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Calculus of Variations

A Liouville theorem for the p-Laplacian and related questions

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Abstract

We prove several classification results for p-Laplacian problems on bounded and unbounded domains, and deal with qualitative properties of sign-changing solutions to p-Laplacian equations on $\mathbb{R}^N$ involving critical nonlinearities. Moreover, on radial domains we characterise the compactness of possibly sign-changing Palais–Smale sequences.

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Contents

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1 Introduction

Throughout the paper we use the following notation:

$$\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u), \quad 1 < p < \infty,$$

$$p^* := Np/(N-p), \quad 1 < p < N,$$

$$\mathcal{D}^{1,p}(\mathbb{R}^N) = \{u \in L^{p^*}(\mathbb{R}^N) : \nabla u \in L^p(\mathbb{R}^N; \mathbb{R}^N)\},$$

$$||u|| = ||\nabla u||_{L^p(\mathbb{R}^N)},$$

$$H = \mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x_N > 0\}.$$

For a smooth possibly unbounded domain $\mathcal{O}$ we denote by $\mathcal{D}^{1,p}_0(\mathcal{O})$ the closure of $\mathcal{D}(\mathcal{O})$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. When $\mathcal{O}$ is bounded we set $\mathcal{D}^{1,p}_0(\mathcal{O}) = W^{1,p}_0(\mathcal{O})$.
Let $1 < p < N$, Ω be a smooth bounded domain of $\mathbb{R}^N$ and $a \in L^{N/p}(\Omega)$. We define on $W^{1,p}_0(\Omega)$

$$\phi(u) = \int_\Omega \Big(\frac{|\nabla u|^p}{p} + a(x)\frac{|u|^p}{p} - \frac{|u|^{p^*}}{p^*}\Big)\mathrm{d}x,$$

and on $\mathcal{D}^{1,p}(\mathbb{R}^N)$

$$\phi_\infty(u) = \int_{\mathbb{R}^N} \Big(\frac{|\nabla u|^p}{p} - \frac{|u|^{p^*}}{p^*}\Big)\mathrm{d}x.$$

Recall that

$$(\phi'(u), h) = \int_\Omega [|\nabla u|^{p-2}\nabla u \cdot \nabla h + a(x)|u|^{p-2}u\, h - |u|^{p^*-2}u\, h]\mathrm{d}x,$$
$$(\phi'_\infty(u), h) = \int_{\mathbb{R}^N} [|\nabla u|^{p-2}\nabla u \cdot \nabla h - |u|^{p^*-2}u\, h]\mathrm{d}x.$$

From [10] a blow-up theory for the Palais–Smale sequences $\{u_n\}_n \subset W^{1,p}_0(\Omega)$ for ϕ is available when $\{u_n\}_n$ is a bounded sequence 'nearby' the positive cone of $W^{1,p}_0(\Omega)$. We assume here that

$$\phi(u_n) \to c \qquad \phi'(u_n) \to 0 \ \text{ in } W^{-1,p'}(\Omega)$$

and

$$\|(u_n)_-\|_{L^{p^*}(\Omega)} \to 0, \quad n \to \infty.$$

Then, a p-Laplacian generalisation of Struwe's global compactness result, see e.g. [9,16,21] holds. In fact by [10], passing if necessary to a subsequence, there exists a possibly nontrivial solution $v_0 \in W^{1,p}_0(\Omega)$ to

$$-\Delta_p u + a(x)u^{p-1} = u^{p^*-1} \text{ in } \Omega,$$
$$u \geq 0 \text{ in } \Omega,$$

k possibly nontrivial solutions $\{v_1, \dots, v_k\} \subset \mathcal{D}^{1,p}(\mathbb{R}^N)$ to

$$
\begin{aligned}
-\Delta_p u &= u^{p^*-1} \text{ in } \mathbb{R}^N,\\
u &\geq 0 \text{ in } \mathbb{R}^N,
\end{aligned}
$$

and k sequences $\{y_n^i\}_n \subset \Omega$ and $\{\lambda_n^i\}_n \subset \mathbb{R}_+$, such that

$$
\begin{aligned}
&\frac{1}{\lambda_n^i} \operatorname{dist}(y_n^i, \partial\Omega) \to \infty, \quad n \to \infty,\\
&\|u_n - v_0 - \sum_{i=1}^{k} (\lambda_n^i)^{(p-N)/p} v_i((\cdot - y_n^i)/\lambda_n^i)\| \to 0, \quad n \to \infty,\\
&\|u_n\|^p \to \sum_{i=0}^{k} \|v_i\|^p, \quad n \to \infty,\\
&\phi(v_0) + \sum_{i=1}^{k} \phi_\infty(v_i) = c.
\end{aligned}
$$

Recent symmetry results of Sciunzi [15] (see also Vétois [20]) together with the uniqueness of the radial positive solutions to $-\Delta_p u = u^{p^*-1}$ on $\mathbb{R}^N$ (see Guedda–Veron [7]), allow to prove that the limiting profiles $\{v_1, \ldots, v_k\}$, when nontrivial, they are necessarily the classical Talenti functions [17]. Among the various applications of this result, it is worth mentioning [11], which extends to the case $p \neq 2$ the classical result of Coron [1].
When considering arbitrary sign-changing Palais–Smale sequences the scenario is much richer. In fact, one may have

$$
\liminf_{n\to\infty} \frac{1}{\lambda_n^i} \operatorname{dist}(y_n^i, \partial\Omega) < \infty
$$

and as a consequence some of the limiting functions v_i may live on a half-space. Ruling out that this situation may occur would yield a complete generalisation of Struwe's result for the p-Laplace operator. More precisely, one may conjecture the following Liouville-type theorem

Conjecture *Let $u \in \mathcal{D}_0^{1,p}(\mathbb{R}_+^N)$ be a weak solution of the equation*

$$
-\Delta_p u = |u|^{p^*-2} u \quad in \quad \mathbb{R}_+^N. \tag{1}
$$

Then $u \equiv 0$.

In [10] it has been shown that this conjecture is true under the additional assumption $u \geq 0$. In a more delicate regularity setting, following [4] the proof in [10] consists in showing that the normal derivative of a nontrivial solution vanishes along the boundary of $\mathbb{R}_+^N$. Therefore u extends by zero to a solution on $\mathbb{R}^N$, contradicting the strong maximum principle [14,19]. However, when dealing with the p-Laplacian operator, a unique continuation principle seems to be a major open question.

1.1 Main results

Among the main results of the present paper we have an a priori quantitative bound on the number of nodal regions for the solutions to (1). More precisely we have the following general result for possibly unbounded domains. Hereafter we refer to the Sobolev constant as defined as

$$S = S(N, p) := \inf \left\{ \int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x,\ u \in \mathcal{D}^{1,p}(\mathbb{R}^N)\ :\ \int_{\mathbb{R}^N} |u|^{p^*} \mathrm{d}x = 1 \right\}, \tag{2}$$

achieved on functions

$$U_{\lambda,x_0} := \left[\frac{\lambda^{\frac{1}{p-1}} N^{\frac{1}{p}} (\frac{N-p}{p-1})^{\frac{p-1}{p}}}{\lambda^{\frac{p}{p-1}} + |\cdot - x_0|^{\frac{p}{p-1}}} \right]^{\frac{N-p}{p}}, \quad \lambda \in \mathbb{R}_+, x_0 \in \mathbb{R}^N,$$

see [17] .

Theorem 1.1 *Let $1 < p < N$, let $\mathcal{O}$ be a smooth domain of $\mathbb{R}^N$ and $u \in \mathcal{D}_0^{1,p}(\mathcal{O})$ be a solution to the equation*

$$-\Delta_p u = |u|^{p^*-2} u \quad in \quad \mathcal{D}'(\mathcal{O}). \tag{3}$$

Then

(i) *for every nodal domain ω of u it holds that*

$$\int_\omega |\nabla u|^p = \int_\omega |u|^{p^*};$$

(ii) *if $u \in \mathcal{D}_0^{1,p}(\mathcal{O}) \setminus \{0\}$ then u has at most a finite number of nodal domains. More precisely let $\mathcal{N}_u$ be the set of nodal domains of u and $\sharp\mathcal{N}_u$ its cardinality, it holds that*

$$\sharp\mathcal{N}_u \le S(N, p)^{-N/p} \int_{\mathcal{O}} |u|^{p^*}$$

where $S(N, p)$ is the best Sobolev constant defined in (2).

As a consequence of the above theorem we have two propositions in a radially symmetric setting, which are of independent interest.
For $R, \mu > 0$ consider the radial problem

$$\begin{cases} -\Delta_p u = \mu |u|^{p^*-2} u & \text{in } B(0, R) \subset \mathbb{R}^N, \\ u \in W_0^{1,p}(B(0, R)). \end{cases} \tag{4}$$

For $p = 2$, 0 is the only solution by the unique continuation principle. When $\frac{2N}{N+2} \le p \le 2$, 0 is the only radial solution, see [10] p. 482. Following a different method, we can now improve this nonexistence result for all $p \in (1, N)$. Set $B = B(0, R)$. We have the following

Proposition 1.2 *Let $1 < p < N$, and let $u \in W_0^{1,p}(B)$ be a possibly sign-changing radial weak solution to the Eq. (4). Then $u \equiv 0$.*

Proposition 1.3 *Let $1 < p < N$ and let $u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\}$ be a possibly sign-changing radial weak solution to the equation*

$$-\Delta_p u = |u|^{p^*-2} u \quad in \quad \mathbb{R}^N. \tag{5}$$

Then, necessarily

$$u \equiv U_\lambda := \pm \left[\frac{\lambda^{\frac{1}{p-1}} N^{\frac{1}{p}} (\frac{N-p}{p-1})^{\frac{p-1}{p}}}{\lambda^{\frac{p}{p-1}} + |\cdot|^{\frac{p}{p-1}}} \right]^{\frac{N-p}{p}} \tag{6}$$

for some $\lambda > 0$.

In Sect. 4 we show that the above propositions yield a precise representation of the Palais–Smale sequences for radial problems, see Propositions 4.1 and 4.2.
In a nonradial setting we have the following classification results by means of the Morse index, we recall its definition in Sect. 5. The assumption $p > 2$ allows to have twice differentiability of the associated energy functionals. In the spirit of [2], the following theorem states that the number of nodal regions of a solution cannot exceed its own index.

Theorem 1.4 *Let $\mathcal{O}$ be a smooth domain of $\mathbb{R}^N$, $2 < p < N$, and let $u \in \mathcal{D}_0^{1,p}(\mathcal{O})$ be a solution to the equation*

$$-\Delta_p u = |u|^{p^*-2}u \quad in \quad \mathcal{D}'(\mathcal{O}).$$

Then

$$\sharp\mathcal{N}_u \leq i(u),$$

where $i(u)$ is the Morse index of u.

As a consequence of the above theorem we have the following classification results which are suitable when studying solutions with min–max methods, see e. g. [21].
In the spirit of [5] we have the following

Theorem 1.5 *Let $2 < p < N$.*

(1) *Let $u \in \mathcal{D}_0^{1,p}(\mathbb{R}^N_+)$ be a weak solution to the equation*

$$-\Delta_p u = |u|^{p^*-2}u \quad in \quad \mathbb{R}^N_+. \tag{7}$$

Then $u \equiv 0$ if and only if $i(u) \leq 1$.

(2) *Let $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ be a weak solution to*

$$-\Delta_p u = |u|^{p^*-2}u \quad in \quad \mathbb{R}^N. \tag{8}$$

If $i(u) \leq 1$, then either $u \equiv 0$ or for some $x_0 \in \mathbb{R}^N$ and $\lambda > 0$ and up to the sign

$$u \equiv U_{\lambda,x_0} := \left[\frac{\lambda^{\frac{1}{p-1}} N^{\frac{1}{p}} \left(\frac{N-p}{p-1}\right)^{\frac{p-1}{p}}}{\lambda^{\frac{p}{p-1}} + |\cdot - x_0|^{\frac{p}{p-1}}}\right]^{\frac{N-p}{p}}.$$

Remark 1 To the best of our knowledge it is not clear whether Talenti's functions have index exactly equal to 1.

For bounded domains we have the following

Theorem 1.6 *Let $2 < p < N$, and Ω is a smooth bounded domain of $\mathbb{R}^N$, starshaped about the origin, namely such that $x \cdot \nu \geq 0$ on $\partial\Omega$, where ν is the exterior normal unit vector. Let $u \in W_0^{1,p}(\Omega)$ be such that*

$$-\Delta_p u = |u|^{p^*-2}u \quad in \quad \mathcal{D}'(\Omega). \tag{9}$$

If $i(u) \leq 1$, then it holds that $u \equiv 0$.

Remark 2 In the case $p = 2$ and $N \geq 3$, this result is well-known without any restriction on the index, as a consequence of the unique continuation principle.

Remark 3 If Ω is an Esteban–Lions type domain (namely a generalisation of Pohozaev's starshaped domains, see [13]), we believe that Theorem 1.6 holds, see [4].

2 The normal derivative vanishes at the boundary

In this section we show that, regardless of their sign, solutions to (1) have vanishing normal derivative along the boundary. Hereafter we set $H = \mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x_N > 0\}$, and $B_R = \{x \in \mathbb{R}^N : |x| < R\}$, for some $R > 0$. Moreover, we denote by $n(\cdot)$ the exterior unit normal to $\partial(H \cap B_R)$ whose N-th component is $n_N(\cdot)$. The N-th partial derivative will be denoted by ∂_N.

Lemma 2.1 *Let $1 < p < N$ and $u \in \mathcal{D}^{1,p}_0(H)$ be a weak solution to Eq. (1). Then $\partial_N u = 0$ everywhere on ∂H.*

Proof The case $1 < p < 2$ had been obtained in [10] without any positivity assumption, while the case $p = 2$ is known from [4].
For $2 < p < N$ we argue as follows. We observe that solutions of (1) are $C^{1,\alpha}_{\mathrm{loc}}(\bar{H})$, see e.g. [3,12,18].
As in [10] we prove the following local Pohozaev's identity, in the spirit of a similar identity proved in [4] in the case $p = 2$:

$$\left(1 - \frac{1}{p}\right) \int_{B_R \cap \partial H} |\partial_N u|^p \mathrm{d}\sigma = \int_{H \cap \partial B_R} [\partial_N u |\nabla u|^{p-2} \nabla u \cdot n(\sigma) - \frac{|\nabla u|^p}{p} n_N(\sigma)] \mathrm{d}\sigma + \int_{H \cap \partial B_R} \frac{|u|^{p^*}}{p^*} n_N(\sigma) \mathrm{d}\sigma. \tag{10}$$

The desired conclusion will be then achieved. Indeed, since $\nabla u \in L^p(H)$ and $u \in L^{p^*}(H)$ the right hand side is bounded by a function $M(R)$ such that for some sequence $R_k \to \infty$, $M(R_k) \to 0$.
In order to prove (10) we use a regularisation argument, see e.g. [3] and [6]. We point out that in [6] a Pohozaev identity for the p-Laplacian is available in the context of Dirichlet problems on bounded domains.
By antireflection with respect to ∂H extend (and still denote by) u to a solution on the whole $\mathbb{R}^N$. Following [p. 833, [3]], we consider u_ε solution to the boundary value problem

$$\begin{aligned} -\mathrm{div} \left(\left(\varepsilon + |\nabla u_\varepsilon|^2 \right)^{\frac{p-2}{2}} \nabla u_\varepsilon \right) &= |u|^{p^*-2} u \qquad \text{in } B_{2R}, \\ u_\varepsilon &= u \qquad \text{on } \partial B_{2R}; \end{aligned}$$

$u_\varepsilon \in C^2(\bar{B}_R)$ and uniformly bounded for $\varepsilon \in (0, 1]$ in $C^{1,\alpha}(\bar{B}_R)$. By the Ascoli–Arzelá theorem for a suitable sequence $\varepsilon \to 0^+$, $u_\varepsilon \to u$ and $\nabla u_\varepsilon \to \nabla u$ uniformly on $\bar{B}_R$.
Consider the vector field

$$v_\varepsilon := \left(\varepsilon + |\nabla u_\varepsilon|^2 \right)^{\frac{p-2}{2}} \partial_N u_\varepsilon \nabla u_\varepsilon.$$

Since

$$\mathrm{div}\, v_\varepsilon = \partial_N u_\varepsilon \,\mathrm{div} \left(\left(\varepsilon + |\nabla u_\varepsilon|^2 \right)^{\frac{p-2}{2}} \nabla u_\varepsilon \right) + \left(\varepsilon + |\nabla u_\varepsilon|^2 \right)^{\frac{p-2}{2}} \nabla u_\varepsilon \cdot \nabla \partial_N u_\varepsilon,$$

by the divergence theorem we obtain

$$\begin{aligned}&\int_{B_R\cap H}\partial_N u_\varepsilon \,\mathrm{div}\left(\left(\varepsilon+|\nabla u_\varepsilon|^2\right)^{\frac{p-2}{2}}\nabla u_\varepsilon\right)\mathrm{d}x\\&\quad=\int_{\partial(H\cap B_R)} v_\varepsilon\cdot n(\sigma)\mathrm{d}\sigma-\int_{B_R\cap H}\left(\varepsilon+|\nabla u_\varepsilon|^2\right)^{\frac{p-2}{2}}\nabla u_\varepsilon\cdot\nabla\partial_N u_\varepsilon \mathrm{d}x\\&\quad=\int_{\partial(H\cap B_R)} v_\varepsilon\cdot n(\sigma)\mathrm{d}\sigma-\int_{\partial(H\cap B_R)}\frac{[\varepsilon+|\nabla u_\varepsilon|^2]^{p/2}}{p}n_N(\sigma)\mathrm{d}\sigma.\end{aligned}$$

Moreover

$$\int_{H\cap B_R}\partial_N u|u|^{p^*-2}u\,\mathrm{d}x=\int_{\partial(H\cap B_R)}\frac{|u|^{p^*}}{p^*}n_N(\sigma)\mathrm{d}\sigma=\int_{H\cap\partial B_R}\frac{|u|^{p^*}}{p^*}n_N(\sigma)\mathrm{d}\sigma.$$

Equating the above two expressions and passing to the limit, as $\varepsilon\to 0^+$ we obtain (10). This concludes the proof. □

3 General facts about nodal regions and proof of Theorem 1.1

The following approximation result will be used to prove Theorems 1.1 and 1.4.

Lemma 3.1 *Let $u\in C^{0,1}_{loc}(\mathbb{R}^N)\cap\mathcal{D}^{1,p}(\mathbb{R}^N)$ and let ω be a nodal domain of u, and $u|_\omega$ its restriction to ω. Then there exists a sequence $\{u_n\}_n\subset C^{0,1}_c(\omega)$ such that*

(i) *$u_n\to u|_\omega$ in $L^{p^*}(\omega)$ and everywhere in ω,*
(ii) *$\nabla u_n\to\nabla u|_\omega$ in $L^p(\omega;\mathbb{R}^N)$ and almost everywhere in ω.*

Proof We can suppose that $u>0$ in ω. Let $f\in C^1(\mathbb{R};\mathbb{R})$ an odd function such that

$$f(t)=\begin{cases}0, & \text{if } |t|\le 1,\\ t, & \text{if } |t|\ge 2,\end{cases}$$

and define for all $n\in\mathbb{N}$, $f_n(t):=\frac{1}{n}f(nt)$. We also define $v:=u|_\omega$, $v_n:=f_n(v)$. It is standard to see that

$$\begin{aligned}&v_n\in C^{0,1}_{\text{loc}}(\omega)\cap L^{p^*}(\omega)\\&\nabla v_n\in L^\infty_{\text{loc}}(\omega;\mathbb{R}^N)\cap L^p(\omega;\mathbb{R}^N)\\&\text{supp}v_n\subseteq\{x\in\omega\mid u(x)\ge 1/n\}\subset\omega,\end{aligned}$$

moreover by the dominated convergence theorem and the definition of f_n we have

(a) $v_n\to u|_\omega$ in $L^{p^*}(\omega)$ and everywhere in ω,
(b) $\nabla v_n\to\nabla u|_\omega$ in $L^p(\omega;\mathbb{R}^N)$ and almost everywhere in ω.

Let now $\theta\in C^1(\mathbb{R}_+)$, with $\theta(t)\in[0,1]$, and such that

$$\theta(t)=\begin{cases}0, & \text{if } t\ge 2\\ 1, & \text{if } 0\le t\le 1\end{cases}$$

and define

$$\theta_n(x):=\theta\Big(\frac{x}{n}\Big).$$

Finally define $u_n:=\theta_n v_n$. It is immediate to verify that (i) and (ii) hold. □

We are in the position to prove Theorem 1.1.

Proof of Theorem 1.1 Since $u \in C^1(\overline{\mathcal{O}})$, extending u by zero outside $\mathcal{O}$ we have that $u \in C^{0,1}_{\text{loc}}(\mathbb{R}^N) \cap \mathcal{D}^{1,p}(\mathbb{R}^N)$.

Proof of (i). Pick (u_n) given by Lemma 3.1 extending by zero outside ω. By a standard density argument for every $n \in \mathbb{N}$ one can test (3) with u_n, namely

$$\int_\omega |\nabla u|^{p-2}\nabla u \nabla u_n = \int_\omega |u|^{p^*-2} u u_n.$$

By Lemma 3.1, as $n \to \infty$ (i) follows.

Proof of ii). For $p \in (1, N)$ and for all $v \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ we write Sobolev's inequality as

$$S(N,p)\Big(\int_{\mathbb{R}^N} |v|^{p^*}\Big)^{p/p^*} \le \int_{\mathbb{R}^N} |\nabla v|^p. \tag{11}$$

Using Sobolev's inequality with $v = u|_\omega$ extended by zero outside ω we have by (i)

$$\int_\omega |\nabla u|^p = \int_{\mathbb{R}^N} |\nabla v|^p \ge S(N,p)\Big(\int_\omega |u|^{p^*}\Big)^{p/p^*} = S(N,p)\Big(\int_\omega |\nabla u|^p\Big)^{p/p^*}, \tag{12}$$

hence

$$\Big(\int_\omega |\nabla u|^p\Big)^{1-\frac{p}{p^*}} \ge S(N,p), \tag{13}$$

namely

$$\int_\omega |\nabla u|^p \ge S(N,p)^{N/p}. \tag{14}$$

It follows that

$$\int_{\mathcal{O}} |u|^{p^*} = \int_{\mathcal{O}} |\nabla u|^p = \sum_{\omega \in \mathcal{N}_u} \int_\omega |\nabla u|^p \ge \sum_{\omega \in \mathcal{N}_u} S(N,p)^{N/p} = S(N,p)^{N/p}\, \sharp \mathcal{N}_u.$$

And this concludes the proof. □

4 Radial problems and proof of Propositions 1.2 and 1.3

Proof of Proposition 1.2 It is standard to see that $u \in C^{1,\alpha}(\overline{B})$, see [3,8,12,18]. Suppose $u \not\equiv 0$. By the strong maximum principle and [6], the solution u must change sign (and so it has a zero in $B \backslash \{0\}$). The nodal regions of u are spherically symmetric, and the number of those is finite, by Theorem 1.1. Now pick a nodal region, say $A = \{x \in B \,:\, R_1 < |x| < R_2\}$ with $0 < R_1 < R_2 \le R$. We can assume that u solves

$$\begin{cases} -\Delta_p u = \mu |u|^{p^*-2} u & \text{in } B(0, R_2) \\ u > 0 & \text{in } A \\ u = 0 & \text{on } \partial A. \end{cases}$$

By Pohozaev's identity, Theorem 1.1 of [6], we have $\nabla u = \mathbf{0}$ on $\partial B(0, R_2)$, and this is in contradiction with Hopf's boundary point lemma, see e.g. [19], since u is positive in A. This concludes the proof. □

Proof of Proposition 1.3 By [7] p.160 and the strong maximum principle, it is enough to prove that u does not change sign. Let us assume that u changes sign. The nodal regions are spherically symmetric and their number is finite. Therefore, by replacing u by $-u$ if necessary, there exists $\overline{R} > 0$ large enough such that

$$\begin{aligned} u &\equiv 0, \quad \text{on } \partial B(0, \overline{R}),\\ u &> 0, \quad \text{on } \mathbb{R}^N \backslash \overline{B(0, \overline{R})}, \end{aligned}$$

and so, by Proposition 1.2

$$\begin{aligned} u &\equiv 0, \quad \text{on } \overline{B(0, \overline{R})},\\ u &> 0, \quad \text{on } \mathbb{R}^N \backslash \overline{B(0, \overline{R})}. \end{aligned}$$

On the other hand by continuity $\nabla u = \mathbf{0}$ on $\partial B(0, \overline{R})$, and this contradicts Hopf's boundary point lemma on $\mathbb{R}^N \backslash \overline{B(0, \overline{R})}$. This concludes the proof. □

Consider the following assumptions:

(A) Ω is the unit ball in $\mathbb{R}^N$, $1 < p < N$, $a \in L^{N/p}_{\text{rad}}(\Omega)$. Assume also
(B)

$$\inf_{\substack{u \in W_0^{1,p}(\Omega)\\ \|\nabla u\|_{L^p} = 1}} \int_\Omega [|\nabla u|^p + a(x)|u|^p]\mathrm{d}x > 0.$$

Define on $W_0^{1,p}(\Omega)$

$$\phi(u) = \int_\Omega \Big(\frac{|\nabla u|^p}{p} + a(x)\frac{|u|^p}{p} - \frac{|u|^{p^*}}{p^*} \Big)\mathrm{d}x,$$

and denote by $W^{1,p}_{0,\text{rad}}(\Omega)$ (resp. $\mathcal{D}^{1,p}_{\text{rad}}(\mathbb{R}^N)$) the space of radial functions in $W_0^{1,p}(\Omega)$ (resp. $\mathcal{D}^{1,p}(\mathbb{R}^N)$). We also define on $\mathcal{D}^{1,p}_{\text{rad}}(\mathbb{R}^N)$

$$\tilde{\phi}_\infty(u) = \int_{\mathbb{R}^N} \Big(\frac{|\nabla u|^p}{p} - \frac{|u|^{p^*}}{p^*} \Big)\mathrm{d}x.$$

Proposition 4.1 *Under assumptions* **(A)***,* **(B)** *let $\{u_n\}_n$ be a sequence in $W^{1,p}_{0,\text{rad}}(\Omega)$ such that*

$$\phi(u_n) \to c \qquad \phi'(u_n) \to 0 \;\; \text{in } (W^{1,p}_{0,\text{rad}}(\Omega))'.$$

Then, passing if necessary to a subsequence, there exists a possibly nontrivial solution $v_0 \in W^{1,p}_{0,\text{rad}}(\Omega)$ to

$$-\Delta_p u + a(x)|u|^{p-2}u = |u|^{p^*-2}u,$$

and k sequences $\{\lambda_n^i\}_n \subset \mathbb{R}_+$, with $\lambda_n^i \to 0$, $n \to \infty$, such that

$$\|u_n - v_0 - \sum_{i=1}^{k} (\lambda_n^i)^{(p-N)/p} v_i(\cdot/\lambda_n^i)\| \to 0,$$

$$\|u_n\|^p \to \sum_{i=0}^{k} \|v_i\|^p,$$

$$\phi(v_0) + \sum_{i=1}^{k} \tilde{\phi}_\infty(v_i) = c,$$

where for $i \geq 1$ v_i is either identically zero, or for some $\lambda > 0$ and up to the sign, it holds that

$$v_i \equiv \left[\frac{\lambda^{\frac{1}{p-1}} N^{\frac{1}{p}} (\frac{N-p}{p-1})^{\frac{p-1}{p}}}{\lambda^{\frac{p}{p-1}} + |\cdot|^{\frac{p}{p-1}}} \right]^{\frac{N-p}{p}},$$

and

$$\tilde{\phi}_\infty(v_i) = \frac{S^{N/p}}{N}.$$

Moreover if $a \equiv 0$, then all weakly convergent subsequences of $\{u_n\}_n$ are weakly convergent to zero in $W^{1,p}_{0,\mathrm{rad}}(\Omega)$. In particular $v_0 \equiv 0$ and hence $\phi(v_0) = 0$.

Proof Let $a \equiv 0$. Since the weak limit of $\{u_n\}_n$, v_0, solves Eq. (4) with $R, \mu = 1$, then by Proposition 1.2 $v_0 \equiv 0$. The rest of the proof follows by Theorem 5.1 in [10] and Proposition 1.3. In this radial setting $\tilde{\phi}_\infty(v_i)$ can be computed explicitly by [17], using Proposition 1.3. And this concludes the proof. □

Now define on $\mathcal{D}^{1,p}(\mathbb{R}^N)$

$$\phi(u) = \int_{\mathbb{R}^N} \Big(\frac{|\nabla u|^p}{p} + a(x)\frac{|u|^p}{p} - \frac{|u|^{p^*}}{p^*} \Big) \mathrm{d}x$$

and

$$\tilde{\phi}_\infty(u) := \int_{\mathbb{R}^N} \Big(\frac{|\nabla u|^p}{p} - \frac{|u|^{p^*}}{p^*} \Big) \mathrm{d}x.$$

We assume
(C) $1 < p < N$, and $a \in L^{N/p}(\mathbb{R}^N)$ is radial such that

$$\inf_{\substack{u \in \mathcal{D}^{1,p}_{\mathrm{rad}}(\mathbb{R}^N) \\ ||\nabla u||_{L^p} = 1}} \int_{\mathbb{R}^N} [|\nabla u|^p + a(x)|u|^p] \mathrm{d}x > 0.$$

Proposition 4.2 *Under assumption* **(C)**, *let $\{u_n\}_n$ be a sequence in $\mathcal{D}^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ such that*

$$\phi(u_n) \to c \qquad \phi'(u_n) \to 0 \quad in\ (\mathcal{D}^{1,p}_{\mathrm{rad}}(\mathbb{R}^N))'.$$

Then, passing if necessary to a subsequence, there exists a possibly nontrivial solution $v_0 \in \mathcal{D}^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ to

$$-\Delta_p u + a(x)|u|^{p-2}u = |u|^{p^*-2}u$$

and k sequences $\{\lambda_n^i\}_n \subset \mathbb{R}_+$*, such that* $\lambda_n^i \to 0$ *or* $\lambda_n^i \to \infty$ *satisfying*

$$\|u_n - v_0 - \sum_{i=1}^{k} (\lambda_n^i)^{(p-N)/p} v_i(\cdot/\lambda_n^i)\| \to 0, \quad n \to \infty,$$

$$\|u_n\|^p \to \sum_{i=0}^{k} \|v_i\|^p, \quad n \to \infty,$$

$$\phi(v_0) + \sum_{i=1}^{k} \tilde{\phi}_\infty(v_i) = c,$$

where for $i \geq 1$ v_i *is either identically zero, or for some* $\lambda > 0$ *and up to the sign, it holds that*

$$v_i \equiv \left[\frac{\lambda^{\frac{1}{p-1}} N^{\frac{1}{p}} (\frac{N-p}{p-1})^{\frac{p-1}{p}}}{\lambda^{\frac{p}{p-1}} + |\cdot|^{\frac{p}{p-1}}} \right]^{\frac{N-p}{p}},$$

and

$$\tilde{\phi}_\infty(v_i) = \frac{S^{N/p}}{N}.$$

Moreover if $a \equiv 0$*, then either* v_0 *is identically zero, or it holds that for some* $\lambda > 0$ *and up to the sign*

$$v_0 \equiv \left[\frac{\lambda^{\frac{1}{p-1}} N^{\frac{1}{p}} (\frac{N-p}{p-1})^{\frac{p-1}{p}}}{\lambda^{\frac{p}{p-1}} + |\cdot|^{\frac{p}{p-1}}} \right]^{\frac{N-p}{p}},$$

and

$$\phi(v_0) = \tilde{\phi}_\infty(v_0) = \frac{S^{N/p}}{N}.$$

Proof It follows by Theorem 5.4 of [10] and Proposition 1.3. When $a \equiv 0$, c can be computed explicitly by [17], using Proposition 1.3. □

5 Finite Morse index solutions

5.1 Bounds on the number of nodal regions and proof of Theorem 1.4

Let $\mathcal{O}$ be a domain of $\mathbb{R}^N$, and $u \in W^{1,p}_{\mathrm{loc}}(\mathcal{O})$ be such that

$$-\Delta_p u = |u|^{p^*-2} u \quad \text{in} \quad \mathcal{D}'(\mathcal{O}).$$

For all $v \in C^1_c(\mathcal{O})$ define

$$\phi''_\infty(u)[v, v] = \int_{\mathbb{R}^N} |\nabla u|^{p-2} |\nabla v|^2 + (p-2)|\nabla u|^{p-4} (\nabla u, \nabla v)^2 - (p^* - 1)|u|^{p^*-2} |v|^2.$$

We say that u has Morse index $i(u)$, see for instance [2,5], if $i(u)$ is the maximal dimension of the subspaces V of $C^1_c(\mathcal{O})$ such that

$$\phi''_\infty(u)[v,v] < 0, \qquad \text{for all } v \in V\setminus\{0\}.$$

Proof of Theorem 1.4 Define on $\mathcal{D}_0^{1,p}(\mathcal{O})$

$$\phi_\infty(u) = \int_{\mathcal{O}} \Big(\frac{|\nabla u|^p}{p} - \frac{|u|^{p^*}}{p^*}\Big)\mathrm{d}x.$$

The linearised functional is

$$\phi''_\infty(u)[v,v] = \int_{\mathcal{O}} |\nabla u|^{p-2}|\nabla v|^2 + (p-2)|\nabla u|^{p-4}(\nabla u,\nabla v)^2 - (p^*-1)|u|^{p^*-2}|v|^2.$$

Pick the sequence (u_n) as given by Lemma 3.1 and with $u|_\omega$ being the restriction of u to ω. Extend by zero outside ω $u|\omega$ and all u_n. By Lemma 3.1 we have

$$\begin{aligned}\phi''_\infty(u)[u_n,u_n] &\longrightarrow \int_\omega |\nabla u|^p + (p-2)|\nabla u|^p - (p^*-1)|u|^{p^*}\\ &= \int_\omega (p-1)|\nabla u|^p - (p^*-1)|u|^{p*}.\end{aligned}$$

This and Lemma 1.1 yields

$$\phi''_\infty(u)[u_n,u_n] \longrightarrow (p-p^*)\int_\omega |u|^{p*} < 0.$$

This means that for every nodal domain ω there exists a direction $u_n \in C_c^{0,1}(\omega)$ (and by density in $C_c^1(\omega)$) such that

$$\phi''_\infty(u)[u_n,u_n] < 0$$

and this concludes the proof. □

5.2 Proof of Theorem 1.5

Proof of Theorem 1.5 (1) If $u \equiv 0$ then $i(u) = 0$.

Assume that $u \not\equiv 0$ and $i(u) \le 1$. By Theorem 1.4 there is exactly one nodal region, say A. If $A = \mathbb{R}^N_+$ we have a contradiction by [10]. If A is a proper subset, we can assume, up to consider $-u$ instead of u, that $A = \{u > 0\}$. Since $u \in C^1(\mathbb{R}^N_+)$ by Lemma 2.1 it follows that $\nabla u = \mathbf{0}$ on ∂A. Pick now any interior point $p \in A$. There exists a ball $B(p,R) \subset A$ centered at p for some radius $R > 0$ such that $B(p,R)$ touches internally ∂A at some points. Let p' be such an intersection point. Since p' is a boundary point and satisfies the interior sphere condition, Hopf's boundary point lemma implies that $\nabla u(p') \ne \mathbf{0}$, which is a contradiction. This concludes the proof of part (1).

(2) Let now $i(u) \le 1$. By Theorem 1.4 u has at most a nodal domain A. If A is a proper subset, the preceding proof of part (1) shows that $u \equiv 0$. Otherwise if $A \equiv \mathbb{R}^N$ then the conclusion follows by the recent classification result [15]. And this concludes the proof of part (2). □

Remark 4 The above proof shows also that u cannot have a nodal domain A surrounded by a region where u is identically zero. Moreover, all nodal domains have always some boundary points satisfying an interior sphere condition.

We observe that in the case of (7) an alternative way to conclude the proof is by the strong maximum principle [19].

5.3 Starshaped domains: Proof of Theorem 1.6

Proof of Theorem 1.6 By a refinement of Moser's iteration, see e.g. Appendix E of [12] and [18], $u \in L^\infty(\Omega)$. By classical regularity results of DiBenedetto [3] and Liebermann [8], we have that $u \in C^{1,\alpha}(\overline{\Omega})$. By Theorem 1.1 of [6] it holds that the normal derivative $u_\nu = 0$ at some point $x_0 \in \partial\Omega$. By Theorem 1.4 u has at most one nodal region. If u were nontrivial, this would be in contrast with Hopf's boundary point lemma, see [19]. It follows that $u \equiv 0$, and this concludes the proof. □

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