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Calculus of Variations

Existence and uniqueness of solutions for Choquard equation involving Hardy–Littlewood–Sobolev critical exponent

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Abstract
In this paper, we first prove that each positive solution of

$$-\Delta u = \big(I_\alpha * |u|^{2_\alpha^*}\big)|u|^{2_\alpha^*-2}u, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$$

is radially symmetric, monotone decreasing about some point and has the form

$$c_\alpha\left(\frac{t}{t^2+|x-x_0|^2}\right)^{\frac{N-2}{2}},$$

where $0<\alpha<N$ if $N=3$ or 4, and $N-4\le\alpha<N$ if $N\ge 5$, $2_\alpha^* := \frac{N+\alpha}{N-2}$ is the upper Hardy–Littlewood–Sobolev critical exponent, $t>0$ is a constant and $c_\alpha>0$ depends only on α and N. Based on this uniqueness result, we then study the following nonlinear Choquard equation

$$-\Delta u + V(x)u = \left(I_\alpha * |u|^{2_\alpha^*}\right)|u|^{2_\alpha^*-2}u, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^N).$$

By using Lions' Concentration-Compactness Principle, we obtain a global compactness result, i.e. we give a complete description for the Palais–Smale sequences of the corresponding energy functional. Adopting this description, we are succeed in proving the existence of at least one positive solution if $\|V(x)\|_{L^{\frac{N}{2}}}$ is suitable small. This result generalizes the result for semilinear Schrödinger equation by Benci and Cerami (J Funct Anal 88:90–117, 1990) to Choquard equation.

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1 Introduction and main results

Recently, the following nonlinear Choquard problem

$$-\Delta u + V(x)u = \big(I_\alpha * |u|^p\big)|u|^{p-2}u, \quad x \in \mathbb{R}^N \tag{1.1}$$

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has been investigated by many authors, where $I_\alpha : \mathbb{R}^N\backslash\{0\} \to \mathbb{R}$ is the Riesz potential defined by

$$I_\alpha(x) := \frac{A_\alpha}{|x|^{N-\alpha}} = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{N/2}2^\alpha|x|^{N-\alpha}}, \quad \alpha \in (0, N)$$

and Γ is the Gamma function, see [31,34].

Equation (1.1) is usually called the nonlinear Choquard or Choquard–Pekar equation. It has several physical motivations. In the physical case $N = 3$, $p = 2$ and $\alpha = 2$, the problem

$$-\Delta u + u = \big(I_2 * |u|^2\big)u, \quad x \in \mathbb{R}^3 \tag{1.2}$$

appeared as early as in 1954, in a work by Pekar describing the quantum mechanics of a polaron at rest [33]. See also [24,30] for more physical background of Eqs. (1.1)–(1.2). In particular, Lieb [21] proved that the ground state solution of Eq. (1.2) is radial and unique up to translations (see also [25]). Later, Wei and Winter [37] showed that the ground state solution is nondegenerate.

Problem (1.1) has a variational structure, setting $V(x) \equiv 1$ for example, the corresponding energy functional is defined by

$$E_{\alpha,p}(u) = \frac{1}{2}\int_{\mathbb{R}^N}\big(|\nabla u|^2+u^2\big)dx - \frac{1}{2p}\int_{\mathbb{R}^N}\big(I_\alpha*|u|^p\big)|u|^p dx, \ \ u \in W^{1,2}(\mathbb{R}^N)\cap L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N). \tag{1.3}$$

It follows by the Hardy–Littlewood–Sobolev inequality that the functional $E_{\alpha,p}(u)$ is well defined and belongs to $\mathcal{C}^1(H^1(\mathbb{R}^N), \mathbb{R})$ if $p \in [\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2}]$. Moreover, the critical points of $E_{\alpha,p}$ are weak solutions of Eq. (1.1).

Theorem A (See [22,23], Hardy–Littlewood–Sobolev inequality) *Suppose $\alpha \in (0, N)$, and $p, r > 1$ with $\frac{1}{p} + \frac{1}{r} = 1 + \frac{\alpha}{N}$. Let $f \in L^p(\mathbb{R}^N)$, $g \in L^r(\mathbb{R}^N)$, then there exists a sharp constant $C(p, r, \alpha, N)$, independent of f and g, such that*

$$\Big|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f(x)g(y)}{|x-y|^{N-\alpha}}dxdy\Big| \le C(p, \alpha, r, N)\|f\|_{L^p}\|g\|_{L^r}, \tag{1.4}$$

where $\|\cdot\|_{L^p} = \big(\int_{\mathbb{R}^N}|u|^p dx\big)^{\frac{1}{p}}$. If $p = r = \frac{2N}{N+\alpha}$, then

$$C(p, r, \alpha, N) = C(N, \alpha) = \pi^{\frac{N-\alpha}{2}}\frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{N+\alpha}{2})}\Big\{\frac{\Gamma(\frac{N}{2})}{\Gamma(N)}\Big\}^{-\frac{\alpha}{N}}. \tag{1.5}$$

In this case, the equality in (1.4) is achieved if and only if $f \equiv (const.)g$ and

$$g(x) = A(\widetilde{\gamma}^2 + |x - \widetilde{a}|^2)^{-\frac{(N+\alpha)}{2}}$$

for some $A \in \mathbb{C}$, $\widetilde{a} \in \mathbb{R}^N$ and $0 \neq \widetilde{\gamma} \in \mathbb{R}$.

For $N \ge 3$, $0 < \alpha < N$, let $2_*^\alpha = \frac{N+\alpha}{N}$ and $2_\alpha^* = \frac{N+\alpha}{N-2}$. By the Sobolev embedding theorem, $W^{1,2}(\mathbb{R}^N) \subset L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$ if and only if $p \in [2_*^\alpha, 2_\alpha^*]$. In [31], Moroz and Van Schaftingen proved that $E_{\alpha,p}(u)$ has no nontrivial critical points when $p \notin \big[2_*^\alpha, 2_\alpha^*\big]$. Hence, 2_*^α and 2_α^* are critical exponents for existence and nonexistence of solutions to Eq. (1.1). In the past few years, there is plenty of work dealt with Eq. (1.1) with $p \in (2_*^\alpha, 2_\alpha^*)$ by variational methods, see for example [2,28–32,37]. When $p = 2_*^\alpha$, Moroz and Van Schaftingen [32] proved the existence of one nontrivial solution to Eq. (1.1) if $V(x)$ satisfies

$$\liminf_{|x|\to+\infty}\big(1 - |x|^2\big)V(x) > \frac{N^2(N-2)}{4(N+1)}.$$

As for the upper Hardy–Littlewood–Sobolev exponent, Gao and Yang [12] considered the following Brézis–Nirenberg type problem on bounded domains

$$-\Delta u = \left(\int_{\Omega} \frac{|u(y)|^{2_\alpha^*}}{|x-y|^{N-\alpha}} dy \right) |u|^{2_\alpha^*-2} u + \lambda u, \quad x \in \Omega, \quad u \in H_0^1(\Omega).$$

In this paper, we first consider

$$-\Delta u = \left(I_\alpha * |u|^{2_\alpha^*} \right) |u|^{2_\alpha^*-2} u, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^N). \tag{1.6}$$

By using Theorem A, one can verify that, up to translations and scalings, the ground state solution of Eq. (1.6) is unique and has the form

$$u(x) = c_\alpha \left(\frac{t}{t^2 + |x - x_0|^2} \right)^{\frac{N-2}{2}} \tag{1.7}$$

where $t > 0$, $x_0 \in \mathbb{R}^N$ and

$$c_\alpha = \frac{[N(N-2)]^{\frac{N-2}{2}}}{\left[C(N,\alpha) A_\alpha S^{\frac{\alpha}{2}} \right]^{\frac{N-2}{4+2\alpha}}}, \tag{1.8}$$

here S is the best Sobolev constant for the embedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$.

A natural question is whether positive solution of Eq. (1.6) is unique and has the form of (1.7). Our result on this aspect can be stated as follows.

Theorem 1.1 *Suppose $0 < \alpha < N$ if $N = 3$ or 4, and $N - 4 \le \alpha < N$ if $N \ge 5$, let $u(x)$ be a positive solution of Eq. (1.6), then $u(x)$ is radially symmetric and monotone decreasing about some point $x_0 \in \mathbb{R}^N$. Moreover, $u(x)$ has the form of (1.7).*

Remark 1.1 (i) If $p < \frac{N+\alpha}{N-2}$, by Pohozaev type identity, the following equation

$$-\Delta u = \left(I_\alpha * |u|^p \right) |u|^{p-2} u, \quad x \in \mathbb{R}^N \tag{1.9}$$

has no nontrivial solution $u \in W^{1,2}(\mathbb{R}^N) \cap L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$ with $\nabla u \in W^{1,2}_{loc}(\mathbb{R}^N) \cap L^{\frac{2Np}{N+\alpha}}_{loc}(\mathbb{R}^N)$.

(ii) We prove Theorem 1.1 by a moving plane method, which was invented by Alexanderov in [1]. Later, it was further developed by Serrin [35], Gidas et al. [14], Caffarelli et al. [5] when classifying the solutions of semilinear elliptic equation

$$-\Delta u = u^{\frac{N+2}{N-2}}, \quad x \in \mathbb{R}^N.$$

Subsequently, Chen and Li [8] and Li [17] simplified the proof, Wei and Xu [38] and Chen et al. [11] generalized the classification result to the solutions of higher order conformally invariant equations

$$(-\Delta)^s u = u^{\frac{N+s}{N-s}}, \quad x \in \mathbb{R}^N, \ 0 < s < N.$$

Li [18] used the method of moving spheres to obtain the same classification result as that in [11]. For other applications, we refer the readers to [7,9,10,16,28].

Based on the uniqueness result, we can investigate the following Choquard equation

$$-\Delta u + V(x) u = \left(I_\alpha * |u|^{2_\alpha^*} \right) |u|^{2_\alpha^*-2} u, \quad x \in \mathbb{R}^N, \ N \ge 3, \tag{1.10}$$

where the potential function $V(x) \in L^{\frac{N}{2}}(\mathbb{R}^N) \cap \mathcal{C}^{\gamma}(\mathbb{R}^N)$ is nonnegative for some $\gamma \in (0,1)$. Define the energy functionals I, I_∞ corresponding to Eqs. (1.10), (1.6) respectively by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} \big(|\nabla u|^2 + V(x)u^2\big)dx - \frac{1}{2\cdot 2^*_\alpha}\int_{\mathbb{R}^N} \big(I_\alpha * |u|^{2^*_\alpha}\big)|u|^{2^*_\alpha}dx, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$$

and

$$I_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{2\cdot 2^*_\alpha}\int_{\mathbb{R}^N} \big(I_\alpha * |u|^{2^*_\alpha}\big)|u|^{2^*_\alpha}dx, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^N).$$

The Nehari manifolds corresponding to I and I_∞ denoted by $\mathcal{N}$ and $\mathcal{N}_\infty$ respectively are

$$\mathcal{N} := \Big\{u \in \mathcal{D}^{1,2}(\mathbb{R}^N)\backslash\{0\} : \langle I'(u), u\rangle = 0\Big\},$$
$$\mathcal{N}_\infty := \Big\{u \in \mathcal{D}^{1,2}(\mathbb{R}^N)\backslash\{0\} : \langle I'_\infty(u), u\rangle = 0\Big\}.$$

Moreover, we define

$$m := \inf_{u\in\mathcal{N}} I(u)$$

and

$$m_\infty := \inf_{u\in\mathcal{N}_\infty} I_\infty(u).$$

Obviously, m is the mountain pass level of the functional I and

$$m = \inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^N)\backslash\{0\}} \max_{t>0} I(tu) > 0.$$

Our main result on Eq. (1.10) can be stated as follows.

Theorem 1.2 *Let $0 < \alpha < N$ if $N = 3$ or 4, and $N - 4 \leq \alpha < N$ if $N \geq 5$, and suppose that $V(x) \in L^{\frac{N}{2}}(\mathbb{R}^N) \cap \mathcal{C}^{\gamma}(\mathbb{R}^N)$ is nonnegative for some $\gamma \in (0,1)$, then $m = m_\infty$ holds and m is not achieved. If $V(x)$ in addition satisfies*

$$0 < \|V(x)\|_{L^{\frac{N}{2}}} := \left(\int_{\mathbb{R}^N} |V(x)|^{\frac{N}{2}}dx\right)^{\frac{2}{N}} < \big(2^{\frac{\alpha+2}{N+\alpha}} - 1\big)S,$$

then Eq. (1.10) possesses at least one positive solution.

We prove Theorem 1.2 by following the variational approach developed by Benci and Cerami [3], in which a similar result was proved for the following Schrödinger equation

$$-\Delta u + V(x)u = u^{\frac{N+2}{N-2}}, \quad x \in \mathbb{R}^N, \ N \geq 3. \tag{1.11}$$

However, we cannot apply this approach directly, several difficulties arise because of the nonlocal nonlinearity with critical exponent. The main obstacle is lack of compactness, even if we get a $(PS)_c$ sequence with $c \in (m_\infty, 2m_\infty)$, we still cannot obtain the strongly convergence of $(PS)_c$ sequence, because the nodal solutions of Eq. (1.6) doesn't possess the double energy property (see [39]), i.e. there may exist nodal solutions of Eq. (1.6) with energy between m_∞ and $2m_\infty$ (see Theorem 3, [13]), but the double energy property is crucial for proving the main result in [3]. We solve this difficulty by using Linking Theorem to seek a nonnegative $(PS)_c$ sequence with $c \in (m_\infty, 2m_\infty)$ and analysing carefully the nonlocal nonlinearity. To this end, a nonlocal version of the Concentration-Compactness Principle (see Lemma 2.1, [27]) is used, which is totally different from the usual local case.

The following splitting result for Palais–Smale sequences is crucial for proving Theorem 1.2, while the local case on bounded domain has been established by Struwe [36].

Theorem 1.3 *Suppose $V(x) \geq 0$ and $V(x) \in L^{\frac{N}{2}}(\mathbb{R}^N)$, let $\{u_n\}$ be a Palais–Smale sequence of I at level c. Then $\{u_n\}$ has a subsequence which converges strongly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, or otherwise, replacing $\{u_n\}$ if necessary by a subsequence, there exists a function $\bar{u} \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ satisfying $u_n \rightharpoonup \bar{u}$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Moreover, there exists a number $k \in \mathbb{N}$, k functions $u^1, \ldots, u^k \in \mathcal{D}^{1,2}(\mathbb{R}^N)$; k sequences of points $\{y_n^i\} \subset \mathbb{R}^N$, $1 \leq i \leq k$ and k sequences of positive numbers $\{\sigma_n^i\}$, $1 \leq i \leq k$, such that*

$$\left\| u_n(\cdot) - \bar{u}(\cdot) - \sum_{i=1}^{k} (\sigma_n^i)^{-\frac{N-2}{2}} u^i \left(\frac{\cdot - y_n^i}{\sigma_n^i} \right) \right\| \to 0, \tag{1.12}$$

where $\bar{u}$ is a nontrivial solution of Eq. (1.10) and u^i, $1 \leq i \leq k$, are the nontrivial solutions of Eq. (1.6). Moreover, as $n \to +\infty$, we have

$$\|u_n\|^2 \to \|\bar{u}\|^2 + \sum_{i=1}^{k} \|u^i\|^2 \tag{1.13}$$

and

$$I(u_n) \to I(\bar{u}) + \sum_{i=1}^{k} I_\infty(u^i). \tag{1.14}$$

where $\|u\|^2 = \int_{\mathbb{R}^N} |\nabla u|^2 dx$ for $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$.

The paper is organized as follows. In Sect. 2, via the moving plane method, we prove that, up to translations and scalings, the positive solution of Eq. (1.6) is unique. In Sect. 3, by studying the behavior of Pslais–Smale sequences, we obtain a global compactness result, which provides a complete description of Palais–Smale sequences. In Sect. 4, we first show that the mountain pass value is not achieved. Then, combining Linking Theorem with Theorem 1.3, we prove the existence of at least one positive solution for Eq. (1.10).

2 Uniqueness of positive solution

In this section, we set $A_\alpha \equiv 1$ for convenience. We will use the moving planes method to show the uniqueness of the positive solution of Eq. (1.6). To do this, we first show the invariance of (1.6) under Kelvin transform. Denote K_u the Kelvin transform of u, that is,

$$K_u(x) = \frac{1}{|x|^{N-2}} u\left(\frac{x}{|x|^2}\right).$$

Lemma 2.1 *Let $u(x)$ be a solution of Eq. (1.6), then, $U = K_u$ is still a solution of Eq. (1.6).*

Proof Note that

$$\Delta K_u(x) = \frac{1}{|x|^{N+2}} \Delta u\left(\frac{x}{|x|^2}\right).$$

On the other hand,

$$\begin{aligned}\left(\frac{1}{|\cdot|^{N-\alpha}} * |K_u|^{2_\alpha^*}\right)(x) &= \int_{\mathbb{R}^N} \frac{\left|u\left(\frac{y}{|y|^2}\right)\right|^{2_\alpha^*}}{|x-y|^{N-\alpha}|y|^{N+\alpha}} dy = \int_{\mathbb{R}^N} \frac{|u(y)|^{2_\alpha^*}}{|x-\frac{y}{|y|^2}|^{N-\alpha}} |y|^{N+\alpha}|y|^{-2N} dy \\ &= |x|^{\alpha-N} \int_{\mathbb{R}^N} \frac{|u(y)|^{2_\alpha^*}}{|\frac{x}{|x|^2}-y|^{N-\alpha}} dy = |x|^{\alpha-N} \left(\frac{1}{|\cdot|^{N-\alpha}} * |u|^{2_\alpha^*}\right)\left(\frac{x}{|x|^2}\right),\end{aligned}$$

where we use the identity

$$|y|\Big|x-\frac{y}{|y|^2}\Big| = |x|\Big|\frac{x}{|x|^2}-y\Big|$$

in the third step. Therefore, we have

$$\Delta K_u(x) = |x|^{-2-N}(-\Delta u)\left(\frac{x}{|x|^2}\right) = |x|^{\alpha-N}\left(\frac{1}{|\cdot|^{N-\alpha}} * |u|^{2_\alpha^*}\right)\left(\frac{x}{|x|^2}\right)$$
$$\times |x|^{-\alpha-2}\Big|u\left(\frac{x}{|x|^2}\right)\Big|^{2_\alpha^*-1}u\left(\frac{x}{|x|^2}\right).$$

This shows that $U = K_u$ is also a solution of Eq. (1.6), which implies that Eq. (1.6) is invariant under Kelvin transform. □

Now, we transform Eq. (1.6) to an equivalent integral system. Let $v(x) = |x|^{-N+\alpha} * |u|^{2_\alpha^*}$. Then, up to a normalization constant, Eq. (1.6) is equivalent to

$$\begin{cases} u(x) = \displaystyle\int_{\mathbb{R}^N} \frac{|u(y)|^{2_\alpha^*-2}u(y)v(y)}{|x-y|^{N-2}}dy, \\ v(x) = \displaystyle\int_{\mathbb{R}^N} \frac{|u(y)|^{2_\alpha^*}}{|x-y|^{N-\alpha}}dy. \end{cases} \tag{2.1}$$

By $u \in L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ and Hardy–Littlewood–Sobolev inequality, we know that $v \in L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$. Making use of the moving plane method in integral forms, we show that each positive solution (u, v) of system (2.1) in $L^{\frac{2N}{N-2}}(\mathbb{R}^N) \times L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$ is radially symmetric and monotone decreasing about some point $x_0 \in \mathbb{R}^N$.

For this purpose, we first introduce some notation. For $x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N$, $\lambda \in \mathbb{R}$, we define $x^\lambda = (2\lambda - x_1, x_2, \dots, x_N)$ and

$$u_\lambda(x) = u(x^\lambda), \quad v_\lambda(x) = v(x^\lambda).$$

Let $\Sigma_\lambda = \{x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N : x_1 \geq \lambda\}$. We set

$$\Sigma_\lambda^u := \{x \in \Sigma_\lambda : u(x) < u_\lambda(x)\}, \quad \overline{\Sigma_\lambda^u} := \{x \in \Sigma_\lambda : u(x) \leq u_\lambda(x)\},$$
$$\Sigma_\lambda^v := \{x \in \Sigma_\lambda : v(x) < v_\lambda(x)\}.$$

Moreover, we denote the complement of Σ_λ in $\mathbb{R}^N$ by Σ_λ^c, and the reflection of Σ_λ^u about the plane $x_1 = \lambda$ by $\left(\Sigma_\lambda^u\right)^*$.

We decompose $u_\lambda(x)$, $u(x)$ in Σ_λ and $v_\lambda(x)$, $v(x)$ in Σ_λ as follows.

Lemma 2.2 *For each positive solution* (u, v) *of system* (2.1)*, we have*

$$u_\lambda(x) - u(x) = \int_{\Sigma_\lambda}\left(\frac{1}{|x-y|^{N-2}} - \frac{1}{|x^\lambda - y|^{N-2}}\right)\left(|u_\lambda(y)|^{2_\alpha^*-1}v_\lambda(y) - |u(y)|^{2_\alpha^*-1}v(y)\right)dy \tag{2.2}$$

and

$$v_\lambda(x) - v(x) = \int_{\Sigma_\lambda}\left(\frac{1}{|x-y|^{N-\alpha}} - \frac{1}{|x^\lambda - y|^{N-\alpha}}\right)\left(|u_\lambda(y)|^{2_\alpha^*} - |u(y)|^{2_\alpha^*}\right)dy. \tag{2.3}$$

Proof By (2.1) and the fact that $|x - y^\lambda| = |x^\lambda - y|$, we then obtain

$$\begin{aligned} u(x) &= \int_{\mathbb{R}^N} \frac{|u(y)|^{2_\alpha^*-1}v(y)}{|x-y|^{N-2}}dy \\ &= \int_{\Sigma_\lambda} \frac{|u(y)|^{2_\alpha^*-1}v(y)}{|x-y|^{N-2}}dy + \int_{\Sigma_\lambda^c} \frac{|u(y)|^{2_\alpha^*-1}v(y)}{|x-y|^{N-2}}dy \\ &= \int_{\Sigma_\lambda} \left(\frac{|u(y)|^{2_\alpha^*-1}v(y)}{|x-y|^{N-2}} + \frac{|u_\lambda(y)|^{2_\alpha^*-1}v_\lambda(y)}{|x^\lambda-y|^{N-2}} \right) dy, \end{aligned} \tag{2.4}$$

which leads to

$$u_\lambda(x) = u(x^\lambda) = \int_{\Sigma_\lambda} \left(\frac{|u(y)|^{2_\alpha^*-1}v(y)}{|x^\lambda-y|^{N-2}} + \frac{|u_\lambda(y)|^{2_\alpha^*-1}v_\lambda(y)}{|x-y|^{N-2}} \right) dy. \tag{2.5}$$

From (2.4) and (2.5), we then get (2.2). By a similar argument, we can also prove (2.3). □

Using the above preliminaries, we then prove the following proposition.

Proposition 2.3 *Suppose* $0 < \alpha < N$ *if* $N = 3$ *or* 4 *and* $N - 4 \le \alpha < N$ *if* $N \ge 5$*, and let* (u, v) *be a positive solution of system* (2.1) *in* $L^{\frac{2N}{N-2}}(\mathbb{R}^N) \times L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$*. Then* u *and* v *are both radially symmetric and decreasing about some point* $x_0 \in \mathbb{R}^N$.

Proof The proof consists of three steps.

Step 1 There exists $l_0 > 0$ such that for any $\lambda < -l_0$, we have

$$u(x) \ge u_\lambda(x) \quad and \quad v(x) \ge v_\lambda(x), \quad \text{for all} \quad x \in \Sigma_\lambda. \tag{2.6}$$

For the sufficiently negative value of λ, we show that both Σ_λ^u and Σ_λ^v must be empty.

In fact, for any $x \in \Sigma_\lambda^u$, we have

$$\begin{aligned} 0 < u_\lambda(x) - u(x) &= \int_{\Sigma_\lambda} \left(\frac{1}{|x-y|^{N-2}} - \frac{1}{|x^\lambda-y|^{N-2}} \right) \big(|u_\lambda(y)|^{2_\alpha^*-1}v_\lambda(y) - |u(y)|^{2_\alpha^*-1}v(y)\big)dy \\ &\le \int_{\Sigma_\lambda \cap \{|u_\lambda|^{2_\alpha^*-1}v_\lambda > |u|^{2_\alpha^*-1}v\}} \frac{1}{|x-y|^{N-2}} \big(|u_\lambda(y)|^{2_\alpha^*-1}v_\lambda(y) - |u(y)|^{2_\alpha^*-1}v(y)\big)dy. \end{aligned}$$

Hence, if $2_\alpha^* \ge 2$, we then get

$$\begin{aligned} &0 < u_\lambda(x) - u(x) \\ &\le \int_{\Sigma_\lambda^u} \frac{(2_\alpha^*-1)|u_\lambda(y)|^{2_\alpha^*-2}v_\lambda(y)(u_\lambda(y)-u(y))}{|x-y|^{N-2}}dy + \int_{\Sigma_\lambda^v} \frac{|u(y)|^{2_\alpha^*-1}(v_\lambda(y)-v(y))}{|x-y|^{N-2}}dy. \end{aligned} \tag{2.7}$$

By Lemma 2.2 and Hölder's inequality, we obtain

$$\begin{aligned} \|u_\lambda - u\|_{L^{\frac{2N}{N-2}}(\Sigma_\lambda^u)} &\le C_1 \|u_\lambda^{2_\alpha^*-2} v_\lambda (u_\lambda - u)\|_{L^{\frac{2N}{N+2}}(\Sigma_\lambda^u)} + C_2 \|u^{2_\alpha^*-1}(v_\lambda - v)\|_{L^{\frac{2N}{N+2}}(\Sigma_\lambda^v)} \\ &\le C_1 \|u_\lambda\|^{2_\alpha^*-2}_{L^{\frac{2N}{N-2}}(\Sigma_\lambda^u)} \|v_\lambda\|_{L^{\frac{2N}{N-\alpha}}(\Sigma_\lambda^u)} \|u_\lambda - u\|_{L^{\frac{2N}{N-2}}(\Sigma_\lambda^u)} \\ &\quad + C_2 \|u\|^{2_\alpha^*-1}_{L^{\frac{2N}{N-2}}(\Sigma_\lambda^v)} \|v_\lambda - v\|_{L^{\frac{2N}{N-\alpha}}(\Sigma_\lambda^v)} \end{aligned} \tag{2.8}$$

and

$$\|v_\lambda - v\|_{L^{\frac{2N}{N-\alpha}}(\Sigma_\lambda^v)} \le C_3 \|u_\lambda\|^{2_\alpha^*-1}_{L^{\frac{2N}{N-2}}(\Sigma_\lambda^u)} \|u_\lambda - u\|_{L^{\frac{2N}{N-2}}(\Sigma_\lambda^u)}. \tag{2.9}$$

Hence, substituting (2.9) into (2.8), we then obtain

$$\begin{aligned}\|u_\lambda - u\|_{L^{\frac{2N}{N-2}}(\Sigma_\lambda^u)} &\le C_1 \|u_\lambda\|^{2_\alpha^*-2}_{L^{\frac{2N}{N-2}}(\Sigma_\lambda^u)} \|v_\lambda\|_{L^{\frac{2N}{N-\alpha}}(\Sigma_\lambda^u)} \|u_\lambda - u\|_{L^{\frac{2N}{N-2}}(\Sigma_\lambda^u)} \\ &\quad + C_4 \|u\|^{2_\alpha^*-1}_{L^{\frac{2N}{N-2}}(\Sigma_\lambda^v)} \|u_\lambda\|^{2_\alpha^*-1}_{L^{\frac{2N}{N-2}}(\Sigma_\lambda^u)} \|u_\lambda - u\|_{L^{\frac{2N}{N-2}}(\Sigma_\lambda^u)}. \end{aligned} \tag{2.10}$$

Recalling that $u \in L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, $v \in L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$, by the dominated convergence theorem that we can choose l_0 sufficiently large such that $\lambda < -l_0$ and

$$\begin{aligned}&C_1 \|u_\lambda\|^{2_\alpha^*-2}_{L^{\frac{2N}{N-2}}(\Sigma_\lambda^u)} \|v_\lambda\|_{L^{\frac{2N}{N-\alpha}}(\Sigma_\lambda^u)} + C_4 \|u\|^{2_\alpha^*-1}_{L^{\frac{2N}{N-2}}(\Sigma_\lambda^v)} \|u_\lambda\|^{2_\alpha^*-1}_{L^{\frac{2N}{N-2}}(\Sigma_\lambda^u)} \\ &\quad \le C_1 \|u_\lambda\|^{2_\alpha^*-2}_{L^{\frac{2N}{N-2}}(\Sigma_\lambda)} \|v\|_{L^{\frac{2N}{N-\alpha}}(\Sigma_\lambda^c)} + C_4 \|u\|^{2_\alpha^*-1}_{L^{\frac{2N}{N-2}}(\Sigma_\lambda)} \|u\|^{2_\alpha^*-1}_{L^{\frac{2N}{N-2}}(\Sigma_\lambda^c)} \le \frac{1}{2}. \end{aligned} \tag{2.11}$$

Thus, it follows by (2.10) and (2.11) that

$$\|u_\lambda - u\|_{L^{\frac{2N}{N-2}}(\Sigma_\lambda^u)} = 0.$$

This implies that Σ_λ^u must be a set with zero measure, hence must be empty up to a set with zero measure. By (2.9), Σ_λ^v must be empty.

Step 2 We move the plane continuously from $\lambda < -l_0$ to the right as long as (2.6) holds. We show that if the procedure stops at $x_1 = \lambda_0$ for some λ_0, then $u(x)$ and $v(x)$ must be symmetric and monotone about the plane $x_1 = \lambda_0$. Otherwise, we can move the plane all the way to the right.

Moving the plane $x_1 = \lambda$ to the right as long as (2.6) holds. Suppose that at some λ_0, we have

$$u(x) \ge u_{\lambda_0}(x) \quad \text{and} \quad v(x) \ge v_{\lambda_0}(x) \quad \text{on } \Sigma_{\lambda_0},$$

but

$$u(x) \not\equiv u_{\lambda_0}(x) \quad \text{or} \quad v(x) \not\equiv v_{\lambda_0}(x) \quad \text{on } \Sigma_{\lambda_0}.$$

In the following, we show that the plane can be moved further to the right. More precisely, there exists $\delta = \delta(N, u, v)$ such that $u(x) \ge u_\lambda(x)$ and $v(x) \ge v_\lambda(x)$ on Σ_λ for all $\lambda \in [\lambda_0, \lambda_0 + \delta)$.

Assume that

$$v(x) \not\equiv v_{\lambda_0}(x) \quad \text{on } \Sigma_{\lambda_0}.$$

By (2.2), we have $u(x) > u_{\lambda_0}(x)$ in the interior of Σ_{λ_0}. Note that

$$meas(\overline{\Sigma_{\lambda_0}^u}) = 0 \quad \text{and} \quad \lim_{\lambda \to \lambda_0} \Sigma_\lambda^u \subseteq \overline{\Sigma_{\lambda_0}^u},$$

where $meas(\overline{\Sigma_{\lambda_0}^u})$ denotes the Lebesgue measure of $\overline{\Sigma_{\lambda_0}^u}$. Since $u \in L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, $v \in L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$ and $meas(\overline{\Sigma_{\lambda_0}^u}) = 0$, then using the dominated convergence theorem, we can choose $\delta > 0$ sufficiently small, such that for all $\lambda \in [\lambda_0, \lambda_0 + \delta)$, we have

$$C_1 \|u_\lambda\|^{2_\alpha^*-2}_{L^{\frac{2N}{N-2}}(\Sigma_\lambda)} \|v\|_{L^{\frac{2N}{N-\alpha}}\left((\Sigma_\lambda^u)^*\right)} + C_4 \|u\|^{2_\alpha^*-1}_{L^{\frac{2N}{N-2}}(\Sigma_\lambda)} \|u\|^{2_\alpha^*-1}_{L^{\frac{2N}{N-2}}\left((\Sigma_\lambda^u)^*\right)} \le \frac{1}{2}.$$

It follows from (2.10) that

$$\|u_\lambda - u\|_{L^{\frac{2N}{N-2}}(\Sigma_\lambda^u)} = 0.$$

Hence, Σ_λ^u must be empty for all $\lambda \in [\lambda_0, \lambda_0 + \delta)$, which also implies that Σ_λ^v is empty for all $\lambda \in [\lambda_0, \lambda_0 + \delta)$.

Assume that

$$u(x) \not\equiv u_{\lambda_0}(x) \quad \text{on} \quad \Sigma_{\lambda_0}.$$

By (2.3), we see $v(x) > v_{\lambda_0}(x)$ in the interior of Σ_{λ_0}. By the above analysis, we know that Σ_λ^u and Σ_λ^v must also be empty for all $\lambda \in [\lambda_0, \lambda_0 + \delta)$. This completes the proof.

Step 3 By step 1, we know that the plane cannot keep moving all the way to the right in Step 2. That is, the plane will eventually stop at some point. In fact, with the similar analysis as that in Step 1 and Step 2, we then assert that there exists a large $\bar{l}$, such that for $\lambda > \bar{l}$,

$$u(x) \le u_\lambda(x) \quad \text{and} \quad v(x) \le v_\lambda(x), \quad \text{for all} \quad x \in \Sigma_\lambda. \tag{2.12}$$

Now we can move the plane continuously from $\lambda > \bar{l}$ to the left as long as the above fact holds. The planes moved from the left and the right will eventually meet at some point. Finally, since the direction of x_1 can be chosen arbitrarily, we deduce that $u(x)$ and $v(x)$ must be radially symmetric and decreasing about some point. □

Now we use the elliptic regularity theory to show the following proposition.

Proposition 2.4 *Assume that u is a positive solution of (1.6). Then u is uniformly bounded in $\mathbb{R}^N$. Furthermore, u is $\mathcal{C}^\infty(\mathbb{R}^N)$ and*

$$\lim_{|x| \to +\infty} |x|^{N-2} u(x) = u_\infty \tag{2.13}$$

for some positive constant u_∞.

Proof **Step 1** We first show that u is uniformly bounded and smooth. For $A > 0$, we define

$$\Omega = \{x \in \mathbb{R}^N : u(x) > A\} \quad \text{and} \quad u_A(x) = \begin{cases} u(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

Hence

$$u - u_A \in L^{\frac{2N}{N-2}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \quad \text{for any } A > 0. \tag{2.14}$$

Since u is a solution of Eq. (1.6), we have

$$u(x) = \int_{\mathbb{R}^N} \frac{\left(|\cdot|^{\alpha-N} * |u|^{2_\alpha^*}\right) |u(y)|^{2_\alpha^*-1}}{|x-y|^{N-2}} dy, \quad \forall x \in \mathbb{R}^N,$$

which implies that for any $x \in \Omega$,

$$\begin{aligned} u_A(x) &= \int_{\mathbb{R}^N} \frac{\left(|\cdot|^{\alpha-N} * |u|^{2_\alpha^*}\right) |u(y)|^{2_\alpha^*-1}}{|x-y|^{N-2}} dy \\ &= \int_{\mathbb{R}^N} \frac{\left(|\cdot|^{\alpha-N} * |u_A|^{2_\alpha^*}\right) |u_A(y)|^{2_\alpha^*-1}}{|x-y|^{N-2}} dy \\ &\quad + \int_{\mathbb{R}^N} \frac{\left(|\cdot|^{\alpha-N} * |u-u_A|^{2_\alpha^*}\right) |u_A(y)|^{2_\alpha^*-1}}{|x-y|^{N-2}} dy \\ &\quad + \int_{\mathbb{R}^N} \frac{\left(|\cdot|^{\alpha-N} * |u_A|^{2_\alpha^*}\right) |u-u_A(y)|^{2_\alpha^*-1}}{|x-y|^{N-2}} dy \end{aligned}$$

$$+\int_{\mathbb{R}^N}\frac{\left(|\cdot|^{\alpha-N}*|u-u_A|^{2^*_\alpha}\right)|u-u_A(y)|^{2^*_\alpha-1}}{|x-y|^{N-2}}dy.$$

Next we divide our argument into three cases.

Case 1 $0<\alpha\le 2$. For any $r\ge\frac{2N}{N-2}$, by Hardy–Littlewood–Sobolev inequality, we see

$$\begin{aligned}\Big\|\int_{\mathbb{R}^N}\frac{(I_\alpha*|u_A(y)|^{2^*_\alpha})|u_A(y)|^{2^*_\alpha-1}}{|x-y|^{N-2}}dy\Big\|_{L^r}&\le\Big[\int_{\mathbb{R}^N}\left(I_\alpha*|u_A|^{2^*_\alpha}\right)^{\frac{Nr}{N+2r}}|u_A(y)|^{(2^*_\alpha-1)\frac{Nr}{N+2r}}dx\Big]^{\frac{N+2r}{Nr}}\\ &\le\Big[\int_{\mathbb{R}^N}\left(I_\alpha*|u_A|^{2^*_\alpha}\right)^{p\frac{Nr}{N+2r}}dx\Big]^{\frac{N+2r}{pNr}}\Big[\int_{\mathbb{R}^N}|u_A(y)|^{q(2^*_\alpha-1)\frac{Nr}{N+2r}}dx\Big]^{\frac{N+2r}{qNr}}\\ &\le\Big[\int_{\mathbb{R}^N}|u_A|^{2^*}dx\Big]^{\frac{2^*_\alpha-1}{2^*}}\Big[\int_{\mathbb{R}^N}|u_A|^{r}dx\Big]^{\frac{1}{r}}\Big[\int_{\mathbb{R}^N}|u_A|^{2^*}dx\Big]^{\frac{2^*_\alpha-1}{2^*}},\end{aligned}$$

where $q=\frac{2N+4r}{r(2+\alpha)}$ and $1/p+1/q=1$. One can easily check that $q>1$ for every $r\ge\frac{2N}{N-2}$ and $0<\alpha\le 2$. Thus, using the Hardy–Littlewood–Sobolev inequality again, one finds

$$\begin{aligned}\|u_A\|_{L^r}&\le C\|u_A\|^{2(2^*_\alpha-1)}_{L^{\frac{2N}{N-2}}}\|u_A\|_{L^r}+C\|u_A\|^{2^*_\alpha-1}_{L^{\frac{2N}{N-2}}}\|u-u_A\|^{2^*_\alpha-1}_{L^{\frac{2N}{N-2}}}\|u-u_A\|_{L^r}\\ &\quad+C\|u_A\|^{2(2^*_\alpha-1)}_{L^{\frac{2N}{N-2}}}\|u-u_A\|_{L^r}+C\|u-u_A\|^{2(2^*_\alpha-1)}_{L^{\frac{2N}{N-2}}}\|u-u_A\|_{L^r}.\end{aligned}\tag{2.15}$$

On one hand, by $u\in L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, we can choose A large enough, such that

$$C\|u_A\|^{2(2^*_\alpha-1)}_{L^{\frac{2N}{N-2}}}\le\frac{1}{2}.\tag{2.16}$$

On the other hand, by $u\in L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ and (2.14), we verify that

$$\begin{aligned}&C\|u_A\|^{2^*_\alpha-1}_{L^{\frac{2N}{N-2}}}\|u-u_A\|^{2^*_\alpha-1}_{L^{\frac{2N}{N-2}}}\|u-u_A\|_{L^r}+C\|u_A\|^{2(2^*_\alpha-1)}_{L^{\frac{2N}{N-2}}}\|u-u_A\|_{L^r}\\ &\quad+C\|u-u_A\|^{2(2^*_\alpha-1)}_{L^{\frac{2N}{N-2}}}\|u-u_A\|_{L^r}\le C(A).\end{aligned}\tag{2.17}$$

Substituting (2.16) and (2.17) into (2.15), we then assert that, for any $r\ge\frac{2N}{N-2}$

$$\|u_A\|_{L^r}\le\frac{1}{2}\|u_A\|_{L^r}+C(A),\tag{2.18}$$

which implies that $u_A\in L^r(\mathbb{R}^N)$ for any $r\ge\frac{2N}{N-2}$. Therefore, we have $u\in L^r(\mathbb{R}^N)$ for any $r\ge\frac{2N}{N-2}$. Using Hardy–Littlewood–Sobolev inequality again, we get

$$-\Delta u=\left(|\cdot|^{\alpha-N}*|u|^{2^*_\alpha}\right)|u|^{2^*_\alpha-2}u\in L^p(\mathbb{R}^N),\quad\text{for any}\quad p\ge\frac{2N}{N-2}.$$

Using the L^p-theory and Sobolev embedding theorem (see Theorem 9.9, [15]), we know that u is uniformly bounded and belongs to $\mathcal{C}^{0,s}(\mathbb{R}^N)$ for all $0<s<1$. In fact, we also conclude $u\in\mathcal{C}^\infty(\mathbb{R}^N)$ from Theorem 4.4.8 in [6].

Case 2 $2<\alpha<N-4$. Let $p=\frac{N+\alpha}{N-2}<2$. First, we claim that $u_A\in L^s$ for every $2^*=\frac{2N}{N-2}\le s\le\frac{Np}{\alpha}$. Set $s_0=2^*$, we assume that $u_A\in L^s$ for every $s\in[2^*,s_n]$ and $s_n<\frac{Np}{\alpha}$. We will prove that $u_A\in L^r$ if $r\ge s_n$ satisfies

$$\frac{1}{r}>\frac{p-1}{s_n}-\frac{2}{N},\tag{2.19}$$

$$\frac{1}{r} < \frac{2p-1}{s_n} - \frac{2+\alpha}{N}. \tag{2.20}$$

Moreover, we compare $r_0 = (\frac{p-1}{s_n} - \frac{2}{N})^{-1}$ with $\frac{Np}{\alpha}$. If $r_0 \geq \frac{Np}{\alpha}$, then the claim is proved. If $r_0 < \frac{Np}{\alpha}$, set $s_{n+1} = r_0$ and proceed again. Since

$$\frac{1}{s_n} - \frac{1}{s_{n+1}} = \frac{2-p}{s_n} + \frac{2}{N} > \frac{1}{N},$$

our argument must terminate at a finite number of steps. We should note that if $s_n < \frac{N}{\alpha}p$,

$$s_n Nr > (N+2r)s_n - (p-1)Nr + s_n r\alpha. \tag{2.21}$$

Then using the Hardy–Littlewood–Sobolev inequality and the condition (2.19)–(2.21), we find

$$\begin{aligned}
\left\| \int_{\mathbb{R}^N} \frac{\big(I_\alpha * |u_A|^p\big)|u_A(y)|^{p-1}}{|x-y|^{N-2}} dy \right\|_{L^r} &\leq \left\| I_\alpha * |u_A|^p \cdot u_A^{p-1} \right\|_{L^{\frac{Nr}{N+2r}}} \\
&\leq \left\| (I_\alpha * |u_A|^p)^{\frac{Nr}{N+2r}} \right\|^{\frac{N+2r}{Nr}}_{L^{\frac{s_n(N+2r)}{s_n(N+2r)-(p-1)Nr}}} \times \left\| u_A^{(p-1)\frac{Nr}{N+2r}} \right\|^{\frac{N+2r}{Nr}}_{L^{\frac{s_n(N+2r)}{(p-1)Nr}}} \\
&\leq \left\| I_\alpha * |u_A|^p \right\|_{L^{\frac{s_n Nr}{s_n(N+2r)-(p-1)Nr}}} \times \|u_A\|^{p-1}_{L^{s_n}}
\end{aligned}$$

and

$$\left\| I_\alpha * |u_A|^p \right\|_{L^{\frac{s_n Nr}{s_n(N+2r)-(p-1)Nr}}} \leq \|u_A\|^p_{L^{\frac{s_n Nrp}{s_n(N+2r)-(p-1)Nr+s_n r\alpha}}}.$$

Setting $t = \frac{s_n Nrp}{s_n(N+2r)-(p-1)Nr+s_n r\alpha}$, we know that $s_n < t < r$. Hence $t = (1-\theta)s_n + \theta r$ where $\theta = \frac{t-s_n}{r-s_n}$. It yields that

$$\|u_A\|^p_{L^t} \leq \|u_A\|^{(1-\theta)p}_{L^{s_n}} \|u_A\|^{\theta p}_{L^r}.$$

Similarly to (2.18), we have

$$\|u_A\|_{L^r} \leq \|u_A\|^{p-1+(1-\theta)p}_{L^{s_n}} \times \|u_A\|^{\theta p}_{L^r} + \|u - u_A\|^{p-1}_{L^{s_n}} \|u_A\|^{(1-\theta)p}_{L^{s_n}} \|u_A\|^{\theta p}_{L^r} + C(A).$$

Then we choose $A > 0$ sufficiently large such that

$$2\|u_A\|_{L^r} \leq \|u_A\|^{\theta p}_{L^r} + C(A). \tag{2.22}$$

Note that $\theta p < 1$. To see this, we only need to prove

$$t - s = \frac{(2p-1)sNr - s^2(N+2r+r\alpha)}{s(N+2r+r\alpha)-(p-1)Nr} < \frac{r-s}{p}, \tag{2.23}$$

which is equivalent to

$$\text{LHS} = 2psNr + Nr^2 < s^2(N+2r+r\alpha) + sr^2(N-2) = \text{RHS}.$$

Since $s \geq \frac{2N}{N-2}$, we compute that

$$\begin{aligned} \text{RHS} &\geq s^2N + s^2r(N-2)(p-1) + 2Nr^2 \\ &\geq s^2N + 2Nsr(p-1) + 2Nr^2 \\ &\geq 2Nsrp + Nr^2 + N(s-r)^2 \\ &> 2Nsrp + Nr^2 = \text{LHS}. \end{aligned}$$

From this, by (2.22) we know

$$\|u_A\|_{L^r} \leq \max\{1, C(A)^{\frac{1}{\theta p}}\}.$$

It follows that $u_A \in L^r(\mathbb{R}^N)$ for $A > 0$ sufficiently large. Thus $u \in L^{\frac{Np}{\alpha}}(\mathbb{R}^N)$ and $I_\alpha * |u|^p \in L^\infty(\mathbb{R}^N)$.

Finally, since u satisfies

$$-\Delta u = (I_\alpha * |u|^{2_\alpha^*})|u|^{2_\alpha^*-2}u, \quad \text{in } \mathbb{R}^N.$$

Then, by standard elliptic regularity theory, $u \in \mathcal{C}^\infty(\mathbb{R}^N)$.

Case 3 $N-4 \leq \alpha < N$. In this case, $2_\alpha^* = \frac{N+\alpha}{N-2} \geq 2$. Then $a(x) := (I_\alpha * |u|^{2_\alpha^*})u^{2_\alpha^*-2} \in L^{N/2}(\mathbb{R}^N)$. The Brézis–Kato theorem [4] implies that $u \in L^t_{loc}(\mathbb{R}^N)$ for all $1 \leq t < \infty$. Thus, $u \in W^{2,t}(\mathbb{R}^N)$ for all $1 \leq t < \infty$. By elliptic regularity theory, $u \in \mathcal{C}^\infty(\mathbb{R}^N)$.

Step 2 We want to prove the asymptotic behavior at infinity of u. We prove it by contradiction. Consider the Kelvin transform:

$$U(x) = \frac{1}{|x|^{N-2}} u\left(\frac{x}{|x|^2}\right) \quad \Rightarrow \quad |x|^{N-2}u(x) = U\left(\frac{x}{|x|^2}\right).$$

Applying Proposition 2.3 to $U(x)$, we conclude that $U(x)$ must be radially symmetric about some point and continuous. Hence

$$\lim_{|x|\to+\infty} |x|^{N-2}u(x) = U(0) > 0,$$

which completes the proof of Proposition 2.4. □

Lemma 2.5 *Let u be a solution of Eq. (1.6), then there exist $\lambda > 0$ and $x \in \mathbb{R}^N$ such that*

$$u(y) = \left(\frac{\lambda}{|y-x|}\right)^{N-2} u\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right). \tag{2.24}$$

Proof Let u be a solution of Eq. (1.6). By Proposition 2.3, we can assume that $u(x)$ is symmetric about the origin, and we prove this lemma with $x = 0$. Moreover, without loss of generality we assume that $\lambda = 1$. Otherwise, we just need to make a translation or a scaling.

By Proposition 2.4, suppose that $\lim_{|x|\to+\infty} |x|^{N-2}u(x) = u_\infty = u(0)$. Let e be any unit vector in $\mathbb{R}^N$. We define

$$w(y) = \left(\frac{1}{|y|}\right)^{N-2} u\left(\frac{y}{|y|^2} - e\right).$$

Obviously, $w(y)$ is the Kelvin transform of $u(y-e)$. By Lemma 2.1, w satisfies the Eq. (1.6) and hence should be radially symmetric about some point $z_0 \in \mathbb{R}^N$. Note that

$$w(0) = u_\infty \quad \text{and} \quad w(e) = u(0).$$

Thus, w must be symmetric about the plane $\Pi = \{x : (x - \frac{e}{2}) \cdot e = 0\}$. Now, choosing $y = \frac{e}{2} - he$ for any $h > 0$, similarly to the proof of Lemma 3.1 in [11], we can prove that

$$w\left(\frac{e}{2} - he\right) = \left(\frac{1}{|\frac{1}{2} - h|}\right)^{N-2} u\left(\frac{\frac{e}{2} + he}{|\frac{1}{2} - h|}\right). \tag{2.25}$$

Taking $y = \frac{e}{2} + he$, $h > 0$, we have

$$w\left(\frac{e}{2} + he\right) = \left(\frac{1}{|\frac{1}{2} + h|}\right)^{N-2} u\left(\frac{\frac{e}{2} - he}{|\frac{1}{2} + h|}\right). \tag{2.26}$$

Combining (2.25) with (2.26) and noticing the radial symmetry of u, we find

$$\left(\frac{1}{|\frac{1}{2} - h|}\right)^{N-2} u\left(\left|\frac{\frac{1}{2} + h}{\frac{1}{2} - h}\right| e\right) = \left(\frac{1}{|\frac{1}{2} + h|}\right)^{N-2} u\left(\left|\frac{\frac{1}{2} - h}{\frac{1}{2} + h}\right| e\right).$$

Let $t = (\frac{1}{2} - h)/(\frac{1}{2} + h)$, then

$$u\left(\frac{e}{|t|}\right) = |t|^{N-2} u(|t|e).$$

Replacing $|t|$, e by $1/|y|$, $y/|y|$, respectively, we obtain

$$u(y) = \frac{1}{|y|^{N-2}} u\left(\frac{y}{|y|^2}\right).$$

Furthermore, we can take a translation transform to obtain (2.24). □

To prove Theorem 1.1, we also need the following proposition from Li and Zhang [19]. Earlier version with stronger assumptions was first proved by Li and Zhu [20].

Proposition 2.6 [19] *Let $f \in C^1(\mathbb{R}^N, \mathbb{R})$, $\lambda > 0$ and $\mu > 0$. Suppose that for every $x \in \mathbb{R}^N$, there exists $\lambda(x) > 0$ such that*

$$f(y) = \left(\frac{\lambda}{|y - x|}\right)^{\mu} f\left(x + \frac{\lambda^2(y - x)}{|y - x|^2}\right), \quad y \in \mathbb{R}^N \backslash \{x\}.$$

Then,

$$f(x) \equiv \pm a\left(\frac{1}{d + |x - \bar{x}|^2}\right)^{\mu/2}$$

for some $a \geq 0$, $d > 0$ and $\bar{x} \in \mathbb{R}^N$.

Proof of Theorem 1.1 Using Lemma 2.5 and Proposition 2.6, we obtain that the solution of Eq. (1.6) must be of form (1.7). □

3 A global compactness result

In this section, we study the behavior of Palais–Smale sequences of the energy functional I and then prove Theorem 1.3. The following result is a Brézis–Lieb's type lemma for problem (1.10), and the proof is similar as Lemma 2.4 in [31].

Lemma 3.1 *Let $N \geq 3$ and $\alpha \in (0, N)$. If $\{u_n\}$ is a bounded sequence in $L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ such that $u_n \to u$ almost everywhere in $\mathbb{R}^N$ as $n \to +\infty$, then*

$$\int_{\mathbb{R}^N} \big(I_\alpha * |u_n|^{2_\alpha^*}\big)|u_n|^{2_\alpha^*} dx - \int_{\mathbb{R}^N} \big(I_\alpha * |u_n - u|^{2_\alpha^*}\big)|u_n - u|^{2_\alpha^*} dx \to \int_{\mathbb{R}^N} \big(I_\alpha * |u|^{2_\alpha^*}\big)|u|^{2_\alpha^*} dx \tag{3.1}$$

and

$$\big(I_\alpha * |u_n|^{2_\alpha^*}\big)|u_n|^{2_\alpha^*-2}u_n - \big(I_\alpha * |u_n - u|^{2_\alpha^*}\big)|u_n - u|^{2_\alpha^*-2}(u_n - u) \rightharpoonup \big(I_\alpha * |u|^{2_\alpha^*}\big)|u|^{2_\alpha^*-2}u, \quad \text{in } \big(\mathcal{D}^{1,2}(\mathbb{R}^N)\big)', \tag{3.2}$$

where $\big(\mathcal{D}^{1,2}(\mathbb{R}^N)\big)'$ is the dual space of $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

In order to prove Theorem 1.3, we need the following concentration principle for Riesz potential.

Lemma 3.2 *Let $\{u_n\} \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$ be a sequence of functions such that*

$$u_n \rightharpoonup 0 \ \text{ weakly in } \ \mathcal{D}^{1,2}(\mathbb{R}^N).$$

Assume that there exist a bounded open set $Q \subset \mathbb{R}^N$ and a positive constant $\varrho > 0$ such that

$$\int_Q |\nabla u_n|^2 dx \geq \varrho \tag{3.3}$$

and

$$\int_Q (I_\alpha * |u_n|^{2_\alpha^*})|u_n|^{2_\alpha^*} dx \geq \varrho. \tag{3.4}$$

Moreover, suppose that

$$\Delta u_n + (I_\alpha * |u_n|^{2_\alpha^*})|u_n|^{2_\alpha^*-2}u_n = \chi_n, \tag{3.5}$$

where $\chi_n \in \big(\mathcal{D}^{1,2}(\mathbb{R}^N)\big)'$ and

$$\langle \chi_n, \psi \rangle \leq \varepsilon_n \|\psi\|, \quad \text{for all} \quad \psi \in C_0^\infty(\Omega), \tag{3.6}$$

with Ω being an open neighborhood of Q and $\{\varepsilon_n\}$ being a sequence of positive numbers converging to 0. Then there exist a sequence of positive numbers $\{\sigma_n\}$ and a sequence of points $\{y_n\} \subset \bar{Q}$ such that

$$v_n(x) := \sigma_n^{\frac{N-2}{2}} u_n(\sigma_n x + y_n)$$

converges weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ to v, which is a nontrivial solution of Eq. (1.6).

Proof Since $u_n \rightharpoonup 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, then by Concentration Compactness Principle II (see Lemma I.1, [26]), we obtain an at most countable index set Γ, a sequence of $\{x_i\}_{i\in\Gamma} \subset \mathbb{R}^N$ and a family of $\{\nu_i\}_{i\in\Gamma} \subset (0, +\infty)$ such that

$$|u_n \phi_\Omega|^{2^*} \rightharpoonup \sum_{i\in\Gamma} \nu_i \delta_{x_i},$$

where $\phi_\Omega(x)$ is a cut-off function with $\phi_\Omega(x) = 1$ in Q; $\phi_\Omega(x) = 0$ in $\mathbb{R}^N \backslash \Omega$ and $0 \leq \phi_\Omega(x) \leq 1$.

For the readers' convenience, we will prove this lemma through three claims.

Claim 1 There is at least one $i_0 \in \Gamma$ such that $x_{i_0} \in \bar{Q}$ with $\nu_{i_0} > 0$.

Proof Otherwise, then $u_n \to 0$ in $L^{2^*}(Q)$, which together with Hardy–Littlewood–Sobolev inequality implies that

$$\int_Q \left(I_\alpha * |u_n|^{2^*_\alpha}\right)|u_n|^{2^*_\alpha} dx \to 0.$$

This is a contradiction to the assumption (3.4) and the claim is proved. □

Now, we define the concentration function

$$G_n(r) := \sup_{z \in \bar{Q}} \int_{B_r(z)} |u_n|^{2^*} dx.$$

For a given small $\tau \in \left(0, \left[\frac{S}{A_\alpha C(N,\alpha)}\right]^{\frac{N}{\alpha+2}}\right)$, we choose $\sigma_n = \sigma_n(\tau) > 0$, $y_n \in \bar{Q}$ such that

$$\int_{B_{\sigma_n}(y_n)} |u_n|^{2^*} dx = G_n(\sigma_n) = \tau. \tag{3.7}$$

Let $v_n(x) := \sigma_n^{\frac{N-2}{2}} u_n(\sigma_n x + y_n)$, then

$$\widetilde{G_n}(r) := \sup_{z \in \bar{Q}_n} \int_{B_r(z)} |v_n|^{2^*} dx = \sup_{z \in \bar{Q}} \int_{B_{\sigma_n r}(z)} |u_n|^{2^*} dx = G_n(\sigma_n r), \tag{3.8}$$

where $\bar{Q}_n := \{x \in \mathbb{R}^N : \sigma_n x + y_n \in \bar{Q}\}$. It follows by (3.7) and (3.8) that

$$\widetilde{G_n}(1) = \int_{B_1(0)} |v_n|^{2*} dx = \int_{B_{\sigma_n}(y_n)} |u_n|^{2^*} dx = G_n(\sigma_n) = \tau. \tag{3.9}$$

Claim 2 There exists some $\tau \in \left(0, \left[\frac{S}{A_\alpha C(N,\alpha)}\right]^{\frac{N}{\alpha+2}}\right)$ such that $\sigma_n(\tau) \to 0$ as $n \to +\infty$.

Proof Assume by contradiction, for any $\varepsilon > 0$, that there exists $r_0 > 0$ such that $\sigma_n(\varepsilon) \geq r_0$. Then a direct calculation shows that

$$\int_{B_{r_0}(x_{i_0})} |u_n|^{2^*} dx \leq \sup_{z \in \bar{Q}} \int_{B_{\sigma_n(\varepsilon)}(z)} |u_n|^{2^*} dx = G_n(\sigma_n(\varepsilon)) = \varepsilon. \tag{3.10}$$

In particular

$$\nu_{i_0} \leq \int_{B_{r_0}(x_{i_0})} |u_n|^{2^*} dx + o_n(1) \leq \varepsilon + o_n(1), \quad \text{for any} \quad \varepsilon > 0, \tag{3.11}$$

where $o_n(1) \to 0$ as $n \to +\infty$. Then, it follows by (3.11) that we have $\nu_{i_0} \leq 0$, which contradicts Claim 1.

By the definition of v_n, we have $\int_{\mathbb{R}^N} |\nabla v_n|^2 dx = \int_{\mathbb{R}^N} |\nabla u_n|^2 dx$, which together with the boundness of $\{u_n\}$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ implies that $\{v_n\}$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Without loss of generality, we may assume that there exists some $v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ such that $v_n \rightharpoonup v$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ up to a subsequence.

Claim 3 v is a nontrivial solution of Eq. (1.6).

Proof In fact, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, we define

$$\widetilde{\varphi}_n(x) = \sigma_n^{\frac{2-N}{2}} \varphi\left(\frac{x - y_n}{\sigma_n}\right). \tag{3.12}$$

Since $\sigma_n \to 0$ and $y_n \in \bar{Q}$, then we assert that $\widetilde{\varphi}_n(x) \in C_0^\infty(\bar{\Omega})$ for n large enough. In virtue of (3.5) and (3.6), we obtain that

$$\begin{aligned} o_n(1)\|\varphi\| = o_n(1)\|\widetilde{\varphi}_n\| &= \int_{\mathbb{R}^N} \nabla u_n \nabla \widetilde{\varphi}_n dx - \int_{\mathbb{R}^N} \left(I_\alpha * |u_n|^{2_\alpha^*}\right)|u_n|^{2_\alpha^*-2} u_n \widetilde{\varphi}_n dx \\ &= \int_{\mathbb{R}^N} \nabla v_n \nabla \varphi dx - \int_{\mathbb{R}^N} \left(I_\alpha * |v_n|^{2_\alpha^*}\right)|v_n|^{2_\alpha^*-2} v_n \varphi dx. \end{aligned} \tag{3.13}$$

Thus, v is a weak solution of Eq. (1.6). Before concluding the proof, we still need to prove $v \neq 0$. To this end, it is sufficient to prove that, up to a subsequence,

$$v_n \to v \quad \text{strongly in} \quad L^{2^*}(B_1(0)). \tag{3.14}$$

Since $v_n \rightharpoonup v$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, by Concentration Compactness Principle (see Lemma I.1 in [26] and Lemma 2.1 in [27]), we may assume that there exist three bounded nonnegative measures $\widetilde{\mu}, \widetilde{\nu}, \widetilde{\omega}$, such that $|\nabla v_n|^2 \rightharpoonup \widetilde{\mu}$, $|v_n|^{2^*} \rightharpoonup \widetilde{\nu}$ and $\left|I_\alpha * |v_n|^{2_\alpha^*}\right|^{\frac{2N}{N-\alpha}} \rightharpoonup \widetilde{\omega}$ weakly in finite measure space $\mathcal{M}(\mathbb{R}^N)$ (see Page 26 in [40]). Moreover,

$$\widetilde{\mu} \geq |\nabla v|^2 + \sum_{j \in \widetilde{\Gamma}} \widetilde{\mu}_j \delta_{x_j}, \quad \widetilde{\nu} = |v|^{2^*} + \sum_{j \in \widetilde{\Gamma}} \widetilde{\nu}_j \delta_{x_j}, \quad \widetilde{\omega} = \left|I_\alpha * |v|^{2_\alpha^*}\right|^{\frac{2N}{N-\alpha}} + \sum_{j \in \widetilde{\Gamma}} \widetilde{\omega}_j \delta_{x_j} \quad \text{in } \mathcal{M}(\mathbb{R}^N) \tag{3.15}$$

and

$$\widetilde{\mu}_j \geq S \widetilde{\nu}_j^{\frac{N-2}{N}}, \quad \widetilde{\nu}_j \geq \left(\frac{1}{A_\alpha C(N,\alpha)}\right)^{\frac{2N}{N+\alpha}} \widetilde{\omega}_j^{\frac{N-\alpha}{N+\alpha}}, \tag{3.16}$$

where $\widetilde{\Gamma}$ is an at most countable index set. In order to prove (3.14), we only need to prove

$$\{x_j\}_{j \in \widetilde{\Gamma}} \cap \overline{B_1(0)} = \emptyset.$$

If not, we suppose that there exists $x_{j_0} \in \overline{B_1(0)}$ for some $j_0 \in \widetilde{\Gamma}$ and define $\phi_\rho(x) := \phi\left(\frac{x - x_{j_0}}{\rho}\right)$, ϕ is a cut-off function which satisfies $\phi = 1$ on $B_1(0)$, $supp\phi \subset B_2(0)$ and $0 \leq \phi \leq 1$. Denote by $\widetilde{\phi}_{\rho,n}(x) = \phi_\rho\left(\frac{x - y_n}{\sigma_n}\right)$, by the facts that $y_n \in \bar{Q}$, $x_{j_0} \in \overline{B_1(0)}$ and $\sigma_n \to 0$, we then observe that $supp\widetilde{\phi}_{\rho,n}(x) \subset B_{2\sigma_n\rho}(y_n + \sigma_n x_{j_0}) \subset \Omega$, which implies $\widetilde{\phi}_{\rho,n}(x)u_n \in \mathcal{D}_0^{1,2}(\Omega)$. A direct calculation yields that

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla(\widetilde{\phi}_{\rho,n} u_n)|^2 dx &\leq C \int_{\mathbb{R}^N} |\nabla \widetilde{\phi}_{\rho,n}|^2 u_n^2 dx + C \int_{\mathbb{R}^N} |\widetilde{\phi}_{\rho,n}|^2 |\nabla u_n|^2 dx \\ &\leq C \left(\int_{B_{2\sigma_n\rho}(y_n + \sigma_n x_{i_0})} |\nabla \widetilde{\phi}_{\rho,n}|^N dx\right)^{\frac{2}{N}} \cdot \left(\int_{\mathbb{R}^N} |u_n|^{2^*} dx\right)^{\frac{N-2}{N}} \\ &\quad + C \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \\ &\leq C. \end{aligned} \tag{3.17}$$

Hence, $\{\widetilde{\phi}_{\rho,n} u_n\}$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and the bound is independent of ρ. Combining (3.5), (3.6) with the fact that $C_0^\infty(\mathbb{R}^N)$ is dense in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, we then get

$$\begin{aligned} &\int_{\mathbb{R}^N} \nabla v_n \nabla(\phi_\rho v_n) dx - \int_{\mathbb{R}^N} (I_\alpha * |v_n|^{2_\alpha^*})|v_n|^{2_\alpha^*-2} v_n (\phi_\rho v_n) dx \\ &= \int_{\mathbb{R}^N} \nabla u_n \cdot \nabla(\widetilde{\phi}_{\rho,n} u_n) dx - \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2_\alpha^*})|u_n|^{2_\alpha^*-2} u_n (\widetilde{\phi}_{\rho,n} u_n) dx = o_n(1). \end{aligned} \tag{3.18}$$

Let $\rho \to 0$, then

$$\limsup_{n\to\infty}\left|\int_{\mathbb{R}^N}(\nabla v_n\cdot\nabla\phi_\rho)v_n dx\right| \le \limsup_{n\to\infty}\left(\int_{\mathbb{R}^N}|\nabla v_n|^2dx\right)^{\frac{1}{2}}\left(\int_{B_{2\rho}(x_{j_0})}v_n^2|\nabla\phi_\rho|^2dx\right)^{\frac{1}{2}}$$
$$\le C\left(\int_{B_{2\rho}(x_{j_0})}|\nabla\phi_\rho|^Ndx\right)^{\frac{1}{N}}\left(\int_{B_{2\rho}(x_{j_0})}|v_n|^{2^*}dx\right)^{\frac{1}{2^*}} \to 0. \tag{3.19}$$

Moreover,

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}|\nabla v_n|^2\phi_\rho dx \ge \int_{\mathbb{R}^N}|\nabla v|^2\phi_\rho dx + \widetilde{\mu}_{j_0} \to \widetilde{\mu}_{j_0} \tag{3.20}$$

and

$$\int_{\mathbb{R}^N}(I_\alpha * |v_n|^{2^*_\alpha})|v_n|^{2^*_\alpha}\phi_\rho dx \le \left(\int_{supp\phi}(I_\alpha * |v_n|^{2^*_\alpha})^{\frac{2N}{N-\alpha}}dx\right)^{\frac{N-\alpha}{2N}}\left(\int_{supp\phi}|v_n|^{2^*}dx\right)^{\frac{N+\alpha}{2N}}$$
$$\to \widetilde{\omega}_{j_0}^{\frac{N-\alpha}{2N}}\widetilde{\nu}_{j_0}^{\frac{N+\alpha}{2N}} \le A_\alpha C(N,\alpha)\widetilde{\nu}_{j_0}^{\frac{N+\alpha}{N}}. \tag{3.21}$$

It follows from (3.18)–(3.20) that

$$S\widetilde{\nu}_{j_0}^{\frac{N-2}{N}} \le \widetilde{\mu}_{j_0} \le A_\alpha C(N,\alpha)\widetilde{\nu}_{j_0}^{\frac{N+\alpha}{N}},$$

then

$$\nu_{j_0} \ge \left[\frac{S}{A_\alpha C(N,\alpha)}\right]^{\frac{N}{\alpha+2}}.$$

Combining the inequality above and (3.9), then we get

$$\left[\frac{S}{A_\alpha C(N,\alpha)}\right]^{\frac{N}{\alpha+2}} \le \nu_{j_0} \le \int_{B_1(0)}|v_n|^{2^*}dx = \tau,$$

which contradicts the assumption $\tau \in \left(0, \left[\frac{S}{A_\alpha C(N,\alpha)}\right]^{\frac{N}{\alpha+2}}\right)$. Therefore, (3.14) is proved. Combining (3.9) and (3.14), we have

$$\int_{B_1(0)}|v|^{2^*}dx = \lim_{n\to+\infty}\int_{B_1(0)}|v_n|^{2^*}dx = \tau > 0,$$

which implies that $v \ne 0$. Thus, combining Claims 1–3, we can complete the proof. □

Lemma 3.3 *Let $\{u_n\}$ be a Palais–Smale sequence for I_∞, such that $u_n \in C_0^\infty(\mathbb{R}^N)$ and*

$$u_n \rightharpoonup 0 \quad \text{weakly in } \mathcal{D}^{1,2}(\mathbb{R}^N); \quad u_n \nrightarrow 0 \text{ strongly in } \mathcal{D}^{1,2}(\mathbb{R}^N).$$

Then there exist a sequence of points $\{y_n\} \subset \mathbb{R}^N$, a sequence of positive numbers $\{\sigma_n\}$ such that

$$v_n(x) := \sigma_n^{\frac{N-2}{2}}u_n(\sigma_n x + y_n)$$

converges weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ to v, which is a nontrivial solution of Eq. (1.6). Moreover,

$$I_\infty(u_n) = I_\infty(v) + I_\infty(v_n - v) + o_n(1); \tag{3.22}$$
$$\|u_n\|^2 = \|v\|^2 + \|v_n - v\|^2 + o_n(1). \tag{3.23}$$

Proof Since $u_n \rightharpoonup 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, then $\{u_n\}$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Furthermore, as $\{u_n\}$ is a Palais–Smale sequence for I_∞, we then know that

$$\Delta u_n - (I_\alpha * |u_n|^{2_\alpha^*})|u_n|^{2_\alpha^*-2}u_n = \chi_n, \tag{3.24}$$

where $\chi_n \in \left(\mathcal{D}^{1,2}(\mathbb{R}^N)\right)'$ satisfies

$$\langle \chi_n, \psi \rangle \le \varepsilon_n \|\psi\|, \quad \text{for all } \ \psi \in C_0^\infty(\Omega). \tag{3.25}$$

Multiplying by u_n on both sides of (3.24) and integrating on $\mathbb{R}^N$, we then have

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 dx = \int_{\mathbb{R}^N} \left(I_\alpha * |u_n|^{2_\alpha^*}\right)|u_n|^{2_\alpha^*} dx + o_n(1). \tag{3.26}$$

Let us decompose $\mathbb{R}^N$ in N-dimensional hypercubes Q_i with unitary sides and vertices with integer coordinates. Next, we assert that for any $n \in \mathbb{N}$, there exists some $\widetilde{\varrho} > 0$ satisfying

$$d_n := \sup_{Q_i} \int_{Q_i} \left(I_\alpha * |u_n|^{2_\alpha^*}\right)|u_n|^{2_\alpha^*} dx \ge \widetilde{\varrho}.$$

If not, then we have $d_n \to 0$ as $n \to +\infty$. A direct calculation shows that

$$\begin{aligned}
\int_{\mathbb{R}^N} \left(I_\alpha * |u_n|^{2_\alpha^*}\right)|u_n|^{2_\alpha^*} dx &\le d_n^{1-\frac{1}{2_\alpha^*}} \sum_i \left(\int_{Q_i} (I_\alpha * |u_n|^{2_\alpha^*})|u_n|^{2_\alpha^*} dx\right)^{\frac{1}{2_\alpha^*}} \\
&\le d_n^{1-\frac{1}{2_\alpha^*}} \left(C(N,\alpha)A_\alpha\right)^{\frac{1}{2_\alpha^*}} \sum_i \left(\int_{Q_i} |u_n|^{2^*} dx\right)^{\frac{1}{2^*}} \left(\int_{\mathbb{R}^N} |u_n|^{2^*} dx\right)^{\frac{1}{2^*}} \\
&\le C d_n^{1-\frac{1}{2_\alpha^*}} \left(C(N,\alpha)A_\alpha\right)^{\frac{1}{2_\alpha^*}} \|u_n\|^2.
\end{aligned} \tag{3.27}$$

Combining (3.26) with (3.27) and letting $d_n \to 0$ as $n \to +\infty$, we observe that $\|u_n\| \to 0$, which leads to a contradiction.

In the following, let $\widetilde{y}_n$ be the center of a hypercube Q_i such that

$$\int_{Q_i} (I_\alpha * |u_n|^{2_\alpha^*})|u_n|^{2_\alpha^*} dx \ge \frac{\widetilde{\varrho}}{2} > 0. \tag{3.28}$$

Set $w_n = u_n(x + \widetilde{y}_n)$, then

$$\int_{Q} (I_\alpha * |w_n|^{2_\alpha^*})|w_n|^{2_\alpha^*} dx \ge \frac{\widetilde{\varrho}}{2} > 0, \tag{3.29}$$

where Q denote a hypercube of unitary side centered at the origin. Using the Hardy–Littlewood–Sobolev inequality and the boundedness of $\{u_n\}$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ again, we get

$$\frac{\widetilde{\varrho}}{2} \le \int_{Q} (I_\alpha * |w_n|^{2_\alpha^*})|w_n|^{2_\alpha^*} dx \le C \left(\int_{Q} |w_n|^{2^*} dx\right)^{\frac{N+\alpha}{N}}.$$

Hence we can deduce that there exists $\bar{\varrho} > 0$ such that

$$\int_{Q} |w_n|^{2^*} dx > \bar{\varrho}.$$

At this point, we have verified the conditions (3.3)–(3.5) in Lemma 3.2 for $\{w_n\}$. The first part of Lemma 3.3 follows from Lemma 3.2. Obviously,

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 dx = \int_{\mathbb{R}^N} |\nabla v_n|^2 dx, \quad \int_{\mathbb{R}^N} \left(I_\alpha * |u_n|^{2_\alpha^*}\right)|u_n|^{2_\alpha^*} dx = \int_{\mathbb{R}^N} \left(I_\alpha * |v_n|^{2_\alpha^*}\right)|v_n|^{2_\alpha^*} dx.$$

Then we can prove (3.23). Similarly, (3.22) follows from (3.1). □

It follows from Theorems A and 1.1 that

$$S_\alpha := \inf_{\mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_\alpha})|u(x)|^{2^*_\alpha} dx\right)^{\frac{N-2}{N+\alpha}}} = \frac{S}{[C(N,\alpha)A_\alpha]^{\frac{N-2}{N+\alpha}}}. \tag{3.30}$$

Now, we are ready to prove Theorem 1.3.

Proof of Theorem 1.3 Since $\{u_n\}$ is a Palais–Smale sequence for I at level c, then it is easy to prove that $\{u_n\}$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and consequently bounded in $L^{2^*}(\mathbb{R}^N)$. Without loss of generality, we may assume that $u_n \rightharpoonup \bar{u}$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and $L^{2^*}(\mathbb{R}^N)$ as $n \to +\infty$. Moreover, $\bar{u}$ is a weak solution of Eq. (1.10). In fact, for any $\varphi_1 \in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned}\langle I'(\bar{u}), \varphi_1\rangle &= \langle I'(u_n), \varphi_1\rangle + \int_{\mathbb{R}^N} V(x)(\bar{u}-u_n)\varphi_1 dx + \int_{\mathbb{R}^N} \nabla(\bar{u}-u_n)\nabla\varphi_1 dx \\ &\quad - \int_{\mathbb{R}^N}(I_\alpha * |\bar{u}|^{2^*_\alpha})|\bar{u}|^{2^*_\alpha-2}\bar{u}\varphi_1 dx + \int_{\mathbb{R}^N}(I_\alpha * |u_n|^{2^*_\alpha})|u_n|^{2^*_\alpha-2}u_n\varphi_1 dx.\end{aligned} \tag{3.31}$$

By Lemma 3.1, we know

$$\int_{\mathbb{R}^N}(I_\alpha * |u_n|^{2^*_\alpha})|u_n|^{2^*_\alpha-2}u_n\varphi_1 dx - \int_{\mathbb{R}^N}(I_\alpha * |\bar{u}|^{2^*_\alpha})|\bar{u}|^{2^*_\alpha-2}\bar{u}\varphi_1 dx = o_n(1). \tag{3.32}$$

Moreover, by Lemma 2.13 [40], we have

$$\int_{\mathbb{R}^N} V(x)(\bar{u}-u_n)^2 dx \to 0, \quad \text{as } n \to +\infty$$

and

$$\int_{\mathbb{R}^N} V(x)(\bar{u}-u_n)\varphi_1 dx \to 0; \quad \int_{\mathbb{R}^N} \nabla(\bar{u}-u_n)\nabla\varphi_1 dx \to 0, \quad \text{as } n \to +\infty. \tag{3.33}$$

Thus, it follows by (3.31)–(3.33) that

$$\langle I'(\bar{u}), \varphi_1\rangle = \langle I'(u_n), \varphi_1\rangle + o_n(1),$$

which leads to $I'(\bar{u}) = 0$, $I(\bar{u}) = I(u_n) - I_\infty(u_n - \bar{u}) + o_n(1)$.

Let $z_n^1 := u_n - \bar{u}$, then $z_n^1 \rightharpoonup 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and $\{z_n^1\}$ is a Palais–Smale sequence for I_∞. In fact, for any $\varphi_2 \in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned}\langle I'_\infty(z_n^1), \varphi_2\rangle &= \langle I'(u_n), \varphi_2\rangle - \langle I'(\bar{u}), \varphi_2\rangle + \int_{\mathbb{R}^N} V(x)(\bar{u}-u_n)\varphi_2 dx \\ &\quad - \int_{\mathbb{R}^N}(I_\alpha * |\bar{u}|^{2^*_\alpha})|\bar{u}|^{2^*_\alpha-2}\bar{u}\varphi_2 dx \\ &\quad - \int_{\mathbb{R}^N}(I_\alpha * |z_n^1|^{2^*_\alpha})|z_n^1|^{2^*_\alpha-2}z_n^1\varphi_2 dx + \int_{\mathbb{R}^N}(I_\alpha * |u_n|^{2^*_\alpha})|u_n|^{2^*_\alpha-2}u_n\varphi_2 dx \\ &= o_n(1)\|\varphi_2\|,\end{aligned}$$

where (3.33) and (3.2) are used. Hence $\{z_n^1\}$ is a Palais–Smale sequence of I_∞.

For any $n \in \mathbb{N}^+$, there exists a sequence $\{u_n^1\} \subset C_0^\infty(\mathbb{R}^N)$ such that

$$\|u_n^1 - z_n^1\| < \frac{1}{n} \text{ and } \|I'_\infty(u_n^1) - I'_\infty(z_n^1)\| < \frac{1}{n}. \tag{3.34}$$

It is not difficult to verify that

$$\|u_n^1\|^2 = \|z_n^1\|^2 + o_n(1); \quad I_\infty(u_n^1) = I_\infty(z_n^1) + o_n(1); \quad I'_\infty(u_n^1) = I'_\infty(z_n^1) + o_n(1). \tag{3.35}$$

Furthermore, one has

$$\|u_n^1\|^2 = \|z_n^1\|^2 + o_n(1) = \|u_n\|^2 - \|\bar{u}\|^2 + o_n(1) \tag{3.36}$$

and

$$I_\infty(u_n^1) = I_\infty(z_n^1) + o_n(1) = I(u_n) - I(\bar{u}) + o_n(1). \tag{3.37}$$

If $u_n^1 \to 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, then we have done. Now we suppose that $u_n^1 \nrightarrow 0$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. From (3.35) that we know that $\{u_n^1\}$ is a Palais–Smale sequence of I_∞ and $\{u_n^1\} \subset C_0^\infty(\mathbb{R}^N)$ satisfies

$$u_n^1 \rightharpoonup 0 \text{ in } \mathcal{D}^{1,2}(\mathbb{R}^N) \text{ and } u_n^1 \nrightarrow 0 \text{ strongly in } \mathcal{D}^{1,2}(\mathbb{R}^N).$$

Applying Lemma 3.3 to $\{u_n^1\}$, we assert that there exist a sequence of points $\{x_n^1\} \subset \mathbb{R}^N$, a sequence of positive numbers $\{\eta_n^1\} \subset \mathbb{R}$ such that

$$v_n^1 := (\eta_n^1)^{\frac{N-2}{2}} u_n^1(\eta_n^1 \cdot + x_n^1)$$

converges weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ to a nontrivial solution u^1 of Eq. (1.6). Moreover,

$$I_\infty(u_n^1) = I_\infty(u^1) + I_\infty(v_n^1 - u^1) + o_n(1) \text{ and } \|u_n^1\|^2 = \|u^1\|^2 + \|v_n^1 - u^1\|^2 + o_n(1). \tag{3.38}$$

Combining (3.38) with (3.35), we obtain that

$$I(u_n) = I(\bar{u}) + I_\infty(u^1) + I_\infty(v_n^1 - u^1) + o_n(1) \tag{3.39}$$

and

$$\|u_n\|^2 = \|\bar{u}\|^2 + \|v_n^1 - u^1\|^2 + \|u^1\|^2 + o_n(1). \tag{3.40}$$

Let $z_n^j = v_n^{j-1} - u^{j-1}$ and repeat the above procedure. If $z_n^j \to 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, we have done. If $z_n^j \nrightarrow 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, the analogously $\{z_n^j\}$ is a Palais–Smale sequence of I_∞, then we can find $\{u_n^j\} \subset C_0^\infty(\mathbb{R}^N)$ such that

$$\|u_n^j - z_n^j\| < \frac{1}{n} \text{ and } \|I'_\infty(u_n^j) - I'_\infty(z_n^j)\| < \frac{1}{n}, \tag{3.41}$$

and there exist a sequence of positive numbers $\{\eta_n^j\} \subset \mathbb{R}$ and a sequence of points $\{x_n^j\} \subset \mathbb{R}^N$ such that

$$v_n^j := (\eta_n^j)^{\frac{N-2}{2}} u_n^j(\eta_n^j \cdot + x_n^j)$$

converges weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ to a nontrivial solution u^j of Eq. (1.6). Moreover, the following properties hold:

$$I_\infty(v_n^j) = I_\infty(u^j) + I_\infty(v_n^j - u^j) + o_n(1) \text{ and } \|v_n^j\|^2 = \|u^j\|^2 + \|v_n^j - u^j\|^2 + o_n(1). \tag{3.42}$$

Furthermore, we deduce that

$$I(u_n) = I(\bar{u}) + \sum_{i=1}^{j-1} I_\infty(u^i) + I_\infty(v_n^j - u^j) + o_n(1) \tag{3.43}$$

and

$$\|u_n\|^2 = \|\bar{u}\|^2 + \sum_{i=1}^{j-1} \|u^i\|^2 + \|v_n^j - u^j\|^2 + o_n(1). \tag{3.44}$$

Since u^j is a nontrivial weak solution of Eq. (1.6), then $\|u^j\|^2 \geq S_\alpha^{\frac{N+\alpha}{\alpha+2}}$, which together with (3.44) and the fact that u_n is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ tells us that the iteration procedure must terminate after finitely-many steps. Therefore, we complete the proof of Theorem 1.3. □

4 Existence of positive bound state solution

In this section, we prove the existence of bound state solutions to Eq. (1.10). Firstly, we show that, providing $V(x) \geq 0$ and $V(x) \in L^{\frac{N}{2}}(\mathbb{R}^N)$, then there is no minimizer for functional I restrict on the Nehari manifold $\mathcal{N}$.

Proposition 4.1 *Assume that $V(x) \geq 0$ and $V(x) \in L^{\frac{N}{2}}(\mathbb{R}^N)$, then $m = m_\infty$ holds and m is not attained.*

Proof Obviously, for $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)\backslash\{0\}$, there exist unique $t_u, s_u > 0$ such that $t_u u \in \mathcal{N}$ $s_u u \in \mathcal{N}_\infty$, moreover $I(t_u u) = \max_{t>0} I(tu)$ and $I_\infty(s_u u) = \max_{s>0} I_\infty(su)$. Especially, if $u \in \mathcal{N}$ and $s_u u \in \mathcal{N}_\infty$, then we have $s_u \in (0, 1]$. Therefore, for $u \in \mathcal{N}$,

$$\begin{aligned} m_\infty \leq I_\infty(s_u u) &= \frac{s_u^2}{2}\|u\|^2 - \frac{s_u^{2\cdot 2_\alpha^*}}{2\cdot 2_\alpha^*}\int_{\mathbb{R}^N}\big(I_\alpha * |u|^{2_\alpha^*}\big)|u|^{2_\alpha^*}dx \\ &\leq \frac{s_u^2}{2}\|u\|^2 + \frac{s_u^2}{2}\int_{\mathbb{R}^N} V(x)u^2dx - \frac{s_u^{2\cdot 2_\alpha^*}}{2\cdot 2_\alpha^*}\int_{\mathbb{R}^N}\big(I_\alpha * |u|^{2_\alpha^*}\big)|u|^{2_\alpha^*}dx \\ &\leq I(u), \end{aligned} \tag{4.1}$$

which implies that $m_\infty \leq m$.

Next, we prove $m \leq m_\infty$. In fact, we consider a sequence $\{u_n := t_n w_n\} \subset \mathcal{N}$, where $w_n(\cdot) = w(\cdot - z_n)$ with w being a positive solution centered at zero to Eq. (1.6), $\{z_n\} \subset \mathbb{R}^N$ satisfying $|z_n| \to +\infty$ as $n \to +\infty$ and $t_n := t_{w_n}$. It follows by the definition of w_n that

$$w_n \rightharpoonup 0 \quad \text{in} \quad \mathcal{D}^{1,2}(\mathbb{R}^N); \quad \|w_n\| = \|w\| \neq 0 \tag{4.2}$$

and

$$\int_{\mathbb{R}^N}(I_\alpha * |w_n|^{2_\alpha^*})|w_n|^{2_\alpha^*}dx = \int_{\mathbb{R}^N}(I_\alpha * |w|^{2_\alpha^*})|w|^{2_\alpha^*}dx, \quad \text{as } n \to +\infty. \tag{4.3}$$

Furthermore, by Lemma 2.13 [40], we know that

$$\int_{\mathbb{R}^N} V(x)w_n^2dx \to 0 \text{ as } n \to +\infty. \tag{4.4}$$

Thus, in virtue of (4.2)–(4.4), we can prove easily that

$$I(u_n) = I(t_n w_n) = \frac{t_n^2}{2}\|w\|^2 + \frac{t_n^2}{2}o_n(1) - \frac{t_n^{2\cdot 2_\alpha^*}}{2\cdot 2_\alpha^*}\int_{\mathbb{R}^N}(I_\alpha * |w|^{2_\alpha^*})|w|^{2_\alpha^*}dx. \tag{4.5}$$

Since $w_n \in \mathcal{N}_\infty$ and $t_n w_n \in \mathcal{N}$, then

$$\|w_n\|^2 = \int_{\mathbb{R}^N}(I_\alpha * |w_n|^{2_\alpha^*})|w_n|^{2_\alpha^*}dx \tag{4.6}$$

and

$$t_n^2\|w_n\|^2 + t_n^2 \int_{\mathbb{R}^N} V(x) w_n^2 dx = t_n^{2\cdot 2_\alpha^*} \int_{\mathbb{R}^N} (I_\alpha * |w_n|^{2_\alpha^*})|w_n|^{2_\alpha^*} dx. \tag{4.7}$$

Combining (4.6) and (4.7), we then have

$$\|w\|^2 + o_n(1) = t_n^{22_\alpha^*-2} \int_{\mathbb{R}^N} \left(I_\alpha * |w|^{2_\alpha^*}\right)|w|^{2_\alpha^*} dx. \tag{4.8}$$

From (4.2), (4.3) and (4.8), then $\{t_n\}$ is bounded and $t_n \to 1$ as $n \to +\infty$. Therefore, we have $I(u_n) \to m_\infty$ as $n \to +\infty$ which implies that $m \le m_\infty$. Thus $m = m_\infty$.

In the following, we prove that m cannot be attained. If not, we suppose that there exists $u_0 \in \mathcal{N}$ such that $I(u_0) = m$ and $s_{u_0} u_0 \in \mathcal{N}_\infty$ with $s_{u_0} \in (0, 1]$. With a direct calculation, we get

$$\begin{aligned} m_\infty \le I_\infty(s_{u_0} u_0) &= \frac{s_{u_0}^2}{2}\|u_0\|^2 - \frac{s_{u_0}^{2\cdot 2_\alpha^*}}{2\cdot 2_\alpha^*} \int_{\mathbb{R}^N} (I_\alpha * |u_0|^{2_\alpha^*})|u_0|^{2_\alpha^*} dx \\ &\le \frac{s_{u_0}^2}{2}\|u_0\|^2 + \frac{s_{u_0}^2}{2} \int_{\mathbb{R}^N} V(x) u_0^2 dx - \frac{s_{u_0}^{2\cdot 2_\alpha^*}}{2\cdot 2_\alpha^*} \int_{\mathbb{R}^N} (I_\alpha * |u_0|^{2_\alpha^*})|u_0|^{2_\alpha^*} dx \\ &\le I(u_0) \le m_\infty, \end{aligned} \tag{4.9}$$

which leads to

$$\int_{\mathbb{R}^N} V(x) u_0^2 dx = 0 \text{ and } s_{u_0} = 1. \tag{4.10}$$

Thus, $u_0 \in \mathcal{N}_\infty$ and $I_\infty(u_0) = m_\infty$. Recalling that u_0 must be of form (1.7) and $u_0 > 0$, then

$$\int_{\mathbb{R}^N} V(x) u_0^2 dx > 0,$$

which contradicts to (4.10). Thus, m is not achieved. □

The following corollaries can be regarded as a direct consequence of Theorem 1.3 and Proposition 4.1.

Corollary 4.2 *Let $\{u_n\} \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$ be a nonnegative Palais–Smale sequence satisfying the assumptions of Theorem 1.3 with $c \in (m, 2m)$, then up to a subsequence, $\{u_n\}$ converges to a nonnegative nontrivial solution $\bar{u}$ of Eq. (1.10) strongly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$.*

Proof Obviously, $u_n \rightharpoonup \bar{u}$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and $\bar{u}$ is nonnegative. Since $c \in (m, 2m)$, we conclude $k \le 1$ in (1.12). If $\bar{u} \ne 0$ and $k = 1$, then $c \ge 2m$ by (1.14). If $\bar{u} = 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and $k = 1$, then u^1 is a nonnegative solution of Eq. (1.6). By using the property of super harmonic function, we deduce that u^1 is positive and $c = m$. This is a contradiction, since $c \in (m, 2m)$. □

Corollary 4.3 *If $\{u_n\}$ is a minimizing sequence for I on $\mathcal{N}$, then there exist a sequence of points $\{y_n\} \subset \mathbb{R}^N$, a sequence of positive numbers $\{\delta_n\} \subset \mathbb{R}^+$ and $\{w_n\} \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$ such that*

$$u_n(x) = w_n(x) + \psi_{\delta_n, y_n}(x), \tag{4.11}$$

where

$$\psi_{\delta_n, y_n}(x) := c_\alpha \left(\frac{\delta_n}{\delta_n^2 + |x - y_n|^2}\right)^{\frac{N-2}{2}}$$

and $w_n \to 0$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

Now, we prove the existence of positive solutions of Eq. (1.10) via classical Linking Theorem. A direct calculation shows that

$$\int_{\mathbb{R}^N} |\nabla \psi_{\delta,y}|^2 dx = \int_{\mathbb{R}^N} \left(I_\alpha * |\psi_{\delta,y}|^{2^*_\alpha}\right)|\psi_{\delta,y}(x)|^{2^*_\alpha} dx = S_\alpha^{\frac{N+\alpha}{\alpha+2}}. \tag{4.12}$$

In order to build a suitable min–max sequence for our problem, we introduce a barycenter type function and define $\mathcal{G} : \mathcal{D}^{1,2}(\mathbb{R}^N) \to \mathbb{R}^N \times \mathbb{R}^+$ by

$$\mathcal{G}(u) = \left(\frac{1}{S_\alpha}\right)^{\frac{N+\alpha}{\alpha+2}} \int_{\mathbb{R}^N} \left(\frac{x}{|x|}, \zeta(x)\right) |\nabla u|^2 dx := (\beta(u), \vartheta(u)),$$

where $\zeta(x)$ is a cut-off function such that

$$\zeta(x) = \begin{cases} 0, & \text{if } |x| < 1; \\ 1, & \text{if } |x| \geq 1. \end{cases} \tag{4.13}$$

Moreover,

$$\beta(u) = \left(\frac{1}{S_\alpha}\right)^{\frac{N+\alpha}{\alpha+2}} \int_{\mathbb{R}^N} \frac{x}{|x|} |\nabla u|^2 dx$$

and

$$\vartheta(u) = \left(\frac{1}{S_\alpha}\right)^{\frac{N+\alpha}{\alpha+2}} \int_{\mathbb{R}^N} \zeta(x) |\nabla u|^2 dx.$$

Lemma 4.4 *If* $|y| \geq \frac{1}{2}$, *then*

$$\beta(\psi_{\delta,y}) = \frac{y}{|y|} + o_n(1) \quad as \quad \delta \to 0.$$

Proof A direct calculation shows that

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B_\varepsilon(y)} |\nabla \psi_{\delta,y}|^2 dx &\leq C\delta^{N-2} \int_{\mathbb{R}^N \setminus B_\varepsilon(y)} \frac{|x-y|^2}{(\delta^2 + |x-y|^2)^N} dx \\ &= C\delta^{N-2} \int_\varepsilon^{+\infty} \frac{\tilde{\rho}^{N+1}}{(\delta^2 + \tilde{\rho}^2)^N} d\tilde{\rho} \leq C\delta^{N-2} \int_\varepsilon^{+\infty} \frac{1}{\tilde{\rho}^{N-1}} d\tilde{\rho}. \end{aligned} \tag{4.14}$$

Then, for each $\varepsilon > 0$, there exists $\delta_0 := \delta_0(\varepsilon)$ such that for any $\delta \in (0, \delta_0]$,

$$\int_{\mathbb{R}^N \setminus B_\varepsilon(y)} |\nabla \psi_{\delta,y}|^2 dx < \varepsilon. \tag{4.15}$$

Furthermore

$$\left|\beta(\psi_{\delta,y}) - \left(\frac{1}{S_\alpha}\right)^{\frac{N+\alpha}{\alpha+2}} \int_{B_\varepsilon(y)} \frac{x}{|x|} |\nabla \psi_{\delta,y}|^2 dx\right| < \varepsilon. \tag{4.16}$$

Let ε be small enough such that for $|y| \geq \frac{1}{2}$, the following property holds

$$\left|\frac{x}{|x|} - \frac{y}{|y|}\right| < \varepsilon \quad \text{for any } x \in B_\varepsilon(y).$$

Then, by (4.15), we have

$$\begin{aligned} &\Big|\frac{y}{|y|}-\left(\frac{1}{S_\alpha}\right)^{\frac{N+\alpha}{\alpha+2}}\int_{B_\varepsilon(y)}\frac{x}{|x|}|\nabla\psi_{\delta,y}|^2dx\Big| \\ &=\left(\frac{1}{S_\alpha}\right)^{\frac{N+\alpha}{\alpha+2}}\Big|\int_{B_\varepsilon(y)}\left(\frac{y}{|y|}-\frac{x}{|x|}\right)|\nabla\psi_{\delta,y}|^2dx+\int_{\mathbb{R}^N\setminus B_\varepsilon(y)}\frac{y}{|y|}|\nabla\psi_{\delta,y}|^2dx\Big| \\ &\le\frac{\varepsilon}{S_\alpha^{\frac{N+\alpha}{\alpha+2}}}\int_{B_\varepsilon(y)}|\nabla\psi_{\delta,y}|^2dx+\varepsilon\le 2\varepsilon. \end{aligned} \tag{4.17}$$

Therefore, it follows by (4.16) and (4.17) that we can easily deduce that

$$\left|\beta(\psi_{\delta,y})-\frac{y}{|y|}\right|\le 3\varepsilon$$

and then we complete the proof of lemma. □

In the sequel, we denote by

$$\mathcal{M}:=\left\{u\in\mathcal{N}:\mathcal{G}(u)=\big(\beta(u),\vartheta(u)\big)=\left(0,\frac{1}{2}\right)\right\}$$

a subset of Nehari manifold $\mathcal{N}$ and define $c_{\mathcal{M}}:=\inf\limits_{u\in\mathcal{M}}I(u)$.

Lemma 4.5 *Let $V(x)\ge 0$ and $V(x)\in L^{\frac{N}{2}}(\mathbb{R}^N)$. Then $c_{\mathcal{M}}>m$.*

Proof Obviously $c_{\mathcal{M}}\ge m$. In order to show the identity cannot hold, we shall argue by contradiction and then assume that there exists a sequence of $\{u_n\}\subset\mathcal{M}$ such that

$$\lim_{n\to+\infty}I(u_n)=m.$$

Moreover, for any $n\in\mathbb{N}$,

$$\beta(u_n)=0\quad\text{and}\quad\vartheta(u_n)=\frac{1}{2}. \tag{4.18}$$

By Corollary 4.3, we deduce that there exist a sequence of points $\{y_n\}\subset\mathbb{R}^N$, a sequence of positive numbers $\{\delta_n\}\subset\mathbb{R}^N$ and a sequence of functions $\{w_n\}\subset\mathcal{D}^{1,2}(\mathbb{R}^N)$ with $w_n\to 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ such that $u_n(x)=w_n(x)+\psi_{\delta_n,y_n}$. By the definition of $\mathcal{G}$, we get

$$\begin{aligned} \mathcal{G}(w_n+\psi_{\delta_n,y_n})&=\left(\frac{1}{S_\alpha}\right)^{\frac{N+\alpha}{\alpha+2}}\int_{\mathbb{R}^N}\left(\frac{x}{|x|},\zeta(x)\right)|\nabla(w_n+\psi_{\delta_n,y_n})|^2dx \\ &=\left(\frac{1}{S_\alpha}\right)^{\frac{N+\alpha}{\alpha+2}}\int_{\mathbb{R}^N}\left(\frac{x}{|x|},\zeta(x)\right)|\nabla w_n|^2dx \\ &\quad+\frac{2}{S_\alpha^{\frac{N+\alpha}{\alpha+2}}}\int_{\mathbb{R}^N}\left(\frac{x}{|x|},\zeta(x)\right)(\nabla w_n\nabla\psi_{\delta_n,y_n})dx \\ &\quad+\left(\frac{1}{S_\alpha}\right)^{\frac{N+\alpha}{\alpha+2}}\int_{\mathbb{R}^N}\left(\frac{x}{|x|},\zeta(x)\right)|\nabla\psi_{\delta_n,y_n}|^2dx \\ &=\left(\frac{1}{S_\alpha}\right)^{\frac{N+\alpha}{\alpha+2}}\int_{\mathbb{R}^N}\left(\frac{x}{|x|},\zeta(x)\right)|\nabla\psi_{\delta_n,y_n}|^2dx+o_n(1)=\mathcal{G}(\psi_{\delta_n,y_n})+o_n(1). \end{aligned}$$

Therefore, by (4.18), we deduce that

$$\beta(\psi_{\delta_n,y_n}) \to 0 \ \text{ and } \ \vartheta(\psi_{\delta_n,y_n}) \to \frac{1}{2} \ \text{ as } \ n \to +\infty. \tag{4.19}$$

There exists a subsequence (δ_n, y_n) such that one of the following cases may happen

(1) $\delta_n \to +\infty$ as $n \to \infty$;
(2) $\delta_n \to \bar{\delta} \neq 0$ as $n \to \infty$;
(3) $\delta_n \to 0$ and $y_n \to \bar{y}$, $|\bar{y}| < \frac{1}{2}$ as $n \to \infty$;
(4) $\delta_n \to 0$ as $n \to \infty$ and $|y_n| \geq \frac{1}{2}$ for n large.

Now we prove that none of the possibilities listed above can be true. Obviously, by Lemma 4.4 and (4.19), case (4) can not happen. If (1) holds, then

$$\begin{aligned} \vartheta(\psi_{\delta_n,y_n}) &= \left(\frac{1}{S_\alpha}\right)^{\frac{N+\alpha}{\alpha+2}} \int_{\mathbb{R}^N} \zeta(x)|\nabla\psi_{\delta_n,y_n}|^2 dx \\ &= \left(\frac{1}{S_\alpha}\right)^{\frac{N+\alpha}{\alpha+2}} \int_{\mathbb{R}^N\setminus B_1(0)} |\nabla\psi_{\delta_n,y_n}|^2 dx \\ &= 1 - \left(\frac{1}{S_\alpha}\right)^{\frac{N+\alpha}{\alpha+2}} \int_{B_1(0)} |\nabla\psi_{\delta_n,y_n}|^2 dx = 1 - o_n(1), \end{aligned}$$

which contradicts to (4.19). If (2) happens, we first assert that $|y_n| \to +\infty$. If not, up to a subsequence, we notice that ψ_{δ_n,y_n} would converge strongly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, so u_n converges strongly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, which is impossible by Proposition 4.1. Thus, for $n \to +\infty$, we have

$$\begin{aligned} \vartheta(\psi_{\delta_n,y_n}) &= \vartheta(\psi_{\bar{\delta},y_n}) + o_n(1) \\ &= \left(\frac{1}{S_\alpha}\right)^{\frac{N+\alpha}{\alpha+2}} \int_{\mathbb{R}^N} \zeta(x)|\nabla\psi_{\bar{\delta},y_n}|^2 dx + o_n(1) \\ &= \left(\frac{1}{S_\alpha}\right)^{\frac{N+\alpha}{\alpha+2}} \int_{\mathbb{R}^N} \zeta(x-y_n)|\nabla\psi_{\bar{\delta},0}|^2 dx + o_n(1) \\ &= 1 - \left(\frac{1}{S_\alpha}\right)^{\frac{N+\alpha}{\alpha+2}} \int_{B_1(y_n)} |\nabla\psi_{\bar{\delta},0}|^2 dx + o_n(1) = 1 + o_n(1), \end{aligned}$$

which is absurd in the sense of (4.19). If (3) is true, then for n large,

$$\begin{aligned} \vartheta(\psi_{\delta_n,y_n}) &= \left(\frac{1}{S_\alpha}\right)^{\frac{N+\alpha}{\alpha+2}} \int_{\mathbb{R}^N\setminus B_1(0)} |\nabla\psi_{\delta_n,y_n}|^2 dx + o_n(1) \\ &\leq \left(\frac{1}{S_\alpha}\right)^{\frac{N+\alpha}{\alpha+2}} \int_{\mathbb{R}^N\setminus B_1(y_n)} |\nabla\psi_{\delta_n,0}|^2 dx = o_n(1), \end{aligned}$$

which is also impossible. Then the proof is completed. □

In the following, we define a mapping $\theta : \mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\} \to \mathcal{N}$ by

$$\theta(u) = t_u u,$$

where t_u is the unique positive number such that $t_u u \in \mathcal{N}$. Also we define the operator $T : \mathbb{R}^N \times (0, +\infty) \to \mathcal{D}^{1,2}(\mathbb{R}^N)$ by

$$T(y, \delta) = \psi_{\delta,y}(x).$$

Then we have the following lemma.

Lemma 4.6 *Assume that $V(x) \geq 0$ and $V(x) \in L^{\frac{N}{2}}(\mathbb{R}^N)$. Then for any $\varepsilon > 0$, there exists $\delta_1 = \delta_1(\varepsilon)$ and $\delta_2 = \delta_2(\varepsilon)$ (without loss of generality, we assume that $\delta_1 \leq \delta_2$) such that*

$$I(\theta \circ T(y, \delta)) < m + \varepsilon$$

for any $\delta \in (0, \delta_1] \cup [\delta_2, +\infty)$ and $y \in \mathbb{R}^N$.

Proof Since $V(x) \in L^{\frac{N}{2}}(\mathbb{R}^N)$, then for any $\varepsilon > 0$, there exists $r > 0$ small enough such that

$$\sup_{y\in\mathbb{R}^N} \left(\int_{B_r(y)} |V(x)|^{\frac{N}{2}} dx \right)^{\frac{2}{N}} < \varepsilon. \tag{4.20}$$

A direct calculation shows that

$$\lim_{\delta\to 0} \int_{B_r(y)} |\psi_{\delta,y}|^{2^*} dx = \int_{\mathbb{R}^N} |\psi_{1,0}|^{2^*} dx = \int_{\mathbb{R}^N} |\psi_{\delta,y}|^{2^*} dx. \tag{4.21}$$

Thus, there exists $\delta_1 = \delta_1(\varepsilon)$ small enough, such that for any $\delta \in (0, \delta_1)$,

$$\left(\int_{\mathbb{R}^N\setminus B_r(y)} |\psi_{\delta,y}|^{2^*} dx \right)^{\frac{N-2}{N}} < \varepsilon. \tag{4.22}$$

From (4.20) and (4.22), we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} V(x)|\psi_{\delta,y}|^2 dx &= \int_{B_r(y)} V(x)|\psi_{\delta,y}|^2 dx + \int_{\mathbb{R}^N\setminus B_r(y)} V(x)|\psi_{\delta,y}|^2 dx \\ &\leq \left(\int_{B_r(y)} |V(x)|^{\frac{N}{2}} dx \right)^{\frac{2}{N}} \left(\int_{B_r(y)} |\psi_{\delta,y}|^{2^*} dx \right)^{\frac{N-2}{N}} \\ &\quad + \left(\int_{\mathbb{R}^N\setminus B_r(y)} |V(x)|^{\frac{N}{2}} dx \right)^{\frac{2}{N}} \left(\int_{\mathbb{R}^N\setminus B_r(y)} |\psi_{\delta,y}|^{2^*} dx \right)^{\frac{N-2}{N}} \\ &\leq C\varepsilon. \end{aligned} \tag{4.23}$$

Using $V(x) \in L^{\frac{N}{2}}(\mathbb{R}^N)$ again, we assume that for any $\varepsilon > 0$, there exists $R > 0$ big enough such that

$$\left(\int_{\mathbb{R}^N\setminus B_R(0)} |V(x)|^{\frac{N}{2}} dx \right)^{\frac{2}{N}} < \varepsilon. \tag{4.24}$$

Recalling that $\lim_{\delta\to+\infty} \sup_{x\in\mathbb{R}^N} |\psi_{\delta,y}| = 0$, then we obtain that

$$\lim_{\delta\to+\infty} \int_{B_R(0)} |\psi_{\delta,y}|^{2^*} dx = 0, \tag{4.25}$$

which implies that there exists $\delta_2 := \delta_2(\varepsilon) > 0$, such that for any $\delta \geq \delta_2$,

$$\left(\int_{B_R(0)} |\psi_{\delta,y}|^{2^*} dx \right)^{\frac{N-2}{N}} \leq \varepsilon. \tag{4.26}$$

In virtue of (4.24) and (4.26), we can prove that that for any $\delta \geq \delta_2$,

$$\int_{\mathbb{R}^N} V(x)|\psi_{\delta,y}|^2 dx = \int_{B_R(0)} V(x)|\psi_{\delta,y}|^2 dx + \int_{\mathbb{R}^N \setminus B_R(0)} V(x)|\psi_{\delta,y}|^2 dx \leq C\varepsilon. \quad (4.27)$$

Thus, combining (4.23) and (4.27) that we can conclude that

$$\int_{\mathbb{R}^N} V(x)|\psi_{\delta,y}|^2 dx < \varepsilon, \quad (4.28)$$

for any $y \in \mathbb{R}^N$ and $\delta \in (0, \delta_1] \cup [\delta_2, \infty)$.

For any $\psi_{\delta,y}$, there exists $t_\psi := t(\psi_{\delta,y}) \geq 1$ such that $t_\psi \psi_{\delta,y} \in \mathcal{N}$. With a similar argument to the proof in (4.6)–(4.8), we prove that for uniformly $y \in \mathbb{R}^N$, $t_\psi \to 1$ as $\delta \to 0$ or $\delta \to +\infty$. Thus, inspired by (4.28), for any $\delta \in (0, \delta_1] \cup [\delta_2, +\infty)$,

$$\begin{aligned} I(\theta \circ T(y,\delta)) &= \frac{t_\psi^2}{2}\|\psi_{\delta,y}\|^2 + \frac{t_\psi^2}{2}\int_{\mathbb{R}^N} V(x)\psi_{\delta,y}^2 dx - \frac{t_\psi^{2\cdot 2_\alpha^*}}{2\cdot 2_\alpha^*}\int_{\mathbb{R}^N} (I_\alpha * |\psi_{\delta,y}|^{2_\alpha^*})|\psi_{\delta,y}|^{2_\alpha^*} dx \\ &= I_\infty(t_\psi \psi_{\delta,y}) + \frac{t_\psi^2}{2}\int_{\mathbb{R}^N} V(x)\psi_{\delta,y}^2 dx < I_\infty(\psi_{\delta,y}) + \varepsilon = m + \varepsilon. \end{aligned}$$

□

Lemma 4.7 *Assume that $V(x) \geq 0$ and $V(x) \in L^{\frac{N}{2}}(\mathbb{R}^N)$. Then for any fixed $\delta > 0$,*

$$\lim_{|y| \to +\infty} I(\theta \circ T(y,\delta)) = m.$$

Proof First, we claim that for any fixed $\delta > 0$,

$$\lim_{|y| \to +\infty} \int_{\mathbb{R}^N} V(x)|\psi_{\delta,y}|^2 dx = 0. \quad (4.29)$$

Indeed, for a given $\varepsilon > 0$, we can choose some $R > 0$ large enough such that

$$\int_{\mathbb{R}^N \setminus B_R(0)} |V(x)|^{\frac{N}{2}} dx < \varepsilon$$

and

$$\int_{\mathbb{R}^N \setminus B_R(y)} |\psi_{\delta,y}|^{2^*} dx = \int_{\mathbb{R}^N \setminus B_R(0)} |\psi_{\delta,0}|^{2^*} dx < \varepsilon.$$

Taking y with $|y| > 2R$, we see

$$\begin{aligned} \int_{\mathbb{R}^N} V(x)|\psi_{\delta,y}|^2 dx &= \int_{\mathbb{R}^N \setminus (B_R(y) \cup B_R(0))} V(x)|\psi_{\delta,y}|^2 dx \\ &\quad + \int_{B_R(y)} V(x)|\psi_{\delta,y}|^2 dx + \int_{B_R(0)} V(x)|\psi_{\delta,y}|^2 dx \\ &\leq \left(\int_{\mathbb{R}^N \setminus B_R(0)} |V(x)|^{\frac{N}{2}} dx\right)^{\frac{2}{N}} \left(\int_{\mathbb{R}^N \setminus B_R(y)} |\psi_{\delta,y}|^{2^*} dx\right)^{\frac{N-2}{N}} \\ &\quad + \left(\int_{\mathbb{R}^N \setminus B_R(0)} |V(x)|^{\frac{N}{2}} dx\right)^{\frac{2}{N}} \|\psi_{\delta,y}\|_{L^{2^*}}^2 \end{aligned}$$

$$
\begin{aligned}
&+\|V(x)\|_{L^{\frac{N}{2}}}\left(\int_{\mathbb{R}^N\setminus B_R(y)}|\psi_{\delta,y}|^{2^*}dx\right)^{\frac{N-2}{N}}\\
&\le C\varepsilon. \qquad (4.30)
\end{aligned}
$$

With a similar argument as the proof in (4.6)–(4.8) again, we can also prove that $t_\psi \to 1$ as $|y| \to +\infty$, where t_ψ satisfies $t_\psi \psi_{\delta,y} \in \mathcal{N}$. Thus, as $|y| \to +\infty$, by (4.30),

$$
\begin{aligned}
m &\le I(\theta \circ T(y,\delta))\\
&= \frac{t_\psi^2}{2}\|\psi_{\delta,y}\|^2 + \frac{t_\psi^2}{2}\int_{\mathbb{R}^N} V(x)|\psi_{\delta,y}|^2dx - \frac{t_\psi^{2\cdot 2^*_\alpha}}{2\cdot 2^*_\alpha}\int_{\mathbb{R}^N}(I_\alpha * |\psi_{\delta,y}|^{2^*_\alpha})|\psi_{\delta,y}|^{2^*_\alpha}dx\\
&= \frac{1}{2}\|\psi_{\delta,y}\|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|\psi_{\delta,y}|^2dx - \frac{1}{2\cdot 2^*_\alpha}\int_{\mathbb{R}^N}(I_\alpha * |\psi_{\delta,y}|^{2^*_\alpha})|\psi_{\delta,y}|^{2^*_\alpha}dx + o(1)\\
&= m + o(1). \qquad (4.31)
\end{aligned}
$$

Thus $\lim_{|y|\to+\infty} I(\theta \circ T(y,\delta)) = m.$ □

From Lemma 4.5, we can deduce that there exists some $\sigma > 0$ such that $m + \sigma < c_{\mathcal{M}}$. In the following, we give some estimates.

Lemma 4.8 *There exists $\delta_1 \in (0, \frac{1}{2})$ such that for any $0 < \delta \le \delta_1$, the following properties hold.*

(a) $I(\theta \circ T(y,\delta)) < m + \sigma$, *for any $y \in \mathbb{R}^N$;*
(b) $\left|\beta(\theta \circ T(y,\delta)) - \frac{y}{|y|}\right| < \frac{1}{4}$, *for any $y \in \mathbb{R}^N$ with $|y| \ge \frac{1}{2}$;*
(c) $\vartheta(\theta \circ T(y,\delta)) < \frac{1}{2}$, *for any $y \in \mathbb{R}^N$ with $|y| < \frac{1}{2}$.*

Proof (a) and (b) are easy to prove. In fact, (a) can be seen as a direct consequence of Lemma 4.6. In Lemma 4.6, we have proved that $t_\psi \to 1$ as $\delta \to 0$, which together with Lemma 4.4 yields (b). Now we only need to prove (c). A direct calculation shows that

$$
\begin{aligned}
\vartheta(\theta \circ T(y,\delta)) &= \frac{t_\psi^2}{S_\alpha^{\frac{N+\alpha}{\alpha+2}}}\int_{\mathbb{R}^N}\zeta(x)|\nabla\psi_{\delta,y}|^2dx = \frac{t_\psi^2}{S_\alpha^{\frac{N+\alpha}{\alpha+2}}}\int_{\mathbb{R}^N\setminus B_1(0)}|\nabla\psi_{\delta,y}|^2dx\\
&= \frac{t_\psi^2}{S_\alpha^{\frac{N+\alpha}{\alpha+2}}}\int_{\mathbb{R}^N\setminus B_1(y)}|\nabla\psi_{\delta,0}|^2dx \to 0, \qquad (4.32)
\end{aligned}
$$

where in the last equality we have used the fact $\int_{\mathbb{R}^N\setminus B_1(y)}|\nabla\psi_{\delta,0}|^2dx \to 0$ for $|y| < \frac{1}{2}$ as $\delta \to 0$. □

Lemma 4.9 *There exist $\delta_2 \in (\frac{1}{2}, +\infty)$ such that for any $\delta \ge \delta_2$, the following properties hold.*

(a) $I(\theta \circ T(y,\delta)) < m + \sigma$, *for any $y \in \mathbb{R}^N$;*
(b) $\vartheta(\theta \circ T(y,\delta)) > \frac{1}{2}$, *for any $y \in \mathbb{R}^N$.*

Proof By Lemma 4.6, (a) is true. Since

$$
\lim_{\delta\to+\infty}\int_{B_1(0)}|\nabla\psi_{\delta,y}|^2dx = 0
$$

and $t_\psi \to 1$ as $\delta \to +\infty$, we obtain

$$\vartheta(\theta \circ T(y,\delta)) = t_\psi^2 \left(1 - \frac{1}{S_\alpha^{\frac{N+\alpha}{\alpha+2}}} \int_{B_1(0)} |\nabla \psi_{\delta,y}|^2 dx\right) \to 1 \text{ as } \delta \to +\infty.$$

Hence (b) holds. □

Lemma 4.10 *There exists some $R > 0$ such that for any $\delta \in [\delta_1, \delta_2]$, the following properties hold.*

(a) $I(\theta \circ T(y,\delta)) < m + \sigma$, *for any $y \in \mathbb{R}^N$ with $|y| \geq R$;*
(b) $\langle \beta(\theta \circ T(y,\delta)), y\rangle > 0$, *for any $y \in \mathbb{R}^N$ with $|y| \geq R$.*

Proof For any fixed δ, let $|y| \to +\infty$ and repeating the argument in the proof of (4.6)–(4.8) again, we know $t_\psi = t(\psi_{\delta,y}) \to 1$, where t_ψ satisfies $t_\psi \psi_{\delta,y} \in \mathcal{N}$. Using Lemma 4.7 and the compactness of $[\delta_1, \delta_2]$, we deduce that there exists some $R_1 > 0$ such that

$$I(\theta \circ T(y,\delta)) < m + \sigma \quad \text{for any } \delta \in [\delta_1, \delta_2] \text{ and } |y| \geq R_1.$$

Let $(\mathbb{R}^N)_y^+ = \{x \in \mathbb{R}^N : \langle x, y\rangle > 0\}$ and $(\mathbb{R}^N)_y^- = \mathbb{R}^N \backslash (\mathbb{R}^N)_y^+$. Since $\delta \in [\delta_1, \delta_2]$, we assert that there exists $R_2 > 0$ large enough and $r \in (0, \frac{1}{4})$ such that the following properties holds: for any y with $|y| \geq R_2$,

$$B_r(\widetilde{y}) = \{x \in \mathbb{R}^N : |x - \widetilde{y}| < r\} \subset (\mathbb{R}^N)_y^+$$

with $|\widetilde{y} - y| = \frac{1}{2}$ and for any $x \in B_r(\widetilde{y})$,

$$|\nabla \psi_{\delta,y}(x)|^2 = K_1 \delta^{N-2} \frac{|x-y|^2}{\left(\delta^2 + |x-y|^2\right)^N} \geq H_1 > 0,$$

where K_1 only depend on N and α, H_1 is a positive constant. Moreover, for each $x \in (\mathbb{R}^N)_y^-$,

$$|\nabla \psi_{\delta,y}(x)|^2 \leq \frac{H_2}{|x-y|^{2N-2}}, \quad H_2 \equiv const.$$

Thus, for any y satisfying $|y| \geq R_2$, we have

$$\begin{aligned}
\langle \beta(\theta \circ T(y,\delta)), y\rangle &= \frac{t_\psi^2}{S_\alpha} \int_{(\mathbb{R}^N)_y^+} |\nabla \psi_{\delta,y}(x)|^2 \frac{\langle x, y\rangle}{|x|} dx + \frac{t_\psi^2}{S_\alpha} \int_{(\mathbb{R}^N)_y^-} |\nabla \psi_{\delta,y}(x)|^2 \frac{\langle x, y\rangle}{|x|} dx \\
&\geq \frac{t_\psi^2}{S_\alpha} \int_{B_r(\widetilde{y})} H_1 \frac{\langle x, y\rangle}{|x|} dx - \frac{t_\psi^2}{S_\alpha} \int_{(\mathbb{R}^N)_y^-} H_2 \frac{|y|}{|x-y|^{2N-2}} dx \\
&\geq H_3 |y| - C \frac{1}{|y|^{N-3}} \int_{\mathbb{R}^{N-1}} \frac{1}{1+|z|^{2N-2}} dz > 0,
\end{aligned}$$

where H_3 is a positive constant. Taking $R = \max\{R_1, R_2\}$, we then complete the proof. □

In the sequel, we define a bounded domain $D \subset \mathbb{R}^N \times \mathbb{R}$ by

$$D := \{(y,\delta) \in \mathbb{R}^N \times \mathbb{R} \; : \; |y| \leq R, \; \delta_1 \leq \delta \leq \delta_2\},$$

where δ_1, δ_2 and R are given in Lemmas 4.8–4.10.

Lemma 4.11 *Define a mapping* $\Upsilon : D \to \mathbb{R}^N \times \mathbb{R}^+$ *by*

$$\Upsilon(y,\delta) = \big(\beta \circ \theta \circ T(y,\delta),\ \ \vartheta \circ \theta \circ T(y,\delta)\big).$$

Then

$$deg\left(\Upsilon, D, \left(0, \frac{1}{2}\right)\right) = 1.$$

Proof Consider the following homotopy

$$\mathscr{F}(y,\delta,s) = (1-s)(y,\delta) + s\Upsilon(y,\delta).$$

Since $deg\big(id, D, \big(0, \frac{1}{2}\big)\big) = 1$, then by the homotopy invariance of topological degree, we can complete the proof. In order to use the homotopy invariance of the topological degree, we must prove

$$\mathscr{F}(y,\delta,s) \neq \left(0, \frac{1}{2}\right) \ \text{ for any } \ (y,\delta) \in \partial D \ \text{ and } \ s \in [0,1]. \tag{4.33}$$

For the readers' convenience, we divide the proof into several cases and discuss them respectively.
Case 1 If $|y| < \frac{1}{2}$ and $\delta = \delta_1$, by Lemma 4.8(c), we know

$$(1-s)\delta_1 + s\vartheta \circ \theta \circ T(y,\delta_1) < \frac{1}{2}$$

for any $s \in [0,1]$.
Case 2 If $\frac{1}{2} \leq |y| \leq R$ and $\delta = \delta_1$, then it follows by Lemma 4.8(b) that

$$\left|\beta(\theta \circ T(y,\delta)) - \frac{y}{|y|}\right| < \frac{1}{4}.$$

Thus

$$\begin{aligned}|(1-s)y + s\beta(\theta \circ T(y,\delta_1))| &\geq \left|(1-s)y + s\frac{y}{|y|}\right| - \left|s\beta(\theta \circ T(y,\delta)) - s\frac{y}{|y|}\right| \\ &\geq s + (1-s)|y| - \frac{s}{4} \geq \frac{1}{4} \neq 0.\end{aligned}$$

Case 3 If $|y| \leq R$ and $\delta = \delta_2$, from Lemma 4.9(b), we know that

$$(1-s)\delta_2 + s\vartheta \circ \theta \circ T(y,\delta_2) > \frac{1}{2}$$

for any $s \in [0,1]$.
Case 4 If $|y| = R$ and $\delta \in [\delta_1, \delta_2]$, by Lemma 4.10(b),

$$\langle (1-s)y + s\beta \circ \theta \circ T(y,\delta),\ \ y\rangle > 0$$

for $s \in [0,1]$. □

Proof of Theorem 1.2 Obviously, the first part of Theorem 1.2 follows from Proposition 4.1. In order to apply the classical Linking Theorem (see [40]), we define

$$\mathcal{H} = \theta \circ T(D)$$

and

$$\mathcal{M} = \left\{u \in \mathcal{N} : \mathcal{G}(u) = (\beta(u), \vartheta(u)) = \left(0, \frac{1}{2}\right)\right\}.$$

We claim that $\mathcal{M}$ and $\partial\mathcal{H}$ is a link, that is

(a) $\partial\mathcal{H} \cap \mathcal{M} = \emptyset$;
(b) $h(\mathcal{H}) \cap \mathcal{M} \neq \emptyset$ for any $h \in \Lambda = \{h \in \mathcal{C}(\mathcal{H}, \mathcal{N}) : h(\partial\mathcal{H}) = id\}$.

In fact, if $u \in \theta \circ T(\partial D)$, then it follows from Lemmas 4.8(a), 4.9(a) and 4.10(a) that

$$I(u) < m + \sigma < c_{\mathcal{M}},$$

which implies $u \notin \mathcal{M}$ and we prove (a).

Next, we prove (b). In fact, for any $h \in \Lambda$, we define a continuous mapping $\widetilde{\eta} : D \to \mathbb{R}^N \times \mathbb{R}^+$ by

$$\widetilde{\eta}(y, \delta) = (\beta \circ h \circ \theta \circ T(y, \delta),\ \vartheta \circ h \circ \theta \circ T(y, \delta)).$$

If $(y, \delta) \in \partial D$, then $\theta \circ T(y, \delta) \in \partial\mathcal{H}$, hence $h \circ \theta \circ T(y, \delta) = \theta \circ T(y, \delta)$. Therefore

$$\widetilde{\eta}(y, \delta) = (\beta \circ \theta \circ T(y, \delta),\ \vartheta \circ \theta \circ T(y, \delta)) = \Upsilon(y, \delta) \ \text{ on } \ \partial D.$$

By the homotopy invariance of the topological degree and Lemma 4.11, we have

$$deg\left(\widetilde{\eta}, D, \left(0, \frac{1}{2}\right)\right) = deg\left(\Upsilon, D, \left(0, \frac{1}{2}\right)\right) = 1,$$

which implies that there exists $(y', \delta') \in D$ such that $h \circ \theta \circ T(y', \delta') \in \mathcal{M}$. Hence (b) holds.

Since $\mathcal{N}$ is a natural constraint for I, with classical minimal arguments we obtain a Palais–Smale sequence for I at level d with

$$d := \inf_{h \in \Lambda} \max_{u \in \mathcal{H}} I(h(u)).$$

From (b) and Lemma 4.5, we have

$$m < c_{\mathcal{M}} \leq d.$$

Moreover, by definition of d and $\mathcal{H}$, we get

$$d \leq \max_{u \in \mathcal{H}} I(u) \leq \sup_{(\delta, y) \in D} I(t_\psi \psi_{\delta,y}),$$

As $t_\psi \psi_{\delta,y} \in \mathcal{N}$, we know that

$$\begin{aligned} I(t_\psi \psi_{\delta,y}) &= \frac{t_\psi^2}{2}\|\psi_{\delta,y}\|^2 + \frac{t_\psi^2}{2}\int_{\mathbb{R}^N} V(x)|\psi_{\delta,y}|^2 dx - \frac{t_\psi^{2\cdot 2_\alpha^*}}{2\cdot 2_\alpha^*}\int_{\mathbb{R}^N}(I_\alpha * |\psi_{\delta,y}|^{2_\alpha^*})|\psi_{\delta,y}|^{2_\alpha^*} dx \\ &= \left(\frac{1}{2} - \frac{1}{2\cdot 2_\alpha^*}\right) t_\psi^{2\cdot 2_\alpha^*} \int_{\mathbb{R}^N}(I_\alpha * |\psi_{\delta,y}|^{2_\alpha^*})|\psi_{\delta,y}|^{2_\alpha^*} dx. \end{aligned} \tag{4.34}$$

On the other hand,

$$\begin{aligned} t_\psi^{2\cdot 2_\alpha^* - 2}\int_{\mathbb{R}^N}(I_\alpha * |\psi_{\delta,y}|^{2_\alpha^*})|\psi_{\delta,y}|^{2_\alpha^*} dx &= \|\psi_{\delta,y}\|^2 + \int_{\mathbb{R}^N} V(x)|\psi_{\delta,y}|^2 dx \\ &\leq \|\psi_{\delta,y}\|^2 + \|V(x)\|_{L^{\frac{N}{2}}} \cdot \|\psi_{\delta,y}\|^2_{L^{2^*}}. \end{aligned} \tag{4.35}$$

Recall that

$$\int_{\mathbb{R}^N}\left(I_\alpha * |\psi_{\delta,y}|^{2_\alpha^*}\right)|\psi_{\delta,y}(x)|^{2_\alpha^*} dx = C(N, \alpha) A_\alpha \|\psi_{\delta,y}\|^{22_\alpha^*}_{L^{2^*}} = S_\alpha^{\frac{N+\alpha}{\alpha+2}}.$$

By (4.35), we obtain that

$$t_\psi^{2\cdot 2_\alpha^* - 2} \leq 1 + \|V(x)\|_{L^{\frac{N}{2}}}\left(\frac{1}{C(N, \alpha) A_\alpha}\right)^{\frac{N-2}{\alpha+2}} S_\alpha^{-1} = 1 + \frac{\|V(x)\|_{L^{\frac{N}{2}}}}{S}, \tag{4.36}$$

which implies that $t_{\psi}^{2\cdot 2_{\alpha}^{*}} \leq \left(1+\frac{\|V(x)\|_{L^{\frac{N}{2}}}}{S}\right)^{\frac{N+\alpha}{\alpha+2}}$. Since $\|V(x)\|_{L^{\frac{N}{2}}} < (2^{\frac{\alpha+2}{N+\alpha}}-1)S$, we have

$$t_{\psi}^{2\cdot 2_{\alpha}^{*}} \leq \left(1+\frac{\|V(x)\|_{L^{\frac{N}{2}}}}{S}\right)^{\frac{N+\alpha}{\alpha+2}} < 2,$$

which combining together with (4.34) and the fact that

$$m_{\infty} = m = \left(\frac{1}{2}-\frac{1}{2\cdot 2_{\alpha}^{*}}\right)\int_{\mathbb{R}^N}(I_{\alpha} * |\psi_{\delta,y}|^{2_{\alpha}^{*}})|\psi_{\delta,y}|^{2_{\alpha}^{*}}dx$$

yields $m < d < 2m$.

We claim that there exists a nonnegative $(PS)_d$ sequence of I with $d \in (m, 2m)$. In fact, we can modify the energy functional I into

$$\widetilde{I}(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2+V(x)u^2\right)dx - \frac{1}{2\cdot 2_{\alpha}^{*}}\int_{\mathbb{R}^N}\left(I_{\alpha} * |u^{+}|^{2_{\alpha}^{*}}\right)|u^{+}|^{2_{\alpha}^{*}}dx, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^N).$$

Suppose $\{u_n\}$ is a $(PS)_d$ sequence of $\widetilde{I}$ with $d \in (m, 2m)$, then $\{u_n\}$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and

$$\langle \widetilde{I}'(u_n), u_n^{-}\rangle = \|u_n^{-}\|^2 = o_n(1).$$

It follows that

$$\widetilde{I}(u_n^{+}) \to d \in (m, 2m), \quad \widetilde{I}'(u_n^{+}) \to 0.$$

Thus, $\{u_n^{+}\}$ is a nonnegative $(PS)_d$ sequence of $\widetilde{I}$ with $d \in (m, 2m)$.

As a direct consequence of Corollary 4.2, up to a subsequence, we may suppose that $u_n^{+} \to u$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, and u is a nonnegative of (1.10). Since $V(x) \in L^{\frac{N}{2}}(\mathbb{R}^N) \cap \mathcal{C}^{\gamma}(\mathbb{R}^N)$ is nonnegative for some $\gamma \in (0, 1)$, by a similar argument as the proof of Proposition 2.4, one can deduce that $u \in \mathcal{C}^{2,\iota}(\mathbb{R}^N)$ for some $0 < \iota < \gamma$. Then, the positivity of u follows from the strong maximum principle. Thus we complete the proof of the Theorem 1.2. □

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