Calculus of Variations

Fast and slow decaying solutions for H^1-supercritical quasilinear Schrödinger equations

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Abstract

We consider the following quasilinear Schrödinger equations of the form

$$\triangle u - \varepsilon V(x)u + u\triangle u^2 + u^p = 0, \ u > 0 \text{ in } \mathbb{R}^N \text{ and } \lim_{|x|\to\infty} u(x) = 0,$$

where $N \geq 3$, $p > \frac{N+2}{N-2}$, $\varepsilon > 0$ and $V(x)$ is a positive function. By imposing appropriate conditions on $V(x)$, we prove that, for $\varepsilon = 1$, the existence of infinity many positive solutions with slow decaying $O(|x|^{-\frac{2}{p-1}})$ at infinity if $p > \frac{N+2}{N-2}$ and, for ε sufficiently small, a positive solution with fast decaying $O(|x|^{2-N})$ if $\frac{N+2}{N-2} < p < \frac{3N+2}{N-2}$. The proofs are based on perturbative approach. To this aim, we also analyze the structure of positive solutions for the zero mass problem.

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1 Introduction

The nonlinear Schrödinger equation

$$i z_t = -\triangle z + W(x)z - [\triangle|z|^2]z - |z|^{p-1}z, \ (t, x) \in (0, \infty) \times \mathbb{R}^N, \tag{1.1}$$

where $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential, has been introduced in [1–3] to study a model of a self-trapped electrons in quadratic or hexagonal lattices (see also [4]). In those references numerical and analytical results have been given.

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Here of particular interest is in the existence of standing wave solutions, that is, solutions of type $z(x, t) = \exp(-iEt)u(x)$, where $E \in \mathbb{R}$. Assuming that the amplitude $u(x)$ is positive and vanishing at infinity, it is well known that z satisfies (1.1) if and only if the function u solves the following equation of quasilinear elliptic type

$$\begin{cases} \triangle u - \varepsilon V(x)u + u\triangle u^2 + u^p = 0, \ x \in \mathbb{R}^N; \\ u > 0 \text{ and } \lim_{|x|\to\infty} u(x) = 0, \end{cases} \tag{1.2}$$

where $V(x) = W(x) - E$ is the new potential function. In the rest of this paper we will assume that $V(x)$ is a bound and positive function.

Because of the presence of the quasilinear term $u\triangle u^2$, we can see that $p = \frac{3N+2}{N-2}$ is the critical exponent for the existence of solutions from the view of variational methods. For the subcritical case, that is, $1 < p < \frac{3N+2}{N-2}$, construction of solutions to this problem by variational methods has been a hot topic during the last decade. A typical result for the Eq. (1.2) is, up to our knowledge, due to Liu et al. [5]. The idea in [5] is to make a change of variable and reduce the quasilinear problem (1.2) to a semilinear one and the Orlicz space framework is used to prove the existence of positive solutions via the mountain pass theorem. Subsequently, the same method of changing of variable is also used in Colin and Jeanjean [6], but the usual Sobolev space $H^1(\mathbb{R}^N)$ is used as the working space. Recently, Shen and Wang [7] study the following generalized quasilinear Schrödinger equation:

$$- \operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = h(u), \ x \in \mathbb{R}^N, \tag{1.3}$$

where $g^2(s) = 1 + \frac{1}{2}(l(s^2)')^2$. By introducing the variable replacement

$$v = G(u) = \int_0^u g(t)\mathrm{d}t, \ u = G^{-1}(v) \tag{1.4}$$

and imposing some conditions on $V(x)$, the authors obtain the positive solution for (1.3) with a general function $l(s)$ when $h(s)$ is superlinear and subcritical. But under the condition

$$\lim_{|x|\to\infty} |x|^2 V(x) = 0, \tag{1.5}$$

the solvability of the Eq. (1.2) with $1 < p < \frac{3N+2}{N-2}$ still remains open.

Subcriticality is a rather essential constraint in the use of many variational methods devised in the literature and many papers [8–12] focused on the subcritical case. Very little is known in the supercritical case since a major technical obstacle in understanding such problems stems from the lack of Sobolev embeddings suitably fit to a weak formulation of this problem. Direct tools of the calculus of variation, very useful in subcritical, and even critical cases, are not appropriate in the supercritical. In the critical case, Liu et al. [5] asked the following open question: are there solutions for (1.2) in the case of $p = \frac{3N+2}{N-2}$? However, generally speaking, except some results relate to the critical exponent, see, for instance, [13–20], there are still no conclusive results about the existence of positive solutions for the problem (1.2) with $p = \frac{3N+2}{N-2}$ or $p > \frac{3N+2}{N-2}$.

In all the papers mentioned above variational methods are used. In this paper, we shall explore the distinctive nature of this problem for having *two* critical exponents, one being $p = \frac{3N+2}{N-2}$ (from the quasilinear term $u\triangle u^2$) and the other being $p = \frac{N+2}{N-2}$ which is H^1-critical (from the term $\triangle u$). We shall concentrate in the problem (1.2) when the exponent p is H^1-supercritical, that is, $p > \frac{N+2}{N-2}$, (which includes $p = \frac{3N+2}{N-2}$), and we establish a new phenomenon from the viewpoint of *singular perturbations*. Noticing that (1.2) is a

quasilinear problem, we adopt the change of variables which enable us to convert the original quasilinear problem (1.2) into a semilinear problem

$$\begin{cases} \triangle v - \varepsilon V(x)\dfrac{G^{-1}(v)}{g(G^{-1}(v))} + f(v) = 0, \ x \in \mathbb{R}^N; \\ v > 0 \text{ and } \lim_{|x|\to\infty} v(x) = 0, \end{cases} \tag{1.6}$$

where $f(v) = \frac{G^{-1}(v)^p}{g(G^{-1}(v))}$ and $g(s) = \sqrt{1 + 2s^2}$. Thus, if v is a solution of (1.6), we have $u = G^{-1}(v)$ is a solution of (1.2).

A solution v to (1.6) is called fast decaying if $v = O(|x|^{2-N})$ at infinity and slow decaying if $v >> O(|x|^{2-N})$. Then, to describe our result about the fast and slow decaying solutions, our starting point is the zero mass problem

$$\begin{cases} \triangle u + u\triangle u^2 + u^p = 0, \ x \in \mathbb{R}^N; \\ u > 0 \text{ and } \lim_{|x|\to\infty} u(x) = 0. \end{cases} \tag{1.7}$$

Applying the change of variables (1.4) again, the quasilinear problem (1.7) can be reduced to the equations of the form

$$\begin{cases} \triangle v + f(v) = 0, \ x \in \mathbb{R}^N; \\ v > 0 \text{ and } \lim_{|x|\to\infty} v(x) = 0. \end{cases} \tag{1.8}$$

Our first result concerns with the structure of positive radial solutions of the zero mass problem (1.7).

Theorem 1.1 *Suppose that $p > 1$. Then*

(1) *there exist no fast decaying solutions to the problem (1.7) if $p \geq \frac{3N+2}{N-2}$ or $1 < p \leq \frac{N+2}{N-2}$;*
(2) *there exist a unique fast decaying radial solution to the problem (1.7) if $\frac{N+2}{N-2} < p < \frac{3N+2}{N-2}$;*
(3) *there exist a one-parameter family of slow decaying radial solutions to the problem (1.7) if $p > \frac{N+2}{N-2}$.*

Remark 1.1 Some cases of the results of Theorem 1.1 are contained in [21,22]. More specifically, similarly to the standard Liouville theorem, if $1 < p < \frac{N+2}{N-2}$, the authors proved the nonexistence results of fast decaying solutions to (1.7) (see [21]). In [22], the authors showed the existence of a unique fast decaying solution and a one-parameter family of slow decaying solutions to (1.7) if $\frac{N+2}{N-2} < p < \frac{3N+2}{N-2}$ via the results introduced in [23]. Moreover, the authors in [22] also pointed out that they did not know whether there are solutions for the Eq. (1.7) with $p = \frac{N+2}{N-2}$. Particularly, in Theorem 1.1, we draw the definite conclusion about this case by using the Pohozaev identity.

Theorem 1.1 shows that the structure of solutions changes along with the variations of the power p and we remark that the solvability of the Eq. (1.7) heavily depends on the power p. Let us explain the main reason for such a rich phenomenon. On one hand, $f(v) \to v^p$ as $v \to 0$. On the other hand, $f(v) \to 2^{\frac{p-3}{4}} v^{\frac{p-1}{2}}$ as $v \to +\infty$. That is, the nonlinearity f is not a pure power of v but f has both H^1-subcritical and H^1-supercritical growth in $v > 0$. In [24], the authors consider a similar model

$$f(u) = \begin{cases} u^p & u \geq 1; \\ u^q & u < 1, \end{cases}$$

where $1 < p < \frac{N+2}{N-2} < q$ and give an almost complete description for the structure of positive radial solutions by a shooting argument.

The following result is about the fast decaying solutions of the Eq. (1.2).

Theorem 1.2 *Assume that*

$$V > 0, \ V \in L^\infty(\mathbb{R}^N) \ and \ V(x) = o(|x|^{-2}) \ as \ |x| \to +\infty \tag{1.9}$$

hold. Then for ε sufficiently small the problem (1.2) *has a positive fast decaying solution if* $\frac{N+2}{N-2} < p < \frac{3N+2}{N-2}$.

Compared with Theorem 1.1, it is natural to ask whether the nonexistence of a fast decaying solution remains true for (1.2) when $p \geq \frac{3N+2}{N-2}$. This may be in general a difficult question to answer if no other conditions imposed on $V(x)$. For the special case $x \cdot \nabla V(x) + 2V(x) \geq 0$, the authors in [25] show the nonexistence results of fast decaying solutions by a Pohozeav identity for the Eq. (1.2) in the case $p \geq \frac{3N+2}{N-2}$ and $\varepsilon = 1$.

Our final result concerns the existence of slow decaying solutions.

Theorem 1.3 *Assume that $\varepsilon = 1$. Then the problem* (1.2) *has a continuum of solutions $u_\lambda(x)$ such that $\lim_{\lambda \to 0} u_\lambda(x) = 0$ uniformly in $\mathbb{R}^N$ provided that either $N \geq 4$, $p > \frac{N+1}{N-3}$ and the condition* (1.9) *holds or $N \geq 3$, $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$ and there exist $C > 0$, $\mu > N$ such that*

$$V(x) \leq C|x|^{-\mu} \ for \ x \in \mathbb{R}^N. \tag{1.10}$$

Remark 1.2 In this theorem, we answer the question raised in [5] for $p = \frac{3N+2}{N-2}$.

The proofs of Theorems 1.2 and 1.3 are based on perturbative approach, introduced by Dávila et al. [26–29] in the study of fast and slow decaying solutions for second order or nonlinear Schrödinger equations and exterior domain problems. Some of our ideas are motivated from these papers.

In the fast-decaying case, we consider the problem (1.6) as small perturbation of the problem (1.8) when $\varepsilon > 0$ is sufficiently small. For a point $\xi \in \mathbb{R}^N$ used as the reference origin, the function $v_f(x + \xi)$ is considered as an initial approximation, where v_f is a solution of (1.8). This function will constitute a good approximation for small ε. By adjusting ξ, we prove that the solutions we want can be achieved.

As for the slow decaying solution of the Eq. (1.2), we set $\varepsilon = 1$ and consider the equation with a parameter λ by means of replacing the variable v in the Eq. (1.6) by $\lambda^{\frac{2}{p-1}} v(\lambda x + \xi)$

$$\begin{cases} \triangle v - V_\lambda(x)\lambda^{-\frac{2}{p-1}} \dfrac{G^{-1}(\lambda^{\frac{2}{p-1}}v)}{g(G^{-1}(\lambda^{\frac{2}{p-1}}v))} + \lambda^{-\frac{2p}{p-1}} f(\lambda^{\frac{2}{p-1}}v) = 0, \ x \in \mathbb{R}^N; \\ v > 0 \ and \ \lim\limits_{|x|\to\infty} v(x) = 0, \end{cases} \tag{1.11}$$

where $\lambda > 0$, $\xi \in \mathbb{R}^N$ and $V_\lambda(x) = \lambda^{-2}V(\frac{x-\xi}{\lambda})$. We observe that $\lambda^{-\frac{2}{p-1}} \dfrac{G^{-1}(\lambda^{\frac{2}{p-1}}v)}{g(G^{-1}(\lambda^{\frac{2}{p-1}}v))} \to v$ and $\lambda^{-\frac{2p}{p-1}} f(\lambda^{\frac{2}{p-1}}v) \to v^p$ as $\lambda \to 0$. Thus the problem may be regarded as small perturbation of the problem

$$\triangle v - V_\lambda v + v^p = 0$$

when $\lambda > 0$ is sufficiently small. Consequently, infinitely many positive solutions with slow decaying $O(|x|^{-\frac{2}{p-1}})$ at infinity can be constructed similar to the perturbative procedure introduced by Dávila et al. [26].

In this paper, we make use of the following notations: the symbol C denotes a positive constant (possibly different) independent with λ. $A \sim B$ if and only if there exist two positive constants a, b such that $aA \leq B \leq bA$. v_f denotes the unique fast decaying solution of (1.8).

2 Proof of Theorem 1.1

In this section, we analyze the structure of positive decaying solutions of (1.7). We first prove the nonexistence of fast-decaying solutions for $p \leq \frac{N+2}{N-2}$ or $p \geq \frac{3N+2}{N-2}$ by using the Pohozaev identity. Then we show the existence of the fast decaying solution for (1.7) by using the classical Berestycki–Lions condition in [30] for $\frac{N+2}{N-2} < p < \frac{3N+2}{N-2}$. Finally we use a perturbative approach to prove the existence of a family of slow-decaying solutions for $p > \frac{N+2}{N-2}$.

To prove the nonexistence results for the Eq. (1.8), we recall the following Pohozaev identity.

Lemma 2.1 (Pohozaev identity) *Suppose $F(x, u, r) \in C^1(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$ satisfies*

$$\operatorname{div} F_r(x, u, \nabla u) = F_u(x, u, \nabla u), \tag{2.1}$$

where

$$F_r(x, u, r) = (F_{r_1}(x, u, r), F_{r_2}(x, u, r), \ldots, F_{r_N}(x, u, r)), \ r = (r_1, r_2, \ldots, r_N),$$
$$F_{r_i}(x, u, r) = \frac{\partial F(x, u, r)}{\partial r_i}, \ i = 1, 2, \ldots, N$$

and

$$F_u(x, u, r) = \frac{\partial F(x, u, r)}{\partial u}.$$

Then, if $F(x, u, \nabla u), x \cdot F_x(x, u, \nabla u)$ and $F_r(x, u, \nabla u) \cdot \nabla u \in L^1(\mathbb{R}^N)$, there holds the following identity

$$N \int_{\mathbb{R}^N} F(x, u, \nabla u) dx + \int_{\mathbb{R}^N} x \cdot F_x(x, u, \nabla u) dx - \int_{\mathbb{R}^N} F_r(x, u, \nabla u) \cdot \nabla u dx = 0. \tag{2.2}$$

We omit the proof of this lemma, since it can be mainly found in [31].

To present the Pohozaev identity associated to (1.7), we rewrite the Eq. (1.7) as

$$\operatorname{div} \left(g^2(u) \nabla u \right) - g(u) g'(u) |\nabla u|^2 + u^p = 0. \tag{2.3}$$

Thus, the integrands in (2.2) can be expressed as

$$F(x, u, \nabla u) = \frac{1}{2} g^2(u) |\nabla u|^2 - \frac{1}{p+1} u^{p+1},$$
$$x \cdot F_x(x, u, \nabla u) = 0$$

and

$$F_r(x, u, \nabla u) \cdot \nabla u = g^2(u) |\nabla u|^2.$$

Consequently, we achieve the following lemma based on Lemma 2.1 under the conditions $|\nabla u|^2, u^2 |\nabla u|^2$ and $u^{p+1} \in L^1(\mathbb{R}^N)$.

Lemma 2.2 *Suppose that $u \in C^2(\mathbb{R}^N)$ is a solution of* (1.7). *Then*

$$\frac{N-2}{2} \int_{\mathbb{R}^N} (1 + 2u^2) |\nabla u|^2 dx = \frac{N}{p+1} \int_{\mathbb{R}^N} |u|^{p+1} dx \qquad (2.4)$$

if $|\nabla u|^2$, $u^2 |\nabla u|^2$ *and* $u^{p+1} \in L^1(\mathbb{R}^N)$.

Equation (1.7) can be rewritten as

$$- \operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 = u^p, \ x \in \mathbb{R}^N, \qquad (2.5)$$

where $g^2(u) = 1 + 2u^2$. By Lemma 2.2, the Pohozaev identity associated to (2.5) is

$$\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + (N-2) \int_{\mathbb{R}^N} u^2 |\nabla u|^2 dx = \frac{N}{p+1} \int_{\mathbb{R}^N} u^{p+1} dx. \qquad (2.6)$$

On the other hand, the classical solution $u \in D^{1,2}(\mathbb{R}^N)$ of (2.5) satisfies

$$\int_{\mathbb{R}^N} [g^2(u)\nabla u \nabla \phi + g(u)g'(u)|\nabla u|^2 \phi] dx = \int_{\mathbb{R}^N} u^p \phi dx.$$

By taking $\phi = u$, we achieve

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx + 4 \int_{\mathbb{R}^N} u^2 |\nabla u|^2 dx = \int_{\mathbb{R}^N} u^{p+1} dx. \qquad (2.7)$$

Consequently, combining (2.6) and (2.7), we have

$$\left[\frac{N-2}{2} - \frac{N}{p+1} \right] \int_{\mathbb{R}^N} |\nabla u|^2 dx + \left[(N-2) - \frac{4N}{p+1} \right] \int_{\mathbb{R}^N} u^2 |\nabla u|^2 dx = 0. \qquad (2.8)$$

If $p \geq \frac{3N+2}{N-2}$, then $(N-2) - \frac{4N}{p+1} \geq 0$ and $\frac{N-2}{2} - \frac{N}{p+1} > 0$. Therefore, (2.8) implies that $u = 0$ under this situation. Similarly, if $p \leq \frac{N+2}{N-2}$, it follows that $(N-2) - \frac{4N}{p+1} < 0$ and $\frac{N-2}{2} - \frac{N}{p+1} \leq 0$. Thus, (2.8) also shows that $u = 0$. So there are no nonzero solutions for (1.7) if $p \leq \frac{N+2}{N-2}$ or $p \geq \frac{3N+2}{N-2}$.

This proves (1) of Theorem 1.1.

Next we prove the existence of fast decaying solutions to (1.7). By the change of variable $u = G^{-1}(v)$ we only need to consider (1.8). To this end we recall the following classical proposition by Berestycki and Lions [30].

Proposition 2.1 *Suppose that the following assumptions hold:*

(F-1) $f(0) = 0$ *and* $\overline{\lim}\limits_{s \to 0^+} \frac{f(s)}{s^l} \leq 0$, *where* $l = \frac{N+2}{N-2}$;

(F-2) *There exists* $\zeta > 0$ *such that* $F(\zeta) > 0$, *where* $F(\zeta) = \int_0^\zeta f(s)ds$;

(F-3) *Let* $\zeta_0 = \inf \{\zeta : \zeta > 0, \ F(\zeta) > 0\}$. *If* $f(s) > 0$ *for all* $s > \zeta_0$, *then* $\lim\limits_{s \to +\infty} \frac{f(s)}{s^l} = 0$.

Then the problem (1.8) *has a positive, spherically symmetric and decreasing (with r) solution v such that* $v \in D^{1,2}(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$.

We now show that $f(s)$ satisfies the conditions (F-1)–(F-3) in Proposition 2.1.

By the definition of $f(s)$, we know that (F-2) is trivial. Noticing that $\lim\limits_{s \to 0} \frac{G^{-1}(s)}{s} = 1$, we have

$$\lim_{s \to 0^+} \frac{f(s)}{s^l} = \lim_{s \to 0^+} \frac{G^{-1}(s)^p}{g(G^{-1}(s))s^l} = \lim_{s \to 0^+} \frac{s^p}{s^l} = 0,$$

which shows that $f(s)$ satisfies the condition (F-1).

To verify the condition (F-3), it suffices to show that

$$\lim_{s \to +\infty} \frac{f(s)}{s^l} = 0$$

since $\zeta_0 = 0$ and $f(s) > 0$ for all $s > 0$. Combining the fact $\lim_{s \to +\infty} \frac{G^{-1}(s)}{\sqrt{s}} = 2^{\frac{1}{4}}$, we deduce that

$$\lim_{s \to +\infty} \frac{f(s)}{s^l} = \lim_{s \to +\infty} \frac{G^{-1}(s)^p}{g(G^{-1}(s))s^l} = \lim_{s \to +\infty} \frac{2^{\frac{p-3}{4}} s^{\frac{p-1}{2}}}{s^l} = 0.$$

This proves (2) of Theorem 1.1.

Finally we prove (3) of Theorem 1.1. To prove the existence of slow decaying solutions, since we are considering the autonomous case, that is, $V(x) \equiv 0$, we can restrict to the radially symmetric case. For this reason, we take $v(x) = v(r)$, where $r = |x|$.

We first consider the problem in the entire space

$$\begin{cases} \triangle u + u^p = 0, \ x \in \mathbb{R}^N; \\ u(0) = 1. \end{cases}$$

It is well known that this problem possesses a unique positive symmetric solution $w(|x|)$ whenever $p > \frac{N+2}{N-2}$. Then all radial solutions to this problem defined in $\mathbb{R}^N$ can be expressed as

$$w_\lambda(|x|) = \lambda^{\frac{2}{p-1}} w(\lambda|x|), \ \lambda > 0$$

and, at a main order, one has

$$w(r) = C_{p,N} r^{-\frac{2}{p-1}} + o(1) \text{ as } r = |x| \to +\infty,$$

which implies that this behavior is actually common to all solutions $w_\lambda(r)$.

Since the problem (1.8) does not carry any parameter explicitly, for $\lambda > 0$, we can make parameters appear by means of replacing the variable v in the equation by $\lambda^{\frac{2}{p-1}} v(\lambda|x|)$, in such a way the problem (1.8) becomes

$$\begin{cases} \triangle v + \lambda^{-\frac{2p}{p-1}} f(\lambda^{\frac{2}{p-1}} v) = 0, \ r \in (0, +\infty); \\ v > 0 \text{ and } \lim_{r \to \infty} v(r) = 0. \end{cases} \tag{2.9}$$

Then, jointly with the properties of $G^{-1}(v) = v + o(1)$ and $g(G^{-1}(v)) = 1 + o(1)$ as $v \to 0$, if v is uniformly bounded, we observe that $\lambda^{-\frac{2p}{p-1}} f(\lambda^{\frac{2}{p-1}} v) \to v^p$ as $\lambda \to 0$. Thus the problem may be regarded as small perturbation of the problem

$$\triangle v + v^p = 0$$

when $\lambda > 0$ is sufficiently small. Consequently, a positive solution with slow decaying $O(|x|^{-\frac{2}{p-1}})$ at infinity can be constructed by asymptotic analysis and Liapunov-Schmidt reduction method. To be more specific, the idea of the proof of Theorem 1.1-(3) is, for λ small, to consider the function $\lambda^{\frac{2}{p-1}} w(\lambda|x|)$ as an initial approximation. This scaling will constitute a good approximation under our situations for λ sufficiently small. Then, by a classical fixed point argument for contraction mappings, we prove that (2.9) possesses solutions as desired. Similar idea has been used in [26,28].

Under appropriate norms

$$\|\phi\|_* = \sup_{|x|\leq 1} |x|^\sigma |\phi(x)| + \sup_{|x|\geq 1} |x|^{\frac{2}{p-1}} |\phi(x)| \tag{2.10}$$

and

$$\|h\|_{**} = \sup_{|x|\leq 1} |x|^{2+\sigma} |h(x)| + \sup_{|x|\geq 1} |x|^{2+\frac{2}{p-1}} |h(x)|, \tag{2.11}$$

where $\sigma > 0$, we first consider the solvability of the linear problem

$$\begin{cases} \phi'' + \frac{N-1}{r}\phi' + pw^{p-1}\phi = h, \ r \in (0, +\infty); \\ \lim_{r\to+\infty} \phi(r) = 0 \end{cases} \tag{2.12}$$

and thus we need the following lemma which is Lemma A. 1 proved by Dávila et al. [26].

Lemma 2.3 *Assume $0 < \sigma < N - 2$ and $p > \frac{N+2}{N-2}$. Then there exists a constant $C > 0$ such that for any h satisfying $\|h\|_{**} < +\infty$, Eq. (2.12) has a solution $\phi = \mathcal{T}(h)$ such that $\mathcal{T}$ define a linear map and*

$$\|\phi\|_* = \|\mathcal{T}(h)\|_* \leq C\|h\|_{**}.$$

Let us look for a solution to (2.9) of the form $v = w + \phi$, which yields the following equation for $\phi = \phi(r)$

$$\begin{cases} \triangle\phi + pw^{p-1}\phi = S(w) + N(\phi), \ r \in (0, +\infty); \\ \lim_{r\to+\infty} \phi(r) = 0, \end{cases} \tag{2.13}$$

where

$$S(w) = -\triangle w - \lambda^{-\frac{2p}{p-1}} f(\lambda^{\frac{2}{p-1}} w)$$

and

$$N(\phi) = \lambda^{-\frac{2p}{p-1}} f(\lambda^{\frac{2}{p-1}} w) + pw^{p-1}\phi - \lambda^{-\frac{2p}{p-1}} f(\lambda^{\frac{2}{p-1}} (w + \phi)).$$

We first estimate the error $\|S(w)\|_{**}$ of the approximate solution. The fact

$$|S(w)| = |\lambda^{-\frac{2p}{p-1}} f(\lambda^{\frac{2}{p-1}} w) - w^p|$$

and the properties of the change of variables (1.4) show that, for $C_p > 0$,

$$S(w) = C_p \lambda^{\frac{4}{p-1}} w^{p+2} + o\left(\lambda^{\frac{4}{p-1}} w^{p+2}\right) \text{ as } \lambda \to 0.$$

Thus, it follows that

$$|S(w)| \leq C\lambda^{\frac{4}{p-1}} |w|^{p+2}.$$

We then conclude

$$\sup_{|x|\leq 1} |x|^{2+\sigma} |S(w)| \leq C\lambda^{\frac{4}{p-1}} \|w\|_\infty^{p+2} \sup_{|x|\leq 1} |x|^{2+\sigma} \leq C\lambda^{\frac{4}{p-1}}. \tag{2.14}$$

On the other hand, recalling that $w(x) \leq C(1 + |x|)^{-\frac{2}{p-1}}$ for $x \in \mathbb{R}^N$, we obtain

$$\sup_{|x|\geq 1} |x|^{2+\frac{2}{p-1}} |S(w)| \leq C\lambda^{\frac{4}{p-1}} \sup_{|x|\geq 1} \left(\frac{|x|}{1+|x|}\right)^{\frac{2(p+2)}{p-1}} \leq C\lambda^{\frac{4}{p-1}}. \tag{2.15}$$

From (2.14) and (2.15), we have

$$\|S(w)\|_{**} \le C\lambda^{\frac{4}{p-1}}. \tag{2.16}$$

In what follows, the proof relies on the contraction mapping theorem. We observe that ϕ solves (2.13) if and only if ϕ is a fixed point for the operator

$$\phi = \mathcal{T}(S(w) + N(\phi)),$$

where $\mathcal{T}$ is introduced in Lemma 2.3. That is to say, ϕ solves (2.13) if and only if ϕ is a fixed point for the operator

$$\mathcal{A}(\phi) := \mathcal{T}(S(w) + N(\phi)).$$

We define

$$\Sigma = \left\{ \phi : \mathbb{R}^N \to \mathbb{R} \mid \|\phi\|_* \le C\lambda^{\frac{4}{p-1}} \right\}$$

and we will prove that $\mathcal{A}$ has a fixed point in Σ.

For any $\phi \in \Sigma$ and $\sigma \in \left(0, \min\left\{ 2, \frac{2}{p-1} \right\} \right)$, according to the arguments given in [26], we have

$$\|N(\phi)\|_{**} \le C[\|\phi\|_*^2 + \|\phi\|_*^p] \tag{2.17}$$

since

$$N(\phi) = w^p + pw^{p-1}\phi - (w + \phi)^p + o(1) \text{ as } \lambda \to 0.$$

Therefore, combining (2.16), (2.17) and Lemma 2.3, it follows that

$$\begin{aligned}
\|\mathcal{A}(\phi)\|_* &\le C[\|S(w)\|_{**} + \|N(\phi)\|_{**}] \\
&\le C[\lambda^{\frac{4}{p-1}} + \lambda^{\frac{8}{p-1}} + \lambda^{\frac{4p}{p-1}}] \le C\lambda^{\frac{4}{p-1}},
\end{aligned} \tag{2.18}$$

which implies that $\mathcal{A}(\Sigma) \subset \Sigma$.

We still have to prove that $\mathcal{A}$ is a contraction mapping in Σ. Let us take $\phi_1, \phi_2 \in \Sigma$. Then we have

$$\|\mathcal{A}(\phi_1) - \mathcal{A}(\phi_2)\|_* \le C\|N(\phi_1) - N(\phi_2)\|_{**}. \tag{2.19}$$

Moreover, noting that

$$|N(\phi_1) - N(\phi_2)| \le C\left(w^{p-2}(|\phi_1| + |\phi_2|) + |\phi_1|^{p-1} + |\phi_2|^{p-1} \right) |\phi_1 - \phi_2|,$$

we have the estimate

$$\begin{aligned}
\|\mathcal{A}(\phi_1) - \mathcal{A}(\phi_2)\|_* &\le C\|N(\phi_1) - N(\phi_2)\|_{**} \\
&\le C\left[\|\phi_1\|_*^{\min\{1, p-1\}} + \|\phi_2\|_*^{\min\{1, p-1\}} \right] \|\phi_1 - \phi_2\|_* \\
&\le \frac{1}{2}\|\phi_1 - \phi_2\|_*
\end{aligned}$$

for suitable small λ. This means that $\mathcal{A}$ is a contraction mapping from Σ into itself, and hence a fixed point ϕ in this region indeed exists. So the function $v_\lambda(|x|) := \lambda^{\frac{2}{p-1}}(w(\lambda|x|) + \phi(\lambda|x|))$ is a continuum solutions of (2.13) satisfying $\lim_{\lambda \to 0} v_\lambda(|x|) = 0$ uniformly in $\mathbb{R}^N$ and $u_\lambda(|x|) = G^{-1}(v_\lambda(|x|))$ is our desired solution. This complete the proof of Theorem 1.1.

3 Proof of Theorem 1.2

In this section, we will construct a fast decaying solution to the problem (1.2) when $\frac{N+2}{N-2} < p < \frac{3N+2}{N-2}$ by the reduction method. The idea of the proof of Theorem 1.2 is, for $\xi \in \mathbb{R}^N$ and ε small, to consider the function $v_f(x + \xi)$ as an initial approximation, where $v_f(x)$ is the unique positive radial solution of the zero mass problem (1.7) stated in Theorem 1.1. These functions will constitute good approximations under our situations for suitable $\xi \in \mathbb{R}^N$ and ε sufficiently small. Then, by adjusting ξ, we prove that (1.2) possesses a solution as desired.

At the beginning, we state some notations which will be used in the following. We consider the initial value problem

$$\begin{cases} v'' + \frac{N-1}{r}v' + f(v) = 0, \ r \in (0, +\infty); \\ v(0) = d > 0, \ v'(0) = 0, \end{cases} \tag{3.1}$$

where $f(s) = \frac{G^{-1}(s)^p}{g(G^{-1}(s))}$. By Theorem 1.1, there exists a unique $d^* > 0$ such that the corresponding solution $v_f(r; d^*)$ is the unique positive fast decaying solution. Moreover, $z_0(r) := \frac{\partial v_f}{\partial d}(r; d^*)$ satisfies the following initial value problem

$$\begin{cases} \phi'' + \frac{N-1}{r}\phi' + f'(v_f)\phi = 0, \ r \in (0, +\infty); \\ \phi(0) = 1 > 0, \ \phi'(0) = 0. \end{cases} \tag{3.2}$$

Then by Lemma 4.4 in [22], we have that v_f is non-degenerate in $D_r^{1,2}(\mathbb{R}^N)$–radial functions in $D^{1,2}$. Our next lemma shows that it is nondegenerate in the class of bounded functions. Let $Z_i = \frac{\partial v_f}{\partial x_i}$ for $1 \le i \le N$. Then we have the following result.

Lemma 3.1 *If ϕ satisfies $|\phi| \le C$ and*

$$\triangle \phi + f'(v_f)\phi = 0, \ x \in \mathbb{R}^N, \tag{3.3}$$

then $\phi \in W = Span\{Z_1, Z_2, \dots, Z_n\}$.

Proof If ϕ is bounded and satisfies (3.3), by bootstrapping, we achieve $\phi(x) = O(|x|^{2-N})$ as $|x| \to +\infty$. Expanding ϕ as

$$\phi(x) = \sum_{k=0}^{\infty} \phi_k(r)\Theta_k(\vartheta),$$

we see that ϕ_k is a solution of

$$\phi_k'' + \frac{N-1}{r}\phi_k' + \left(f'(v_f) - \frac{\lambda_k}{r^2}\right)\phi_k = 0 \text{ for all } r > 0 \text{ and } k \ge 0. \tag{3.4}$$

For mode 0, noticing that $\lambda_0 = 0$, we know $\phi_0(r)$ is a solution of (3.4) and, by Lemma 4.2 in [22], $\phi_0(r)$ satisfies

$$r^{\lambda^*}\phi_0(r) \to -\infty \text{ as } r \to \infty,$$

where $\lambda^* = \begin{cases} \frac{N-1}{2} & \text{if } N \ge 4; \\ \frac{1}{2} & \text{if } N = 3. \end{cases}$ Thus, if $\phi_0(r) \in D_r^{1,2}(\mathbb{R}^N)$, we conclude that

$$r^{\lambda^*}z_0(r) = \begin{cases} O(r^{\frac{3-N}{2}}), & \text{if } N \ge 4; \\ O(r^{-\frac{1}{2}}), & \text{if } N = 3, \end{cases}$$

which is a contradiction. For mode k with $k > 1$, according to Lemma A. 3 in [26], we conclude that the solution ϕ_k to (3.4) is zero by the maximum principle. Consequently, jointly with the nondegeneracy in radial class, we have

$$\phi = \phi_1 \in \text{Span}\{Z_1, Z_2, \ldots, Z_N\}.$$

□

We introduce appropriate norms

$$\|\phi\|_{*,\xi} = \sup_{x \in \mathbb{R}^N} < x - \xi >^\sigma |\phi(x)| \tag{3.5}$$

and

$$\|h\|_{**,\xi} = \sup_{x \in \mathbb{R}^N} < x - \xi >^{2+\sigma} |h(x)|, \tag{3.6}$$

where $< \cdot > := \left(1 + |\cdot|^2\right)^{\frac{1}{2}}$ and $0 < \sigma < N - 2$. We first solve the linear problem

$$\begin{cases} \triangle\phi + f'(v_f)\phi = h + \sum_{i=1}^N c_i f'(v_f) Z_i, \ x \in \mathbb{R}^N; \\ \int_{\mathbb{R}^N} f'(v_f)\phi Z_i = 0, \ i = 1, 2, \ldots, N; \\ \lim_{|x| \to +\infty} \phi(x) = 0. \end{cases} \tag{3.7}$$

Lemma 3.2 *Let $\Lambda > 0$ and $|\xi| \leq \Lambda$. Assume $\frac{N+2}{N-2} < p < \frac{3N+2}{N-2}$ and $\sigma < N - 2$. Then there is a linear map $(\phi, c_1, \ldots, c_N) = \mathcal{T}(h)$ defined whenever $\|h\|_{**,\xi} < \infty$ such that $(\phi, c_1, \ldots, c_N)$ satisfies (3.7) and*

$$\|\phi\|_{*,\xi} + \sum_{i=1}^N |c_i| \leq C\|h\|_{**,\xi}. \tag{3.8}$$

Moreover, $c_i = 0$ for all $1 \leq i \leq N$ if and only if

$$\int_{\mathbb{R}} h \frac{\partial v_f}{\partial x_i} = 0 \text{ for } 1 \leq i \leq N. \tag{3.9}$$

Proof We will divide the proof into two steps.

Step 1. A priori estimate

By taking $h = h^{(1)} + h^{(2)}$ in (3.7), where $h^{(1)} \in W_1 = \{f'(v_f)Z_1, f'(v_f)Z_2, \ldots, f'(v_f)Z_N\}$ and $h^{(2)} \in W_1^\perp$, we have

$$\triangle\phi + f'(v_f)\phi = h^{(1)} + h^{(2)} + \sum_{i=1}^N c_i f'(v_f) Z_i. \tag{3.10}$$

If we take $h^{(1)} = -\sum_{i=1}^N c_i f'(v_f) Z_i$, that is,

$$c_i = -\frac{\int_{\mathbb{R}^N} h^{(1)} Z_i}{\int_{\mathbb{R}^N} f'(v_f)|Z_i|^2} \text{ for } i = 1, 2, \ldots, N, \tag{3.11}$$

it follows from (3.10) that

$$\triangle\phi + f'(v_f)\phi = h^{(2)} \tag{3.12}$$

and $c_i = 0$ for all $1 \leq i \leq N$ if and only if

$$\int_{\mathbb{R}} h \frac{\partial v_f}{\partial x_i} = 0 \text{ for } 1 \leq i \leq N.$$

So, in what follows, we consider

$$\begin{cases} \triangle \phi + f'(v_f)\phi = h^{(2)}, \ x \in \mathbb{R}^N; \\ \int_{\mathbb{R}^N} f'(v_f)\phi Z_i = 0, \ i = 1, 2, \dots, N; \\ \lim_{|x| \to +\infty} \phi(x) = 0. \end{cases} \tag{3.13}$$

We first prove the priori estimates (3.8) by using the contradiction argument. Suppose that there exist ϕ_n, $h_n^{(2)}$ such that $\|\phi_n\|_{*,\xi} = 1$ and $\|h_n^{(2)}\|_{**,\xi} = o(1)$ as $n \to +\infty$. By the definition of $\|\phi_n\|_{*,\xi}$, we can take $x_n \in \mathbb{R}^N$ with the property

$$< x_n - \xi >^\sigma |\phi_n(x_n)| \geq \frac{1}{2}. \tag{3.14}$$

Then, we again have to distinguish two possibilities. Along a subsequence, it follows that $x_n \to x_0 \in \mathbb{R}^N$ or $|x_n| \to +\infty$.

If $x_n \to x_0$, standard elliptic estimates show that $\phi_n \to \phi$ uniformly on compact sets of $\mathbb{R}^N$. Moreover, ϕ is a solution to (3.13) with $h^{(2)} = 0$ satisfying

$$< x_0 - \xi >^\sigma |\phi(x_0)| \geq \frac{1}{2} \tag{3.15}$$

and $|\phi(x)| < +\infty$. Thus Lemma 3.1 shows that

$$\phi = \phi_1 \in \text{Span} \{Z_1, Z_2, \dots, Z_N\}.$$

Then the facts $\int_{\mathbb{R}^N} \nabla \phi \cdot \nabla Z_i = 0$ for $i = 1, 2, \dots, N$ show that $\nabla \phi = 0$. We achieve a contradiction to (3.15) since $\lim_{|x| \to +\infty} \phi(x) = 0$.

If $x_n \to +\infty$, We consider $\tilde{\phi}_n(y) = |x_n|^\sigma \phi_n(|x_n|y + x_n + \xi)$ and observe that $\tilde{\phi}_n$ satisfies

$$\triangle \tilde{\phi}_n + |x_n|^2 f'(v_{f,n}) \tilde{\phi}_n = \tilde{h}_n^{(2)}, \ y \in \mathbb{R}^N,$$

where $v_{f,n}(y) = v_f(|x_n|y + x_n + \xi)$ and $\tilde{h}_n^{(2)}(y) = |x_n|^{2+\sigma} h_n^{(2)}(|x_n|y + x_n + \xi)$. Noticing that $\|\phi_n\|_{*,\xi} = 1$, we have

$$|\tilde{\phi}_n(y)| \leq \frac{1}{(y + \hat{x}_n)^\sigma}, \ \forall y \in \mathbb{R}^N \setminus \{-\hat{x}_n\}, \tag{3.16}$$

where $\hat{x}_n := \frac{x_n}{|x_n|}$. So $\tilde{\phi}_n$ is uniformly bounded on compact sets of $\mathbb{R}^N \setminus \{-\hat{x}_n\}$. Similarly, considering that

$$|\tilde{h}_n^{(2)}(y)| \leq \frac{1}{(y + \hat{x}_n)^\sigma} \|h_n^{(2)}\|_{**,\xi}, \ \forall y \in \mathbb{R}^N \setminus \{-\hat{x}_n\},$$

we obtain $\tilde{h}_n^{(2)} \to 0$ uniformly on compact sets of $\mathbb{R}^N \setminus \{-\hat{x}_n\}$ as $n \to +\infty$. Thus, by elliptic estimates, we have $\tilde{\phi}_n \to \tilde{\phi}$ uniformly on compact sets of $\mathbb{R}^N \setminus \{\hat{e}\}$ and $\tilde{\phi}$ satisfies

$$\begin{cases} \triangle \tilde{\phi} = 0, \ y \in \mathbb{R}^N \setminus \{\hat{e}\}; \\ |\tilde{\phi}(y)| \leq \frac{1}{|y - \hat{e}|^\sigma}, \ \forall y \in \mathbb{R}^N \setminus \{\hat{e}\}, \end{cases}$$

where $\hat{e} = -\lim\limits_{n\to+\infty}\frac{x_n}{|x_n|}$. By the maximum principle, we conclude that $\tilde{\phi} = 0$ which is impossible since $\tilde{\phi}(-\hat{e}) \neq 0$.

Step 2. Existence

We first want to solve (3.7) on a bounded domain $B_R(\xi)$. Let us consider the subspace

$$X = \left\{\phi \in D_0^{1,2}(B_R(\xi)) \text{ and } \int_{B_R(\xi)} \psi f'(v_f)Z_i = 0, \ i = 1, 2, \ldots, N\right\}.$$

Then, according the arguments in [32], finding solution to (3.7) in this case is equivalent to finding $\phi \in X$ such that

$$\int_{B_R(\xi)} \nabla\phi\nabla\psi - \int_{B_R(\xi)} f'(v_f)\phi\psi + \int_{B_R(\xi)} h\psi = 0 \text{ for all } \psi \in X. \tag{3.17}$$

Now, for h satisfying $\|h\|_{**,\xi} < +\infty$, let us denote by $\phi = A(h)$ the unique solution of the problem

$$\int_{B_R(\xi)} \nabla\phi\nabla\psi + \int_{B_R(\xi)} h\psi = 0 \text{ for all } \psi \in X.$$

Thus, (3.17) can be written as

$$\phi - A(f'(v_f)\phi) = A(h) \text{ for } \phi \in X$$

and, by the compactness of Sobolev's embedding, the map $\phi \to f'(v_f)\phi$ is compact.

Hence, we conclude the existence of the solution by the Fredholm alternative since the priori estimate (3.8) implies that the only solution of this equation is $\phi = 0$ when $h = 0$. Finally, thanks to the priori estimate again, we can let $R \to +\infty$ and obtain the existence in the whole space. □

Now we begin to prove Theorem 1.2. We look for a solution of the form $v = v_f + \phi$ to the Eq. (1.6) and thus acieve the following equation for ϕ

$$\begin{cases} \triangle\phi + f'(v_f)\phi = E(v_f) + F(\phi) + M(\phi), \ x \in \mathbb{R}^N; \\ \lim\limits_{|x|\to+\infty} \phi(x) = 0, \end{cases} \tag{3.18}$$

where

$$E(v_f) = \varepsilon V(x - \xi)\frac{G^{-1}(v_f)}{g(G^{-1}(v_f))},$$

$$F(\phi) = f(v_f) + f'(v_f)\phi - f(v_f + \phi)$$

and

$$M(\phi) = \varepsilon V(x - \xi)\left[\frac{G^{-1}(v_f + \phi)}{g(G^{-1}(v_f + \phi))} - \frac{G^{-1}(v_f)}{g(G^{-1}(v_f))}\right].$$

However, the problem (3.18) may not be solvable under our situation unless ξ can be chosen in a very special way. So instead of solving (3.18), we consider the following projected problem

$$\begin{cases} \triangle\phi + f'(v_f)\phi = E(v_f) + F(\phi) + M(\phi) + \sum_{i=1}^N c_i f'(v_f)Z_i, \ x \in \mathbb{R}^N; \\ \lim\limits_{|x|\to+\infty} \phi(x) = 0, \end{cases} \tag{3.19}$$

where c_i are constants.

For $\frac{2}{p-1} < \sigma < N - 2$, we first estimate the error $\|E(v_f)\|_{**,\xi}$ of the approximate solution v_f. Considering that

$$\left| \frac{G^{-1}(v_f)}{g(G^{-1}(v_f))} \right| \leq v_f$$

and

$$|v_f| \leq C(1 + |x|)^{2-N} \text{ for all } x \in \mathbb{R}^N,$$

we have

$$
\begin{aligned}
\|E(v_f)\|_{**,\xi} &= \sup_{x \in \mathbb{R}^N} < x - \xi >^{2+\sigma} |E(v_f)| \\
&\leq \varepsilon \sup_{x \in \mathbb{R}^N} < x - \xi >^{2+\sigma} V(x - \xi)|v_f| \\
&\leq C\varepsilon \sup_{x \in \mathbb{R}^N} \left(\frac{< x - \xi >}{1 + |x|} \right)^{\sigma} (1 + |x|)^{2-N+\sigma} \\
&\leq C\varepsilon.
\end{aligned}
$$

In what follows, by applying the Banach fixed point theorem, we can prove that (3.19) is indeed solvable and achieve a solution $(\phi_\varepsilon, c_1, \ldots, c_N)$. We then obtain a solution of the problem (3.18) if $c_i = 0$ for all $i = 1, 2, \ldots, N$.

Based on the description of Lemma 3.2, solving (3.19) reduces now to a fixed point problem. Namely, we need to find a fixed point for the map

$$(\phi, c_1, c_2, \ldots, c_N) = \mathcal{A}(\phi, c_1, c_2, \ldots, c_N) := \mathcal{T}(N_1(\phi) + N_2(\phi)).$$

Here, we will restrict ϕ to be small enough such that the function $v_f + \phi$ is always positive and we define the set

$$\Theta = \left\{ (\phi, c_1, c_2, \ldots, c_N) \in \mathbb{R}^{N+1} \mid \|\phi\|_{*,\xi} + \sum_{i=1}^{N} |c_i| \leq C\varepsilon \right\}.$$

We now prove that $\mathcal{A}$ has a fixed point in Θ.

For any $(\phi, c_1, c_2, \ldots, c_N) \in \Theta$, we first estimate $M(\phi)$. Note that

$$\left(\frac{G^{-1}(s)}{g(G^{-1}(s))} \right)' = \frac{1}{g^4(G^{-1}(s))} \leq 1 \text{ for all } s \geq 0.$$

We have

$$\left| \frac{G^{-1}(v_f + \phi)}{g(G^{-1}(v_f + \phi))} - \frac{G^{-1}(v_f)}{g(G^{-1}(v_f))} \right| \leq |\phi|$$

and then

$$
\begin{aligned}
\|M(\phi)\|_{**,\xi} &= \sup_{x \in \mathbb{R}^N} < x - \xi >^{2+\sigma} |M(\phi)| \\
&\leq \varepsilon \sup_{x \in \mathbb{R}^N} < x - \xi >^{2+\sigma} V(x - \xi)|\phi| \qquad (3.20) \\
&\leq C\varepsilon \|\phi\|_{*,\xi}.
\end{aligned}
$$

To estimate $F(\phi)$, we need the following fact: if $1 < p < 2$, then

$$|f'(s_1) - f'(s_2)| \le C|s_1 - s_2|^{p-1} \text{ for all } s_1, s_2 \ge 0 \text{ and } |s_1 - s_2| \le 1. \qquad (3.21)$$

Indeed, since

$$f'(s) = (p-1)\frac{G^{-1}(s)^{p-1}}{g^2(G^{-1}(s))} + \frac{G^{-1}(s)^{p-1}}{g^4(G^{-1}(s))},$$

we have

$$\begin{aligned}
|f'(s_1) - f'(s_2)| &\le (p-1)\left|\frac{G^{-1}(s_1)^{p-1}}{g^2(G^{-1}(s_1))} - \frac{G^{-1}(s_2)^{p-1}}{g^2(G^{-1}(s_2))}\right| \\
&+ \left|\frac{G^{-1}(s_1)^{p-1}}{g^4(G^{-1}(s_1))} - \frac{G^{-1}(s_2)^{p-1}}{g^4(G^{-1}(s_2))}\right|.
\end{aligned} \qquad (3.22)$$

Then, noticing that $|s_1 - s_2| \le 1$, we have

$$\begin{aligned}
&\left|\frac{G^{-1}(s_1)^{p-1}}{g^2(G^{-1}(s_1))} - \frac{G^{-1}(s_2)^{p-1}}{g^2(G^{-1}(s_2))}\right| \\
&= \frac{\left|(1 + 2G^{-1}(s_2)^2)G^{-1}(s_1)^{p-1} - (1 + 2G^{-1}(s_1)^2)G^{-1}(s_2)^{p-1}\right|}{g^2(G^{-1}(s_1))g^2(G^{-1}(s_2))} \\
&\le \frac{\left|G^{-1}(s_1)^{p-1} - G^{-1}(s_2)^{p-1}\right|}{g^2(G^{-1}(s_1))g^2(G^{-1}(s_2))} + 2\frac{\left|G^{-1}(s_2)^2 G^{-1}(s_1)^{p-1} - G^{-1}(s_1)^2 G^{-1}(s_2)^{p-1}\right|}{g^2(G^{-1}(s_1))g^2(G^{-1}(s_2))} \\
&\le C|G^{-1}(s_1) - G^{-1}(s_2)|^{p-1} + \frac{2G^{-1}(s_1)^{p-1}G^{-1}(s_2)^{p-1}}{g^2(G^{-1}(s_1))g^2(G^{-1}(s_2))}|G^{-1}(s_2)^{3-p} - G^{-1}(s_1)^{3-p}| \\
&\le C|s_1 - s_2|^{p-1} + \frac{2(3-p)G^{-1}(s_{12})^{2-p}G^{-1}(s_1)^{p-1}G^{-1}(s_2)^{p-1}}{g(G^{-1}(s_{12}))g^2(G^{-1}(s_1))g^2(G^{-1}(s_2))}|s_1 - s_2| \\
&\le C|s_1 - s_2|^{p-1} \qquad (3.23)
\end{aligned}$$

since

$$\frac{2(3-p)G^{-1}(s_{12})^{2-p}G^{-1}(s_1)^{p-1}G^{-1}(s_2)^{p-1}}{g(G^{-1}(s_{12}))g^2(G^{-1}(s_1))g^2(G^{-1}(s_2))} \le C \text{ for all } s_1, s_2 \ge 0,$$

where s_{12} belongs to the segment jointing s_1 and s_2. On the other hand, by a similar strategy as the proof of the inequality (3.23), we conclude that

$$\left|\frac{G^{-1}(s_1)^{p-1}}{g^4(G^{-1}(s_1))} - \frac{G^{-1}(s_2)^{p-1}}{g^4(G^{-1}(s_2))}\right| \le C|s_1 - s_2|^{p-1}$$

and thus show the inequality (3.21).

Since ϕ is small, based on the fact (3.21), we observe that

$$\begin{aligned}
|F(\phi)| &= |f(v_f) + f'(v_f)\phi - f(v_f + \phi)| \\
&\le |f'(v_1)\phi - f'(v_f)\phi| \\
&\le \begin{cases} |f''(v_2)(v_1 - v_f)||\phi|, & \text{if } p \ge 2; \\ C|(v_1 - v_f)^{p-1}\phi|, & \text{if } 1 < p < 2 \end{cases} \\
&\le \begin{cases} C|v_2|^{p-2}|\phi|^2, & \text{if } p \ge 2; \\ C|\phi|^p, & \text{if } 1 < p < 2, \end{cases}
\end{aligned} \qquad (3.24)$$

where v_1 lies in the segment jointing v_f, $v_f + \phi$ and $v_2 = tv_f + (1-t)v_1$ with $t \in [0, 1]$. Thus, jointly with the fact $|v_2| \leq C(1+|x|)^{-\sigma}$ for all $x \in \mathbb{R}^N$, we have

$$
\begin{aligned}
\|F(\phi)\|_{**,\xi} &= \sup_{x \in \mathbb{R}^N} <x-\xi>^{2+\sigma} |F(\phi)| \\
&\leq \begin{cases} C \sup_{x \in \mathbb{R}^N} <x-\xi>^{2+\sigma} |v_2|^{p-2}|\phi|^2, & \text{if } p \geq 2; \\ C \sup_{x \in \mathbb{R}^N} <x-\xi>^{2+\sigma} |\phi|^p, & \text{if } 1 < p < 2, \end{cases} \\
&\leq \begin{cases} C \sup_{x \in \mathbb{R}^N} <x-\xi>^{2-(p-1)\sigma} \left(\frac{<x-\xi>}{1+|x|}\right)^{(p-2)\sigma} \|\phi\|_{*,\xi}^2, & \text{if } p \geq 2; \\ C \sup_{x \in \mathbb{R}^N} <x-\xi>^{2-(p-1)\sigma} \|\phi\|_{*,\xi}^p, & \text{if } 1 < p < 2, \end{cases} \\
&\leq C\|\phi\|_{*,\xi}^{\gamma},
\end{aligned}
\tag{3.25}
$$

where $\gamma = \min\{2, p\}$.

Therefore, by (3.20) and (3.25), jointly with Lemma 3.2, it follows that

$$
\begin{aligned}
\|\mathcal{A}(\phi, c_1, c_2, \ldots, c_N)\|_{*,\xi} &\leq C(\|E(v_f)\|_{**,\xi} + \|M(\phi)\|_{**,\xi} + \|F(\phi)\|_{**,\xi}) \\
&\leq C(\varepsilon + \varepsilon\|\phi\|_{*,\xi} + \|\phi\|_{*,\xi}^{\gamma}) \\
&\leq C\varepsilon,
\end{aligned}
\tag{3.26}
$$

which shows $\mathcal{A}(\Theta) \subset \Theta$.

We still have to prove that $\mathcal{A}$ is a contraction mapping in Θ. If we take

$$(\phi_1, c_{1,1}, c_{2,1}, \ldots, c_{N,1}), (\phi_2, c_{1,2}, c_{2,2}, \ldots, c_{N,2}) \in \Theta,$$

then we have

$$
\begin{aligned}
&\|\mathcal{A}(\phi_1, c_{1,1}, c_{2,1}, \ldots, c_{N,1}) - \mathcal{A}(\phi_2, c_{1,2}, c_{2,2}, \ldots, c_{N,2})\|_{*,\xi} \\
&\leq C[\|M(\phi_1) - M(\phi_2)\|_{**,\xi} + \|F(\phi_1) - F(\phi_2)\|_{**,\xi}].
\end{aligned}
\tag{3.27}
$$

To estimate $\|M(\phi_1) - M(\phi_2)\|_{**,\xi}$, we note that

$$|M(\phi_1) - M(\phi_2)| = |D_\phi M(\bar{\phi})(\phi_1 - \phi_2)|, \tag{3.28}$$

where $\bar{\phi}$ lies in the segment joining ϕ_1 and ϕ_2. Moreover, a direct calculation shows

$$
\begin{aligned}
|D_\phi M(\phi)| &= \left| \frac{\varepsilon V(x-\xi)}{g^2(G^{-1}(v_f+\phi))} \left[1 - \frac{2G^{-1}(v_f+\phi)^2}{g^2(G^{-1}(v_f+\phi))} \right] \right| \\
&= \left| \frac{\varepsilon V(x-\xi)}{g^4(G^{-1}(v_f+\phi))} \right| \leq \varepsilon V(x-\xi).
\end{aligned}
\tag{3.29}
$$

Then,

$$
\begin{aligned}
\sup_{x \in \mathbb{R}^N} <x-\xi>^{2+\sigma} |M(\phi_1) - M(\phi_2)| &\leq \sup_{x \in \mathbb{R}^N} <x-\xi>^2 |D_\phi N_1(\bar{\phi})| \|\phi_1 - \phi_2\|_{*,\xi} \\
&\leq C\varepsilon\|\phi_1 - \phi_2\|_{*,\xi} \sup_{x \in \mathbb{R}^N} <x-\xi>^2 V(x-\xi) \\
&\leq C\varepsilon\|\phi_1 - \phi_2\|_{*,\xi}.
\end{aligned}
\tag{3.30}
$$

Thus, we have

$$\|M(\phi_1) - M(\phi_2)\|_{**,\xi} \leq C\varepsilon\|\phi_1 - \phi_2\|_{*,\xi}. \tag{3.31}$$

Now we estimate $\|F(\phi_1) - F(\phi_2)\|_{**,\xi}$. We note that

$$|F(\phi_1) - F(\phi_2)| = |D_\phi N_2(\bar{\phi})(\phi_1 - \phi_2)|, \tag{3.32}$$

where $\bar{\phi}$ lies in the segment joining ϕ_1 and ϕ_2. Moreover,

$$|D_\phi N_2(\phi)| = |f'(v_f) - f'(v_f + \phi)|$$
$$\leq \begin{cases} C|v_1|^{p-2}|\phi|, & \text{if } p \geq 2; \\ C|\phi|^{p-1}, & \text{if } 1 < p < 2, \end{cases} \tag{3.33}$$

where $v_1 = tv_f + (1-t)(v_f + \phi)$ with $t \in [0,1]$. Then, similarly as the proof of (3.25), we have

$$\|F(\phi_1) - F(\phi_2)\|_{**,\xi} \leq C\|\bar{\phi}\|_{*,\xi}^{\min\{1,p-1\}}\|\phi_1 - \phi_2\|_{*,\xi} \leq C\varepsilon^{\min\{1,p-1\}}\|\phi_1 - \phi_2\|_{*,\xi}. \tag{3.34}$$

Thus, under our situation, combining (3.27), (3.31) and (3.34), we have that $\mathcal{A}$ is a contraction mapping in Θ and hence there indeed exists a fixed point $(\phi_\varepsilon, c_1, c_2, \ldots, c_N)$.

In what follows of this section, we will complete the proof of Theorem 1.2.

We have found a solution $(\phi_\varepsilon, c_1, c_2, \ldots, c_N)$ to (3.19) satisfying

$$\|\phi_\varepsilon\|_{*,\xi} + \sum_{i=1}^{N} |c_i| \leq C\varepsilon.$$

To prove the result contained in Theorem 1.2, it suffices to show that the point ξ can be adjust so that the constants $c_1, c_2, \ldots, c_N$ are all contemporarily equal to zero. Combining Lemma 3.2, we only need to show

$$\int_{\mathbb{R}^N} (E(v_f) + M(\phi_\varepsilon) + F(\phi_\varepsilon)) \frac{\partial v_f}{\partial x_j} = 0 \text{ for } j = 1, 2, \ldots, N. \tag{3.35}$$

We first define

$$F_j(\varepsilon, \xi) = \int_{\mathbb{R}^N} (E(v_f) + M(\phi_\varepsilon) + F(\phi_\varepsilon)) \frac{\partial v_f}{\partial x_j}. \tag{3.36}$$

The subordinate terms in (3.36) are $\int_{\mathbb{R}^N} F(\phi_\varepsilon) \frac{\partial v_f}{\partial y_j}$ and $\int_{\mathbb{R}^N} M(\phi_\varepsilon) \frac{\partial v_f}{\partial y_j}$. Indeed, we have the following estimates

$$\left| \int_{\mathbb{R}^N} F(\phi_\varepsilon) \frac{\partial v_f}{\partial y_j} \right| \leq \|F(\phi_\varepsilon)\|_{**,\xi} \int_{\mathbb{R}^N} <x-\xi>^{-2-\sigma} \left| \frac{\partial v_f}{\partial y_j} \right| = O(\varepsilon^\gamma) \tag{3.37}$$

and

$$\left| \int_{\mathbb{R}^N} M(\phi_\varepsilon) \frac{\partial v_f}{\partial y_j} \right| \leq \|M(\phi_\varepsilon)\|_{**,\xi} \int_{\mathbb{R}^N} <x-\xi>^{-2-\sigma} \left| \frac{\partial v_f}{\partial y_j} \right| = O(\varepsilon^2). \tag{3.38}$$

Noticing that there exist $k_0 \in (0, 1)$ and $k_1 > 1$ such that $k_0 v_f \leq \frac{G^{-1}(v_f)}{g(G^{-1}(v_f))} \leq k_1 v_f$, the dominant term in (3.36) satisfies

$$
\begin{aligned}
\int_{\mathbb{R}^N} E(v_f) \frac{\partial v_f}{\partial y_j} &= \varepsilon \int_{\mathbb{R}^N} V(y - \xi) \frac{G^{-1}(v_f)}{g(G^{-1}(v_f))} \frac{\partial v_f}{\partial y_j} \sim \varepsilon \int_{\mathbb{R}^N} V(y - \xi) v_f \frac{\partial v_f}{\partial y_j} \\
&= \varepsilon \int_{\mathbb{R}^N} V(y - \xi) v_f \frac{\partial v_f}{\partial y_j} \\
&= \frac{\varepsilon}{2} \int_{\mathbb{R}^N} v_f^2 \frac{\partial V}{\partial \xi_j}(y - \xi) \\
&= \frac{\varepsilon}{2} \frac{\partial}{\partial \xi_j} \int_{\mathbb{R}^N} v_f^2 V(y - \xi).
\end{aligned}
\tag{3.39}
$$

Combining (3.37)–(3.39), we achieve

$$
F_j(\varepsilon, \xi) \sim \frac{\varepsilon}{2} \frac{\partial}{\partial \xi_j} \int_{\mathbb{R}^N} v_f^2 V(y - \xi) + o(\varepsilon) \text{ for } j = 1, 2, \ldots, N.
$$

Thus, if we set $G(\xi) = \int_{\mathbb{R}^N} v_f^2 V(y - \xi)$, then $G(0) > 0$ and $\lim_{|x| \to +\infty} G(\xi) = 0$. This implies that G attains a global maximum point $\xi_0 \in B_M(0)$ for some $M > 0$. By the definition of stable critical point [33], G has a stable critical point in $B_M(0)$ and as a result, we deduce that, for ε small, $F(\varepsilon, \xi) = (F_1(\varepsilon, \xi), F_2(\varepsilon, \xi), \ldots, F_N(\varepsilon, \xi))$ has a zero point in $B_M(0)$. Consequently, $c_j = 0$ for $j = 1, 2, \ldots, N$.

4 Proof of Theorem 1.3

In this section, we will construct slow decaying solutions to the problem (1.2) with $\varepsilon = 1$. The results of Theorem 1.3 are based on a suitable linear theory devised for the linearized operator associated to the Eq. (1.2) at $u = w$ in the entire space $\mathbb{R}^N$ and in the application of perturbation arguments. We consider w as an approximation for a solution of (1.2), provided that $\lambda > 0$ is chosen small enough. To this aim, we need to know the solvability of the operator $\triangle - V_\lambda + pw^{p-1}$ in suitable weighted L^∞ space.

Let

$$
Z_i = \eta \frac{\partial w}{\partial x_i}, i = 1, 2, \ldots, N,
$$

where $\eta \in C_0^\infty(\mathbb{R}^N)$ satisfies $0 \leq \eta \leq 1$. Moreover, $\eta(x) = 1$ if $|x| \leq R_0$ and $\eta(x) = 0$ if $|x| \geq R_0 + 1$ for a fixed number $R_0 > 0$ large enough.

Under appropriate norms

$$
\|\phi\|_{*,\xi} = \sup_{|x - \xi| \leq 1} |x - \xi|^\sigma |\phi(x)| + \sup_{|x - \xi| \geq 1} |x - \xi|^{\frac{2}{p-1}} |\phi(x)|
\tag{4.1}
$$

and

$$
\|h\|_{**,\xi} = \sup_{|x - \xi| \leq 1} |x - \xi|^{2+\sigma} |h(x)| + \sup_{|x - \xi| \geq 1} |x - \xi|^{2+\frac{2}{p-1}} |h(x)|,
\tag{4.2}
$$

where $\sigma > 0$ and $\xi \in \mathbb{R}^N$, we first consider the solvability of the linear problem

$$\begin{cases} \triangle\phi - V_\lambda(x)\phi + pw^{p-1}\phi = h + \sum_{i=1}^N c_i Z_i, \ x \in \mathbb{R}^N; \\ \lim_{|x|\to+\infty} \phi(x) = 0 \end{cases} \tag{4.3}$$

and thus we need the following lemma which is proved by Dávila et al. [26].

Lemma 4.1 *Let $|\xi| \leq \Lambda$. Suppose V satisfies (1.9) and $\|h\|_{**,\xi} < \infty$. Then, for $\lambda > 0$ sufficiently small,*

(1) *if $N \geq 4$, $p > \frac{N+1}{N-3}$, Eq. (4.3) with $c_i = 0$ for $1 \leq i \leq N$ and $\xi = 0$ has a solution $\phi = \mathcal{T}_\lambda(h)$ which depends linearly on h and there exist a constant C independent with λ such that*

$$\|\mathcal{T}_\lambda(h)\|_{*,0} \leq C\|h\|_{**,0};$$

(2) *if $N \geq 3$, $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$ and V also satisfies (1.10), Eq. (4.3) has a solution $(\phi, c_1, c_2, \ldots, c_N) = \mathcal{T}_\lambda(h)$ which depends linearly on h and there exist a constant C independent with λ such that*

$$\|\phi\|_{*,\xi} + \max_{1\leq i\leq N}|c_i| \leq C\|h\|_{**,\xi}.$$

Moreover, $c_i = 0$ for all $1 \leq i \leq N$ if and only if

$$\int_{\mathbb{R}^N} h\frac{\partial w}{\partial x_i} = 0 \text{ for } 1 \leq N \leq N. \tag{4.4}$$

Based on Lemma 4.1, we can prove Theorem 1.3. We look for a solution of the form $v = w + \phi$ to the Eq. (1.11) and, for $S(w)$, $N(\phi)$ defined in Sect. 2 and $l(s) := \frac{G^{-1}(s)}{g(G^{-1}(s))}$, we achieve the following equation

$$\begin{cases} \triangle\phi - V_\lambda(x)\phi + pw^{p-1}\phi = S_1(w) + N(\phi) + P(\phi), \ x \in \mathbb{R}^N; \\ \lim_{|x|\to+\infty} \phi(x) = 0, \end{cases} \tag{4.5}$$

where

$$S_1(w) = S(w) + V_\lambda(x)\lambda^{-\frac{2}{p-1}}l(\lambda^{\frac{2}{p-1}}w)$$

and

$$P(\phi) = V_\lambda(x)[\lambda^{-\frac{2}{p-1}}(l(\lambda^{\frac{2}{p-1}}(w + \phi)) - l(\lambda^{\frac{2}{p-1}}w)) - \phi].$$

The case $p > \frac{N+1}{N-3}$

In this case, we rescale $v(x)$ as $\lambda^{\frac{2}{p-1}}v(\lambda x)$, that is, $\xi = 0$ in the previous paragraph. Computations show that

$$\lambda^{-\frac{2}{p-1}}l(\lambda^{\frac{2}{p-1}}w) = w + o(1) \text{ as } \lambda \to 0.$$

According to the arguments in [26], we know

$$\|V_\lambda(x)\lambda^{-\frac{2}{p-1}}l(\lambda^{\frac{2}{p-1}}w)\|_{**,0} := \delta(\lambda) = o(1) \text{ as } \lambda \to 0.$$

Thus, for $\sigma \in \left(0, \min\left\{2, \frac{2}{p-1}\right\}\right)$, the error of the approximate solution in the norm (4.2) is

$$\|S_1(w)\|_{**,0} = \|S(w) + V_\lambda(x)\lambda^{-\frac{2}{p-1}}l(\lambda^{\frac{2}{p-1}}w)\|_{**,0} \leq C\rho(\lambda), \tag{4.6}$$

where $\rho(\lambda) := \lambda^{\frac{4}{p-1}} + \delta(\lambda)$. Consequently, for the operator $\mathcal{A}_\lambda(\phi) := \mathcal{T}_\lambda(S_1(w) + N(\phi) + P(\phi))$, where $\mathcal{T}_\lambda$ is given in Lemma 4.1-(1), we can use the contraction mapping theorem on

$$\Sigma_\lambda = \left\{ \phi : \mathbb{R}^N \to \mathbb{R} \big| \; \|\phi\|_{*,0} \leq C\rho(\lambda) \right\}$$

and we will prove that $\mathcal{A}_\lambda$ has a fixed point in Σ_λ.

For any $\phi \in \Sigma_\lambda$, we first give the estimate of $\|P(\phi)\|_{**,0}$. We observe that, for a number $C_3 > 0$,

$$\lambda^{-\frac{2}{p-1}}(l(\lambda^{\frac{2}{p-1}}(w+\phi)) - l(\lambda^{\frac{2}{p-1}}w)) - \phi = -C_3\lambda^{\frac{4}{p-1}}[(w+\phi)^3 - w^3] \\ + o\left(\lambda^{\frac{4}{p-1}}\right) \text{ as } \lambda \to 0. \tag{4.7}$$

Thus, we have

$$\begin{aligned} \sup_{|x|\leq 1} |x|^{2+\sigma}|P(\phi)| &\leq C \sup_{|x|\leq 1} |x|^{2+\sigma} V_\lambda(x) \lambda^{\frac{4}{p-1}}(|w^2\phi| + |\phi|^3) \\ &\leq C\lambda^{\frac{4}{p-1}} \sup_{|x|\leq 1} |x|^\sigma (|w^2\phi| + |\phi|^3) \\ &\leq C\lambda^{\frac{4}{p-1}} \|\phi\|_{*,0}. \end{aligned} \tag{4.8}$$

On the other hand, combining

$$|\phi(x)| \leq C|x|^{-\frac{2}{p-1}} \|\phi\|_{*,0} \text{ for } |x| \geq 1 \tag{4.9}$$

and

$$w(x) \leq C(1+|x|)^{-\frac{2}{p-1}} \text{ for } x \in \mathbb{R}^N, \tag{4.10}$$

we have

$$\begin{aligned} \sup_{|x|\geq 1} |x|^{2+\frac{2}{p-1}}|P(\phi)| &\leq C\lambda^{\frac{4}{p-1}} \sup_{|x|\geq 1} |x|^{2+\frac{2}{p-1}} V_\lambda(x)(|w^2\phi| + |\phi|^3) \\ &\leq C\lambda^{\frac{4}{p-1}} \sup_{|x|\geq 1} \left((1+|x|)^{-\frac{4}{p-1}} \|\phi\|_{*,0} + |x|^{-\frac{4}{p-1}} \|\phi\|_{*,0}^3 \right) \\ &\leq C\lambda^{\frac{4}{p-1}} (\|\phi\|_{*,0} + \|\phi\|_{*,0}^3). \end{aligned} \tag{4.11}$$

Consequently, combining (4.8) and (4.11), we have

$$\|P(\phi)\|_{**,0} \leq C\lambda^{\frac{4}{p-1}} (\|\phi\|_{*,0} + \|\phi\|_{*,0}^3). \tag{4.12}$$

Thus, jointly with the estimate of $\|N(\phi)\|_{**,0}$ in Sect. 2, we conclude

$$\begin{aligned} \|\mathcal{A}_\lambda(\phi)\|_{*,0} &\leq C\|S_1(w) + N(\phi) + P(\phi)\|_{**,0} \\ &\leq C[\rho(\lambda) + \rho(\lambda)^2 + \rho(\lambda)^p + \rho(\lambda) + \rho(\lambda)^3] \\ &\leq C\rho(\lambda) \text{ for any } \phi \in \Sigma_\lambda. \end{aligned}$$

That is, $\mathcal{A}_\lambda(\Sigma_\lambda) \subset \Sigma_\lambda$.

For any $\phi_1, \phi_2 \in \Sigma_\lambda$, we want to estimate $\|P(\phi_1) - P(\phi_2)\|_{**,0}$. We note that

$$|P(\phi_1) - P(\phi_2)| = |D_\phi P(\bar{\phi})(\phi_1 - \phi_2)|, \tag{4.13}$$

where $\bar{\phi}$ lies in the segment joining ϕ_1 and ϕ_2. Then, it follows that

$$|x|^{2+\sigma}|P(\phi_1) - P(\phi_2)| \leq |x|^2|D_\phi P(\bar{\phi})|\|\phi_1 - \phi_2\|_{*,0} \text{ for } |x| \leq 1$$

and

$$|x|^{2+\frac{2}{p-1}}|P(\phi_1) - P(\phi_2)| \leq |x|^2|D_\phi P(\bar{\phi})|\|\phi_1 - \phi_2\|_{*,0} \text{ for } |x| \geq 1.$$

Thus, we have

$$\|P(\phi_1) - P(\phi_2)\|_{**,0} \leq C \sup_{x \in \mathbb{R}^N} \left(|x|^2|D_\phi P(\bar{\phi})|\right) \|\phi_1 - \phi_2\|_{*,0}. \tag{4.14}$$

Moreover, a direct calculation shows

$$D_\phi P(\phi) = V_\lambda(x) \left(\frac{1}{g^2(G^{-1}(\lambda^{\frac{2}{p-1}}(w+\phi)))} - \frac{2(G^{-1}(\lambda^{\frac{2}{p-1}}(w+\phi)))^2}{g^4(G^{-1}(\lambda^{\frac{2}{p-1}}(w+\phi)))} - 1 \right)$$

$$= V_\lambda(x) \left[1 - 4\lambda^{\frac{4}{p-1}}(w+\phi)^2 + o(\lambda^{\frac{4}{p-1}}(w+\phi)^2) - 1 \right] \text{ as } \lambda \to 0. \tag{4.15}$$

To go a step further, based on the arguments in the previous paragraph, we conclude that

$$\sup_{x \in \mathbb{R}^N} \left(|x|^2|D_\phi P(\bar{\phi})|\right) \leq C\lambda^{\frac{4}{p-1}}. \tag{4.16}$$

Consequently, combining (4.14) and (4.16), it follows that

$$\|P(\phi_1) - P(\phi_2)\|_{**,0} \leq \frac{1}{4}\|\phi_1 - \phi_2\|_{*,0} \tag{4.17}$$

for λ sufficiently small.

It is straightforward to show that

$$\|\mathcal{A}_\lambda(\phi_1) - \mathcal{A}_\lambda(\phi_2)\|_{*,0} \leq C[\|N(\phi_1) - N(\phi_2)\|_{**,0} + \|P(\phi_1) - P(\phi_2)\|_{**,0}]$$
$$\leq \frac{1}{2}\|\phi_1 - \phi_2\|_{*,0} \text{ for } \phi_1, \phi_2 \in \Sigma_\lambda \tag{4.18}$$

since we can achieve that

$$\|N(\phi_1) - N(\phi_2)\|_{**,0} \leq \frac{1}{4}\|\phi_1 - \phi_2\|_{*,0}$$

according to Sect. 2 for λ sufficiently small. This means that $\mathcal{A}_\lambda$ is a contraction mapping from Σ_λ into itself and hence a fixed point ϕ_λ indeed exists. So the function $v_\lambda(x) := \lambda^{\frac{2}{p-1}}(w(\lambda x) + \phi_\lambda(\lambda x))$ is a continuum solutions of (1.11) satisfying $\lim_{\lambda \to 0} v_\lambda(x) = 0$ uniformly in $\mathbb{R}^N$ and $u_\lambda(x) = G^{-1}(v_\lambda(x))$ is our desired solution to (1.2).

The case $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$

In this case, the problem (4.5) may not be solvable under our situation unless ξ is chosen in a very special way. So, instead of solving (4.5), we consider the following projected problem

$$\begin{cases} \Delta\phi - V_\lambda(x)\phi + pw^{p-1}\phi = S_1(w) + N(\phi) + P(\phi) + \sum_{i=1}^N c_i Z_i, \ x \in \mathbb{R}^N; \\ \lim_{|x|\to+\infty} \phi(x) = 0, \end{cases} \tag{4.19}$$

where c_i are constants. Moreover, we will slightly change the previous definition of the norms as

$$\|\phi\|_{*,\xi}^{(\theta)} = \sup_{|x-\xi|\leq 1} |x - \xi|^\theta|\phi(x)| + \sup_{|x-\xi|\geq 1} |x - \xi|^{\frac{2}{p-1}}|\phi(x)|$$

and

$$\|h\|_{**,\xi}^{(\theta)} = \sup_{|x-\xi|\leq 1} |x-\xi|^{2+\theta}|h(x)| + \sup_{|x-\xi|\geq 1} |x-\xi|^{2+\frac{2}{p-1}}|h(x)|.$$

Just as the case $p > \frac{N+1}{N-3}$, we can prove that (4.19) is indeed solvable and achieve a solution $(\phi(\lambda,\xi), c_1(\lambda,\xi), c_2(\lambda,\xi), \ldots, c_N(\lambda,\xi))$. We then obtain a solution of the problem (4.5) if $c_i(\lambda,\xi) = 0$ for all $i = 1, 2, \ldots, N$.

Here, we also fix $\sigma \in \left(0, \min\left\{2, \frac{2}{p-1}\right\}\right)$ and find the error of the approximate solution is

$$\|S_1(w)\|_{**,\xi}^{(\sigma)} \leq C\rho(\lambda), \tag{4.20}$$

where $\rho(\lambda) = o(1)$ as $\lambda \to 0$. So we can define

$$\Sigma_{\lambda,\sigma} = \left\{ (\phi, c_1, c_2, \ldots, c_N) \in \mathbb{R}^{N+1} \mid \|\phi\|_{*,\xi}^{(\sigma)} + \sum_{i=1}^{N} |c_i| \leq C\rho(\lambda) \right\}.$$

Similarly, as the proof of the previous case, jointly with Lemma 4.1-(2), we conclude that the operator $(\phi, c_1, c_2, \ldots, c_N) = \mathcal{A}_\lambda(\phi, c_1, c_2, \ldots, c_N) := \mathcal{T}_\lambda(S_1(w) + N(\phi) + P(\phi))$ is a contraction mapping in $\Sigma_{\lambda,\sigma}$ and hence achieve a fixed point

$$(\phi(\lambda,\xi), c_1(\lambda,\xi), c_2(\lambda,\xi) \ldots, c_N(\lambda,\xi)) \in \Sigma_{\lambda,\sigma},$$

which satisfies the Eq. (4.19). Moreover, under the condition (1.10), we observe that $\rho(\lambda)$ can be taken as λ^θ in (4.20) for any $\theta \in \left(0, \frac{4}{p-1}\right)$. That is,

$$\|\lambda^{-\frac{2}{p-1}} V_\lambda l(\lambda^{\frac{2}{p-1}} w)\|_{**,\xi}^{(\theta)} \leq C\lambda^\theta \text{ for } \theta \in (0, N-2), \tag{4.21}$$

$$\|S(w)\|_{**,\xi}^{(\theta)} \leq C\lambda^\theta \text{ for } \theta \in \left(0, \frac{4}{p-1}\right) \tag{4.22}$$

and

$$\|\phi(\lambda,\xi)\|_{*,\xi}^{(\theta)} + \max_{1\leq i\leq N} |c_i(\lambda)| \leq C\lambda^\theta \text{ for } \theta \in \left(0, \frac{4}{p-1}\right). \tag{4.23}$$

Thus, to complete our proof, by Lemma 4.1-(2) we need to find $\xi = \xi_\lambda$ such that

$$\int_{\mathbb{R}^N} \left(S(w) + V_\lambda \lambda^{-\frac{2}{p-1}} l(\lambda^{\frac{2}{p-1}} w) + N(\phi(\lambda,\xi)) + P(\phi(\lambda,\xi)) \right) \frac{\partial w}{\partial x_j} = 0, \ 1 \leq j \leq N. \tag{4.24}$$

Combining the arguments in [26] and noticing that $\frac{4}{p-1} < N-2$, we know

$$\int_{\mathbb{R}^N} \lambda^{-\frac{2}{p-1}} V_\lambda l(\lambda^{\frac{2}{p-1}} w) \frac{\partial w}{\partial x_j} = o(\lambda^{\frac{4}{p-1}}) \text{ as } \lambda \to 0 \tag{4.25}$$

and

$$\int_{\mathbb{R}^N} N(\phi(\lambda,\xi)) \frac{\partial w}{\partial x_j} = o(\lambda^{\frac{4}{p-1}}) \text{ as } \lambda \to 0. \tag{4.26}$$

Moreover, noticing that

$$|P(\phi(\lambda,\xi))| \leq C\lambda^{\frac{4}{p-1}} V_\lambda(|w^2\phi(\lambda,\xi)| + |\phi(\lambda,\xi)|^3),$$

we have

$$\left| \int_{\mathbb{R}^N} P(\phi(\lambda, \xi)) \frac{\partial w}{\partial x_j} \right| \leq C \lambda^{\frac{4}{p-1}} \left| \int_{\mathbb{R}^N} V_\lambda \phi(\lambda, \xi)(w^2 + \phi^2(\lambda, \xi)) \frac{\partial w}{\partial x_j} \right|$$

$$\leq C \lambda^{\frac{4}{p-1}} \left| \int_{\mathbb{R}^N} V_\lambda \phi(\lambda, \xi) \frac{\partial w}{\partial x_j} \right|$$

$$= o(\lambda^{\frac{4}{p-1}}) \text{ as } \lambda \to 0. \tag{4.27}$$

Now, we claim that the dominant term in (4.24) is

$$\int_{\mathbb{R}^N} S(w) \frac{\partial w}{\partial x_j}. \tag{4.28}$$

Note that

$$S(w) = C_p \lambda^{\frac{4}{p-1}} w^{p+2} + o(\lambda^{\frac{4}{p-1}} w^{p+2}) \text{ as } \lambda \to 0.$$

We have

$$\int_{\mathbb{R}^N} S(w) \frac{\partial w}{\partial x_j} = C_p \lambda^{\frac{4}{p-1}} \int_{\mathbb{R}^N} w^{p+2}(x+\xi) \frac{\partial w}{\partial x_j}(x+\xi) \mathrm{d}x + o(\lambda^{\frac{4}{p-1}}) \text{ as } \lambda \to 0. \tag{4.29}$$

If we define

$$F_\lambda^{(j)}(\xi) = \int_{\mathbb{R}^N} \left(S(w) + V_\lambda \lambda^{-\frac{2}{p-1}} l(\lambda^{\frac{2}{p-1}} w) + N(\phi(\lambda, \xi)) + P(\phi(\lambda, \xi)) \right) \frac{\partial w}{\partial x_j}$$

and $F_\lambda(\xi) = (F_\lambda^{(1)}(\xi), F_\lambda^{(2)}(\xi), \ldots, F_\lambda^{(N)}(\xi))$. Then, by (4.25)–(4.27) and (4.29), we achieve that

$$F_\lambda^{(j)}(\xi) = C_p \lambda^{\frac{4}{p-1}} \int_{\mathbb{R}^N} w^{p+2}(x+\xi) \frac{\partial w}{\partial x_j}(x+\xi) \mathrm{d}x + o(\lambda^{\frac{4}{p-1}}) \text{ as } \lambda \to 0$$

and so we can show the existence of a solution ξ_λ to (4.24) since 0 is a critical point of w. Thus, we conclude that

$$\langle F_\lambda(\xi), \xi \rangle < 0 \text{ for } |\xi| = \delta,$$

where δ is a fixed small constant. Using this fact and degree theory we obtain the existence of ξ_λ such that $F_\lambda(\xi_\lambda) = 0$ in B_δ. This complete the proof of Theorem 1.3.

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