Calc. Var. (2019) 58:166
https://doi.org/10.1007/s00526-019-1608-y

Calculus of Variations

Local well-posedness of the vacuum free boundary of 3-D compressible Navier–Stokes equations

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Received: 3 May 2019 / Accepted: 31 July 2019 / Published online: 9 September 2019

Abstract
In this paper, we consider the 3-D motion of viscous gas with the vacuum free boundary. We use the conormal derivative to establish local well-posedness of this system. One of important advantages in the paper is that we do not need any strong compatibility conditions on the initial data in terms of the acceleration.

Mathematics Subject Classification 35K65, 35R35, 76N10

1 Introduction

1.1 Formulation in Eulerian coordinates

In the paper, we consider a 3-D viscous compressible fluid in a moving domain $\Omega(t)$ with an upper free surface $\Gamma(t)$ and a fixed bottom Γ_b. This model can be expressed by the 3-D compressible Navier–Stokes equations(CNS)

Communicated by L. Caffarelli.

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$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0 \quad \text{in} \quad \Omega(t), \\ \rho(\partial_t u + u \cdot \nabla u) + \nabla p - \nabla \cdot \mathbb{S}(u) = 0 \quad \text{in} \quad \Omega(t), \\ \rho > 0 \quad \text{in} \quad \Omega(t), \quad \rho = 0 \quad \text{on} \quad \Gamma(t), \\ \mathcal{V}(\Gamma(t)) = u \cdot n \quad \text{on} \quad \Gamma(t), \\ (\mathbb{S}(u) - p\,\mathbb{I})n = 0 \quad \text{on} \quad \Gamma(t), \\ u|_{\Gamma_b} = 0 \quad \text{on} \quad \Gamma_b, \\ (\rho, u)|_{t=0} = (\rho_0, u_0) \quad \text{in} \quad \Omega(0), \quad \Omega(0) = \Omega_0, \end{cases} \tag{1.1}$$

where $\mathcal{V}(\Gamma(t))$ denotes the normal velocity of the free surface $\Gamma(t)$, and $n = n(t)$ is the exterior unit normal vector of $\Gamma(t)$, the vector-field u denotes the Eulerian velocity field, ρ is the density of the fluid, and $p = p(\rho)$ denotes the pressure function. The stress tensor $\mathbb{S}(u)$ is defined by $\mathbb{S}(u) = \mu\mathbb{D}(u)+\lambda(\nabla\cdot u)\mathbb{I}$, where the strain tensor $\mathbb{D}(u) = \nabla u+\nabla u^T$ and dynamic viscosity μ and bulk viscosity ν are constants which satisfy the following relationship

$$\mu > 0, \; \lambda + \frac{2}{3}\mu \geq 0. \tag{1.2}$$

The deviatoric (trace-free) part of the strain tensor $\mathbb{D}(u)$ is then $\mathbb{D}^0(u) = \mathbb{D}(u) - \frac{2}{3}\text{div}\, u\,\mathbb{I}$. The viscous stress tensor in fluid is then given by $\mathbb{S}(u) = \mu\,\mathbb{D}^0(u) + (\lambda + \frac{2}{3}\mu)(\nabla \cdot u)\,\mathbb{I}$. Moreover, the pressure obeys the γ-law: $p(\rho) = K\rho^\gamma$, where K is an entropy constant and $\gamma > 1$ is the adiabatic gas exponent.

Equation $(1.1)_1$ is the conservation of mass; Eq. $(1.1)_2$ means the momentum conserved; the boundary condition $(1.1)_3$ states that the pressure (and hence the density function) vanishes along the moving boundary $\Gamma(t)$, which indicates that the vacuum state appears on the boundary $\Gamma(t)$; the kinematic boundary condition $(1.1)_4$ states that the vacuum boundary $\Gamma(t)$ is moving with speed equal to the normal component of the fluid velocity; $(1.1)_5$ means the fluid satisfies the kinetic boundary condition on the free boundary, $(1.1)_6$ denotes the fluid is no-slip, no-penetrated on the fixed bottom boundary, and $(1.1)_7$ are the initial conditions for the density, velocity, and domain.

In the paper, we assume the bottom $\Gamma_b = \{y_3 = b(y_h)\}$, and the moving domain $\Omega(t)$ is horizontal periodic by setting $\mathbb{T}^2_{y_h}$ with $y_h := (y_1, y_2)^T$ for $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

1.2 Known results

Whether or not the appearance of vacuum state is related to the regularity of the solution to the compressible Navier–Stokes equations. Even if there is no vacuum in initial data, it cannot guarantee that vacuum state will be not generated in finite time in high-dimensional system. Whence initial data is close to a non-vacuum equilibrium in some functional space, Matsumura and Nishida [35,36] proved global well-posedness of strong solutions to the 3-D CNS. Moreover, for the one dimensional case, Hoff and Smoller [17] proved that if the vacuum is not included at the beginning, no vacuum will occur in the future. Hoff and Serre [16] showed some physical weak solution does not have to depend continuously on their initial data when vacuum occurs.

When the initial density may vanish in open sets or on the (part of) boundary of the domain, the flow density may contain a vacuum, the equation of velocity becomes a strong degenerate hyperbolic-parabolic system and the degeneracy is one of major difficulties in study of regularity and the solution's behavior, which is completely different from the non-vacuum case. For the existence of solutions for arbitrary data (the far field density is vacuum, that is, $\rho(t,x) \to 0$ as $x \to \infty$), the major breakthrough is due to Lions [27] (also see [8, 14,22]), where he obtains global existence of weak solutions, defined as solutions with finite energy with suitable γ. Recently, Li and Xin [26] and Vasseur and Yu [39] independently studied global existence of weak solutions of CNS whence the viscosities depend on the density and satisfy the Bresch and Desjardins relation [1]. Yet little is known on the structure of such weak solutions except for the case that some additional assumptions are added (see [15] for example). Indeed, the works of Xin etc. [24,40] showed that the homogeneous Sobolev space is as crucial as studying the well-posedness for the Cauchy problem of compressible Navier–Stokes equations in the presence of a vacuum at far fields even locally in time. Adding some compatible condition on initial data, Cho and Kim [3] develop local well-posedness for strong solutions. Moreover, if initial energy is small, Huang et al. [18] showed the global existence of classical solutions but with large oscillations to CNS.

Physically, the vacuum problem appears extensively in the fundamental free boundary hydrodynamical setting: for instance, the evolving boundary of a viscous gaseous star, formation of shock waves, vortex sheets, as well as phase transitions.

For free boundary problem of the multi-dimensional Navier–Stokes equations with non-vacuum state, there are many results concerning its local and global strong solutions, one may refer to [43,44] and references therein.

But when the vacuum (in particular, the physical vacuum [28]) appears, the system becomes much harder. To understand the difficulty of the vacuum, we introduce the sound speed $c := \sqrt{p'(\rho)}(= \sqrt{K\gamma}\rho^{\frac{\gamma-1}{2}}$ for polytropic gases) of the gas or fluid to describe the behavior of the smoothness of the density connecting to vacuum boundary. A vacuum boundary $\Gamma(t)$ is called physical vacuum if there holds

$$-\infty < \frac{\partial c^2}{\partial n} < 0 \tag{1.3}$$

near the boundary $\Gamma(t)$, where n is the outward unit normal to the free surface. The physical vacuum condition (1.3) implies the pressure (or the enthalpy c^2) accelerates the boundary in the normal direction. Thus, the initial physical vacuum condition (1.3) is equivalent to the requirement that

$$-\infty < \partial_n(\rho_0^{\gamma-1}) < 0 \quad \text{on} \quad \Gamma(0) \tag{1.4}$$

which means that $\rho_0^{\gamma-1}(x) \sim dist(x,\Gamma(0))$, in other words, the initial sound speed c_0 is only $C^{\frac{1}{2}}$-Hölder continuous near the interface $\Gamma(0)$.

Due to lack of sufficient smoothness of the enthalpy c^2 at the vacuum boundary, a rigorous understanding of the existence of physical vacuum states in compressible fluid dynamics has been a challenging problem, especially in multi-dimensional cases.

Recently, the local well-posedness theory for compressible Euler system with physical vacuum singularity was established in [4,20,21], and also global existence of smooth solutions for the physical vacuum free boundary problem of the 3-D spherically symmetric compressible Euler equations with damping was showed in [32]. And more recently, Hadzic and Jang [13] proved global nonlinear stability of the affine solutions to the compressible Euler system with physical vacuum, and Guo et al. [9] constructed an infinite dimensional family of collapsing solutions to the Euler-Poisson system whose density is in general space inhomogeneous and undergoes gravitational blowup along a prescribed space-time surface, with continuous mass absorption at the origin.

The study of vacuum is important in understanding viscous surface flows [30]. Very little is rigorously known about well-posedness theories available about free boundary problems of CNS with physical vacuum boundary. For 1-D problem, global regularity for weak solutions to the vacuum free boundary problem of CNS was obtained in [30], which is further generalized by Zeng [45] which established the strong solutions. For the multidimensional case, regularity results related to spherically symmetric motions. Guo et al. [11] obtain a global weak solution to the problem with spherically symmetric motions and a jump density connects to vacuum. Later Liu [29] gives the existence of global solutions with small energy in spherically symmetric motions with the density connected to vacuum continuously or discontinuously. Anyway, almost all the well-posedness results require additional strongly singular compatibility conditions on initial data in terms of the acceleration for gaining more regularities of the velocity. Some related works can refer to [2,6,7,12,19,25,28,30,31,37,41,42] and references therein.

The purpose of this paper is to establish the local well-posedness of the 3-D compressible Navier–Stokes equations (1.1) with physical vacuum boundary condition without any compatibility conditions, more precisely, we do not need any initial condition on the material derivative $D_t u$ or its derivatives. For simplicity, we set $\gamma = 2$ and $K = 1$ in this paper.

As mentioned above, the main difficulty in obtaining regularity for the vacuum free boundary problem (1.1) lies in the degeneracy of the system near vacuum boundaries. In order to solve the system (1.1), the first idea is that we use Lagrangian coordinates to transform it to a system with fixed domain. One of advantage of Lagrangian coordinates is that the density ρ is solved directly by initial data and we only focus on the equation of velocity with coefficients related to Lagrangian coordinates.

The second and also key idea in our paper is that we use the conormal derivatives to obtain the high-order regularity. Because the density vanishes on the boundary, we can not close the energy estimates if we directly take normal derivatives to the system. So another choose is to take time derivatives in [4,21] solving the compressible Euler equations with the physical vacuum, where high-order enough time-derivative estimates as long as spatial-derivative estimates allow us to close the energy estimates and then get the local-in-time existence of the strong solution of the Euler system. This high-order energy estimate in it is reasonable since the pressure term may cancel the singularity near the vacuum boundary when consider compatibility conditions on initial data in terms of the acceleration and its derivatives. However, this method may not work for the Navier–Stokes system (1.1) with constant viscosity coefficients. In fact, a strong singular compatibility conditions on initial data in terms of the acceleration and its derivatives will appear in it when we consider the high-order energy estimate, which is mainly due to the non-degenerate of the viscosity, but it seems very hard to find such kind of initial data satisfying these compatibility conditions. In order to get rid of this difficulty, our strategy is that we use conormal Sobolev space introduced in [34] to get the tangential regularity. Based on that, we multiply $\partial_t v$ on the both sides of equations of v to get the estimates of $\rho^{\frac{1}{2}} \partial_t v$ which implies the two-order derivative on the normal direction. Form this, together with high-order tangential derivatives estimates, we get the $W^{1,\infty}$ estimates of v and its conormal derivatives, which in turn guarantees the propagation of conormal regularities of the velocity.

1.3 Derivation of the system in Lagrangian coordinates and main result

In this paper, we consider the case that the upper boundary does not touch the bottom which means that

$$dist(\Gamma(0), \Gamma_b) > 0.$$

Take $\Omega = \{x \in \mathbb{T}^2 \times \mathbb{R}| \quad 0 < x_3 < 1\}$ as the domain of equilibrium. Let $\eta(t, x)$ be the position of the gas particle x at time t so that

$$\begin{cases} \partial_t \eta(t, x) = u(t, \eta(t, x)) \quad \text{for} \quad t > 0, \\ \eta(0, x) = \eta_0(x) \quad \text{in} \quad \Omega. \end{cases} \tag{1.5}$$

Here η_0 is a diffeomorphism from Ω to the initial moving domain $\Omega(0)$ which satisfies that $\Gamma(0) = \eta_0(\{x_3 = 1\})$ and $\Gamma_b = \eta_0(\{x_3 = 0\})$. It is easy to construct a invertible transform η_0 which satisfies that

$$\det(D\eta_0) > 0.$$

Due to (1.5), we introduce the displacement $\xi(t, x) \overset{\text{def}}{=} \eta(t, x) - x$ which satisfies the following ODE

$$\begin{cases} \partial_t \xi(t, x) = u(t, x + \xi(t, x)) \quad \text{for} \quad t > 0, \\ \xi(0, x) = \xi_0(x) := \eta_0(x) - x \quad \text{in} \quad \Omega. \end{cases} \tag{1.6}$$

We define the following Lagrangian quantities:

$$v(t, x) := u(t, \eta(t, x)), \quad f(t, x) := \rho(t, \eta(t, x)),$$
$$\mathcal{A} := [D\eta]^{-1}, \quad J := \det(D\eta), \quad \mathcal{N} := J\mathcal{A}e_3.$$

Then, the system (1.1) is reformulated in Lagrangian coordinates as follows

$$\begin{cases} \partial_t \xi = v \quad \text{in} \quad \Omega, \\ \partial_t f + f \nabla_{\mathcal{A}} \cdot v = 0 \quad \text{in} \quad \Omega, \\ f \partial_t v + \nabla_{\mathcal{A}}(f^2) - \nabla_{\mathcal{A}} \cdot \mathbb{S}_{\mathcal{A}}(v) = 0 \quad \text{in} \quad \Omega \end{cases} \tag{1.7}$$

with boundary conditions

$$\begin{cases} f = 0 \quad \text{on} \quad \Gamma, \\ \mathbb{S}_{\mathcal{A}}(v)\mathcal{N} = 0, \quad \text{on} \quad \Gamma, \\ v|_{x_3=0} = 0 \end{cases} \tag{1.8}$$

and initial data

$$(\xi, f, v)|_{t=0} = (\xi_0, \rho_0, u_0). \tag{1.9}$$

One may readily check from the definition of J that

$$\partial_t J = \nabla_{J\mathcal{A}} \cdot v,$$

which together with the equation of f in (1.7) yields

$$\partial_t (fJ) = J\partial_t f + f\partial_t J = -Jf\nabla_{\mathcal{A}} \cdot v + fJ\nabla_{\mathcal{A}} \cdot v = 0.$$

Hence, we find

$$Jf(t, x) = (Jf)(0, x) = \det(D\eta_0)\rho_0(\eta_0), \tag{1.10}$$

where ρ_0 is a given initial density function. We are interested in the initial density ρ_0 satisfying

$$\rho_0(\eta_0)\det(D\eta_0) = \overline{\rho}(x) \quad \text{in} \quad \Omega, \tag{1.11}$$
$$C^{-1}d(x) \le \overline{\rho}(x) \le C\,d(x) \quad \text{in} \quad \Omega, \tag{1.12}$$
$$|\nabla\overline{\rho}| \le C, \ |\overline{\rho}^{-1}\nabla_h^k\overline{\rho}| \le C_k \quad \text{in} \quad \Omega \tag{1.13}$$

with some given function $\overline{\rho}(x)$ $(x \in \Omega)$, for any $k \in \mathbb{N}$ with $\nabla_h = (\partial_1, \partial_2)$, where $d(x)$ is the distance function to the boundary $\{x_3 = 1\}$.

Thus, it follows from (1.10) that

$$Jf = \overline{\rho}(x), \tag{1.14}$$

which implies that

$$f = J^{-1}\overline{\rho}, \quad q = f^2 = J^{-2}\overline{\rho}^2. \tag{1.15}$$

Remark 1.1 For any smooth subdomain $\mathcal{O}$ of Ω, we know that $\eta_0(\mathcal{O})$ is a subdomain of $\Omega(0)$ if η_0 is a diffeomorphism from Ω to $\Omega(0)$. Hence, by using change of variables, we get

$$\int_{\eta_0(\mathcal{O})} \rho_0(y)\,dy = \int_{\mathcal{O}} \rho_0(\eta_0)\ \det(D\eta_0)\,dx. \tag{1.16}$$

Hence, the assumption (1.11) is equivalent to the mass conservation law

$$\int_{\eta_0(\mathcal{O})} \rho_0(y)\,dy = \int_{\mathcal{O}} \overline{\rho}\,dx \quad \forall\ \mathcal{O} \subset \Omega. \tag{1.17}$$

Multiplying the both side of equation v by J, we obtain the equivalent form of the system (1.7)–(1.9) as follows

$$\begin{cases} \partial_t \xi = v \quad \text{in} \quad \Omega, \\ \overline{\rho}\partial_t v + \nabla_{J\mathcal{A}}(J^{-2}\overline{\rho}^2) - \nabla_{J\mathcal{A}} \cdot \mathbb{S}_{\mathcal{A}}(v) = 0 \quad \text{in} \quad \Omega, \\ \mathbb{S}_{\mathcal{A}}(v)\mathcal{N} = 0, \quad \text{on} \quad \Gamma, \\ v|_{x_3=0} = 0, \\ \xi|_{t=0} = \xi_0, \quad v|_{t=0} = v_0 \quad \text{in} \quad \Omega. \end{cases} \tag{1.18}$$

Next, we give some useful equations which we often use in what follows. Since $\mathcal{A}[D\eta] = I$, one obtains that

$$\partial_t \mathcal{A}_i^k = -\mathcal{A}_i^s \partial_s v^r \mathcal{A}_r^k, \quad \partial_l \mathcal{A}_i^k = -\mathcal{A}_i^s \partial_s \partial_l \eta^r \mathcal{A}_r^k. \tag{1.19}$$

Differentiating the Jacobian determinant, we get

$$\partial_t J = J\mathcal{A}_r^s \partial_s v^r, \quad \partial_l J = J\mathcal{A}_r^s \partial_s \partial_l \eta^r. \tag{1.20}$$

Moreover, the following Piola identity holds:

$$\partial_j (J\mathcal{A}_i^j) = 0, \tag{1.21}$$

for any $i = 1, 2, 3$.

1.4 Main results

Before we state our main results, we give some definitions of functional spaces. First, define the operators:

$$Z_1 \overset{\text{def}}{=} \partial_1, \quad Z_2 \overset{\text{def}}{=} \partial_2, \quad Z_3 \overset{\text{def}}{=} \overline{\rho}\partial_3. \tag{1.22}$$

Using Z^m to denote $Z_3^{m_2} Z_h^{m_1} = Z_3^{m_2} Z_1^{m_{11}} Z_2^{m_{12}}$ with $m_1 = (m_{11}, m_{12})$ and $|m|$ to denote $|m| = |m_1| + m_2 = m_{11} + m_{12} + m_2$. Moreover, we use $Z_3^{m_2}$ to denote $\overline{\rho}^{m_2}\partial_3^{m_2}$. By (1.11)–(1.13), it is easy to see

$$[\partial_3, Z^m] \sim Z^{m-1}\partial_3, \ [Z_h, Z_3] \sim Z_3.$$

We recall the following conormal Sobolev space introduced by Masmoudi and Rousset [34].

$$\|f\|^2_{X^N_\alpha} := \sum_{|m|=0}^{N} \|\overline{\rho}^\alpha Z^m f\|^2_{L^2}, \quad \|f\|^2_{\dot{X}^N_\alpha} := \sum_{|m|=1}^{N} \|\overline{\rho}^\alpha Z^m f\|^2_{L^2},$$

where $\alpha \in \mathbb{R}$. In particular, when $\alpha = 0$, we the spaces X^N_α and $\dot{X}^N_\alpha$ will be denoted by X^N and $\dot{X}^N$ respectively for simplicity.

For $T > 0$, we define the energy space E_T as

$$E_T \overset{\text{def}}{=} C([0,T]; X^{12}_{\frac{1}{2}} \cap H^1(\Omega))$$

with the instantaneous energy $\mathcal{E}(t)$ (in terms to the velocity v)

$$\mathcal{E}(t) \overset{\text{def}}{=} \|v\|^2_{X^{12}_{\frac{1}{2}}} + \|v\|^2_{H^1},$$

and the dissipation $\mathcal{D}(t)$

$$\mathcal{D}(t) \overset{\text{def}}{=} \|\nabla v\|^2_{X^{12}} + \|\overline{\rho}^{\frac{1}{2}} \partial_t v\|^2_{L^2}.$$

Given $\kappa > 0$, we also introduce the space F_κ in terms to the flow map η as follows:

$$\mathcal{F}_\kappa = \mathcal{F}_\kappa(\Omega) \overset{\text{def}}{=} \{\xi \in X^{12} \cap H^1(\Omega) |\ \nabla\xi \in X^{12},\ \ \overline{\rho}^{-\frac{1}{2}+\kappa} \Delta\xi \in L^2\}$$

equipped with the norm

$$\|\xi\|_{\mathcal{F}_\kappa} \overset{\text{def}}{=} \|\xi\|_{X^{12}} + \|\nabla\xi\|_{X^{12}} + \|\overline{\rho}^{-\frac{1}{2}+\kappa} \Delta\xi\|_{L^2}.$$

Now, we are in the position to state our main results.

Theorem 1.2 *Under the assumptions (1.11)–(1.13), assume that there exists a positive number σ_0 such that*

$$dist(\Gamma(0), \Gamma_b) > 0, \tag{1.23}$$

$$2\sigma_0 \le J_0 \le 3\sigma_0. \tag{1.24}$$

If the initial data $(v_0, \eta_0) \in (X^{12}_{\frac{1}{2}} \cap H^1(\Omega)) \times \mathcal{F}_\kappa(\Omega)$ for some constant $\kappa \in (0, \frac{1}{16})$, then the system (1.18) is locally well-posed. More precisely, there exists a positive time $T > 0$ such that the system (1.18) has a unique solution $(v, \eta) \in C([0,T]; X^{12}_{\frac{1}{2}} \cap H^1(\Omega)) \times C([0,T]; \mathcal{F}_\kappa(\Omega))$ depending continuously on initial data $(v_0, \eta_0) \in (X^{12}_{\frac{1}{2}} \cap H^1(\Omega)) \times \mathcal{F}_\kappa(\Omega)$, and there hold

$$\begin{aligned} &\sup_{t\in[0,T]} (\|v\|^2_{X^{12}_{\frac{1}{2}}} + \|v\|^2_{H^1}) + \int_0^T \left(\|\nabla v\|^2_{X^{12}} + \|\overline{\rho}^{\frac{1}{2}} \partial_t v\|^2_{L^2} \right) ds \le C, \\ &\sup_{t\in[0,T]} \|\xi(t)\|^2_{\mathcal{F}_\kappa} \le C, \quad \sigma_0 \le \sup_{(t,x)\in[0,T]\times\Omega} J(t,x) \le 4\sigma_0, \end{aligned} \tag{1.25}$$

where C depends on initial data.

Remark 1.3 The assumption (1.11)–(1.13) on ρ_0 is reasonable. In fact, if $\Omega(0) = \Omega := \{x \in \mathbb{T}^2 \times \mathbb{R} |\ \ 0 < x_3 < 1\}$ and $\rho_0 = dist(x, \partial\Omega) \sim x_3(1-x_3)$, then the assumptions (1.11)–(1.13) are automatically satisfied .

Remark 1.4 In this paper, we consider the case that $\gamma = 2$. But our method may still work for all the cases $\gamma > 1$.

Remark 1.5 For any $t \in [0, T]$, since $\sigma_0 \leq \sup_{(t,x)\in[0,T]\times\Omega} J(t, x) \leq 4\sigma_0$, the flow-map $\eta(t, x)$ defines a diffeomorphism from the equilibrium domain Ω to the moving domain $\Omega(t)$ with the boundary $\Gamma(t)$. From this, together with the fact that η_0 is a diffeomorphism from the equilibrium domain Ω to the initial domain $\Omega(0)$, we deduce a diffeomorphism from the initial domain $\Omega(0)$ to the evolving domain $\Omega(t)$ for any $t \in [0, T]$. Denote the inverse of the flow map $\eta(t, x)$ by $\eta^{-1}(t, y)$ for $t \in [0, T]$ so that if $y = \eta(t, x)$ for $y \in \Omega(t)$ and $t \in [0, T]$, then $x = \eta^{-1}(t, y) \in \Omega$.

For the strong solution (η, v) obtained in Theorem 1.2, and for $y \in \Omega(t)$ and $t \in [0, T]$, we denote that

$$\rho(t,\ y) := J^{-1}(t, \eta^{-1}(t, y))\overline{\rho}_0(\eta^{-1}(t, y)), \quad u(t,\ y) := v(t,\ \eta^{-1}(t, y)). \tag{1.26}$$

Then the triple $(\rho(t,\ y), u(t, y), \Omega(t))$ $(t \in [0, T])$ defines a strong solution to the free boundary problem (1.1). Furthermore, we obtain the following theorem.

Theorem 1.6 *Under the assumptions in Theorem 1.2, the free boundary problem (1.1) is locally well-posed, and the triple $(\rho(t,\ y), u(t, y), \Omega(t))$ $(t \in [0, T])$ defined in Remark 1.5 and (1.26) is the unique strong solution to the free boundary problem (1.1) satisfying $\eta - Id \in C([0, T],\ \mathcal{F}_\kappa)$.*

The rest of the paper is organized as follows. In Sect. 2, we derive some preliminary estimates. Some necessary *a priori* estimates are obtained in Sect. 3. Finally in Sect. 4, the proof of Theorem 1.2 is proved.

Let us complete this section with some notations that we use in this context.

Notations Let A, B be two operators, we denote $[A, B] = AB - BA$, the commutator between A and B. For $a \lesssim b$, we mean that there is a uniform constant C, which may be different on different lines, such that $a \leq Cb$ and C_0 denotes a positive constant depending on the initial data only.

2 Preliminary estimates

In what follows, we denote by C a positive constant which may depend on initial data (v_0, η_0) if we don't make a special explanation in it. This notation is allowed to change from one inequality to the next.

We first introduce the following inequality which we heavily use throughout the paper.

Lemma 2.1 (Hardy inequality, [23]) *For any* $\varepsilon > 0$, *there holds that*

$$\|\overline{\rho}^{-\frac{1}{2}+\varepsilon} f\|_{L^2(\Omega)} \leq C(\|\overline{\rho}^{\frac{1}{2}+\varepsilon} f\|_{L^2(\Omega)} + \|\overline{\rho}^{\frac{1}{2}+\varepsilon}\nabla f\|_{L^2(\Omega)}).$$

With Hardy inequality in hand, we may get the following interpolation equalities.

Lemma 2.2 *For any* $\kappa \in (0, \frac{1}{16})$, *there hold that, for* $0 \leq \ell \leq 6$,

$$\|Z^\ell \nabla f\|_{L^\infty_{x_3}(L^2_h)} \leq C(\|\nabla f\|_{X^{12}} + \|\overline{\rho}^{-\frac{1}{2}+\kappa}\triangle f\|_{L^2}). \tag{2.1}$$

and for $0 \leq \ell \leq 4$,

$$\|Z^\ell \nabla f\|_{L^\infty} \leq C(\|\nabla f\|_{X^{12}} + \|\overline{\rho}^{-\frac{1}{2}+\kappa}\triangle f\|_{L^2}). \tag{2.2}$$

Proof For $0 \le \ell \le 6$, thanks to the Sobolev embedding theorem and Lemma 2.1, we have

$$
\begin{aligned}
\|Z^\ell \nabla f\|_{L^\infty_{x_3}(L^2_h)} &\le C_0(\|\overline{\rho}^{-\frac{21}{44}} Z^\ell \nabla f\|_{L^2_{x_3}(L^2_h)} + \|\overline{\rho}^{\frac{21}{44}} \partial_3 Z^\ell \nabla f\|_{L^2_{x_3}(L^2_h)}) \\
&\le C_0(\|\overline{\rho}^{\frac{23}{44}} Z^\ell \nabla f\|_{L^2} + \|\overline{\rho}^{\frac{23}{44}} \nabla Z^\ell \nabla f\|_{L^2} \\
&\quad + \sum_{i=0}^{\ell} (\|\overline{\rho}^{\frac{21}{44}} Z^{i+1} \nabla f\|_{L^2} + \|\overline{\rho}^{\frac{21}{44}} Z^i \triangle f\|_{L^2}) \\
&\le C\|\nabla f\|_{X^{12}} + C\sum_{i=0}^{\ell} \|\overline{\rho}^{\frac{21}{44}} Z^i \triangle f\|_{L^2}.
\end{aligned} \tag{2.3}
$$

According to the fact $|Z\bar{\rho}| \le C\bar{\rho}$, we deduce from integration by parts that

$$
\begin{aligned}
\|\overline{\rho}^{\frac{21}{44}} Z^i \triangle f\|_{L^2} &\le C\|\overline{\rho}^{-\frac{1}{2}+\kappa} \triangle f\|_{L^2}^{1-\frac{7}{11}} \|\triangle f\|_{\dot{X}_1^{11}}^{\frac{7}{11}} \\
&\le C(\|\overline{\rho}^{-\frac{1}{2}+\kappa} \triangle f\|_{L^2} + \|\triangle f\|_{\dot{X}_1^{11}}), \quad \forall\ i \le 5,
\end{aligned} \tag{2.4}
$$

where we used that $\frac{14}{22} + (-\frac{1}{2}+\kappa)\frac{4}{11} \le \frac{21}{44} < \frac{1}{2}$ with $\kappa \in (0, \frac{1}{16})$.

While by using integration by parts again, one can see that

$$
\begin{aligned}
\|\overline{\rho}^{\frac{21}{44}} Z^6 \triangle f\|_{L^2}^2 &= \int_\Omega \bar{\rho}^{\frac{21}{22}} Z^6 \triangle f \cdot Z^6 \triangle f\, dx \\
&= -\int_\Omega \bar{\rho}^{\frac{21}{22}} Z\triangle f \cdot Z^{11} \triangle f\, dx - \int_\Omega [\rho^{\frac{21}{22}};\ Z^5] \triangle f \cdot Z^6 \triangle f\, dx,
\end{aligned}
$$

which follows from the fact $|Z\bar{\rho}| \le C\bar{\rho}$ that

$$
\|\overline{\rho}^{\frac{21}{44}} Z^6 \triangle f\|_{L^2}^2 \le C\|\triangle f\|_{X_1^{11}} (\|\bar{\rho}^{-\frac{1}{22}} \triangle f\|_{L^2} + \|\bar{\rho}^{-\frac{1}{22}} Z\triangle f\|_{L^2}).
$$

Next, we deal with the last term in the above inequality. In fact, we may get from integration by parts that

$$
\begin{aligned}
\|\bar{\rho}^{-\frac{1}{22}} Z\triangle f\|_{L^2}^2 &= \int_\Omega \bar{\rho}^{-\frac{1}{11}} Z\triangle f \cdot Z\triangle f\, dx \le C\|\rho^{-\frac{1}{2}+\kappa} \triangle f\|_{L^2} \sum_{k=0}^{2} \|\rho^{\frac{1}{2}-\frac{1}{11}-\kappa} Z^k \triangle f\|_{L^2} \\
&\le C\|\rho^{-\frac{1}{2}+\kappa} \triangle f\|_{L^2} (\|\triangle f\|_{X_1^{11}} + C_0\|\overline{\rho}^{-\frac{1}{2}+\kappa} \triangle f\|_{L^2}),
\end{aligned}
$$

which implies

$$
\|\bar{\rho}^{-\frac{1}{22}} Z\triangle f\|_{L^2} \le C(\|\triangle f\|_{X_1^{11}} + \|\overline{\rho}^{-\frac{1}{2}+\kappa} \triangle f\|_{L^2}).
$$

Hence, one has

$$
\|\overline{\rho}^{\frac{21}{44}} Z^6 \triangle f\|_{L^2}^2 \le C\|\triangle f\|_{X_1^{11}} (\|\bar{\rho}^{-\frac{1}{22}} \triangle f\|_{L^2} + \|\overline{\rho}^{-\frac{1}{2}+\kappa} \triangle f\|_{L^2} + \|\triangle f\|_{X_1^{11}}). \tag{2.5}
$$

Inserting (2.4–2.5) into (2.3) ensures that for $0 \le \ell \le 6$

$$
\|Z^\ell \nabla f\|_{L^\infty_{x_3}(L^2_h)} \le C(\|\nabla f\|_{X^{12}} + \|\overline{\rho}^{-\frac{1}{2}+\kappa} \triangle f\|_{L^2}),
$$

that is, the inequality (2.1) holds.

The second inequality (2.2) comes from the Sobolev embedding theorem and (2.1):

$$
\|Z^\ell \nabla f\|_{L^\infty} \le C\,\|Z^\ell \nabla f\|_{L^\infty_{x_3} H^2_h} \le C(\|\nabla f\|_{X^{12}} + \|\overline{\rho}^{-\frac{1}{2}+\kappa} \triangle f\|_{L^2})
$$

for $0 \le \ell \le 4$, which ends the proof of Lemma 2.2. □

To deal with nonlinear term, we need the following product laws in the spaces X^N.

Lemma 2.3 *There hold true that*

$$\|g\,f\|_{X^{12}} \le C\|g\|_{X^{12}} \sum_{|\ell|\le 6} \|Z^\ell f\|_{L^\infty_{x_3}(L^2_h)} + C\|f\|_{X^{12}} \sum_{|\ell|\le 6} \|Z^\ell g\|_{L^\infty_{x_3}(L^2_h)}, \tag{2.6}$$

and

$$\sum_{0\le|j|\le 1} \|g\,Z^j f\|_{X^{11}} \le C\|g\|_{X^{11}} \sum_{|\ell|\le 6} \|Z^\ell f\|_{L^\infty_{x_3}(L^2_h)} + C\|f\|_{X^{12}} \sum_{|\ell|\le 6} \|Z^\ell g\|_{L^\infty_{x_3}(L^2_h)}. \tag{2.7}$$

Proof By the Leibnitz formula, one can see that

$$\|g\,f\|_{X^{12}} \le C \sum_{|m_1|+|m_2|=0}^{12} \|Z^{m_1} g\, Z^{m_2} f\|_{L^2}.$$

Now, we focus only on the proof of the most difficulty case: $|m_1|+|m_2| = 12$. The others can be treated by a similar way. In fact, we divide its proof into three cases.

- *Case 1.* $8 \le |m_1| \le 12$. By Hölder's inequality, we prove

$$\begin{aligned}\|Z^{m_1} g\, Z^{m_2} f\|_{L^2} &\le \|Z^{m_1} g\|_{L^2}\|Z^{m_2} f\|_{L^\infty} \le C\|Z^{m_1} g\|_{L^2}\|Z^{m_2} f\|_{L^\infty_{x_3}(H^2_h)}\\ &\le C\|g\|_{X^{12}} \sum_{|\ell|\le 6} \|Z^\ell f\|_{L^\infty_{x_3}(L^2_h)},\end{aligned}$$

  where we used $|m_2| + 2 \le 6$.
- *Case 2.* $6 \le |m_1| \le 7$. Thanks to the Sobolev embedding theorem and Hölder's inequality, one can obtain that

$$\begin{aligned}\|Z^{m_1} g\, Z^{m_2} f\|_{L^2} &\le \|Z^{m_1} g\|_{L^2_{x_3}(L^\infty_h)}\|Z^{m_2} f\|_{L^\infty_{x_3}(L^2_h)}\\ &\le C\|g\|_{X^{12}} \sum_{|\ell|\le 6} \|Z^\ell f\|_{L^\infty_{x_3}(L^2_h)}.\end{aligned}$$

- *Case 3.* $0 \le |m_1| \le 5$. For this case, we only need to exchange the position of f and g and apply the same argument as in the above two cases to get that

$$\|Z^{m_1} g\, Z^{m_2} f\|_{L^2} \le C\|f\|_{X^{12}} \sum_{|\ell|\le 6} \|Z^\ell g\|_{L^\infty_{x_3}(L^2_h)}.$$

Collecting all the above cases together, we obtain

$$\|g\,f\|_{X^{12}} \le C\|g\|_{X^{12}} \sum_{|\ell|\le 6} \|Z^\ell f\|_{L^\infty_{x_3}(L^2_h)} + C \sum_{|\ell|\le 6} \|Z^\ell g\|_{L^\infty_{x_3}(L^2_h)}\|f\|_{X^{12}},$$

which follows (2.6).

Next, since we the highest order in (2.7) is 11, we may readily verify (2.7) by the same process above, which ends the proof of Lemma 2.3. □

We introduce a new quantity $\mathfrak{D}(v)(t)$ which controls $\|\nabla v\|_{L^\infty}$ according to Lemma 2.2:

$$\mathfrak{D}(v)(t) \overset{\text{def}}{=} \|\nabla v(t)\|_{X^{12}} + \|\overline{\rho}^{-\frac{1}{2}+\kappa}\Delta v(t)\|_{L^2}. \tag{2.8}$$

In what follows, $\mathcal{P}(\cdot)$ stands for some polynomial function which coefficients may depend on initial data.

Lemma 2.4 *Assume that*

$$\xi_0 \in \mathcal{F}_\kappa, \quad \|\mathfrak{D}(v)\|_{L^2(0,T)} \le \mathfrak{C}, \quad \sigma_0 \le J \le 4\sigma_0.$$

Then there hold that for any $t \in [0, T]$

$$\|\nabla v : \nabla v(t)\|_{X^{12}} \le C\mathfrak{D}(v)^2(t), \tag{2.9}$$

and

$$\begin{aligned} &\sum_{0\le|\ell|\le 6} \|Z^\ell (J\mathcal{A})(t)\|_{L^\infty_{x_3}(L^2_h)} \le C(1 + t^{\frac12}\mathcal{P}(\mathfrak{C})), \\ &\sum_{0\le|\ell|\le 6} \|Z^\ell \mathcal{A}(t)\|_{L^\infty_{x_3}(L^2_h)} \le C(1 + t^{\frac12}\mathcal{P}(\mathfrak{C})), \\ &\|J\mathcal{A}(t)\|_{X^{12}} \le C(1 + t^{\frac12}\mathcal{P}(\mathfrak{C})), \quad \|\mathcal{A}(t)\|_{X^{12}} \le C(1 + t^{\frac12}\mathcal{P}(\mathfrak{C})), \end{aligned} \tag{2.10}$$

where the constant C *depends on* $\|\xi_0\|_{\mathcal{F}_\kappa}$ *and* σ_0.

Proof Before giving the proof of this lemma, we state some estimates as preliminary.

First, taking $f = g = \nabla v$ in (2.6), we obtain

$$\|\nabla v : \nabla v\|_{X^{12}} \le C\|\nabla v\|_{X^{12}} \sum_{|\ell|\le 6} \|Z^\ell \nabla v\|_{L^\infty_{x_3}(L^2_h)}. \tag{2.11}$$

While by Lemma 2.2, one can prove that

$$\begin{aligned} \sum_{0\le|\ell|\le 6} \|Z^\ell(\nabla v : \nabla v)\|_{L^\infty_{x_3}(L^2_h)} &\le C \sum_{0\le|\ell|\le 6} \|Z^\ell \nabla v\|_{L^\infty_{x_3}(L^2_h)} \sum_{0\le|\ell|\le 4} \|Z^\ell \nabla v\|_{L^\infty} \\ &\le C(\|\nabla v\|_{X^{12}} + \|\overline{\rho}^{-\frac12+\kappa}\triangle v\|_{L^2})^2 \le C\mathfrak{D}(v)^2, \end{aligned} \tag{2.12}$$

which along with (2.11) ensures (2.9).

Now we are in the position to prove the estimates in terms of $J\mathcal{A}$ and $\mathcal{A}$. Notice that

$$J\mathcal{A} = (D\eta)^{-1} = \left(\nabla\eta_0 + \int_0^t \nabla v\, ds\right)^{-1},$$

and every entry in $J\mathcal{A}$ is a linear combination of

$$\nabla\eta_0,\ \nabla\eta_0 \int_0^t \nabla v ds,\ \left(\int_0^t \nabla v ds\right)^2.$$

Then, thanks to Lemmas 2.2–2.3, (2.12) and Minkowski's inequality, one has

$$\begin{aligned} &\sum_{0\le|\ell|\le 6} \|Z^\ell (J\mathcal{A})\|_{L^\infty_{x_3}(L^2_h)} \\ &\le \sum_{0\le|\ell|\le 6} \|Z^\ell \nabla\eta_0\|_{L^\infty_{x_3}(L^2_h)} + \sum_{0\le|\ell|\le 6} \|Z^\ell \left(\nabla\eta_0 \int_0^t \nabla v ds\right)\|_{L^\infty_{x_3}(L^2_h)} \\ &\quad + \sum_{0\le|\ell|\le 6} \|Z^\ell \left(\left(\int_0^t \nabla v ds\right)^2\right)\|_{L^\infty_{x_3}(L^2_h)} \\ &\le C\,\|\xi_0\|_{\mathcal{F}_\kappa} + C\,\|\xi_0\|_{\mathcal{F}_\kappa}\, t^{\frac12}\|\mathfrak{D}(v)\|_{L^2_t} + Ct\|\mathfrak{D}(v)\|^2_{L^2_t} \le C\,(1 + t^{\frac12}\,\mathcal{P}(\mathfrak{C})), \end{aligned} \tag{2.13}$$

which proves the first inequality in (2.10).

Similarly, we deduce that

$$\sum_{0\le|\ell|\le 6}\|Z^\ell(\nabla\eta_0\int_0^t\nabla v ds)\|_{L^\infty_{x_3}(L^2_h)}+\sum_{0\le|\ell|\le 6}\|Z^\ell\left(\left(\int_0^t\nabla v\, ds\right)^2\right)\|_{L^\infty_{x_3}(L^2_h)}$$
$$\le C\,t^{\frac12}\|\mathfrak{D}(v)\|_{L^2_t}+C\,t\|\mathfrak{D}(v)\|^2_{L^2_t}\le C\,t^{\frac12}\mathcal{P}(\mathfrak{C}).\tag{2.14}$$

Recalling the definition of J: $J=\det(\nabla\eta_0+\int_0^t\nabla v ds)$, J is a linear combination of the terms

$$(\nabla\eta_0)^3,\ \nabla\eta_0\left(\int_0^t\nabla v ds\right)^2,\ (\nabla\eta_0)^2\int_0^t\nabla v ds,\ \left(\int_0^t\nabla v ds\right)^3.$$

Hence, similar to the proof of the first inequality in (2.10) in terms of $J\mathcal{A}$, we may obtain

$$\sum_{0\le|\ell|\le 6}\|Z^\ell J\|_{L^\infty_{x_3}(L^2_h)}\le C\,(1+t^{\frac12}\mathcal{P}(\mathfrak{C})).\tag{2.15}$$

Owing to the fact $J\ge\sigma_0$ and the formula to the composition of two functions, we obtain

$$\sum_{0\le|\ell|\le 6}\|Z^\ell(J^{-1})\|_{L^\infty_{x_3}(L^2_h)}\le C\sum_{0\le|\ell|\le 6}\|\prod_{\sum_j|k_j|m_j\le|\ell|}(Z^{k_j}J)^{m_j}\|_{L^\infty_{x_3}(L^2_h)}.$$

We put $\|\cdot\|_{L^\infty_{x_3}(L^2_h)}$ on the highest order term $Z^{k_j}J$ and put $\|\cdot\|_{L^\infty}$ to other lower terms (not more than order 4) with similar process to (2.12). It follows from Lemma 2.2 and (2.15) that

$$\sum_{0\le|\ell|\le 6}\delta^{|\ell|}\|Z^\ell(J^{-1})\|_{L^\infty_{x_3}(L^2_h)}\le C\,(1+t^{\frac12}\mathcal{P}(\mathfrak{C})).\tag{2.16}$$

Therefore, due to (2.13) and (2.16), we find

$$\begin{aligned}\sum_{|\ell|\le 6}\|Z^\ell\mathcal{A}\|_{L^\infty_{x_3}(L^2_h)}\le& C+C\sum_{|\ell|\le 6}\|Z^\ell(J\mathcal{A})\|_{L^\infty_{x_3}(L^2_h)}\sum_{|\ell|\le 4}\|Z^\ell(J^{-1})\|_{L^\infty}\\&+C\sum_{|\ell|\le 6}\|Z^\ell(J^{-1})\|_{L^\infty_{x_3}(L^2_h)}\sum_{|\ell|\le 4}\|Z^\ell(J\mathcal{A})\|_{L^\infty}\\\le& C\,(1+t^{\frac12}\mathcal{P}(\mathfrak{C})).\end{aligned}$$

For the high order estimate, similar to the proof of (2.9), by using Lemma 2.2, we achieve

$$\|\nabla\eta_0\left(\int_0^t\nabla v ds\right)^2\|_{X^{12}}+\|(\nabla\eta_0)^2\int_0^t\nabla v ds\|_{X^{12}}+\|\left(\int_0^t\nabla v ds\right)^3\|_{X^{12}}\le C\,t^{\frac12}\mathcal{P}(\mathfrak{C}),\tag{2.17}$$

and then

$$\begin{aligned}\|(J,\ J\mathcal{A})\|_{X^{12}}\le& C\Big(\|\nabla\eta_0\|_{X^{12}}+\|\nabla\eta_0\Big(\int_0^t\nabla v ds\Big)^2\|_{X^{12}}\\&+\|(\nabla\eta_0)^2\int_0^t\nabla v ds\|_{X^{12}}+\|\Big(\int_0^t\nabla v ds\Big)^3\|_{X^{12}}\Big)\\\le& C+C\,t^{\frac12}\|\mathfrak{D}(v)\|_{L^2_t}+C\,t\|\mathfrak{D}(v)\|^2_{L^2_t}+C\,t^{\frac32}\|\mathfrak{D}(v)\|^3_{L^2_t}\\\le& C\,(1+t^{\frac12}\mathcal{P}(\mathfrak{C})).\end{aligned}\tag{2.18}$$

While by virtue of (2.15), (2.18) and Lemma 2.3, we deduce that

$$\|J^{-1}\|_{X^{12}} \le C\,(1+t^{\frac{1}{2}}\mathcal{P}(\mathfrak{E})),$$

and

$$\|\mathcal{A}\|_{X^{12}} = \|J\mathcal{A}\,J^{-1}\|_{X^{12}} \le C\,(1+t^{\frac{1}{2}}\mathcal{P}(\mathfrak{E})),$$

which completes the proof of Lemma 2.4. □

Based on the above lemma, we may get the following estimates.

Lemma 2.5 *Under the assumptions in Lemma 2.4, there hold*

$$\begin{aligned}
\sum_{0\le|j|\le1} \|Z^j(J\mathcal{A})\,\nabla v\|_{X^{11}} &\le C\,\|\nabla v\|_{X^{11}} + t^{\frac{1}{2}}\mathcal{P}(\mathfrak{E})\mathfrak{D}(v),\\
\sum_{0\le|j|\le1} \|Z^j(\mathcal{A})\,\nabla v\|_{X^{11}} &\le C\,\|\nabla v\|_{X^{11}} + t^{\frac{1}{2}}\mathcal{P}(\mathfrak{E})\mathfrak{D}(v),\\
\|\nabla_{\mathcal{A}} v\|_{X^{12}} &\le C\,\|\nabla v\|_{X^{12}} + t^{\frac{1}{2}}\mathcal{P}(\mathfrak{E})\mathfrak{D}(v),\\
\|\mathrm{S}_{J\mathcal{A}}(v)\|_{X^{12}} &\le C\,\|\nabla v\|_{X^{12}} + t^{\frac{1}{2}}\mathcal{P}(\mathfrak{E})\mathfrak{D}(v).
\end{aligned} \tag{2.19}$$

Proof We mainly utilize Lemmas 2.3, 2.4 to prove (2.19). So one may focus only on the proof of the first inequality in (2.19), and the proofs of the others are the same as it, whose details will be omitted here.

First, by the definition of $J\mathcal{A}$, we split $\sum_{0\le|j|\le1}\|\nabla v\, Z^j(J\mathcal{A})\|_{X^{11}}$ into three parts:

$$\begin{aligned}
&\sum_{0\le|j|\le1} \|Z^j(J\mathcal{A})\,\nabla v\|_{X^{11}}\\
&\le C\sum_{0\le|j|\le1}\|\nabla v\,Z^j\nabla\eta_0\|_{X^{11}} + C\sum_{0\le|j|\le1}\|\nabla v\,Z^j\left(\nabla\eta_0\int_0^t\nabla v ds\right)\|_{X^{11}}\\
&\quad + C\sum_{0\le|j|\le1}\|\nabla v\,Z^j\left(\int_0^t\nabla v ds\right)^2\|_{X^{11}} \triangleq \sum_{i=1}^{3} I_i.
\end{aligned} \tag{2.20}$$

For I_1, we have

$$I_1 \le C\,\|\nabla v\|_{X^{11}}. \tag{2.21}$$

For I_2, taking $g=\nabla v$ and $f=J\mathcal{A}$ in (2.7) in Lemma 2.3 to obtain that

$$\begin{aligned}
I_2 &\le C\|\nabla v\|_{X^{11}}\sum_{|\ell|\le6}\|Z^j\nabla\eta_0\int_0^t\nabla v ds\|_{L^\infty_{x_3}(L^2_h)}\\
&\quad + C\sum_{|\ell|\le6}\|Z^\ell\nabla v\|_{L^\infty_{x_3}(L^2_h)}\|Z^j\nabla\eta_0\int_0^t\nabla v ds\|_{X^{12}}.
\end{aligned}$$

Applying Lemma 2.2 and (2.14), (2.17) in Lemma 2.4 to get

$$I_2 \le t^{\frac{1}{2}}\mathcal{P}(\mathfrak{E})\|\nabla v\|_{X^{11}} + Ct^{\frac{1}{2}}\mathcal{P}(\mathfrak{E})\mathfrak{D}(v) \le\ t^{\frac{1}{2}}\mathcal{P}(\mathfrak{E})\mathfrak{D}(v). \tag{2.22}$$

Similarly, we have

$$I_3 \le t^{\frac{1}{2}}\mathcal{P}(\mathfrak{E})\mathfrak{D}(v). \tag{2.23}$$

Plugging the estimates (2.21)–(2.23) into (2.20), we prove

$$\sum_{0\le|j|\le 1} \|Z^j(J\mathcal{A})\,\nabla v\|_{X^{11}} \le C\,\|\nabla v\|_{X^{11}} + t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C})\mathfrak{D}(v),$$

which ends our proof. □

Next we recall a version of Korn's inequality involving only the deviatoric part $\mathbb{D}^0$.

Lemma 2.6 (Korn's lemma, Theorem 1.1 in [5]) *Let $n \ge 3$ and U be a Lipschitz domain in $\mathbb{R}^n$, then there exists a constant C, independent of f, such that*

$$\|f\|_{H^1(U)} \le C\,(\|\mathbb{D}^0(f)\|_{L^2(U)} + \|f\|_{L^2(U)})$$

for all $f \in H^1(U)$.

3 A priori estimates

In this section, we give a priori estimates of the system (1.18). The main result of the section is as follows:

Proposition 3.1 *Assume (ξ, v) is a smooth solution of system (1.18) on $[0, \bar{T}]$ with initial data $(\xi_0, v_0) \in \mathcal{F}_\kappa \times (X^{12}_{\frac{1}{2}} \cap H^1)$ and $0 < 2\sigma_0 \le J_0 \le 3\sigma_0$, and $\overline{\rho}$ satisfies (1.11)–(1.13). Then, there exists a positive constant $T \le \bar{T}$ which depends on the initial data such that*

$$\sup_{t\in[0,T]} \mathcal{E}(t) + \int_0^T \mathcal{D}(s)ds \le 2\mathcal{E}(0).$$

Here, we use the bootstrap argument to prove this proposition. Now, we define a T such that there holds that

$$\|\mathfrak{D}(v)\|_{L^2(0,T)} \le \mathfrak{C}, \quad \sigma_0 \le \sup_{t\in[0,T]} J \le 4\sigma_0. \tag{3.1}$$

Before, we give the proof of the proposition, we prove some useful lemmas.

Lemma 3.2 *Under the assumption of Proposition 3.1, we have*

$$\|\nabla v\|_{L^1(0,t;L^\infty)} \le t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C}), \quad \|(J, A)(t)\|_{L^\infty} \le C\,(1 + t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C})), \quad \forall\ t \in [0, T).$$

Proof It is a direct result from Lemma 2.2 and Lemma 2.4. □

Lemma 3.3 *Under the assumption of Proposition 3.1, the following holds*

$$\|\nabla v\|_{X^N} \le C\,(\|\mathbb{D}^0(v)\|_{X^N} + \|v\|_{X^N_{\frac{1}{2}}}). \tag{3.2}$$

Proof Thanks to Korn's lemma (Lemma 2.6), we have

$$\|v\|_{H^1} \le C_0(\|\mathbb{D}^0(v)\|_{L^2} + \|v\|_{L^2}).$$

For any function $f(s)$, by Lemma 2.1, we have

$$\int_0^1 f^2 ds \le C_0 \int_0^1 s^2(f^2 + f'^2)ds$$

By scaling, we have

$$\int_{1-\varepsilon}^{1} f^2 ds \le \frac{C_0}{\varepsilon^2}\int_{1-\varepsilon}^{1}(1-s)^2 f^2 ds + C_0\int_{1-\varepsilon}^{1}(1-s)^2 f'^2 ds.$$

Then (1.12) gives that

$$\|f\|_{L^2}^2 \le C\,\|\overline{\rho}^{-1}\|_{L^\infty(0\le x_3\le 1-\varepsilon)}\int_\Omega \overline{\rho} f^2 dx + C\,\varepsilon^2\|f\|_{H^1}^2 \le \frac{C}{\varepsilon}\int_\Omega \overline{\rho} f^2 dx + C\,\varepsilon^2\|f\|_{H^1}^2. \tag{3.3}$$

Taking ε small enough and $f := v$, we combine with Lemma 2.6 to get that

$$\|v\|_{H^1} \le C\,(\|\mathbb{D}^0(v)\|_{L^2} + \|\overline{\rho}^{\frac{1}{2}} v\|_{L^2}).$$

For given $m \in \mathbb{N}^3$: $1 \le |m| \le N$,

$$\begin{aligned}\|Z^m v\|_{H^1} \le& C\,(\|\mathbb{D}^0(Z^m v)\|_{L^2} + \|\overline{\rho}^{\frac{1}{2}} Z^m v\|_{L^2})\\ \le& C\,(\|Z^m\mathbb{D}^0 v\|_{L^2} + \|[\mathbb{D}^0, Z^m]v\|_{L^2} + \|\overline{\rho}^{\frac{1}{2}} Z^m v\|_{L^2}),\end{aligned}$$

which follows from the fact $[\mathbb{D}^0, Z^m]v \sim Z^{m-1}\nabla v$ that

$$\|Z^m v\|_{H^1} \le C\,(\|Z^m\mathbb{D}^0 v\|_{L^2} + \|Z^{m-1}\nabla v\|_{L^2} + \|\overline{\rho}^{\frac{1}{2}} Z^m v\|_{L^2}).$$

Therefore, by a standard inductive argument in terms of $m = 0, 1, \ldots, N$ and the definition of space X^N, we prove (3.2). □

Lemma 3.4 *Let the initial flow map $\eta_0 = Id + \xi_0 : \Omega \to \Omega(0)$ satisfy its Jacobian $2\sigma_0 \le J_0 \le 3\sigma_0$ and $\xi_0 \in \mathcal{F}_\kappa$, and its inverse map $\eta_0^{-1} : \Omega(0) \to \Omega$, $v(x) = \widetilde{u}(\eta_0(x))$ with $x \in \Omega$ and $\widetilde{u}(y) = v(\eta_0^{-1}(y))$ with $y \in \Omega(0)$, then there is a positive constant $C_1 \ge 1$ such that*

$$C_1^{-1}(1+\|\xi_0\|_{\mathcal{F}_\kappa}^2)^{-1}\int_\Omega |\nabla v|^2\,dx \le \int_{\Omega(0)} |\nabla_y \widetilde{u}(y)|^2\,dy \le C_1(1+\|\xi_0\|_{\mathcal{F}_\kappa}^2)\int_\Omega |\nabla v|^2\,dx. \tag{3.4}$$

Proof First, taking changes of variables $y = \eta_0(x)$, we have

$$\int_{\Omega(0)} |\nabla_y \widetilde{u}(y)|^2\,dy = \int_\Omega |\nabla_y v(x)|^2\,d(\eta_0(x)) = \int_\Omega |(D_y(\eta_0^{-1}))(\eta_0(x))\nabla_x v(x)|^2\,J_0 dx,$$

which along with the assumptions $2\sigma_0 \le J_0 \le 3\sigma_0$, $\xi_0 \in \mathcal{F}_\kappa$, and (2.2) implies

$$\begin{aligned}\int_{\Omega(0)} |\nabla_y \widetilde{u}(y)|^2\,dy &\le C\|(D_y(\eta_0^{-1}))(\eta_0(x))\|_{L^\infty}^2\int_\Omega |\nabla_x v(x)|^2\,dx\\ &\le C_1(1+\|\xi_0\|_{\mathcal{F}_\kappa}^2)\int_\Omega |\nabla v|^2\,dx.\end{aligned}$$

Similarly, one may readily check

$$\begin{aligned}\int_\Omega |\nabla_x v(x)|^2\,dx &= \int_{\Omega(0)} |(D_x\eta_0)(\eta_0^{-1}(y))\nabla_y \widetilde{u}(y)|^2\,J_0^{-1} dy\\ &\le C_1(1+\|\xi_0\|_{\mathcal{F}_\kappa}^2)\int_{\Omega(0)} |\nabla_y \widetilde{u}(y)|^2\,dy.\end{aligned}$$

Therefore, we get (3.4), and complete the proof of Lemma 3.4. □

Lemma 3.5 *Under the assumption of Proposition 3.1, if (3.1) holds, then we have*

$$(c_0 - t^2\mathcal{P}(\mathfrak{C}))\|\nabla v\|_{L^2}^2 - C_0\|\overline{\rho}^{\frac{1}{2}} v\|_{L^2}^2 \le \|\mathbb{D}_{\mathcal{A}}^0 v\|_{L^2}^2 \le C_0(1 + t^2\mathcal{P}(\mathfrak{C}))\|\nabla v\|_{L^2}^2.$$

Moreover, if T is small enough such that $T^{\frac{1}{2}}\mathcal{P}(\mathfrak{C}) < \frac{c_0}{2}$, then we have

$$\int_\Omega J\mathbb{S}_{\mathcal{A}} v : \nabla_{\mathcal{A}} v\, dx \ge c_1\|\mathbb{D}_{\mathcal{A}}^0 v\|_{L^2}^2 \ge \frac{c_0 c_1}{2}\|\nabla v\|_{L^2}^2 - C_0\|\overline{\rho}^{\frac{1}{2}} v\|_{L^2}^2.$$

Proof We first to prove the first result. According to the fact

$$J\mathcal{A} - J_0\mathcal{A}_0 \sim \left(\int_0^t \nabla v ds\right)^2, \tag{3.5}$$

and $\mathcal{A}_0^{-1} = D\eta_0$, combining Lemmas 2.2, 3.2 with (3.1) , we have

$$\|J\mathcal{A} - J_0\mathcal{A}_0\|_{L^\infty} \le C\,\|\nabla v\|_{L_t^1 L^\infty}^2 \le C\, t\mathcal{P}(\mathfrak{C}), \quad \|(\mathcal{A}_0^{-1}, \mathcal{A}_0)\|_{L^\infty} \le C\,(1 + t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C})), \tag{3.6}$$

which imply that

$$\|\mathbb{D}_{J\mathcal{A}-J_0\mathcal{A}_0}^0(v)\|_{L^2}^2 \le C\,\|J\mathcal{A} - J_0\mathcal{A}_0\|_{L^\infty}^2\|\nabla v\|_{L^2}^2 \le t^2\mathcal{P}(\mathfrak{C})\|\nabla v\|_{L^2}^2, \tag{3.7}$$

$$\|\mathbb{D}_{J_0\mathcal{A}_0}^0(v)\|_{L^2}^2 \le C\,\|\nabla v\|_{L^2}^2. \tag{3.8}$$

On the other hand, we use (3.1), the coordinate transformation from Ω to $\Omega(0)$ and Lemmas 2.6, 3.4 to get that

$$\begin{aligned}\int_\Omega |\mathbb{D}_{\mathcal{A}_0}^0(v)|^2\, J_0\, dx &= \int_{\Omega(0)} |\mathbb{D}^0(\widetilde{u})|^2\, dx \ge c_1\|\nabla\,\widetilde{u}\|_{L^2(\Omega(0))}^2 - C_1\|\widetilde{u}\|_{L^2(\Omega(0))}^2\\ &\ge c_0\|v\|_{H^1}^2 - C_0\|v\|_{L^2}^2,\end{aligned}$$

where $\widetilde{u} = v \circ \eta_0^{-1}$. Hence, according to (3.1) and (3.3), we obtain that

$$\|\mathbb{D}_{J_0\mathcal{A}_0}^0(v)\|_{L^2}^2 \ge c_0\|v\|_{H^1}^2 - C_0\|\overline{\rho}^{\frac{1}{2}} v\|_{L^2}^2,$$

which combining with (3.7) gives rise to

$$(c_0 - t^2\mathcal{P}(\mathfrak{C}))\|\nabla v\|_{L^2}^2 - C_0\|\overline{\rho}^{\frac{1}{2}} v\|_{L^2}^2 \le \|\mathbb{D}_{\mathcal{A}}^0 v\|_{L^2}^2 \le (C_0 + t^2\mathcal{P}(\mathfrak{C}))\|\nabla v\|_{L^2}^2,$$

which we complete the first result. For the second one, we deduce

$$\begin{aligned}\int_\Omega J\mathbb{S}_{\mathcal{A}} v : \nabla_{\mathcal{A}} v dx &= \frac{1}{2}\int_\Omega (\frac{\mu}{2}|\mathbb{D}_{\mathcal{A}}^0 v|^2 + (\lambda + \frac{2}{3}\mu)|\nabla_{\mathcal{A}} \cdot v|^2)\, J dx\\ &\ge c_1\|\mathbb{D}_{\mathcal{A}}^0 v\|_{L^2}^2 \ge (c_0 c_1 - t^2\mathcal{P}(\mathfrak{C}))\|\nabla v\|_{L^2}^2 - C_0\|\overline{\rho}^{\frac{1}{2}} v\|_{L^2}^2,\end{aligned}$$

here we used (3.1) in the last step and assumption $\mu > 0,\ \lambda + \frac{2}{3}\mu \ge 0$. Combining with the first result, we finish this proof. □

Zeroth-order estimate of *v*

Now, we are in a position to give a priori estimates. First, multiplying by v on the first equation of (1.18) and integrating over Ω, from the Piola identity (1.21) and boundary conditions, we get the basic energy estimate:

Proposition 3.6 *Assume v is a smooth solution of system* (1.18) *on* $[0, T]$. *Then, we have*

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega \overline{\rho}|v|^2 dx+2\int_\Omega \overline{\rho}^2 J^{-1} dx\right)+\frac{1}{2}\int_\Omega\left(\frac{\mu}{2}|\mathbb{D}^0_{\mathcal{A}}v|^2+\left(\lambda+\frac{2}{3}\mu\right)|\nabla_{\mathcal{A}}\cdot v|^2\right)Jdx=v0.$$

First-order estimate of *v*

Here, to get the higher regularity of the v. We multiply $\partial_t v$ on the both sides of (1.18) to get that

Proposition 3.7 *Assume that* (3.1) *holds and v is a smooth solution of system* (1.18) *on* $[0, T]$, *then there holds that for* $t \in [0, T]$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega\left(\frac{\mu}{2}|\mathbb{D}^0_{\mathcal{A}}v|^2+\left(\lambda+\frac{2}{3}\mu\right)|\nabla_{\mathcal{A}}\cdot v|^2\right)J\,dx+\|\overline{\rho}^{\frac{1}{2}}\partial_t v\|^2_{L^2}$$
$$\leq (C+t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C}))(\mathfrak{D}(v)\|\nabla v\|^2_{L^2}+1).$$

Proof Taking L^2 product with $\partial_t v$ to the first equation of (1.18) to get that

$$\|\overline{\rho}^{\frac{1}{2}}\partial_t v\|^2_{L^2}+\int_\Omega \nabla_{J\mathcal{A}}(\overline{\rho}^2 J^{-2})\cdot\partial_t v dx-\int_\Omega \nabla_{J\mathcal{A}}\cdot\mathbb{S}_{\mathcal{A}}(v)\cdot\partial_t v dx=0.$$

Due to the Piola identity (1.21) and the boundary condition $\mathrm{S}_{\mathcal{A}}(v)\cdot\mathcal{N}|_{x_3=1}=0$ and $v|_{x_3=0}=0$, integration by parts yields

$$-\int_\Omega \nabla_{J\mathcal{A}}\cdot\mathbb{S}_{\mathcal{A}}(v)\cdot\partial_t v dx=\int_\Omega \mathbb{S}_{J\mathcal{A}}(v):\partial_t(\nabla_{\mathcal{A}}v)dx-\int_\Omega \mathbb{S}_{J\mathcal{A}}(v):\nabla_{\partial_t\mathcal{A}}v dx.$$

Since $\mathbb{D}_{\mathcal{A}}(v)$ and $(\nabla_{\mathcal{A}}\cdot v)\mathbb{I}$ are symmetric, it implies that

$$\begin{aligned}\int_\Omega \mathbb{S}_{J\mathcal{A}}(v):\partial_t(\nabla_{\mathcal{A}}v)dx&=\int_\Omega(\mu\mathbb{D}_{J\mathcal{A}}(v)+\lambda(\nabla_{J\mathcal{A}}\cdot v)\mathbb{I}):\partial_t(\nabla_{\mathcal{A}}v)dx\\&=\frac{\mu}{2}\int_\Omega \mathbb{D}_{J\mathcal{A}}(v):\partial_t\mathbb{D}_{\mathcal{A}}(v)dx+\lambda\int_\Omega \nabla_{J\mathcal{A}}\cdot v\,\partial_t(\nabla_{\mathcal{A}}\cdot v)\\&=\frac{1}{2}\frac{d}{dt}\int_\Omega J\Big(\frac{\mu}{2}|\mathbb{D}^0_{\mathcal{A}}v|^2+\Big(\lambda+\frac{2}{3}\mu\Big)|\nabla_{\mathcal{A}}\cdot v|^2\Big)dx\\&\quad-\frac{1}{2}\int_\Omega \partial_t J\Big(\frac{\mu}{2}|\mathbb{D}_{\mathcal{A}}(v)|^2+\lambda|\nabla_{\mathcal{A}}\cdot v|^2\Big)dx\\&=\frac{1}{2}\frac{d}{dt}\int_\Omega \mathbb{S}_{J\mathcal{A}}(v):\nabla_{\mathcal{A}}v\,dx-\frac{1}{2}\int_\Omega \mathbb{S}_{\mathcal{A}}(v):\nabla_{\mathcal{A}}v\partial_t J\,dx,\end{aligned}$$

which gives that

$$\begin{aligned}-\int_\Omega \nabla_{J\mathcal{A}}\cdot\mathbb{S}_{\mathcal{A}}(v)\cdot\partial_t v dx&=\frac{1}{2}\frac{d}{dt}\int_\Omega J\left(\frac{\mu}{2}|\mathbb{D}^0_{\mathcal{A}}v|^2+\left(\lambda+\frac{2}{3}\mu\right)|\nabla_{\mathcal{A}}\cdot v|^2\right)dx\\&\quad-\frac{1}{2}\int_\Omega \mathbb{S}_{\mathcal{A}}(v):\nabla_{\mathcal{A}}v\partial_t J\,dx-\int_\Omega \mathbb{S}_{J\mathcal{A}}(v):\nabla_{\partial_t\mathcal{A}}v\,dx.\end{aligned}$$

To estimate the last two terms of right hand of the above equation, we recall that formula (1.19)–(1.20), Lemmas 2.2 and 3.2 to get that

$$\|\partial_t J,\ \partial_t \mathcal{A}\|_{L^\infty} \le C\,\|\mathcal{A}\|_{L^\infty}^2 \|\nabla v\|_{L^\infty} \le (C + t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C}))\mathfrak{D}(v),$$

which implies that

$$\begin{aligned}&\left|\int_\Omega \mathbb{S}_{\mathcal{A}}(v) : \nabla_{\mathcal{A}} v \partial_t J\, dx\right| + \left|\int_\Omega \mathbb{S}_{J\mathcal{A}}(v) : \nabla_{\partial_t \mathcal{A}} v\, dx\right|\\ &\le (C + t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C}))\mathfrak{D}(v)\|J\mathcal{A}\|_{L^\infty}\|\nabla v\|_{L^2}^2 \le (C + t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C}))\mathfrak{D}(v)\|\nabla v\|_{L^2}^2.\end{aligned}$$

For the pressure term, we notice it contains $\overline{\rho}^2$. Thus, we have

$$\overline{\rho}^{-\frac{1}{2}}\nabla_{J\mathcal{A}}(\overline{\rho}^2 J^{-2}) = \overline{\rho}^{-\frac{1}{2}}\partial_k(J^{-1}\mathcal{A}_i^k\overline{\rho}^2) = \overline{\rho}^{-\frac{1}{2}}\Big(J^{-1}\mathcal{A}_i^k\partial_k\overline{\rho}^2 + \partial_k(J^{-2})J\mathcal{A}_i^k\overline{\rho}^2\Big),$$

which implies that for all $t \in [0, T]$, we have

$$\begin{aligned}\|\overline{\rho}^{-\frac{1}{2}}\nabla_{J\mathcal{A}}(\overline{\rho}^2 J^{-2})\|_{L^2} &\le \|\overline{\rho}^{\frac{1}{2}}\overline{\rho}'\|_{L^\infty}(C + t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C}))\|\mathcal{A}\|_{L^2}\\ &\quad + (C + t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C}))\|ZJ\|_{L^\infty}\|J\mathcal{A}\|_{L^2}\\ &\le C + t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C}),\end{aligned}$$

where we used Lemma 2.4. Thus, by Hölder's inequality, we get

$$\begin{aligned}\left|\int_\Omega \nabla_{J\mathcal{A}}(\overline{\rho}^2 J^{-2})\ \partial_t v dx\right| &\le \|\overline{\rho}^{\frac{1}{2}}\partial_t v\|_{L^2}\|\overline{\rho}^{-\frac{1}{2}}\nabla_{J\mathcal{A}}(\overline{\rho}^2 J^{-2})\|_{L^2}\\ &\le (C + t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C}))\|\overline{\rho}^{\frac{1}{2}}\partial_t v\|_{L^2} \le C + t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C}) + \frac{1}{2}\|\overline{\rho}^{\frac{1}{2}}\partial_t v\|_{L^2}^2.\end{aligned}$$

This ends the proof of Proposition 3.7. □

High-order estimates of *v*

In this subsection, we use the conormal derivative to get the regularity of the horizontal direction. For this, we recall the conormal Sobolev space with a parameter δ introduced by Masmoudi and Rousset [34].

$$\|f\|_{X_{\alpha,\delta}^N}^2 := \sum_{|m|=0}^{N} \delta^{2|m|}\|\overline{\rho}^\alpha Z^m f\|_{L^2}^2, \quad \|f\|_{\dot{X}_{\alpha,\delta}^N}^2 := \sum_{|m|=1}^{N} \delta^{2|m|}\|\overline{\rho}^\alpha Z^m f\|_{L^2}^2,$$

where δ is a small positive constant which will be determined later on and $\alpha \in \mathbb{R}$. In particular, when $\delta = 1$, the spaces $X_{\alpha,\delta}^N$ and $\dot{X}_{\alpha,\delta}^N$ will be denoted by X_α^N and $\dot{X}_\alpha^N$ respectively for simplicity.

For $T > 0$, $\delta > 0$, and $t \in [0, T]$, we define the modified instantaneous energy $\mathcal{E}_\delta(t)$ (in terms to the velocity v)

$$\mathcal{E}_\delta(t) \stackrel{\text{def}}{=} \|v\|_{X_{\frac{1}{2},\delta}^{12}}^2 + \|v\|_{H^1}^2,$$

and the modified dissipation $\mathcal{D}_\delta(t)$

$$\mathcal{D}_\delta(t) \stackrel{\text{def}}{=} \|\nabla v\|_{X_{0,\delta}^{12}}^2 + \|\overline{\rho}^{\frac{1}{2}}\partial_t v\|_{L^2}^2.$$

In particular, if $\delta = 1$, then $\mathcal{E}_\delta(t)$ and $\mathcal{D}_\delta(t)$ become the usual instantaneous energy $\mathcal{E}(t)$ and the dissipation $\mathcal{D}(t)$ respectively.

Let's now state our main results of this subsection:

Proposition 3.8 *Assume that* (3.1) *holds and* v *is a smooth solution of system* (1.18) *on* $[0, T]$, *then it holds that*

$$\frac{d}{dt}\|v\|^2_{X^{12}_{\frac{1}{2},\delta}} + \Big(c_0 - \delta(C_0 + t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C}))\Big)\|\nabla v\|^2_{X^{12}_{0,\delta}}$$
$$\leq t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C})\mathfrak{D}^2(v) + C_0\|v\|^2_{X^{12}_{\frac{1}{2},\delta}} + C_0 + t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C}),$$

where the positive constants c_0 *and* C_0 *are independent of* δ, *and* $\mathcal{P}(\mathfrak{C})$ *may depend on* δ.

Proof Acting Z^m on the first equation of (1.18) and taking L^2 inner product with $\delta^{2|m|}Z^m v$, then summing $\sum_{|m|=0}^{12}$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{X^{12}_{\frac{1}{2},\delta}} - \sum_{|m|=0}^{12}\delta^{2|m|}\int_\Omega Z^m\big(\nabla_{J\mathcal{A}}\cdot\mathbb{S}_\mathcal{A}v\big)\cdot Z^m v\,dx = I_1 + I_2$$

with

$$I_1 = \sum_{|m|=1}^{12}\delta^{2|m|}\int_\Omega[\overline{\rho}, Z^m]\partial_t v\cdot Z^m v\,dx,$$
$$I_2 = -\sum_{|m|=0}^{12}\delta^{2|m|}\int_\Omega Z^m\big(\nabla_{J\mathcal{A}}(J^{-2}\overline{\rho}^2)\big)\cdot Z^m v\,dx.$$

Estimate of dissipation term. For the dissipation term, by using integration by parts, we split it into three parts:

$$\begin{aligned}
&-\sum_{|m|=0}^{12}\delta^{2|m|}\int_\Omega Z^m\big(\nabla_{J\mathcal{A}}\cdot\mathbb{S}_\mathcal{A}v\big)\cdot Z^m v dx\\
&=\sum_{|m|=0}^{12}\delta^{2|m|}\int_\Omega J\mathbb{S}_\mathcal{A}(Z^m v):\nabla_\mathcal{A}Z^m v\,dx + \sum_{|m|=1}^{12}\delta^{2|m|}\int_\Omega[Z^m,\mathbb{S}_\mathcal{A}]v:\nabla_{J\mathcal{A}}(Z^m v)dx\\
&\quad-\sum_{|m|=1}^{12}\delta^{2|m|}\Big(\int_{x_3=1}\mathcal{N}\cdot Z_h^m\mathbb{S}_\mathcal{A}v\cdot Z_h^m v dS + \int_\Omega[Z^m,\nabla_{J\mathcal{A}}]\cdot\mathbb{S}_\mathcal{A}v\cdot Z^m v\,dx\Big)\\
&=: I_3 + I_4 + I_5.
\end{aligned}$$

Next, we deal with the commutators I_3, I_4 and I_5 step by step.

- *Estimates of* I_3. Thanks to Lemma 3.5, one can see that for any $m : |m| = 0, 1, \ldots, 12$

$$\begin{aligned}
&\int_\Omega J\mathbb{S}_\mathcal{A}(Z^m v):\nabla_\mathcal{A}Z^m v\,dx\\
&=\int_\Omega\left(\frac{\mu}{4}|\mathbb{D}^0_\mathcal{A}Z^m v|^2 + \frac{\lambda+\frac{2}{3}\mu}{2}|\nabla_\mathcal{A}\cdot Z^m v|^2\right)J\,dx\\
&\geq c_1\|\mathbb{D}^0_\mathcal{A}Z^m v\|^2_{L^2}\geq c_1\Big((c_0 - t^2\mathcal{P}(\mathfrak{C}))\|\nabla(Z^m v)\|^2_{L^2} - C_0\|\overline{\rho}^{\frac{1}{2}}Z^m v\|^2_{L^2}\Big),
\end{aligned}$$

which implies

$$\sum_{|m|=0}^{12} \delta^{2|m|} \int_\Omega J\mathbb{S}_{\mathcal{A}}(Z^m v) : \nabla_{\mathcal{A}} Z^m v \, dx \geq \sum_{|m|=0}^{12} \delta^{2|m|} c_1 \Big(\frac{1}{2}(c_0 - t^2\mathcal{P}(\mathfrak{C})) \|Z^m \nabla v\|_{L^2}^2 - \frac{1}{2}(c_0 + t^2\mathcal{P}(\mathfrak{C})) \|[\nabla, Z^m] v\|_{L^2}^2 - C_0 \|\overline{\rho}^{\frac{1}{2}} Z^m v\|_{L^2}^2 \Big), \tag{3.9}$$

For $|m| \geq 1$, by a direct calculation, we have

$$[\nabla, Z^m] = m \nabla \overline{\rho} Z^{m-1} \partial_3, \tag{3.10}$$

which implies that

$$\sum_{|m|=1}^{12} \delta^{2|m|} \frac{c_1}{2} (c_0 + t^2\mathcal{P}(\mathfrak{C})) \|[\nabla, Z^m] v\|_{L^2}^2 \leq (C_0 + t^2\mathcal{P}(\mathfrak{C})) \delta^2 \|\nabla v\|_{X_{0,\delta}^{11}}^2. \tag{3.11}$$

Plugging (3.11) into (3.9) shows

$$\begin{aligned} &\sum_{|m|=0}^{12} \delta^{2|m|} \int_\Omega J\mathbb{S}_{\mathcal{A}}(Z^m v) : \nabla_{\mathcal{A}} Z^m v \, dx \\ &\geq (c_2 - t^2\mathcal{P}(\mathfrak{C})) \|\nabla v\|_{X_{0,\delta}^{12}}^2 - (C_0 + t^2\mathcal{P}(\mathfrak{C})) \delta^2 \|\nabla v\|_{X_{0,\delta}^{11}}^2 - C_0 \|v\|_{X_{\frac{1}{2},\delta}^{12}}^2. \end{aligned}$$

- *Estimates of I_4*. For $|m| \geq 1$, by a direct calculation, we have

$$\begin{aligned} [Z^m, \mathbb{D}_{\mathcal{A}}] v &= Z^m \Big(\mathcal{A}_i^k \partial_k v_j + \mathcal{A}_j^k \partial_k v_i \Big) - \Big(\mathcal{A}_i^k \partial_k (Z^m v_j) + \mathcal{A}_j^k \partial_k (Z^m v_i) \Big) \\ &= \mathcal{A}_i^k [Z^m, \partial_k] v_j + \mathcal{A}_j^k [Z^m, \partial_k] v_i \\ &\quad + \sum_{\substack{|m_1|+|m_2|=|m|, \\ |m_1| \geq 1}} (Z^{m_1} \mathcal{A}_i^k Z^{m_2} \partial_k v_j + Z^{m_1} \mathcal{A}_j^k Z^{m_2} \partial_k v_i) \\ &= m \partial_k \overline{\rho} \mathcal{A}_i^3 Z^{m-1} \partial_3 v_j + m \partial_k \overline{\rho} \mathcal{A}_j^3 Z^{m-1} \partial_3 v_i \\ &\quad + \sum_{\substack{|m_1|+|m_2|=|m|, \\ |m_1| \geq 1}} (Z^{m_1} \mathcal{A}_i^k Z^{m_2} \partial_k v_j + Z^{m_1} \mathcal{A}_j^k Z^{m_2} \partial_k v_i). \end{aligned}$$

By Lemmas 2.5, 3.2, we have

$$\begin{aligned} \delta^{|m|} \|[Z^m, \mathbb{D}_{\mathcal{A}}] v\|_{L^2} &\leq \delta \Big((C_0 + t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C})) \|\nabla v\|_{X_{0,\delta}^{11}} + C_0 \|\nabla v\|_{X_{0,\delta}^{11}} + t^{\frac{1}{2}} \mathcal{P}(\mathfrak{C}) \mathfrak{D}(v) \Big) \\ &\leq \delta \Big(C_0 \|\nabla v\|_{X_{0,\delta}^{11}} + t^{\frac{1}{2}} \mathcal{P}(\mathfrak{C}) \mathfrak{D}(v) \Big). \end{aligned}$$

By the same argument, we have

$$\delta^{|m|} \|[Z^m, \mathrm{div}_{\mathcal{A}}] v\|_{L^2} \leq \delta \Big(C_0 \|\nabla v\|_{X_{0,\delta}^{11}} + t^{\frac{1}{2}} \mathcal{P}(\mathfrak{C}) \mathfrak{D}(v) \Big).$$

Combining the above two estimates, we have

$$\begin{aligned} I_4 \leq& \delta (C_0 + t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C})) (\delta \|\nabla v\|_{X_{0,\delta}^{11}} + \|\nabla v\|_{X_{0,\delta}^{12}}) (C_0 \|\nabla v\|_{X_{0,\delta}^{11}} + t^{\frac{1}{2}} \mathcal{P}(\mathfrak{C}) \mathfrak{D}(v)) \\ \leq& \delta (C_0 + t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C})) \|\nabla v\|_{X_{0,\delta}^{12}}^2 + t^{\frac{1}{2}} \mathcal{P}(\mathfrak{C}) \mathfrak{D}(v)^2. \end{aligned}$$

- *Estimates of I_5*. A direct calculation gives that

$$[Z^m, \nabla_{J\mathcal{A}}] \cdot \mathbb{S}_{\mathcal{A}} v = Z^m (J\mathcal{A}_i^k \partial_k (\mathbb{S}_{\mathcal{A}} v)^i) - \partial_k (J\mathcal{A}_i^k (Z^m (\mathbb{S}_{\mathcal{A}} v))^i)$$
$$= \partial_k \Big(Z^m (J\mathcal{A}_i^k (\mathbb{S}_{\mathcal{A}} v)^i) - J\mathcal{A}_i^k (Z^m (\mathbb{S}_{\mathcal{A}} v))^i \Big) + [Z^m, \partial_k](J\mathcal{A}_i^k (\mathbb{S}_{\mathcal{A}} v)^i). \tag{3.12}$$

For the commutator term, we see

$$[Z^m, \partial_3] = -m\partial_3 \overline{\rho} Z^{m-1} \partial_3 \sim Z^{m-1}\partial_3, \ [Z^m, \partial_h] = -m\partial_h \overline{\rho} Z^{m-1} \partial_3 \sim Z^m, \tag{3.13}$$

where we used (1.13). Then one has

$$\Big| \int_\Omega [Z^m, \partial_k](J\mathcal{A}_i^k (\mathbb{S}_{\mathcal{A}} v)^i) \cdot Z^m v dx \Big|$$
$$\leq C_0 \Big| \int_\Omega Z^{m-1} \partial_3 (J\mathcal{A}_i^k (\mathbb{S}_{\mathcal{A}} v)^i) \cdot Z_3 Z^{m-1} v dx \Big|$$
$$+ C_0 \Big| \int_\Omega Z^m (J\mathcal{A}_i^k (\mathbb{S}_{\mathcal{A}} v)^i) \cdot Z^m v dx \Big| \leq C_0 \Big| \int_\Omega Z^m (J\mathcal{A}_i^k (\mathbb{S}_{\mathcal{A}} v)^i) \cdot Z^{m-1} \nabla v dx \Big|,$$

which combining with Lemma 2.5 follows

$$\Big| \sum_{|m|=1}^{12} \delta^{2|m|} \int_\Omega [Z^m, \partial_k](J\mathcal{A}_i^k (\mathbb{S}_{\mathcal{A}} v)^i) \cdot Z^m v dx \Big|$$
$$\leq \delta (C_0 \|\nabla v\|_{X^{12}_{0,\delta}} + t^{\frac{1}{2}} \mathcal{P}(\mathfrak{C}) \mathfrak{D}(v)) \|\nabla v\|_{X^{11}_{0,\delta}}.$$

Now, we deal with the first term of the right hand of (3.12). By using integration by parts, one has

$$\sum_{|m|=1}^{12} \delta^{2|m|} \int_\Omega \partial_k \Big(Z^m (J\mathcal{A}_i^k (\mathbb{S}_{\mathcal{A}}(v))^i) - J\mathcal{A}_i^k (Z^m (\mathbb{S}_{\mathcal{A}}(v)))^i \Big) \cdot Z^m v dx$$
$$= - \sum_{|m|=1}^{12} \delta^{2|m|} \int_\Omega \Big(Z^m (J\mathcal{A}_i^k (\mathbb{S}_{\mathcal{A}}(v))^i) - J\mathcal{A}_i^k (Z^m (\mathbb{S}_{\mathcal{A}}(v)))^i \Big) \cdot \partial_k Z^m v dx$$
$$+ \sum_{|m|=1}^{12} \delta^{2|m|} \int_{x_3=1} \Big(Z_h^m (J\mathcal{A}_i^3 e_3 (\mathbb{S}_{\mathcal{A}}(v))^i) - J\mathcal{A}_i^3 e_3 (Z_h^m (\mathbb{S}_{\mathcal{A}}(v)))^i \Big) \cdot Z_h^m v dS.$$

Because of $\mathbb{S}_{\mathcal{A}}(v)\mathcal{N} = 0$ on the boundary $\{x_3 = 1\}$, $J\mathcal{A}_i^3 e_3 = \mathcal{N}$, and $Z_h^m(\mathbb{S}_{\mathcal{A}} v \mathcal{N}) = 0$ on $\{x_3 = 1\}$, the second term on the above equality plus the second term of I_5 is zero:

$$\sum_{|m|=1}^{12} \delta^{2|m|} \int_{x_3=1} \Big(Z_h^m (\mathcal{N}(\mathbb{S}_{\mathcal{A}}(v))) - \mathcal{N}(Z_h^m (\mathbb{S}_{\mathcal{A}}(v))) \Big) Z_h^m v dS$$
$$+ \sum_{|m|=1}^{12} \delta^{2|m|} \int_{x_3=1} \mathcal{N} Z_h^m \mathbb{S}_{\mathcal{A}}(v) \ Z_h^m v dS = 0.$$

Hence, all we left is to deal with the commutator

$$\int_\Omega \Big(Z^m (J\mathcal{A}_i^k (\mathbb{S}_{\mathcal{A}}(v))^i) - J\mathcal{A}_i^k (Z^m (\mathbb{S}_{\mathcal{A}}(v)))^i \Big) \cdot \partial_k Z^m v dx.$$

By the same arguments as I_4 and using Lemma 2.2–2.5, we deduce that

$$\Big|\sum_{|m|=1}^{12}\delta^{2|m|}\int_\Omega\Big(Z^m(J\mathcal{A}_i^k(\mathbb{S}_\mathcal{A}(v))^i)-J\mathcal{A}_i^k(Z^m(\mathbb{S}_\mathcal{A}(v)))^i\Big)\cdot\partial_k Z^m v dx\Big|$$
$$\leq\delta(C_0+t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C}))\|\nabla v\|^2_{X^{12}_{0,\delta}}+t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C})\mathfrak{D}(v)^2.$$

Combining all the above estimates, we get that

$$I_5\leq\delta(C_0+t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C}))\|\nabla v\|^2_{X^{12}_{0,\delta}}+t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C})\mathfrak{D}(v)^2.$$

So far, we obtain

$$-\sum_{|m|=0}^{12}\delta^{2|m|}\int_\Omega Z^m\big(\nabla_{J\mathcal{A}}\cdot\mathbb{S}_\mathcal{A}v\big)\cdot Z^m v dx$$
$$\geq\Big(c_2-\delta(C_0+t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C}))\Big)\|\nabla v\|^2_{X^{12}_{0,\delta}}-C_0\|v\|^2_{X^{12}_{\frac{1}{2},\delta}}-t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C})\mathfrak{D}(v)^2.$$

Estimate of I_2. Now, we deal with the pressure.

$$I_2=\sum_{|m|=0}^{12}\delta^{2|m|}\int_\Omega\partial_k Z^m\big(\mathcal{A}_i^k J^{-1}\overline{\rho}^2\big)\cdot Z^m v^i dx$$
$$+\sum_{|m|=1}^{12}\delta^{2|m|}\int_\Omega[Z^m,\partial_k]\big(\mathcal{A}_i^k J^{-1}\overline{\rho}^2\big)\cdot Z^m v^i\,dx\triangleq I_{21}+I_{22}.$$

- *Estimates of* I_{22}. Since $Z^m\overline{\rho}^2\sim\overline{\rho}^2$ for any m, we use (3.10) and Lemmas 2.3–2.4 to get

$$I_{22}\leq\delta(C_0+t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C}))\|\nabla v\|_{X^{11}_{0,\delta}}.$$

- *Estimates of* I_{21}. Because of $\overline{\rho}|_{x_3=1}=0$, the boundary terms vanish when we integrate by parts. By the same argument as I_5, it is easy to see I_{21} is bounded by

$$I_{21}\leq(C_0+t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C}))(\|\nabla v\|_{X^{12}_{0,\delta}}+\delta\|\nabla v\|_{X^{11}_{0,\delta}}).$$

Combining the two estimates, we get

$$I_2\leq(C_0+t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C}))(\|\nabla v\|_{X^{12}_{0,\delta}}+\delta\|\nabla v\|_{X^{11}_{0,\delta}}).$$

Estimate of I_1. For $m\geq 1$, it holds that

$$[\overline{\rho},Z^m]\sim\sum_{k=0}^{m-1}f_k Z^k(\overline{\rho}\cdot).$$

where f_k are smooth functions which are defined by $\overline{\rho}$. Thus

$$I_1\leq C_0\sum_{|m|=1}^{12}\sum_{k=0}^{m-1}\delta^{2|m|}\Big|\int_\Omega Z^k(\overline{\rho}\partial_t v)\cdot Z^m v dx\Big|$$
$$=C_0\sum_{|m|=1}^{12}\sum_{k=0}^{m-1}\delta^{2|m|}\Big|\int_\Omega Z^k(-\nabla_{J\mathcal{A}}(J^{-2}\overline{\rho}^2)+\nabla_{J\mathcal{A}}\cdot\mathbb{S}_\mathcal{A}v)\cdot Z^m v dx\Big|.$$

From the formula above, I_1 can be regarded as lower term to I_2 plus dissipation term with the highest order 11. Since $k \leq m-1$, extra δ is left. Thus, we have

$$I_1 \leq \delta(C_0 + t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C}))(\|\nabla v\|_{X^{12}_{0,\delta}} + \delta\|\nabla v\|_{X^{11}_{0,\delta}})$$
$$+ \delta(C_0 + t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C}))(C_0\|\nabla v\|_{X^{11}_{0,\delta}} + t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C})\mathfrak{D}(v))\|\nabla v\|_{X^{12}_{0,\delta}}.$$

Collecting all estimates together, we finally obtain

$$\frac{d}{dt}\|v\|^2_{X^{12}_{\frac{1}{2},\delta}} + \left(c_0 - \delta(C_0 + t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C}))\right)\|\nabla v\|^2_{X^{12}_{0,\delta}}$$
$$\leq t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C})\mathfrak{D}^2(v) + C_0\|v\|^2_{X^{12}_{\frac{1}{2},\delta}} + C_0 + t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C}),$$

which implies the desired results. □

Estimate for $\mathfrak{D}(v)$

To close the energy estimates, all we left is the estimate of $\mathfrak{D}(v)$ which should be controlled by the energy.

Lemma 3.9 *Assume that (3.1) holds. Then there exists $0 < T \leq \bar{T}$ and $\delta_0 > 0$ which depend on the initial data, σ_0 and $\mathfrak{C}$ such that for any $t \in [0, T]$ and $\delta \in (0, \delta_0)$, there holds that*

$$\mathfrak{D}(v) \leq C\mathcal{D}_\delta^{\frac{1}{2}}(t) + (C + t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C}))(1 + t^{\frac{1}{2}}\mathfrak{D}(v)).$$

Proof Here we only need to control the term $\|\overline{\rho}^{\kappa-\frac{1}{2}}\triangle v\|_{L^2}$. To do that, we go back to the equation of v. Since

$$\overline{\rho}^{-\frac{1}{2}+\kappa}\nabla_{J_0\mathcal{A}_0}\cdot\mathbb{S}_{\mathcal{A}_0}(v) = \overline{\rho}^{-\frac{1}{2}+\kappa}\nabla_{J\mathcal{A}}\cdot\mathbb{S}_{\mathcal{A}}(v) + \overline{\rho}^{-\frac{1}{2}+\kappa}\big(\nabla_{J_0\mathcal{A}_0}\cdot\mathbb{S}_{\mathcal{A}_0}(v) - \nabla_{J\mathcal{A}}\cdot\mathbb{S}_{\mathcal{A}}(v)\big)$$
$$= \overline{\rho}^{-\frac{1}{2}+\kappa}\Big(-\overline{\rho}\partial_t v - \nabla_{J\mathcal{A}}(J^{-2}\overline{\rho}^2)\Big)$$
$$+ \overline{\rho}^{-\frac{1}{2}+\kappa}\big(\nabla_{J_0\mathcal{A}_0}\cdot\mathbb{S}_{\mathcal{A}_0}(v) - \nabla_{J\mathcal{A}}\cdot\mathbb{S}_{\mathcal{A}}(v)\big),$$

which implies that

$$\|\overline{\rho}^{-\frac{1}{2}+\kappa}\nabla_{J_0\mathcal{A}_0}\cdot\mathbb{S}_{\mathcal{A}_0}(v)\|_{L^2} \leq \|\overline{\rho}^{\frac{1}{2}}\partial_t v\|_{L^2} + \|\overline{\rho}^{-\frac{1}{2}+\kappa}\nabla_{J\mathcal{A}}(J^{-2}\overline{\rho}^2)\|_{L^2}$$
$$+ \|\overline{\rho}^{-\frac{1}{2}+\kappa}\nabla_{J_0\mathcal{A}_0-J\mathcal{A}}\cdot\mathbb{S}_{\mathcal{A}_0}(v)\|_{L^2}$$
$$+ \|\overline{\rho}^{-\frac{1}{2}+\kappa}\nabla_{J\mathcal{A}}\cdot\mathbb{S}_{\mathcal{A}_0-\mathcal{A}}(v)\|_{L^2}$$
$$\triangleq \|\overline{\rho}^{\frac{1}{2}}\partial_t v\|_{L^2} + I_1 + I_2 + I_3.$$

Owing to Lemma 2.4, we have

$$I_1 \leq C_0 + t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C}).$$

For I_2, by Lemma 2.1, Lemma 2.4 and (3.5)–(3.6), we have

$$I_2 \leq \|J_0\mathcal{A}_0 - J\mathcal{A}\|_{L^\infty}\|\overline{\rho}^{-\frac{1}{2}+\kappa}\nabla\cdot\mathbb{S}_{\mathcal{A}_0}(v)\|_{L^2}$$
$$\leq t^{\frac{1}{2}}(C_0 + t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C}))(\|\overline{\rho}^{-\frac{1}{2}+\kappa}\nabla v\|_{L^2} + \|\overline{\rho}^{-\frac{1}{2}+\kappa}\nabla^2 v\|_{L^2})$$

$$\le t^{\frac{1}{2}}(C_0+t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C}))(\|\overline{\rho}^{-\frac{1}{2}+\kappa}\triangle v\|_{L^2}+\|\overline{\rho}^{-\frac{1}{2}+\kappa}Z_h\partial_3 v\|_{L^2}+\|\overline{\rho}^{-\frac{1}{2}+\kappa}Z_h^2 v\|_{L^2}+\mathfrak{D}(v))$$
$$\le t^{\frac{1}{2}}(C_0+t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C}))(\|\overline{\rho}^{-\frac{1}{2}+\kappa}\triangle v\|_{L^2}+\|\overline{\rho}^{\frac{1}{2}+\kappa}Z_h\triangle v\|_{L^2}+\|\overline{\rho}^{\frac{1}{2}+\kappa}Z_h^2\nabla v\|_{L^2}+\mathfrak{D}(v))$$
$$\le t^{\frac{1}{2}}(C+t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C}))\mathfrak{D}(v).$$

Similarly, by the fact that

$$\mathcal{A}-\mathcal{A}_0=(\mathcal{A}J-\mathcal{A}_0J_0)J^{-1}+J^{-1}(J_0-J)\mathcal{A}_0$$

and

$$J-J_0=\int_0^t\partial_t J ds=\int_0^t J\nabla_{\mathcal{A}}v ds,$$

combine (3.5) with Lemma 3.2 to get

$$I_3\le t^{\frac{1}{2}}(C+t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C}))\mathfrak{D}(v).$$

Collecting all above estimates to obtain

$$\|\overline{\rho}^{-\frac{1}{2}+\kappa}\nabla_{J_0\mathcal{A}_0}\cdot\mathbb{S}_{\mathcal{A}_0}(v)\|_{L^2}\le\|\overline{\rho}^{\frac{1}{2}}\partial_t v\|_{L^2}+(C+t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C}))(1+t^{\frac{1}{2}}\mathfrak{D}(v)). \tag{3.14}$$

Next, we give the relationship between $\triangle v$ and $\nabla_{J_0\mathcal{A}_0}\cdot\mathbb{S}_{\mathcal{A}_0}(v)$. It is easy to find that

$$\nabla_{J_0\mathcal{A}_0}\cdot\mathbb{S}_{\mathcal{A}_0}(v)=\begin{pmatrix}\mu J_0^{-1}\partial_3^2 v_1\\ \mu J_0^{-1}\partial_3^2 v_2\\ (2\mu+\lambda)J_0^{-1}\partial_3^2 v_3\end{pmatrix}+\text{some terms likes } Z\nabla v.$$

By Lemma 2.1 and the interpolation inequality, we have

$$\begin{aligned}\|\overline{\rho}^{-\frac{1}{2}+\kappa}Z_h\nabla v\|_{L^2}&\le C_0\|\overline{\rho}^{\frac{1}{2}+\kappa}Z_h\nabla v\|_{L^2}+C_0\|\overline{\rho}^{\frac{1}{2}+\kappa}Z_h\nabla^2 v\|_{L^2}\\&\le C_0\|\nabla v\|_{L^2_{x_3}(H_h^2)}+C_0\|\overline{\rho}^{-\frac{1}{2}+\kappa}\triangle v\|_{L^2}^{\theta}\|\overline{\rho}\triangle v\|_{L^2}^{1-\theta}\\&\le C_\varepsilon\|\nabla v\|_{L^2_{x_3}(H_h^2)}+\varepsilon\|\overline{\rho}^{-\frac{1}{2}+\kappa}\triangle v\|_{L^2},\end{aligned}\tag{3.15}$$

where we use Young inequality in the last step and $\theta\in(0,1)$.

Taking ε small enough and using (3.1), (3.14), (3.15), we have

$$\|\overline{\rho}^{-\frac{1}{2}+\kappa}\triangle v\|_{L^2}\le\|\overline{\rho}^{\frac{1}{2}}\partial_t v\|_{L^2}+(C+t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C}))(1+t^{\frac{1}{2}}\mathfrak{D}(v))+\frac{C_0}{\delta^2}\|\nabla v\|_{X^{12}_{0,\delta}}. \tag{3.16}$$

Combining (3.14) and (3.16), we obtain the desired results. □

3.1 Proof of Proposition 3.1

Now, from Propositions 3.7 to 3.8, we obtain that

$$\begin{aligned}&\Big(c_0-\delta(C_0+t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C}))\Big)\Big(\sup_{\tau\in[0,t]}\mathcal{E}_\delta(\tau)+\int_0^t\mathcal{D}_\delta(\tau)\Big)\\&\quad\le\Big(c_0-\delta(C_0+t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C}))\Big)\mathcal{E}(0)+t^{\frac{1}{2}}\mathcal{P}(\mathfrak{C})(1+\sup_{\tau\in[0,t]}\mathcal{E}_\delta(\tau)).\end{aligned}$$

Now, we give the estimates of J. By the definition of J, we have

$$J - J_0 = \int_0^t \partial_t J ds = \int_0^t J \nabla_{\mathcal{A}} v ds,$$

which implies that

$$|J - J_0| \le \|J\|_{L^\infty} \|\mathcal{A}\|_{L^\infty} \|\nabla v\|_{L_t^1 L^\infty} \le t^{\frac{1}{2}} (C + t^{\frac{1}{2}} \mathcal{P}(\mathfrak{C})).$$

Then by the Lemma 3.9 and standard bootstrap argument, we finish the proof of Proposition 3.1.

4 Local well-posedness

In this section, we will first give existence and uniqueness of strong solutions of system (1.18), which is motivated by the method in [10]. First, we give some definitions of functional spaces. Given $T > 0$, let $\widetilde{Y}_T$ and Y_T are defined by

$$\widetilde{Y}_T \triangleq C([0, T], X^0_{\frac{1}{2}}) \cap L^2([0, T], H^1),$$
$$Y_T \triangleq \{v \in \widetilde{Y}_T \cap C([0, T], X^{12}_{\frac{1}{2}} \cap H^1) : \|v\|_{Y_T} < +\infty\},$$

where $\|v\|^2_{\widetilde{Y}_T} := \sup_{t\in[0,T]} \|\overline{\rho}^{\frac{1}{2}} v\|^2_{L^2} + \|v\|^2_{L^2_T(H^1)}$ and $\|v\|^2_{Y_T} := \sup_{t\in[0,T]}(\|v\|^2_{X^{12}_{\frac{1}{2}}} + \|\nabla v\|^2_{L^2}) + \|\overline{\rho}^{-\frac{1}{2}+\kappa} \Delta v\|^2_{L^2_T(L^2)} + \|\nabla v\|^2_{L^2_T(X^{12})} + \|\overline{\rho}^{\frac{1}{2}} \partial_t v\|^2_{L^2_T(L^2)}$.

Now, we define map $\Theta : Y_T \to Y_T$ as follows. For any given $\widetilde{v} \in Y_T$, $v := \Theta(\widetilde{v})$ is the solution of the following linear $\mathcal{A}$-equations:

$$\begin{cases} \overline{\rho}\partial_t v + \nabla_{\widetilde{J}\widetilde{\mathcal{A}}}((\widetilde{J})^{-2}\overline{\rho}^2) - \nabla_{\widetilde{J}\widetilde{\mathcal{A}}} \cdot \mathbb{S}_{\widetilde{\mathcal{A}}}(v) = 0 \quad \text{in} \quad \Omega, \\ \mathbb{S}_{\widetilde{\mathcal{A}}}(v)\,\widetilde{\mathcal{N}} = 0, \quad \text{on} \quad \Gamma, \\ v|_{x_3=0} = 0, \\ v|_{t=0} = v_0 \quad \text{in} \quad \Omega. \end{cases} \tag{4.1}$$

4.1 Existence and uniqueness of the strong solution to (4.1).

Our aim in this subsection is to construct strong solutions to linear $\mathcal{A}$-equations (4.1).

Lemma 4.1 *Assume that $\overline{\rho}^{\frac{1}{2}} v_0, \nabla v_0 \in L^2$ and $\widetilde{v} \in Y_T$, then there exists a positive time $T_1 \in (0, T]$ such that the system (4.1) has a unique strong solution v with*

$$\overline{\rho}^{\frac{1}{2}} v \in C([0, T_1], L^2), \quad v \in C([0, T_1], H^1),$$
$$\overline{\rho}\partial_t v \in L^2(0, T_1; (H^1)^*), \quad \overline{\rho}^{\frac{1}{2}} \partial_t v \in L^2([0, T_1], L^2), \quad \overline{\rho}^{-\frac{1}{2}+\kappa} \Delta v \in L^2([0, T_1], L^2).$$

Moreover, the solution satisfies the following estimate

$$\sup_{t\in[0,T_1]} (\|\overline{\rho}^{\frac{1}{2}} v\|^2_{L^2} + \|\nabla v\|^2_{L^2}) + \|v\|^2_{L^2_{T_1}(H^1)} + \|\overline{\rho}^{\frac{1}{2}} \partial_t v\|^2_{L^2_{T_1}(L^2)}$$
$$+ \|\overline{\rho}^{-\frac{1}{2}+\kappa} \Delta v\|^2_{L^2_{T_1}(L^2)} + \|\overline{\rho}\partial_t v\|^2_{L^2_{T_1}(H^1)^*} \le C_0 \|\overline{\rho}^{\frac{1}{2}} v_0\|^2_{L^2} + C_0 \|\nabla v_0\|^2_{L^2} + C_0(1 + T_1).$$

Proof We split the proof of the lemma into four steps.

Step 1: Galerkin approximation. We first use Galerkin method to construct approximate solutions of the system (4.1). Let $\{w_k\}_{k=1}^{\infty}$ are orthonormal basis of $H^1(\Omega)$ which satisfy boundary condition $\mathrm{S}_{\widetilde{\mathcal{A}}}(w_k)\,\widetilde{\mathcal{N}}|_{x_3=1} = 0$ and $w_k|_{x_3=0} = 0$ and set approximate solution with the form

$$v^m(t,x) := \sum_{k=1}^{m} d_k^m(t) w_k(x), \quad d_k^m(t) \quad \text{will be determined later on,}$$

which solves the linear system

$$\begin{cases} \overline{\rho}\partial_t v^m + \nabla_{\widetilde{J}\widetilde{\mathcal{A}}}((\widetilde{J})^{-2}\overline{\rho}^2) - \nabla_{\widetilde{J}\widetilde{\mathcal{A}}} \cdot \mathbb{S}_{\widetilde{\mathcal{A}}}(v^m) = 0. \quad \text{in} \quad \Omega, \\ \mathrm{S}_{\widetilde{\mathcal{A}}}(v^m)\,\widetilde{\mathcal{N}} = 0, \quad \text{on} \quad \Gamma, \\ v^m|_{x_3=0} = 0, \\ v^m|_{t=0} = v_0^m = \sum_{k=1}^{m} d_k^m(0) w_k(x) \quad \text{in} \quad \Omega \end{cases} \tag{4.2}$$

in the sense of the distribution, where $d_k^m(0) = \int_\Omega v_0 w_k$ for $k = 1, \ldots, m$.

Taking the test function $\phi = w_\ell$, $\ell = 1, \ldots, m$, from the weak formula of the system (4.2), we obtain the following ordinary differential equations

$$\begin{cases} \sum_{k=1}^{m} \int_\Omega \overline{\rho} w_k w_\ell dx \; d_k^m(t)' + \sum_{k=1}^{m} \int_\Omega \mathbb{S}_{\widetilde{\mathcal{A}}} w_k : \nabla_{\widetilde{J}\widetilde{\mathcal{A}}} w_\ell dx \; d_k^m(t) \\ \quad = \int_\Omega \overline{\rho}^2 \widetilde{J}^{-2} \nabla_{\widetilde{J}\widetilde{\mathcal{A}}} \cdot w_\ell dx, \\ d_k^m(0) = \int_\Omega v_0 w_k. \end{cases} \tag{4.3}$$

Notice that the matrix $\left(\int_\Omega \overline{\rho} w_k w_\ell dx\right)_{m\times m}$ is invertible for any $m \geq 1$, and the coefficient $\int_\Omega \mathbb{S}_{\widetilde{\mathcal{A}}} w_k : \nabla_{\widetilde{J}\widetilde{\mathcal{A}}} w_\ell dx$ (in front of $d_k^m(t)$) is continuous in terms of $t \in [0, T]$ because of $\widetilde{v} \in Y_T$, we know that (4.3) is a non-generate linear ODE system with continuous coefficients. Due to the classical theory of ODE, we find solutions $d_k^m(t) \in C^1([0, T])$, $k = 1, \ldots, m$, which means approximate solutions $v^m(t, x)$ exist and belong to the space $C^1([0, T], H^1(\Omega))$.

Step 2: Uniform estimates for v^m. Multiplying $d_\ell^m(t)$ on the both sides of (4.3) and taking the summation in terms of $\ell = 1, \ldots, m$, one has

$$\int_\Omega \overline{\rho}\partial_t v^m \cdot v^m + \int_\Omega \mathbb{S}_{\widetilde{\mathcal{A}}}(v^m) : \nabla_{\widetilde{J}\widetilde{\mathcal{A}}} v^m dx = \int_\Omega \overline{\rho}^2 \widetilde{J}^{-2} \nabla_{\widetilde{J}\widetilde{\mathcal{A}}} \cdot v^m dx.$$

Then Lemmas 2.4 and 3.3 give that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\overline{\rho}^{\frac{1}{2}} v^m\|_{L^2}^2 + c_0\|v^m\|_{H^1}^2 \leq & \int_\Omega \overline{\rho}^2 \widetilde{J}^{-2} \nabla_{\widetilde{J}\widetilde{\mathcal{A}}} \cdot v^m dx + C_0\|\overline{\rho}^{\frac{1}{2}} v^m\|_{L^2}^2 \\ \leq & C_0\|\nabla v^m\|_{L^2} + C_0\|\overline{\rho}^{\frac{1}{2}} v^m\|_{L^2}^2, \end{aligned} \tag{4.4}$$

for t small enough.

By Gronwall's inequality, we know there exists $T_1 > 0$ independent of m such that

$$\sup_{t\in[0,T_1]} \|\overline{\rho}^{\frac{1}{2}} v^m\|_{L^2}^2 + \int_0^{T_1} \|v^m\|_{H^1}^2 ds \leq C_0\|\overline{\rho}^{\frac{1}{2}} v_0^m\|_{L^2}^2 + C_0 T_1. \tag{4.5}$$

For any test function $\phi \in C([0,T],H^1)$ with $\phi|_{x_3=0}=0$ and $\|\phi\|_{L^2_T H^1} \le 1$, owing to the weak formula of the system (4.2), we deduce from (4.5) that

$$\begin{aligned}
|\int_0^{T_1} \langle \overline{\rho}\partial_t v^m,\ \phi\,\rangle\, ds| = & | -\int_0^{T_1}\int_\Omega \mathbb{S}_{\widetilde{\mathcal{A}}}(v^m) : \nabla_{\widetilde{J}\widetilde{\mathcal{A}}}\phi\, dxds + \int_0^{T_1}\int_\Omega \overline{\rho}^2 \widetilde{J}^{-2}\nabla_{\widetilde{J}\widetilde{\mathcal{A}}}\cdot\phi\, dxds| \\
\le & (C_0 + C_0\|\nabla v^m\|_{L^2_{T_1}L^2})\|\phi\|_{L^2_T H^1} \le (C_0(1+T_1^{\frac{1}{2}}) + C_0\|\overline{\rho}^{\frac{1}{2}} v_0^m\|_{L^2})\|\phi\|_{L^2_T H^1},
\end{aligned}$$

which follows from the dual argument that

$$\|\overline{\rho}\partial_t v^m\|_{L^2_{T_1}(H^1)^*} \le C_0(1+T_1^{\frac{1}{2}}) + C_0\|\overline{\rho}^{\frac{1}{2}} v_0^m\|_{L^2}. \tag{4.6}$$

Multiplying $d_\ell^m(t)'$ on the both sides of (4.3) and taking the summation in terms of $\ell = 1,\ldots,m$, we have

$$\int_\Omega \overline{\rho}|\partial_t v^m|^2 + \int_\Omega \mathbb{S}_{\widetilde{\mathcal{A}}}(v^m) : \nabla_{\widetilde{J}\widetilde{\mathcal{A}}}\partial_t v^m dx = \int_\Omega \overline{\rho}^2\widetilde{J}^{-2}\nabla_{\widetilde{J}\widetilde{\mathcal{A}}}\cdot\partial_t v^m dx.$$

Similar estimate in Proposition 3.7 implies that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \widetilde{J}\mathbb{S}_{\widetilde{\mathcal{A}}}(v^m) : \nabla_{\widetilde{\mathcal{A}}} v^m dx + \|\overline{\rho}^{\frac{1}{2}}\partial_t v^m\|_{L^2}^2 \\
&\quad \le \Big|\frac{1}{2}\int_\Omega \mathbb{S}_{\widetilde{\mathcal{A}}}(v^m) : \nabla_{\widetilde{\mathcal{A}}} v^m \partial_t \widetilde{J} dx\Big| + \Big|\int_\Omega \widetilde{J}\mathbb{S}_{\widetilde{\mathcal{A}}}(v^m) : \nabla_{\partial_t\widetilde{\mathcal{A}}} v^m dx\Big| \\
&\qquad + \Big|\int_\Omega \overline{\rho}^2\widetilde{J}^{-2}\nabla_{\widetilde{J}\widetilde{\mathcal{A}}}\cdot\partial_t v^m dx\Big|.
\end{aligned}$$

Since $\widetilde{v} \in Y_T$, we infer that

$$\Big|\frac{1}{2}\int_\Omega \mathbb{S}_{\widetilde{\mathcal{A}}}(v^m) : \nabla_{\widetilde{\mathcal{A}}} v^m \partial_t \widetilde{J} dx\Big| + \Big|\int_\Omega \widetilde{J}\mathbb{S}_{\widetilde{\mathcal{A}}}(v^m) : \nabla_{\partial_t\widetilde{\mathcal{A}}} v^m\, dx\Big| \le C\|\nabla v^m\|_{L^2}^2 \mathfrak{D}(\widetilde{v})$$

and

$$\Big|\int_\Omega \overline{\rho}^2\widetilde{J}^{-2}\nabla_{\widetilde{J}\widetilde{\mathcal{A}}}\cdot\partial_t v^m dx\Big| \le C_0\|\overline{\rho}^{\frac{1}{2}}\partial_t v^m\|_{L^2}.$$

As a result, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \widetilde{J}\mathbb{S}_{\widetilde{\mathcal{A}}}(v^m) : \nabla_{\widetilde{\mathcal{A}}} v^m dx + \|\overline{\rho}^{\frac{1}{2}}\partial_t v^m\|_{L^2}^2 \le C\|\nabla v^m\|_{L^2}^2\mathfrak{D}(\widetilde{v}) + C_0\|\overline{\rho}^{\frac{1}{2}}\partial_t v^m\|_{L^2}.$$

Integrating time from 0 to T_1 and using $\widetilde{v} \in Y_T$ and Lemma 3.9, we obtain

$$\begin{aligned}
&\sup_{t\in[0,T_1]}\|\nabla v^m\|_{L^2}^2 + \|\overline{\rho}^{\frac{1}{2}}\partial_t v^m\|_{L^2_{T_1}L^2}^2 \\
&\quad \le C_0\|\nabla v_0^m\|_{L^2}^2 + C_0 T_1^{\frac{1}{2}}\sup_{t\in[0,T_1]}\|\nabla v^m\|_{L^2}^2 \Big(\int_0^{T_1}\mathfrak{D}(\widetilde{v})^2 ds\Big)^{\frac{1}{2}} + C_0 T_1.
\end{aligned}$$

Taking T_1 small enough such that the second term on the right hand side absorbed by the left hand side, we obtain

$$\sup_{t\in[0,T_1]}\|\nabla v^m\|_{L^2}^2 + c_0\|\overline{\rho}^{\frac{1}{2}}\partial_t v^m\|_{L^2_{T_1}L^2}^2 \le 2C_0\|\nabla v_0^m\|_{L^2}^2 + C_0 T_1. \tag{4.7}$$

Combining estimate (4.5), (4.6) and (4.7) together, there holds that

$$\begin{aligned}&\sup_{t\in[0,T_1]}(\|\overline{\rho}^{\frac{1}{2}}v^m\|_{L^2}^2+\|\nabla v^m\|_{L^2}^2)+\|v^m\|_{L^2_{T_1}H^1}^2+\|\overline{\rho}^{\frac{1}{2}}\partial_t v^m\|_{L^2_{T_1}L^2}^2+\|\overline{\rho}\partial_t v^m\|_{L^2_{T_1}(H^1)^*}^2\\&\leq 2C_0\|\overline{\rho}^{\frac{1}{2}}v_0^m\|_{L^2}^2+2C_0\|\nabla v_0^m\|_{L^2}^2+C_0(1+T_1).\end{aligned}\tag{4.8}$$

Step 3: Passing to the limit. Since

$$\sup_{t\in[0,T_1]}(\|\overline{\rho}^{\frac{1}{2}}v^m\|_{L^2}^2+\|\nabla v^m\|_{L^2}^2)+\|v^m\|_{L^2_{T_1}H^1}^2+\|\rho^{\frac{1}{2}}\partial_t v^m\|_{L^2_{T_1}L^2}^2+\|\overline{\rho}\partial_t v^m\|_{L^2_{T_1}(H^1)^*}^2$$

is uniformly bounded, up to the extraction of a subsequence, we know as $m\to\infty$

$$\begin{cases}\overline{\rho}^{\frac{1}{2}}v^m\rightharpoonup^*\overline{\rho}^{\frac{1}{2}}v & \text{in } L^\infty_{T_1}L^2,\\ \nabla v^m\rightharpoonup^*\nabla v & \text{in } L^\infty_{T_1}L^2,\\ \overline{\rho}\partial_t v^m\rightharpoonup\overline{\rho}\partial_t v & \text{in } L^2_{T_1}(H^1)^*,\\ v^m\rightharpoonup v & \text{in } L^2_{T_1}H^1.\end{cases}\tag{4.9}$$

By lower semicontinuity and energy estimate (4.8), we use the fact $\|v^m(0)-v_0\|_{L^2(\Omega)}\to 0$ as $m\to\infty$ to infer that

$$\begin{aligned}&\sup_{t\in[0,T_1]}(\|\overline{\rho}^{\frac{1}{2}}v\|_{L^2}^2+\|\nabla v\|_{L^2}^2)+\|v\|_{L^2_{T_1}H^1}^2+\|\rho^{\frac{1}{2}}\partial_t v\|_{L^2_{T_1}L^2}+\|\overline{\rho}\partial_t v\|_{L^2_{T_1}(H^1)^*}^2\\&\leq 4C_0\|\overline{\rho}^{\frac{1}{2}}v_0\|_{L^2}^2+4C_0\|\nabla v_0\|_{L^2}^2+C_0(1+T_1),\end{aligned}\tag{4.10}$$

and v is a weak solution to the linear $\mathcal{A}$-equations (4.1). Moreover, according to (4.10), we may obtain from Aubin–Lions's lemma [38] that $v\in C([0,T_1],X^0_{\frac{1}{2}}\cap H^1)$.

Step 4: The strong solution. Now, we prove the above weak solution v is a strong one. In fact, for a.e $t\in[0,T]$, $v(t)$ is a weak solution to the elliptic system in the sense of

$$\int_\Omega \mathbb{S}_{\widetilde{\mathcal{A}}}(v):\nabla_{\widetilde{J}\widetilde{\mathcal{A}}}\phi\,dx=\int_\Omega\Big(\nabla_{\widetilde{J}\widetilde{\mathcal{A}}}(\overline{\rho}^2\widetilde{J}^{-2})-\overline{\rho}\partial_t v\Big)\phi dx\tag{4.11}$$

for $\phi\in H^1$. Since $\overline{\rho}^{-\frac{1}{2}}\Big(\nabla_{\widetilde{J}\widetilde{\mathcal{A}}}(\overline{\rho}^2\widetilde{J}^{-2})-\overline{\rho}\partial_t v\Big)\in L^2$ for a.e $t\in[0,T]$, by elliptic regularity theory, we know this system admires a strong solution v solving (4.1) with $\rho^{-\frac{1}{2}+\kappa}\triangle v\in L^2([0,T],L^2)$. The uniqueness comes from energy estimates with zero initial data. □

4.2 High regularity of *v*

In this subsection, we prove when $\widetilde{v}\in Y_T$, so does $v:=\Theta(\widetilde{v})$. It is mainly based on the priori estimates in Sect. 3.

Lemma 4.2 *Assume that v is a strong solution obtained in Lemma 4.1 and $\widetilde{v}\in Y_T$ with initial data $v_0\in Y_0$, then we have $v\in Y_{T_1}$ and satisfies*

$$\|v\|_{Y_{T_1}}\leq CT_1+C_0\|v_0\|_{Y_0},$$

where the constant C depends on $\|\widetilde{v}\|_{Y_T}$.

Proof We take $\widetilde{\mathcal{A}}$, $\widetilde{J}$ instead of $\mathcal{A}$, J respectively in those estimates in Propositions 3.7 and 3.8. System (4.1) is a linear system due to $\widetilde{\mathcal{A}}$, $\widetilde{J}$ are regarded as known quantities, so for small $T_1 > 0$, it is easy to arrive at the following estimate:

$$\|v^m\|_{Y_{T_1}} \leq CT_1 + C_0\|v_0^m\|_{Y_0}.$$

Passing to the limit, we get the desired results. □

Remark 4.3 By Lemma 4.2, we know that $\Theta : Y_{T_1} \to Y_{T_1}$ is well-defined.

4.3 Contraction

By Lemmas 4.1 and 4.2, we know that if $\widetilde{v} \in Y_T$ with $T > 0$ sufficiently small , we can find a unique strong solution of equation (4.1) with regular $v = \Theta(\widetilde{v}) \in Y_T$. In order to construct the solution to (1.18), we need to construct approximate solutions. The approximate solutions $\{\xi^{(n)}, v^{(n)}\}_{n=1}^{\infty}$ we defined are iterated as follows:

$$\begin{cases} \partial_t \xi^{(n)} = v^{(n)} \quad \text{in} \quad \Omega, \\ \overline{\rho}\partial_t v^{(n)} + \nabla_{J^{(n-1)}\mathcal{A}^{(n-1)}}((J^{(n-1)})^{-2}\overline{\rho}^2) - \nabla_{J^{(n-1)}\mathcal{A}^{(n-1)}} \cdot \mathbb{S}_{\mathcal{A}^{(n-1)}}(v^{(n)}) = 0 \quad \text{in} \quad \Omega, \\ \mathrm{S}_{\mathcal{A}^{(n-1)}}(v^{(n)})\, N^{(n-1)} = 0, \quad \text{on} \quad \Gamma, \\ v^{(n)}|_{x_3=0} = 0, \\ (\xi^{(n)}, v^{(n)}|_{t=0} = (\xi_0, v_0) \quad \text{in} \quad \Omega. \end{cases} \tag{4.12}$$

with $\{\xi^{(1)}, v^{(1)}\}$ be the solution of linear equation

$$\begin{cases} \partial_t \xi^{(1)} = v^{(1)} \quad \text{in} \quad \Omega, \\ \overline{\rho}\partial_t v^{(1)} + \nabla_{J_0\mathcal{A}_0}(\overline{\rho}^2 J_0^{-1}) - \nabla_{J_0\mathcal{A}_0} \cdot \mathbb{S}_{\mathcal{A}_0}(v^{(1)}) = 0 \quad \text{in} \quad \Omega, \\ \mathrm{S}_{\mathcal{A}_0}(v^{(1)})\, \mathcal{N}_0 = 0, \quad \text{on} \quad \Gamma, \\ v^{(1)}|_{x_3=0} = 0, \\ (\xi^{(1)}, v^{(1)}|_{t=0} = (\xi_0, v_0) \quad \text{in} \quad \Omega, \end{cases} \tag{4.13}$$

where $\mathcal{A}_0$, J_0 are given by $\eta_0(x) = x + \xi_0(x)$ and $\mathcal{N}_0 = \partial_1\eta_0 \times \partial_2\eta_0$ on $\{x_3 = 1\}$. Since (4.12) is a decouple linear system in terms of $\xi^{(n)}$ and $v^{(n)}$, we need only to solve first $v^{(n)}$ then $\xi^{(n)}$ according to the first equation in (4.12). Notice that (4.13) is linear, the assumption on initial data $\|v_0\|_{Y_0}^2 := \|v_0\|_{X_{\frac{1}{2}}^{12}}^2 + \|\nabla v_0\|_{L^2}^2 \leq \frac{M}{2C_0}$ guarantees that $v^{(1)} \in Y_T$ with bound $\|v^{(1)}\|_{Y_T}^2 \leq M$. By Lemma 4.2, we obtain $\{v^{(n)}\}_{n=1}^{\infty} \subset Y_T$ for any $n \geq 1$.

Next, our goal in this subsection is to prove sequence $\{v^{(n)}\}_{n=1}^{\infty}$ is contracted under norm $\widetilde{Y}_T$.

First of all, we deduce $\sigma(v^{(n)}) \triangleq v^{(n+1)} - v^{(n)}$ satisfies the following equation

$$\begin{cases} \overline{\rho}\partial_t\sigma(v^{(n)}) - \left(\nabla_{J^{(n)}\mathcal{A}^{(n)}} \cdot \mathbb{S}_{\mathcal{A}^{(n)}} v^{(n+1)} - \nabla_{J^{(n-1)}\mathcal{A}^{(n-1)}} \cdot \mathbb{S}_{\mathcal{A}^{(n-1)}} v^{(n)}\right) \\ \quad + \left(\nabla_{J^{(n)}\mathcal{A}^{(n)}}\left((J^{(n)})^{-2}\overline{\rho}^2\right) - \nabla_{J^{(n-1)}\mathcal{A}^{(n-1)}}\left((J^{(n-1)})^{-2}\overline{\rho}^2\right)\right) = 0 \quad \text{in} \quad \Omega, \\ \mathrm{S}_{\mathcal{A}^{(n)}} v^{(n+1)} \mathcal{N}^{(n)} - \mathrm{S}_{\mathcal{A}^{(n-1)}} v^{(n)} \mathcal{N}^{(n-1)} = 0, \quad \text{on} \quad \Gamma, \\ \sigma(v^{(n)})|_{x_3=0} = 0, \\ \sigma(v^{(n)})|_{t=0} = v_0^{(n+1)} - v_0^{(n)} = 0 \quad \text{in} \quad \Omega. \end{cases} \tag{4.14}$$

Lemma 4.4 *Assume that* $\{v^{(n)}\}_{n=1}^{\infty}$ *be the solutions of Eq.* (4.12) *with bound* $\|v^{(n)}\|_{Y_T}^2 \leq M$ *for each* $n \geq 1$. *It holds that*

$$\frac{d}{dt}\|\overline{\rho}^{\frac{1}{2}}\sigma(v^{(n)})\|_{L^2}^2 + \|\sigma(v^{(n)})\|_{H^1}^2 \leq Ct\|\sigma(v^{(n-1)})\|_{L_t^2L^2}^2(1 + \mathfrak{D}(\sigma(v^{(n)}))^2).$$

Moreover, taking T *small enough, the sequence* $v^{(n)}$ *is a Cauchy sequence in the space* $\widetilde{Y}_T$.

Proof Taking L^2 inner product between (4.14) and $\sigma(v^{(n)})$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\overline{\rho}^{\frac{1}{2}}\sigma(v^{(n)})\|_{L^2}^2 - \int_\Omega \left(\nabla_{J^{(n)}\mathcal{A}^{(n)}} \cdot \mathbb{S}_{\mathcal{A}^{(n)}} v^{(n+1)} - \nabla_{J^{(n-1)}\mathcal{A}^{(n-1)}} \cdot \mathbb{S}_{\mathcal{A}^{(n-1)}} v^{(n)}\right) \sigma(v^{(n)})dx \\ &\quad = -\int_\Omega \left(\nabla_{J^{(n)}\mathcal{A}^{(n)}}\left((J^{(n)})^{-2}\overline{\rho}^2\right) - \nabla_{J^{(n-1)}\mathcal{A}^{(n-1)}}\left((J^{(n-1)})^{-2}\overline{\rho}^2\right)\right) \sigma(v^{(n)})dx. \end{aligned}$$

Estimate of dissipation term. Since

$$e_3 J^{(n)}(\mathcal{A}^{(n)})_i^3 = \mathcal{N}^{(n)}, \quad e_3 J^{(n-1)}(\mathcal{A}^{(n-1)})_i^3 = \mathcal{N}^{(n-1)}$$

and

$$\mathrm{S}_{\mathcal{A}^{(n)}}(v^{(n+1)})\, \mathcal{N}^{(n)} - \mathrm{S}_{\mathcal{A}^{(n-1)}}(v^{(n)})\, \mathcal{N}^{(n-1)} = 0 \quad \text{on} \quad \Gamma,$$

we get by using integration by parts that

$$\begin{aligned} &-\int_\Omega \left(\nabla_{J^{(n)}\mathcal{A}^{(n)}} \cdot \mathbb{S}_{\mathcal{A}^{(n)}} v^{(n+1)} - \nabla_{J^{(n-1)}\mathcal{A}^{(n-1)}} \cdot \mathbb{S}_{\mathcal{A}^{(n-1)}} v^{(n)}\right) \sigma(v^{(n)})dx \\ &\quad = \int_\Omega \left(J^{(n)}(\mathcal{A}^{(n)})_i^k(\mathbb{S}_{\mathcal{A}^{(n)}} v^{(n+1)})_l^i - J^{(n-1)}(\mathcal{A}^{(n-1)})_i^k(\mathbb{S}_{\mathcal{A}^{(n-1)}} v^{(n)})_l^i\right) \partial_k\sigma(v_l^{(n)})dx. \\ &\quad = \int_\Omega J^{(n)}\mathcal{A}^{(n)}\mathbb{S}_{\mathcal{A}^{(n)}}\sigma(v^{(n)}) \cdot \partial_k\sigma(v_l^{(n)})dx \\ &\qquad + \int_\Omega (J^{(n)}\mathcal{A}^{(n)} - J^{(n-1)}\mathcal{A}^{(n-1)})\mathbb{S}_{\mathcal{A}^{(n)}}(v^{(n)}) \cdot \partial_k\sigma(v_l^{(n)})dx \\ &\qquad + \int_\Omega J^{(n-1)}\mathcal{A}^{(n-1)} \cdot \mathbb{S}_{\left(\mathcal{A}^{(n)}-\mathcal{A}^{(n-1)}\right)} v^{(n)} \cdot \partial_k\sigma(v_l^{(n)})dx \end{aligned}$$

Under the assumption $\|v^{(n)}\|_{Y_T}^2 \le M$, we have

$$
\begin{aligned}
&-\int_\Omega \left(\nabla_{J^{(n)}\mathcal{A}^{(n)}}\cdot \mathbb{S}_{\mathcal{A}^{(n)}} v^{(n+1)} - \nabla_{J^{(n-1)}\mathcal{A}^{(n-1)}}\cdot \mathbb{S}_{\mathcal{A}^{(n-1)}} v^{(n)}\right)\sigma(v^{(n)})dx\\
&\ge c_0\|\sigma(v^{(n)})\|_{H^1}^2 - C_0\|\overline{\rho}^{\frac12}\sigma(v^{(n)})\|_{L^2}^2\\
&\quad -\left|\int_\Omega (J^{(n)}\mathcal{A}^{(n)} - J^{(n-1)}\mathcal{A}^{(n-1)})\mathbb{S}_{\mathcal{A}^{(n)}}(v^{(n)}) : \nabla\sigma(v^{(n)})dx\right|\\
&\quad -\left|\int_\Omega J^{(n-1)}\mathcal{A}^{(n-1)}\cdot \mathbb{S}_{\left(\mathcal{A}^{(n)}-\mathcal{A}^{(n-1)}\right)} v^{(n)} : \nabla\sigma(v^{(n)})dx\right|\\
&\triangleq c_0\|\sigma(v^{(n)})\|_{H^1}^2 - C_0\|\overline{\rho}^{\frac12}\sigma(v^{(n)})\|_{L^2}^2 - I_1 - I_2,
\end{aligned}
$$

where we use $|J^{(n)}| \ge \sigma_0$ and Lemma 3.5 for $\mathcal{A}^{(n)}$.

For I_1, owing to

$$
\begin{aligned}
J^{(n)}\mathcal{A}^{(n)} - J^{(n-1)}\mathcal{A}^{(n-1)} &= \left(\nabla(\eta^{(n)} - \eta^{(n-1)})\right)^*\\
&= \left(\int_0^t \nabla\sigma(v^{(n-1)})ds\right)^* \sim \left(\int_0^t \nabla\sigma(v^{(n-1)})ds\right)^2,
\end{aligned}
$$

then

$$
\begin{aligned}
&\|J^{(n)}\mathcal{A}^{(n)} - J^{(n-1)}\mathcal{A}^{(n-1)}\|_{L^2}\\
&\le Ct\|\nabla\sigma(v^{(n-1)})\|_{L_t^2L^2}\|\mathfrak{D}(\sigma(v^{(n-1)}))\|_{L_t^2} \le Ct\|\nabla\sigma(v^{(n-1)})\|_{L_t^2L^2}.
\end{aligned}
$$

Applying Holder inequality and Lemma 3.2 to $\mathcal{A}^{(n)}$, one has

$$
\begin{aligned}
I_1 &\le \|J^{(n)}\mathcal{A}^{(n)} - J^{(n-1)}\mathcal{A}^{(n-1)}\|_{L^2}\|\mathcal{A}^{(n)}\|_{L^\infty}\|\nabla v^{(n)}\|_{L^\infty}\|\nabla\sigma(v^{(n)})\|_{L^2}\\
&\le Ct\|\nabla\sigma(v^{(n-1)})\|_{L_t^2L^2}\mathfrak{D}(v^{(n)})\|\nabla\sigma(v^{(n)})\|_{L^2}.
\end{aligned}
$$

Similarly, we have

$$
I_2 \le Ct\|\nabla\sigma(v^{(n-1)})\|_{L_t^2L^2}\mathfrak{D}(v^{(n)})\|\nabla\sigma(v^{(n)})\|_{L^2}.
$$

Combining all above estimates, we obtain

$$
\begin{aligned}
&-\int_\Omega \left(\nabla_{J^{(n)}\mathcal{A}^{(n)}}\cdot \mathbb{S}_{\mathcal{A}^{(n)}} v^{(n+1)} - \nabla_{J^{(n-1)}\mathcal{A}^{(n-1)}}\cdot \mathbb{S}_{\mathcal{A}^{(n-1)}} v^{(n)}\right)\sigma(v^{(n)})dx\\
&\ge \frac34 c_0\|\sigma(v^{(n)})\|_{H^1}^2 - C_0\|\overline{\rho}^{\frac12}\sigma(v^{(n)})\|_{L^2}^2 - Ct^2\|\nabla\sigma(v^{(n-1)})\|_{L_t^2L^2}^2\mathfrak{D}(v^{(n)})^2.
\end{aligned}
$$

Estimate of pressure term. Integrating by parts and using $\overline{\rho}|_{x_3=1} = 0,\ \sigma(v^{(n)})|_{x_3=0} = 0$, we prove that

$$
\begin{aligned}
&-\int_\Omega \left(\nabla_{J^{(n)}\mathcal{A}^{(n)}}\left((J^{(n)})^{-2}\overline{\rho}^2\right) - \nabla_{J^{(n-1)}\mathcal{A}^{(n-1)}}\left((J^{(n-1)})^{-2}\overline{\rho}^2\right)\right)\sigma(v^{(n)})dx\\
&= \int_\Omega \left(\mathcal{A}^{(n)}\left((J^{(n)})^{-1}\overline{\rho}^2\right) - \mathcal{A}^{(n-1)}\left((J^{(n-1)})^{-1}\overline{\rho}^2\right)\right) : \nabla\sigma(v^{(n)})dx\\
&= \int_\Omega \left(\mathcal{A}^{(n)} - \mathcal{A}^{(n-1)}\right)(J^{(n)})^{-1}\overline{\rho}^2 : \nabla\sigma(v^{(n)})dx
\end{aligned}
$$

$$
\begin{aligned}
&+\int_\Omega \mathcal{A}^{(n-1)}\big((J^{(n)})^{-1}-(J^{(n-1)})^{-1}\big)\overline{\rho}^2 : \nabla\sigma(v^{(n)})dx\\
&\le Ct^{\frac12}\|\nabla\sigma(v^{(n-1)})\|_{L^2_tL^2}\|\nabla\sigma(v^{(n)})\|_{L^2}.
\end{aligned}
$$

Collecting all above estimates together, we finally obtain

$$
\begin{aligned}
&\frac{d}{dt}\|\overline{\rho}^{\frac12}\sigma(v^{(n)})\|_{L^2}^2+\frac{c_0}{2}\|\nabla\sigma(v^{(n)})\|_{L^2}^2\\
&\quad\le Ct\|\nabla\sigma(v^{(n-1)})\|_{L^2_tL^2}^2(1+\mathfrak{D}(v^{(n)})^2)+C_0\|\overline{\rho}^{\frac12}\sigma(v^{(n)})\|_{L^2}^2.
\end{aligned}
\tag{4.15}
$$

Integrating (4.15) in $t\in[0,T]$ and taking T small enough, we have

$$
\begin{aligned}
&\sup_{t\in[0,T]}\|\overline{\rho}^{\frac12}\sigma(v^{(n)}(t))\|_{L^2}^2+\frac{c_0}{2}\int_0^T\|\sigma(v^{(n)}(t))\|_{H^1}^2dt\\
&\le\|\overline{\rho}^{\frac12}\sigma(v^{(n)}(0))\|_{L^2}^2+CT\|\nabla\sigma(v^{(n-1)})\|_{L^2_TL^2}^2(T+\int_0^T\mathfrak{D}(v^{(n)})^2dt),
\end{aligned}
\tag{4.16}
$$

and then

$$
\begin{aligned}
&\sup_{t\in[0,T]}\|\overline{\rho}^{\frac12}\sigma(v^{(n)}(t))\|_{L^2}^2+\frac{c_0}{2}\int_0^T\|\sigma(v^{(n)}(t))\|_{H^1}^2dt\\
&\quad\le CT(T+M)\|\nabla\sigma(v^{(n-1)}\|_{L^2_TL^2}^2\le CT\|\nabla\sigma(v^{(n-1)})\|_{L^2_TL^2}^2.
\end{aligned}
$$

By now, we get that when T takes small enough, then we get

$$
\begin{aligned}
&\sup_{t\in[0,T]}\|\overline{\rho}^{\frac12}\sigma(v^{(n)}(t))\|_{L^2}^2+\|\sigma(v^{(n)}(t))\|_{L^2_TH^1}^2\\
&\quad\le\frac12(\sup_{t\in[0,T]}\|\overline{\rho}^{\frac12}\sigma(v^{(n-1)}(t))\|_{L^2}^2+\|\sigma(v^{(n-1)}(t))\|_{L^2_TH^1}^2),
\end{aligned}
$$

which completes this Lemma. □

4.4 Proof of Theorem 1.2.

From Lemma 4.4, we know $\{v^{(n)}\}_{n=1}^\infty$ is Cauchy sequence in the space $\widetilde{Y}_T$. So as $n\to\infty$,

$$
\begin{cases}
\overline{\rho}^{\frac12}v^{(n)}\to\overline{\rho}^{\frac12}v \quad \text{in} \quad C([0,T],L^2),\\
v^{(n)}\to v \quad \text{in} \quad L^2([0,T],H^1).
\end{cases}
\tag{4.17}
$$

Due to Lemma 4.2 that $\|v^{(n)}\|_{Y_T}^2\le M$ uniformly in $n\ge1$, sequence $\{v^{(n)}\}_{n=1}^\infty$ have weakly convergent subsequence. Along with strong convergence (4.17), we infer that as $n\to0$

$$
\begin{cases}
v^{(n)}\rightharpoonup^* v \quad \text{in} \quad L^\infty([0,T],X^{12}_{\frac12}),\\
v^{(n)}\rightharpoonup v, \quad \nabla v^{(n)}\rightharpoonup\nabla v \quad \text{in} \quad L^2([0,T],X^{12}),\\
\nabla v^{(n)}\rightharpoonup^*\nabla v \quad \text{in} \quad L^\infty([0,T],L^2),\\
\overline{\rho}^{\frac12}\partial_t v^{(n)}\rightharpoonup\overline{\rho}^{\frac12}\partial_t v \quad \text{in} \quad L^2([0,T],L^2).
\end{cases}
$$

So the function v satisfies equation (1.18) in weak sense. On the other hand, lower semicontinuity gives bound $\|v\|^2_{Y_T} \le 2M$, and then (1.25) holds. As a result, thanks to Aubin–Lions's lemma [38], we get that $(v, \eta) \in C([0,T]; X^{12}_{\frac{1}{2}} \cap H^1(\Omega)) \times C([0,T]; \mathcal{F}_\kappa(\Omega))$ by using a standard procedure (cf. the proof of Theorem 3.5 in [33]), which is a strong solution to (1.18). The uniqueness comes from L^2 energy estimates with zero initial data. More precise, let (ξ_1, v_1) and (ξ_2, v_2) are solutions to (1.18) with same initial data. The same process in Lemma 4.4 deduce that

$$\|v_1 - v_2\|^2_{\tilde{Y}_T} \le \frac{1}{2}\|v_1 - v_2\|^2_{\tilde{Y}_T},$$

which implies $v_1 = v_2$ and then $\xi_1 = \xi_2$ on the time interval $[0, T]$. Furthermore, applying (4.16) to the system (1.18), we may readily prove that the solution $(v, \eta) \in C([0,T]; X^{12}_{\frac{1}{2}} \cap H^1(\Omega)) \times C([0,T]; \mathcal{F}_\kappa(\Omega))$ depends continuously on the initial data $(v_0, \eta_0) \in (X^{12}_{\frac{1}{2}} \cap H^1(\Omega)) \times \mathcal{F}_\kappa(\Omega)$. This finishes the proof of Theorem 1.2. □

Acknowledgements G. Gui is partially supported by the National Natural Science Foundation of China under Grants 11571279 and 11931013. C. Wang is partially supported by NSF of China under Grant 11701016. Y. Wang is partially supported by China Postdoctoral Science Foundation 8206200009.

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