



Fully anisotropic elliptic problems with minimally integrable data

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Abstract

We investigate nonlinear elliptic Dirichlet problems whose growth is driven by a general anisotropic N -function, which is not necessarily of power-type and need not satisfy the Δ_2 nor the ∇_2 -condition. Fully anisotropic, non-reflexive Orlicz–Sobolev spaces provide a natural functional framework associated with these problems. Minimal integrability assumptions are detected on the datum on the right-hand side of the equation ensuring existence and uniqueness of weak solutions. When merely integrable, or even measure, data are allowed, existence of suitably further generalized solutions—in the approximable sense—is established. Their maximal regularity in Marcinkiewicz-type spaces is exhibited as well. Uniqueness of approximable solutions is also proved in case of L^1 -data.

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1 Introduction

This paper concerns Dirichlet problems for elliptic equations of the form

$$\begin{cases} -\operatorname{div} a(x, \nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open set in \mathbb{R}^n , $n \geq 2$, $a : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a Carathéodory function and the function $f : \Omega \rightarrow \mathbb{R}$ is assigned.

Second-order elliptic equations, in divergence form, are a very classical theme in the theory of partial differential equations, and have been extensively investigated in the literature. The punctum of the present contribution is in that, besides the standard monotonicity assumption

$$(a(x, \xi) - a(x, \eta)) \cdot (\xi - \eta) > 0 \quad \text{for every } \xi, \eta \in \mathbb{R}^n \text{ such that } \xi \neq \eta, \tag{1.2}$$

for a.e. $x \in \Omega$, the function a is subject to very general coercivity and growth conditions, that embrace and considerably extend customary instances. The leading hypotheses on a amount to requiring that there exists a (possibly fully anisotropic) N -function $\Phi : \mathbb{R}^n \rightarrow [0, \infty)$ such that, for a.e. $x \in \Omega$,

$$a(x, \xi) \cdot \xi \geq \Phi(\xi) \quad \text{for every } \xi \in \mathbb{R}^n, \tag{1.3}$$

and

$$\tilde{\Phi}(c_\Phi a(x, \xi)) \leq \Phi(\xi) + h(x) \quad \text{for every } \xi \in \mathbb{R}^n \tag{1.4}$$

for some positive constant c_Φ and some nonnegative function $h \in L^1(\Omega)$. Here, $\tilde{\Phi}$ denotes the Young conjugate of Φ . Of course, there is no loss of generality in assuming that $c_\Phi \in (0, 1)$. In particular, condition (1.4) is fulfilled if $a(x, \xi)$ satisfies the stronger inequality obtained on replacing the left-hand side of (1.4) by $c_\Phi \tilde{\Phi}(a(x, \xi))$.

An N -function is an even convex function, vanishing at zero, decaying faster than linearly near zero and growing faster than linearly near infinity. Its Young conjugate is also an N -function and comes into play in an Hölder-type inequality for the Orlicz norm defined in terms of Φ . Precise definitions of N -function and Young conjugate can be found in the next

section, where a number of notions and properties concerning the unconventional functional framework associated with our analysis are recalled or proved.

Let us just stress here that $\Phi(\xi)$ does not have to depend on ξ just through its length $|\xi|$, thus allowing for full anisotropy in the differential operator. Moreover, in contrast with the assumptions imposed on p -Laplace-type equations, Φ need not have a polynomial growth. In fact, Φ is not even supposed to fulfill the so-called Δ_2 -condition, nor the ∇_2 -condition, that are usually required as a replacement for homogeneity of Φ . The lack of these conditions on Φ results in the non-reflexivity and non-separability of the Orlicz–Sobolev space $W_0^1 L^\Phi(\Omega)$ built upon Φ , a natural function space associated with problem (1.1).

We are concerned with existence, uniqueness and regularity of solutions to the Dirichlet problem (1.1). Our analysis initiates by discussing weak solutions to (1.1), namely solutions u that belong to the Orlicz–Sobolev space $W_0^1 L^\Phi(\Omega)$, or, more precisely, to the corresponding Orlicz–Sobolev class. Due to the generality of the situation under consideration and, specifically, to the anisotropy and non-reflexivity of the involved function spaces, standard methods do not apply. Our approach combines various techniques, including approximation via isotropic operators, comparison with solutions to symmetrized problems, the use of sharp embedding theorems for Orlicz–Sobolev spaces. This enables us to exhibit an optimal integrability assumption on the datum f , depending on the growth of Φ near infinity, for the existence of a (unique) weak solution to problem (1.1). The relevant optimal assumption on f amounts to its membership in a space of Orlicz–Lorentz-type, which arises as an associate space of the optimal rearrangement-invariant target space in an anisotropic Orlicz–Sobolev embedding. This is the content of Theorem 3.2. Let us emphasize that this result is new even in the isotropic case, that is when Φ is a radial function.

When f is affected by poor integrability properties, existence of weak solutions to problem (1.1) is not guaranteed. This is well known even in the linear situation when the differential operator is the Laplacian. In particular, solutions that do exist in a yet weaker sense—for instance, merely distributional solutions—typically do not belong to the pertaining Sobolev space. Also, they need not be unique, as shown in [53].

In this connection, after disposing the issue of existence of weak solutions, we drop any extra regularity on f besides plain integrability in Ω , and address the question of existence of solutions to the Dirichlet problem (1.1) in a suitably generalized sense. Our result with this regard is stated in Theorem 3.7. Under the mere assumption that $f \in L^1(\Omega)$, it asserts the existence and uniqueness of solutions, called *approximable solutions* throughout, that are limits of weak solutions to approximating problems with regular right-hand sides. Importantly, Theorem 3.7 also provides us with maximal regularity of the solution u and of its gradient ∇u . Such a regularity is properly described in terms of Marcinkiewicz-type spaces, depending on Φ . An anisotropic Orlicz–Sobolev embedding, with optimal Orlicz target space, is critical in dictating the form of these Marcinkiewicz-type spaces.

Our approach to problem (1.1) with right-hand side in $L^1(\Omega)$ carries over, in fact, to the case when f is replaced by a measure with finite total variation in Ω . The relevant result is stated in Theorem 3.10. Let us point out that, though existence and regularity of solutions hold exactly under the same conditions as for data in $L^1(\Omega)$, their uniqueness is uncertain. As far as we know, this is an open problem even in case of standard isotropic nonlinear operators, such as the p -Laplacian.

The literature on elliptic equations, under such a broad ellipticity condition as that defined in terms of N -functions Φ , is quite limited—see e.g. [2–5, 19, 24, 39, 40]. Our results answer some questions in their general theory, and provide a unified framework for results available for functions Φ of special forms.

So-called operators with p -growth, modelled upon the p -Laplacian, correspond to the choice

$$\Phi(\xi) = |\xi|^p \quad \text{for } \xi \in \mathbb{R}^n, \tag{1.5}$$

with $p > 1$. The theory of equations governed by this kind of nonlinearity has been thoroughly developed since the sixties of the last century. The analysis of solutions that are well suited to allow for right-hand sides in L^1 is more recent. Their systematic study was initiated with the papers [14,45]. Other contributions in this direction include [6,7,11,28,29,36,50].

Existence and sharp regularity results for equations with non-polynomial growth and L^1 or measure data, but still in the isotropic and reflexive setting where

$$\Phi(\xi) = A(|\xi|) \quad \text{for } \xi \in \mathbb{R}^n \tag{1.6}$$

for some classical N -function of one variable satisfying both the Δ_2 and ∇_2 -condition, are presented in [27]. Previous researches along this direction can be found in [12,32,33]. Results concerning this kind of ellipticity, but involving more regular operators a , or right-hand sides f enjoying stronger integrability properties, are the subject of [9,10,20–22,31,37,38,43,44,49,58].

Elliptic problems with growth of the form

$$\Phi(\xi) = \sum_{i=1}^n |\xi_i|^{p_i} \quad \text{for } \xi \in \mathbb{R}^n, \tag{1.7}$$

where $\xi = (\xi_1, \dots, \xi_n)$, $1 < p_i < \infty$, $i = 1, \dots, n$, provide a basic framework for physical models in the presence of anisotropies. They are the topic of diverse contributions, including [17,18,35,41,46,57,61]. The case of L^1 right-hand sides was considered in [15] under some restrictions on the exponents p_i . Note that functions as in (1.7) are particular examples of those given by

$$\Phi(\xi) = \sum_{i=1}^n A_i(|\xi_i|) \quad \text{for } \xi \in \mathbb{R}^n, \tag{1.8}$$

where A_i are N -functions of one variable, which fall within the frames of the present discussion.

As an application of Theorems 3.2, 3.7 and 3.10, stated in Sect. 3, optimal results are offered in the specific instances mentioned above. However, let us again emphasize that our discussion covers more general situations than those described so far and, importantly, allows for functions Φ that do not necessarily admit the split form (1.8). Examples which generalize one from [60] are provided by N -functions Φ of the form

$$\Phi(\xi) = \sum_{k=1}^K A_k \left(\left| \sum_{i=1}^n \alpha_i^k \xi_i \right| \right) \quad \text{for } \xi \in \mathbb{R}^n, \tag{1.9}$$

where A_k are N -functions of one variable, $K \in \mathbb{N}$ and the coefficients $\alpha_i^k \in \mathbb{R}$ are arbitrary. A possible instance, when $n = 2$, corresponds to the function

$$\Phi(\xi) = |\xi_1 - \xi_2|^p + |\xi_1|^q \log(c + |\xi_1|)^\alpha \quad \text{for } \xi \in \mathbb{R}^2, \tag{1.10}$$

where either $q \geq 1$ and $\alpha > 0$, or $q = 1$ and $\alpha > 0$, the exponent $p > 1$, and c is a sufficiently large constant for Φ to be convex. Another example amounts to the function

$$\Phi(\xi) = |\xi_1 + 3\xi_2|^p + e^{|\xi_1 - \xi_2|^\beta} - 1 \quad \text{for } \xi \in \mathbb{R}^2, \tag{1.11}$$

with $p > 1$ and $\beta > 1$.

2 Function spaces

Assume that Ω is a measurable subset of \mathbb{R}^n , with $n \geq 1$, having finite Lebesgue measure $|\Omega|$. Given $m \in \mathbb{N}$, we set

$$\mathcal{M}(\Omega; \mathbb{R}^m) = \{U : U \text{ is a measurable function from } \Omega \text{ into } \mathbb{R}^m\}.$$

When $m = 1$, we shall make use of the abridged notation $\mathcal{M}(\Omega)$ for $\mathcal{M}(\Omega; \mathbb{R})$. An analogous simplification will be employed in the notation of other function spaces without further mentioning.

Given $u \in \mathcal{M}(\Omega)$, we define the *distribution function* $\mu_u : [0, \infty) \rightarrow [0, \infty)$ as

$$\mu_u(t) = |\{x \in \Omega : |u(x)| > t\}| \quad \text{for } t \geq 0, \tag{2.1}$$

and the *decreasing rearrangement* $u^* : [0, \infty) \rightarrow [0, \infty)$ as

$$u^*(s) = \inf\{t \geq 0 : \mu_u(t) \leq s\} \quad \text{for } s \geq 0. \tag{2.2}$$

The function u^* is equimeasurable with u and right-continuous. The function $u^{**} : (0, \infty) \rightarrow [0, \infty]$, called the *maximal rearrangement* of u^* and given by

$$u^{**}(s) = \frac{1}{s} \int_0^s u^*(r) dr \quad \text{for } s > 0, \tag{2.3}$$

is non-increasing, and satisfies $u^* \leq u^{**}$.

A *Banach function space* $X(\Omega)$ (in the sense of Luxemburg [13]) of functions in $\mathcal{M}(\Omega)$ is called a *rearrangement-invariant* space if its norm $\|\cdot\|_{X(\Omega)}$ satisfies

$$\|u\|_{X(\Omega)} = \|v\|_{X(\Omega)} \quad \text{whenever } u^* = v^*. \tag{2.4}$$

If $X(\Omega)$ is a rearrangement-invariant space, then

$$L^\infty(\Omega) \rightarrow X(\Omega) \rightarrow L^1(\Omega), \tag{2.5}$$

where \rightarrow stands for a continuous embedding.

Let $X(\Omega)$ be a rearrangement-invariant space. Its *associate space* is the rearrangement-invariant space $X'(\Omega)$ equipped with the norm given by

$$\|u\|_{X'(\Omega)} = \sup \left\{ \int_\Omega |u(x)v(x)| dx : \|v\|_{X(\Omega)} \leq 1 \right\}. \tag{2.6}$$

The space $X'(\Omega)$ is contained in the topological dual of $X(\Omega)$, denoted by $X(\Omega)^*$, but need not coincide with the latter.

Let $\varrho : (0, |\Omega|) \rightarrow (0, \infty)$ be a continuous increasing function. We denote by $L^{\varrho(\cdot), \infty}(\Omega)$ the *Marcinkiewicz-type space* associated with ϱ , and defined as

$$L^{\varrho(\cdot), \infty}(\Omega) = \left\{ u \in \mathcal{M}(\Omega) : \text{there exists } \lambda > 0 \text{ such that } \sup_{s \in (0, |\Omega|)} \frac{u^*(s)}{\varrho^{-1}(\lambda/s)} < \infty \right\}.$$

Note that $L^{\varrho(\cdot), \infty}(\Omega)$ is not always a normed space. Special choices of the function ϱ recover standard spaces of weak-type. For instance, if $\varrho(s) = s^q$ for some $q > 0$, then $L^{\varrho(\cdot), \infty}(\Omega) = L^{q, \infty}(\Omega)$, the customary weak- $L^q(\Omega)$ space. When $\varrho(s)$ behaves like $s^q (\log s)^\beta$ near infinity

for some $q > 0$ and $\beta \in \mathbb{R}$, we shall adopt the notation $L^{q,\infty}(\log L)^\beta(\Omega)$ for $L^{q(\cdot),\infty}(\Omega)$. The meaning of the notation $L^{q,\infty}(\log L)^\beta(\log \log L)^{-1}(\Omega)$ is analogous.

Orlicz and Orlicz–Lorenz spaces generalize Lebesgue and Lorentz spaces, respectively, and are classical instances of rearrangement-invariant spaces. Together with their anisotropic counterpart and with the associated Sobolev-type spaces, they play a critical role in our discussion. Their definitions and basic properties are recalled in what follows.

2.1 Orlicz, Orlicz–Lorentz and Orlicz–Sobolev spaces

We say that a function $A : [0, \infty) \rightarrow [0, \infty]$ is a *Young function* if it is convex, vanishes at 0, and is neither identically equal to 0, nor to infinity. A Young function A which is finite-valued, vanishes only at 0 and satisfies the additional growth conditions

$$\lim_{t \rightarrow 0} \frac{A(t)}{t} = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{A(t)}{t} = \infty, \tag{2.7}$$

is called an *N-function*.

The *Young conjugate* of a Young function A is the Young function \tilde{A} defined by

$$\tilde{A}(t) = \sup\{st - A(s) : s \geq 0\} \quad \text{for } t \geq 0.$$

Hence,

$$st \leq A(s) + \tilde{A}(t) \quad \text{for } s, t \geq 0. \tag{2.8}$$

Note that $\tilde{\tilde{A}} = A$ for any Young function A . The class of *N-functions* is closed under the operation of Young conjugation. One has that

$$t \leq \tilde{A}^{-1}(t)A^{-1}(t) \leq 2t \quad \text{for } t \geq 0, \tag{2.9}$$

where A^{-1} stands for the (generalized) left-continuous inverse of A . Hence,

$$\frac{A(t)}{t} \leq \tilde{A}^{-1}(A(t)) \leq 2 \frac{A(t)}{t} \quad \text{for } t \geq 0. \tag{2.10}$$

A Young function A fulfils the Δ_2 -condition near infinity if A is finite-valued and there exist constants $c > 0$ and $t_0 \geq 0$ such that $A(2t) \leq cA(t)$ for $t \geq t_0$.

A function A satisfies the ∇_2 -condition near infinity if there exist constants $c > 2$ and $t_0 \geq 0$ such that $A(2t) \geq cA(t)$ for $t \geq t_0$.

We shall also write “ $A \in \Delta_2$ near infinity” and “ $A \in \nabla_2$ near infinity” to denote these properties.

One has that $A \in \Delta_2$ near infinity if and only if $\tilde{A} \in \nabla_2$ near infinity.

We say that a Young function A dominates another Young function B near infinity if there exist constants $c > 0$ and $t_0 \geq 0$ such that $B(t) \leq A(ct)$ if $t \geq t_0$. If two Young functions A and B dominate each other near infinity, then we say that they are *equivalent near infinity*.

A Young function A is said to *increase essentially faster* than B near infinity if

$$\lim_{t \rightarrow +\infty} \frac{A^{-1}(t)}{B^{-1}(t)} = 0. \tag{2.11}$$

Let Ω be a measurable set in \mathbb{R}^n , $n \geq 1$, with $|\Omega| < \infty$, and let A be a Young function. The *Orlicz class* $\mathcal{L}^A(\Omega)$ is defined as

$$\mathcal{L}^A(\Omega) = \left\{ u \in \mathcal{M}(\Omega) : \int_{\Omega} A(|u|) \, dx < \infty \right\}. \tag{2.12}$$

The set $\mathcal{L}^A(\Omega)$ is convex, but it is not a linear space in general. The Orlicz space $L^A(\Omega)$ is the set of all functions $u \in \mathcal{M}(\Omega)$ such that the Luxemburg norm

$$\|u\|_{L^A(\Omega)} = \inf \left\{ \lambda > 0 : \int_{\Omega} A\left(\frac{1}{\lambda}|u|\right) dx \leq 1 \right\} \tag{2.13}$$

is finite. The space $L^A(\Omega)$ equipped with this norm is a Banach space. It is the smallest vector space containing $\mathcal{L}^A(\Omega)$. In particular, one has that $L^A(\Omega) = L^p(\Omega)$ if $A(t) = t^p$ for some $p \in [1, \infty)$, and $L^A(\Omega) = L^\infty(\Omega)$ if $A(t) = \infty \chi_{(1, \infty)}(t)$. Here, and in what follows, χ_E stands for the characteristic function of a set E .

A Hölder-type inequality in the Orlicz setting reads

$$\int_{\Omega} |uv| dx \leq 2\|u\|_{L^A(\Omega)} \|v\|_{L^{\tilde{A}}(\Omega)} \tag{2.14}$$

for every $u \in L^A(\Omega)$ and $v \in L^{\tilde{A}}(\Omega)$.

Let A and B be two Young functions. Then

$$L^A(\Omega) \rightarrow L^B(\Omega) \quad \text{if and only if} \quad A \text{ dominates } B \text{ near infinity.} \tag{2.15}$$

In particular, $L^A(\Omega) \rightarrow L^1(\Omega)$ for any Young function A . Hence,

$$L^A(\Omega) = L^B(\Omega) \quad \text{if and only if} \quad A \text{ is equivalent to } B \text{ near infinity,} \tag{2.16}$$

where the equality has to be interpreted up to equivalent norms.

Let us next set

$$E^A(\Omega) = \left\{ u \in \mathcal{M}(\Omega) : \int_{\Omega} A\left(\frac{1}{\lambda}|u|\right) dx < \infty \text{ for every } \lambda > 0 \right\}. \tag{2.17}$$

The space $E^A(\Omega)$ agrees with the closure in $L^A(\Omega)$, in the norm topology, of the space of functions which are bounded in Ω and have bounded support. Trivially,

$$E^A(\Omega) \subset \mathcal{L}^A(\Omega) \subset L^A(\Omega). \tag{2.18}$$

Both inclusions hold as equalities in (2.18) if and only if A satisfies the Δ_2 -condition near infinity.

If A increases essentially faster than B near infinity, then

$$L^A(\Omega) \rightarrow E^B(\Omega). \tag{2.19}$$

The alternative notation $A(L)(\Omega)$ will also be employed, when convenient, to denote the Orlicz space associated with any Young function equivalent to A near infinity. For instance, if $\alpha > 0$, then $\exp L^\alpha(\Omega)$ stands for the Orlicz space built upon a Young function equivalent to e^{t^α} near infinity. Moreover, if either $p > 1$ and $\alpha \in \mathbb{R}$, or $p = 1$ and $\alpha \geq 0$, then the space $L^p \log^\alpha L(\Omega)$ denotes the Orlicz space associated with a Young function equivalent to $t^p \log^\alpha t$ near infinity. We refer to the monographs [1,51,52] for comprehensive treatments of Orlicz spaces.

Given a Young function A and $r \in (-\infty, \infty) \setminus \{0\}$, we denote by $L[A, r](\Omega)$ the Orlicz–Lorentz-type space of those functions $u \in \mathcal{M}(\Omega)$ such that the quantity

$$\|u\|_{L[A,r](\Omega)} = \|s^{\frac{1}{r}} u^{**}(s)\|_{L^A(0,|\Omega|)} \tag{2.20}$$

is finite. Here, and in what follows, we use the convention that $\frac{1}{\infty} = 0$. The space $L[A, r](\Omega)$ is a rearrangement-invariant space. It is non-trivial, namely it contains functions that do not

vanish identically, if $\|s^{\frac{1}{r}}\|_{L^A(0,|\Omega|)} < \infty$. In analogy with $E^A(\Omega)$, we define

$$E[A, r](\Omega) = \left\{ u \in \mathcal{M}(\Omega) : \int_0^{|\Omega|} A\left(\frac{1}{\lambda} s^{\frac{1}{r}} u^{**}(s)\right) ds < \infty \text{ for every } \lambda > 0 \right\}. \tag{2.21}$$

Similarly, we denote by $L(A, r)(\Omega)$ the Orlicz–Lorentz-type space of all functions $u \in \mathcal{M}(\Omega)$ for which the expression

$$\|u\|_{L(A,r)(\Omega)} = \|s^{\frac{1}{r}} u^*(s)\|_{L^A(0,|\Omega|)} \tag{2.22}$$

is finite. The space $E(A, r)(\Omega)$ is defined accordingly. Under suitable assumptions on A and r the functional defined by (2.22) is a norm, and, consequently, $L(A, r)(\Omega)$ is a rearrangement-invariant space equipped with this norm—see [25]. In particular, for any Young function A , formula (2.22) defines a norm provided that $r < -1$.

The special instance corresponding to $L^A(0, |\Omega|) = L^q(0, |\Omega|)$ yields

$$L(A, r)(\Omega) = E(A, r)(\Omega) = L^{p,q}(\Omega), \tag{2.23}$$

up to equivalent norms, provided that p, q and r are suitably related. Here, $L^{p,q}(\Omega)$ denotes the customary Lorentz space of those functions $u \in \mathcal{M}(\Omega)$ making the quantity

$$\|u\|_{L^{p,q}(\Omega)} = \|s^{\frac{1}{p}-\frac{1}{q}} u^*(s)\|_{L^q(0,|\Omega|)} \tag{2.24}$$

finite. Also, with a proper choice of p and r ,

$$L[A, r](\Omega) = E[A, r](\Omega) = L^{[p,q]}(\Omega), \tag{2.25}$$

up to equivalent norms, where $L^{[p,q]}(\Omega)$ denotes the Lorentz space equipped with the norm given by

$$\|u\|_{L^{[p,q]}(\Omega)} = \|s^{\frac{1}{p}-\frac{1}{q}} u^{**}(s)\|_{L^q(0,|\Omega|)} \tag{2.26}$$

for $u \in \mathcal{M}(\Omega)$.

When $L^A(0, |\Omega|) = L^q \log^\alpha L(0, |\Omega|)$, where either $q \in (1, \infty]$ and $\alpha \geq 0$, or $q = 1$ and $\alpha \geq 0$, one has that

$$L[A, r](\Omega) = E[A, r](\Omega) = L^{[p,q]}(\log L)^\alpha(\Omega), \tag{2.27}$$

up to equivalent norms, again with a suitable choice of p and r —see e.g. [13, Lemma 6.12, Chapter 4]. Here, $L^{[p,q]}(\log L)^\alpha(\Omega)$ denotes the Lorentz–Zygmund space equipped with the norm defined as

$$\|u\|_{L^{[p,q]}(\log L)^\alpha(\Omega)} = \|s^{\frac{1}{p}-\frac{1}{q}} \log^\alpha \left(1 + \frac{|\Omega|}{s}\right) u^{**}(s)\|_{L^q(0,|\Omega|)} \tag{2.28}$$

for $u \in \mathcal{M}(\Omega)$. An analogous relation links the spaces $L(A, r)(\Omega)$, $E(A, r)(\Omega)$ and $L^{p,q}(\log L)^\alpha(\Omega)$, where the latter is defined as the set of all functions $u \in \mathcal{M}(\Omega)$ which render the right-hand side of Eq. (2.28), with u^{**} replaced by u^* , finite.

Assume now that Ω is an open set in \mathbb{R}^n , $n \geq 2$, with $|\Omega| < \infty$. We define the *Orlicz–Sobolev class*

$$W_0^1 \mathcal{L}^A(\Omega) = \{u \in \mathcal{M}(\Omega) : \text{the continuation of } u \text{ by } 0 \text{ outside } \Omega \text{ is weakly differentiable in } \mathbb{R}^n \text{ and } |\nabla u| \in \mathcal{L}^A(\Omega)\}. \tag{2.29}$$

The Orlicz–Sobolev space $W_0^1 L^A(\Omega)$ is defined analogously, on replacing $\mathcal{L}^A(\Omega)$ by $L^A(\Omega)$ on the right-hand side of definition (2.29). The space $W_0^1 L^A(\Omega)$, endowed with the norm

$$\|u\|_{W_0^1 L^A(\Omega)} = \|\lvert \nabla u \rvert\|_{L^A(\Omega)}, \tag{2.30}$$

is a Banach space. Note that, thanks to a Poincaré-type inequality—see [59, Lemma 3]—the norm defined by (2.30) is equivalent to the norm given by $\|u\|_{L^A(\Omega)} + \|\lvert \nabla u \rvert\|_{L^A(\Omega)}$.

In the case when $L^A(\Omega) = L^p(\Omega)$ for some $p \in [1, \infty)$ and $\partial\Omega$ is regular enough, the above definition of $W_0^1 L^A(\Omega)$ reproduces the usual space $W_0^{1,p}(\Omega)$ defined as the closure in $W^{1,p}(\Omega)$ of the space $C_0^\infty(\Omega)$ of smooth compactly supported functions in Ω . On the other hand, the set of smooth bounded functions is dense in $L^A(\Omega)$ if and only if A satisfies the Δ_2 -condition, and hence, for arbitrary A , our definition of $W_0^1 L^A(\Omega)$ yields a space which can be larger than the closure of $C_0^\infty(\Omega)$ with respect to the norm in (2.30). A systematic study of Orlicz–Sobolev spaces was initiated in [32]. An account of more recent developments can be found in [51,52].

2.2 Anisotropic Orlicz and Orlicz–Sobolev spaces

A function $\Phi : \mathbb{R}^n \rightarrow [0, \infty]$ is called an *n-dimensional Young function* if it is convex, $\Phi(0) = 0$, $\Phi(\xi) = \Phi(-\xi)$ for $\xi \in \mathbb{R}^n$, and $\{\xi \in \mathbb{R}^n : \Phi(\xi) \leq t\}$ is a compact set containing 0 in its interior for every $t > 0$.

The function Φ is called an *n-dimensional N-function* if, in addition, Φ is finite-valued, vanishes only at 0, and

$$\lim_{\xi \rightarrow 0} \frac{\Phi(\xi)}{|\xi|} = 0 \quad \text{and} \quad \lim_{|\xi| \rightarrow \infty} \frac{\Phi(\xi)}{|\xi|} = \infty. \tag{2.31}$$

Notice that, for technical reasons and ease of presentation, in the case when $n = 1$ we are distinguishing Young functions or *N-functions*, as defined on $[0, \infty)$ as in the previous subsection, from 1-dimensional Young functions or 1-dimensional *N-functions* defined on the whole of \mathbb{R} here. However, extending a Young function to an even function on the entire \mathbb{R} results in a 1-dimensional Young function; conversely, the restriction of a 1-dimensional Young function to $[0, \infty)$ is a Young function. Thus, any definition or result concerning Young functions or *N-functions* translates into a corresponding definition or result for 1-dimensional Young functions or *N-functions*, and viceversa.

In what follows, Young or *N-functions* will be denoted by latin capital letters, whereas *n-dimensional* Young or *N-functions* will be denoted by greek capital letters. Thus, there will be no ambiguity if we simply write Young function or *N-function* when referring to an *n-dimensional* function.

The *Young conjugate* of a Young function Φ is the Young function $\tilde{\Phi}$ defined as

$$\tilde{\Phi}(\xi) = \sup\{\eta \cdot \xi - \Phi(\eta) : \eta \in \mathbb{R}^n\} \quad \text{for } \xi \in \mathbb{R}^n.$$

Here, the dot “ \cdot ” denotes scalar product in \mathbb{R}^n . By the very definition of $\tilde{\Phi}$, one has that $\xi \cdot \eta \leq \Phi(\xi) + \tilde{\Phi}(\eta)$ for $\xi, \eta \in \mathbb{R}^n$. One has that $(\tilde{\tilde{\Phi}}) = \Phi$ for any Young function Φ . The class of *N-functions* is closed under the operation of Young conjugation.

A Young function Φ is said to satisfy the *Δ_2 -condition near infinity*, briefly $\Phi \in \Delta_2$ near infinity, if it is finite-valued and there exist positive constants c and M such that $\Phi(2\xi) \leq c\Phi(\xi)$ if $|\xi| \geq M$.

A Young function Φ is said to satisfy the *∇_2 -condition near infinity*, briefly $\Phi \in \nabla_2$ near infinity, if there exist constants $c > 2$ and $M > 0$ such that $\Phi(2\xi) \geq c\Phi(\xi)$ if $|\xi| \geq M$.

A Young function Φ is said to *dominate* another Young function Ψ *near infinity* if there exist positive constants c and M such that $\Psi(\xi) \leq \Phi(c\xi)$ if $|\xi| \geq M$. Equivalence of Young functions is defined accordingly.

Let Ω be a measurable set in \mathbb{R}^n , $n \geq 1$, with $|\Omega| < \infty$, and let Φ be an n -dimensional Young function. The *anisotropic Orlicz space* $L^\Phi(\Omega; \mathbb{R}^n)$ is the set of all vector-valued functions $U \in \mathcal{M}(\Omega; \mathbb{R}^n)$ such that the norm

$$\|U\|_{L^\Phi(\Omega; \mathbb{R}^n)} = \inf \left\{ \lambda > 0 : \int_\Omega \Phi\left(\frac{1}{\lambda}U\right) dx \leq 1 \right\}$$

is finite. The space $L^\Phi(\Omega; \mathbb{R}^n)$, equipped with this norm, is a Banach space. The Orlicz class $\mathcal{L}^\Phi(\Omega; \mathbb{R}^n)$ and the space $E^\Phi(\Omega; \mathbb{R}^n)$ are defined in analogy with definitions (2.12) and (2.17), respectively. One has that $E^\Phi(\Omega; \mathbb{R}^n)$ agrees with the closure in $L^\Phi(\Omega; \mathbb{R}^n)$ of the space of bounded functions in Ω with bounded support. Clearly,

$$E^\Phi(\Omega; \mathbb{R}^n) \subset \mathcal{L}^\Phi(\Omega; \mathbb{R}^n) \subset L^\Phi(\Omega; \mathbb{R}^n), \tag{2.32}$$

and both inclusions hold as equalities if and only if $\Phi \in \Delta_2$ near infinity. The Hölder-type inequality

$$\int_\Omega |U \cdot V| dx \leq 2\|U\|_{L^\Phi(\Omega; \mathbb{R}^n)} \|V\|_{L^{\tilde{\Phi}}(\Omega; \mathbb{R}^n)} \tag{2.33}$$

holds for every $U \in L^\Phi(\Omega; \mathbb{R}^n)$ and $V \in L^{\tilde{\Phi}}(\Omega; \mathbb{R}^n)$.

If Φ and Ψ are Young functions, then

$$L^\Phi(\Omega; \mathbb{R}^n) \rightarrow L^\Psi(\Omega; \mathbb{R}^n) \text{ if and only if } \Phi \text{ dominates } \Psi \text{ near infinity.}$$

In particular, $L^\Phi(\Omega; \mathbb{R}^n) \rightarrow L^1(\Omega; \mathbb{R}^n)$ for any Young function Φ . Moreover,

$$L^\Phi(\Omega; \mathbb{R}^n) = L^\Psi(\Omega; \mathbb{R}^n) \text{ if and only if } \Phi \text{ and } \Psi \text{ are equivalent near infinity.}$$

By [54, Corollary 7.2], given any N -function Φ , the space $L^\Phi(\Omega; \mathbb{R}^n)$ is reflexive if and only if $\Phi \in \Delta_2 \cap \nabla_2$ near infinity. In general, if Φ is an arbitrary n -dimensional N -function, then

$$\text{the dual of } E^\Phi(\Omega; \mathbb{R}^n) \text{ is isomorphic and homeomorphic to } L^{\tilde{\Phi}}(\Omega; \mathbb{R}^n), \tag{2.34}$$

see [3, Proposition 2.4]. Orlicz spaces of vector-valued functions are studied in detail in [55,56], as special cases of more general Musielak–Orlicz spaces; the analysis of the paper [54] also includes Orlicz spaces of functions defined on infinite dimensional spaces.

Assume that Ω is an open set in \mathbb{R}^n , $n \geq 2$, with $|\Omega| < \infty$. Let Φ be an n -dimensional Young function. The *anisotropic Orlicz–Sobolev class* is defined as

$$W_0^1 \mathcal{L}^\Phi(\Omega) = \{u \in \mathcal{M}(\Omega): \text{the continuation of } u \text{ by } 0 \text{ outside } \Omega \\ \text{is weakly differentiable in } \mathbb{R}^n \text{ and } \nabla u \in \mathcal{L}^\Phi(\Omega; \mathbb{R}^n)\}. \tag{2.35}$$

The *anisotropic Orlicz–Sobolev space* $W_0^1 L^\Phi(\Omega)$ is defined accordingly, on replacing $\mathcal{L}^\Phi(\Omega; \mathbb{R}^n)$ by $L^\Phi(\Omega; \mathbb{R}^n)$ on the right-hand side of Eq. (2.35). One has that $W_0^1 L^\Phi(\Omega)$, equipped with the norm

$$\|u\|_{W_0^1 L^\Phi(\Omega)} = \|\nabla u\|_{L^\Phi(\Omega; \mathbb{R}^n)},$$

is a Banach space. The Orlicz–Sobolev space $W_0^1 L^\Phi(\Omega)$ is reflexive if and only if $\Phi \in \Delta_2 \cap \nabla_2$ near infinity. Classical contributions on Orlicz–Sobolev spaces are [42,60].

The use of sets of functions, whose truncations belong to an Orlicz–Sobolev space, is crucial in dealing with approximable solutions. Given any $t > 0$, let $T_t : \mathbb{R} \rightarrow \mathbb{R}$ denote the function defined by

$$T_t(s) = \begin{cases} s & \text{if } |s| \leq t, \\ t \operatorname{sign}(s) & \text{if } |s| > t. \end{cases} \tag{2.36}$$

We set

$$\mathcal{T}_0^{1,\Phi}(\Omega) = \{u \in \mathcal{M}(\Omega) : T_t(u) \in W_0^1 L^\Phi(\Omega) \text{ for every } t > 0\}. \tag{2.37}$$

The space $\mathcal{T}_0^{1,\Phi}(\Omega)$ is the anisotropic counterpart of the space introduced in [11] and associated with the standard Sobolev space $W_0^{1,p}(\Omega)$ corresponding to the choice $\Phi(\xi) = |\xi|^p$.

If $u \in \mathcal{T}_0^{1,\Phi}(\Omega)$, then there exists a (unique) measurable function $Z_u : \Omega \rightarrow \mathbb{R}^n$ such that

$$\nabla T_t(u) = \chi_{\{|u|<t\}} Z_u \quad \text{a.e. in } \Omega \tag{2.38}$$

for every $t > 0$. This is a consequence of [11, Lemma 2.1]. One has that $u \in W_0^1 L^\Phi(\Omega)$ if and only if $u \in \mathcal{T}_0^{1,\Phi}(\Omega)$ and $Z_u \in L^\Phi(\Omega; \mathbb{R}^n)$. In the latter case, $Z_u = \nabla u$ a.e. in Ω . With an abuse of notation, for every $u \in \mathcal{T}_0^{1,\Phi}(\Omega)$ we denote Z_u simply by ∇u throughout.

2.3 Auxiliary functions associated with Φ

Let Φ be an n -dimensional Young function. By $\Phi_\circ : [0, \infty) \rightarrow [0, \infty)$ we denote the Young function obeying

$$|\{\xi \in \mathbb{R}^n : \Phi_\circ(|\xi|) \leq t\}| = |\{\xi \in \mathbb{R}^n : \Phi(\xi) \leq t\}| \quad \text{for } t \geq 0. \tag{2.39}$$

The function $\mathbb{R}^n \ni \xi \mapsto \Phi_\circ(|\xi|)$ can be regarded as a kind of ‘‘average in measure’’ of Φ . It can be used to define the radially increasing symmetral $\Phi_\star : \mathbb{R}^n \rightarrow [0, \infty)$ of Φ by

$$\Phi_\star(\xi) = \Phi_\circ(|\xi|) \quad \text{for } \xi \in \mathbb{R}^n.$$

Since Φ_\star is radially symmetric, the function $\Phi_\diamond : [0, \infty) \rightarrow [0, \infty)$, defined by

$$\Phi_\diamond(|\xi|) = \widetilde{(\Phi_\star)}(\xi) \quad \text{for } \xi \in \mathbb{R}^n, \tag{2.40}$$

is a Young function. Moreover, the function Φ_\diamond is equivalent to Φ_\circ , and there exist constants $c_1 = c_1(n)$ and $c_2 = c_2(n)$ such that

$$\Phi_\circ(c_1 t) \leq \Phi_\diamond(t) \leq \Phi_\circ(c_2 t) \quad \text{for } t \geq 0, \tag{2.41}$$

see [42, Lemma 7]. Note that if Φ is an n -dimensional N -function, then the functions Φ_\circ and Φ_\diamond are 1-dimensional N -functions and Φ_\star is an n -dimensional N -function.

Two more functions associated with Φ , denoted by Φ_n and $\widehat{\Phi}_\circ$, will be introduced in the next section in connection with Orlicz–Sobolev-type embeddings.

Some auxiliary functions depending on Φ will still be needed. We denote by $\Psi_\circ : [0, \infty) \rightarrow [0, \infty)$ the increasing function given by

$$\Psi_\circ(t) = \frac{\Phi_\circ(t)}{t} \quad \text{for } t > 0 \quad \text{and} \quad \Psi_\circ(0) = 0. \tag{2.42}$$

Also, we call $\Psi_\diamond : [0, \infty) \rightarrow [0, \infty)$ the increasing function given by

$$\Psi_\diamond(t) = \frac{\Phi_\diamond(t)}{t} \quad \text{for } t > 0 \quad \text{and} \quad \Psi_\diamond(0) = 0. \tag{2.43}$$

The function $\Theta : \mathbb{R}^n \rightarrow [0, \infty)$ is defined as

$$\Theta(\xi) = \widetilde{\Phi}_\diamond^{-1}(\Phi(\xi)) \quad \text{for } \xi \in \mathbb{R}^n, \tag{2.44}$$

and the function $\Theta_\diamond : [0, \infty) \rightarrow [0, \infty)$ as

$$\Theta_\diamond(t) = \widetilde{\Phi}_\diamond^{-1}(\Phi_\diamond(t)) \quad \text{for } t \geq 0. \tag{2.45}$$

Relations among the functions introduced above are the subject of the following lemma.

Lemma 2.1 *Let $\Phi : \mathbb{R}^n \rightarrow [0, \infty)$ be an n -dimensional N -function, and let $\Phi_\diamond, \Psi_\diamond, \Theta,$ and Θ_\diamond be the functions associated with Φ as in (2.40), (2.43), (2.44) and (2.45), respectively. Then*

- (i) $\Phi_\diamond \circ \Theta_\diamond^{-1} = \widetilde{\Phi}_\diamond,$
- (ii) $\Phi_\diamond \circ \Theta_\diamond^{-1} \circ \Theta = \Phi,$
- (iii) $\Phi_\diamond^{-1}(t\Psi_\diamond^{-1}(t)) = \Psi_\diamond^{-1}(t) \quad \text{for } t \geq 0,$
- (iv) $\Theta_\diamond(\Psi_\diamond^{-1}(t)) \leq 2t \quad \text{for } t \geq 0.$
- (v) $\Phi_\diamond(\Psi_\diamond^{-1}(t/2)) \leq \widetilde{\Phi}_\diamond(t) \leq \Phi_\diamond(\Psi_\diamond^{-1}(t)) \quad \text{for } t \geq 0.$

Proof Equations (i) and (ii) are straightforward consequences of definitions (2.44) and (2.45). Equation (iii) easily follows on replacing t by $\Psi_\diamond^{-1}(t)$ in the definition of Ψ_\diamond . As for inequality (iv), recall that, since Φ_\diamond is a Young function, then, by (2.9),

$$t \leq \Phi_\diamond^{-1}(t)\widetilde{\Phi}_\diamond^{-1}(t) \leq 2t \quad \text{for } t \geq 0.$$

By (iii) and the second inequality above we get

$$\begin{aligned} \Theta_\diamond(\Psi_\diamond^{-1}(t)) &= \widetilde{\Phi}_\diamond^{-1}(\Phi_\diamond(\Psi_\diamond^{-1}(t))) = \widetilde{\Phi}_\diamond^{-1}(\Phi_\diamond(\Phi_\diamond^{-1}(t\Psi_\diamond^{-1}(t)))) = \widetilde{\Phi}_\diamond^{-1}(t\Psi_\diamond^{-1}(t)) \\ &\leq \frac{2t\Psi_\diamond^{-1}(t)}{\Phi_\diamond^{-1}(t\Psi_\diamond^{-1}(t))} = \frac{2t\Psi_\diamond^{-1}(t)}{\Psi_\diamond^{-1}(t)} = 2t \quad \text{for } t \geq 0. \end{aligned} \tag{2.46}$$

Finally, property (v) follows via Eq. (2.10) applied with A replaced by Ψ_\diamond . □

2.4 Sobolev embeddings

The sharp embeddings for anisotropic Orlicz–Sobolev spaces collected in this subsection are pivotal in our analysis.

Let Φ be an n -dimensional Young function. A basic anisotropic Poincaré-type inequality tells us that there exists a constant $\kappa_1 = \kappa_1(n)$ such that

$$\int_\Omega \Phi_\diamond(\kappa_1|\Omega|^{-\frac{1}{n}}|u|) \, dx \leq \int_\Omega \Phi(\nabla u) \, dx, \tag{2.47}$$

for every $u \in W_0^1\mathcal{L}^\Phi(\Omega)$, and

$$\|u\|_{\mathcal{L}^{\Phi_\diamond}(\Omega)} \leq \kappa_1^{-1}|\Omega|^{\frac{1}{n}}\|\nabla u\|_{\mathcal{L}^\Phi(\Omega)} \tag{2.48}$$

for every $u \in W_0^1 L^\Phi(\Omega)$ —see [8, Proposition 3.2].

The statement of optimal anisotropic Sobolev inequalities requires some further definitions. Assume that

$$\int_0^t \left(\frac{\tau}{\Phi_\circ(\tau)} \right)^{\frac{1}{n-1}} d\tau < \infty. \tag{2.49}$$

If

$$\int_0^\infty \left(\frac{\tau}{\Phi_\circ(\tau)} \right)^{\frac{1}{n-1}} d\tau = \infty, \tag{2.50}$$

then we denote by $\Phi_n : [0, \infty) \rightarrow [0, \infty]$ the Sobolev conjugate of Φ introduced in [23]. Namely, Φ_n is the Young function defined as

$$\Phi_n(t) = \Phi_\circ(H^{-1}(t)) \quad \text{for } t \geq 0, \tag{2.51}$$

where $H : [0, \infty) \rightarrow [0, \infty)$ is given by

$$H(t) = \left(\int_0^t \left(\frac{\tau}{\Phi_\circ(\tau)} \right)^{\frac{1}{n-1}} d\tau \right)^{\frac{n-1}{n}} \quad \text{for } t \geq 0. \tag{2.52}$$

Here, H^{-1} denotes the generalized left-continuous inverse of H .

By [23, Theorem 1 and Remark 1], there exists a constant $\kappa_2 = \kappa_2(n)$ such that

$$\int_\Omega \Phi_n \left(\frac{|u|}{\kappa_2 \left(\int_\Omega \Phi(\nabla u) dy \right)^{\frac{1}{n}}} \right) dx \leq \int_\Omega \Phi(\nabla u) dx \tag{2.53}$$

for every $u \in W_0^1 \mathcal{L}^\Phi(\Omega)$, and

$$\|u\|_{L^{\Phi_n}(\Omega)} \leq \kappa_2 \|\nabla u\|_{L^\Phi(\Omega)} \tag{2.54}$$

for every $u \in W_0^1 L^\Phi(\Omega)$. Moreover, $L^{\Phi_n}(\Omega)$ is the optimal, i.e. the smallest possible, Orlicz space which renders (2.54) true for all n -dimensional Young functions Φ with prescribed Φ_\circ .

This result can be still improved if embeddings of $W_0^1 L^\Phi(\Omega)$ into the broader class of rearrangement-invariant target spaces are considered. Indeed, denote by $\phi_\circ : [0, \infty) \rightarrow [0, \infty)$ the non-decreasing, left-continuous function such that

$$\Phi_\circ(t) = \int_0^t \phi_\circ(\tau) d\tau \quad \text{for } t \geq 0,$$

and let $\widehat{\Phi}_\circ$ be the Young function given by

$$\widehat{\Phi}_\circ(t) = \int_0^t \widehat{\phi}_\circ(\tau) d\tau \quad \text{for } t \geq 0, \tag{2.55}$$

where $\widehat{\phi}_\circ : [0, \infty) \rightarrow [0, \infty)$ is the non-decreasing, left-continuous function defined via

$$(\widehat{\phi}_\circ)^{-1}(t) = \left(\int_{\phi_\circ^{-1}(t)}^\infty \left(\int_0^r \left(\frac{1}{\phi_\circ(t)} \right)^{\frac{1}{n-1}} dt \right)^{-n} \frac{dr}{\phi_\circ(r)^{\frac{n}{n-1}}} \right)^{\frac{1}{1-n}} \quad \text{for } t \geq 0, \tag{2.56}$$

and ϕ_\circ^{-1} and $\widehat{\phi}_\circ^{-1}$ are the (generalized) left-continuous inverses of ϕ_\circ and $\widehat{\phi}_\circ$, respectively.

Let $L(\widehat{\Phi}_\circ, -n)(\Omega)$ be the Orlicz–Lorentz-type space defined as in (2.22). By [25], there exists a constant $\kappa_3 = \kappa_3(n)$ such that

$$\int_0^{|\Omega|} \widehat{\Phi}_\circ(\kappa_3^{-1} s^{-\frac{1}{n}} u^*(s)) ds \leq \int_\Omega \Phi(\nabla u) dx \tag{2.57}$$

for every $u \in W_0^1 L^\Phi(\Omega)$, and

$$\|u\|_{L(\widehat{\Phi}_\circ, -n)(\Omega)} \leq \kappa_3 \|\nabla u\|_{L^\Phi(\Omega)} \tag{2.58}$$

for every $u \in W_0^1 L^\Phi(\Omega)$. Moreover, $L(\widehat{\Phi}_\circ, -n)(\Omega)$ is the optimal, i.e. the smallest possible, rearrangement-invariant space which renders inequality (2.58) true for all n -dimensional Young functions Φ with prescribed Φ_\circ .

Let us notice that the Orlicz–Lorentz-type space $L(\widetilde{\Phi}_\circ, n)(\Omega)$, defined as in (2.20), is the associate space of $L(\widehat{\Phi}_\circ, -n)(\Omega)$ (up to equivalent norms). Moreover, as shown in [25, Inequality (4.46)],

$$\begin{aligned} \int_\Omega |uv| dx &\leq \int_0^{|\Omega|} u^*(s)v^*(s) ds \\ &\leq C \left(\int_0^{|\Omega|} \widehat{\Phi}_\circ(s^{-\frac{1}{n}} u^*(s)) ds + \int_0^{|\Omega|} \widetilde{\Phi}_\circ(s^{\frac{1}{n}} v^{**}(s)) ds \right) \end{aligned} \tag{2.59}$$

for some constant $C = C(n)$, and for every $u, v \in \mathcal{M}(\Omega)$.

When Φ_\circ grows so fast near infinity that condition (2.50) fails, namely

$$\int^\infty \left(\frac{t}{\Phi_\circ(t)} \right)^{\frac{1}{n-1}} dt < \infty, \tag{2.60}$$

then there exists a constant $\kappa_4 = \kappa_4(\Phi, n, |\Omega|)$ such that

$$\|u\|_{L^\infty(\Omega)} \leq \kappa_4 \|\nabla u\|_{L^\Phi(\Omega)} \tag{2.61}$$

for every $u \in W_0^1 L^\Phi(\Omega)$.

2.5 Modular approximation

One obstacle to be faced when dealing with Orlicz and Orlicz–Sobolev spaces built upon Young functions that do not satisfy the Δ_2 -condition is the lack of separability of these spaces. In particular, functions in these spaces cannot be approximated in norm by smooth functions. Substitutes for this property are certain approximation results in integral form, usually referred to as “modular approximability” in the theory of Orlicz spaces, which are well fitted for applications to partial differential equations. This kind of approximation is well known for isotropic Orlicz and Orlicz–Sobolev spaces, and goes back to [38]. On the other hand, a counterpart in the more general anisotropic framework seems not to be completely settled yet. In this subsection, we recall a few definitions and state the approximation properties that are needed in view of our main results. Their proofs present some additional difficulty with respect to the isotropic case, and are given in Sect. 5.

Let Φ be an n -dimensional Young function and let Ω be a measurable set in \mathbb{R}^n with $|\Omega| < \infty$. A sequence $\{U_k\} \subset L^\Phi(\Omega; \mathbb{R}^n)$ is said to converge modularly to U in $L^\Phi(\Omega; \mathbb{R}^n)$ if there exists $\lambda > 0$ such that

$$\lim_{k \rightarrow \infty} \int_{\Omega} \Phi \left(\frac{U_k - U}{\lambda} \right) dx = 0. \tag{2.62}$$

Note that if $U_k \rightarrow U$ modularly, then $U_k \rightarrow U$ in measure.

The following proposition links modular convergence to a kind of weak convergence against test functions in the associate space.

Proposition 2.2 *Let Φ be an n -dimensional N -function and let Ω be a measurable set in \mathbb{R}^n with $|\Omega| < \infty$. Let $U \in L^\Phi(\Omega; \mathbb{R}^n)$. Assume that the sequence $\{U_k\} \subset L^\Phi(\Omega; \mathbb{R}^n)$ and that $U_k \rightarrow U$ modularly in $L^\Phi(\Omega; \mathbb{R}^n)$. Then there exists a subsequence of $\{U_k\}$, still indexed by k , such that*

$$\lim_{k \rightarrow \infty} \int_{\Omega} U_k \cdot V dx = \int_{\Omega} U \cdot V dx \quad \text{for every } V \in L^{\tilde{\Phi}}(\Omega; \mathbb{R}^n). \tag{2.63}$$

The next result concerns the modular density of simple functions in anisotropic Orlicz spaces.

Proposition 2.3 *Let Φ be an n -dimensional N -function and let Ω be a measurable set in \mathbb{R}^n with $|\Omega| < \infty$. Assume that $U \in L^\Phi(\Omega; \mathbb{R}^n)$. Then there exists a sequence of simple functions $\{U_k\}$ such that $U_k \rightarrow U$ modularly in $L^\Phi(\Omega; \mathbb{R}^n)$.*

We conclude with a modular smooth approximation property in anisotropic Orlicz–Sobolev spaces on bounded Lipschitz domains. Recall that an open set Ω is called a *Lipschitz domain* if each point of $\partial\Omega$ has a neighborhood \mathcal{U} such that $\Omega \cap \mathcal{U}$ is the subgraph of a Lipschitz continuous function of $n - 1$ variables.

Proposition 2.4 *Let Φ be an n -dimensional N -function and let Ω be a bounded Lipschitz domain in \mathbb{R}^n . Assume that $u \in W_0^1 L^\Phi(\Omega) \cap L^\infty(\Omega)$. Then there exist a constant $C = C(\Omega)$ and a sequence $\{u_k\} \subset C_0^\infty(\Omega)$ such that*

$$u_k \rightarrow u \quad \text{a.e. in } \Omega, \tag{2.64}$$

$$\|u_k\|_{L^\infty(\Omega)} \leq C \|u\|_{L^\infty(\Omega)} \quad \text{for every } k \in \mathbb{N}, \tag{2.65}$$

$$\nabla u_k \rightarrow \nabla u \quad \text{modularly in } L^\Phi(\Omega; \mathbb{R}^n). \tag{2.66}$$

Remark 2.5 In the isotropic case, namely when $\Phi(\xi) = A(|\xi|)$ for $\xi \in \mathbb{R}^n$, for some N -function A , properties (2.64) and (2.66) in Proposition 2.4 are known to hold even if the assumption $u \in L^\infty(\Omega)$ is dropped—see [38, Theorem 4].

2.6 Some classical theorems of functional analysis

We conclude this section by recalling a few well-known results of functional analysis, formulated in the anisotropic Orlicz space framework. In their statements, Ω is assumed to be a measurable set in \mathbb{R}^n with $|\Omega| < \infty$.

Theorem 2.6 (Vitali) *Assume that the sequence $\{U_k\} \subset \mathcal{M}(\Omega; \mathbb{R}^n)$ is uniformly integrable in Ω , and there exists a function $U : \Omega \rightarrow \mathbb{R}^n$ such that $\lim_{k \rightarrow \infty} U_k = U$ a.e. in Ω and $|U| < \infty$ a.e. in Ω . Then $U \in L^1(\Omega; \mathbb{R}^n)$ and $\lim_{k \rightarrow \infty} U_k = U$ in $L^1(\Omega; \mathbb{R}^n)$.*

Theorem 2.7 (Dunford–Pettis) *A family $\{U_\sigma\}_{\sigma \in \Sigma}$ of functions in $\mathcal{M}(\Omega; \mathbb{R}^n)$ is uniformly integrable in $L^1(\Omega; \mathbb{R}^n)$ if and only if it is relatively compact in the weak topology.*

Theorem 2.8 (Anisotropic De La Vallée Poussin) *Let Φ be an n -dimensional N -function. Assume that $\{U_\sigma\}_{\sigma \in \Sigma}$ is a family of functions in $\mathcal{M}(\Omega; \mathbb{R}^n)$ such that $\sup_{\sigma \in \Sigma} \int_\Omega \Phi(U_\sigma) dx < \infty$. Then the family $\{U_\sigma\}$ is uniformly integrable.*

The next result follows from the customary version of the Banach–Alaoglu theorem, owing to property (2.34) applied to Φ and $\tilde{\Phi}$. Notice that, in view of that property, a sequence $\{U_k\} \subset L^\Phi(\Omega; \mathbb{R}^n)$ weakly- $*$ converges to $U \in L^\Phi(\Omega; \mathbb{R}^n)$ in $L^\Phi(\Omega; \mathbb{R}^n)$ if

$$\lim_{k \rightarrow \infty} \int_\Omega U_k \cdot V dx = \int_\Omega U \cdot V dx$$

for every $V \in E^{\tilde{\Phi}}(\Omega; \mathbb{R}^n)$. Weak- $*$ convergence in $L^{\tilde{\Phi}}(\Omega; \mathbb{R}^n)$ can be characterized on exchanging the roles of Φ and $\tilde{\Phi}$.

Theorem 2.9 (Banach–Alaoglu in anisotropic Orlicz spaces) *Let Φ be an n -dimensional N -function. Then the closed unit ball in $L^\Phi(\Omega; \mathbb{R}^n)$ and the closed unit ball in $L^{\tilde{\Phi}}(\Omega; \mathbb{R}^n)$ are weakly- $*$ compact in the respective spaces.*

3 Main results

This is a section where definitions of solutions to the Dirichlet problem (1.1) are introduced and the pertaining existence, uniqueness, and regularity results are stated. In what follows, when referring to assumptions (1.2)–(1.4), we mean that they are fulfilled for some N -function Φ , some function $h \in L^1(\Omega)$, and some constant $c_\Phi \in (0, 1)$.

3.1 Weak solutions

Our first purpose is to detect a minimal integrability condition on the datum f for a weak solution to problem (1.1) to exist. In order to allow for the largest possible class of admissible functions f , in the definition of weak solution that will be adopted the function f is a priori assumed to be just integrable in Ω . The class of test functions is thus accordingly chosen for the weak formulation of the problem to be well posed for any such f .

Definition 3.1 (Weak solution) *Let $f \in L^1(\Omega)$. Under assumptions (1.2)–(1.4), a function $u \in W_0^1 \mathcal{L}^\Phi(\Omega)$ is called a weak solution to the Dirichlet problem (1.1) if*

$$\int_\Omega a(x, \nabla u) \cdot \nabla \varphi dx = \int_\Omega f \varphi dx \tag{3.1}$$

for every $\varphi \in W_0^1 \mathcal{L}^\Phi(\Omega) \cap L^\infty(\Omega)$.

Observe that both sides of equality (3.1) are well defined if f , u and φ are as in definition 3.1. In particular, the integral on the left-hand side of (3.1) is convergent by the Hölder inequality (2.33), since, owing to assumption (1.4), $a(x, \nabla u) \in L^{\tilde{\Phi}}(\Omega; \mathbb{R}^n)$ provided that $u \in W_0^1 \mathcal{L}^\Phi(\Omega)$.

Our main result about weak solutions is contained in Theorem 3.2. Its assumptions in connection with the existence (and uniqueness) of these solutions take a form of an alternative, depending on a threshold on the growth near infinity of the function Φ . More precisely, what is relevant is the growth of its “average” Φ_\circ , defined as in (2.39), and the alternative corresponds to the two complementary conditions (2.50) and (2.60). Indeed, if Φ_\circ grows

fast enough near infinity for the latter condition to hold, then any integrable function f is admissible. On the other hand, if (2.60) fails, and hence the former condition is in force, then a proper degree of integrability has to be imposed on f . A natural ambient space for f is the largest rearrangement-invariant space ensuring that the integral on the right-hand side of Eq. (3.1) is convergent for every (non-necessarily bounded) test function $\varphi \in W_0^1 \mathcal{L}^\Phi(\Omega)$ (or even $\in W_0^1 L^\Phi(\Omega)$). This corresponds to the associate space $L[\widehat{\Phi}_\circ, n](\Omega)$ of the optimal rearrangement-invariant target space $L(\widehat{\Phi}_\circ, -n)(\Omega)$ for embeddings of $W_0^1 L^\Phi(\Omega)$ —see (2.58). Theorem 3.2 asserts that the Dirichlet problem (1.1) does actually admit a unique weak solution provided that f belongs to the separable counterpart $E[\widetilde{\Phi}_\circ, n](\Omega)$ of $L[\widehat{\Phi}_\circ, n](\Omega)$, defined as in (2.21).

As will be clear from Example 1 in the next section, in the classical case of p -Laplacian-type problems, the two alternatives discussed above correspond to the situations when $p \leq n$ or $p > n$. In particular, if $p < n$, our assumption amounts to requiring that f belongs to the Lorentz space $L^{[\frac{np}{np+p-n}, p']}(\Omega)$, where $p' = \frac{p}{p-1}$, thus weakening the customary condition that $f \in L^{\frac{np}{np+p-n}}(\Omega)$.

Theorem 3.2 (Existence of weak solutions) *Let Ω be a bounded Lipschitz domain in \mathbb{R}^n . Assume that conditions (1.2)–(1.4) are in force, and let Φ_\circ be the function associated with Φ as in (2.39). If either*

$$\Phi_\circ \text{ grows so slowly that (2.50) holds and } f \in E[\widetilde{\Phi}_\circ, n](\Omega), \tag{3.2}$$

or

$$\Phi_\circ \text{ grows so fast that (2.60) holds and } f \in L^1(\Omega), \tag{3.3}$$

then there exists a unique weak solution $u \in W_0^1 \mathcal{L}^\Phi(\Omega)$ to the Dirichlet problem (1.1).

In some applications, we need to make use of the solution u itself as a test function φ in equation (3.1) in the definition of weak solution to problem (1.1). This requires u to be bounded. An optimal condition on f for this property to hold is exhibited in the next result.

Proposition 3.3 (Boundedness of weak solutions) *Assume, in addition to the assumptions of Theorem 3.2, that*

$$\int_0^{|\Omega|} s^{-\frac{1}{n'}} \Psi_\circ^{-1}(\lambda s^{\frac{1}{n}} f^{**}(s)) ds < \infty \tag{3.4}$$

for every $\lambda > 0$, where Ψ_\circ is defined as in (2.42). Then $u \in L^\infty(\Omega)$, and there exists a constant $C = C(n)$ such that

$$\|u\|_{L^\infty(\Omega)} \leq C \int_0^{|\Omega|} s^{-\frac{1}{n'}} \Psi_\circ^{-1}(C s^{\frac{1}{n}} f^{**}(s)) ds. \tag{3.5}$$

Remark 3.4 Owing to Eq. (2.41), condition (3.4) can be equivalently formulated with Ψ_\circ replaced by the function Ψ_\diamond defined by (2.43). The use of the latter function allows for an explicit sharp value of the constant λ in corresponding condition. Actually, the weak solution u to the Dirichlet problem (1.1)–(1.4) is bounded provided that

$$\int_0^{|\Omega|} s^{-\frac{1}{n'}} \Psi_\diamond^{-1}\left(\frac{s^{\frac{1}{n}}}{n\omega_n^{1/n}} f^{**}(s)\right) ds < \infty. \tag{3.6}$$

Moreover,

$$\|u\|_{L^\infty(\Omega)} \leq \frac{1}{n\omega_n^{1/n}} \int_0^{|\Omega|} s^{-\frac{1}{n'}} \Psi_\diamond^{-1} \left(\frac{s^{1/n}}{n\omega_n^{1/n}} f^{**}(s) \right) ds. \tag{3.7}$$

Both condition (3.6) and the bound given by (3.7) are sharp. The sufficiency of condition (3.6), and the validity of estimate (3.7) are apparent from a close inspection of the proof of Proposition 3.3. Their sharpness is due to the fact that equality holds in (3.7) if u is the solution to a suitable symmetric problem in a ball, which is stated in Eq. (6.14) below.

Remark 3.5 If condition (3.6), or even (3.4), is dropped, boundedness of the weak solution u to problem (1.1) u is not guaranteed. In this case, sharp integrability properties of u can be derived via [26, Proposition 3.7].

3.2 Approximable solutions

When neither of conditions (3.2) and (3.3) holds, weak solutions to problem (1.1) do not necessarily exist. This calls for the use of some notion of solution, still weaker than that of weak solution, which enables to deal with arbitrary right-hand sides $f \in L^1(\Omega)$, and yet with measure data, whatever Φ is. Merely distributional solutions are not satisfactory, since even for linear equations this class of solutions does not guarantee uniqueness and permits well-known pathologies [53]. These drawbacks can be overcome if, instead, solutions obtained as limits of solutions to approximating problems with regularized right-hand sides are introduced. Such a notion of solution has been extensively exploited, more or less explicitly, for nonlinear problems with isotropic growth—see e.g. [11,14,28,29,47,48]. It restores uniqueness and, importantly, is well suited to analyze regularity.

Approximable solutions to problem (1.1) under the present assumptions on the differential operator, and with right-hand side in $L^1(\Omega)$, can be defined as follows.

Definition 3.6 (Approximable solution with L^1 data) Let $f \in L^1(\Omega)$. Under assumptions (1.2)–(1.4), a function $u \in \mathcal{T}_0^{1,\Phi}(\Omega)$ is called an approximable solution to problem (1.1) if there exists a sequence $\{f_k\} \subset L^\infty(\Omega)$ such that $f_k \rightarrow f$ in $L^1(\Omega)$, and the sequence of weak solutions $\{u_k\} \subset W_0^1\mathcal{L}^\Phi(\Omega)$ to problems

$$\begin{cases} -\operatorname{div} a(x, \nabla u_k) = f_k & \text{in } \Omega \\ u_k = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.8}$$

satisfies

$$u_k \rightarrow u \quad \text{a.e. in } \Omega. \tag{3.9}$$

Despite its apparent mildness, this definition gives grounds for an adequate generalized notion of solution u to problem (1.1). Indeed, although the function u is a priori just assumed to be the pointwise limit of the solutions u_k to the approximating problems (3.8), its “surrogate gradient” ∇u , in the sense of (2.38), turns out to be the pointwise limit of the weak gradients ∇u_k , and hence $a(x, \nabla u_k) \rightarrow a(x, \nabla u)$ a.e. in Ω as well. This fact, together with the uniqueness of the approximable solution u and its regularity, are the subject of the next theorem. Information about regularity amounts to membership of u and $\Phi(\nabla u)$ in Marcinkiewicz-type spaces associated with the functions $\vartheta_n, \varrho_n : (0, \infty) \rightarrow (0, \infty)$ defined by

$$\vartheta_n(t) = \frac{\Phi_n(t^{1/n'})}{t} \quad \text{and} \quad \varrho_n(t) = \frac{t}{\Phi_n^{-1}(t)^{n'}} \quad \text{for } t > 0, \tag{3.10}$$

respectively. Here, Φ_n denotes the Sobolev conjugate of Φ given by (2.51).

Theorem 3.7 (Well-posedness and regularity with L^1 data) *Let Ω be a bounded Lipschitz domain in \mathbb{R}^n and let $f \in L^1(\Omega)$. Assume that conditions (1.2)–(1.4) and (2.50) are in force. Then there exists a unique approximable solution $u \in T_0^{1,\Phi}(\Omega)$ to the Dirichlet problem (1.1). If $\{u_k\}$ is any sequence as in the definition of approximable solution, then $\nabla u_k \rightarrow \nabla u$ a.e. in Ω , where ∇u has to be understood in the sense of Eq. (2.38). Moreover,*

$$u \in L^{\vartheta_n(\cdot),\infty}(\Omega) \quad \text{and} \quad \Phi(\nabla u) \in L^{\varrho_n(\cdot),\infty}(\Omega), \tag{3.11}$$

where ϑ_n and ϱ_n are the functions defined as in (3.10).

Remark 3.8 Theorem 3.7 is relevant, and therefore stated, only under assumption (2.50). Actually, if Φ_\circ grows so fast near infinity that (2.50) is violated, and hence (2.60) is satisfied, then a weak solution certainly exists by Theorem 3.2, and, by their uniqueness, it agrees with the approximable one.

We conclude this section by considering the still more general situation when the function f in problem (1.1) is replaced by a signed Radon measure μ with finite total variation $\|\mu\|(\Omega)$. Approximable solutions to the corresponding Dirichlet problem

$$\begin{cases} -\operatorname{div} a(x, \nabla u) = \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.12}$$

can be defined in analogy with Definition 3.6, provided that convergence of the approximating sequence $\{f_k\}$ to f in $L^1(\Omega)$ is replaced by weak- $*$ convergence in the space of measures. Recall that a sequence of functions $\{f_k\} \subset L^1(\Omega)$ is said to weak- $*$ converge to μ in the space of measures if

$$\lim_{k \rightarrow \infty} \int_{\Omega} \varphi f_k \, dx = \int_{\Omega} \varphi \, d\mu \tag{3.13}$$

for every function $\varphi \in C_0(\Omega)$. Here, $C_0(\Omega)$ denotes the space of continuous functions with compact support in Ω .

Definition 3.9 (Approximable solution with measure data) *Let μ be a signed Radon measure with finite total variation on Ω . Under assumptions (1.2)–(1.4), a function $u \in T_0^{1,\Phi}(\Omega)$ is called an approximable solution to problem (3.12) if there exists a sequence $\{f_k\} \subset L^\infty(\Omega)$ weakly- $*$ converging to μ in the space of measures, such that the sequence of weak solutions $\{u_k\} \subset W_0^1\mathcal{L}^\Phi(\Omega)$ to problems (3.8) satisfies*

$$u_k \rightarrow u \quad \text{a.e. in } \Omega.$$

Apart from uniqueness, an analogue to Theorem 3.7 for approximable solutions u with measure data can be established via essentially the same proof. In particular, a.e. convergence of gradients, and hence of the nonlinear coefficient of the differential operator, as well as regularity of u and ∇u hold exactly as in the case of data in $L^1(\Omega)$.

Theorem 3.10 (Existence and regularity with measure data) *Let Ω be a bounded Lipschitz domain in \mathbb{R}^n and let μ be a signed Radon measure with finite total variation on Ω . Assume that conditions (1.2)–(1.4) are in force. Then there exists an approximable solution $u \in T_0^{1,\Phi}(\Omega)$ to the Dirichlet problem (3.12). If $\{u_k\}$ is the sequence in the definition of approximable solution then $\nabla u_k \rightarrow \nabla u$ a.e. in Ω . Moreover, u and ∇u fulfill property (3.11).*

4 Special instances

In this section we implement the results stated above in cases when the N -function Φ takes one of the forms given by (1.5)–(1.11). Model equations whose nonlinearities are driven by these specific functions Φ are also exhibited.

In what follows, the relation $\phi_1 \approx \phi_2$ between two functions $\phi_i : I \rightarrow [0, \infty]$, $i = 1, 2$, where I is either \mathbb{R}^n or $[0, \infty)$, means that there exist positive constants c_1 and c_2 such that $\phi_1(c_1x) \leq \phi_2(x) \leq \phi_1(c_2x)$ for every $x \in I$. If these inequalities hold for $|x|$ larger than some positive constant M , we shall write that $\phi_1 \approx \phi_2$ near infinity.

Example 1 A prototypical equation with a power growth in the gradient is the p -Laplace equation. In a slightly generalized form, involving a non-necessarily smooth coefficient, the corresponding Dirichlet problem reads

$$\begin{cases} -\operatorname{div}(b(x)|\nabla u|^{p-2}\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $1 < p < \infty$ and $b \in L^\infty(\Omega)$ is such that $b(x) \geq c$ for some positive constant c . Without loss of generality, here, and in similar circumstances in the following examples, we assume for simplicity that $c = 1$. Plainly, assumptions (1.3) and (1.4) are now fulfilled with Φ obeying (1.5), namely $\Phi(\xi) = |\xi|^p$. Note that, with this choice of Φ , assumption (1.4) agrees with the classical growth condition

$$|a(x, \xi)| \leq c(|\xi|^{p-1} + g(x)) \quad \text{for a.e. } x \in \Omega \text{ and every } \xi \in \mathbb{R}^n,$$

for some function $g \in L^{p'}(\Omega)$ and some constant $c > 0$. Existence and regularity of weak and approximable solutions to problem (4.1) are discussed below in items (A) and (B), respectively.

- (A) Theorem 3.2 implies that problem (4.1) has a unique weak solution u in each of the following cases:

$$1 < p < n \text{ and} \quad f \in L^{[\frac{np}{np+p-n}, p']}(\Omega), \tag{4.2}$$

$$p = n \text{ and} \quad f \in L^{[1, n']}(\Omega), \tag{4.3}$$

$$p > n \text{ and} \quad f \in L^1(\Omega). \tag{4.4}$$

Case (4.2) extends a standard result on the existence of weak solutions under the assumption that $f \in L^{\frac{np}{np+p-n}}(\Omega)$, since the latter space is strictly contained in $L^{[\frac{np}{np+p-n}, p']}(\Omega)$. As far as we know, the result in the borderline situation (4.3) is new. The conclusion under (4.4) is classical.

- (B) Assume now that $f \in L^1(\Omega)$ and $1 < p \leq n$. Theorem 3.7 yields the existence and uniqueness of an approximable solution u to problem (4.1). The existence of such a solution is guaranteed by Theorem 3.10 even if f is replaced by a signed measure μ with finite total variation on Ω . In both cases, if $1 < p < n$, then

$$u \in L^{\frac{n(p-1)}{n-p}, \infty}(\Omega) \quad \text{and} \quad |\nabla u| \in L^{\frac{n(p-1)}{n-1}, \infty}(\Omega). \tag{4.5}$$

In the limiting case when $p = n$, the approximable solution in question fulfills

$$u \in \exp L(\Omega) \quad \text{and} \quad |\nabla u| \in L^{\ell(\cdot), \infty}(\Omega), \tag{4.6}$$

where $\varrho(t) \approx \frac{t^n}{\log t}$ near infinity. Property (4.5) is nowadays classical—see [11]. Equation (4.6) is a special case of [27, Example 3.4]. In [30] it is shown that, indeed, $|\nabla u| \in L^{n,\infty}(\Omega)$ when $p = n$. This stronger piece of information is derived via ad hoc sophisticated techniques, exploiting the fact that the differential operator has exactly an n -growth.

Example 2 Consider next the case when problem (1.1) has still an isotropic growth, but not necessarily of power-type. A model with this regard is provided by the problem

$$\begin{cases} -\operatorname{div} \left(b(x) \frac{A(|\nabla u|)}{|\nabla u|^2} \nabla u \right) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.7}$$

where A is an N -function and $b \in L^\infty(\Omega)$ is such that $b(x) \geq 1$. Clearly, problem (4.7) reduces to (4.1) when $A(t) = t^p$ for some $p > 1$. Assumptions (1.3) and (1.4) are satisfied with Φ given by (1.6), i.e. $\Phi(\xi) = A(|\xi|)$ for $\xi \in \mathbb{R}^n$. In particular, owing to the first inequality in (2.10), assumption (1.4) is equivalent to

$$|a(x, \xi)| \leq c(A(|\xi|)/|\xi| + g(x)) \quad \text{for a.e. } x \in \Omega \text{ and every } \xi \in \mathbb{R}^n,$$

for some function $g \in L^{\tilde{A}}(\Omega)$ and some constant. This agrees with a growth condition typically imposed under the Δ_2 -condition on A . Of course, here the expression $A(|\xi|)/|\xi|$ has to be understood as 0 if $\xi = 0$. Since $\Phi_\circ(t) = A(t)$ in the situation at hand, our conclusions about weak solutions and approximable solutions to problem (4.7) can be deduced from Theorems 3.2 and 3.7 just on replacing Φ_\circ by A in all relevant occurrences.

For instance, consider the case when

$$A(t) \approx t^p (\log t)^\alpha \quad \text{near infinity,} \tag{4.8}$$

where either $p > 1$ and $\alpha \in \mathbb{R}$, or $p = 1$ and $\alpha > 0$.

The conclusions described below can be derived via our general results. Equation (2.27) is also exploited for such a derivation. In what follows, $E[\exp L^{\frac{1}{\alpha}}, n](\Omega)$ denotes the space defined as in (2.21), with $A(t) \approx e^{t^{1/\alpha}}$ near infinity.

(A) Theorem 3.2 tells us that problem (4.7) admits a unique weak solution u under any of the following assumptions:

$$p = 1 \text{ and } \alpha > 0, \quad \text{and } f \in E[\exp L^{\frac{1}{\alpha}}, n](\Omega), \tag{4.9}$$

$$\begin{cases} \text{either } 1 < p < n, \alpha \in \mathbb{R}, \\ \text{or } p = n, \alpha \leq n - 1, \end{cases} \quad \text{and } f \in L^{[\frac{np}{np+p-n}, p']}(\log L)^{-\frac{\alpha}{p-1}}(\Omega), \tag{4.10}$$

$$\begin{cases} \text{either } p > n, \\ \text{or } p = n \text{ and } \alpha > n - 1, \end{cases} \quad \text{and } f \in L^1(\Omega). \tag{4.11}$$

(B) If $f \in L^1(\Omega)$, then Theorem 3.7 provides us with the existence and uniqueness of an approximable solution u to problem (4.7). When f is replaced by a signed measure μ with finite total variation, Theorem 3.10 applies to ensure the existence of a solution of the same kind. Moreover, in both cases:

(i) if $1 \leq p < n$, then

$$u \in L^{\vartheta(\cdot), \infty}(\Omega) \quad \text{and} \quad \nabla u \in L^{\varrho(\cdot), \infty}(\Omega),$$

$$\begin{aligned} \text{where } \vartheta(t) &\approx t^{\frac{n(p-1)}{n-p}} (\log t)^{\frac{n\alpha}{n-p}} \text{ and} \\ \varrho(t) &\approx t^{\frac{n(p-1)}{n-1}} (\log t)^{\frac{n\alpha}{n-1}} \text{ near infinity;} \end{aligned} \tag{4.12}$$

(ii) if $p = n$ and $\alpha < n - 1$, then

$$\begin{aligned} u \in \exp L^{\frac{n-1}{n-1-\alpha}}(\Omega) \text{ and } \nabla u \in L^{\varrho(\cdot),\infty}(\Omega) \\ \text{where } \varrho(t) \approx t^n (\log t)^{\frac{\alpha n}{n-1}-1} \text{ near infinity;} \end{aligned} \tag{4.13}$$

(iii) if $p = n$ and $\alpha = n - 1$, then

$$\begin{aligned} u \in \exp \exp L(\Omega) \text{ and } \nabla u \in L^{\varrho(\cdot),\infty}(\Omega), \\ \text{where } \varrho(t) \approx t^n (\log t)^{n-1} (\log \log t)^{-1} \text{ near infinity.} \end{aligned} \tag{4.14}$$

Properties (4.12), (4.13) and (4.14) were established in [27, Example 3.4], except for the case when $p = 1$ in (4.12), which is new. This case involves an N -function A that does not satisfy the ∇_2 -condition near infinity, a situation that is not contemplated in [27].

Example 3 Pattern anisotropic problems have the form

$$\begin{cases} -\sum_{i=1}^n (b_i(x)|u_{x_i}|^{p_i-2}u_{x_i})_{x_i} = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.15}$$

where u_{x_i} denotes the partial derivative of u with respect to the variable x_i , the functions $b_i \in L^\infty(\Omega)$ are such that $b_i(x) \geq 1$, and $p_i > 1$ for $i = 1, \dots, n$. Here, assumptions (1.3) and (1.4) are fulfilled with Φ as in (1.7), namely $\Phi(\xi) = \sum_{i=1}^n |\xi_i|^{p_i}$ for $\xi \in \mathbb{R}^n$. One has that

$$\Phi_o(t) \approx t^{\bar{p}} \quad \text{for } t \geq 0, \tag{4.16}$$

where \bar{p} denotes the harmonic mean of the exponents p_i . Namely,

$$\bar{p} = \frac{1}{\frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}}. \tag{4.17}$$

Equation (4.16) is a special case of (4.21) below.

Our results with regard to problem (4.15) can be described as follows.

- (A) Owing to Theorem 3.2, a unique weak solution to problem (4.15) exists under the same conditions as in (4.2)–(4.4), with p replaced by \bar{p} .
- (B) When $f \in L^1(\Omega)$ and $1 < \bar{p} \leq n$, Theorem 3.7 yields the existence and uniqueness of an approximable solution u to problem (4.15). An approximable solution also exists, owing to Theorem 3.10, if a signed measure μ with finite total variation replaces f in problem (4.15). Moreover, if $1 < \bar{p} < n$, then

$$u \in L^{\frac{n(\bar{p}-1)}{n-\bar{p}},\infty}(\Omega) \quad \text{and} \quad u_{x_i} \in L^{\frac{p_i n(\bar{p}-1)}{(n-1)\bar{p}},\infty}(\Omega) \quad \text{for } i = 1, \dots, n, \tag{4.18}$$

whereas, if $\bar{p} = n$, then

$$u \in \exp L(\Omega) \quad \text{and} \quad u_{x_i} \in L^{\varrho_i(\cdot),\infty}(\Omega), \quad \text{where } \varrho_i(t) \approx \frac{t^{p_i}}{\log t} \text{ near infinity.} \tag{4.19}$$

Property (4.18) extends and enhances a result of [15], proved only for $p_i \geq 2, i = 1, \dots, n$, and yielding the weaker piece of information that $u_{x_i} \in L^q(\Omega)$ for every $q < \frac{p_i n(\bar{p}-1)}{(n-1)\bar{p}}$.

Example 4 Problem (4.15) is a distinguished member of a more general class of problems taking the form

$$\begin{cases} -\sum_{i=1}^n \left(b_i(x) \frac{A_i(|u_{x_i}|)}{|u_{x_i}|^2} u_{x_i} \right)_{x_i} = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.20}$$

where A_i are N -functions, and $b_i \in L^\infty(\Omega)$ are such that $b_i(x) \geq 1$, for $i = 1, \dots, n$. A choice of the function Φ that renders assumptions (1.3) and (1.4) true is now (1.8), i.e. $\Phi(\xi) = \sum_{i=1}^n A_i(|\xi_i|)$ for $\xi \in \mathbb{R}^n$. One can show that

$$\Phi_\circ(t) \approx \bar{A}(t) \quad \text{near infinity,}$$

where \bar{A} is the N -function obeying

$$\bar{A}^{-1}(\tau) = \left(\prod_{i=1}^n A_i^{-1}(\tau) \right)^{\frac{1}{n}} \quad \text{for } \tau \geq 0, \tag{4.21}$$

see [23, Eq. 1.9]. Thus, our results about weak and approximable solutions to problem (4.20) follow from Theorems 3.2, 3.7, and 3.10 on replacing Φ_\circ by \bar{A} throughout.

To give the flavor of the conclusions that can be derived from these theorems, let us test them on the example given by choosing

$$A_i(t) \approx t^{p_i} (\log t)^{\alpha_i} \quad \text{near infinity,} \tag{4.22}$$

where either $p_i > 1$ and $\alpha_i \in \mathbb{R}$, or $p_i = 1$ and $\alpha_i > 0$, for $i = 1, \dots, n$. Let \bar{p} be given by (4.17), and let $\bar{\alpha}$ be defined as

$$\bar{\alpha} = \frac{\bar{p}}{n} \sum_{i=1}^n \frac{\alpha_i}{p_i}.$$

One can verify via (4.21) that

$$\bar{A}(t) \approx t^{\bar{p}} (\log t)^{\bar{\alpha}} \quad \text{near infinity.}$$

Then we have what follows.

- (A) The existence and uniqueness of a weak solution to problem (4.20), with A_i given by (4.22), depends on the exponents p_i and α_i only through \bar{p} and $\bar{\alpha}$, according to the same assumptions as in (4.9)–(4.11), with p and α replaced by \bar{p} and $\bar{\alpha}$.
- (B) Theorem 3.7 or Theorem 3.10 ensure that an approximable solution u to problem (4.20), with A_i given by (4.22), exists whenever $f \in L^1(\Omega)$, or f is replaced by a signed measure with finite total variation, respectively. In the former case, the uniqueness of the solution is also assured. In both cases:
 - (i) if $1 \leq \bar{p} < n$, then

$$\begin{aligned} & u \in L^{\vartheta(\cdot), \infty}(\Omega) \quad \text{and} \quad u_{x_i} \in L^{\varrho_i(\cdot), \infty}(\Omega) \quad \text{for } i = 1, \dots, n, \\ & \text{where } \vartheta(t) \approx t^{\frac{n(\bar{p}-1)}{n-\bar{p}}} (\log t)^{\frac{n\bar{\alpha}}{n-\bar{p}}} \quad \text{and} \quad \varrho_i(t) \approx t^{\frac{p_i n(\bar{p}-1)}{(n-1)\bar{p}}} (\log t)^{\frac{n(\alpha_i(\bar{p}-1)+\bar{\alpha})}{(n-1)\bar{p}}} \quad \text{near infinity;} \end{aligned} \tag{4.23}$$

(ii) if $\bar{p} = n$ and $\bar{\alpha} < n - 1$, then

$$u \in \exp L^{\frac{n-1}{n-1-\bar{\alpha}}}(\Omega) \quad \text{and} \quad u_{x_i} \in L^{\varrho_i(\cdot), \infty}(\Omega) \quad \text{for } i = 1, \dots, n,$$

where $\varrho_i(t) \approx t^{p_i} (\log t)^{\frac{\alpha_i(n-1)+\bar{\alpha}}{n-1}-1}$ near infinity;

(4.24)

(iii) if $\bar{p} = n$ and $\bar{\alpha} = n - 1$, then

$$u \in \exp \exp L(\Omega) \quad \text{and} \quad u_{x_i} \in L^{\varrho_i(\cdot), \infty}(\Omega) \quad \text{for } i = 1, \dots, n,$$

where $\varrho_i(t) \approx t^{p_i} (\log t)^{\alpha_i-1} (\log \log t)^{-1}$ near infinity.

(4.25)

Example 5 Assume that $\Omega \subset \mathbb{R}^2$, and consider any Dirichlet problem

$$\begin{cases} -\operatorname{div} a(x, \nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.26}$$

under assumptions (1.2)–(1.4), with Φ given by (1.10), namely $\Phi(\xi) = |\xi_1 - \xi_2|^p + |\xi_1|^q \log(c + |\xi_1|)^\alpha$ for $\xi \in \mathbb{R}^2$, with $p > 1$ and either $q \geq 1$ and $\alpha > 0$, or $q = 1$ and $\alpha > 0$. Let Φ_2 be the function associated with this Φ as in (2.51), with $n = 2$. One has that

- (i) if $pq < p + q$, then $\Phi_2(t) \approx s^{\frac{2pq}{p+q-pq}} \log^{\frac{p\alpha}{p+q-pq}}(t)$ near infinity,
- (ii) if $pq = p + q$ and $p\alpha < p + q$, then $\Phi_2(t) \approx \exp\left(t^{\frac{2(p+q)}{p+q-p\alpha}}\right)$ near infinity,
- (iii) if $pq = p\alpha = p + q$, then $\Phi_2(t) \approx \exp(\exp(t^2))$ near infinity,
- (iv) if either $pq > p + q$, or $pq = p + q$ and $\alpha > q$, then condition (2.60) holds,

see [23, Sect. 1]. Thus the following conclusions hold.

(A) Owing to Theorem 3.2, problem (4.26) admits a unique weak solution u under any of the following assumptions:

$$\begin{cases} \text{either } pq < p + q, \\ \text{or } pq = p + q \text{ and } \alpha \leq q, \end{cases} \quad \text{and} \quad f \in L^{\left[\frac{2pq}{3pq-p-q}, \frac{2pq}{2pq-p-q}\right]} (\log L)^{-\frac{\alpha p}{2pq-p-q}}(\Omega), \tag{4.27}$$

$$\begin{cases} \text{either } pq > p + q, \\ \text{or } pq = p + q \text{ and } \alpha > q, \end{cases} \quad \text{and} \quad f \in L^1(\Omega). \tag{4.28}$$

(B) Problem (4.26) has an approximable solution u if either $f \in L^1(\Omega)$, or f is replaced by a measure μ with finite total variation. In the former case, the solution is also unique. These assertions are consequences of Theorems 3.7 and 3.10. Also,

(i) if $pq < p + q$, then

$$u \in L^{\vartheta(\cdot), \infty}(\Omega),$$

where $\vartheta(t) \approx t^{\frac{pq}{p+q-pq}-1} (\log t)^{\frac{\alpha p}{p+q-pq}}$ near infinity,

(4.29)

$$u_{x_1} \in L^{\varrho_1(\cdot), \infty}(\Omega) \quad \text{and} \quad u_{x_1} - u_{x_2} \in L^{\varrho_2(\cdot), \infty}(\Omega),$$

where $\varrho_1(t) \approx t^{q(2-\frac{1}{p})-1} (\log t)^{\alpha(2-\frac{1}{p})}$ and

$$\varrho_2(t) \approx t^{q(2-\frac{1}{p})-1} (\log t)^{\frac{\alpha}{q}} \text{ near infinity;}$$
(4.30)

(ii) if $pq = p + q$ and $\alpha < q$, then

$$u \in \exp L^{\frac{q}{q-\alpha}}(\Omega), \quad u_{x_1} \in L^{\varrho_1(\cdot), \infty}(\Omega), \quad \text{and} \quad u_{x_1} - u_{x_2} \in L^{\varrho_2(\cdot), \infty}(\Omega),$$

$$\text{where } \varrho_1(t) \approx t^q (\log t)^{\frac{\alpha}{q}} \quad \text{and} \quad \varrho_2(t) \approx t^p (\log t)^{\frac{\alpha}{q}-1} \quad \text{near infinity}; \tag{4.31}$$

(iii) if $pq = p + q$ and $\alpha = q$, then

$$u \in \exp \exp L(\Omega), \quad u_{x_1} \in L^{\varrho_1(\cdot), \infty}(\Omega), \quad \text{and} \quad u_{x_1} - u_{x_2} \in L^{\varrho_2(\cdot), \infty}(\Omega),$$

$$\text{where } \varrho_1(t) \approx t^q (\log t)^\alpha (\log \log t)^{-1} \quad \text{and}$$

$$\varrho_2(t) \approx t^p (\log \log t)^{-1} \quad \text{near infinity}. \tag{4.32}$$

Example 6 Assume that $\Omega \subset \mathbb{R}^2$, and consider any Dirichlet problem as in (4.26), with Φ now given by (1.11), namely $\Phi(\xi) = |\xi_1 + 3\xi_2|^p + e^{|\xi_1 - \xi_2|^\beta} - 1$ for $\xi \in \mathbb{R}^2$, where $p > 1$ and $\beta > 1$. An analogous argument as in [23, Sect. 1] shows that

$$\Phi_\circ(t) \approx t^{2p} \log^{-\frac{p}{\beta}}(1+t) \quad \text{near infinity}.$$

Hence, condition (2.60) is in force. Theorem 3.7 then tells us that there exists a unique weak solution to problem (4.26) for every $f \in L^1(\Omega)$.

5 Proofs of approximation theorems

Here, we are concerned with proofs of the results stated in Sect. 2.5.

Proof of Proposition 2.2 By our assumption, there exists $\lambda_1 > 0$ such that $\int_\Omega \Phi((U_k - U)/\lambda_1) dx \rightarrow 0$ as $k \rightarrow \infty$, namely, $\Phi((U_k - U)/\lambda_1) \rightarrow 0$ in $L^1(\Omega)$. Hence, there exists a subsequence of $\{U_k\}$, still indexed by k , such that $U_k \rightarrow U$ a.e. in Ω , and the sequence of functions $\Phi((U_k - U)/\lambda_1)$ is pointwise bounded by a function in $L^1(\Omega)$ independent of k . Given any function $V \in L^{\tilde{\Phi}}(\Omega; \mathbb{R}^n)$, there exists $\lambda_2 > 0$ such that $\tilde{\Phi}(V/\lambda_2) \in L^1(\Omega)$. The definition of Young’s conjugate implies that

$$\frac{|V \cdot (U_k - U)|}{\lambda_1 \lambda_2} \leq \Phi\left(\frac{U_k - U}{\lambda_1}\right) + \tilde{\Phi}\left(\frac{V}{\lambda_2}\right) \quad \text{a.e. in } \Omega.$$

Hence, Eq. (2.63) follows, via the dominated convergence theorem. □

Proof of Proposition 2.3 Fix any $U \in L^\Phi(\Omega; \mathbb{R}^n)$. Set, for $\ell \in \mathbb{N}$,

$$\Omega_\ell = \{x \in \Omega : |U(x)| \leq \ell\}.$$

By Tchebyshev inequality, $|\Omega \setminus \Omega_\ell| \leq \|U\|_{L^1(\Omega; \mathbb{R}^n)}/\ell$. Next, define $U_\ell = U \chi_{\Omega_\ell}$, and notice that $|U_\ell(x)| \leq |U(x)|$ and $\Phi(U_\ell(x)) \leq \Phi(U(x))$ for $x \in \Omega$. Thus, if $\lambda \geq \|U\|_{L^\Phi(\Omega; \mathbb{R}^n)}/2$, then

$$\lim_{\ell \rightarrow \infty} \int_\Omega \Phi\left(\frac{U_\ell - U}{2\lambda}\right) dx = \lim_{\ell \rightarrow \infty} \int_{\Omega \setminus \Omega_\ell} \Phi\left(\frac{U}{2\lambda}\right) dx = 0. \tag{5.1}$$

Let \tilde{U}_ℓ denote the representative of the function U_ℓ which is defined everywhere in Ω as the limit of its averages on balls at each Lebesgue point, and by 0 elsewhere. Fix any $\ell, k \in \mathbb{N}$, and set $Q = [-\ell, \ell]^n$. We split Q into a family of $N(k)$ cubes Q_i^k of diameter $\frac{1}{k}$ defined as follows. Consider a dyadic decomposition of Q , and distribute the boundaries of the dyadic cubes Q_i^k

in such a way that they are pairwise disjoint, and $Q = \cup_{i=1}^{N(k)} Q_i^k$. Define $y_i = \operatorname{argmin} \Phi|_{Q_i^k}$ for $i = 1, \dots, N(k)$. On setting $E_i^k = \tilde{U}_\ell^{-1}(Q_i^k)$, we have that $\Omega = \cup_{i=1}^{N(k)} E_i^k$. Since Q_i^k is a Borel set and $\tilde{U}_\ell \in \mathcal{M}(\Omega; \mathbb{R}^n)$, the set E_i^k is measurable. Therefore, the family $\{E_i^k : i = 1, \dots, N(k)\}$ is a partition of Ω into pairwise disjoint measurable sets. Next, define the function $U_{\ell,k} : \Omega \rightarrow \mathbb{R}^n$ as

$$U_{\ell,k} = \sum_{i=1}^{N(k)} y_i \chi_{E_i^k}.$$

We have that $\lim_{k \rightarrow \infty} U_{\ell,k}(x) = \tilde{U}_\ell(x)$ for every $x \in \Omega$. Indeed, $U_{\ell,k}(x) = y_i$ for every $x \in E_i^k$, whence $|y_i - \tilde{U}_\ell(x)| \leq \operatorname{diam} Q_i^k \leq \frac{1}{k}$ for every such x . As a consequence, $\lim_{k \rightarrow \infty} U_{\ell,k}(x) = U_\ell(x)$ for a.e. $x \in \Omega$. On the other hand, $\Phi(U_{\ell,k}(x)/\lambda) = \Phi(y_i/\lambda) \leq \Phi(U_\ell(x)/\lambda)$ for every $\ell, k \in \mathbb{N}$, and $x \in E_i^k$. Hence, owing to Jensen's inequality,

$$\int_\Omega \Phi\left(\frac{U_{\ell,k} - U_\ell}{2\lambda}\right) dx \leq \frac{1}{2} \int_\Omega \Phi\left(\frac{U_{\ell,k}}{\lambda}\right) dx + \frac{1}{2} \int_\Omega \Phi\left(\frac{U_\ell}{\lambda}\right) dx \leq \int_\Omega \Phi\left(\frac{U_\ell}{\lambda}\right) dx$$

for every $\ell, k \in \mathbb{N}$. Therefore, thanks to the dominated convergence theorem,

$$\lim_{k \rightarrow \infty} \int_\Omega \Phi\left(\frac{U_{\ell,k} - U_\ell}{2\lambda}\right) dx = 0 \tag{5.2}$$

for every $\ell \in \mathbb{N}$. By the convexity of Φ ,

$$\int_\Omega \Phi\left(\frac{U_{\ell,k} - U}{4\lambda}\right) dx \leq \frac{1}{2} \int_\Omega \Phi\left(\frac{U_{\ell,k} - U_\ell}{2\lambda}\right) dx + \frac{1}{2} \int_\Omega \Phi\left(\frac{U_\ell - U}{2\lambda}\right) dx \tag{5.3}$$

for every $\ell, k \in \mathbb{N}$. Owing to equations (5.1) and (5.2), the left-hand side of (5.3) tends to 0 as $k \rightarrow \infty$. A diagonal argument then completes the proof. \square

With Proposition 2.3 at our disposal, we are ready to prove Proposition 2.4. The proof to be presented is based on ideas of that of [39, Theorem 2.2].

Proof of Proposition 2.4 Assume, for the time being, that Ω is starshaped with respect to the ball $B_r(0)$, centered at 0 and with radius r . This means that Ω is starshaped with respect to every point in $B_r(0)$. Without loss of generality, we may assume that $r < 2$. Let $k \in \mathbb{N}$ be so large that $\frac{1}{k} \in (0, \frac{r}{4})$, and set $\gamma_k = 1 - \frac{2}{rk} < 1$. For any such k , we define the set

$$\Omega_k = \gamma_k \Omega + \frac{1}{k} B_1(0). \tag{5.4}$$

Our choice of k and γ_k ensures that $\Omega_k \subset\subset \Omega$. Let $m \in \mathbb{N}$, and let $U \in L^1_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m)$ be such that $U = 0$ in $\mathbb{R}^n \setminus \Omega$. Define $U_k : \Omega \rightarrow \mathbb{R}^m$ as

$$U_k(x) = \int_{\mathbb{R}^n} \rho_k(x - y) U(y/\gamma_k) dy \quad \text{for } x \in \Omega, \tag{5.5}$$

where $\rho_k(x) = \rho(kx)k^n$ is a standard smoothing kernel on \mathbb{R}^n , i.e. ρ is a nonnegative radially decreasing function, $\rho \in C^\infty(\mathbb{R}^n)$, $\operatorname{supp} \rho \subset\subset B_1(0)$ and $\int_{\mathbb{R}^n} \rho(x) dx = 1$. Since $U(y/\gamma_k) = 0$ if $y \notin \gamma_k \Omega$, one has that $U_k \in C^\infty_0(\Omega; \mathbb{R}^m)$. Moreover, if $U \in L^\infty(\Omega; \mathbb{R}^m)$, then

$$\|U_k\|_{L^\infty(\Omega; \mathbb{R}^m)} \leq \|U\|_{L^\infty(\Omega; \mathbb{R}^m)}. \tag{5.6}$$

We claim that, if $m = n$, then

$$\int_{\Omega} \Phi(U_k) dx \leq \int_{\Omega} \Phi(U) dx \tag{5.7}$$

for k as above. Indeed,

$$\begin{aligned} \int_{\Omega} \Phi(U_k(x)) dx &= \int_{\mathbb{R}^n} \Phi \left(\int_{\mathbb{R}^n} \rho_k(x - y) U(y/\gamma_k) dy \right) dx \\ &\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho_k(x - y) \Phi(U(y/\gamma_k)) dy dx \\ &= \int_{\mathbb{R}^n} \Phi(U(y/\gamma_k)) \int_{\mathbb{R}^n} \rho_k(x - y) dx dy = \gamma_k^n \int_{\mathbb{R}^n} \Phi(U(z)) dz \\ &= \gamma_k^n \int_{\Omega} \Phi(U(z)) dz \\ &\leq \int_{\Omega} \Phi(U(z)) dz, \end{aligned}$$

where the first equality holds since $U_k = 0$ in $\mathbb{R}^n \setminus \Omega$ and $\Phi(0) = 0$, the first inequality follows from Jensen’s inequality, and the third equality is due to the fact that $\int_{\mathbb{R}^n} \rho_k(x - y) dx = 1$ for every $y \in \mathbb{R}^n$.

Assume now that $u \in W_0^1 L^\Phi(\Omega)$. As observed above, the function u_k , defined as in (5.5), belongs to $C_0^\infty(\Omega)$. Moreover, since the continuation of u to \mathbb{R}^n by 0 outside Ω is weakly differentiable in \mathbb{R}^n , the function $\Omega \ni x \mapsto u(x/\gamma_k)$ is weakly differentiable in Ω . Thus,

$$(\nabla u)_k = \nabla u_k \quad \text{in } \Omega, \tag{5.8}$$

where $(\nabla u)_k$ is defined as in (5.5), with $U = \nabla u$. We shall show that there exists $\lambda > 0$ such that

$$\lim_{k \rightarrow \infty} \int_{\Omega} \Phi \left(\frac{\nabla u_k - \nabla u}{\lambda} \right) dx = 0. \tag{5.9}$$

Owing to (5.8), Eq. (5.9) will follow if we prove that

$$\lim_{k \rightarrow \infty} \int_{\Omega} \Phi \left(\frac{(\nabla u)_k - \nabla u}{\lambda} \right) dx = 0 \quad \text{for some } \lambda > 0. \tag{5.10}$$

Fix any $\sigma > 0$. By Propositions 2.3, there exist $\lambda > 0$ and a simple function $V : \Omega \rightarrow \mathbb{R}^n$ such that

$$\int_{\Omega} \Phi \left(\frac{\nabla u - V}{\frac{1}{3}\lambda} \right) dx < \sigma. \tag{5.11}$$

The convexity of Φ ensures that

$$\begin{aligned} \int_{\Omega} \Phi \left(\frac{(\nabla u)_k - \nabla u}{\lambda} \right) dx &= \int_{\Omega} \Phi \left(\frac{(\nabla u)_k - V_k + V_k - V + V - \nabla u}{\lambda} \right) dx \\ &\leq \frac{1}{3} \int_{\Omega} \Phi \left(\frac{(\nabla u)_k - V_k}{\frac{1}{3}\lambda} \right) dx + \frac{1}{3} \int_{\Omega} \Phi \left(\frac{V_k - V}{\frac{1}{3}\lambda} \right) dx + \frac{1}{3} \int_{\Omega} \Phi \left(\frac{V - \nabla u}{\frac{1}{3}\lambda} \right) dx. \end{aligned} \tag{5.12}$$

By (5.7) and (5.11),

$$\int_{\Omega} \Phi \left(\frac{(\nabla u)_k - V_k}{\frac{1}{3}\lambda} \right) dx = \int_{\Omega} \Phi \left(\frac{(\nabla u - V)_k}{\frac{1}{3}\lambda} \right) dx < \sigma. \tag{5.13}$$

On the other hand, owing to Jensen’s inequality and Fubini’s theorem

$$\begin{aligned} \int_{\Omega} \Phi \left(\frac{V_k - V}{\frac{1}{3}\lambda} \right) dx &= \int_{\Omega} \Phi \left(\frac{3}{\lambda} \int_{B_1(0)} \rho(y) (V((x - y/k)/\gamma_k) - V(x)) dy \right) dx \\ &\leq \int_{B_1(0)} \rho(y) \int_{\Omega} \Phi \left(\frac{3}{\lambda} (V((x - y/k)/\gamma_k) - V(x)) \right) dx dy. \end{aligned} \tag{5.14}$$

Therefore

$$\lim_{k \rightarrow \infty} \Phi \left(\frac{3}{\lambda} (V((x - y/k)/\gamma_k) - V(x)) \right) = 0 \quad \text{for a.e. } x \in \Omega \text{ and every } y \in B_1(0).$$

Moreover,

$$\Phi \left(\frac{3}{\lambda} (V((x - y/k)/\gamma_k) - V(x)) \right) \leq C$$

for some constant C , and for every $x \in \Omega$, $y \in B_1(0)$ and k such that $\frac{1}{k} \in (0, \frac{r}{4})$. Hence, by the dominated convergence theorem,

$$\lim_{k \rightarrow \infty} \int_{\Omega} \Phi \left(\frac{3}{\lambda} (V((x - y/k)/\gamma_k) - V(x)) \right) dx = 0 \quad \text{for every } y \in B_1(0).$$

Furthermore,

$$\int_{\Omega} \Phi \left(\frac{3}{\lambda} (V((x - y/k)/\gamma_k) - V(x)) \right) dx \leq C|\Omega|$$

for every $y \in B_1(0)$ and every k such that $\frac{1}{k} \in (0, \frac{r}{4})$. Consequently, the rightmost side of (5.14) converges to zero as $k \rightarrow \infty$, thanks to the dominated convergence theorem again, whence

$$\lim_{k \rightarrow \infty} \int_{\Omega} \Phi \left(\frac{V_k - V}{\frac{1}{3}\lambda} \right) dx = 0. \tag{5.15}$$

Inequality (5.10) follows from (5.11)–(5.13) and (5.15), owing to the arbitrariness of σ . This completes the proof in the case when Ω is a starshaped domain.

Assume now that Ω is any bounded Lipschitz domain in \mathbb{R}^n . Then, there exist a finite family of open sets $\omega_1, \dots, \omega_J$ and a corresponding family of balls B_1, \dots, B_J , with radii r_1, \dots, r_J , such that $\Omega = \cup_{k=1}^J \omega_j$, and every set ω_j is starshaped with respect to the ball B_j . Let us introduce a partition of unity θ_j subordinated to the family $\{\omega_j\}$. Any function $u \in W_0^1 L^\Phi(\Omega)$ admits the decomposition

$$u(x) = \sum_{j=1}^J \theta_j(x)u(x) \quad \text{for } x \in \Omega. \tag{5.16}$$

If $u \in W_0^1 L^\Phi(\Omega)$, then $\nabla u \in L^\Phi(\Omega; \mathbb{R}^n)$ and $u \in L^\infty(\Omega)$, whence $\nabla(\theta_j u) = (u \nabla \theta_j + \theta_j \nabla u) \in L^\Phi(\Omega; \mathbb{R}^n)$. Therefore, $\theta_j u \in W_0^1 L^\Phi(\omega_j)$. Property (2.66) then follows on applying to each function $\theta_j u$ the result for domains starshaped with respect to balls.

Inequality (2.65) is a consequence of inequality (5.6) and of the representation formula (5.16).

As far as property (2.64) is concerned, choose any $\lambda > 0$ such that

$$\lim_{k \rightarrow \infty} \int_{\Omega} \Phi \left(\frac{\nabla u_k - \nabla u}{\lambda} \right) dx = 0. \tag{5.17}$$

By inequality (2.47),

$$\int_{\Omega} \Phi_{\circ} \left(\frac{\kappa_1 |u_k - u|}{|\Omega|^{\frac{1}{n}} \lambda} \right) dx \leq \int_{\Omega} \Phi \left(\frac{\nabla u_k - \nabla u}{\lambda} \right) dx \tag{5.18}$$

for every $k \in \mathbb{N}$. From (5.17) and an application of Jensen’s inequality to the integral on the left-hand side of inequality (5.18) we infer that $u_k \rightarrow u$ in $L^1(\Omega)$. Hence, Eq. (2.64) follows on taking a subsequence if necessary. \square

6 Weak solutions: Proof of Theorem 3.2

The present section is split into subsections, corresponding to subsequent steps towards a proof of Theorem 3.2.

6.1 Regularized problems

We begin by constructing a sequence of problems approximating (1.1), and whose principal part satisfies isotropic ellipticity and growth conditions.

Let $A : [0, \infty) \rightarrow [0, \infty)$ be a strictly convex N -function such that $A \in C^1([0, \infty))$. In particular, $A'(0) = 0$. Hence, the function

$$\mathbb{R}^n \ni \xi \mapsto A(|\xi|) \in [0, \infty)$$

is a continuously differentiable radially increasing n -dimensional N -function, whose gradient agrees with $A'(|\xi|) \frac{\xi}{|\xi|}$ for $\xi \in \mathbb{R}^n$, with the convention that the latter expression has to be interpreted as 0 when $\xi = 0$. The equality case in Young’s inequality yields

$$tA'(t) = A(t) + \tilde{A}(A'(t)) \quad \text{for } t \geq 0. \tag{6.1}$$

Moreover, since A is strictly convex,

$$\left(A'(|\xi|) \frac{\xi}{|\xi|} - A'(|\eta|) \frac{\eta}{|\eta|} \right) \cdot (\xi - \eta) > 0 \quad \text{for every } \xi \neq \eta. \tag{6.2}$$

Given $\varepsilon \in (0, 1)$, we define $a^\varepsilon : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$a^\varepsilon(x, \xi) = a(x, \xi) + \varepsilon A'(|\xi|) \frac{\xi}{|\xi|} \quad \text{for } x \in \Omega \text{ and } \xi \in \mathbb{R}^n, \tag{6.3}$$

and consider the problem

$$\begin{cases} -\operatorname{div} a^\varepsilon(x, \nabla u^\varepsilon) = f & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.4}$$

We shall show that the function $a^\varepsilon(x, \cdot)$ satisfies isotropic ellipticity and growth conditions, that allow to make use of an existence theory available in the literature. A priori estimates for u^ε , independent of $\varepsilon \in (0, 1)$, will then be derived.

Proposition 6.1 (Existence of solutions to regularized problems) *Let Ω be a bounded Lipschitz domain in \mathbb{R}^n . Assume that $a : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a Carathéodory function satisfying assumptions (1.2)–(1.4) for some n -dimensional N -function Φ . Let $A(t)$ be any continuously differentiable strictly convex N -function in $[0, \infty)$ that grows essentially faster than t^q near infinity for some $q > n$, and such that*

$$A(|\xi|) \geq \Phi(\xi) \quad \text{for } \xi \in \mathbb{R}^n. \tag{6.5}$$

Let $\varepsilon \in (0, 1)$ and let a^ε be defined as in (6.3). If $f \in L^1(\Omega)$, then there exists a weak solution $u^\varepsilon \in W_0^1 \mathcal{L}^A(\Omega) \cap L^\infty(\Omega)$ to problem (6.4).

The following function spaces will come into play in the proof of Proposition 6.1. Let us denote by $\mathcal{W}_0^1 L^A(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^1 L^A(\Omega)$ with respect to the weak topology $\sigma(L^A \times L^A, E^{\tilde{A}} \times E^{\tilde{A}})$. One has that

$$\mathcal{W}_0^1 L^A(\Omega) \subset W_0^1 L^A(\Omega), \tag{6.6}$$

see [38]. Moreover, we shall consider the space of distributions defined as

$$W^{-1} E^{\tilde{A}}(\Omega) = \left\{ f \in \mathcal{D}'(\Omega) : f = f_0 - \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}, \quad f_i \in E^{\tilde{A}}(\Omega), \quad i = 0, \dots, n \right\}. \tag{6.7}$$

Proof of Proposition 6.1 We begin by showing that, under condition (6.5), the function a^ε fulfills the assumptions required in [37, Sect. 5]. Besides being a Carathéodory’s function, those assumptions on a^ε amount to a monotonicity condition that immediately follows from (6.2) and (1.2), and to an estimate of the form

$$|a^\varepsilon(x, \xi)| \leq c\tilde{A}^{-1}(cA(|c\xi|)) + c\tilde{A}^{-1}(ch(x)) \quad \text{for a.e. } x \in \Omega \text{ and for } \xi \in \mathbb{R}^n, \tag{6.8}$$

for some positive constant c . To verify inequality (6.8), observe that, by inequality (2.8),

$$a^\varepsilon(x, \xi) \cdot \xi \leq A\left(\left|\frac{2}{c\Phi}\xi\right|\right) + \tilde{A}\left(\left|\frac{c\Phi}{2}a^\varepsilon(x, \xi)\right|\right) \tag{6.9}$$

for a.e. $x \in \Omega$ and for $\xi \in \mathbb{R}^n$. Inequality (6.5) implies that $\tilde{A}(|\xi|) \leq \tilde{\Phi}(\xi)$ for $\xi \in \mathbb{R}^n$. Hence, via inequalities (1.3) and (1.4),

$$\begin{aligned} a^\varepsilon(x, \xi) \cdot \xi &\geq \Phi(\xi) + \varepsilon A(|\xi|) + \varepsilon \tilde{A}(A'(|\xi|)) \geq \tilde{\Phi}(c\Phi a(x, \xi)) + \tilde{A}(\varepsilon A'(|\xi|)) - h(x) \\ &\geq 2\left(\frac{1}{2}\tilde{A}(c\Phi |a(x, \xi)|) + \frac{1}{2}\tilde{A}(c\Phi \varepsilon A'(|\xi|))\right) - h(x) \\ &\geq 2\tilde{A}\left(\frac{c\Phi}{2}|a^\varepsilon(x, \xi)|\right) - h(x) \end{aligned} \tag{6.10}$$

for a.e. $x \in \Omega$ and for $\xi \in \mathbb{R}^n$. Combining inequalities (6.9) and (6.10) tells us that

$$\tilde{A}\left(\frac{c\Phi}{2}|a^\varepsilon(x, \xi)|\right) \leq A\left(\left|\frac{2}{c\Phi}\xi\right|\right) + h(x)$$

for a.e. $x \in \Omega$ and for $\xi \in \mathbb{R}^n$. Therefore, thanks to the monotonicity of the function \tilde{A}^{-1} , we obtain that

$$|a^\varepsilon(x, \xi)| \leq \frac{2}{c\Phi}\tilde{A}^{-1}\left(A\left(\left|\frac{2}{c\Phi}\xi\right|\right) + h(x)\right) \leq \frac{2}{c\Phi}\tilde{A}^{-1}\left(2A\left(\left|\frac{2}{c\Phi}\xi\right|\right)\right) + \frac{2}{c\Phi}\tilde{A}^{-1}(2h(x)) \tag{6.11}$$

for a.e. $x \in \Omega$ and for $\xi \in \mathbb{R}^n$. Hence, (6.8) follows.

Now, since $q > n$, we have that $q' < n'$. Then there exists a function $F \in L^{q'}(\Omega; \mathbb{R}^n)$, with $F = (F_1, \dots, F_n)$, such that

$$\operatorname{div} F = f - f_\Omega \quad \text{in } \Omega, \tag{6.12}$$

where $f_\Omega = \frac{1}{|\Omega|} \int_\Omega f(x) dx$, the mean value of f over Ω . This follows, for instance, from the use of the Bogowskii operator, and the boundedness of the latter from $L^1(\Omega)$ into $L^{q'}(\Omega)$ —see [16]. Inasmuch as $A(t)$ grows essentially faster than t^q near infinity, the function $t^{q'}$ grows essentially faster than $\tilde{A}(t)$ near infinity. Thus, $L^{q'}(\Omega) \subset E^{\tilde{A}}(\Omega)$, and hence f is a distribution of the form $f = f_\Omega - \sum_{i=1}^n \frac{\partial F_i}{\partial x_i}$ with $F_i \in E^{\tilde{A}}(\Omega)$ for $i = 1, \dots, n$. Therefore, $f \in W^{-1}E^{\tilde{A}}(\Omega)$. As a consequence, the results in [37, Sect. 5] ensure that there exists a function $u^\varepsilon \in \mathcal{W}_0^1 L^A(\Omega)$ such that $a^\varepsilon(x, \nabla u^\varepsilon) \in L^{\tilde{A}}(\Omega)$ and

$$\int_\Omega a^\varepsilon(x, \nabla u^\varepsilon) \cdot \nabla \varphi \, dx = \int_\Omega f \varphi \, dx \tag{6.13}$$

for every $\varphi \in \mathcal{W}_0^1 L^A(\Omega)$. By (6.6), $u^\varepsilon \in W_0^1 L^A(\Omega)$. Moreover, an inspection of the proof of [37, Section 5] reveals that $\int_\Omega A(\nabla u^\varepsilon) \, dx < \infty$, whence $u^\varepsilon \in W_0^1 \mathcal{L}^A(\Omega)$. Since the function $A(t)$ grows faster than t^q near infinity, one has that $W_0^1 L^A(\Omega) \rightarrow W_0^{1,q}(\Omega) \rightarrow L^\infty(\Omega)$, and hence $u^\varepsilon \in L^\infty(\Omega)$ as well.

It remains to show that Eq. (6.13) holds not only for $\varphi \in \mathcal{W}_0^1 L^A(\Omega)$, but also for every $\varphi \in W_0^1 L^A(\Omega)$, a space containing $W_0^1 \mathcal{L}^A(\Omega)$. Fix any such φ and observe that, by the embedding mentioned above, one has in fact that $\varphi \in L^\infty(\Omega)$ as well. An application of Proposition 2.4 (in the isotropic case) ensures that there exists a sequence $\{\varphi_k\} \subset C_0^\infty(\Omega)$ such that $\varphi_k \rightarrow \varphi$ a.e. in Ω , $\|\varphi_k\|_{L^\infty(\Omega)} \leq C\|\varphi\|_{L^\infty(\Omega)}$ for some constant C and for every $k \in \mathbb{N}$, and $\nabla \varphi_k \rightarrow \nabla \varphi$ modularly in $L^A(\Omega)$. Since $a^\varepsilon(x, \nabla u^\varepsilon) \in L^{\tilde{A}}(\Omega)$, by Proposition 2.2 and the dominated convergence theorem one can pass to the limit in Eq. (6.13) applied with φ replaced by φ_k , and infer that Eq. (6.13) holds for φ as well. This fact amounts to saying that u^ε is actually a weak solution to problem (6.4). \square

A priori bounds for the solution u^ε to problem (6.4), independent of $\varepsilon \in (0, 1)$, are established in Proposition 6.2 below. They are critical in obtaining a weak solution to problem (1.1) as the limit of u^ε as $\varepsilon \rightarrow 0^+$.

Proposition 6.2 (Uniform estimates in approximating problems) *Let Ω, a, Φ and A be as in Proposition 6.1. Suppose that f satisfies either of assumptions (3.2) and (3.3). Given $\varepsilon \in (0, 1)$, let u^ε be a weak solution to problem (6.4) exhibited in Proposition 6.1. Then:*

- (i) *the family $\{u^\varepsilon\}$ is uniformly bounded in $W_0^1 L^\Phi(\Omega)$,*
- (ii) *the family $\{\varepsilon A(A'(|\nabla u^\varepsilon|))\}$ is uniformly bounded in $L^1(\Omega)$,*
- (iii) *the family $\{a(x, \nabla u^\varepsilon)\}$ is uniformly bounded in $L^{\tilde{\Phi}}(\Omega; \mathbb{R}^n)$.*

Proof We shall make use of a comparison principle established in [26], that links the solution u^ε to the solution v to the symmetrized problem

$$\begin{cases} -\operatorname{div} \left(\frac{\Phi_\diamond(|\nabla v|)}{|\nabla v|^2} \nabla v \right) = f^\star(x) & \text{in } \Omega^\star \\ v = 0 & \text{on } \partial\Omega^\star, \end{cases} \tag{6.14}$$

where Φ_\diamond is defined in (2.40), Ω^\star denotes the open ball centered at the origin such that $|\Omega^\star| = |\Omega|$, and f^\star stands for the radially decreasing symmetral of f . Recall that

$f^\star(x) = f^*(\omega_n|x|^n)$ for $x \in \Omega^\star$, where ω_n denotes Lebesgue measure of the unit ball in \mathbb{R}^n . According to [26, Theorem 3.1], our alternate assumptions (3.2) or (3.3) on f ensure that problem (6.14) admits a weak solution v , given by

$$v(x) = \int_{\omega_n|x|^n}^{|\Omega|} \frac{1}{n\omega_n^{1/n}s^{1/n'}} \Psi_\diamond^{-1}\left(\frac{s^{1/n}}{n\omega_n^{1/n}} f^{**}(s)\right) ds \quad \text{for } x \in \Omega^\star, \tag{6.15}$$

where Ψ_\diamond is the function defined as in (2.43). Indeed, by (6.15),

$$|\nabla v(x)| = \Psi_\diamond^{-1}\left(\frac{(\omega_n|x|^n)^{1/n}}{n\omega_n^{1/n}} f^{**}(\omega_n|x|^n)\right) \quad \text{for a.e. } x \in \Omega^\star. \tag{6.16}$$

Thus

$$\int_{\Omega^\star} G(|\nabla v|) dx = \int_0^{|\Omega|} G\left(\Psi_\diamond^{-1}\left(\frac{s^{1/n}}{n\omega_n^{1/n}} f^{**}(s)\right)\right) ds \tag{6.17}$$

for every continuous function $G : [0, \infty) \rightarrow [0, \infty)$. Equation (6.17), with $G = \Phi_\diamond$, combined with property (v) of Lemma 2.1 and (2.41), tells us that

$$\int_{\Omega^\star} \Phi_\diamond(|\nabla v|) dx \leq \int_0^{|\Omega|} \widetilde{\Phi}_\diamond(cs^{1/n} f^{**}(s)) ds \tag{6.18}$$

for some constant c depending on n . If (3.2) is in force, then the last integral converges, owing to the very definition of the space $E[\widetilde{\Phi}_\diamond, n](\Omega)$. Suppose that, instead, (3.3) holds. Then, owing to the inequality

$$f^{**}(s) \leq \frac{1}{s} \int_0^{|\Omega|} f^*(r) dr = \frac{1}{s} \|f\|_{L^1(\Omega)} \quad \text{for } s \in (0, |\Omega|),$$

the convergence of the integral on the right-hand side of inequality (6.18) is a consequence of the fact that

$$\int_0^\infty \widetilde{\Phi}_\diamond(\lambda s^{-\frac{1}{n'}}) ds < \infty \quad \text{for every } \lambda > 0. \tag{6.19}$$

The validity of condition (6.19) can be verified via a change of variables in the integral, which tells us that it can be rewritten as

$$\int_0^\infty \frac{\widetilde{\Phi}_\diamond(t)}{t^{n'+1}} dt < \infty. \tag{6.20}$$

Condition (6.20) turns to be equivalent—see [25, Lemma 4.1]—to condition (2.60) appearing in (3.3). Altogether, we have shown that

$$\int_{\Omega^\star} \Phi_\diamond(|\nabla v|) dx < \infty \tag{6.21}$$

under either assumption (3.2) or (3.3). This implies that $v \in W_0^1 \mathcal{L}^{\Phi_\diamond}(\Omega^\star)$, and hence it is indeed a weak solution to problem (6.14).

On making use of the solution u^ε as a test function in the weak formulation of problem (6.4), and recalling assumption (1.3) we deduce that

$$\int_{\Omega} \Phi(\nabla u^\varepsilon) dx + \int_{\Omega} \varepsilon A(|\nabla u^\varepsilon|) dx + \int_{\Omega} \varepsilon \tilde{A}(A'(|\nabla u^\varepsilon|)) dx \leq \int_{\Omega} f u^\varepsilon dx. \tag{6.22}$$

In particular, inequality (6.22) ensures that $u^\varepsilon \in W_0^1 \mathcal{L}^\Phi(\Omega)$, and hence [26, Theorem 3.1] can be exploited to infer that

$$(u^\varepsilon)^*(s) \leq v^*(s) \text{ for } s \in (0, |\Omega|). \tag{6.23}$$

We now distinguish between the cases when either assumption (3.2) or (3.3) holds.

Assume first that condition (3.2) is in force. Let us replace, if necessary, Φ_\circ in the definition of $\widehat{\Phi}_\circ$ in (2.55) by another Young function Φ_\bullet fulfilling condition (2.49) and such that $\Phi_\bullet(t) = \Phi_\circ(t)$ if $t \geq 1$. For instance, one can define Φ_\bullet in such a way that it is linear in $[0, 1]$. Therefore, there exists a constant $t_1 > 0$ such that $\widetilde{\Phi}_\bullet(t) = \widetilde{\Phi}_\circ(t)$ if $t \geq t_1$. Denote by $\widehat{\Phi}_\bullet$ the function defined as in (2.55) and (2.56), with Φ_\circ replaced by Φ_\bullet . Let λ be a positive number to be fixed later. By inequality (2.59), with Φ_\circ replaced by Φ_\bullet ,

$$\int_{\Omega} f u^\varepsilon dx \leq C \left(\int_0^{|\Omega|} \widetilde{\Phi}_\bullet(\lambda s^{\frac{1}{n}} f^{**}(s)) ds + \int_0^{|\Omega|} \widehat{\Phi}_\bullet\left(\frac{1}{\lambda} s^{-\frac{1}{n}} (u^\varepsilon)^*(s)\right) ds \right). \tag{6.24}$$

Choose $\lambda = \kappa_3/c_1$, where κ_3 and c_1 are the constants appearing in inequalities (2.57) and (2.41), respectively. The following chain holds:

$$\begin{aligned} & \int_0^{|\Omega|} \widehat{\Phi}_\bullet\left(\frac{1}{\lambda} s^{-\frac{1}{n}} (u^\varepsilon)^*(s)\right) ds \leq \int_0^{|\Omega|} \widehat{\Phi}_\bullet\left(\frac{1}{\lambda} s^{-\frac{1}{n}} v^*(s)\right) ds \\ & \leq \int_{\Omega^\star} \Phi_\bullet\left(\frac{\kappa_3}{\lambda} |\nabla v|\right) dx \leq \int_0^{|\Omega|} \Phi_\bullet\left(\frac{\kappa_3}{\lambda} \left(\Psi_\diamond^{-1}\left(\frac{s^{1/n}}{n\omega_n^{1/n}} f^{**}(s)\right)\right)\right) ds \\ & \leq \int_{s_0}^{|\Omega|} \Phi_\bullet\left(\frac{\kappa_3}{\lambda} \left(\Psi_\diamond^{-1}\left(\frac{s^{1/n}}{n\omega_n^{1/n}} f^{**}(s)\right)\right)\right) ds + \int_0^{s_0} \Phi_\circ\left(\frac{\kappa_3}{\lambda} \left(\Psi_\diamond^{-1}\left(\frac{s^{1/n}}{n\omega_n^{1/n}} f^{**}(s)\right)\right)\right) ds \\ & \leq |\Omega| \Phi_\circ(1) + \int_0^{|\Omega|} \Phi_\diamond\left(\frac{\kappa_3}{c_1 \lambda} \left(\Psi_\diamond^{-1}\left(\frac{s^{1/n}}{n\omega_n^{1/n}} f^{**}(s)\right)\right)\right) ds \\ & = |\Omega| \Phi_\circ(1) + \int_0^{|\Omega|} \Phi_\diamond\left(\Psi_\diamond^{-1}\left(\frac{s^{1/n}}{n\omega_n^{1/n}} f^{**}(s)\right)\right) ds \\ & \leq |\Omega| \Phi_\circ(1) + \int_0^{|\Omega|} \widetilde{\Phi}_\diamond\left(\frac{2s^{1/n}}{n\omega_n^{1/n}} f^{**}(s)\right) ds \leq |\Omega| \Phi_\circ(1) + \int_0^{|\Omega|} \widetilde{\Phi}_\circ\left(\frac{2s^{1/n}}{c_1 n\omega_n^{1/n}} f^{**}(s)\right) ds. \end{aligned} \tag{6.25}$$

Note that the first inequality is due to (6.23), the second to (2.57) (with Φ_\circ replaced by Φ_\bullet), the third by (6.17), the fourth by the definition of Φ_\bullet , where $s_0 \in [0, |\Omega|]$ is chosen in such a way that

$$s_0 = \inf \left\{ s \in [0, |\Omega|] : \frac{\kappa_3}{\lambda} \left(\Psi_\diamond^{-1}\left(\frac{s^{1/n}}{n\omega_n^{1/n}} f^{**}(s)\right)\right) \leq 1 \right\},$$

the fifth by (2.41), the equality holds owing to the very choice of λ , the sixth inequality is a consequence of property (v) of Lemma 2.1, and the last one follows via (2.41) again.

On the other hand,

$$\begin{aligned}
 \int_0^{|\Omega|} \widetilde{\Phi}_\bullet(\lambda s^{\frac{1}{n}} f^{**}(s)) \, ds &= \int_0^{|\Omega|} \widetilde{\Phi}_\bullet\left(\frac{\kappa_3}{c_1} s^{\frac{1}{n}} f^{**}(s)\right) \, ds \\
 &\leq \int_{s_1}^{|\Omega|} \widetilde{\Phi}_\bullet\left(\frac{\kappa_3}{c_1} s^{\frac{1}{n}} f^{**}(s)\right) \, ds + \int_0^{|\Omega|} \widetilde{\Phi}_\circ\left(\frac{\kappa_3}{c_1} s^{\frac{1}{n}} f^{**}(s)\right) \, ds \\
 &\leq |\Omega| \widetilde{\Phi}_\circ(t_1) + \int_0^{|\Omega|} \widetilde{\Phi}_\circ\left(\frac{\kappa_3}{c_1} s^{\frac{1}{n}} f^{**}(s)\right) \, ds, \tag{6.26}
 \end{aligned}$$

where

$$s_1 = \inf \left\{ s \in [0, |\Omega|] : \frac{\kappa_3}{c_1} s^{\frac{1}{n}} f^{**}(s) \leq t_1 \right\}.$$

The rightmost sides in inequalities (6.25) and (6.26) are finite, owing to assumption (3.2), and only depend on f , n and Φ . From inequalities (6.22) and (6.24) one thus deduces that there exists a constant C , depending on these data, such that

$$\int_\Omega \Phi(\nabla u^\varepsilon) \, dx + \int_\Omega \varepsilon A(|\nabla u^\varepsilon|) \, dx + \int_\Omega \varepsilon \widetilde{A}(A'(|\nabla u^\varepsilon|)) \, dx \leq C \tag{6.27}$$

for $\varepsilon \in (0, 1)$. Assertions (i)–(ii) follow from (6.27). Assertion (iii) follows on coupling inequality (6.27) with assumption (1.4).

Assume next that condition (3.3) holds. Then $W_0^1 L^\Phi(\Omega) \rightarrow L^\infty(\Omega)$, and from Eqs. (6.22), (6.23), (2.61), (6.16) and (2.41) we obtain that

$$\begin{aligned}
 \int_\Omega \Phi(\nabla u^\varepsilon) \, dx + \int_\Omega \varepsilon A(|\nabla u^\varepsilon|) \, dx + \int_\Omega \varepsilon \widetilde{A}(A'(|\nabla u^\varepsilon|)) \, dx &\leq \|f\|_{L^1(\Omega)} \|u^\varepsilon\|_{L^\infty(\Omega)} \\
 &\leq \|f\|_{L^1(\Omega)} \|v\|_{L^\infty(\Omega^\star)} \leq C \|f\|_{L^1(\Omega)} \|\nabla v\|_{L^{\Phi_\circ}(\Omega^\star)} \\
 &\leq C \|f\|_{L^1(\Omega)} \left\| \Psi_\diamond^{-1} \left(\frac{s^{1/n}}{n\omega_n^{1/n}} f^{**}(s) \right) \right\|_{L^{\Phi_\circ}(0,|\Omega|)} \\
 &\leq C' \|f\|_{L^1(\Omega)} \left\| \Psi_\diamond^{-1} \left(\frac{s^{1/n}}{n\omega_n^{1/n}} f^{**}(s) \right) \right\|_{L^{\Phi_\diamond}(0,|\Omega|)} \tag{6.28}
 \end{aligned}$$

for $\varepsilon \in (0, 1)$, and for some constants C and C' depending on n , Φ_\circ and $|\Omega|$. We claim that the last norm on the rightmost side of inequality (6.28) is finite, since $f \in L^1(\Omega)$. This is a consequence of the fact that $s^{1/n} f^{**}(s) \leq s^{-1/n'} \|f\|_{L^1(\Omega)}$ for $s \in (0, |\Omega|)$, of property (v) of Lemma 2.1, of Eq. (2.41), and of (6.19), which is equivalent to (2.60). Therefore, inequality (6.27) holds also in this case. One can then conclude as above. \square

6.2 A Browder–Minty-type result

The following proposition provides us with an anisotropic version of a classical result, known as the Browder–Minty monotonicity trick. It will be applied later, in the identification of limits of certain nonlinear expressions in an approximation process.

Proposition 6.3 (A monotonicity trick) *Let Ω be a measurable set in \mathbb{R}^n with $|\Omega| < \infty$. Assume that the Carathéodory function $a : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies condition (1.4) for some N -function Φ . Suppose that there exist functions*

$$Y \in L^{\widetilde{\Phi}}(\Omega; \mathbb{R}^n) \quad \text{and} \quad U \in L^\Phi(\Omega; \mathbb{R}^n) \tag{6.29}$$

such that

$$\int_{\Omega} (Y - a(x, V)) \cdot (U - V) \, dx \geq 0 \quad \text{for every } V \in L^{\infty}(\Omega; \mathbb{R}^n). \tag{6.30}$$

Then

$$a(x, U(x)) = Y(x) \quad \text{for a.e. } x \in \Omega. \tag{6.31}$$

Proof Define the increasing family $\{\Omega_j\}$ of invading subsets of Ω as $\Omega_j = \{x \in \Omega : |U(x)| \leq j\}$ for $j \in \mathbb{N}$. Fix any $j, k \in \mathbb{N}$ with $j < k$. An application of inequality (6.30), with $V = U\chi_{\Omega_k} + \sigma Z\chi_{\Omega_j}$ for any $\sigma \in (0, 1)$ and any function $Z \in L^{\infty}(\Omega; \mathbb{R}^n)$, yields

$$\int_{\Omega} (Y - a(x, U\chi_{\Omega_k} + \sigma Z\chi_{\Omega_j})) \cdot (U - U\chi_{\Omega_k} - \sigma Z\chi_{\Omega_j}) \, dx \geq 0.$$

The last inequality is equivalent to

$$\int_{\Omega \setminus \Omega_k} (Y - a(x, 0)) \cdot U \, dx + \sigma \int_{\Omega_j} (a(x, U + \sigma Z) - Y) \cdot Z \, dx \geq 0. \tag{6.32}$$

The first integral on the left-hand side of inequality (6.32) tends to zero as $k \rightarrow \infty$. Indeed, assumption (1.4) implies that $(Y - a(x, 0)) \cdot U \in L^1(\Omega)$, and hence the convergence follows owing to assumption (6.29) and Hölder’s inequality (2.33). Thus, passing to the limit as $k \rightarrow \infty$ in inequality (6.32) and dividing by σ the resultant inequality tells us that

$$\int_{\Omega_j} (a(x, U + \sigma Z) - Y) \cdot Z \, dx \geq 0.$$

Clearly,

$$\lim_{\sigma \rightarrow 0^+} a(x, U + \sigma Z) = a(x, U) \quad \text{for a.e. } x \in \Omega_j. \tag{6.33}$$

Moreover, by (1.4),

$$\sup_{\sigma \in (0,1)} \int_{\Omega_j} \tilde{\Phi}(c_{\Phi} a(x, U + \sigma Z)) \, dx \leq \int_{\Omega_j} \sup_{\sigma \in (0,1)} \Phi(U + \sigma Z) \, dx + \int_{\Omega_j} h(x) \, dx. \tag{6.34}$$

The integral on the right-hand side of (6.34) is finite, since the function $\sup_{\sigma \in (0,1)} (U + \sigma Z)$, and hence also the function $\sup_{\sigma \in (0,1)} \Phi(U + \sigma Z)$, is bounded in Ω_j . By Theorem 2.8, the family of functions $\{a(x, U + \sigma Z)\}_{\sigma \in (0,1)}$ is uniformly integrable in Ω_j . Hence, owing to Theorem 2.6,

$$\lim_{\sigma \rightarrow 0^+} a(x, U + \sigma Z) = a(x, U) \quad \text{in } L^1(\Omega_j; \mathbb{R}^n).$$

Thus,

$$\lim_{j \rightarrow \infty} \int_{\Omega_j} (a(x, U + \sigma Z) - Y) \cdot Z \, dx = \int_{\Omega_j} (a(x, U) - Y) \cdot Z \, dx.$$

Consequently,

$$\int_{\Omega_j} (a(x, U) - Y) \cdot Z \, dx \geq 0$$

for every $Z \in L^\infty(\Omega; \mathbb{R}^n)$. The choice of

$$Z = \begin{cases} -\frac{a(x,U)-Y}{|a(x,U)-Y|} & \text{if } a(x,U) - Y \neq 0 \\ 0 & \text{if } a(x,U) - Y = 0, \end{cases}$$

ensures that

$$\int_{\Omega_j} |a(x,U) - Y| dx \leq 0,$$

whence

$$a(x, U(x)) = Y(x) \quad \text{for a.e. } x \in \Omega_j.$$

Equation (6.31) follows, owing to the arbitrariness of j . □

6.3 Proof of existence of weak solutions

We are now ready to accomplish the proofs of Theorem 3.2 and of Proposition 3.3.

Proof of Theorem 3.2 Let A be an N -function as in Propositions 6.1 and 6.2, and let $\{u^\varepsilon\} \subset W_0^1 \mathcal{L}^A(\Omega) \cap L^\infty(\Omega)$ be the family of solutions to problems (6.4) for $\varepsilon \in (0, 1)$. By property (i) of Proposition 6.2, this family is bounded in $W_0^1 L^\Phi(\Omega)$, and hence in $W_0^{1,1}(\Omega)$. Therefore, it is compact in $L^1(\Omega)$, and consequently there exists a function $u \in L^1(\Omega)$ and a sequence $\{u^{\varepsilon_k}\}$ such that $u^{\varepsilon_k} \rightarrow u$ in $L^1(\Omega)$ and a.e. in Ω . Property (i) of Proposition 6.2 and Theorem 2.9 then ensure that the family of functions $\{\nabla u^{\varepsilon_k}\}$ is weakly-* compact in $L^\Phi(\Omega; \mathbb{R}^n)$. Since $u^{\varepsilon_k} \rightarrow u$ in $L^1(\Omega)$, we have that u is weakly differentiable, and its gradient agrees with the weak-* limit of $\{\nabla u^{\varepsilon_k}\}$ in $L^\Phi(\Omega; \mathbb{R}^n)$. Similarly, property (iii) of Proposition 6.2 and Theorem 2.9 again imply that the family of functions $\{a(x, \nabla u^{\varepsilon_k})\}$ is weakly-* compact in $L^{\tilde{\Phi}}(\Omega; \mathbb{R}^n)$. Finally, property (i) of Proposition 6.2 implies, via Theorems 2.8 and 2.7, that the family $\{\nabla u^\varepsilon\}$ is weakly compact in $L^1(\Omega; \mathbb{R}^n)$. Altogether, there exist a decreasing sequence $\{\varepsilon_k\}$, fulfilling $\varepsilon_k \rightarrow 0^+$, and functions $u \in W_0^1 L^\Phi(\Omega)$ and $Y \in L^{\tilde{\Phi}}(\Omega; \mathbb{R}^n)$ such that

$$u^{\varepsilon_k} \rightarrow u \quad \text{in } L^1(\Omega) \text{ and a.e. in } \Omega, \tag{6.35}$$

$$u^{\varepsilon_k} \rightharpoonup u \quad \text{weakly in } W^{1,1}(\Omega), \tag{6.36}$$

$$\nabla u^{\varepsilon_k} \overset{*}{\rightharpoonup} \nabla u \quad \text{weakly } - * \text{ in } L^\Phi(\Omega; \mathbb{R}^n), \tag{6.37}$$

$$a(x, \nabla u^{\varepsilon_k}) \overset{*}{\rightharpoonup} Y \quad \text{weakly } - * \text{ in } L^{\tilde{\Phi}}(\Omega; \mathbb{R}^n). \tag{6.38}$$

By the weak formulation of problem (6.4) with $\varepsilon = \varepsilon_k$,

$$\int_{\Omega} a(x, \nabla u^{\varepsilon_k}) \cdot \nabla \varphi + \varepsilon_k A'(|\nabla u^{\varepsilon_k}|) \frac{\nabla u^{\varepsilon_k}}{|\nabla u^{\varepsilon_k}|} \cdot \nabla \varphi dx = \int_{\Omega} f \varphi dx \tag{6.39}$$

for every $\varphi \in W_0^1 \mathcal{L}^A(\Omega)$. Notice that any such φ is automatically bounded by the classical Sobolev embedding, since our assumptions on A imply that $A(t) \geq t^q$ near infinity for some $q > n$. We begin by observing that

$$\lim_{k \rightarrow \infty} \int_{\Omega} \varepsilon_k A'(|\nabla u^{\varepsilon_k}|) \frac{\nabla u^{\varepsilon_k}}{|\nabla u^{\varepsilon_k}|} \cdot \nabla \varphi dx = 0 \quad \text{for every } \varphi \in C_0^\infty(\Omega). \tag{6.40}$$

To verify this assertion, consider, for fixed $j \in \mathbb{N}$, the set

$$\Omega_j^{\varepsilon_k} = \{x \in \Omega : |\nabla u^{\varepsilon_k}| \leq j\}.$$

Plainly,

$$\begin{aligned} & \int_{\Omega} \varepsilon_k A'(|\nabla u^{\varepsilon_k}|) \frac{\nabla u^{\varepsilon_k}}{|\nabla u^{\varepsilon_k}|} \cdot \nabla \varphi \, dx \\ &= \int_{\Omega_j^{\varepsilon_k}} \varepsilon_k A'(|\nabla u^{\varepsilon_k}|) \frac{\nabla u^{\varepsilon_k}}{|\nabla u^{\varepsilon_k}|} \cdot \nabla \varphi \, dx + \int_{\Omega \setminus \Omega_j^{\varepsilon_k}} \varepsilon_k A'(|\nabla u^{\varepsilon_k}|) \frac{\nabla u^{\varepsilon_k}}{|\nabla u^{\varepsilon_k}|} \cdot \nabla \varphi \, dx. \end{aligned} \tag{6.41}$$

Inasmuch as A' is a non-decreasing function,

$$\limsup_{k \rightarrow \infty} \left| \int_{\Omega_j^{\varepsilon_k}} \varepsilon_k A'(|\nabla u^{\varepsilon_k}|) \frac{\nabla u^{\varepsilon_k}}{|\nabla u^{\varepsilon_k}|} \cdot \nabla \varphi \, dx \right| \leq |\Omega| \|\nabla \varphi\|_{L^\infty(\Omega)} A'(j) \lim_{k \rightarrow \infty} \varepsilon_k = 0. \tag{6.42}$$

On the other hand, since the sequence $\{|\nabla u^{\varepsilon_k}|\}$ is uniformly integrable in $L^1(\Omega)$, there exists a constant C , such that

$$\sup_{k \in \mathbb{N}} |\Omega \setminus \Omega_j^{\varepsilon_k}| \leq \frac{C}{j}. \tag{6.43}$$

Furthermore, since \tilde{A} is an N -function, one has that $\tilde{A}(\lambda t) \leq \lambda \tilde{A}(t)$, provided that $t \geq 0$ and $\lambda \in (0, 1)$. Thereby, $\tilde{A}(\varepsilon_k A'(|\nabla u^{\varepsilon_k}|)) \leq \varepsilon_k \tilde{A}(A'(|\nabla u^{\varepsilon_k}|))$, and hence, by property (ii) of Proposition 6.2, the sequence $\{\varepsilon_k A'(|\nabla u^{\varepsilon_k}|)\}$ is uniformly bounded in $L^{\tilde{A}}(\Omega)$. Thanks to Theorem 2.8, the sequence $\{\varepsilon_k A'(|\nabla u^{\varepsilon_k}|)\}$ is uniformly integrable in Ω . Coupling this piece of information with (6.43) implies that

$$\begin{aligned} & \limsup_{j \rightarrow \infty} \left(\sup_{k \in \mathbb{N}} \left| \int_{\Omega \setminus \Omega_j^{\varepsilon_k}} \varepsilon_k A'(|\nabla u^{\varepsilon_k}|) \frac{\nabla u^{\varepsilon_k}}{|\nabla u^{\varepsilon_k}|} \cdot \nabla \varphi \, dx \right| \right) \\ & \leq \|\nabla \varphi\|_{L^\infty(\Omega)} \lim_{j \rightarrow \infty} \left(\sup_{k \in \mathbb{N}} \int_{\Omega \setminus \Omega_j^{\varepsilon_k}} \varepsilon_k |A'(|\nabla u^{\varepsilon_k}|)| \, dx \right) = 0. \end{aligned} \tag{6.44}$$

Equation (6.40) follows from (6.41), (6.42) and (6.44).

Thanks to (6.38) and (6.40), choosing $\varphi \in C_0^\infty(\Omega)$ in (6.39) and passing to the limit as $k \rightarrow \infty$ yield

$$\int_{\Omega} Y \cdot \nabla \varphi \, dx = \int_{\Omega} f \varphi \, dx. \tag{6.45}$$

Since $u^{\varepsilon_k} \in W_0^1 L^\Phi(\Omega) \cap L^\infty(\Omega)$, for each $k \in \mathbb{N}$ the function u^{ε_k} can be approximated by a sequence of functions from $C_0^\infty(\Omega)$ as in Proposition 2.4. On making use of Eq. (6.45) with φ replaced by the functions approximating u^{ε_k} , passing to the limit in the approximating sequence, and recalling that $Y \in L^\Phi(\Omega; \mathbb{R}^n)$ and that the sequence of approximating functions is uniformly bounded in $L^\infty(\Omega)$ by $C\|u^{\varepsilon_k}\|_{L^\infty(\Omega)}$ we infer that

$$\int_{\Omega} Y \cdot \nabla u^{\varepsilon_k} \, dx = \int_{\Omega} f u^{\varepsilon_k} \, dx \tag{6.46}$$

for every $k \in \mathbb{N}$. Inasmuch as u^{ε_k} belongs to $W_0^1 \mathcal{L}^A(\Omega) \cap L^\infty(\Omega)$, it can be used as a test function in the weak formulation of problem (6.4) with $\varepsilon = \varepsilon_k$. Therefore,

$$\int_{\Omega} a(x, \nabla u^{\varepsilon_k}) \cdot \nabla u^{\varepsilon_k} + \varepsilon_k A'(|\nabla u^{\varepsilon_k}|) |\nabla u^{\varepsilon_k}| dx = \int_{\Omega} f u^{\varepsilon_k} dx \tag{6.47}$$

for every $k \in \mathbb{N}$. Since the second term in the integral on the left-hand side of (6.47) is nonnegative, Eqs. (6.46), (6.47) and (6.37) imply that

$$\limsup_{k \rightarrow \infty} \int_{\Omega} a(x, \nabla u^{\varepsilon_k}) \cdot \nabla u^{\varepsilon_k} dx \leq \int_{\Omega} Y \cdot \nabla u dx. \tag{6.48}$$

Now, given any function $V \in L^\infty(\Omega; \mathbb{R}^n)$, we have, by assumption (1.2),

$$\begin{aligned} 0 &\leq \int_{\Omega} (a(x, V) - a(x, \nabla u^{\varepsilon_k})) \cdot (V - \nabla u^{\varepsilon_k}) dx \\ &\leq \int_{\Omega} a(x, V) \cdot V dx - \int_{\Omega} a(x, V) \cdot \nabla u^{\varepsilon_k} dx \\ &\quad - \int_{\Omega} a(x, \nabla u^{\varepsilon_k}) \cdot V dx + \int_{\Omega} a(x, \nabla u^{\varepsilon_k}) \cdot \nabla u^{\varepsilon_k} dx. \end{aligned} \tag{6.49}$$

Passing to the limit as $k \rightarrow \infty$ on the rightmost side of (6.49), and making use of (6.36), (6.38) and (6.48) imply that

$$\int_{\Omega} (a(x, V) - Y) \cdot (V - \nabla u) dx \geq 0. \tag{6.50}$$

Therefore, we are in a position to apply Proposition 6.3, with $U = \nabla u$, and deduce that

$$a(x, \nabla u(x)) = Y(x) \quad \text{for a.e. } x \in \Omega. \tag{6.51}$$

Hence, in particular, $a(x, \nabla u) \in L^{\tilde{\Phi}}(\Omega; \mathbb{R}^n)$. Fix any test function $\varphi \in C_0^\infty(\Omega)$. On passing to the limit as $k \rightarrow \infty$ in Eq. (6.39), and exploiting (6.38), (6.51) and (6.40) one concludes that

$$\int_{\Omega} a(x, \nabla u) \cdot \nabla \varphi dx = \int_{\Omega} f \varphi dx \tag{6.52}$$

for every $\varphi \in C_0^\infty(\Omega)$. Equation (6.52) continues to hold for any test function $\varphi \in W_0^1 \mathcal{L}^\Phi(\Omega) \cap L^\infty(\Omega)$ as in the definition of weak solution to problem (1.1). Actually, let $\{\varphi_k\} \subset C_0^\infty(\Omega)$ be a sequence approximating φ as in Proposition 2.4. Then, from Eq. (6.52) with φ replaced by φ_k , we have that

$$\int_{\Omega} a(x, \nabla u) \cdot \nabla \varphi dx = \lim_{k \rightarrow \infty} \int_{\Omega} a(x, \nabla u) \cdot \nabla \varphi_k dx = \lim_{k \rightarrow \infty} \int_{\Omega} f \varphi_k dx = \int_{\Omega} f \varphi dx,$$

where the first equality holds by properties (2.66) and (2.63), and the last equality since $\|\varphi_k\|_{L^\infty(\Omega)} \leq C \|\varphi\|_{L^\infty(\Omega)}$ for some constant $C = C(n)$ and every $k \in \mathbb{N}$.

Finally, we have that

$$\int_{\Omega} \Phi(\nabla u) dx < \infty. \tag{6.53}$$

Indeed, since Φ is an n -dimensional N -function, inequality (6.53) follows, via semicontinuity, from the convergence in (6.35) and estimate (6.22), whose right-hand side is uniformly bounded as $\varepsilon \rightarrow 0^+$. Equation (6.53) ensures that, in fact, $u \in W_0^1 \mathcal{L}^\Phi(\Omega)$.

The uniqueness of the solution u can be established along the same lines as in the case of approximable solutions—see Step 6 of the proof of Theorem 3.7 in Sect. 7.2. We shall not reproduce it here, for brevity. \square

Proof of Proposition 3.3 Let u^{ε_k} be as in the proof of Theorem 3.2. By property (6.35), one has that $u^{\varepsilon_k} \rightarrow u$ a.e. in Ω . Moreover, inequality (6.23) implies that $\|u^{\varepsilon_k}\|_{L^\infty(\Omega)} \leq \|v\|_{L^\infty(\Omega)}$. The norm $\|v\|_{L^\infty(\Omega)}$ can be estimated on making use of equation (6.15). Thanks to assumption (3.4), passing to the limit as $k \rightarrow \infty$ in the resultant estimate yields inequality (3.5). \square

7 Approximable solutions: Proof of Theorems 3.7 and 3.10

Proofs of Theorems 3.7 and 3.10 are presented in Sect. 7.2 below. Their outline is reminiscent of that of the diverse contributions on approximable solutions mentioned above, and, in particular, it is patterned on that of [11]. However, some of the specific steps require substantially new ingredients, due to the nonstandard functional setting at hand. This is especially apparent in some fundamental a priori bounds that are the subject of the next subsection.

7.1 A priori estimates

A fundamental step in the proof of Theorem 3.7 amounts to an a priori anisotropic gradient bound for the solution u_k to the approximating problem (3.8) by the L^1 norm of f_k . Of course, we need such estimate to be independent of k . This is a consequence of the following proposition.

Proposition 7.1 (A gradient estimate by the L^1 norm of the datum) *Let Ω be an open set in \mathbb{R}^n with $|\Omega| < \infty$. Assume that assumptions (1.2)–(1.4) hold for some N -function Φ . Let Θ be the function associated with Φ as in (2.44). Assume that $f \in L^1(\Omega)$ and that there exists a weak solution u to problem (1.1). Then*

$$\int_{\Omega} \Theta(\nabla u) dx \leq c |\Omega|^{1/n} \|f\|_{L^1(\Omega)} \tag{7.1}$$

for some constant $c = c(n)$.

Proof Standard properties of truncations of weakly differentiable functions ensure that, since $u \in W_0^1 \mathcal{L}^\Phi(\Omega)$, the function $T_\tau(u - T_t(u))$ is weakly differentiable for every $t, \tau > 0$, and belongs to $W_0^1 \mathcal{L}^\Phi(\Omega) \cap L^\infty(\Omega)$. Thus, the function $T_\tau(u - T_t(u))$ can be used as a test function in Eq. (3.1).

This choice of test functions is the point of departure to derive [26, Inequalities (5.5) and (5.6)], which tell us that

$$\frac{1}{-\mu'_u(t)} \leq \frac{1}{n\omega_n^{1/n} \mu_u^{1/n'}(t)} \Psi_{\diamond}^{-1} \left(\frac{-\frac{d}{dt} \int_{\{|u|>t\}} \Phi(\nabla u) dx}{\omega_n^{1/n} \mu_u^{1/n'}(t)} \right) \text{ for a.e. } t > 0. \tag{7.2}$$

Here, μ_u is the distribution function of u defined as in (2.1). Multiplying through inequality (7.2) by $-\frac{d}{dt} \int_{\{|u|>t\}} \Phi(\nabla u) dx$ results in

$$\begin{aligned} & \frac{-\frac{d}{dt} \int_{\{|u|>t\}} \Phi(\nabla u) dx}{-\mu'_u(t)} \\ & \leq \frac{-\frac{d}{dt} \int_{\{|u|>t\}} \Phi(\nabla u) dx}{n\omega_n^{1/n} \mu_u^{1/n'}(t)} \Psi_{\diamond}^{-1} \left(\frac{-\frac{d}{dt} \int_{\{|u|>t\}} \Phi(\nabla u) dx}{n\omega_n^{1/n} \mu_u^{1/n'}(t)} \right) \text{ for a.e. } t > 0. \end{aligned} \tag{7.3}$$

Now, Lemma 2.1 (i) ensures that the function $\Phi_{\diamond} \circ \Theta_{\diamond}^{-1}$ is convex. Thereby, an application of Jensen’s inequality and Lemma 2.1 (ii) yield

$$\begin{aligned} \Phi_{\diamond} \circ \Theta_{\diamond}^{-1} \left(\frac{\frac{1}{h} \int_{\{t<|u|<t+h\}} \Theta(\nabla u) dx}{\frac{1}{h}(-\mu_u(t+h) + \mu_u(t))} \right) & \leq \frac{\frac{1}{h} \int_{\{t<|u|<t+h\}} \Phi_{\diamond} \circ \Theta_{\diamond}^{-1}(\Theta(\nabla u)) dx}{\frac{1}{h}(-\mu_u(t+h) + \mu_u(t))} \\ & = \frac{\frac{1}{h} \int_{\{t<|u|<t+h\}} \Phi(\nabla u) dx}{\frac{1}{h}(-\mu_u(t+h) + \mu_u(t))} \text{ for } t, h > 0. \end{aligned} \tag{7.4}$$

Passing to the limit as $h \rightarrow 0^+$ in (7.4) tells us that

$$\Phi_{\diamond} \circ \Theta_{\diamond}^{-1} \left(\frac{-\frac{d}{dt} \int_{\{|u|>t\}} \Theta(\nabla u) dx}{-\mu'_u(t)} \right) \leq \frac{-\frac{d}{dt} \int_{\{|u|>t\}} \Phi(\nabla u) dx}{-\mu'_{u_{\kappa}}(t)} \text{ for a.e. } t > 0. \tag{7.5}$$

On the other hand, [26, Inequality (5.5)] implies that

$$-\frac{d}{dt} \int_{\{|u|>t\}} \Phi(\nabla u) dx \leq \int_0^{\mu_u(t)} f^*(s) ds \text{ for a.e. } t > 0. \tag{7.6}$$

From (7.5), (7.3), Lemma 2.1 (iii), and (7.6) one deduces that

$$\begin{aligned} & \Theta_{\diamond}^{-1} \left(\frac{-\frac{d}{dt} \int_{\{|u|>t\}} \Theta(\nabla u) dx}{-\mu'_u(t)} \right) \\ & \leq \Phi_{\diamond}^{-1} \left(\frac{-\frac{d}{dt} \int_{\{|u|>t\}} \Phi(\nabla u) dx}{-\mu'_u(t)} \right) \\ & \leq \Phi_{\diamond}^{-1} \left(\frac{-\frac{d}{dt} \int_{\{|u|>t\}} \Phi(\nabla u) dx}{n\omega_n^{1/n} \mu_u^{1/n'}(t)} \Psi_{\diamond}^{-1} \left(\frac{-\frac{d}{dt} \int_{\{|u|>t\}} \Phi(\nabla u) dx}{n\omega_n^{1/n} \mu_u^{1/n'}(t)} \right) \right) \\ & = \Psi_{\diamond}^{-1} \left(\frac{-\frac{d}{dt} \int_{\{|u|>t\}} \Phi(\nabla u) dx}{n\omega_n^{1/n} \mu_u^{1/n'}(t)} \right) \leq \Psi_{\diamond}^{-1} \left(\frac{\int_0^{\mu_u(t)} f^*(s) ds}{n\omega_n^{1/n} \mu_u^{1/n'}(t)} \right) \end{aligned} \tag{7.7}$$

for a.e. $t > 0$. Hence,

$$-\frac{d}{dt} \int_{\{|u|>t\}} \Theta(\nabla u) dx \leq -\mu'_u(t) \Theta_{\diamond} \circ \Psi_{\diamond}^{-1} \left(\frac{\int_0^{\mu_u(t)} f^*(s) ds}{n\omega_n^{1/n} \mu_u^{1/n'}(t)} \right) \text{ for a.e. } t > 0. \tag{7.8}$$

Now, notice that

$$\begin{aligned} \int_{\{|u|>t\}} \Theta(\nabla u) dx & = \int_{\Omega} \chi_{\{\nabla u=0\}} \Theta(\nabla u) dx + \int_{\Omega} \chi_{\{\nabla u \neq 0\}} \Theta(\nabla u) dx \\ & = \int_{\Omega} \chi_{\{\nabla u \neq 0\}} \frac{\Theta(\nabla u)}{|\nabla u|} |\nabla u| dx = \int_t^{\infty} \int_{\{|u|=\tau\}} \frac{\Theta(\nabla u)}{|\nabla u|} d\mathcal{H}^{n-1} d\tau \text{ for } t > 0, \end{aligned} \tag{7.9}$$

where the second equality holds since $\Theta(0) = 0$, and the last one by the coarea formula for Sobolev functions. Here, \mathcal{H}^{n-1} denotes the $(n - 1)$ -dimensional Hausdorff measure. Therefore, the function $[0, \infty) \ni t \mapsto \int_{\{|u|>t\}} \Theta(\nabla u) dx$ is absolutely continuous. Combining this fact with inequalities (7.8) and Lemma 2.1 (iv) ensures that

$$\begin{aligned} \int_{\Omega} \Theta(\nabla u) dx &= \int_0^\infty \left(-\frac{d}{dt} \int_{\{|u|>t\}} \Theta(\nabla u) dx \right) dt \\ &\leq \int_0^\infty (-\mu'_u(t)) \Theta_\diamond \left(\Psi_\diamond^{-1} \left(\frac{\int_0^{\mu_u(t)} f^*(s) ds}{n\omega_n^{1/n} \mu_{u_k}^{1/n'}(t)} \right) \right) dt \\ &\leq \int_0^{|\Omega|} \Theta_\diamond \left(\Psi_\diamond^{-1} \left(\frac{\int_0^r f^*(s) ds}{n\omega_n^{1/n} r^{1/n'}} \right) \right) dr \leq \int_0^{|\Omega|} \frac{2}{n\omega_n^{1/n} r^{1/n'}} \int_0^r f^*(s) ds dr \\ &\leq \frac{2}{n\omega_n^{1/n}} \|f\|_{L^1(\Omega)} \int_0^{|\Omega|} r^{-1/n'} dr = 2\omega_n^{-1/n} |\Omega|^{1/n} \|f\|_{L^1(\Omega)}. \end{aligned}$$

Inequality (7.1) is thus established. □

The next two propositions provide us with superlevel set estimates for functions $u \in \mathcal{T}_0^{1,\Phi}(\Omega)$ and for their gradients ∇u depending of the decay of the integrals of $\Phi(\nabla u)$ over the sublevel sets of u .

Proposition 7.2 (Superlevel set estimate for u) *Let Ω be an open set in \mathbb{R}^n with $|\Omega| < \infty$. Let Φ be an N -function fulfilling conditions (2.49) and (2.50). Assume that $u \in \mathcal{T}_0^{1,\Phi}(\Omega)$ and there exist constants $K > 0$ and $t_0 \geq 0$ such that*

$$\int_{\{|u|<t\}} \Phi(\nabla u) dx \leq Kt \quad \text{for } t > t_0. \tag{7.10}$$

Then

$$|\{|u| \geq t\}| \leq \frac{Kt}{\Phi_n(\kappa_2 t^{\frac{1}{n'}} K^{-\frac{1}{n}})} \quad \text{for } t > t_0, \tag{7.11}$$

where Φ_n and κ_2 are the Young function and the constant appearing in the Sobolev inequality (2.53).

If condition (2.49) is not satisfied, then an analogous statement holds, provided that Φ_n is defined as in (2.51)–(2.52), with Φ modified near 0 in such a way that (2.49) is fulfilled. In this case, the constant κ_2 in (7.11) has to be replaced by another constant depending also on Φ . Furthermore, in (7.11) the constant t_0 has to be replaced by another constant depending also on Φ , and the constant K has to be replaced by another constant depending on the constant K appearing in (7.10), on Φ and on $|\Omega|$.

In any case, irrespective of whether (2.49) holds or does not, for every $\varepsilon > 0$, there exists $\bar{t} = \bar{t}(\varepsilon, K, t_0, n, \Phi)$ such that

$$|\{|u| \geq t\}| < \varepsilon \quad \text{if } t > \bar{t}. \tag{7.12}$$

Proof Assume first that assumption (2.49) is in force. Thanks to the definition of T_t and to property (2.38),

$$\int_{\Omega} \Phi(\nabla T_t(u)) dx = \int_{\{|u|<t\}} \Phi(\nabla u) dx \quad \text{and} \quad \{|T_t(u)| \geq t\} = \{|T_t(u)| = t\} = \{|u| \geq t\}$$

for $t > 0$. We have that

$$\begin{aligned} | \{ |u| \geq t \} | \Phi_n \left(\frac{t}{\kappa_2 \left(\int_{\{|u|<t\}} \Phi(\nabla u) dy \right)^{\frac{1}{n}}} \right) &\leq \int_{\{|u| \geq t\}} \Phi_n \left(\frac{|T_t(u)|}{\kappa_2 \left(\int_{\{|u|<t\}} \Phi(\nabla u) dy \right)^{\frac{1}{n}}} \right) dx \\ &\leq \int_{\Omega} \Phi_n \left(\frac{|T_t(u)|}{\kappa_2 \left(\int_{\{|u|<t\}} \Phi(\nabla u) dy \right)^{\frac{1}{n}}} \right) dx \leq \int_{\Omega} \Phi_n \left(\frac{|T_t(u)|}{\kappa_2 \left(\int_{\Omega} \Phi(\nabla T_t(u)) dy \right)^{\frac{1}{n}}} \right) dx. \end{aligned} \tag{7.13}$$

By inequality (2.53) applied to $T_t(u)$,

$$\int_{\Omega} \Phi_n \left(\frac{|T_t(u)|}{\kappa_2 \left(\int_{\Omega} \Phi(\nabla T_t(u)) dy \right)^{\frac{1}{n}}} \right) dx \leq \int_{\Omega} \Phi(\nabla T_t(u)) dx = \int_{\{|u|<t\}} \Phi(\nabla u) dx. \tag{7.14}$$

Combining inequalities (7.13), (7.14) and (7.10) yields

$$| \{ |u| \geq t \} | \Phi_n \left(\frac{t}{\kappa_2 (Kt)^{\frac{1}{n}}} \right) \leq Kt \quad \text{for } t > t_0,$$

an equivalent formulation of (7.11).

Assume next that condition (2.49) fails. Consider the n -dimensional Young function $\bar{\Phi} : \mathbb{R}^n \rightarrow [0, \infty)$ defined as

$$\bar{\Phi}(\xi) = \begin{cases} \Xi(\xi) & \text{if } \xi \in \{\Phi \leq 1\}, \\ \Phi(\xi) & \text{if } \xi \in \{\Phi > 1\}, \end{cases} \tag{7.15}$$

where Ξ is the (unique) function, which vanishes at 0, is linear along each half-line issued from 0, and agrees with Φ on $\{\Phi = 1\}$. Clearly, $\Phi(\xi) \leq \bar{\Phi}(\xi)$ for $\xi \in \mathbb{R}^n$, and condition (2.49) is satisfied if Φ is replaced by $\bar{\Phi}$. One has that

$$\begin{aligned} \int_{\{|u|<t\}} \bar{\Phi}(\nabla u) dx &\leq \int_{\{|u|<t, \Phi(\nabla u)>1\}} \Phi(\nabla u) dx + \int_{\{|u|<t, \Phi(\nabla u)\leq 1\}} \bar{\Phi}(\nabla u) dx \\ &\leq \int_{\{|u|<t, \Phi(\nabla u)>1\}} \Phi(\nabla u) dx + | \{ |u| < t \} | \leq t(K + |\Omega|), \end{aligned} \tag{7.16}$$

if $t > \max\{t_0, 1\}$. Therefore, the function u satisfies assumption (7.10) with Φ replaced by $\bar{\Phi}$, K replaced by $K + |\Omega|$, and t_0 replaced by $\max\{t_0, 1\}$. Consequently, inequality (7.11) holds with Φ_n replaced by $(\bar{\Phi})_n$, K replaced by $K + |\Omega|$, and t_0 replaced by $\max\{t_0, 1\}$.

Finally, in the light of (7.11), inequality (7.12) will follow if we show that

$$\lim_{t \rightarrow \infty} \frac{\Phi_n(t^{\frac{1}{n'}})}{t} = \infty. \tag{7.17}$$

By the definitions of Φ_n and Φ_o , Eq. (7.17) is equivalent to

$$\lim_{t \rightarrow \infty} \frac{\Phi_o(t)}{\int_0^t \left(\frac{\tau}{\Phi_o(\tau)} \right)^{\frac{1}{n-1}} d\tau} = \infty. \tag{7.18}$$

On the other hand, since Φ_\circ is an N -function, there exist constants $c > 0$ and $\widehat{t} > 0$ such that

$$\int_0^t \left(\frac{\tau}{\Phi_\circ(\tau)} \right)^{\frac{1}{n-1}} d\tau \leq c + t \quad \text{if } t > \widehat{t}, \tag{7.19}$$

whence Eq. (7.18) follows, owing to the behavior of N -functions near infinity. □

Proposition 7.3 (Superlevel set estimate for $\Phi(\nabla u)$) *Let Ω be an open set in \mathbb{R}^n with $|\Omega| < \infty$. Let Φ be an N -function fulfilling conditions (2.49) and (2.50). Assume that $u \in \mathcal{T}_0^{1,\Phi}(\Omega)$ and fulfills inequality (7.10) for some constants $K > 0$ and $t_0 \geq 0$. Then there exist constants $c_1 = c_1(n, K)$ and $s_0 = s_0(t_0, \Phi, n, K)$ such that*

$$|\{\Phi(\nabla u) > s\}| \leq c_1 \frac{\Phi_n^{-1}(s)^{n'}}{s} \quad \text{for } s > s_0. \tag{7.20}$$

If condition (2.49) is not satisfied, then an analogous statement holds, provided that Φ_n is defined as in (2.51)–(2.52), with Φ modified near 0 in such a way that (2.49) is fulfilled. In this case, the constant c_1 in (7.20) depends also on Φ .

Proof Inequality (7.10) implies that

$$|\{\Phi(\nabla u) > s, |u| < t\}| \leq \frac{1}{s} \int_{\{\Phi(\nabla u) > s, |u| < t\}} \Phi(\nabla u) dx \leq K \frac{t}{s} \quad \text{for } t > t_0 \text{ and } s > 0. \tag{7.21}$$

On the other hand,

$$|\{\Phi(\nabla u) > s\}| \leq |\{|u| \geq t\}| + |\{\Phi(\nabla u) > s, |u| < t\}| \quad \text{for } t > 0 \text{ and } s > 0. \tag{7.22}$$

From (7.21) (7.22) and (7.11) one deduces that

$$|\{\Phi(\nabla u) > s\}| \leq \frac{Kt}{\Phi_n(ct^{1/n'} / K^{1/n})} + K \frac{t}{s} \quad \text{for } t > t_0 \text{ and } s > 0.$$

Choosing $t = (K^{1/n} \Phi_n^{-1}(s)/c)^{n'}$ in this inequality yields

$$|\{\Phi(\nabla u) > s\}| \leq 2 \left(\frac{K}{c} \right)^{n'} \frac{(\Phi_n^{-1}(s))^{n'}}{s} \quad \text{for } s > \Phi_n(ct_0^{1/n'} / K^{1/n}),$$

whence (7.20) follows.

If condition (2.49) is not fulfilled, the conclusion follows on modifying the function Φ near 0, via an argument analogous to that of the proof of Proposition 7.2. □

7.2 Proof of existence of approximable solutions

The proofs of the common parts of the statements of Theorems 3.7 and 3.10 are very similar. We shall provide details on the former, and just briefly comment on the minor variants needed for the latter.

Proof of Theorem 3.7 For clarity of presentation, we split the proof into steps.

Step 1 Approximating problems with smooth data.

Let $\{f_k\} \subset L^\infty(\Omega)$ be a sequence such that

$$f_k \rightarrow f \quad \text{in } L^1(\Omega) \quad \text{and} \quad \|f_k\|_{L^1(\Omega)} \leq 2\|f\|_{L^1(\Omega)}. \tag{7.23}$$

By Theorem 3.2, there exists a (unique) weak solution $u_k \in W_0^1 \mathcal{L}^\Phi(\Omega)$ to problem (3.8). In particular, the very definition of weak solution tells us that

$$\int_{\Omega} a(x, \nabla u_k) \cdot \nabla \varphi \, dx = \int_{\Omega} f_k \varphi \, dx \tag{7.24}$$

for every $\varphi \in W_0^1 \mathcal{L}^\Phi(\Omega) \cap L^\infty(\Omega)$.

Step 2 A priori estimates.

The following inequality holds for every $k \in \mathbb{N}$ and for every $t > 0$:

$$\int_{\Omega} \Phi(\nabla T_t(u_k)) \, dx \leq 2t \|f\|_{L^1(\Omega)}. \tag{7.25}$$

Inequality (7.25) is a consequence of the following chain, that relies upon assumption (1.4) and on the use of the test function $\varphi = T_t(u_k)$ in Eq. (7.24):

$$\begin{aligned} \int_{\Omega} \Phi(\nabla T_t(u_k)) \, dx &\leq \int_{\Omega} a(x, \nabla T_t(u_k)) \nabla T_t(u_k) \, dx \\ &= \int_{\Omega} a(x, \nabla u_k) \nabla T_t(u_k) \, dx = \int_{\Omega} f_k T_t(u_k) \, dx \leq 2t \|f\|_{L^1(\Omega)}. \end{aligned}$$

Step 3 Almost everywhere convergence of solutions.

There exists a function $u \in \mathcal{M}(\Omega)$ such that (up to subsequences)

$$u_k \rightarrow u \quad \text{a.e. in } \Omega. \tag{7.26}$$

Indeed, let $t, \tau > 0$. Then

$$|\{|u_k - u_m| > \tau\}| \leq |\{|u_k| > t\}| + |\{|u_m| > t\}| + |\{|T_t(u_k) - T_t(u_m)| > \tau\}| \tag{7.27}$$

for $k, m \in \mathbb{N}$. Fix any $\varepsilon > 0$. Inequality (7.25) ensures, via inequality (7.27) of Proposition 7.2, that

$$|\{|u_k| > t\}| + |\{|u_m| > t\}| < \varepsilon \tag{7.28}$$

for every $k, m \in \mathbb{N}$, provided t is sufficiently large. Moreover, inequality (7.25) again ensures that the sequence $\nabla T_t(u_k)$ is bounded in $L^1(\Omega)$. Hence, the sequence $T_t(u_k)$ is bounded in $W_0^{1,1}(\Omega)$ and since the latter space is compactly embedded into $L^1(\Omega)$, there exists a subsequence, still denoted by $\{u_k\}$, such that $T_t(u_k)$ converges to some function in $L^1(\Omega)$. In particular, it is a Cauchy sequence in measure, and hence

$$|\{|T_t(u_k) - T_t(u_m)| > \tau\}| < \varepsilon \tag{7.29}$$

if k and m are large enough. From inequalities (7.27)–(7.29) we infer that (up to subsequences) $\{u_k\}$ is a Cauchy sequence in measure, whence (7.26) follows.

Step 4 $\{\nabla u_k\}$ is a Cauchy sequence in measure.

An application of Proposition 7.1 with f and u replaced by f_k and u_k yields, via (7.23),

$$\int_{\Omega} \Theta(\nabla u_k) \, dx \leq c |\Omega|^{1/n} \|f\|_{L^1(\Omega)} \tag{7.30}$$

for some constant $c = c(n)$ and every $k \in \mathbb{N}$. Here, Θ is the function given by (2.44). Define the function $\Theta_- : [0, \infty) \rightarrow [0, \infty)$ by

$$\Theta_-(s) = \inf_{|\xi|=s} \Theta(\xi). \tag{7.31}$$

Namely, Θ_- is the largest radially symmetric minorant of Θ . Note that Θ_- is a strictly increasing function vanishing at 0. Let $\varepsilon > 0$. Given any $t, \tau, s > 0$, one has that

$$|\{\Theta_- (|\nabla u_k - \nabla u_m|) > t\}| \leq |\{\Theta_- (|\nabla u_k|) > \tau\}| + |\{\Theta_- (|u_m|) > \tau\}| + |\{ |u_k - u_m| > s \}| + |\{ |u_k - u_m| \leq s, \Theta_- (|\nabla u_k|) \leq \tau, \Theta_- (|\nabla u_m|) \leq \tau, \Theta_- (|\nabla u_k - \nabla u_m|) > t \}|. \tag{7.32}$$

Owing to inequality (7.30),

$$t|\{\Theta_- (|\nabla u_k|) > t\}| \leq \int_{\Omega} \Theta_- (|\nabla u_k|) dx \leq \int_{\Omega} \Theta (\nabla u_k) dx \leq c|\Omega|^{1/n} \|f\|_{L^1(\Omega)} \tag{7.33}$$

for $k \in \mathbb{N}$. Thus,

$$|\{\Theta_- (|\nabla u_k|) > \tau\}| + |\{\Theta_- (|\nabla u_m|) > \tau\}| < \varepsilon \tag{7.34}$$

for every $k, m \in \mathbb{N}$, provided that τ is large enough. Next, set

$$G = \{ |u_k - u_m| \leq s, \Theta_- (|\nabla u_k|) \leq \tau, \Theta_- (|\nabla u_m|) \leq \tau, \Theta_- (|\nabla u_k - \nabla u_m|) > t \}, \tag{7.35}$$

and define

$$S = \{ (\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n : |\xi| \leq \tau, |\eta| \leq \tau, |\xi - \eta| \geq t \},$$

a compact set. Consider the function $\psi : \Omega \rightarrow [0, \infty)$ given by

$$\psi(x) = \inf_{(\xi, \eta) \in S} [(a(x, \xi) - a(x, \eta)) \cdot (\xi - \eta)].$$

The monotonicity assumption (1.2) and the continuity of the function $\xi \mapsto a(x, \xi)$ for a.e. $x \in \Omega$ on the compact set S ensure that $\psi \geq 0$ in Ω and $|\{\psi(x) = 0\}| = 0$. Moreover,

$$\begin{aligned} \int_G \psi(x) dx &\leq \int_G (a(x, \nabla u_k) - a(x, \nabla u_m)) \cdot (\nabla u_k - \nabla u_m) dx \\ &\leq \int_{\{|u_k - u_m| \leq s\}} (a(x, \nabla u_k) - a(x, \nabla u_m)) \cdot (\nabla u_k - \nabla u_m) dx \\ &= \int_{\Omega} (a(x, \nabla u_k) - a(x, \nabla u_m)) \cdot (\nabla T_s(u_k - u_m)) dx \\ &= \int_{\Omega} (f_k - f_m) T_s(u_k - u_m) dx \leq 4s \|f\|_{L^1(\Omega)}, \end{aligned} \tag{7.36}$$

where the last but one equality follows on making use of the test function $T_s(u_k - u_m)$ in (3.8) and in the corresponding equation with k replaced by m , and subtracting the resultant equations. Inequality (7.36) and the properties of the function ψ ensure that, if s is chosen sufficiently small, then

$$|\{ |u_k - u_m| \leq s, \Theta_- (|\nabla u_k|) \leq \tau, \Theta_- (|\nabla u_m|) \leq \tau, \Theta_- (|\nabla u_k - \nabla u_m|) > t \}| < \varepsilon. \tag{7.37}$$

On the other hand, since $\{u_k\}$ is a Cauchy sequence in measure,

$$|\{ |u_k - u_m| > s \}| < \varepsilon, \tag{7.38}$$

if k and m are sufficiently large. From inequalities (7.32), (7.34), (7.37), and (7.38), we infer that $\{\nabla u_k\}$ is a Cauchy sequence in measure.

Step 5 Almost everywhere convergence of gradients.

Our aim here is to show that the function u obtained in Step 3 belongs to the class $\mathcal{T}_0^{1,\Phi}(\Omega)$, and that $\nabla u_k \rightarrow \nabla u$ a.e. in Ω (up to subsequences), where ∇u denotes the “generalized gradient” of u in the sense of the function Z_u appearing in (2.38).

Since $\{\nabla u_k\}$ is a Cauchy sequence in measure, there exist a subsequence (still indexed by k) and a function $W \in \mathcal{M}(\Omega; \mathbb{R}^n)$ such that

$$\nabla u_k \rightarrow W \quad \text{a.e. in } \Omega. \tag{7.39}$$

We have to show that

$$\nabla u = W \tag{7.40}$$

and

$$\chi_{\{|u|<t\}} W \in L^\Phi(\Omega; \mathbb{R}^n) \quad \text{for every } t > 0. \tag{7.41}$$

To this purpose, observe that estimate (7.25) ensures that, for each fixed $t > 0$, the sequence $\{\nabla T_t(u_k)\}$ is bounded in $L^\Phi(\Omega; \mathbb{R}^n)$. Hence, by Theorem 2.7, the sequence $\{\nabla T_t(u_k)\}$ is compact in $L^\Phi(\Omega; \mathbb{R}^n)$ with respect to the weak- $*$ convergence. Since $T_t(u_k) \rightarrow T_t(u)$ in $L^1(\Omega)$, the function $T_t(u)$ is weakly differentiable, and its gradient agrees with the weak- $*$ limit of $\{\nabla T_t(u_k)\}$.

Thus, for each fixed $t > 0$, there exists a subsequence of $\{u_k\}$, still indexed by k , such that

$$\lim_{k \rightarrow \infty} \nabla T_t(u_k) = \lim_{k \rightarrow \infty} \chi_{\{|u_k|<t\}} \nabla u_k = \chi_{\{|u|<t\}} W \quad \text{a.e. in } \Omega, \tag{7.42}$$

and

$$\lim_{k \rightarrow \infty} \nabla T_t(u_k) = \nabla T_t(u) \quad \text{weakly-} * \text{ in } L^\Phi(\Omega; \mathbb{R}^n). \tag{7.43}$$

Therefore, $\nabla T_t(u) = \chi_{\{|u|<t\}} W$ a.e. in Ω , whence Eqs. (7.40) and (7.41) follow, owing to (2.38).

Step 6 Uniqueness of the solution.

Suppose that u and \bar{u} are approximable solutions to problem (1.1). Thus, there exist sequences $\{f_k\}$ and $\{\bar{f}_k\}$ in $L^\infty(\Omega)$, such that $f_k \rightarrow f$ and $\bar{f}_k \rightarrow \bar{f}$ in $L^1(\Omega)$ and weak solutions u_k to (3.8) and \bar{u}_k to

$$\begin{cases} -\operatorname{div} a(x, \nabla \bar{u}_k) = \bar{f}_k & \text{in } \Omega \\ \bar{u}_k(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.44}$$

such that $u_k \rightarrow u$ and $\bar{u}_k \rightarrow \bar{u}$ a.e. in Ω .

Fix any $t > 0$, make use of $\varphi = T_t(u_k - \bar{u}_k)$ as a test function in (3.8) and (7.44), and subtract the resultant equations to obtain

$$\int_{\{|u_k - \bar{u}_k| \leq t\}} (a(x, \nabla u_k) - a(x, \nabla \bar{u}_k)) \cdot (\nabla u_k - \nabla \bar{u}_k) \, dx = \int_{\Omega} (f_k - \bar{f}_k) T_t(u_k - \bar{u}_k) \, dx \tag{7.45}$$

for every $k \in \mathbb{N}$. The right-hand side of (7.45) tends to 0 as $k \rightarrow \infty$, since $|T_t(u_k - \bar{u}_k)| \leq t$. As shown in Steps 3–5, one has that $u, \bar{u} \in \mathcal{T}_0^{1,\Phi}(\Omega)$, and $\{\nabla u_k\}$ and $\{\nabla \bar{u}_k\}$ converge (up to subsequences) a.e. in Ω to the generalized gradients ∇u and $\nabla \bar{u}$, respectively. Thus, by assumption (1.2) and Fatou’s lemma, passing to the limit in (7.45) tells us that

$$\int_{\{|u - \bar{u}| \leq t\}} (a(x, \nabla u) - a(x, \nabla \bar{u})) \cdot (\nabla u - \nabla \bar{u}) \, dx = 0.$$

Consequently, by (1.2) again, $\nabla u = \nabla \bar{u}$ a.e. in $\{|u - \bar{u}| \leq t\}$ for every $t > 0$, whence

$$\nabla u = \nabla \bar{u} \quad \text{a.e. in } \Omega. \tag{7.46}$$

Fix any $t, \tau > 0$. Inequality (2.47), applied to the function $T_\tau(u - T_t(\bar{u}))$, and Eq. (7.46) tell us that

$$\int_\Omega \Phi_\circ(c|T_\tau(u - T_t(\bar{u}))|) dx \leq \left(\int_{\{|t < |u| < t + \tau\}} \Phi(\nabla u) dx + \int_{\{|t - \tau < |u| < t\}} \Phi(\nabla u) dx \right), \tag{7.47}$$

where $c = \kappa_1 |\Omega|^{-\frac{1}{n}}$, and κ_1 is the constant appearing in (2.47). We claim that, for each $\tau > 0$, the right-hand side of (7.47) converges to 0 as $t \rightarrow \infty$. To verify this claim, choose the test function $\varphi = T_\tau(u_k - T_t(u_k))$ in Eq. (7.24) and deduce that

$$\int_{\{|t < |u_k| < t + \tau\}} \Phi(\nabla u_k) dx \leq \int_{\{|t < |u_k| < t + \tau\}} a(x, \nabla u_k) \cdot \nabla u_k dx \leq \tau \int_{\{|u_k| > t\}} |f_k| dx. \tag{7.48}$$

Passing to the limit as $k \rightarrow \infty$ in (7.48) yields, by Fatou’s lemma,

$$\int_{\{|t < |u| < t + \tau\}} \Phi(\nabla u) dx \leq \tau \int_{\{|u| > t\}} |f| dx. \tag{7.49}$$

Thereby, the first integral on the right-hand side of (7.47) approaches 0 as $t \rightarrow \infty$. An analogous argument implies that also the last integral in (7.47) tends to 0 as $t \rightarrow \infty$. On the other hand,

$$\lim_{t \rightarrow \infty} T_\tau(u - T_t(\bar{u})) = T_\tau(u - \bar{u}) \quad \text{a.e. in } \Omega.$$

From (7.47), via Fatou’s lemma, we thus infer that

$$\int_\Omega \Phi_\circ(c|T_\tau(u - \bar{u})|) dx = 0 \tag{7.50}$$

for every $\tau > 0$. Since Φ_\circ vanishes only at 0, Eq. (7.50) ensures that $T_\tau(u - \bar{u}) = 0$ a.e. in Ω for every $\tau > 0$, whence $u = \bar{u}$ a.e. in Ω .

Step 7 Property (3.11) holds.

Choosing $t = 0$ in inequality (7.49) tells us that u satisfies assumption (7.10) of Proposition 7.2 with $K = \|f\|_{L^1(\Omega)}$. By Propositions 7.2 and 7.3, the solution u fulfills inequalities (7.11) and (7.20). These inequalities in turn imply (3.11). \square

Proof of Theorem 3.10 The proof follows exactly along the same lines as Steps 1–5 and 7 of the proof of Theorem 3.7. One has just to begin with a sequence $\{f_k\} \subset L^\infty(\Omega)$, which is weakly-* convergent to μ in the space of measures, and such that $\|f_k\|_{L^1(\Omega)} \leq 2\|\mu\|(\Omega)$. Such a sequence can be defined, for instance, as in (5.5), with $U(y)dy$ replaced by $d\mu(y)$. Of course, the quantity $\|f\|_{L^1(\Omega)}$ has then to be replaced by $\|\mu\|(\Omega)$ throughout.

Let us just point out that the proof of uniqueness, namely of Step 6 of Theorem 3.7, fails in the present situation since, for instance, it is not guaranteed that the right-hand side of equation (7.45) approaches 0 as $k \rightarrow \infty$. \square

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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