+)$ with $\varphi = 0$ on $\partial^+ U$, we can extend it to a function belonging to $H_0^1(U \cap \Omega)$. Therefore

$$\int_{U^+} \nabla u \nabla \varphi \, dx dy = \int_\Gamma \psi \varphi^+ \, dx \tag{3.8}$$

for every $\varphi \in H^1(U^+)$ with $\varphi = 0$ on $\partial^+ U$. Similarly we prove that

$$\int_{U^-} \nabla u \nabla \varphi \, dx dy = -\int_\Gamma \psi \varphi^- \, dx \tag{3.9}$$

for every $\varphi \in H^1(U^-)$ with $\varphi = 0$ on $\partial^- U$, where φ^- denotes the trace of φ on Γ from below. By taking the sum we get

$$\int_{U^+ \cup U^-} \nabla u \nabla \varphi \, dx dy = \int_\Gamma \psi [\varphi] \, dx \tag{3.10}$$

for every $\varphi \in H^1(U^+ \cup U^-)$ with $\varphi = 0$ on $\partial^+ U \cup \partial^- U$.

Let ω_k be a sequence in $C_c^\infty(U \cap \Omega)$, with $0 \leq \omega_k \leq 1$, such that $\omega_k \to 1$ a.e. on Γ. Given $\varphi \in H_{0,D}^1(\Omega \setminus \Gamma)$ we set $\varphi_k := \omega_k \varphi$ and $\hat{\varphi}_k := (1 - \omega_k)\varphi$. Then $\varphi_k \in H^1(U^+ \cup U^-)$, $\varphi_k = 0$ on $\partial^+ U \cup \partial^- U$, and $\hat{\varphi}_k \in H_{0,D}^1(\Omega \setminus \Gamma)$. Moreover $[\varphi_k] \to [\varphi]$ strongly in $L^1(\Gamma)$. Since $\varphi = \varphi_k + \hat{\varphi}_k$ we have

$$\int_{\Omega \setminus \Gamma} \nabla u \nabla \varphi \, dx dy = \int_{U^+ \cup U^-} \nabla u \nabla \varphi_k dx dy + \int_{\Omega \setminus \Gamma} \nabla u \nabla \hat{\varphi}_k dx dy. \tag{3.11}$$

By (3.10) we have

$$\int_{U^+ \cup U^-} \nabla u \nabla \varphi_k dx dy \to \int_\Gamma \psi [\varphi] \, dx, \tag{3.12}$$

while (3.5) gives

$$\left| \int_{\Omega \setminus \Gamma} \nabla u \nabla \hat{\varphi}_k \, dx dy \right| \leq \int_{\Gamma_s^b} |[\hat{\varphi}_k]| \, dx \to 0. \tag{3.13}$$

Equality (3.4) follows from (3.11)–(3.13). □

Lemma 3.2 *Let $v, w \in H^1(\Omega \setminus \Gamma)$, let $s \in [a, b]$, let $g := [v]$, and let u be the minimiser of (3.3). Then the function ψ introduced in Lemma 3.1 satisfies $\psi = -1$ a.e. on $\{[u] > g\} \cap \Gamma_s^b$ and $\psi = 1$ a.e. on $\{[u] < g\} \cap \Gamma_s^b$.*

Proof Since U^- has Lipschitz boundary, there exist $\tilde{u}, \tilde{v} \in H^1(U \cap \Omega)$ such that $\tilde{u} = u$ and $\tilde{v} = v$ in U^-. Let $\hat{u}, \hat{v} \in H^1(U \cap (\Omega \setminus \Gamma))$ be defined by $\hat{u} := u - \tilde{u}$ and $\hat{v} := v - \tilde{v}$, so that $\hat{u}^+ = [\hat{u}] = [u]$ and $\hat{v}^+ = [\hat{v}] = [v] = g$ on Γ, while $\hat{u} = \hat{v} = 0$ in U^-.

Let $A := \{[u] > [v]\} \cap \Gamma_s^b = \{\hat{u}^+ > \hat{v}^+\} \cap \Gamma_s^b$. To prove that $\psi = -1$ a.e. on A it is enough to show that

$$\int_A \psi \varphi^+ dx + \int_A \varphi^+ dx = 0 \tag{3.14}$$

for every $\varphi \in H^1(U^+)$ with $\varphi = 0$ on $\partial^+ U$. Let us fix such a φ and for every k let $\varphi_k := (\varphi \wedge (k\omega)) \vee (-k\omega) \in H^1(U^+)$, where $\omega := (\hat{u} - \hat{v}) \vee 0$. We extend φ_k to $\Omega \setminus \Gamma$ by setting $\varphi_k = 0$ on $\Omega \setminus (\Gamma \cup U^+)$. Since $\varphi_k = 0$ on $\partial^+ U$, the extended function satisfies $\varphi_k \in H^1_{0,D}(\Omega \setminus \Gamma)$. For every ε with $|\varepsilon| < \frac{1}{k}$ we have $|\varepsilon[\varphi_k]| \le |\varepsilon \varphi_k^+| \le \omega^+ = (\hat{u}^+ - \hat{v}^+) \vee 0 = ([u] - [v]) \vee 0$ a.e. on Γ_s^b. It follows that

$$\int_{\Gamma_s^b} |[u] - [v] + \varepsilon[\varphi_k]| dx - \int_{\Gamma_s^b} |[u] - [v]| dx = \int_{\Gamma_s^b} \varepsilon[\varphi_k] dx = \int_{\Gamma_s^b} \varepsilon \varphi_k^+ dx.$$

By the minimality of u, repeating the argument at the beginning of the proof of Lemma 3.1 we obtain

$$\frac{\varepsilon^2}{2} \int_{\Omega \setminus \Gamma} |\nabla \varphi_k|^2 dxdy + \varepsilon \int_{\Omega \setminus \Gamma} \nabla u \nabla \varphi_k dxdy + \int_{\Gamma_s^b} \varepsilon \varphi_k^+ dx \ge 0$$

for every $\varepsilon \in (-\frac{1}{k}, \frac{1}{k})$. Taking the derivative at $\varepsilon = 0$ and using (3.4) we obtain

$$\int_{\Gamma_s^b} \psi \varphi_k^+ \, dx + \int_{\Gamma_s^b} \varphi_k^+ \, dx = 0.$$

Since $\{\varphi_k^+ \neq 0\} \cap \Gamma_s^b \subset A$, we obtain

$$\int_A \psi \varphi_k^+ \, dx + \int_A \varphi_k^+ \, dx = 0.$$

Passing to the limit as $k \to \infty$ we obtain (3.14).

The proof on the set $\{[u] < g\} \cap \Gamma_s^b$ is similar. □

Lemma 3.3 *Let $w \in H^1(\Omega \setminus \Gamma)$, $s \in [a, b]$, $g \in L^1(\Gamma)$, and let $u \in H^1(\Omega \setminus \Gamma)$ with $u = w$ on $\partial_D \Omega$. Suppose that there exists $\psi \in L^\infty(\Gamma)$ satisfying (3.4) such that*

$$\psi = 0 \text{ a.e. on } \Gamma_a^s, \tag{3.15}$$

$$\psi = -1 \text{ a.e. on } \{[u] > g\} \cap \Gamma_s^b, \tag{3.16}$$

$$\psi = 1 \text{ a.e. on } \{[u] < g\} \cap \Gamma_s^b, \tag{3.17}$$

$$|\psi| \le 1 \text{ a.e. on } \{[u] = g\} \cap \Gamma_s^b. \tag{3.18}$$

Then u is the minimiser of (3.3).

Proof Let us fix $v \in H^1(\Omega \setminus \Gamma)$ with $v = w$ on $\partial_D \Omega$ and let $\varphi := v - u$. Then $\varphi \in H^1_{0,D}(\Omega \setminus \Gamma)$. For every $\varepsilon \in [0, 1]$ we define

$$f(\varepsilon) := \frac{1}{2} \int_{\Omega \setminus \Gamma} |\nabla(u + \varepsilon \varphi)|^2 dxdy + \int_{\Gamma_s^b} |[u] - g + \varepsilon[\varphi]| dx,$$

and we set

$$f_r'(0) := \lim_{\varepsilon \to 0+} \frac{f(\varepsilon) - f(0)}{\varepsilon}.$$

By convexity the limit exists and

$$f(1) - f(0) \geq f_r'(0). \tag{3.19}$$

Since

$$\begin{aligned} f(1) - f(0) = \frac{1}{2} \int_{\Omega \setminus \Gamma} |\nabla v|^2 dx dy + \int_{\Gamma_s^b} |[v] - g| dx \\ - \frac{1}{2} \int_{\Omega \setminus \Gamma} |\nabla u|^2 dx dy - \int_{\Gamma_s^b} |[u] - g| dx, \end{aligned}$$

by (3.19) the minimality is proved if we show that

$$f_r'(0) \geq 0. \tag{3.20}$$

By taking the derivative with respect to ε in the first term of the definition of f we obtain

$$f_r'(0) = \int_{\Omega \setminus \Gamma} \nabla u \nabla \varphi \, dx dy + \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \int_{\Gamma_s^b} \big(|[u] - g + \varepsilon[\varphi]| - |[u] - g| \big) dx. \tag{3.21}$$

By (3.16) on $\{[u] > g\} \cap \Gamma_s^b$ we have

$$\lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \big(|[u] - g + \varepsilon[\varphi]| - |[u] - g| \big) = [\phi] = -\psi[\varphi]. \tag{3.22}$$

By (3.17) on $\{[u] < g\} \cap \Gamma_s^b$ we have

$$\lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \big(|[u] - g + \varepsilon[\varphi]| - |[u] - g| \big) = -[\phi] = -\psi[\varphi]. \tag{3.23}$$

Finally, by (3.18) on $\{[u] = g\} \cap \Gamma_s^b$ we have

$$\lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \big(|[u] - g + \varepsilon[\varphi]| - |[u] - g| \big) = |[\varphi]| \geq -\psi[\varphi]. \tag{3.24}$$

By the triangle inequality we have

$$\frac{1}{\varepsilon} \big(|[u] - g + \varepsilon[\varphi]| - |[u] - g| \big) \geq -|[\varphi]|$$

for every $\varepsilon \in (0, 1]$. We can now apply the Fatou Lemma and from (3.22)–(3.24) we obtain

$$\lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \int_{\Gamma_s^b} \big(|[u] - g + \varepsilon[\varphi]| - |[u] - g| \big) dx \geq - \int_{\Gamma_s^b} \psi[\varphi] \, dx.$$

Using this inequality, together with (3.4), (3.15), and (3.21), we obtain (3.20). □

The following technical result will be used in the proof of Lemma 4.5, which is crucial to obtain our main result on the jerky crack growth. Let us fix a sequence Ω_k of open subsets of Ω with boundary of class C^∞ such that $\Omega_k \subset\subset \Omega_{k+1}$ for every k and $\Omega \setminus \Gamma = \cup_k \Omega_k$. For every k we set (see Fig. 2)

$$S_k := \Omega \setminus (\overline{\Omega}_k \cup \Gamma). \tag{3.25}$$

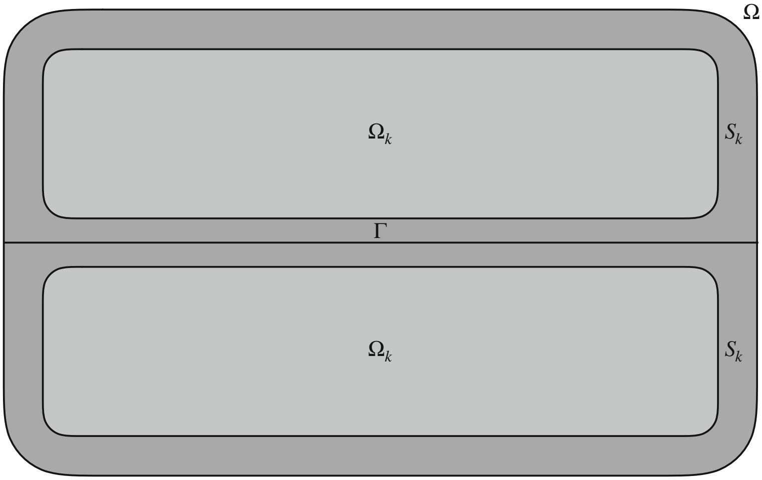

Fig. 2 The sets Ω_k and S_k

We now prove the convergence to zero in L^∞ of the sequence of harmonic functions z_k on S_k which satisy the homogeneous Dirichlet condition on $\partial\Omega_k$, the homogeneous Neumann condition on $\partial\Omega$, and the nonhomogeneous boundary condition $\frac{\partial z_k}{\partial\nu} = 1$ on both sides of Γ. For every $R > 0$ let $B_R = \{(x, y) \in \mathbb{R}^2 : x^2+y^2 < R^2\}$ and $B_R^\pm = \{(x, y) \in B_R : \pm y > 0\}$.

Lemma 3.4 *For every k let S_k be as in (3.25) and let z_k be the solution of*

$$\begin{cases} z_k \in H^1(S_k), \quad z_k = 0 \text{ a.e. on } \partial\Omega_k, \\ \displaystyle\int_{S_k} \nabla z_k \nabla\varphi\, dxdy = \int_\Gamma (\varphi^+ + \varphi^-)\, dx \\ \text{for every } \varphi \in H^1(S_k) \text{ with } \varphi = 0 \text{ a.e. on } \partial\Omega_k. \end{cases} \tag{3.26}$$

We extend z_k by setting $z_k := 0$ in Ω_k. Then $z_k \to 0$ strongly in $L^\infty(\Omega \setminus \Gamma)$.

Proof We first prove that

$$z_k \to 0 \quad \text{strongly in } H^1(\Omega \setminus \Gamma). \tag{3.27}$$

By taking $\varphi := z_k$ in (3.26) we obtain

$$\int_{\Omega\setminus\Gamma} |\nabla z_k|^2 dxdy = \int_{S_k} |\nabla z_k|^2 dxdy = \int_\Gamma (z_k^+ + z_k^-)\, dx. \tag{3.28}$$

Since $z_k = 0$ in Ω_k, the Trace Inequality, together with the Poincaré Inequality, gives a constant $c > 0$ such that

$$\int_\Gamma (z_k^+ + z_k^-)\, dx \le c\Big(\int_{\Omega\setminus\Gamma} |\nabla z_k|^2 dxdy\Big)^{1/2}$$

for k large enough. Together with (3.28) this implies that ∇z_k is bounded in $L^2(\Omega \setminus \Gamma)$, hence z_k is bounded in $H^1(\Omega \setminus \Gamma)$. Since $z_k = 0$ in Ω_k, we deduce that $z_k \rightharpoonup 0$ weakly in $H^1(\Omega \setminus \Gamma)$. This implies that $z_k^+ + z_k^- \to 0$ strongly in $L^2(\Gamma)$, and (3.28) gives $\nabla z_k \to 0$ strongly in $L^2(\Omega \setminus \Gamma)$. Since $z_k = 0$ in Ω_k, this proves (3.27).

By the maximum principle we have

$$z_k \geq 0 \quad \text{in } S_k. \tag{3.29}$$

Indeed, if we take $\varphi := z_k \wedge 0$ in (3.26) we obtain

$$\int_{S_k} |\nabla(z_k \wedge 0)|^2 dxdy = \int_{S_k} \nabla z_k \nabla(z_k \wedge 0)\, dxdy = \int_{\Gamma} ((z_k^+ \wedge 0) + (z_k^- \wedge 0))\, dx \leq 0.$$

This inequality, together with the boundary condition on $\partial\Omega_k$, implies that $z_k \wedge 0 = 0$ in S_k, which proves (3.29). Since $z_k \in C^\infty(S_k \cup \partial\Omega_k)$ by the regularity theory of elliptic equations, (3.29) implies that $\frac{\partial z_k}{\partial \nu} \leq 0$ on $\partial\Omega_k$, where ν is the outer unit normal to S_k. Hence

$$\int_{S_k} \nabla z_k \nabla \varphi\, dxdy = \int_{\partial\Omega_k} \frac{\partial z_k}{\partial \nu} \varphi\, ds \leq 0 \tag{3.30}$$

for every $\varphi \in H_0^1(\Omega \setminus \Gamma)$ with $\varphi \geq 0$ in $\Omega \setminus \Gamma$.

Let us prove that

$$\int_{\Omega\setminus\Gamma} \nabla z_k \nabla \varphi\, dxdy \leq \int_{\Gamma} (\varphi^+ + \varphi^-)\, dx \tag{3.31}$$

for every $\varphi \in H^1(\Omega \setminus \Gamma)$ with $\varphi \geq 0$. Let us fix such a φ and let $\omega \in C_0^\infty(\Omega \setminus \Gamma)$ with $\omega \geq 0$ in $\Omega \setminus \Gamma$ and $\omega = 1$ in $\overline{\Omega}_k$. Then we have

$$\int_{\Omega\setminus\Gamma} \nabla z_k \nabla \varphi\, dxdy = \int_{\Omega\setminus\Gamma} \nabla z_k \nabla(\omega\varphi)\, dxdy + \int_{\Omega\setminus\Gamma} \nabla z_k \nabla((1-\omega)\varphi)\, dxdy \tag{3.32}$$

By (3.30) we have

$$\int_{\Omega\setminus\Gamma} \nabla z_k \nabla(\omega\varphi)\, dxdy \leq 0. \tag{3.33}$$

Since $(1-\omega)\varphi = 0$ on $\partial\Omega_k$ and $(1-\omega)\varphi^\pm = \varphi^\pm$ on Γ, by (3.26) we have

$$\int_{\Omega\setminus\Gamma} \nabla z_k \nabla((1-\omega)\varphi)\, dxdy = \int_{\Gamma} (\varphi^+ + \varphi^-)\, dx. \tag{3.34}$$

Inequality (3.31) follows from (3.32)–(3.34).

By the maximum principle we have

$$\|z_k\|_{L^\infty(\Omega\setminus\Gamma)} \leq \|z_k^+ + z_k^-\|_{L^\infty(\Gamma)}. \tag{3.35}$$

Indeed, if $M := \|z_k^+ + z_k^-\|_{L^\infty(\Gamma)}$ and we take $\varphi := (z_k - M) \vee 0$ in (3.31) we obtain

$$\int_{\Omega\setminus\Gamma} |\nabla((z_k - M) \vee 0)|^2 dxdy = \int_{\Omega\setminus\Gamma} \nabla z_k \nabla((z_k - M) \vee 0)\, dxdy \leq 0,$$

which, together with the boundary condition on $\partial\Omega_k$, implies that $(z_k - M) \vee 0 = 0$ in $\Omega \setminus \Gamma$. This proves (3.35).

Therefore, to prove the lemma it is enough to show that $z_k^+ + z_k^- \to 0$ in $L^\infty(\Gamma)$. We shall prove only that $z_k^+ \to 0$ in $L^\infty(\Gamma)$, since the result for z_k^- can be proved in the same way. Let us prove first that z_k is uniformly small in the intersection between U^+ and a suitable neighbourhood of $(a, 0)$. Since U^+ has Lipschitz boundary, there exist an open neighbourhood V of $(a, 0)$, a constant $R > 0$, and a bi-Lipschitz map $\Phi\colon B_R \to V$ such that $\Phi(B_R^+) = U^+ \cap V$. To simplify the exposition we assume $a = 0$. Since part of the boundary of U^+ near $(a,0)=(0,0)$ is rectilinear, we may assume that there exists $\alpha>0$ such that Φ is the identity map in the sector $\{(x,y)\in B_R : 0\le y<\alpha x\}$ and that $(b,0)\notin B_R$.

Let $v_k(x,y):=z_k(\Phi(x,y))$. By (3.31) and by well known properties of elliptic equations, there exists a symmetric $2{\times}2$ matrix (a_{ij}) of functions in $L^\infty(B_R^+)$, satisfying the uniform ellipticity condition, such that

$$\sum_{i,j=1}^{2}\int_{B_R^+} a_{ij}\partial_j v_k \partial_i\varphi\, dxdy \le \int_{\Gamma_0^R}\varphi\, dx \tag{3.36}$$

for every $\varphi\in H^1(B_R^+)$ with $\varphi\ge 0$ in B_R^+ and $\varphi=0$ on $\partial^+B_R:=\partial B_R\cap\partial B_R^+$, where $\partial_1=\frac{\partial}{\partial x}$ and $\partial_2=\frac{\partial}{\partial y}$.

Let $H:\mathbb{R}\to\mathbb{R}$ be the Heaviside function defined by $H(x)=1$ for $x>0$ and $H(x)=0$ for $x<0$. Since

$$\int_{B_R^+} H\partial_2\varphi\, dxdy = -\int_{\Gamma_0^R}\varphi\, dx$$

for every $\varphi\in H^1(B_R^+)$ with $\varphi=0$ on ∂^+B_R, from (3.36) we obtain that

$$\sum_{i,j=1}^{2}\int_{B_R^+} a_{ij}\partial_j v_k \partial_i\varphi\, dxdy \le -\int_{B_R^+} H\partial_2\varphi\, dxdy \tag{3.37}$$

for every $\varphi\in H^1(B_R^+)$ with $\varphi\ge 0$ in B_R^+ and $\varphi=0$ on ∂^+B_R.

For every $(x,y)\in B_R^-$ we define $v_k(x,y):=v_k(x,-y)$, $a_{ij}(x,y):=a_{ij}(x,-y)$ for $i=j$, $a_{ij}(x,y):=-a_{ij}(x,-y)$ for $i\ne j$. Note that $v_k\in H^1(B_R)$, $a_{ij}\in L^\infty(B_R)$, and that the matrix (a_{ij}) is uniformly elliptic in B_R. Moreover, we define $F\in L^\infty(B_R)$ as

$$F(x,y):=\begin{cases}-H(x) & \text{if } (x,y)\in B_R^+,\\ H(x) & \text{if } (x,y)\in B_R^-.\end{cases}$$

For every $\varphi\in H^1(B_R^-)$, with $\varphi\ge 0$ in B_R^- and $\varphi=0$ on $\partial^-B_R:=\partial B_R\cap\partial B_R^-$, we have

$$\sum_{i,j=1}^{2}\int_{B^-} a_{ij}\partial_j v_k \partial_i\varphi\, dxdy = \sum_{i,j=1}^{2}\int_{B^+} a_{ij}\partial_j v_k \partial_i\hat\varphi\, dxdy$$
$$\int_{B_R^-} F\partial_2\varphi\, dxdy = -\int_{B_R^+} H\partial_2\hat\varphi\, dxdy.$$

where $\hat\varphi(x,y):=\varphi(x,-y)$. Therefore (3.37) yields

$$\sum_{i,j=1}^{2}\int_{B_R} a_{ij}\partial_j v_k \partial_i\varphi\, dxdy \le \int_{B_R} F\partial_2\varphi\, dxdy, \tag{3.38}$$

for every $\varphi\in H_0^1(B_R)$ with $\varphi\ge 0$.

Given $0<r<R$, let $v^{(r)}$ be the solution of the problem

$$\begin{cases} v^{(r)}\in H_0^1(B_r),\\ \displaystyle\sum_{i,j=1}^{2}\int_{B_r} a_{ij}\partial_j v^{(r)}\partial_i\varphi\, dxdy = \int_{B_r} F\partial_2\varphi\, dxdy \quad \text{for every } \varphi\in H^1(B_r).\end{cases} \tag{3.39}$$

Then $v_k = v^{(r)} + v_k^{(r)}$, where $v_k^{(r)} \in H^1(B_r)$ and

$$\sum_{i,j=1}^{2} \int_{B_r} a_{ij} \partial_j v_k^{(r)} \partial_i \varphi \, dxdy \leq 0, \tag{3.40}$$

for every $\varphi \in H_0^1(B_r)$ with $\varphi \geq 0$.

By (3.39) and by the global estimates for solutions of Dirichlet problems for elliptic equations with bounded measurable coefficients (see [16, Théorème 4.2]) for every $p > 2$ there exists a constant $K_p > 0$, independent of r, such that

$$\sup_{B_r} |v^{(r)}| \leq K_p \|F\|_{L^p(B_r)} r^{1-\frac{2}{p}}. \tag{3.41}$$

By (3.40) and by the local estimates for sub-solutions of elliptic equations with bounded measurable coefficients (see [16, Théorème 5.1]) there exists a constant $K > 0$, independent of k, such that

$$\sup_{B_{r/2}} v_k^{(r)} \leq K \Big(\frac{1}{\pi r^2} \int_{B_r} |v_k^{(r)}|^2 dxdy \Big)^{1/2}.$$

Since $v_k = v^{(r)} + v_k^{(r)}$, from these inequalities we get

$$\sup_{B_{r/2}} v_k \leq K \Big(\frac{1}{\pi r^2} \int_{B_r} |v_k|^2 dxdy \Big)^{1/2} + K_p (K+1) \|F\|_{L^p(B_r)} r^{1-\frac{2}{p}}. \tag{3.42}$$

Since $v_k(x, y) := z_k(\Phi(x, y)) \geq 0$ on B_R^+, by (3.27) we have that $v_k \to 0$ strongly in $L^2(B_R^+)$ and by (3.42) we have

$$\limsup_{k\to\infty} \sup_{V_{r/2}} |z_k| \leq K_p (K+1) \|F\|_{L^p(B_r)} r^{1-\frac{2}{p}},$$

where $V_{r/2} := \Phi(B_{r/2}^+)$. Therefore, for every $\varepsilon > 0$ there exist k_0 and a neighbourhood W of $(a, 0)$ such that

$$\sup_{W\cap U^+} |z_k| \leq \varepsilon \tag{3.43}$$

for every $k \geq k_0$. In a similar way we can prove the same result in a neighbourhood of $(b, 0)$.

For every $a < x < b$ the local estimates at the boundary for solutions to Neumann problems, together with (3.27), imply that there exist k_0 and a neighbourhood W of $(x, 0)$ such that (3.43) holds. By a covering argument we conclude that $z_k^+ \to 0$ in $L^\infty(\Gamma)$. □

We now use the previous lemma to show that the displacement u corresponding to a quasistatic evolution is bounded in L^∞ provided the same property holds for the boundary value w.

Corollary 3.5 *Let $T > 0$, $s_0 \in [a, b)$, $w \in AC([0, T]; H^1(\Omega \setminus \Gamma_a^{s_0}))$, and let (u, s) be a quasistatic evolution with boundary value w on $\partial_D\Omega$ according to Definition 2.1. Assume that $w(t)$ is bounded in $L^\infty(\Omega)$ uniformly with respect to t. Then there exists a constant $M > 0$ such that*

$$\|u(t)\|_{L^\infty(\Omega\setminus\Gamma)} \leq M \tag{3.44}$$

for every $t \in [0, T]$.

Proof Let us fix k and let Ω_k, S_k, and z_k be as in Lemma 3.4. Since $u(t)$ is harmonic in $\Omega \setminus \Gamma$, by (2.4) and by the Mean Value Theorem there exists a constant M_k such that

$$\max_{\overline{\Omega}_k} |u(t)| \leq M_k \tag{3.45}$$

for every $t \in [0, T]$. It is not restrictive to assume that

$$M_k \geq \|w(t)\|_{L^\infty(\Omega)} \quad \text{for every } t \in [0, T]. \tag{3.46}$$

Using the standard argument that leads to the maximum principle we now prove that

$$|u(t)| \leq M_k + z_k \quad \text{in } S_k. \tag{3.47}$$

By the equilibrium condition (b) in Definition 2.1 and by Lemma 3.1 for every $t \in [0, T]$ there exists $\psi(t) \in L^\infty(\Gamma)$, with $\|\psi(t)\|_{L^\infty(\Gamma)} \leq 1$, such that

$$\int_{S_k} \nabla u(t) \nabla \varphi \, dx dy = \int_\Gamma \psi(t)[\varphi] \, dx$$

for every $\varphi \in H^1(S_k)$ with $\varphi = 0$ on $\partial\Omega_k \cup \partial_D \Omega$. By (3.26) we have

$$\int_{S_k} \nabla (M_k + z_k) \nabla \varphi \, dx dy = \int_\Gamma (\varphi^+ + \varphi^-) \, dx$$

for every $\varphi \in H^1(S_k)$ with $\varphi = 0$ on $\partial\Omega_k$. Subtracting the first equality from the second one we get

$$\int_{S_k} \nabla (M_k + z_k - u(t)) \nabla \varphi \, dx dy = \int_\Gamma \big((1 - \psi(t))\varphi^+ + (1 + \psi(t))\varphi^-\big) \, dx \tag{3.48}$$

for every $\varphi \in H^1(S_k)$ with $\varphi = 0$ on $\partial\Omega_k \cup \partial_D \Omega$. Let us take $\varphi := (M_k + z_k - u(t)) \wedge 0$. Since $z_k = 0$ on $\partial\Omega_k$ and $M_k - u(t) \geq 0$ on $\partial\Omega_k$ by (3.45), we have that $\varphi = 0$ on $\partial\Omega_k$. Since $z_k \geq 0$ on $\partial_D\Omega$ by (3.29) and $M_k - u(t) = M_k - w(t) \geq 0$ on $\partial_D\Omega$ by (3.46), we have also $\varphi = 0$ on $\partial_D\Omega$. Therefore (3.48) gives

$$\int_{S_k} \nabla (M_k + z_k - u(t)) \nabla \big((M_k + z_k - u(t)) \wedge 0\big) \, dx dy \leq 0.$$

This gives $(M_k + z_k - u(t)) \wedge 0 = 0$ in S_k, which implies $u(t) \leq M_k + z_k$ in S_k. In the same way we prove that $-u(t) \leq M_k + z_k$, obtaining (3.47). This inequality together with (3.45) yields (3.44), since $z_k \in L^\infty(S_k)$ by Lemma 3.4. □

4 The jerky growth of the cracks

In this section we prove the main result of the paper: under suitable continuity assumptions on the boundary datum w, for every quasistatic evolution (u, s) the nondecreasing function s is piecewise constant. In other words, the crack grows only through sudden jumps. More precisely, we obtain the following result.

Theorem 4.1 *Let $T > 0$, $s_0 \in [a, b)$, and $w \in AC([0, T]; H^1(\Omega \setminus \Gamma_a^{s_0})) \cap C^0([0, T]; L^\infty(\Omega))$. Let (u, s) be a quasistatic evolution with boundary value w on $\partial_D\Omega$, according to Definition 2.1. Then there exist a finite number of times $t_0, t_1, \dots, t_m$, with $0 = t_0 < t_1 < \dots < t_{m-1} < t_m = T$, and a finite number $s_1, s_2, \dots, s_m$ of elements of $[s_0, b]$, with*

$s_0 \le s_1 < s_2 < \cdots < s_{m-1} < s_m \le b$, such that for every $j = 1, \ldots, m$ we have $s(t) = s_j$ for every $t \in (t_{j-1}, t_j)$.

Remark 4.2 The previous theorem does not exclude that $s_1 \neq s(0)$, i.e., the constant value of $s(t)$ in the interval $[0, t_1]$ might be different from $s(0)$. This means that a jump of the crack might occur at $t = 0$. However, if we take $u(0) = 0$, the energy-dissipation condition (c) in Definition 2.1 gives

$$\begin{aligned} &\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t)|^2 dxdy + s(t) - s(0) + \int_{\Gamma^b_{s(t)}} |[u(t)]| dx \\ &\le \int_0^t \Big(\int_{\Omega\setminus\Gamma} \nabla u(\tau)\nabla \dot{w}(\tau) dxdy \Big) d\tau, \end{aligned} \tag{4.1}$$

which implies, by (2.3) and Theorem 4.1,

$$s_1 - s(0) = s(t) - s(0) \le M_1^{1/2} \int_0^t \Big(\int_{\Omega\setminus\Gamma} |\nabla \dot{w}(\tau)|^2 dxdy \Big)^{1/2} d\tau,$$

for every $t \in (0, t_1)$. Taking the limit as $t \to 0+$ we obtain that $s_1 = s(0)$. Therefore, Theorem 4.1 implies that, if $u(0) = 0$, then $s(t) = s(0)$ for every $t \in [0, t_1)$.

We now fix the notation we are going to use in the lemmas that will lead to the proof of Theorem 4.1. Let (u, s) be a quasistatic evolution with boundary value w on $\partial_D\Omega$, according to Definition 2.1. For every $t_1, t_2 \in [0, T]$, with $t_1 < t_2$, we define

$$\begin{aligned} \omega_{1,2} = \omega(t_1, t_2) &:= \int_{t_1}^{t_2} \Big(\int_{\Omega\setminus\Gamma} \nabla u(t)\nabla \dot{w}(t) dxdy \Big) dt \\ &- \frac{1}{2}\int_{\Omega\setminus\Gamma} (\nabla u(t_2) + \nabla u(t_1))(\nabla w(t_2) - \nabla w(t_1)) dxdy. \end{aligned} \tag{4.2}$$

Note that $\omega_{1,2}$ can be interpreted as the difference between the integral on $[t_1, t_2]$ of the function $t \mapsto \int_{\Omega\setminus\Gamma} \nabla u(t)\nabla \dot{w}(t) dxdy$ and its approximation obtained by replacing $\nabla u(t)$ with $(\nabla u(t_2) + \nabla u(t_1))/2$.

To simplify the notation we set

$$u_i = u(t_i),\ \ w_i = w(t_i),\ \ s_i = s(t_i). \tag{4.3}$$

By the equilibrium condition (b) we can apply Lemma 3.1 and we obtain that for $i = 1, 2$ there exists $\psi_i \in L^\infty(\Gamma)$ such that

$$\psi_i = 0 \text{ a.e. on } \Gamma_a^{s_i} \text{ and } |\psi_i| \le 1 \text{ a.e. on } \Gamma^b_{s_i}, \tag{4.4}$$

$$\int_{\Omega\setminus\Gamma} \nabla u_i \nabla \varphi \, dxdy = \int_\Gamma \psi_i [\varphi] \, dx \quad \text{for every } \varphi \in H^1_{0,D}(\Omega \setminus \Gamma). \tag{4.5}$$

The first step in the proof of Theorem 4.1 is given by the following result.

Lemma 4.3 *Under the assumptions of Theorem 4.1, let $0 \le t_1 < t_2 \le T$, and let u_i, w_i, s_i, ψ_i, and $\omega_{1,2}$ be as in (4.2)–(4.5). Then*

$$\begin{aligned} &\frac{1}{2}\int_{\Gamma^{s_2}_{s_1}} \psi_1 [u_2 - u_1] \, dx \\ &+ \frac{1}{2}\int_{\Gamma^b_{s_2}} (\psi_1 + \psi_2)[u_2 - u_1] \, dx + s_2 - s_1 + \int_{\Gamma^b_{s_2}} |[u_2 - u_1]| dx \le \omega_{1,2}. \end{aligned} \tag{4.6}$$

Moreover, there exists a constant M, independent of t_1, t_2, s_1, and s_2, such that

$$-\frac{1}{2}\int_{\Gamma_{s_1}^{s_2}} |[u_2-u_1]|dx + s_2 - s_1 \leq \omega_{1,2} \tag{4.7}$$

$$\frac{1}{2}\int_{\Gamma_{s_2}^b}(\psi_1+\psi_2)[u_2-u_1]\,dx + \int_{\Gamma_{s_2}^b}|[u_2-u_1]|\,dx \leq M(s_2-s_1)+\omega_{1,2}. \tag{4.8}$$

Proof By the energy-dissipation inequality (condition (c) in Definition 2.1) we have

$$\frac{1}{2}\int_{\Omega\setminus\Gamma}(\nabla u_2+\nabla u_1)(\nabla u_2-\nabla u_1)dxdy + s_2 - s_1 + \int_{\Gamma_{s_2}^b}|[u_2]-[u_1]|dx$$
$$\leq \int_{t_1}^{t_2}\Big(\int_{\Omega\setminus\Gamma}\nabla u(t)\nabla\dot{w}(t)dxdy\Big)dt. \tag{4.9}$$

We set $\varphi := (u_2-u_1)-(w_2-w_1)$ and observe that $\varphi \in H^1_{0,D}(\Omega\setminus\Gamma)$ and that $[\varphi]=[u_2-u_1]$ on $\Gamma_{s_1}^b$. By (4.4) and (4.5) we obtain

$$\begin{aligned}\frac{1}{2}&\int_{\Omega\setminus\Gamma}\nabla u_2(\nabla u_2-\nabla u_1)dxdy\\ &=\frac{1}{2}\int_{\Omega\setminus\Gamma}\nabla u_2\nabla\varphi\,dxdy+\frac{1}{2}\int_{\Omega\setminus\Gamma}\nabla u_2(\nabla w_2-\nabla w_1)dxdy\\ &=\frac{1}{2}\int_{\Gamma_{s_2}^b}\psi_2[u_2-u_1]\,dx+\frac{1}{2}\int_{\Omega\setminus\Gamma}\nabla u_2(\nabla w_2-\nabla w_1)dxdy.\end{aligned} \tag{4.10}$$

In the same way we obtain

$$\begin{aligned}\frac{1}{2}&\int_{\Omega\setminus\Gamma}\nabla u_1(\nabla u_2-\nabla u_1)dxdy\\ &=\frac{1}{2}\int_{\Gamma_{s_1}^b}\psi_1[u_2-u_1]\,dx+\frac{1}{2}\int_{\Omega\setminus\Gamma}\nabla u_1(\nabla w_2-\nabla w_1)dxdy.\end{aligned}$$

This equality, together with (4.2), (4.9), and (4.10), gives (4.6).

Since $|\psi_i|\leq 1$ a.e. on Γ for $i=1,2$, we have $\frac{1}{2}(\psi_1+\psi_2)[u_2-u_1]+|[u_2-u_1]|\geq 0$ and $\psi_1[u_2-u_1]\geq -|[u_2-u_1]|$ a.e. on Γ. Therefore (4.6) implies (4.7).

By Corollary 3.5 there exists a constant M, independent of t_1, t_2, s_1, and s_2, such that

$$|[u_2-u_1]|\leq 2M \quad \text{on } \Gamma.$$

Since $|\psi_1|\leq 1$ a.e. on $\Gamma_{s_1}^b$, this implies that

$$\frac{1}{2}\int_{\Gamma_{s_1}^{s_2}}\psi_1[u_2-u_1]\,dx+s_2-s_1\geq(-M+1)(s_2-s_1)\geq -M(s_2-s_1).$$

This inequality, together with (4.6), gives (4.8). □

To continue the proof of Theorem 4.1 we want to show that under suitable assumptions on t_1, t_2, s_1, s_2 we have

$$|[u_2-u_1]|\leq 1 \quad \text{a.e. on } \Gamma. \tag{4.11}$$

This inequality, together with (4.7), gives

$$s_2-s_1\leq 2\omega_{1,2}, \tag{4.12}$$

which is an important intermediate result in the proof. The next lemma is the first step in the proof of (4.11).

Lemma 4.4 *Under the assumptions of Theorem 4.1, for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\|w_2 - w_1\|_{L^\infty(\Omega)} < \varepsilon, \tag{4.13}$$

$$\|u_2 - u_1\|_{H^1(\Omega\setminus\Gamma)} < \varepsilon, \tag{4.14}$$

whenever

$$0 \leq t_1 < t_2 \leq T \quad and \quad t_2 - t_1 < \delta, \tag{4.15}$$

$$s_0 \leq s_1 \leq s_2 \leq b \quad and \quad s_2 - s_1 < \delta, \tag{4.16}$$

where u_i, w_i, and s_i, $i = 1, 2$, are defined as in (4.3).

Proof Given a pair of sequences $t_1^n, t_2^n \in [0, T]$, with $t_1^n < t_2^n$, we set $u_i^n := u(t_i^n)$, $w_i^n := w(t_i^n)$, $s_i^n := s(t_i^n)$, and $\omega_{1,2}^n := \omega(t_1^n, t_2^n)$, where ω is defined in (4.2). To prove the lemma it is enough to show that

$$w_2^n - w_1^n \to 0 \quad \text{strongly in } L^\infty(\Omega), \tag{4.17}$$

$$u_2^n - u_1^n \to 0 \quad \text{strongly in } H^1(\Omega \setminus \Gamma), \tag{4.18}$$

assuming that $t_2^n - t_1^n \to 0$ and $s_2^n - s_1^n \to 0$. The convergence of $w_2^n - w_1^n$ follows from the fact that $w \in C^0([0, T]; L^\infty(\Omega))$. Note that by Remark 2.2, the convergence $t_2^n - t_1^n \to 0$ implies that $\omega_{1,2}^n \to 0$, since $w \in AC([0, T]; H^1(\Omega \setminus \Gamma_a^{s_0}))$. By compactness we may also assume that there exists $t_* \in [0, T]$ and $s_* \in [s_0, b]$ such that $t_1^n \to t_*$, $t_2^n \to t_*$, $s_1^n \to s_*$, and $s_2^n \to s_*$.

By Remark 2.2 a suitable subsequence satisfies $u_i^n \rightharpoonup u_i^*$ weakly in $H^1(\Omega \setminus \Gamma)$ for some $u_i^* \in H^1(\Omega \setminus \Gamma)$, for $i = 1, 2$. This implies in particular that $[u_i^n] \to [u_i^*]$ strongly in $L^2(\Gamma)$. Since $w_i^n \to w_* := w(t_*)$ strongly in $H^1(\Omega \setminus \Gamma_a^{s_0})$, from the minimality of u_i^n and Lemma 2.8 we deduce that $u_i^n \to u_i^*$ strongly in $H^1(\Omega \setminus \Gamma)$ and that

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u_i^*|^2\, dxdy \leq \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla v|^2\, dxdy + \int_{\Gamma_{s_*}^b} |[v] - [u_i^*]|dx$$

for every $v \in H^1(\Omega \setminus \Gamma)$ with $v = w_*$ on $\partial_D\Omega$.

By the Euler condition (see Lemma 3.1) we obtain that there exist $\psi_i^n \in L^\infty(\Gamma)$, with $|\psi_i^n| \leq 1$ a.e. on Γ and $\psi_i^* \in L^\infty(\Gamma)$, with $|\psi_i^*| \leq 1$ a.e. on Γ, such that for $i = 1, 2$ we have

$$\int_{\Omega\setminus\Gamma} \nabla u_i^n \nabla\varphi\, dxdy = \int_\Gamma \psi_i^n[\varphi]\, dx \quad \text{for every } \varphi \in H^1_{0,D}(\Omega \setminus \Gamma) \text{ and every } n,$$

$$\int_{\Omega\setminus\Gamma} \nabla u_i^* \nabla\varphi\, dxdy = \int_\Gamma \psi_i^*[\varphi]\, dx \quad \text{for every } \varphi \in H^1_{0,D}(\Omega \setminus \Gamma). \tag{4.19}$$

Therefore the convergence of u_i^n to u_i^* in $H^1(\Omega \setminus \Gamma)$ implies that $\psi_i^n \rightharpoonup \psi_i^*$ weakly* in $L^\infty(\Gamma)$.

Since $\frac{1}{2}(\psi_1^n + \psi_2^n)[u_2^n - u_1^n] + |[u_2^n - u_1^n]| \geq 0$ (which follows from the fact that $|\psi_i^n| \leq 1$) and $\omega_{1,2}^n \to 0$, by (4.8) we have

$$\frac{1}{2}\int_\Gamma (\psi_1^n + \psi_2^n)[u_2^n - u_1^n]\, dx + \int_\Gamma |[u_2^n - u_1^n]|\, dx \to 0.$$

This implies that

$$\frac{1}{2}\int_{\Gamma}(\psi_1^*+\psi_2^*)[u_2^*-u_1^*]\,dx+\int_{\Gamma}|[u_2^*-u_1^*]|\,dx=0.$$

Since $\frac{1}{2}(\psi_1^*+\psi_2^*)[u_2^*-u_1^*]+|[u_2^*-u_1^*]|\geq 0$, we deduce that $\frac{1}{2}(\psi_1^*+\psi_2^*)[u_2^*-u_1^*]+|[u_2^*-u_1^*]|=0$ a.e. on Γ. Using the inequality $|\psi_i^*|\leq 1$ a.e. on Γ, we obtain $\psi_1^*=\psi_2^*$ on $\{[u_2^*-u_1^*]\neq 0\}$.

As $u_1^*=u_2^*=w^*$ on $\partial_D\Omega$, we have $u_2^*-u_1^*\in H^1_{0,D}(\Omega\setminus\Gamma)$. By (4.19) we have

$$\int_{\Omega\setminus\Gamma}\nabla(u_2^*-u_1^*)\nabla\varphi\,dxdy=\int_{\Gamma}(\psi_2^*-\psi_1^*)[\varphi]\,dx\quad\text{for every }\varphi\in H^1_{0,D}(\Omega\setminus\Gamma).$$

Taking $\varphi=u_2^*-u_1^*$ we deduce that $\nabla u_1^*=\nabla u_2^*$, which implies $u_1^*=u_2^*$, since $u_1^*=u_2^*=w^*$ on $\partial_D\Omega$ and (2.1) holds. Therefore the strong convergence of u_i^n to u_i^* implies (4.18). Since the result does not depend on the subsequence, (4.18) holds for the whole sequence. □

We now complete the proof of (4.11).

Lemma 4.5 *Under the assumptions of Theorem 4.1, there exists $\delta_0>0$ such that (4.11) holds whenever*

$$0\leq t_1<t_2\leq T\quad\textit{and}\quad t_2-t_1<\delta_0, \tag{4.20}$$

$$s_0\leq s_1\leq s_2\leq b\quad\textit{and}\quad s_2-s_1<\delta_0, \tag{4.21}$$

where u_i and s_i, $i=1,2$, are defined as in (4.3).

Proof Let Ω_k be a sequence of open subsets of Ω with boundary of class C^∞ such that $\Omega_k\subset\subset\Omega_{k+1}$ for every k and $\Omega\setminus\Gamma=\cup_k\Omega_k$. For every k let S_k and z_k be defined by (3.25) and (3.26). By Lemma 3.4 there exists k_0 such that

$$\|z_{k_0}\|_{L^\infty(S_{k_0})}\leq 1/8. \tag{4.22}$$

We fix $\rho>0$ such that $B_\rho(x_0,y_0)\subset\Omega\setminus\Gamma$ for every $(x_0,y_0)\in\overline{\Omega}_{k_0}$. By the Mean Value Theorem we have

$$|(u_2-u_1)(x_0,y_0)|\leq\frac{1}{\pi\rho^2}\int_{B_\rho(x_0,y_0)}|u_2-u_1|\,dxdy\leq\frac{1}{\pi^{1/2}\rho}\|u_2-u_1\|_{L^2(\Omega\setminus\Gamma)} \tag{4.23}$$

for every $(x_0,y_0)\in\overline{\Omega}_{k_0}$. We now fix $0<\varepsilon_0<1/4$ such that $\varepsilon_0/(\pi^{1/2}\rho)<1/4$. The constant δ given by Lemma 4.4 for $\varepsilon=\varepsilon_0$ will be denoted by δ_0. If (4.20) and (4.21) hold, by (4.23) we have

$$|(u_2-u_1)(x_0,y_0)|<\tfrac{1}{4}\quad\text{for every }(x_0,y_0)\in\overline{\Omega}_{k_0}. \tag{4.24}$$

Using the standard argument that leads to the maximum principle we now prove that

$$|u_2-u_1|\leq\tfrac{1}{4}+2z_{k_0}\quad\text{in }S_{k_0}. \tag{4.25}$$

By (4.5) we have

$$\int_{S_{k_0}}\nabla(u_2-u_1)\nabla\varphi\,dxdy=\int_{\Gamma}(\psi_2-\psi_1)[\varphi]\,dx$$

for every $\varphi\in H^1(S_{k_0})$ with $\varphi=0$ on $\partial\Omega_{k_0}\cup\partial_D\Omega$. By (3.26) we have

$$\int_{S_{k_0}}\nabla(\tfrac{1}{4}+2z_{k_0})\nabla\varphi\,dxdy=\int_{\Gamma}2(\varphi^++\varphi^-)\,dx$$

for every $\varphi \in H^1(S_{k_0})$ with $\varphi = 0$ on $\partial\Omega_{k_0}$. Subtracting the terms of the first equality from those of the second one we get

$$\int_{S_{k_0}} \nabla(\tfrac{1}{4} + 2z_{k_0} - u_2 + u_1)\nabla\varphi\, dxdy = \int_\Gamma \big((2 - \psi_2 + \psi_1)\varphi^+ + (2 + \psi_2 - \psi_1)\varphi^-\big) d\mathcal{H}^1 \tag{4.26}$$

for every $\varphi \in H^1(S_{k_0})$ with $\varphi = 0$ on $\partial\Omega_{k_0} \cup \partial_D\Omega$. Let us take $\varphi := (\frac{1}{4} + 2z_{k_0} - u_2 + u_1) \wedge 0$. Since $z_{k_0} = 0$ on $\partial\Omega_{k_0}$ and $\frac{1}{4} - u_2 + u_1 \geq 0$ on $\partial\Omega_{k_0}$ by (4.24), we have that $\varphi = 0$ on $\partial\Omega_{k_0}$. Since $z_{k_0} \geq 0$ on $\partial_D\Omega$ by (3.29) and $\frac{1}{4} - u_2 + u_1 = \frac{1}{4} - w_2 + w_1 \geq 0$ on $\partial_D\Omega$ by (4.13), we have also $\varphi = 0$ on $\partial_D\Omega$. Therefore (4.26) gives

$$\int_{S_{k_0}} \nabla(\tfrac{1}{4} + 2z_{k_0} - u_2 + u_1)\nabla\big((\tfrac{1}{4} + 2z_{k_0} - u_2 + u_1) \wedge 0\big)\, dxdy \leq 0.$$

This gives $(\frac{1}{4} + 2z_{k_0} - u_2 + u_1) \wedge 0 = 0$ in S_{k_0}, which implies $u_2 - u_1 \leq \frac{1}{4} + 2z_{k_0}$ in S_{k_0}. In the same way we prove that $u_1 - u_2 \leq \frac{1}{4} + 2z_{k_0}$, obtaining (4.25).

By (4.22), (4.24), and (4.25) we obtain $|u_2 - u_1| \leq \frac{1}{2}$ in $\Omega \setminus \Gamma$. This implies that $|u_2^+ - u_1^+| \leq \frac{1}{2}$ and $|u_2^- - u_1^-| \leq \frac{1}{2}$ on Γ, which give (4.11). □

Inequality (4.11), together with (4.7), gives the following result.

Corollary 4.6 *Under the assumptions of Theorem 4.1, let $\delta_0 > 0$ be the constant introduced in Lemma 4.5. Let $t_1, t_2 \in [0, T]$, with $t_1 < t_2$, and let s_1, s_2, and $\omega_{1,2}$ be defined by (4.2) and (4.3). If (4.20) and (4.21) hold, then (4.12) is satisfied.*

We now consider an interval $[\tau_1, \tau_2]$, with no restriction on its length. Iterating estimate (4.12) on the intervals of a suitable subdivision we obtain an estimate on the difference $s(\tau_2) - s(\tau_1)$.

Lemma 4.7 *Under the assumptions of Theorem 4.1, let $\delta_0 > 0$ be the constant introduced in Lemma 4.5 and let $[\tau_1, \tau_2] \subset [0, T]$. Suppose that there exists a finite subdivision $\tau_1 = t_0 < t_1 < \cdots < t_m = \tau_2$ of the interval $[\tau_1, \tau_2]$ such that*

$$t_j - t_{j-1} < \delta_0 \quad \text{and} \quad s(t_j) - s(t_{j-1}) < \delta_0 \tag{4.27}$$

for every $j = 1, \ldots, m$. Then

$$s(\tau_2) - s(\tau_1) \leq 2\sum_{j=1}^{m} \omega(t_{j-1}, t_j), \tag{4.28}$$

where ω is defined by (4.2).

Proof It is enough to apply Corollary 4.6 to each interval $[t_{j-1}, t_j]$. □

To conclude the proof of Theorem 4.1 we have to show that, under suitable assumptions, it is possible to find a subdivision such that (4.27) holds and the right-hand side of (4.28) is arbitrarily small. We shall see (Corollary 4.11) that the latter property is related to the approximation of a Lebesgue integral by its Riemann sums.

As for (4.27), it is clear that the second inequality follows from an estimate on $t_j - t_{j-1}$ when s is continuous. The following lemma shows that this happens even if s is discontinuous provided it is nondecreasing and its jumps have an amplitude less than δ_0. For every nondecreasing real-valued function s defined on an interval $[\tau_1, \tau_2]$ and for every $t \in [\tau_1, \tau_2]$ we define

$$[s](t) := s(t+) - s(t-),$$

where $s(t+)$ and $s(t-)$ are the right and left limits of s at t, with the convention $s(\tau_1-) = s(\tau_1)$ and $s(\tau_2+) = s(\tau_2)$.

Lemma 4.8 *Let $\tau_1 < \tau_2$ and let $s\colon [\tau_1, \tau_2] \to \mathbb{R}$ be a nondecreasing function. Let $\delta_0 > 0$ be such that $[s](t) < \delta_0$ for every $t \in [\tau_1, \tau_2]$. Then there exists $\eta_0 \in (0, \delta_0]$ such that*

$$s(t_2) - s(t_1) < \delta_0 \quad \textit{for every } t_1, t_2 \in [\tau_1, \tau_2] \textit{ with } 0 < t_2 - t_1 < \eta_0.$$

Proof Let J be the set of jump points of s, which is at most countable. Let us prove that

$$\sup_{t\in J}[s](t) < \delta_0. \tag{4.29}$$

This is trivial if the supremum is zero. Otherwise we fix $0 < \delta_1 < \sup_{t\in J}[s](t)$ and we observe that

$$\sup_{t\in J}[s](t) = \max_{t\in F_1}[s](t) < \delta_0,$$

where F_1 is the finite set defined by $F_1 := \{t \in [\tau_1, \tau_2] : [s](t) > \delta_1\}$. This concludes the proof of (4.29).

Let δ_2 be such that

$$\sup_{t\in J}[s](t) + 2\delta_2 < \delta_0. \tag{4.30}$$

Let us decompose s as $s = s_j + s_c$, where s_j is the pure jump component of s defined by

$$s_j(t) = s(t) - s(t-) + \sum_{\tau\in J, \tau<t} [s](\tau), \tag{4.31}$$

while s_c is its continuous component.

Let $\eta_1 > 0$ be such that

$$s_c(t_2) - s_c(t_1) < \delta_2 \tag{4.32}$$

whenever $0 < t_2 - t_1 < \eta_1$. On the other hand there exists a finite set $F_2 \subset J$ such that

$$\sum_{t\in J\setminus F_2} [s](t) < \delta_2. \tag{4.33}$$

Since F_2 is finite, the distance between any two distinct points in F_2 is larger than some constant $\eta_2 > 0$.

Set $\eta_0 := \eta_1 \wedge \eta_2 \wedge \delta_0$ and let $t_1, t_2 \in [\tau_1, \tau_2]$ with $0 < t_2 - t_1 < \eta_0$. First of all we note that $[t_1, t_2]$ contains at most one point $\tau \in F_2$. Then, by (4.31)–(4.33), we have

$$s(t_2) - s(t_1) = s_j(t_2) - s_j(t_1) + s_c(t_2) - s_c(t_1) \leq \delta_2 + [s](\tau) + \delta_2.$$

By (4.30) the conclusion follows. □

Combining Lemmas 4.7 and 4.8 we obtain the following result.

Lemma 4.9 *Under the assumptions of Theorem 4.1, let $\delta_0 > 0$ be the constant given by Lemma 4.5, and let F be the finite set defined by*

$$F := \{t \in [0, T] : [s](t) \geq \delta_0\}.$$

Let $\tau_1, \tau_2 \in [0, T]$, with $\tau_1 < \tau_2$, be such that

$$[\tau_1, \tau_2] \cap F = \emptyset,$$

and let $\tau_1 = t_0 < t_1 < \cdots < t_m = \tau_2$ be a subdivision of the interval $[\tau_1, \tau_2]$ such that

$$t_j - t_{j-1} < \eta_0 \tag{4.34}$$

for every $j = 1, \ldots, m$, where η_0 is the constant introduced in Lemma 4.8 corresponding to δ_0. Then (4.28) holds.

Proof By Lemma 4.8 the inequality (4.34) implies the second condition in (4.27), so that the conclusion follows from Lemma 4.7. □

The following proposition, related to the approximation of a Lebesgue integral by suitable Riemann sums, will be used to show that the right-hand side of (4.28) can be made arbitrarily small by a suitable choice of the subdivision.

Proposition 4.10 *Let H be a Hilbert space, let $\tau_1 < \tau_2$, and let $f, g : [\tau_1, \tau_2] \to H$ be Bochner integrable functions. Assume that there exists a constant $M > 0$ such that $\|f(t)\| \leq M$ for every $t \in [\tau_1, \tau_2]$, where $\|\cdot\|$ denotes the norm in H. Then for every integer $k \geq 1$ there exists a subdivision $\tau_1 = t_0^k < t_1^k < \cdots < t_{m_k}^k = \tau_2$ such that $t_j^k - t_{j-1}^k \leq \frac{1}{k}$ for every $1 \leq j \leq m_k$ and*

$$\begin{aligned} \int_{\tau_1}^{\tau_2} (f(t), g(t))dt &= \lim_{k\to\infty} \sum_{j=1}^{m_k} \int_{t_{j-1}^k}^{t_j^k} (f(t_j^k), g(t))dt \\ &= \lim_{k\to\infty} \sum_{j=1}^{m_k} \int_{t_{j-1}^k}^{t_j^k} (f(t_{j-1}^k), g(t))dt, \end{aligned} \tag{4.35}$$

where $(\cdot, \cdot)$ denotes the scalar product in H.

Proof A direct proof of (4.35) can be obtained by adapting the proof in [8, page 63]. We provide here a short proof based on [4, Lemma 4.12], which guarantees for every $k \geq 1$ the existence of a subdivision $\tau_1 = t_0^k < t_1^k < \cdots < t_{m_k}^k = \tau_2$ such that $t_j^k - t_{j-1}^k \leq \frac{1}{k}$ for every $1 \leq j \leq m_k$ and

$$\lim_{k\to\infty} \sum_{j=1}^{m_k} \int_{t_{j-1}^k}^{t_j^k} \|f(t) - f(t_j^k)\|dt \to 0. \tag{4.36}$$

Let us define $F_k : [\tau_1, \tau_2) \to H$ by

$$F_k(t) := f(t) - \sum_{j=1}^{m_k} f(t_j^k)\chi_{[t_{j-1}^k, t_j^k)}(t) = \sum_{j=1}^{m_k} (f(t) - f(t_j^k))\chi_{[t_{j-1}^k, t_j^k)}(t).$$

By (4.36) we have $F_k \to 0$ in $L^1([\tau_1, \tau_2); H)$. Since $\|F_k(t)\| \leq 2M$ for every $t \in [\tau_1, \tau_2)$ and g is Bochner integrable, we obtain that

$$\int_{\tau_1}^{\tau_2} (F_k(t), g(t))dt \to 0, \tag{4.37}$$

which gives the first equality in (4.35). The second one can be proved in the same way. □

Corollary 4.11 *Under the assumptions of Theorem 4.1, let $\tau_1, \tau_2 \in [0, T]$, with $\tau_1 < \tau_2$, and let ω be defined by (4.2). Then there exists a sequence of subdivisions $\tau_1 = t_0^k < t_1^k < \cdots < t_{m_k}^k = \tau_2$ such that $t_j^k - t_{j-1}^k \leq \frac{1}{k}$ for every $1 \leq j \leq m_k$ and*

$$\lim_{k\to\infty} \sum_{j=1}^{m_k} \omega(t_{j-1}^k, t_j^k) = 0.$$

Proof It is enough to apply the previous proposition with $X := L^2(\Omega \setminus \Gamma; \mathbb{R}^2)$, $f(t) := \nabla u(t)$, and $g(t) := \nabla \dot{w}(t)$. □

Proof Let $\delta_0 > 0$ be the constant introduced in Lemma 4.5, let $\eta_0 > 0$ be the constant introduced in Lemma 4.8 related to δ_0, and let F be the finite set defined by

$$F := \{t \in [0, T] : [s](t) \geq \delta_0\} \cup \{0, T\}.$$

Let $\tau_1, \tau_2 \in [0, T]$ be such that $\tau_1 < \tau_2$ and $[\tau_1, \tau_2] \cap F = \emptyset$. By Corollary 4.11, for every $\varepsilon > 0$ we can find a finite subdivision $\tau_1 = t_0 < t_1 < \cdots < t_m = \tau_2$ of the interval $[\tau_1, \tau_2]$ such that $t_j - t_{j-1} < \eta_0$ for every $j = 1, \ldots, m$ and

$$2 \sum_{j=1}^{m} \omega(t_{j-1}, t_j) < \varepsilon.$$

By Lemma 4.9 we obtain $s(\tau_2) - s(\tau_1) < \varepsilon$. By the arbitrariness of ε we deduce that $s(\tau_2) \leq s(\tau_1)$ and by monotonicity we deduce that s is constant on the interval $[\tau_1, \tau_2]$. It follows that s is constant in each connected component of $[0, T] \setminus F$. This concludes the proof. □

To prove the regularity of u on $[0, T] \setminus \{t_0, t_1, \ldots, t_m\}$, it is convenient to introduce a different notion of quasistatic evolution in which the crack does not grow.

Definition 4.12 Let $T > 0$, $s_0 \in [a, b)$, and $w \in AC([0, T]; H^1(\Omega \setminus \Gamma_a^{s_0}))$. A quasistatic evolution with fixed crack and boundary value w on $\partial_D \Omega$ is a function $u\colon [0, T] \to H^1(\Omega \setminus \Gamma)$ such that

(a$_0$) (measurability) $u\colon [0, T] \to H^1(\Omega \setminus \Gamma)$ is measurable;
(b$_0$) (equilibrium) for every $t \in [0, T]$ we have $u(t) = w(t)$ on $\partial_D \Omega$ and

$$\frac{1}{2} \int_{\Omega\setminus\Gamma} |\nabla u(t)|^2 dx dy \leq \frac{1}{2} \int_{\Omega\setminus\Gamma} |\nabla \hat{u}|^2 dx dy + \int_{\Gamma_{s_0}^b} |[\hat{u}] - [u(t)]| dx,$$

for every $\hat{u} \in H^1(\Omega \setminus \Gamma)$ with $\hat{u} = w(t)$ on $\partial_D \Omega$;
(c$_0$) (energy-dissipation inequality) for every $t_1, t_2 \in [0, T]$, with $t_1 < t_2$, we have

$$\begin{aligned} &\frac{1}{2} \int_{\Omega\setminus\Gamma} |\nabla u(t_2)|^2 dx dy + \int_{\Gamma_{s_0}^b} |[u(t_2)] - [u(t_1)]| dx \\ &\leq \frac{1}{2} \int_{\Omega\setminus\Gamma} |\nabla u(t_1)|^2 dx dy + \int_{t_1}^{t_2} \Big(\int_{\Omega\setminus\Gamma} \nabla u(\tau) \nabla \dot{w}(\tau) dx dy \Big) d\tau. \end{aligned}$$

Remark 4.13 Taking $\hat{u} = w(t)$ in condition (b_0) above we obtain that there exists a constant $M > 0$ such that

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(t)|^2 dxdy \le M + \int_{\Gamma^b_{s_0}} |[u(t)]|dx \quad \text{for every } t \in [0,T]. \tag{4.38}$$

By (2.1) and (2.2), the Trace Inequality, combined with the Poincaré Inequality, implies that there exists a constant $c > 0$ such that

$$\int_{\Gamma^b_{s_0}} |[u(t)]|dx \le c\Big(\int_{\Omega\setminus\Gamma} |\nabla u(t)|^2 dxdy\Big)^{1/2} + c.$$

This inequality and (4.38) imply that $\nabla u(t)$ is bounded in L^2 uniformly with respect to t. Together with the measurability of $t \mapsto u(t)$ this ensures that the last integral in condition (c_0) above is well defined.

Theorem 4.14 *Let $T > 0$, $s_0 \in [a,b)$, and $w \in AC([0,T]; H^1(\Omega \setminus \Gamma^{s_0}_a))$, and let $u\colon [0,T] \to H^1(\Omega\setminus\Gamma)$ be a quasistatic evolution with fixed crack and boundary value w. Then $u \in AC([0,T]; H^1(\Omega \setminus \Gamma^{s_0}_a))$ and*

$$\Big(\int_{\Omega\setminus\Gamma} |\nabla u(\tau_2) - \nabla u(\tau_1)|^2 dxdy\Big)^{1/2} \le \int_{\tau_1}^{\tau_2} \Big(\int_{\Omega\setminus\Gamma} |\nabla \dot{w}(\tau)|^2 dxdy\Big)^{1/2} d\tau \tag{4.39}$$

for every $\tau_1, \tau_2 \in [0,T]$ with $\tau_1 < \tau_2$.

Proof The proof is taken from [3, Theorem 5.2], with obvious simplifications. Let us fix $\tau_1, \tau_2 \in [0,T]$ with $\tau_1 < \tau_2$. From the energy-dissipation condition (c_0) between τ_1 and τ_2 we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(\tau_2)|^2 dxdy + \int_{\Gamma^b_{s_0}} |[u(\tau_2)] - [u(\tau_1)]|dx \\
&\le \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(\tau_1)|^2 dxdy + \int_{\tau_1}^{\tau_2} \Big(\int_{\Omega\setminus\Gamma} \nabla u(\tau)\nabla \dot{w}(\tau) dxdy\Big) d\tau.
\end{aligned} \tag{4.40}$$

The Euler equation corresponding to the equilibrium condition (b_0) of Definition 4.12 (see Lemma 3.1) implies that

$$-\int_{\Omega\setminus\Gamma} \nabla u(\tau_1)\nabla\varphi\, dxdy \le \int_{\Gamma^b_{s_0}} |[\varphi]|\, dx \quad \text{for every } \varphi \in H^1_{0,D}(\Omega\setminus\Gamma).$$

Taking $\varphi := u(\tau_2) - u(\tau_1) - (w(\tau_2) - w(\tau_1))$ we obtain

$$\begin{aligned}
&-\int_{\Omega\setminus\Gamma} \nabla u(\tau_1)\nabla u(\tau_2)\, dxdy + \int_{\Omega\setminus\Gamma} |\nabla u(\tau_1)|^2 dxdy \\
&\le -\int_{\Omega\setminus\Gamma} \nabla u(\tau_1)(\nabla w(\tau_2) - \nabla w(\tau_1))\, dxdy + \int_{\Gamma^b_{s_0}} |[u(\tau_2)] - [u(\tau_1)]|dx,
\end{aligned} \tag{4.41}$$

where we have used the fact that $[w(\tau_1)] = [w(\tau_2)] = 0$ on $\Gamma^b_{s_0}$. Adding (4.40) and (4.41) we get

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u(\tau_2) - \nabla u(\tau_1)|^2 dxdy \le \int_{\tau_1}^{\tau_2} \Big(\int_{\Omega\setminus\Gamma} (\nabla u(\tau) - \nabla u(\tau_1))\nabla\dot{w}(\tau) dxdy\Big) d\tau.$$

Since this holds for every $\tau_2 \in (\tau_1, T]$, by the Gronwall Inequality we obtain (4.39) for every $\tau_2 \in (\tau_1, T]$. This inequality, together with the integrability of the function $\tau \mapsto$

$\left(\int_{\Omega\setminus\Gamma} |\nabla \dot{w}(\tau)|^2 dxdy\right)^{1/2}$, implies that $\nabla u \in AC([0,T]; L^2(\Omega\setminus\Gamma;\mathbb{R}^2))$. Since $u(t) = w(t)$ on $\partial_D\Omega$, by (2.1) and the Poincaré Inequality we conclude that $u \in AC([0,T]; H^1(\Omega\setminus\Gamma))$. □

Theorem 4.15 *Under the assumptions of Theorem 4.1, for every $j = 1,\dots,m$ there exists $u^j \in AC([t_{j-1},t_j]; H^1(\Omega\setminus\Gamma))$ such that $u(t) = u^j(t)$ for every $t \in (t_{j-1},t_j)$.*

Proof Let us fix $1 \le j \le m$. By Theorem 4.1, we have $s(t) = s_j$ for every $t \in (t_{j-1},t_j)$. Therefore for every $\tau_1,\tau_2 \in (t_{j-1},t_j)$ with $\tau_1 < \tau_2$, the function u is a quasistatic evolution with fixed crack in the sense of Definition 4.12 on the interval $[\tau_1,\tau_2]$.

By Theorem 4.14 we obtain (4.39) for every $[\tau_1,\tau_2] \subset (t_{j-1},t_j)$. This shows that the restriction of u to the open interval (t_{j-1},t_j) can be extended to an absolutely continuous function $u^j : [t_{j-1},t_j] \to H^1(\Omega\setminus\Gamma)$. □

Remark 4.16 Besides the assumptions of Theorem 4.1, suppose also that $w(0) = u(0) = 0$ and that $s(0) = s_0$. Then there exists $u^1 \in AC([0,t_1]; H^1(\Omega\setminus\Gamma))$ such that $u(t) = u^1(t)$ for every $t \in [0,t_1)$. Indeed, (4.1) implies that $u(t) \to 0$ strongly in $H^1(\Omega\setminus\Gamma)$ as $t \to 0+$.

Theorem 4.17 *Let $T > 0$, $s_0 \in [a,b)$, and $w \in AC([0,T]; H^1(\Omega\setminus\Gamma_a^{s_0}))$. Let u_1, u_2 be two quasistatic evolutions with fixed crack and boundary condition w on $\partial_D\Omega$. If $u_1(0) = u_2(0)$ then $u_1(t) = u_2(t)$ for every $t \in [0,T]$.*

Proof The proof is taken from [3, Theorem 5.9], with obvious simplifications. Since $u_2 \in AC([0,T]; H^1(\Omega\setminus\Gamma))$ by Theorem 4.15, from the energy-dissipation condition (c_0) (dividing by $t_2 - t_1$, and passing to the limit as $t_1,t_2 \to t$), we obtain

$$\int_{\Omega\setminus\Gamma} \nabla u_2(t)\big(\nabla\dot{u}_2(t) - \nabla\dot{w}(t)\big)dxdy \le -\int_{\Gamma^b_{s_0}} |[\dot{u}_2(t)]|dx \tag{4.42}$$

for a.e. $t \in (0,T)$.

On the other hand, for every $t \in [0,T]$, the Euler equation (see Lemma 3.1) for the equilibrium condition (b_0) for u_1 gives that there exists $\psi_1(t) \in L^\infty(\Gamma^b_{s_0})$ with $|\psi_1(t)| \le 1$, such that

$$\int_{\Omega\setminus\Gamma} \nabla u_1(t)\nabla\varphi\, dxdy = \int_{\Gamma^b_{s_0}} \psi_1(t)[\varphi]dx \quad \text{for every } \varphi \in H^1_{0,D}(\Omega\setminus\Gamma). \tag{4.43}$$

Since $u_2(t) = w(t)$ on $\partial_D\Omega$ for every $t \in [0,T]$ and $u_2 \in AC([0,T]; H^1(\Omega\setminus\Gamma))$, we have that $\dot{u}_2(t) - \dot{w}(t) \in H^1_{0,D}(\Omega\setminus\Gamma)$ for a.e. $t \in (0,T)$. Using $\varphi = -(\dot{u}_2(t) - \dot{w}(t))$ in (4.43) we obtain

$$-\int_{\Omega\setminus\Gamma} \nabla u_1(t)\big(\nabla\dot{u}_2(t) - \nabla\dot{w}(t)\big)dxdy = -\int_{\Gamma^b_{s_0}} \psi_1(t)[\dot{u}_2(t)]dx \tag{4.44}$$

for a.e. $t \in (0,T)$. Since $|\psi_1(t)| \le 1$, adding (4.42) and (4.44) we get

$$\int_{\Omega\setminus\Gamma} \big(\nabla u_2(t) - \nabla u_1(t)\big)\big(\nabla\dot{u}_2(t) - \nabla\dot{w}(t)\big)dxdy \le 0 \tag{4.45}$$

for a.e. $t \in (0,T)$.

In a similar way we obtain

$$\int_{\Omega\setminus\Gamma} \big(\nabla u_1(t) - \nabla u_2(t)\big)\big(\nabla\dot{u}_1(t) - \nabla\dot{w}(t)\big)dxdy \le 0 \tag{4.46}$$

for a.e. $t \in (0, T)$. Adding (4.45) and (4.46) we have

$$\int_{\Omega\setminus\Gamma} \big(\nabla u_2(t) - \nabla u_1(t)\big)\big(\nabla \dot{u}_2(t) - \nabla \dot{u}_1(t)\big)dxdy \le 0 \tag{4.47}$$

for a.e. $t \in (0, T)$. This implies that the absolutely continuous function $t \mapsto \int_{\Omega\setminus\Gamma} |\nabla u_2(t) - \nabla u_1(t)|^2$ has a nonpositive derivative a.e. in $[0, T]$, thus it is nonincreasing. Since it is 0 at $t = 0$ we conclude that $\nabla u_1(t) = \nabla u_2(t)$ for every $t \in [0, T]$. Since $u_1(t) = u_2(t) = w(t)$ on $\partial_D\Omega$ we deduce that $u_1(t) = u_2(t)$ for every $t \in [0, T]$ by (2.1). □

The following corollary is an immediate consequence of Remark 4.16 and Theorem 4.17.

Corollary 4.18 *Besides the assumptions of Theorem 4.1, suppose also that $w(0) = u(0) = 0$ and that $s(0) = s_0$. Then u is uniquely determined in the interval $[0, t_1)$.*

5 An example

In this section we describe an example of quasistatic evolution (u, s) where s is not constant. We consider here

$$\Omega := (a, b) \times (-h, h), \quad \Gamma := [a, b] \times \{0\}, \quad \partial_D\Omega := [a, b] \times \{-h, h\}, \tag{5.1}$$

for some $h > 0$. Therefore we have $\Omega^+ = (a, b) \times (0, h)$ and $\partial_D^+\Omega = [a, b] \times \{h\}$. The boundary condition at time $t \in [0, T]$ will be $u(t) = t$ on $[a, b] \times \{h\}$ and $u(t) = -t$ on $[a, b] \times \{-h\}$. This leads to the following choice for $w(t) \in H^1(\Omega)$:

$$w(t)(x, y) := t\frac{y}{h}. \tag{5.2}$$

Let $z_0 \in H^1(\Omega^+)$ be the solution of the problem

$$\begin{cases} \Delta z_0 = 0 & \text{in } \Omega^+, \\ z_0 = 0 & \text{if } y = h, \\ \frac{\partial z_0}{\partial x} = 0 & \text{if } x = a \text{ and } x = b, \\ \frac{\partial z_0}{\partial y} = 0 & \text{if } a < x < s_0 \text{ and } y = 0, \\ \frac{\partial z_0}{\partial y} = 1 & \text{if } s_0 < x < b \text{ and } y = 0. \end{cases} \tag{5.3}$$

We shall prove that $z_0 \in C^0(\overline{\Omega}^+)$ (see Remark 5.6 and Lemma 5.7 below). We define z_0 in Ω^- by

$$z_0(x, y) := -z_0(x, -y) \quad \text{for every } (x, y) \in \Omega^-.$$

Theorem 5.1 *Let Ω, Γ, $\partial_D\Omega$, and w be as in (5.1) and (5.2). Let $T > 0$, $s_0 \in (a, b)$, and let (u, s) be a quasistatic evolution with boundary condition w on $\partial_D\Omega$ and initial conditions $u(0) = 0$ and $s(0) = s_0$. Assume that*

$$T > T_* := -\inf_{\Omega^+} z_0 \quad \text{and} \quad \int_{\Omega^+} |\nabla z_0|^2 dxdy + s_0 > b, \tag{5.4}$$

where z_0 is defined by (5.3). Then $s(t)$ takes at least two distinct values in two nondegenerate intervals.

Remark 5.2 Since $z_0(x, y) \to y$ as $s_0 \to a+$, the second inequality in (5.4) is surely satisfied if $h > 1$ and s_0 is sufficiently close to a.

To prove Theorem 5.1 we shall construct a quasistatic evolution u_* with fixed crack and boundary condition w such that $u_*(0) = 0$ and

$$u_*(t) = \begin{cases} t + z_0 & \text{in } \Omega^+ \\ -t + z_0 & \text{in } \Omega^- \end{cases} \tag{5.5}$$

for every $t > T_*$. If we had $s(t) = s_0$ for every $t \in [0, T]$, by the uniqueness result proved in Theorem 4.17 we would have $u(t) = u_*(t)$ for every $t \in [0, T]$. On the other hand, we shall see that, if $\int_{\Omega^+} |\nabla z_0|^2 dxdy + s_0 > b$ and condition (5.5) holds, then $(u_*(t), s_0)$ does not satisfy the equilibrium condition (b) in Definition 2.1. This contradiction shows that s cannot be constantly equal to s_0.

The construction of u_* requires a careful analysis of the properties of the solutions of some auxiliary minimum problems. Due to the symmetry of the data we shall work in Ω^+. This is justified by the following remark.

Remark 5.3 Since $w(t)$ is odd with respect to y, a function $u_*\colon [0, T] \to H^1(\Omega \setminus \Gamma)$ is a quasistatic evolution with fixed crack and boundary condition w if and only if it is odd with respect to y and satisfies the following conditions

(a_0) (measurability) $u_*\colon [0, T] \to H^1(\Omega^+)$ is measurable;
(b_0) (equilibrium) for every $t \in [0, T]$ we have $u_*(t) = t$ on $\partial_D^+ \Omega$, and

$$\frac{1}{2} \int_{\Omega^+} |\nabla u_*(t)|^2 dxdy \le \frac{1}{2} \int_{\Omega^+} |\nabla \hat{u}|^2 dxdy + \int_{\Gamma_{s_0}^b} |\hat{u}^+ - u_*^+(t)| dx \tag{5.6}$$

for every $\hat{u} \in H^1(\Omega^+)$ with $\hat{u} = t$ on $\partial_D^+ \Omega$.
(c_0) (energy-dissipation inequality) for every $t_1, t_2 \in [0, T]$, with $t_1 < t_2$, we have

$$\frac{1}{2} \int_{\Omega^+} |\nabla u_*(t_2)|^2 dxdy + \int_{\Gamma_{s_0}^b} |u_*^+(t_2) - u_*^+(t_1)| dx$$
$$\le \frac{1}{2} \int_{\Omega^+} |\nabla u_*(t_1)|^2 dxdy + \int_{t_1}^{t_2} \Big(\int_{\Omega^+} \frac{\partial u_*(\tau)}{\partial y} dxdy \Big) d\tau.$$

Indeed, the oddness of $u_*(t)$ with respect to y follows from the uniqueness of the solutions of problems of the form (3.3) and from the oddness of the data.

To prove (5.5) we need a detailed study of the properties of the solutions of (5.6), which uses the Euler conditions introduced in Lemmas 3.1–3.3. This analysis requires the results of the following two lemmas, which give a precise description of the singularities of some solutions of the Laplace equation with suitable boundary conditions.

For every $R > 0$ let $\Gamma_R^- = (-R, 0) \times \{0\}$, and $\Gamma_R^+ = (0, R) \times \{0\}$. In the next lemmas we identify the point (x, y) with the complex number $z = x + iy$.

Lemma 5.4 *Let $R > 0$ and let $u \in H^1(B_R^+)$ be such that $\Delta u = 0$ in B_R^+, $\frac{\partial u}{\partial y} = 0$ on Γ_R^-, and $u = 0$ on Γ_R^+. Let S_0 be defined by*

$$S_0(z) := \mathfrak{Im}(\sqrt{z}),$$

where for $y \ge 0$ we use the determination of $\sqrt{z}$ such that $\sqrt{-1} = i$. Then

$$u = cS_0 + u_{reg} \tag{5.7}$$

for some $c \in \mathbb{R}$ *and* $u_{reg} \in C^1(\overline{B}_r^+)$ *for every* $0 < r < R$.

Proof Using Schwarz symmetrization principle we may assume that u is harmonic in $B_R \setminus \Gamma_R^-$ and satisfies the homogeneous Neumann boundary condition on both sides of Γ_R^-. By using the conformal map $z \mapsto \sqrt{z}$ we can write

$$u(z) = v(\sqrt{z}), \tag{5.8}$$

where v is harmonic on $B_{\sqrt{R}} \cap \{(x, y) : x > 0\}$, belongs to $H^1(B_{\sqrt{R}} \cap \{(x, y) : x > 0\})$, and satisfies $\frac{\partial v}{\partial \nu} = 0$ on $\{(0, y) : -\sqrt{R} < y < \sqrt{R}\}$ and $v = 0$ on $\{(x, 0) : 0 < x < \sqrt{R}\}$. We now extend v to the whole ball $B_{\sqrt{R}}$ by reflection and we obtain a function, still denoted by v, which is harmonic on $B_{\sqrt{R}}$ and satisfies $v = 0$ on $\{(x, 0) : -\sqrt{R} < x < \sqrt{R}\}$.

Therefore, there exists a holomorphic function f defined on $B_{\sqrt{R}}$ such that

$$v(z) = \mathfrak{Im} f(z) \quad \text{for every } z \in B_{\sqrt{R}}. \tag{5.9}$$

We may assume $f(0) = 0$. Since f is real on the real axis we can write

$$f(z) = \sum_{k=1}^{\infty} a_k z^k,$$

where $a_k \in \mathbb{R}$ and the series converges uniformly on compact subsets of $B_{\sqrt{R}}$. Let g be the holomorphic function on $B_{\sqrt{R}}$ defined by

$$g(z) = \sum_{k=2}^{\infty} a_k z^k,$$

Therefore (5.8) and (5.9) imply (5.7) with $c = a_1$ and

$$u_{reg}(z) = \mathfrak{Im}\big(g(\sqrt{z})\big). \tag{5.10}$$

Let us fix $0 < r < R$. It remains to prove that $u_{reg} \in C^1(\overline{B}_r^+)$. Since

$$\nabla u_{reg}(x, y) = \Big(\mathfrak{Im}\big(\frac{g'(\sqrt{z})}{2\sqrt{z}}\big), \mathfrak{Re}\big(\frac{g'(\sqrt{z})}{2\sqrt{z}}\big)\Big) \tag{5.11}$$

it is enough to prove that

$$z \mapsto \frac{g'(\sqrt{z})}{\sqrt{z}} \tag{5.12}$$

is continuous on $\overline{B}_r^+$. Since

$$g'(z) = \sum_{k=2}^{\infty} k a_k z^{k-1}, \tag{5.13}$$

the function $h(z) := g'(z)/z$ is holomorphic on $B_{\sqrt{R}}$. Therefore we have $g'(\sqrt{z})/\sqrt{z} = h(\sqrt{z})$, which gives the continuity of (5.12) and concludes the proof. □

Lemma 5.5 *Let* $R > 0$ *and let* $u \in H^1(B_R^+)$ *be such that* $\Delta u = 0$ *in* B_R^+, $\frac{\partial u}{\partial y} = 0$ *on* Γ_R^-, *and* $\frac{\partial u}{\partial y} = 1$ *on* Γ_R^+. *Let* S_1 *be defined by*

$$S_1(z) := \frac{1}{\pi} \mathfrak{Re}(z \log(-z)).$$

Then $u = S_1 + u_{reg}$ *with* $u_{reg} \in C^\infty(\overline{B}_r^+)$ *for every* $r < R$.

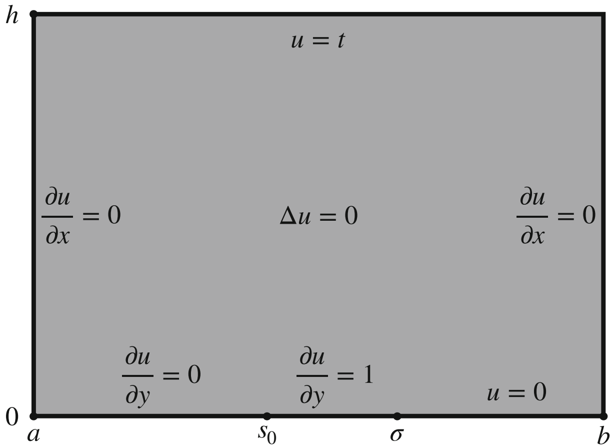

Fig. 3 The boundary value problem for u^t_σ

Proof By direct computation we see that $S_1 \in H^1(B^+_R)$, it is harmonic on B^+_R and satisfies the boundary conditions $\frac{\partial S_1}{\partial y} = 0$ on Γ^-_R, and $\frac{\partial S_1}{\partial y} = 1$ on Γ^+_R. Therefore $u_{reg} := u - S_1 \in H^1(B^+_R)$, it is harmonic and satisfies the homogeneous Neumann boundary condition on $\Gamma^-_R \cup \Gamma^+_R$, and hence on $(-R, R) \times \{0\}$. The conclusion follows from the regularity theory for elliptic equations with Neumann boundary condition. □

The quasistatic evolution $u_*(t)$ will be constructed by using the solutions of some auxiliary boundary value problems depending on a parameter σ, and then by choosing a particular value σ_t of this parameter. For every $t \geq 0$ and for every $\sigma \in [s_0, b]$ we consider the solution $u^t_\sigma \in H^1(\Omega^+)$ of the problem (see Fig. 3)

$$\begin{cases} \Delta u = 0 & \text{in } \Omega^+, \\ u = t & \text{if } y = h, \\ \frac{\partial u}{\partial x} = 0 & \text{if } x = a \text{ or } x = b, \\ \frac{\partial u}{\partial y} = 0 & \text{for } a < x < s_0 \text{ and } y = 0, \\ \frac{\partial u}{\partial y} = 1 & \text{for } s_0 < x < \sigma \text{ and } y = 0, \\ u = 0 & \text{for } \sigma < x < b \text{ and } y = 0. \end{cases} \tag{5.14}$$

By the continuous dependence on the data, the function u^t_σ is continuous in $H^1(\Omega^+)$ with respect to t and σ.

Remark 5.6 In the particular case $\sigma = b$ we have $u^t_b = t + z_0$ where $z_0 \in H^1(\Omega^+)$ is the solution of (5.3).

The following two lemmas give some important properties of u^t_σ, which will be used in our construction of $u_*(t)$.

Lemma 5.7 *For every $t \geq 0$ and $\sigma \in [s_0, b]$ we have $u^t_\sigma \in C^\infty(\overline{\Omega}^+ \setminus \{(s_0, 0), (\sigma, 0)\}) \cap C^0(\overline{\Omega}^+)$.*

Proof The result follows from the regularity theory for elliptic equations; the regularity near the vertices of the rectangle can be easily obtained by extending u^t_σ through a suitable reflection, while the continuity at the points $(s_0, 0)$ and $(\sigma, 0)$ follows from Lemmas 5.4 and 5.5. □

Lemma 5.8 *Let $t \geq 0$ and let $\sigma \in [s_0, b]$ be such that $u^t_\sigma \geq 0$ in Ω^+. Then*

$$\frac{\partial u^t_\sigma}{\partial x} \leq 0 \quad in\ \Omega^+. \tag{5.15}$$

Proof Let $v := \dfrac{\partial u^t_\sigma}{\partial x}$. By Lemma 5.7 we have that $v \in C^\infty(\overline{\Omega}^+ \setminus \{(s_0, 0), (\sigma, 0)\})$ and satisfies

$$\begin{cases} \Delta v = 0 & \text{in } \Omega^+, \\ v = 0 & \text{if } y = h, \\ v = 0 & \text{if } x = a \text{ or } x = b, \\ \frac{\partial v}{\partial y} = 0 & \text{if } a < x < s_0 \text{ and } y = 0, \\ \frac{\partial v}{\partial y} = 0 & \text{if } s_0 < x < \sigma \text{ and } y = 0, \\ v = 0 & \text{if } \sigma < x < b \text{ and } y = 0. \end{cases} \tag{5.16}$$

Case $s_0 = \sigma$. Let us consider the behaviour of the function u^t_σ near $(s_0, 0)$. By Lemma 5.4 we can write

$$u^t_\sigma(x, y) = c\sqrt{\rho}\sin(\theta/2) + u_{reg}(x, y) \tag{5.17}$$

for some constant c and some function $u_{reg} \in C^1(\overline{\Omega}^+)$, where ρ, θ are polar coordinates around $(s_0, 0)$, with $\theta \in [0, \pi]$. We observe that $0 = u^t_\sigma(x, 0) = u_{reg}(x, 0)$ for every $s_0 < x < b$. This implies that $u_{reg}(s_0, 0) = 0$ and $\frac{\partial u_{reg}}{\partial x}(s_0, 0) = 0$.

By (5.17) we have $u^t_\sigma(x, 0) = c\sqrt{s_0 - x} + u_{reg}(x, 0)$ for every $a < x < s_0$, while the properties of u_{reg} imply that $|u_{reg}(x, 0)| \leq M|x - s_0|$ for a suitable constant M. Hence the inequality $c < 0$ would lead to $u^t_\sigma(x, 0) < 0$ for $x < s_0$, x close to s_0, in contradiction with the assumption $u^t_\sigma(x, 0) \geq 0$. This shows that $c \geq 0$.

Since

$$v(x, y) = \frac{\partial u^t_\sigma}{\partial x}(x, y) = -\frac{c}{2\sqrt{\rho}}\sin(\theta/2) + \frac{\partial u_{reg}}{\partial x}(x, y), \tag{5.18}$$

we have

$$\limsup_{(x,y)\to(s_0,0)} v(x, y) \leq 0.$$

Therefore, if v is positive at some point of Ω^+, by the maximum principle v attains its maximum on $\overline{\Omega}^+$ at a point of $\partial\Omega^+ \setminus \{(s_0, 0)\}$ where v has a positive value. By (5.16) this point must be of the form $(x_0, 0)$ with $a < x_0 < s_0$. By the Hopf Maximum Principle we should have $\frac{\partial v}{\partial y}(x_0, 0) < 0$, which contradicts (5.16). This shows that we must have $v \leq 0$ in Ω^+.

Case $s_0 < \sigma < b$. We have to study the behaviour of the function u^t_σ near the points $(s_0, 0)$ and $(\sigma, 0)$. By Lemma 5.5 and (5.14), near $(s_0, 0)$ we have

$$u^t_\sigma(x, y) = \frac{1}{\pi}\big((x - s_0)\log\rho - y(\theta - \pi)\big) + u^{reg}_{s_0}(x, y), \tag{5.19}$$

where $u_{s_0}^{reg}$ is C^∞ in a neighbourhood of $(s_0, 0)$ in $\overline{\Omega}^+$ and ρ, θ are polar coordinates around $(s_0, 0)$, with $\theta \in [0, \pi]$. Hence

$$v(x, y) = \frac{1}{\pi}(\log \rho + 1) + \frac{\partial u_{s_0}^{reg}}{\partial x}(x, y), \tag{5.20}$$

and this implies that

$$\lim_{(x,y)\to(s_0,0)} v(x, y) = -\infty. \tag{5.21}$$

By Lemma 5.4 and (5.14), using polar coordinates r, ϕ around $(\sigma, 0)$, with $\phi \in [0, \pi]$, we can write

$$u_\sigma^t(x, y) = c\sqrt{r}\sin(\phi/2) + u_\sigma^{reg}(x, y), \tag{5.22}$$

where $c \in \mathbb{R}$ and u_σ^{reg} is C^1 in a neighbourhood of $(\sigma, 0)$ in $\overline{\Omega}^+$. This gives

$$v(x, y) = -\frac{c}{2\sqrt{r}}\sin(\phi/2) + \frac{\partial u_\sigma^{reg}}{\partial x}(x, y). \tag{5.23}$$

Arguing as in the case $s_0 = \sigma$ we can prove that $c \geq 0$ and that $u_\sigma^{reg}(x, 0) = 0$ for every $\sigma < x < b$. Since $\frac{\partial u_\sigma^{reg}}{\partial x}(\sigma, 0) = 0$ we have

$$\limsup_{(x,y)\to(\sigma,0)} v(x, y) \leq 0. \tag{5.24}$$

By (5.21) and (5.24), the subharmonic function $v \vee 0$ can be extended to a continuous function on $\overline{\Omega}^+$ which satisfies

$$(v \vee 0)(s_0, 0) = (v \vee 0)(\sigma, 0) = 0. \tag{5.25}$$

Therefore, if v is positive at some point of Ω^+, by the maximum principle for subharmonic functions $v \vee 0$ attains its maximum on $\overline{\Omega}^+$ at a point of $\partial\Omega^+$ where v has a positive value. By (5.16) and (5.25) this point must be of the form $(x_0, 0)$ with $a < x_0 < s_0$ or $s_0 < x_0 < \sigma$. By the Hopf Maximum Principle we should have $\frac{\partial v}{\partial y}(x_0, 0) < 0$, which contradicts (5.16). This concludes the proof of (5.15) for $s_0 < \sigma < b$.

Case $\sigma = b$. In this case the only singular point of v is $(s_0, 0)$ and we can repeat the argument of the previous case with obvious simplifications. □

For $t \geq 0$ we define

$$\sigma_t := \max\{\sigma \in [s_0, b] : u_\sigma^t \geq 0 \text{ in } \Omega^+\} \quad \text{and} \quad u_*(t) := u_{\sigma_t}^t. \tag{5.26}$$

The existence of the maximum follows easily from the continuous dependence of u_σ^t on σ. It is easy to see that for $t = 0$ we have $\sigma_0 = s_0$.

The results of following three lemmas will be used to prove Lemma 5.12, which shows that $u_*(t)$ is a quasistatic evolution.

Lemma 5.9 *Let* $t \geq 0$. *Then*

$$0 \leq \frac{\partial u_*(t)}{\partial y}(x, 0) \leq 1 \tag{5.27}$$

for every $\sigma_t < x < b$.

Proof It is not restrictive to assume $\sigma_t < b$. By (5.26) and Lemma 5.8 for every $y \in (0, h)$ the function $x \mapsto u_*(t)(x, y)$ is nonnegative and nonincreasing in (a, b). Since $u_*(t)(x, 0) = 0$ for $x \in (\sigma_t, b)$, the function $x \mapsto (u_*(t)(x, y) - u_*(t)(x, 0))/y$ is nonnegative and nonincreasing in (σ_t, b) for every $y \in (0, h)$. Taking the limit as $y \to 0+$ we deduce that $x \mapsto \frac{\partial u_*(t)}{\partial y}(x, 0)$ is nonnegative and nonincreasing in (σ_t, b).

It remains to prove the second inequality in (5.27). If it is not satisfied, by the monotonicity of $x \mapsto \frac{\partial u_*(t)}{\partial y}(x, 0)$ there exists $\varepsilon \in (0, b - \sigma_t)$ such that

$$\frac{\partial u_*(t)}{\partial y}(x, 0) > 1 \quad \text{for every } x \in (\sigma_t, \sigma_t + \varepsilon).$$

Let $\sigma \in (\sigma_t, \sigma_t + \varepsilon)$. We want to prove that $u^t_\sigma \geq u^t_{\sigma_t}$ in Ω^+. Setting $v := u^t_\sigma - u^t_{\sigma_t}$ we have that $v \in H^1(\Omega^+)$ and satisfies

$$\begin{cases} \Delta v = 0 & \text{in } \Omega^+, \\ v = 0 & \text{if } y = h, \\ \frac{\partial v}{\partial x} = 0 & \text{if } x = a \text{ or } x = b, \\ \frac{\partial v}{\partial y} = 0 & \text{if } a < x < s_0 \text{ and } y = 0, \\ \frac{\partial v}{\partial y} = 0 & \text{if } s_0 < x < \sigma_t \text{ and } y = 0, \\ \frac{\partial v}{\partial y} < 0 & \text{if } \sigma_t < x < \sigma \text{ and } y = 0, \\ v = 0 & \text{if } \sigma < x < b \text{ and } y = 0. \end{cases}$$

Integrating by parts we obtain the weak formulation

$$\int_{\Omega^+} \nabla v \nabla \varphi \, dx dy = - \int_{\Gamma^\sigma_{\sigma_t}} \frac{\partial v}{\partial y} \varphi^+ dx$$

for every $\varphi \in H^1(\Omega^+)$ with $\varphi = 0$ on $\Gamma^b_\sigma \cup \partial^+_D \Omega$. Taking $\varphi := v \wedge 0$ we obtain

$$\int_{\Omega^+} |\nabla(v \wedge 0)|^2 = \int_{\Omega^+} \nabla v \nabla(v \wedge 0) \, dx dy = - \int_{\Gamma^\sigma_{\sigma_t}} \frac{\partial v}{\partial y}(v^+ \wedge 0) dx \leq 0,$$

which gives $\nabla(v \wedge 0) = 0$. Taking into account the boundary condition $v = 0$ on $\partial^+_D \Omega$ we get $v \wedge 0 = 0$ in Ω^+. This implies $v \geq 0$, so that $u^t_\sigma \geq u^t_{\sigma_t}$ in Ω^+. Therefore $u^t_\sigma \geq 0$ in Ω^+, which contradicts the maximality of σ_t (see (5.26)), thus concluding the proof of the second inequality in (5.27). □

Lemma 5.10 *For every* $0 \leq t_1 \leq t_2$ *we have* $u_*(t_1) \leq u_*(t_2)$ *in* Ω^+.

Proof Let us fix $0 \leq t_1 \leq t_2$. By the maximum principle we have

$$u_*(t_1) = u^{t_1}_{\sigma_{t_1}} \leq u^{t_2}_{\sigma_{t_1}} \quad \text{in } \Omega^+. \tag{5.28}$$

By (5.26) this implies $\sigma_{t_1} \leq \sigma_{t_2}$. Let $v := u_*(t_2) - u^{t_2}_{\sigma_{t_1}} = u^{t_2}_{\sigma_{t_2}} - u^{t_2}_{\sigma_{t_1}} \in H^1(\Omega^+)$. By (5.26) we have $u^{t_2}_{\sigma_{t_2}}(x, 0) \geq 0$ for $x \in (\sigma_{t_1}, \sigma_{t_2})$, while by the last line in (5.14) we have $u^{t_2}_{\sigma_{t_1}}(x, 0) = 0$ for $x \in (\sigma_{t_1}, \sigma_{t_2})$. Hence $v(x, 0) \geq 0$ for $x \in (\sigma_{t_1}, \sigma_{t_2})$. Thus v satisfies

$$\begin{cases} \Delta v = 0 & \text{in } \Omega^+ \\ v = 0 & \text{if } y = h, \\ \frac{\partial v}{\partial x} = 0 & \text{if } x = a \text{ or } x = b, \\ \frac{\partial v}{\partial y} = 0 & \text{if } a < x < s_0 \text{ and } y = 0, \\ \frac{\partial v}{\partial y} = 0 & \text{if } s_0 < x < \sigma_{t_1} \text{ and } y = 0, \\ v \geq 0 & \text{if } \sigma_{t_1} < x < \sigma_{t_2} \text{ and } y = 0, \\ v = 0 & \text{if } \sigma_{t_2} < x < b \text{ and } y = 0. \end{cases}$$

By using the Maximum Principle (see also the proof of Lemma 5.9) we can prove that $v \geq 0$ in Ω^+. Together with (5.28), this concludes the proof. □

Lemma 5.11 *For every* $0 \leq t_1 \leq t_2$ *the function* $u_*(t_2)$ *is the solution of the minimum problem*

$$\min_{\substack{u \in H^1(\Omega^+) \\ u = t_2 \text{on } \partial_D^+ \Omega}} \Big(\frac{1}{2} \int_{\Omega^+} |\nabla u|^2 dx dy + \int_{\Gamma_{s_0}^b} |u^+ - u_*(t_1)^+| dx \Big). \tag{5.29}$$

Proof Let us fix $0 \leq t_1 \leq t_2$. By (5.14), (5.26), and Lemma 5.9 the function $u_*(t_2)$ satisfies

$$\left| \frac{\partial u_*(t_2)}{\partial y}(x, 0) \right| \leq 1 \quad \text{for every } x \in (s_0, b) \setminus \{\sigma_{t_2}\}.$$

Moreover, by (5.14) we have

$$\frac{\partial u_*(t_2)}{\partial y}(x, 0) = 1 \quad \text{for every } x \in (s_0, b) \text{ such that } u_*(t_2)(x, 0) > u_*(t_1)(x, 0),$$

since $\{x \in (s_0, b) : u_*(t_2)(x, 0) > u_*(t_1)(x, 0)\} \subset \{x \in (s_0, b) : u_*(t_2)(x, 0) > 0\} \subset (s_0, \sigma_{t_2})$. By Lemma 3.3 (applied to the odd extension of $u_*(t_2)$ to $\Omega \setminus \Gamma$) these properties of $\frac{\partial u_*(t_2)}{\partial y}$ on Γ, together with the boundary conditions of (5.14), imply that $u_*(t_2)$ is the solution of (5.29). □

Lemma 5.12 *The odd extension to* $\Omega \setminus \Gamma$ *of the function* u_* *defined by* (5.26) *is a quasistatic evolution with fixed crack and boundary condition* w *on each interval* $[0, \hat{T}]$ *with* $\hat{T} > 0$.

Proof Let us fix $\hat{T} > 0$. By Lemma 5.11 for every t the odd extension of $u_*(t)$ to $\Omega \setminus \Gamma$ is the solution of the minimum problem (2.23) with $w = w(t)$, with $w(t)$ defined by (5.2), $s = s_0$, and $g = 0$. From Lemma 2.8 we deduce that $u_* : [0, \hat{T}] \to H^1(\Omega^+)$ is continuous.

To conclude the proof we have to show that $u_*(t)$ satisfies also conditions (b$_0$) and (c$_0$) in Remark 5.3. Condition (b$_0$) follows from Lemma 5.11. To prove (c$_0$) we fix $\tau_1, \tau_2 \in [0, \hat{T}]$, with $\tau_1 < \tau_2$ and a sequence of subdivisions $\tau_1 = t_0^k < t_1^k < \cdots < t_{m_k}^k = \tau_2$ such that $t_j^k - t_{j-1}^k \leq \frac{1}{k}$ for every $1 \leq j \leq m_k$. By Lemma 5.11 for every j we have

$$\begin{aligned} &\frac{1}{2} \int_{\Omega^+} |\nabla u_*(t_j^k)|^2 dx dy + \int_{\Gamma_{s_0}^b} |u_*(t_j^k)^+ - u_*(t_{j-1}^k)^+| dx \\ &\leq \frac{1}{2} \int_{\Omega^+} |\nabla \hat{u}|^2 dx dy + \int_{\Gamma_{s_0}^b} |\hat{u}^+ - u_*(t_{j-1}^k)^+| dx \end{aligned}$$

for every $\hat{u} \in H^1(\Omega^+)$ with $\hat{u} = t_j^k$ on $\partial_D^+ \Omega$. Taking $\hat{u} = u_*(t_{j-1}^k) + w(t_j^k) - w(t_{j-1}^k)$, where w is defined by (5.2), and using Lemma 5.10 we obtain

$$\frac{1}{2}\int_{\Omega^+}|\nabla u_*(t_j^k)|^2dxdy+\int_{\Gamma_{s_0}^b}\big(u_*(t_j^k)^+-u_*(t_{j-1}^k)^+\big)dx$$
$$\leq\frac{1}{2}\int_{\Omega^+}|\nabla u_*(t_{j-1}^k)|^2dxdy+(t_j^k-t_{j-1}^k)\int_{\Omega^+}\frac{\partial u_*(t_{j-1}^k)}{\partial y}dxdy+\frac{1}{2}(t_j^k-t_{j-1}^k)^2\frac{b-a}{h}.$$

Summing for $j=1,\dots,m_k$ we obtain

$$\frac{1}{2}\int_{\Omega^+}|\nabla u_*(\tau_2)|^2dxdy+\int_{\Gamma_{s_0}^b}\big(u_*(\tau_2)^+-u_*(\tau_1)^+\big)dx$$
$$\leq\frac{1}{2}\int_{\Omega^+}|\nabla u_*(\tau_1)|^2dxdy+\sum_{j=1}^{m_k}(t_j^k-t_{j-1}^k)\int_{\Omega^+}\frac{\partial u_*(t_{j-1}^k)}{\partial y}dxdy+\frac{1}{2}\frac{b-a}{kh}.$$

Since $u_*\colon[0,\hat{T}]\to H^1(\Omega^+)$ is continuous, taking the limit as $k\to\infty$ we obtain (c$_0$). □

The following lemma will be used to prove Lemma 5.14, which shows that (5.5) holds.

Lemma 5.13 *Let $z_0\in H^1(\Omega^+)$ be the solution of problem (5.3). Then for every $t\geq 0$ we have $u_*(t)\geq t+z_0$ in Ω^+.*

Proof Let us fix $t\geq 0$ and let $v:=u_*(t)-(t+z_0)$. Then $v\in H^1(\Omega^+)$ and, by Lemma 5.9, it satisfies

$$\begin{cases}\Delta v=0 & \text{in } \Omega^+,\\ v=0 & \text{if } y=h,\\ \frac{\partial v}{\partial x}=0 & \text{if } x=a \text{ or } x=b,\\ \frac{\partial v}{\partial y}=0 & \text{if } a<x<s_0 \text{ and } y=0,\\ \frac{\partial v}{\partial y}=0 & \text{if } s_0<x<\sigma_t \text{ and } y=0,\\ \frac{\partial v}{\partial y}\leq 0 & \text{if } \sigma_t<x<b \text{ and } y=0.\end{cases}$$

Integrating by parts we obtain the weak formulation

$$\int_{\Omega^+}\nabla v\nabla\varphi\,dxdy=-\int_{\Gamma_{\sigma_t}^b}\frac{\partial v}{\partial y}\varphi^+dx$$

for every $\varphi\in H^1(\Omega^+)$ with $\varphi=0$ on $\partial_D^+\Omega$. Taking $\varphi:=v\wedge 0$ and arguing as in Lemma 5.9 we can prove that $v\geq 0$ in Ω^+. Hence $u_*(t)\geq t+z_0$ in Ω^+. □

Lemma 5.14 *Let $T_*:=-\inf_{\Omega^+}z_0$, where z_0 is the solution of (5.3). If $t>T_*$, then $u_*(t)=t+z_0$ in Ω^+.*

Proof Let us fix $t>T_*$. By Lemma 5.13 there exists $\eta>0$ such that $u_{\sigma_t}^t=u_*(t)\geq t+z_0\geq\eta$ in Ω^+, hence $u_{\sigma_t}^t(x,0)\geq\eta$ for every $x\in(\sigma_t,b)$. Since $u_{\sigma_t}^t(x,0)=0$ for every $x\in(\sigma_t,b)$ by the last condition in (5.14), we deduce that $\sigma_t=b$. The conclusion follows from Remark 5.6. □

Proof of Theorem 5.1 By Theorem 4.1 and Remark 4.2 we have $s(t)=s_1=s_0$ for every $t\in[0,t_1)$. Moreover, if $t_1<T$ we have also $s(t)=s_2$ for every $t\in(t_1,t_2)$. To prove the theorem it is enough to show that $t_1<T$.

Assume, by contradiction, that $t_1=T$. Since $s(t)=s_0$ for every $t\in[0,T)$, the function u is a quasistatic evolution with fixed crack and boundary condition w on each interval $[0,\hat{T}]$

with $0 < \hat{T} < T$. Let u_* be the function defined by (5.26). By Lemma 5.12 the odd extension (with respect to y) of u_* is a quasistatic evolution with fixed crack and boundary condition w on each interval $[0, \hat{T}]$ with $\hat{T} > 0$. Since $u(0) = 0 = u_*(0)$, by the uniqueness result proved in Theorem 4.17 we have $u(t) = u_*(t)$ for every $t \in [0, T)$. Let us fix $t \in (T_*, T)$. By Lemma 5.14 we have $u_*(t) = t + z_0$ in Ω^+, which implies (5.5). Taking $\hat{s} = b$, $\hat{u} = t$ in Ω^+, and $\hat{u} = -t$ in Ω^- in the equilibrium condition (b) of Definition 2.1, we obtain

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla z_0|^2 dxdy + s_0 \leq b,$$

which contradicts the second inequality in (5.4). This proves that $t_1 < T$ and concludes the proof of the theorem. □

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