Calc. Var. (2019) 58:152
https://doi.org/10.1007/s00526-019-1600-6

Calculus of Variations

Check for updates

Geometric properties of harmonic mappings in stable Riemannian domains

Friedrich Sauvigny[1]

Received: 29 March 2019 / Accepted: 30 June 2019 / Published online: 27 July 2019

Abstract

H. Kneser (Jahresber Dt Math Vereinigung 35:123–124, 1926) showed by an ingenious method that plane harmonic mappings on the unit disc B, which attribute the circumference ∂B in a topological way to a convex curve Γ, necessarily yield a diffeomorphism of B onto the interior G of the contour Γ and a homeomorphism between their closures. E. Heinz has generalized this method to solutions of nonlinear elliptic systems [see Chap. 13, Sect. 6 of Sauvigny (Partial differential equations. 1. Foundations and integral representations; 2. Functional analytic methods; with consideration of lectures by E. Heinz. Springer, London, 2012], however, this reasoning is restricted to the local situation and requires Lipschitz conditions for certain linear combinations of their coefficient functions. These Lewy-Heinz-systems comprise the equations for harmonic mappings with respect to a Riemannian metric and were utilized by Jost (J Reine Angew Math 342:141–153, 1981) to prove univalency for harmonic mappings between Riemannian surfaces. A global result is achieved by reconstruction of the solution for the Dirichlet problem, since this problem is uniquely determined by the uniqueness result of Jäger and Kaul (Manuscr Math 28:269–291, 1979). Here we shall adapt the original method of H. Kneser for harmonic mappings with respect to Riemannian metrics in order to receive harmonic diffeomorphisms from B onto stable Riemannian domains Ω. We construct a global nonlinear auxiliary function associated with an embedding into a field of geodesics. In the special case of planar harmonic mappings under semi-free boundary conditions, this procedure already appears in Proposition 3 of Hildebrandt and Sauvigny (J Reine Angew Math 422:69–89, 1991). By our present method to show univalency and to obtain a diffeomorphism between the domains, we can dispense of the uniqueness for the associate Dirichlet problem. The crucial idea consists of the notion *stable Riemannian domains* Ω, which possess a family of non-intersecting geodesic rays emanating from each boundary point and furnish a simple covering of the whole domain. Furthermore, we establish a convex hull property for harmonic mappings within Ω. On the basis of investigations by Hildebrandt et al. (Acta Math 138:1–16, 1977), we construct harmonic embeddings within the hemisphere by direct variational methods.

Communicated by J. Jost.

✉ Friedrich Sauvigny
sauvigny@b-tu.de

1 Fachgebiet Analysis im Institut für Mathematik, Brandenburgische Technische Universität Cottbus – Senftenberg, Platz der Deutschen Einheit 1, 03046 Cottbus, Germany

Mathematics Subject Classification 35J60 · 53C21

1 The Dirichlet problem of harmonic mappings

For the coordinates (x^1, x^2) we define the domain

$$\Omega_1 := \{X = (x^1, x^2) \in \mathbb{R}^2 \, : \, |X| < 1\}$$

and introduce the unit disc

$$B := \{w = u + iv \in \mathbb{C} \, : \, |w| < 1\}$$

with the parameters $u + iv \cong (u, v)$. Now we prescribe the Riemannian metric

$$\begin{aligned} ds^2 &= \sum_{j,k=1,2} g_{jk}(x^1, x^2)\, dx^j\, dx^k \\ &= g_{11}(x^1, x^2)\,(dx^1)^2 + 2g_{12}(x^1, x^2)\, dx^1\, dx^2 + g_{22}(x^1, x^2)\,(dx^2)^2 \end{aligned} \tag{1.1}$$

on the disc Ω_1. Here we require our coefficients to satisfy

$$\begin{aligned} g_{jk} = g_{jk}(x^1, x^2) \in C^{1+\alpha}(\overline{\Omega}_1, \mathbb{R}) \quad &\text{for} \quad j, k = 1, 2, \\ g_{12}(x^1, x^2) = g_{21}(x^1, x^2) \quad &\text{in} \quad \overline{\Omega}_1, \end{aligned} \tag{1.2}$$

$$\begin{aligned} \lambda\, |\xi|^2 \le \sum_{j,k=1,2} g_{jk}(x^1, x^2)\xi^j \xi^k \le \frac{1}{\lambda}\, |\xi|^2 \\ \text{for all} \quad \xi = (\xi^1, \xi^2) \in \mathbb{R}^2 \quad \text{and} \quad (x^1, x^2) \in \overline{\Omega}_1, \end{aligned} \tag{1.3}$$

with the Hölder constant $\alpha \in (0, 1)$ and the quantity $\lambda \in (0, 1]$.

By a continuity method the following profound result is established:

Theorem 1 (Conformal mappings w. r. t. Riemannian metrics) *For the Riemannian metric (1.1), (1.2), (1.3) there exists a* $C^{2+\alpha}(\overline{B}, \overline{\Omega}_1)$*-diffeomorphic, positive-oriented mapping*

$$X = X(u, v) = (x^1(u, v), x^2(u, v)) : \overline{B} \to \overline{\Omega}_1 \in C^{2+\alpha}(\overline{B}, \overline{\Omega}_1)$$

satisfying the weighted conformality relations

$$\begin{aligned} &\sum_{j,k=1,2} x_u^j(u, v) g_{jk}(x^1(u, v), x^2(u, v)) x_v^k(u, v) = 0 \\ &\sum_{j,k=1,2} x_u^j(u, v) g_{jk}(x^1, x^2) x_u^k(u, v) = \sum_{j,k=1,2} x_v^j(u, v) g_{jk}(x^1, x^2) x_v^k(u, v) \text{ in } B. \end{aligned} \tag{1.4}$$

Proof See our uniformization theorem from [12] Chap. 12 the Theorem 8.2. □

Due to Proposition 7.1 of [12] Chap. 12, the function X then satisfies the nonlinear elliptic system

$$\Delta x^l + \sum_{j,k=1,2} \Gamma^l_{jk}(x_u^j x_u^k + x_v^j x_v^k) = 0 \quad \text{in} \quad B \quad \text{for} \quad l = 1, 2. \tag{1.5}$$

Here we use the Christoffel symbols

$$\Gamma^l_{jk} := \frac{1}{2} \sum_{i=1,2} g^{li}(g_{ki,x^j} + g_{ij,x^k} - g_{jk,x^i}), \quad j,k,l = 1,2 \tag{1.6}$$

with the inverse matrix $(g^{jk})_{j,k=1,2} := (g_{jk})^{-1}_{j,k=1,2}$. Therefore, X represents a one-to-one harmonic mapping of the disc $\{B, (\delta_{jk})\}$ with the Euclidean metric $(\delta_{jk})_{j,k=1,2}$ onto the disc $\{\Omega_1, (g_{jk})\}$. On account of well-known regularity results, the associate boundary function

$$\Phi(u,v) := X(u,v), \quad (u,v) \in \partial B \quad \text{with} \quad \Phi\colon \partial B \to \partial\Omega_1 \in C^{2+\alpha}(\partial B, \partial\Omega_1) \tag{1.7}$$

appearing within this approximation and selection procedure, yields a positive-oriented $C^{2+\alpha}(\partial B, \partial\Omega_1)$-diffeomorphism between the circumferences ∂B and $\partial\Omega_1$. This weighted-conformal mapping is uniquely determined by a three-point-condition on the boundary. Of course, this boundary representation optimally appears for these weighted-conformal mappings and cannot be prescribed!

Remark 1 Starting with an analogous result to Theorem 1 above, Jost [7] has constructed harmonic diffeomorphisms, for arbitrary convex boundary data, by deformation of the boundary values via a topological method. This has been combined with a priori estimates for their Jacobian by E. Heinz. With the aid of the maximum principle by Jäger and Kaul [6], then Jost obtained the diffeomorphic character of harmonic maps by reconstruction.

In Sect. 4 we shall see directly the one-to-one character of our harmonic maps, established in Theorem 2 below, and may dispense of the uniqueness for the associate Dirichlet problem. Here we prescribe the Riemannian metric (1.1) on the whole plane $\mathbb{R}^2$, which is Euclidean outside of the disc

$$\Omega_M := \{X = (x^1, x^2) \in \mathbb{R}^2 : |X| < M\}$$

of a fixed radius $0 < M < +\infty$. More precisely, we assume that our coefficients satisfy the following conditions with the Hölder constant $\alpha \in (0,1)$ and a positive number $\lambda \in (0,1]$ as follows:

$$\begin{aligned} & g_{jk} = g_{jk}(x^1,x^2) \in C^{1+\alpha}(\mathbb{R}^2,\mathbb{R}) \quad \text{for} \quad j,k = 1,2, \\ & g_{12}(x^1,x^2) = g_{21}(x^1,x^2) \quad \text{in} \quad \mathbb{R}^2, \\ & g_{jk}(x^1,x^2) = \delta_{jk} \quad \text{in} \quad \mathbb{R}^2 \backslash \Omega_M \quad \text{for} \quad j,k = 1,2, \end{aligned} \tag{1.8}$$

and

$$\begin{aligned} & \lambda\,|\xi|^2 \le \sum_{j,k=1,2} g_{jk}(x^1,x^2)\xi^j\xi^k \le \frac{1}{\lambda}\,|\xi|^2 \\ & \text{for all} \quad \xi = (\xi^1,\xi^2) \in \mathbb{R}^2 \quad \text{and} \quad (x^1,x^2) \in \mathbb{R}^2. \end{aligned} \tag{1.9}$$

Furthermore, we require *that the metric ds^2 possesses a moderate deviation in the disc Ω_M from the Euclidean metric with the constant $a \in (0, \frac{1}{2M})$ in the following sense: The associate Christoffel symbols* (1.6) *satisfy the estimate*

$$\begin{aligned} & \sqrt{\Big(\sum_{j,k=1,2} \Gamma^1_{jk}\xi^j\xi^k\Big)^2 + \Big(\sum_{j,k=1,2} \Gamma^2_{jk}\xi^j\xi^k\Big)^2} \le a|\xi|^2 \\ & \text{for all} \quad \xi = (\xi^1,\xi^2) \in \mathbb{R}^2 \quad \text{and} \quad (x^1,x^2) \in \mathbb{R}^2. \end{aligned} \tag{1.10}$$

By the Leray–Schauder degree of mapping we can establish the following

Theorem 2 (Dirichlet problem for moderate harmonic mappings) *Let the Riemannian metric* (1.1)*,* (1.8)*,* (1.9) *be given with a moderate deviation* (1.10) *by the constant* $a \in (0, \frac{1}{2M})$ *from the Euclidean metric. For each boundary function*

$$\Phi \in C^{2+\alpha}(\partial B, \mathbb{R}^2) \quad with \quad |\Phi(u, v)| \le M, \ \forall\, (u, v) \in \partial B$$

there exists a solution

$$\begin{aligned} X = X(u, v) = (x^1(u, v), x^2(u, v)) : \overline{B} \to \mathbb{R}^2 \in C^{2+\alpha}(\overline{B}, \mathbb{R}^2) \\ with \quad |X(u, v)| \le M \quad for\ all \quad (u, v) \in \overline{B} \end{aligned} \tag{1.11}$$

for the system (1.5)*,* (1.6) *of harmonic mappings under the boundary condition*

$$X(u, v) = \Phi(u, v) \quad for\ all \quad (u, v) \in \partial B. \tag{1.12}$$

Proof From Theorem 4.4 of [12] Chap. 12 we deduce the existence of a harmonic mapping (1.11) under the boundary conditions (1.12). □

Remark 2 Due to the geometric maximum principle by E. Heinz (see Theorem 1.4 in [12] Chap. 12), the solution X of Theorem 2 is subject to the inequality

$$\sup_{(u,v)\in\overline{B}} |X(u, v)| \le \sup_{(u,v)\in\partial B} |X(u, v)|. \tag{1.13}$$

When the boundary values satisfy $|\Phi(u, v)| < M, \ \forall (u, v) \in \partial B$, then the estimate

$$\sup_{(u,v)\in\overline{B}} |X(u, v)| < M \tag{1.14}$$

follows, and $X : \overline{B} \to \Omega_M$ represents an *inner solution* of the system (1.5), (1.6), briefly an *inner harmonic mapping*.

2 Geodesically stable Riemannian domains

We begin our considerations with the central

Definition 1 We call the disc Ω_M of radius $0 < M < +\infty$ endowed with a Riemannian metric (1.1), (1.8), (1.9) *a geodesically stable Riemannian domain* or simply *a stable Riemannian domain*, if each geodesic—in unit velocity—emanating from an arbitrary boundary point $X_0 \in \partial\Omega_M$ into an interior direction $\xi \in S^1$—within the disc Ω_M -

$$\Big\{ Y(t) = Y(t; \xi, X_0) \in \overline{\Omega_M}, \quad 0 \le t \le \tau(X_0, \xi) \Big\} \tag{2.1}$$

of the lenght $\tau(X_0, \xi) > 0$ does not contain conjugate points.

Remark 3 Let the Riemannian metric (1.1), (1.8), (1.9) of the regularity class C^3 be given, such that their Gaussian curvature K satisfies

$$K(x^1, x^2) \le \kappa, \quad \forall\, (x^1, x^2) \in \mathbb{R}^2 \tag{2.2}$$

with the barrier $\kappa \in [0, +\infty)$. Furthermore, let the diameter of the Riemannian domain be bounded by the constant $\frac{\pi}{\sqrt{\kappa}} \in (0, +\infty]$ as follows:

$$\tau(X_0, \xi) < \frac{\pi}{\sqrt{\kappa}} \quad \text{for all} \quad X_0 \in \partial\Omega_M \quad \text{and every interior direction} \quad \xi \in S^1. \tag{2.3}$$

Then this Riemannian domain is necessarily stable.

In this context, we refer our readers to the comparison theorem of J. C. F. Sturm in Satz 3 of Kapitel VII, § 7 from our treatise *Analysis* [13].

Of central importance is the subsequent

Lemma 1 (Geodesic central fields) *Let the domain Ω_M be endowed with a stable Riemannian metric from Definition 1, and a boundary point $X_0 \in \partial\Omega_M$ be chosen arbitrarily. Then the family of geodesics*

$$\mathbf{Y}(t, s) = \mathbf{Y}(t, s; X_0), \quad 0 < t \leq \tau(X_0, s), \quad -\frac{\pi}{2} < s < +\frac{\pi}{2} \tag{2.4}$$

with their initial position

$$\mathbf{Y}(0+, s; X_0) = X_0, \quad -\frac{\pi}{2} < s < +\frac{\pi}{2} \tag{2.5}$$

and their initial velocity

$$\mathbf{Y}_t(0+, s; X_0) = -\exp(is) \cdot |X_0|^{-1} X_0, \quad -\frac{\pi}{2} < s < +\frac{\pi}{2} \tag{2.6}$$

yields a simple covering of the pointed disc $\overline{\Omega_M} \backslash \{X_0\}$.

Proof For arbitrary $-\frac{\pi}{2} < s_1 < s_2 < +\frac{\pi}{2}$ we consider the geodesic

$$\mathbf{Y}(t, s_1) = \mathbf{Y}(t, s_1; X_0), \quad 0 \leq t \leq \tau(X_0, s_1), \tag{2.7}$$

the circular arc

$$\mathbf{Z}(s; X_0) = \mathbf{Y}(\tau(X_0, s), s; X_0), \quad s_1 \leq s \leq s_2, \tag{2.8}$$

and the geodesic

$$\mathbf{Y}(t, s_2) = \mathbf{Y}(\tau(X_0, s_2) - t, s_2; X_0), \quad 0 \leq t \leq \tau(X_0, s_2). \tag{2.9}$$

Since our geodesics do not contain conjugate points within $\overline{\Omega_M}$, the arcs (2.7) and (2.8) and (2.9) consecutively constitute a Jordan contour $\Gamma(s_1, s_2; X_0)$ for parameters $s_1 < s_2$ chosen sufficiently near. They form a Jordan curve $\Gamma(s_1, s_2; X_0)$ for arbitrary parameters $-\frac{\pi}{2} < s_1 < s_2 < +\frac{\pi}{2}$ as well, since the mapping

$$\mathbf{Z}(s; X_0) = \mathbf{Y}(\tau(X_0, s), s; X_0), \quad -\frac{\pi}{2} < s < -\frac{\pi}{2} \tag{2.10}$$

is strictly monotonic. Therefore, the interior of the Jordan curves $\Gamma(s_1, s_2; X_0)$ exhausts the domain Ω_M for $s_1 \to -\frac{\pi}{2}+$ and $s_2 \to +\frac{\pi}{2}-$. These contours $\Gamma(s_1, s_2; X_0)$ cover $\partial\Omega_M$ in the limit $s_1 = -\frac{\pi}{2}$ and $s_2 = +\frac{\pi}{2}$, where the singularity X_0 remains fixed. □

As in [13] Kap. VII, Sect. 5 we introduce

Definition 2 The *Riemannian inner product* of the planar vector fields $Y(t) = (y_1(t), y_2(t))$ and $Z(t) = (z_1(t), z_2(t))$ along the plane curve $X(t) = (x_1(t), x_2(t))$ with the parameter $a < t < b$ is determined as follows:

$$\Big[Y(t), Z(t)\Big]_{X(t)} := \sum_{j,k=1,2} g_{jk}(X(t))y_j(t)z_k(t), \quad a < t < b. \tag{2.11}$$

Remark 4 From the Gauß–Riemann-Lemma (see Satz 2 in [13] Kap. VII, Sect. 4), we realize the following identities for our geodesic central field in Lemma 1 above:

$$\Big[\mathbf{Y}_t(t,s), \mathbf{Y}_t(t,s)\Big]_{\mathbf{Y}(t,s)} = 1, \quad G(t,s) := \Big[\mathbf{Y}_s(t,s), \mathbf{Y}_s(t,s)\Big]_{\mathbf{Y}(t,s)} > 0,$$
$$\Big[\mathbf{Y}_t(t,s), \mathbf{Y}_s(t,s)\Big]_{\mathbf{Y}(t,s)} = 0 \;\; ; \;\; 0 < t \le \tau(X_0, s), \quad -\frac{\pi}{2} < s < +\frac{\pi}{2}. \tag{2.12}$$

Definition 3 Let the stable Riemannian domain Ω_M of Definition 1 with the geodesic central fields (2.4) of Lemma 1 be given. For all points $X \in \Omega_M$ with their unique representation

$$X = \mathbf{Y}(t,s) = \mathbf{Y}(t,s;X_0), \quad 0 < t < \tau(X_0, s), \quad -\frac{\pi}{2} < s < +\frac{\pi}{2} \tag{2.13}$$

we define the *lifted vector fields*

$$\widehat{\mathbf{Y}_t}(x^1, x^2; X_0) = \widehat{\mathbf{Y}_t}(X; X_0) := \mathbf{Y}_t(t,s), \quad X = (x^1, x^2) \in \Omega_M,$$
$$\widehat{\mathbf{Y}_s}(x^1, x^2; X_0) = \widehat{\mathbf{Y}_s}(X; X_0) := \mathbf{Y}_s(t,s), \quad X = (x^1, x^2) \in \Omega_M \tag{2.14}$$

and the *lifted Gaussian fundamental coefficient*

$$\widehat{G}(x^1, x^2; X_0) = \widehat{G}(X; X_0) := G(t,s), \quad X = (x^1, x^2) \in \Omega_M. \tag{2.15}$$

We obtain with

$$\left\{\widehat{\mathbf{Y}_t}(X; X_0), \frac{\widehat{\mathbf{Y}_s}(X; X_0)}{\sqrt{\widehat{G}(X; X_0)}}\right\}, \quad X = (x^1, x^2) \in \Omega_M \tag{2.16}$$

the *Gaussian geodesic frame*, which constitutes an orthonormal, positive-oriented system of vectors—with respect to the Riemannian inner product (2.11)—on account of Remark 4 above.

In stable Riemannian domains, we can conveniently characterize the convex hull of arbitrary compact sets $F \subset \Omega_M$ with the subsequent

Definition 4 Let the stable Riemannian domain Ω_M of Definition 1 with the geodesic central fields (2.4) of Lemma 1 be given. For all $-\frac{\pi}{2} < s_0 < +\frac{\pi}{2}$ we introduce the *geodesic region*

$$\Theta(s_0; X_0) := \left\{\mathbf{Y}(t,s;X_0) \in \Omega_M \,\middle|\, 0 < t < \tau(X_0, s), \quad -\frac{\pi}{2} < s \le s_0\right\}. \tag{2.17}$$

This region is closed by the geodesic arc

$$\mathbf{Y}(\tau(X_0, s_0) - t, \; s_0) = \mathbf{Y}(\tau(X_0, s_0) - t, s_0; X_0), \quad 0 \le t \le \tau(X_0, s_0) \tag{2.18}$$

and furthermore bounded by the circular arc

$$\mathbf{Z}(s; X_0) = \mathbf{Y}(\tau(X_0, s), s; X_0), \quad -\frac{\pi}{2} < s < s_0, \tag{2.19}$$

where these curves constitute with $\partial\Theta(s_0; X_0)$ a positive-oriented Jordan contour. For an arbitrary compact set $F \subset \Omega_M$, we define the *convex hull $\mathcal{H}(F)$ of F within the stable Riemannian domain Ω_M* as follows:

$$\mathcal{H}(F) := \bigcap \left\{ \Theta(s_0; X_0) \middle| X_0 \in \partial\Omega_M,\ s_0 \in (-\frac{\pi}{2}, +\frac{\pi}{2}) :\ F \subset \Theta(s_0; X_0) \right\}. \quad (2.20)$$

Finally, we introduce the important geodesic function with

Definition 5 Let the stable Riemannian domain Ω_M of Definition 1 with the geodesic central fields (2.4) of Lemma 1 be given. For all points $X \in \overline{\Omega_M} \backslash \{X_0\}$ with their unique representation

$$X = \mathbf{Y}(t, s) = \mathbf{Y}(t, s; X_0), \quad 0 < t \leq \tau(X_0, s), \quad -\frac{\pi}{2} < s < +\frac{\pi}{2} \quad (2.21)$$

we define *the geodesic function*

$$\Psi(x^1, x^2; X_0) = \Psi(X; X_0) := s \in (-\frac{\pi}{2}, +\frac{\pi}{2}), \quad X = (x^1, x^2) \in \overline{\Omega_M} \backslash \{X_0\}. \quad (2.22)$$

Remark 5 Obviously, the equation $\Psi(X; X_0) = s_0,\ X \in \Omega_M$ describes the geodesic

$$\mathbf{Y}(t, s_0; X_0), \quad 0 < t < \tau(X_0, s_0)$$

for all $-\frac{\pi}{2} < s_0 < +\frac{\pi}{2}$, and the following characterization is valid:

$$\Theta(s_0; X_0) = \left\{ X \in \Omega_M \middle| \Psi(X; X_0) \leq s_0 \right\}. \quad (2.23)$$

3 Pseudoharmonic nonlinear combination for harmonic mappings

We refer to the covariant differentiation $\frac{\nabla}{dt}$ from [13] Kap. VII, § 5 and begin with

Definition 6 Let us consider the Riemannian metric (1.1), (1.8), (1.9) with its inverse tensor

$$g^{ij} = g^{ij}(x^1, x^2) \in C^{1+\alpha}(\mathbb{R}^2, \mathbb{R}) \quad \text{for} \quad i, j = 1, 2 \quad \text{satisfying}$$
$$\sum_{j=1,2} g_{ij}(x^1, x^2)\, g^{jk}(x^1, x^2) = \delta_{ik}, \quad (x^1, x^2) \in \mathbb{R}^2 \quad \text{for} \quad i, k = 1, 2. \quad (3.1)$$

Now we define the *cogradient of the function* $\Psi(\,.\,, X_0)$ from Definition 5 as follows

$$\nabla\Psi(X; X_0) := \Big(\sum_{j=1,2} g^{ij}(X) \Psi_{x^j}(X; X_0) \Big)_{i=1,2} =: \Big(f^i(X) \Big)_{i=1,2}$$
$$\text{for all} \quad X = (x^1, x^2) \in \Omega_M, \quad (3.2)$$

where the boundary point $X_0 \in \partial\Omega_M$ is arbitrary.

Remark 6 Differentiation of the identity (3.1) yields

$$-\sum_{j=1,2} \frac{\partial g_{ij}(x^1, x^2)}{\partial x^l}\, g^{jk}(x^1, x^2) = \sum_{j=1,2} g_{ij}(x^1, x^2)\, \frac{\partial g^{jk}(x^1, x^2)}{\partial x^l},$$
$$\text{for all} \quad (x^1, x^2) \in \mathbb{R}^2 \quad \text{and} \quad i, k, l = 1, 2. \quad (3.3)$$

When an arbitrary mapping

$$X(u,v) = (x^1(u,v), x^2(u,v)) \colon B \to \Omega_M \in C^2(B, \mathbb{R}^2) \tag{3.4}$$

is given, we consider the associate auxiliary function

$$\psi(u,v) := \Psi(X(u,v); X_0) = \Psi(x^1(u,v), x^2(u,v); X_0), \quad (u,v) \in B. \tag{3.5}$$

We immediately comprehend the following *covariant chain rule*

$$\begin{aligned} \psi_u(u,v) &= \Big[\nabla\Psi(X(u,v); X_0),\ X_u(u,v)\Big]_{X(u,v)}, \quad (u,v) \in B, \\ \psi_v(u,v) &= \Big[\nabla\Psi(X(u,v); X_0),\ X_v(u,v)\Big]_{X(u,v)}, \quad (u,v) \in B. \end{aligned} \tag{3.6}$$

With the aid of the covariant product rule (see Satz 2 in [13] Kap. VII, § 5) we calculate

$$\begin{aligned} \psi_{uu}(u,v) = &\Big[\frac{\nabla\,\nabla\Psi(X(u,v); X_0)}{du},\ X_u(u,v)\Big]_{X(u,v)} \\ &+\Big[\nabla\Psi(X(u,v); X_0),\ \frac{\nabla\, X_u(u,v)}{du}\Big]_{X(u,v)}, \quad (u,v) \in B. \end{aligned} \tag{3.7}$$

Here $\dfrac{\nabla}{du}$ denotes the covariant derivative of the respective vector field along the curve $X(.,v)$ due to Definition 1 in [13] Kap. VII, § 5. In order to evaluate the first bracket term in (3.7), we determine the partial derivative

$$\begin{aligned} \frac{d}{du}\Big(f^i(X(u,v))\Big)_{i=1,2} &= \frac{d}{du}\nabla\Psi(X(u,v); X_0) \\ &= \frac{d}{du}\Big(\sum_{j=1,2} g^{ij}(x^1(u,v), x^2(u,v))\Psi_{x^j}(x^1(u,v), x^2(u,v); X_0)\Big)_{i=1,2} \\ &= \Big(\sum_{j,k=1,2} g^{ij}(x^1(u,v), x^2(u,v))\Psi_{x^j x^k}(x^1(u,v), x^2(u,v); X_0)x_u^k\Big)_{i=1,2} \\ &\quad +\Big(\sum_{j,k=1,2} \frac{\partial g^{ij}(x^1(u,v), x^2(u,v))}{\partial x^k}\Psi_{x^j}(x^1(u,v), x^2(u,v); X_0)x_u^k\Big)_{i=1,2} \\ &\quad \text{for all} \quad (u,v) \in B. \end{aligned} \tag{3.8}$$

Now we utilize the *Christoffel symbols of the first kind* (see (1.6) and the formula (10) in [13] Kap. VII, § 3):

$$\begin{aligned} \gamma_{mjk} &:= \frac{1}{2}(g_{km,x^j} + g_{mj,x^k} - g_{jk,x^m}) = \frac{1}{2}\sum_{i,l=1,2} g_{ml}g^{li}(g_{ki,x^j} + g_{ij,x^k} - g_{jk,x^i}) \\ &= \sum_{l=1,2} g_{ml}\Gamma^l_{jk} \quad \text{for} \quad j,k,m = 1,2. \end{aligned} \tag{3.9}$$

With the aid of the identities (3.8) and (3.9) and the Remark 6, we determine the first bracket term in (3.7):

$$\begin{aligned} &\Big[\frac{\nabla\,\nabla\Psi(X(u,v); X_0)}{du},\ X_u(u,v)\Big]_{X(u,v)} \\ &= \Big[\frac{\nabla\Big(f^i(X(u,v))\Big)_{i=1,2}}{du},\ \Big(x_u^l(u,v)\Big)_{l=1,2}\Big]_{X(u,v)} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{l,k=1,2} \Psi_{x^l x^k}(x^1(u,v), x^2(u,v); X_0) x_u^l x_u^k \\
&\quad + \sum_{i,j,k,l=1,2} g_{li}(X(u,v)) \frac{\partial g^{ij}(X(u,v))}{\partial x^k} \Psi_{x^j}(X(u,v); X_0) x_u^l x_u^k \\
&\quad + \sum_{l,j,k=1,2} \gamma_{ljk}(X(u,v)) x_u^l \, f^j(X(u,v)) \, x_u^k \\
&= \sum_{l,k=1,2} \Psi_{x^l x^k}(x^1(u,v), x^2(u,v); X_0) x_u^l x_u^k \\
&\quad - \sum_{i,j,k,l=1,2} g_{li,x^k}(X(u,v)) g^{ij}(X(u,v)) \Psi_{x^j}(X(u,v); X_0) x_u^l x_u^k \\
&\quad + \sum_{l,j,k=1,2} \gamma_{ljk}(X(u,v)) x_u^l \, f^j(X(u,v)) \, x_u^k \\
&= \sum_{l,k=1,2} \Psi_{x^l x^k}(x^1(u,v), x^2(u,v); X_0) x_u^l x_u^k \\
&\quad - \sum_{i,k,l=1,2} g_{li,x^k}(X(u,v)) f^i(X(u,v)) x_u^l x_u^k \\
&\quad + \sum_{l,i,k=1,2} \gamma_{lik}(X(u,v)) x_u^l \, f^i(X(u,v)) \, x_u^k \\
&= \sum_{l,k=1,2} \Psi_{x^l x^k}(x^1(u,v), x^2(u,v); X_0) x_u^l x_u^k \\
&\quad + \sum_{l,i,k=1,2} \widetilde{\gamma_{lik}}(X(u,v)) x_u^l \, f^i(X(u,v)) \, x_u^k, \quad (u,v) \in B.
\end{aligned} \tag{3.10}
$$

Here we use the *modified Christoffel symbols of the first kind*

$$
\widetilde{\gamma_{lik}} := \frac{1}{2}(g_{kl,x^i} - g_{li,x^k} - g_{ik,x^l}) = -\frac{1}{2}(g_{li,x^k} + g_{ik,x^l} - g_{kl,x^i}) = -\gamma_{ikl} \quad \text{for} \quad i,k,l = 1,2. \tag{3.11}
$$

Definition 7 We define the covariant Hessian bilinear form

$$
\begin{aligned}
&\Big[X_u(u,v), \nabla^2 \Psi(X(u,v); X_0), \ X_u(u,v)\Big]_{X(u,v)} \\
&:= \sum_{l,k=1,2} \Psi_{x^l x^k}(x^1(u,v), x^2(u,v); X_0) x_u^l x_u^k \\
&\quad - \sum_{i,k,l=1,2} \gamma_{ikl}(X(u,v)) \, f^i(X(u,v)) \, x_u^k \, x_u^l, \quad (u,v) \in B.
\end{aligned} \tag{3.12}
$$

The combination of (3.7) and (3.10) – (3.12) yields the identity

$$
\begin{aligned}
\psi_{uu}(u,v) &= \Big[X_u(u,v), \nabla^2 \Psi(X(u,v); X_0), \ X_u(u,v)\Big]_{X(u,v)} \\
&\quad + \Big[\nabla \Psi(X(u,v); X_0), \ \frac{\nabla \, X_u(u,v)}{du}\Big]_{X(u,v)}, \quad (u,v) \in B.
\end{aligned} \tag{3.13}
$$

Analogously, we derive the identity

$$\psi_{vv}(u,v) = \Big[X_v(u,v), \nabla^2\,\Psi(X(u,v);X_0),\ X_v(u,v)\Big]_{X(u,v)}$$
$$+\Big[\nabla\Psi(X(u,v);X_0),\ \frac{\nabla\, X_v(u,v)}{dv}\Big]_{X(u,v)}, \quad (u,v)\in B. \tag{3.14}$$

With the aid of Lemma 1 we shall see that the bilinear form in Definition 7 vanishes at each point on an appropriate one-dimensional space. More precisely, we have the

Lemma 2 (Covariant derivatives of the geodesic function)
For the geodesic function Ψ in Definition 5 the cogradient satisfies the equations

$$\Big[\nabla\Psi(\mathbf{Y}(t,s;X_0);X_0), \mathbf{Y}_t(t,s;X_0)\Big]_{\mathbf{Y}(t,s;X_0)} = 0 \quad \textit{and}$$
$$\Big[\nabla\Psi(\mathbf{Y}(t,s;X_0);X_0), \mathbf{Y}_s(t,s;X_0)\Big]_{\mathbf{Y}(t,s;X_0)} = 1$$
$$\textit{for all}\quad 0<t<\tau(X_0,s), \quad -\frac{\pi}{2}<s<+\frac{\pi}{2}. \tag{3.15}$$

Moreover, the identity

$$\nabla\Psi(X;X_0) = \frac{\widehat{\mathbf{Y}_s}(X;X_0)}{\widehat{G}(X;X_0)}, \quad X\in\Omega_M \tag{3.16}$$

holds true. Finally, the covariant Hessian form (3.12) of the second derivatives vanishes as follows:

$$\Big[\mathbf{Y}_t(t,s;X_0), \nabla^2\,\Psi(\mathbf{Y}(t,s;X_0);X_0),\ \mathbf{Y}_t(t,s;X_0)\Big]_{\mathbf{Y}(t,s;X_0)} = 0$$
$$\textit{for all}\quad 0<t<\tau(X_0,s), \quad -\frac{\pi}{2}<s<+\frac{\pi}{2}. \tag{3.17}$$

Here the boundary point $X_0\in\partial\Omega_M$ is chosen arbitrarily.

Proof 1. We consider the auxiliary function

$$\psi(t,s) := \Psi(\mathbf{Y}(t,s;X_0);X_0) = s, \quad 0<t<\tau(X_0,s),\ -\frac{\pi}{2}<s<+\frac{\pi}{2}. \tag{3.18}$$

With the covariant chain rule (3.6) we determine the derivatives

$$0 = \psi_t(t,s) = \Big[\nabla\Psi(\mathbf{Y}(t,s;X_0);X_0),\ \mathbf{Y}_t(t,s;X_0)\Big]_{\mathbf{Y}(t,s;X_0)},$$
$$1 = \psi_s(t,s) = \Big[\nabla\Psi(\mathbf{Y}(t,s;X_0);X_0),\ \mathbf{Y}_s(t,s;X_0)\Big]_{\mathbf{Y}(t,s;X_0)},$$
$$\text{for all}\quad 0<t<\tau(X_0,s), \quad -\frac{\pi}{2}<s<+\frac{\pi}{2}, \tag{3.19}$$

which yields the Eq. (3.15). With the aid of the Gaussian geodesic frame (2.16), we deduce the identity (3.16) from the Eq. (3.15).

2. Since the curve $\mathbf{Y}(.,s;X_0)$ represents a geodesic, we have the identity

$$\frac{\nabla\,\mathbf{Y}_t(t,s;X_0)}{dt} = 0, \quad 0<t<\tau(X_0,s), \quad -\frac{\pi}{2}<s<+\frac{\pi}{2}. \tag{3.20}$$

Now the Eq. (3.13) yields

$$
\begin{aligned}
0 = \psi_{tt}(t,s) &= \Big[\mathbf{Y}_t(t,s;X_0), \boldsymbol{\nabla}^2\,\Psi(\mathbf{Y}(t,s;X_0);X_0),\ \mathbf{Y}_t(t,s;X_0)\Big]_{\mathbf{Y}(t,s;X_0)} \\
&\quad + \Big[\boldsymbol{\nabla}\Psi(\mathbf{Y}(t,s;X_0);X_0),\ \frac{\nabla\,\mathbf{Y}_t(t,s;X_0)}{dt}\Big]_{\mathbf{Y}(t,s;X_0)} \\
&= \Big[\mathbf{Y}_t(t,s;X_0)\,,\ \boldsymbol{\nabla}^2\,\Psi(\mathbf{Y}(t,s;X_0);X_0),\ \mathbf{Y}_t(t,s;X_0)\Big]_{\mathbf{Y}(t,s;X_0)} \\
&\quad \text{for all} \quad 0 < t < \tau(X_0,s)\,, \quad -\frac{\pi}{2} < s < +\frac{\pi}{2}\,,
\end{aligned} \tag{3.21}
$$

which implies the statement (3.17). □

Now we present the principal device of our investigations within

Lemma 3 (Pseudoharmonic nonlinear combination for harmonic maps)
Let the mapping $X(u,v)$ from (3.4) be harmonic, i.e. the Eqs. (1.5), (1.6) hold true. Then the geodesic auxiliary function $\psi(u,v)$ in (3.5) satisfies the elliptic partial differential equation

$$
\Delta\psi(u,v) + a(u,v)\psi_u(u,v) + b(u,v)\psi_v(u,v) = 0\,, \quad (u,v)\in B \tag{3.22}
$$

with the continuous functions $a = a(u,v)\colon B \to \mathbb{R}$ and $b = b(u,v)\colon B \to \mathbb{R}$. The gradient $\nabla\psi$ possesses only isolated zeroes in B and allows expansions of Hartman–Wintner-type (see Theorem 1.2 in [12] Chap. 9) there. Since this function ψ shares important properties with harmonic functions, we may address ψ as being **pseudoharmonic**.

Proof 1. The mapping X is harmonic, and we have the identity

$$
\frac{\nabla\,X_u(u,v)}{du} + \frac{\nabla\,X_v(u,v)}{dv} = 0\,, \quad (u,v)\in B. \tag{3.23}
$$

Now we add the Eqs. (3.13) and (3.14), and we obtain the following identity for our auxiliary function $\psi(u,v)$, $(u,v)\in B$ on account of (3.23):

$$
\begin{aligned}
\Delta\psi(u,v) &= \psi_{uu}(u,v) + \psi_{vv}(u,v) \\
&= \Big[X_u(u,v)\,,\ \boldsymbol{\nabla}^2\,\Psi(X(u,v);X_0),\ X_u(u,v)\Big]_{X(u,v)} \\
&\quad + \Big[X_v(u,v)\,,\ \boldsymbol{\nabla}^2\,\Psi(X(u,v);X_0),\ X_v(u,v)\Big]_{X(u,v)} \\
&\quad + \Big[\boldsymbol{\nabla}\Psi(X(u,v);X_0),\ \frac{\nabla\,X_u(u,v)}{du} + \frac{\nabla\,X_v(u,v)}{dv}\Big]_{X(u,v)} \\
&= \Big[X_u(u,v)\,,\ \boldsymbol{\nabla}^2\,\Psi(X(u,v);X_0),\ X_u(u,v)\Big]_{X(u,v)} \\
&\quad + \Big[X_v(u,v)\,,\ \boldsymbol{\nabla}^2\,\Psi(X(u,v);X_0),\ X_v(u,v)\Big]_{X(u,v)}\,, \quad (u,v)\in B.
\end{aligned} \tag{3.24}
$$

2. With the aid of the Gaussian geodesic frame (2.16) and the identity (3.16), we expand the vector $X_u(u,v)$ via the covariant chain rule (3.6) as follows:

$$
\begin{aligned}
X_u(u,v) &= \Big[X_u(u,v), \widehat{\mathbf{Y}}_t(X(u,v);X_0)\Big]_{X(u,v)} \widehat{\mathbf{Y}}_t(X(u,v);X_0) \\
&\quad + \Big[X_u(u,v), \frac{\widehat{\mathbf{Y}_s}(X(u,v);X_0)}{\sqrt{\widehat{G}(X(u,v);X_0)}}\Big]_{X(u,v)} \frac{\widehat{\mathbf{Y}_s}(X(u,v);X_0)}{\sqrt{\widehat{G}(X(u,v);X_0)}}
\end{aligned}
$$

$$
\begin{aligned}
&= \Big[X_u(u,v), \widehat{\mathbf{Y}}_t(X(u,v);X_0) \Big]_{X(u,v)} \widehat{\mathbf{Y}}_t(X(u,v);X_0) \\
&\quad + \Big[X_u(u,v), \frac{\widehat{\mathbf{Y}_s}(X(u,v);X_0)}{\widehat{G}(X(u,v);X_0)} \Big]_{X(u,v)} \widehat{\mathbf{Y}_s}(X(u,v);X_0) \\
&= \Big[X_u(u,v), \widehat{\mathbf{Y}}_t(X(u,v);X_0) \Big]_{X(u,v)} \widehat{\mathbf{Y}}_t(X(u,v);X_0) \\
&\quad + \Big[X_u(u,v), \nabla\Psi(X(u,v);X_0) \Big]_{X(u,v)} \widehat{\mathbf{Y}_s}(X(u,v);X_0) \\
&= \Big[X_u(u,v), \widehat{\mathbf{Y}}_t(X(u,v);X_0) \Big]_{X(u,v)} \widehat{\mathbf{Y}}_t(X(u,v);X_0) \\
&\quad + \psi_u(u,v)\, \widehat{\mathbf{Y}_s}(X(u,v);X_0)\,, \quad (u,v) \in B.
\end{aligned} \tag{3.25}
$$

Proceeding in the same way for the derivative with respect to v, we arrive at the following equations:

$$
\begin{aligned}
X_u(u,v) &= \Big[X_u(u,v), \widehat{\mathbf{Y}}_t(X(u,v);X_0) \Big]_{X(u,v)} \widehat{\mathbf{Y}}_t(X(u,v);X_0) \\
&\quad + \psi_u(u,v)\, \widehat{\mathbf{Y}_s}(X(u,v);X_0)\,, \quad (u,v) \in B\,; \\
X_v(u,v) &= \Big[X_v(u,v), \widehat{\mathbf{Y}}_t(X(u,v);X_0) \Big]_{X(u,v)} \widehat{\mathbf{Y}}_t(X(u,v);X_0) \\
&\quad + \psi_v(u,v)\, \widehat{\mathbf{Y}_s}(X(u,v);X_0)\,, \quad (u,v) \in B.
\end{aligned} \tag{3.26}
$$

3. When we insert the vectors $X_u(u,v)$ and $X_v(u,v)$ from (3.26) into the covariant Hessian forms within (3.24) and observe the property (3.17), we receive the representation (3.22) with continuous coefficient functions. □

Remark 7 Similar arguments for the Euclidean situation under semi-free boundary conditions have been established in [5] Proposition 3 within my joint investigation with Hildebrandt.

4 Convex hull property, univalency and transversality for harmonic mappings

We start with the central definition and assume that setting throughout this section.

Definition 8 Let ds^2 denote a stable Riemannian metric (1.1), (1.8), (1.9) on the disc Ω_M of radius $0 < M < +\infty$ with a moderate deviation (1.10) by the constant $a \in (0, \frac{1}{2M})$ from the Euclidean metric. For each continuous boundary function

$$
\Phi \in C^0(\partial B, \mathbb{R}^2) \quad \text{with} \quad |\Phi(u,v)| \le M,\ \forall\,(u,v) \in \partial B
$$

we call the function

$$
\begin{aligned}
X = X(u,v) = (x^1(u,v), x^2(u,v)) : \overline{B} \to \mathbb{R}^2 \in C^2(B,\mathbb{R}^2) \cap C^0(\overline{B},\mathbb{R}^2) \\
\text{with} \quad |X(u,v)| \le M \quad \text{for all} \quad (u,v) \in \overline{B}
\end{aligned} \tag{4.1}
$$

a *solution of the Dirichlet problem* $\mathcal{P}(\Omega_M, ds^2; \Phi)$, when the function X satisfies the system (1.5), (1.6) of harmonic mappings and fulfills the boundary condition

$$
X(u,v) = \Phi(u,v) \quad \text{for all} \quad (u,v) \in \partial B. \tag{4.2}
$$

Theorem 3 (Convex hull property for harmonic mappings)
Let the continuous function $\Phi\colon \partial B \to \Omega_M \in C^0(\partial B)$ with the boundary point set $F := \Phi(\partial B) \subset \Omega_M$ and its convex hull $\mathcal{H}(F) \subset \Omega_M$ due to Definition 4 be given. For each solution

$$X = X(u,v) = (x^1(u,v), x^2(u,v)) \in \mathcal{P}(\Omega_M, ds^2; \Phi)$$

of the Dirichlet problem we have the following inclusion:

$$X(u,v) \in \mathcal{H}(F) \quad \textit{for all} \quad (u,v) \in \overline{B}. \tag{4.3}$$

Proof 1. The boundary point set $F := \Phi(\partial B) \subset \Omega_M$ is compact in Ω_M, and the convex hull of the boundary values $\mathcal{H}(F) \subset \Omega_M$ as well. Therefore, we can find a unique number $\sigma(X_0, F) \in (-\frac{\pi}{2}, +\frac{\pi}{2})$, such that

$$\begin{aligned} &\bigcap \left\{ \Theta(s_0; X_0) \middle| s_0 \in (-\frac{\pi}{2}, +\frac{\pi}{2}) : F \subset \Theta(s_0; X_0) \right\} = \Theta(\sigma(X_0, F); X_0) \\ &= \left\{ X \in \Omega_M \middle| \Psi(X; X_0) \le \sigma(X_0, F) \right\} \quad \text{for each point} \quad X_0 \in \partial\Omega_M. \end{aligned} \tag{4.4}$$

Here we have utilized the characterization (2.23) for the last identity. Now we determine the convex hull of the boundary point set as follows:

$$\begin{aligned} \mathcal{H}(F) &= \bigcap \left\{ \Theta(s_0; X_0) \middle| X_0 \in \partial\Omega_M,\ s_0 \in (-\frac{\pi}{2}, +\frac{\pi}{2}) : F \subset \Theta(s_0; X_0) \right\} \\ &= \bigcap_{X_0 \in \partial\Omega_M} \Theta(\sigma(X_0, F); X_0) = \bigcap_{X_0 \in \partial\Omega_M} \left\{ X \in \Omega_M \middle| \Psi(X; X_0) \le \sigma(X_0, F) \right\}. \end{aligned} \tag{4.5}$$

2. With the aid of the geometric maximum principle by E. Heinz, we can see as in Remark 2 that the inclusion $X(\partial B) \subset \Omega_M$ implies the property $X(B) \subset \Omega_M$. For arbitrary points $X_0 \in \partial\Omega_M$ we consider the geodesic auxiliary function

$$\psi(u,v) := \Psi(X(u,v); X_0), \quad (u,v) \in \overline{B}. \tag{4.6}$$

Since the inclusion $F \subset \Theta(\sigma(X_0, F); X_0)$ for all $X_0 \in \partial\Omega_M$ holds true, we receive

$$\psi(u,v) \le \sigma(X_0, F) \quad \text{for all} \quad (u,v) \in \partial B. \tag{4.7}$$

Now Lemma 3 implies that the function ψ is subject to the maximum principle, which gives us the following statement:

$$\Psi(X(u,v); X_0) \le \sigma(X_0, F), \quad (u,v) \in \overline{B} \quad \text{for all points} \quad X_0 \in \partial\Omega_M. \tag{4.8}$$

On account of (4.5), we obtain that $X(\overline{B}) \subset \mathcal{H}(F)$ holds true. □

Definition 9 A **Jordan contour** $\Gamma \subset \Omega_M$ is called **convex in** Ω_M, when the following properties are fulfilled:

(i) The Jordan contour Γ coincides with the boundary $\partial\mathcal{H}(\Gamma)$ of its convex hull, and the interior $I(\Gamma)$ of the contour Γ corresponds to the open kernel of the convex hull $\mathcal{H}(\Gamma)$.

(ii) A geodesic $\mathbf{Y}(t, s_0; X_0)$, $0 \le t \le \tau(X_0, s_0)$ for the parameter $s_0 \in (-\frac{\pi}{2}, +\frac{\pi}{2})$, such that $\mathbf{Y}(., s_0; X_0)$ meets the interior $I(\Gamma)$ at an inner point $Y_0 \in I(\Gamma)$, shall decompose the Jordan curve into the closed Jordan arcs

$$\Gamma^-(X_0, s_0) := \Gamma \cap \Theta(s_0, X_0) \quad \text{and} \quad \Gamma^+(X_0, s_0) := \overline{\Gamma \backslash \Theta(s_0, X_0)}.$$

These arcs meet at their end points on the geodesic $\mathbf{Y}(., s_0; X_0)$ above.

With the original method by Kneser [9] for the Euclidean plane, which we adapt to the Riemannian situation here, we shall establish the subsequent

Theorem 4 (Univalency for harmonic mappings)
Let the convex Jordan contour $\Gamma \subset \Omega_M$ and the topological boundary function $\Phi\colon \partial B \to \Gamma \in C^0(\partial B, \mathbb{R}^2)$ be given. Then each solution

$$X = X(u, v) = (x^1(u, v), x^2(u, v)) \in \mathcal{P}(\Omega_M, ds^2; \Phi)$$

of the Dirichlet problem furnishes a topological mapping of $\overline{B}$ onto $\overline{I(\Gamma)}$ and a C^2-diffeomorphism of B onto $I(\Gamma)$.

Proof 1. From Theorem 3 and Definition 9, (i) we infer the inclusion

$$X(\overline{B}) \subset \mathcal{H}(\Gamma) = I(\Gamma) \cup \Gamma. \tag{4.9}$$

Moreover, the strict inclusion

$$X(B) \subset I(\Gamma) \tag{4.10}$$

is valid, which we deduce as follows:
If the statement (4.10) were violated, there exists a point $(u_0, v_0) \in B$ with $Y_0 = X(u_0, v_0) \in \Gamma$. On account of Definition 9 we can find a point $X_0 \in \partial\Omega_M$ and a value $-\frac{\pi}{2} < s_0 < +\frac{\pi}{2}$, such that

$$\Gamma \subset \Theta(s_0; X_0) \quad \text{and} \quad Y_0 \in \Gamma \cap \partial\Theta(s_0; X_0) \tag{4.11}$$

holds true. Now we consider the auxiliary function

$$\psi(u, v) := \Psi(X(u, v); X_0) \le s_0, \; (u, v) \in \overline{B} \text{ with } \psi(u_0, v_0) = \Psi(Y_0; X_0) = s_0. \tag{4.12}$$

From Lemma 3 we see that ψ is a pseudoharmonic function and cannot attain a strict maximum within B. Consequently, the equality

$$\psi(u, v) = s_0 \quad \text{for all} \quad (u, v) \in \overline{B} \tag{4.13}$$

holds true, which yields an evident contradiction. Therefore, the strict inclusion (4.10) is valid.

2. Now we show indirectly that the Jacobian of the mapping X does not vanish:

$$J_X(u, v) := \frac{\partial(x^1(u, v), x^2(u, v))}{\partial(u\,,\,v)} = \begin{vmatrix} x^1_u(u, v), x^1_v(u, v) \\ x^2_u(u, v), x^2_v(u, v) \end{vmatrix} \ne 0\,, \forall\,(u, v) \in B. \tag{4.14}$$

If the statement (4.14) were violated, there exists a point

$$(u_0, v_0) \in B \quad \text{with} \quad Y_0 := X(u_0, v_0) \in I(\Gamma)\,,$$

such that the vectors $\{X_u(u_0, v_0),\; X_v(u_0, v_0)\}$ are linearly dependent. Consequently, we find a unit vector Z_0 orthogonal to these vectors as follows:

$$\begin{aligned} &Z_0 \in \mathbb{R}^2 \backslash \{(0, 0)\} \quad \text{with} \quad \Big[Z_0\,,\, Z_0\Big]_{X(u_0,v_0)} = 1\,, \\ &\Big[Z_0\,,\, X_u(u_0, v_0)\Big]_{X(u_0,v_0)} = 0 = \Big[Z_0\,,\, X_v(u_0, v_0)\Big]_{X(u_0,v_0)}. \end{aligned} \tag{4.15}$$

3. Now the Gaussian geodesic frame at the fixed point $Y_0 \in \Omega_M$

$$\left\{ \widehat{\mathbf{Y}}_t(Y_0; X_0)\,,\ \frac{\widehat{\mathbf{Y}}_s(Y_0; X_0)}{\sqrt{\widehat{G}(Y_0; X_0)}} \right\}, \quad X_0 \in \partial\Omega_M \tag{4.16}$$

performs one positive-oriented and continuous rotation, when X_0 traverses the circumference $\partial\Omega_M$ once in positive orientation. This results from the construction of the geodesic vector fields, which depend continuously on the point $X_0 \in \partial\Omega_M$ together with their nonvanishing derivatives. Therefore, we can choose a point $X_0 \in \partial\Omega_M$ such that

$$\frac{\widehat{\mathbf{Y}}_s(Y_0; X_0)}{\sqrt{\widehat{G}(Y_0; X_0)}} = Z_0 \tag{4.17}$$

holds true. With the aid of (3.16) and (4.17), we obtain the following representation for the cogradient of the geodesic function Ψ

$$\nabla\Psi(Y_0; X_0) = \frac{\widehat{\mathbf{Y}}_s(Y_0; X_0)}{\widehat{G}(Y_0; X_0)} = \lambda\, Z_0 \quad \text{with} \quad \lambda := \frac{1}{\sqrt{\widehat{G}(Y_0; X_0)}}. \tag{4.18}$$

4. Let us now consider the geodesic auxiliary function

$$\psi(u, v) := \Psi(X(u, v); X_0)\,, \quad (u, v) \in \overline{B}. \tag{4.19}$$

With the aid of (4.15) and (4.18) we derive

$$\begin{aligned} \psi_u(u_0, v_0) &= \Big[\nabla\Psi(Y_0; X_0),\ X_u(u_0, v_0)\Big]_{X(u_0,v_0)} \\ &= \lambda\Big[Z_0, X_u(u_0, v_0)\Big]_{X(u_0,v_0)} = 0\,; \\ \psi_v(u_0, v_0) &= \Big[\nabla\Psi(Y_0; X_0),\ X_v(u_0, v_0)\Big]_{X(u_0,v_0)} \\ &= \lambda\Big[Z_0, X_v(u_0, v_0)\Big]_{X(u_0,v_0)} = 0. \end{aligned} \tag{4.20}$$

Since the function ψ is pseudoharmonic due to Lemma 3 and $\nabla\psi(u_0, v_0) = (0, 0)$ holds true, now ψ represents a saddle point near (u_0, v_0). This behavior propagates to the boundary ∂B on account of the maximum/minimum principle. This yields a contradiction to the behavior of the function $\psi : \partial B \to \mathbb{R}$ on the boundary, which only possesses two points for the level s_0 due to Definition 9, (ii) Consequently, the Jacobian J_X is not allowed to vanish within B, and the statement (4.14) holds true. For an exact proof, we can follow the arguments for harmonic functions in Lemma 2 and Lemma 3 of our book on *Minimal Surfaces* [2] within Section 4.9. These arguments remain valid for the pseudoharmonic function ψ, due to the asymptotic expansions of P. Hartman and A. Wintner (see Theorem 1.2 in [12] Chap. 9.) at their critical points.
5. With the monodromy principle (see Lemma 1 in [2], Sect. 4.9) we can infer the topological character of the mapping

$$X : \overline{B} \to \overline{I(\Gamma)} \subset \Omega_M$$

from (4.14) and the property that the boundary representation $X : \partial B \to \Gamma$ is topological. Alternatively, we can use an index-argument from [11] Hilfssatz 7 in order to show that the mapping $X : \overline{B} \to \overline{I(\Gamma)}$ is one-to-one. □

Remark 8 In the Euclidean situation, we find this result by T. Radó and H. Kneser in § 398 of J. C. C. Nitsche's monograph [10] *Vorlesungen über Minimalflächen*.

Furthermore, we refer to Proposition 4.2 in my joint treatise [4] with S. Hildebrandt.

The following statement contains the *transversality of harmonic mappings to the boundary*. More precisely, we shall establish

Theorem 5 (Existence of $C^{2+\alpha}(\overline{B}, \overline{\Omega_M})$-diffeomorphisms for $\mathcal{P}(\Omega_M, ds^2; \Phi)$) *Let the $C^{2+\alpha}(\partial B, \partial\Omega_M)$-diffeomorphic boundary function $\Phi\colon \partial B \to \partial\Omega_M$ be given. Then there exists a $C^{2+\alpha}(\overline{B}, \overline{\Omega_M})$-diffeomorphism*

$$X = X(u, v) = (x^1(u, v), x^2(u, v))\colon \overline{B} \to \overline{\Omega}_M\,,$$

which furnishes a solution of the Dirichlet Problem $\mathcal{P}(\Omega_M, ds^2; \Phi)$.

Proof 1. We build upon our exsistence result in Theorem 2, and we receive a solution $X = X(u, v) \in C^{2+\alpha}(\overline{B}, \overline{\Omega_M})$ for the Dirichlet problem $\mathcal{P}(\Omega_M, ds^2; \Phi)$. By the geometric maximum principle of E. Heinz the function

$$\chi(u, v) := |X(u, v)|^2\,, \quad (u, v) \in \overline{B} \quad \text{satisfies} \quad \Delta\chi(u, v) \geq 0\,, \quad (u, v) \in B.$$

The boundary point lemma of E. Hopf implies the following inequality for the derivative w. r. t. the exterior normal ν to B:

$$0 < \frac{d}{d\nu}\chi(u_1, v_1) = 2\,X(u_1, v_1) \cdot \frac{d}{d\nu}X(u_1, v_1) \quad \text{for all points} \quad (u_1, v_1) \in \partial B. \tag{4.21}$$

This property (4.21) together with the arguments in [11] Satz 2 yield that our mapping X is transversal in the following sense:

$$J_X(u, v) \neq 0 \quad \text{for all} \quad (u, v) \in \partial B. \tag{4.22}$$

2. Now we follow the parts (2)–(4) in the proof of Theorem 4, in order to exclude zeroes of the Jacobian J_X within B. When the geodesic field $\mathbf{Y}(t, s; X_0)$ has the center $X_0 \in \partial\Omega_M$, we exempt from Ω_M a disc about this singularity for a sufficiently small number $\epsilon > 0$. With the domain

$$\Omega_M^\epsilon(X_0) := \Big\{X \in \Omega_M \Big|\, |X - X_0| > \epsilon\Big\}$$

we modify the arguments in part (4) within the proof of Theorem 4, and we consider alternatively the auxiliary function

$$\psi(u, v) := \Psi(X(u, v); X_0), (u, v) \in \overline{B}_\epsilon := \Big\{(u, v) \in \overline{B} \Big| X(u, v) \in \overline{\Omega_M^\epsilon(X_0)}\Big\}. \tag{4.23}$$

Thus we can exclude each zero of the Jacobian in the interior of the disc B. With the part (5) in the proof of Theorem 4, we complete the derivation of our result above. □

Remark 9 In order to show that a conformally parametrized H-surface represents a graph, one has to prove that the *associate plane mapping is one-to-one*. Here the investigation [11] contains as the decisive step that *transversal mappings yield necessarily a diffeomorphism*. There we need a *stability condition* in the sense that the *second variation of the associate parametric integral is nonnegative*.

5 Harmonic embeddings within the hemisphere

For all radii $0 < M < +\infty$ with their associate discs

$$\Omega_M := \Big\{X = (x^1, x^2) \in \mathbb{R}^2 : \ |X| < M\Big\}$$

we consider the upper hemisphere S_M^+ in the following representation

$$Z(x^1, x^2) := \Big(x^1, x^2, \sqrt{M^2 - |X|^2}\Big), \quad X = (x^1, x^2) \in \Omega_M. \tag{5.1}$$

Then we derive

$$Z_{x^i}(x^1, x^2) = \Big(\delta_{1i}, \delta_{2i}, \frac{-x^i}{\sqrt{M^2 - |X|^2}}\Big), \quad X = (x^1, x^2) \in \Omega_M, \ i = 1, 2 \tag{5.2}$$

and determine their first fundamental form (1.1) as follows

$$g_{ij} := Z_{x^i} \cdot Z_{x^j}(x^1, x^2) = \delta_{1i}\delta_{1j} + \delta_{2i}\delta_{2j} + \frac{x^i x^j}{M^2 - |X|^2} = \delta_{ij} + \frac{x^i x^j}{M^2 - |X|^2} \tag{5.3}$$
$$\text{for all} \quad X = (x^1, x^2) \in \Omega_M \quad \text{and} \quad i, j = 1, 2.$$

We denote the *hemispherical metric* (1.1), (5.3) by $ds^2(M)$. This metric becomes singular near the boundary $\partial\Omega_M$, and our Theorem 2 is not applicable globally. S. Hildebrandt, H. Kaul and K. Widman have constructed harmonic mappings into complete Riemannian manifolds with positive sectional curvature by direct variational methods (see [3]). Since this result is especially valid for hemispheres, we receive the following

Theorem 6 (Dirichlet problem for hemispherical harmonic mappings) *Let a radius* $0 < M < +\infty$ *be chosen arbitrarily. For each boundary function*

$$\Phi \in C^0(\partial B, \Omega_M) \quad \textit{possessing a} \quad W^{1,2}(B, \mathbb{R}^2) - \textit{extension} \tag{5.4}$$

there exists a solution

$$X = X(u, v) = (x^1(u, v), x^2(u, v)) : \overline{B} \to \Omega_M \in C^{2+\alpha}(B, \mathbb{R}^2) \cap C^0(\overline{B}, \mathbb{R}^2) \tag{5.5}$$

for the Dirichlet problem $\mathcal{P}(\Omega_M, ds^2(M); \Phi)$ *of the harmonic mapping associated with the hemispherical metric* $ds^2(M)$.

Proof See the Theorems 1–4 in [3]. □

We construct a field of geodesics, which emanates from an arbitrary *equatorial point*

$$Z_\vartheta = \Big(M\cos\vartheta, M\sin\vartheta, 0\Big) \in \partial S_M^+, \quad 0 \le \vartheta \le 2\pi \tag{5.6}$$

and simply covers the hemisphere. We begin with the great circle on S_M^+

$$\Big(M\cos\Big(\frac{t}{M}\Big), 0, M\sin\Big(\frac{t}{M}\Big)\Big)^*, \quad 0 < t < M\pi. \tag{5.7}$$

This circle represents a geodesic without interior conjugate points; it starts at the point $Z_0 = (M, 0, 0)$ and ends at the antipodal point $Z_\pi = (-M, 0, 0)$, which is conjugate to Z_0. We use the *rotation by the angle s about the x^1-axis*

$$D_s^1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos s & -\sin s \\ 0 & \sin s & \cos s \end{pmatrix}, \quad -\frac{\pi}{2} < s < +\frac{\pi}{2} \tag{5.8}$$

and the *rotation by the angle ϑ about the x^3-axis*

$$D^3_\vartheta := \begin{pmatrix} \cos\vartheta & -\sin\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad 0 \le \vartheta \le 2\pi. \tag{5.9}$$

We obtain the field of geodesics

$$\begin{aligned} &\mathbf{Z}(t,s;\vartheta) := D^3_\vartheta \circ D^1_s \circ \Big(M\cos\Big(\frac{t}{M}\Big), 0, M\sin\Big(\frac{t}{M}\Big)\Big)^*, \\ &0 < t < M\pi, \ -\frac{\pi}{2} < s < +\frac{\pi}{2} \quad \text{for all angles} \quad 0 \le \vartheta \le 2\pi. \end{aligned} \tag{5.10}$$

Via the projection from the Euclidean space onto the plane

$$\Pi^3(Z) = \Pi^3(x^1,x^2,x^3) := (x^1,x^2) = X \in \mathbb{R}^2, \quad Z = (x^1,x^2,x^3) \in \mathbb{R}^3 \tag{5.11}$$

we see from the construction above that the family of functions

$$\mathbf{Y}(t,s;\vartheta) := \Pi^3 \circ \mathbf{Z}(t,s;\vartheta), \quad 0 < t < M\pi, \ -\frac{\pi}{2} < s < +\frac{\pi}{2} \tag{5.12}$$

constitutes a central field of geodesics for the hemispherical metric (1.1), (5.3). This central field $\mathbf{Y}(.,.;\vartheta)$ simply covers the disc Ω_M and emanates from the singular point $X_0 = \Pi^3(X_\vartheta) \in \partial\Omega_M$, where $0 \le \vartheta \le 2\pi$ denotes an arbitrary angle.

With the aid of this central field of geodesics, we introduce geodesic regions and convex hulls for compact sets within Ω_M as in Definition 4. Furthermore, we can define the geodesic function parallel to Definition 5 and receive the fundamental Lemma 3 for the hemispherical metric $ds^2(M)$. Finally, we characterize convex Jordan contours $\Gamma \subset \Omega_M$ as in Definition 9 with respect of the hemispherical metric. By the arguments in the proofs for Theorem 3 and Theorem 4, we can establish

Theorem 7 (Harmonic embeddings within the hemisphere) *Let the convex Jordan contour $\Gamma \subset \Omega_M$ and the topological boundary function $\Phi\colon \partial B \to \Gamma$ as in (5.4) be given. Then each solution $X(u,v) = (x^1(u,v), x^2(u,v))$ of the regularity (5.5) for the Dirichlet problem $\mathcal{P}(\Omega_M, ds^2(M); \Phi)$ of the hemispherical metric $ds^2(M)$ furnishes a topological mapping of $\overline{B}$ onto $\overline{I(\Gamma)}$ and a $C^{2+\alpha}$-diffeomorphism of B onto $I(\Gamma)$.*

Remark 10 Let $\mathcal{M}$ denote a 2-dimensional, geodesically complete, connected and oriented Riemannian manifold of the class C^3 without boundary, whose Gaussian curvature is bounded from above by the constant $\kappa \in [0, +\infty)$ as follows:

$$K(X) \le \kappa \quad \text{for all} \quad X \in \mathcal{M}. \tag{5.13}$$

On this manifold we choose an arbitrary point $P \in \mathcal{M}$ and a radius $0 < M < \dfrac{\pi}{2\sqrt{\kappa+}}$ such that the geodesic disc

$$\mathcal{B}_M(P) := \Big\{ Q \in \mathcal{M} \Big| \operatorname{dist}(Q,P) \le M \Big\} \tag{5.14}$$

satisfies a *cut-locus-condition* (see the treatise [3] by S. Hildebrandt, H. Kaul and K. Widman). Then we can solve the Dirichlet problem for harmonic mappings in the interior of $\mathcal{B}_M(P)$ by these investigations using direct variational methods. Here we can apply our methods from above in order to obtain harmonic diffeomorphisms.

Example 1 Harmonic diffeomorphisms in the Poincaré half-plane.

We consider *the Poincaré half-plane* (see [8] 5.1.3 in the *Lectures* by W. Klingenberg) with the following coefficients for their first fundamental form ds^2 from (1.1):

$$g_{ij}(X) := \frac{1}{x_2^2}\,\delta_{ij},\ X \in \mathbb{R}^2_+ := \left\{X = (x^1, x^2) \in \mathbb{R}^2 \,\middle|\, x_2 > 0\right\} \quad \text{for } i, j = 1, 2. \tag{5.15}$$

Due to [8] Satz 5.1.7, the geodesics in the Poincaré half-plane with their Gaussian curvature $K \equiv -1$ consist of all circular arcs within $\mathbb{R}^2_+$ meeting the x_1-axis perpendicularly and the rays emanating orthogonally from the x_1-axis, which we address as *orthocircles*. From [1] §§ 81–84 we see that the geodesic discs $\mathcal{B}_M(P) \subset \mathbb{R}^2_+$ with their center P on the positive x_2-axis possess a convex circumference

$$\partial\mathcal{B}_M(P) := \left\{Q \in \mathcal{M} \,\middle|\, \operatorname{dist}(Q, P) = M\right\} \tag{5.16}$$

with the constant geodesic curvature

$$\kappa_g(X) > 0 \quad \text{for all} \quad X \in \partial\mathcal{B}_M(P). \tag{5.17}$$

Due to Figure 14 in [1] § 84 of the *Grundlehren* by W. Blaschke and K. Leichtweiß the circumferences for these geodesic discs constitute the orthogonal trajectories of the orthocircles. The geodesic discs $\mathcal{B}_M(P_M)$ exhaust the Poincaré half-plane for $M \to +\infty$. Here we also refer to Abb. 5.1 in [8] 5.1.

Each boundary point $X_0 \in \partial\mathcal{B}_M(P_M)$ possesses a central field of geodesics, which emanates from X_0 and foliates $\mathcal{B}_M(P_M)$. Consequently, the variational solution $X(u, v)$, $(u, v) \in \overline{B}$ of the Dirichlet problem for harmonic mappings by Hildebrandt, Kaul and Widman [3] exists within the geodesic discs of all radii $M > 0$. Then we can apply the methods from Sects. 2 to 4 above, and we see that this solution X shares the convex-hull property. Furthermore, this variational solution X yields a diffeomorphism in B and a topological mapping on $\overline{B}$ for topological boundary representations onto convex Jordan contours Γ, which are contained in the interior of the disc $\mathcal{B}_M(P_M)$. Thus we receive an analogue of Theorem 7 within the Poincaré half-plane.

References

1. Blaschke, W., Leichtweiß, K.: Elementare Differentialgeometrie. Grundlehren der mathematischen Wissenschaften 1, 5. Auflage (vollständige Neubearbeitung). Springer, Berlin (1973)
2. Dierkes, U., Hildebrandt, S., Sauvigny, F.: Minimal Surfaces. Grundlehren der mathematischen Wissenschaften 339. Springer, Berlin (2010)
3. Hildebrandt, S., Kaul, H., Widman, K.: An existence theorem for harmonic mappings of Riemannian manifolds. Acta Math. **138**, 1–16 (1977)
4. Hildebrandt, S., Sauvigny, F.: Embeddedness and uniquenes of minimal surfaces solving a partially free boundary value problem. J. Reine Angew. Math. **422**, 69–89 (1991)
5. Hildebrandt, S., Sauvigny, F.: On one-to-one harmonic mappings and minimal surfaces. Nachrichten der Akademie der Wissenschaften in Göttingen, II. Math. Phys. Klasse, Jahrgang Nr.3 (1992)
6. Jäger, W., Kaul, H.: Uniqueness and stability of harmonic maps and their Jacobi fields. Manuscr. Math. **28**, 269–291 (1979)
7. Jost, J.: Univalency of harmonic mappings between surfaces. J. Reine Angew. Math. **342**, 141–153 (1981)
8. Klingenberg, W.: Eine Vorlesung über Differentialgeometrie. Heidelberger Taschenbücher. Springer, Berlin (1973)
9. Kneser, H.: Lösung der Aufgabe 41. Jahresber. Dt. Math. Vereinigung **35**, 123–124 (1926)
10. Nitsche, J.C.C.: Vorlesungen über Minimalflächen. Grundlehren der mathematischen Wissenschaften 199. Springer, Berlin (1975)
11. Sauvigny, F.: Flächen vorgeschriebener mittlerer Krümmung mit eineindeutiger Projektion auf eine Ebene. Math. Z. **180**, 41–67 (1982)

12. Sauvigny, F.: Partial Differential Equations. 1. Foundations and Integral Representations; 2. Functional Analytic Methods; With Consideration of Lectures by E. Heinz. Springer Universitext, 2nd edn. Springer, London (2012)
13. Sauvigny, F.: Analysis - Grundlagen, Differentiation, Integrationstheorie, Differentialgleichungen, Variationsmethoden. Springer, Berlin (2014)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.