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Calculus of Variations

An optimal result for global existence in a three-dimensional Keller–Segel–Navier–Stokes system involving tensor-valued sensitivity with saturation

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Abstract
This paper focuses on the following Keller–Segel–Navier–Stokes system with rotational flux:

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (nS(x,n,c)\nabla c), & x \in \Omega, \quad t > 0, \\ c_t + u \cdot \nabla c = \Delta c - c + n, & x \in \Omega, \quad t > 0, \\ u_t + \kappa (u \cdot \nabla) u + \nabla P = \Delta u + n \nabla \phi, & x \in \Omega, \quad t > 0, \\ \nabla \cdot u = 0, & x \in \Omega, \quad t > 0 \end{cases} \qquad (KSNF)$$

in a bounded domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary, where $\kappa \in \mathbb{R}$ is a given constant, $\phi \in W^{1,\infty}(\Omega)$, $|S(x,n,c)| \leq C_S(1+n)^{-\alpha}$, and the parameter $\alpha \geq 0$. If $\alpha > \frac{1}{3}$, then, for all reasonable regular initial data, a corresponding initial-boundary value problem for $(KSNF)$ possesses a globally defined weak solution. This result improves upon the result of Wang (Math Models Methods Appl Sci 27(14):2745–2780, 2017), in which the global very weak solution for the system $(KSNF)$ is obtained. In comparison with the result of the corresponding fluid-free system, the optimal condition on the parameter α for global (weak) existence is established. Our proofs rely on a variant of the natural gradient-like energy functional.

Mathematics Subject Classification 35K55 · 35Q92 · 35Q35 · 92C17

1 Introduction

Chemotaxis, the biased movement of cells (or organisms) in response to chemical gradients, plays an important role in coordinating cell migration in many biological phenomena (see Hillen and Painter [8]). Let n denote the density of the cells and c present the concentration

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of the chemical signal. In the 1970s, Keller and Segel [12] proposed a mathematical system for chemotaxis through a system of parabolic equations. The mathematical model reads as

$$\begin{cases} n_t = \Delta n - \nabla \cdot (nS(x,n,c)\nabla c), & x \in \Omega, \quad t > 0, \\ c_t = \Delta c - c + n, & x \in \Omega, \quad t > 0, \end{cases} \tag{1.1}$$

where S is a given chemotactic sensitivity function, which can either be a scalar function or, more generally, a tensor-valued function (see, e.g., Xue and Othmer [42]). During the past four decades, Keller–Segel models (1.1) and their variants have attracted extensive attention, withthe main issue of investigation focusing on whether the solutions of the models are bounded or blow up (see Winkler et al. [1], Hillen and Painter [8] and Horstmann [9]). For instance, if $S := S(n)$ is a scalar function satisfying $S(s) \leq C(1+s)^{-\alpha}$ for all $s \geq 1$, $\alpha > 1 - \frac{2}{N}$, and $C > 0$, then all solutions to the corresponding Neumann problem are global and uniformly bounded (see Horstmann and Winkler [10]). However, if $N \geq 2$, Ω (a ball) $\subset R^N$, and $S(s) > cs^{-\alpha}$ for some $\alpha < 1 - \frac{2}{N}$ and $c > 0$, then the solution to problem (1.1) may blow up (see Horstmann and Winkler [10]). Therefore,

$$\alpha = 1 - \frac{2}{N} \tag{1.2}$$

is the critical blow-up exponent, which is related to the presence of a so-called volume-filling effect. For related works in this direction, we mention that a corresponding quasilinear version, the logistic damping or the signal consumed by the cells, has been deeply investigated by Cieślak and Stinner [4,5], Tao and Winkler [20,31,41] and Zheng et al. [44–46,50,51].

As in the classical Keller–Segel model where the chemoattractant is produced by bacteria, the corresponding chemotaxis–fluid model then becomes the following Keller–Segel(–Navier)–Stokes system:

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (nS(x,n,c)\nabla c), & x \in \Omega, \quad t > 0, \\ c_t + u \cdot \nabla c = \Delta c - c + n, & x \in \Omega, \quad t > 0, \\ u_t + \kappa (u \cdot \nabla) u + \nabla P = \Delta u + n \nabla \phi, & x \in \Omega, \quad t > 0, \\ \nabla \cdot u = 0, & x \in \Omega, \quad t > 0, \\ (\nabla n - nS(x,n,c)) \cdot \nu = \nabla c \cdot \nu = 0, \ u = 0, & x \in \partial\Omega, \quad t > 0, \\ n(x,0) = n_0(x), c(x,0) = c_0(x), \ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.3}$$

where n and c are defined as before and $\Omega \subset \mathbb{R}^3$ is a bounded domain with a smooth boundary. Here u, P, ϕ, and $\kappa \in \mathbb{R}$ denote, respectively, the velocity field, the associated pressure of the fluid, the potential of the gravitational field, and the strength of nonlinear fluid convection. $S(x,n,c)$ is a chemotactic sensitivity tensor satisfying

$$S \in C^2(\bar{\Omega} \times [0,\infty)^2; \mathbb{R}^{3\times 3}) \tag{1.4}$$

and

$$|S(x,n,c)| \leq C_S(1+n)^{-\alpha} \quad \text{for all} \quad (x,n,c) \in \Omega \times [0,\infty)^2 \tag{1.5}$$

with some $C_S > 0$ and $\alpha > 0$. Problem (1.3) is proposed to describe the chemotaxis-fluid interaction in cases when the evolution of the chemoattractant is essentially dominated by production through cells (see Winkler et al. [1] and Hillen and Painter [8]).

Before delving into our mathematical analysis, we recall some important progress on system (1.3) and its variants. The following chemotaxis-fluid model,which is closely related to the variation of (1.3), was proposed by Tuval et al. [24]:

$$
\begin{cases}
n_t + u\cdot\nabla n = \Delta n - \nabla\cdot(nS(x,n,c)\nabla c), & x\in\Omega,\quad t>0,\\
c_t + u\cdot\nabla c = \Delta c - nf(c), & x\in\Omega,\quad t>0,\\
u_t + \kappa(u\cdot\nabla)u + \nabla P = \Delta u + n\nabla\phi, & x\in\Omega,\quad t>0,\\
\nabla\cdot u = 0, & x\in\Omega,\quad t>0,
\end{cases}
\tag{1.6}
$$

where $f(c)$ is the consumption rate of oxygen by the cells. In the past few years, by making use of the energy-type functionals, system (1.6) and its variants have attracted extensive attention (see, e.g., Chae et al. [3], Duan et al. [6], Liu and Lorz [13,15], Tao and Winkler [23,33,34,37], Zhang and Zheng [43] and references therein). For example, Winkler [37] established the global existence of a weak solution in a three-dimensional domain when $S(x,n,c)\equiv 1$ and $\kappa\neq 0$. Recently, if $S(x,n,c):=S(c)$, the long-term behavior of eventual smoothness of the weak solution was investigated by Winkler [38], in which the weak solution became smooth on some interval $[T,\infty)$ and uniformly converged in the large-time limit. For more literature related to this model, we can refer to Tao and Winkler [21,22,39] and the references therein. For example, Winkler [39] proved that the chemotaxis–Stokes system (with nonlinear diffusion) admits a global bounded weak solution under the assumption $m>\frac{9}{8}$. Furthermore, he also showed that the obtained solution approached the spatially homogeneous steady state in the large-time limit.

If the chemotactic sensitivity $S(x,n,c)$ is regarded as a tensor rather than a scalar one (see Xue and Othmer [42]), (1.6) turns into a chemotaxis(–Navier)–Stokes system with rotational flux. Owing to the presence of the tensor-valued sensitivity, the corresponding chemotaxis–Stokes system loses some energy structure, which has played a key role in previous studies for the scalar sensitivity case (see Cao [2] and Winkler [36]). Therefore, very few results appear to be available on chemotaxis-Stokes systems with such tensor-valued sensitivities (see, e.g., Ishida [11], Wang et al. [26,28] and Winkler [36]). In fact, assuming that $f(c)=c$ and that (1.4) and (1.5) hold, Ishida [11] proved that (1.6) admits a bounded global weak solution in two dimensions with nonlinear diffusion, whereas, in three dimensions, Winkler [36] showed that the chemotaxis-Stokes system [$\kappa=0$ in the first equation of (1.6)] with nonlinear diffusion (where the coefficient of diffusion satisfies $m>\frac{7}{6}$) possesses at least one bounded weak solution that stabilizes to a spatially homogeneous equilibrium $(\frac{1}{|\Omega|}\int_\Omega n_0, 0, 0)$.

In contrast to the large number of existing results ofor (1.6), the mathematical analysis of (1.3) with regard to global and bounded solutions is far from trivial. On the one hand, as its subsystem, the Navier–Stokes system lacks a complete existence theory (see Wiegner [30]). On the other hand, the previously mentioned properties for the Keller–Segel system can still emerge (see Wang et al. [17,25,27–29] and Zheng [48,49]). In fact, in two dimensions, if $S=S(x,n,c)$ is a tensor-valued sensitivity fulfilling (1.4) and (1.5), Wang and Xiang [28] proved that the Stokes version [$\kappa=0$ in the first equation of (1.3)] of system (1.3) admits a unique global classical solution that is bounded. Recently, Wang et al. [27] extended the above result [28] to the Navier–Stokes version ($\kappa\neq 0$ in the first equation of (1.3)). In both papers [27] and [28], the condition $\alpha>0$, corresponding to the condition (1.2) with $n=2$, is optimal for the existence of the solution. Furthermore, similar results are also valid for the three-dimensional Stokes version [$\kappa=0$ in the first equation of (1.3)] of system (1.3) with $\alpha>\frac{1}{2}$ (see Wang and Xiang [29]). In the three dimensional case, Wang and Liu [14] showed that the Keller–Segel–Navier–Stokes [$\kappa\neq 0$ in the first equation of (1.3)) system (1.3] admits a global weak solution for tensor-valued sensitivity $S(x,n,c)$ satisfying (1.4) and (1.5) with $\alpha>\frac{3}{7}$. Recently, because of the lack of enough regularity and compactness properties for the first equation, by using the idea proposed by Winkler [35], Wang [25] presented the existence of global very weak solutions for the system (1.3) under the assumption that S satisfies (1.4) and (1.5) with $\alpha > \frac{1}{3}$, which, in light of the known results for the fluid-free system mentioned above, is an optimal restriction on α [see (1.2)]. However, the existence of global (stronger than the result of [25]) weak solutions is still open. In this paper, we try to obtain enough regularity and compactness properties (see Lemmas 3.4, 5.1, and 5.2), then show that system (1.3) possesses a globally defined weak solution (see Definition 2.1), which improves the result of [25].

Throughout this paper, we assume that

$$\phi \in W^{2,\infty}(\Omega) \tag{1.7}$$

and that the initial data (n_0, c_0, u_0) fulfill

$$\begin{cases} n_0 \in C^\kappa(\bar{\Omega}) \text{ for certain } \kappa > 0 \text{ with } n_0 \geq 0 \text{ in } \Omega, \\ c_0 \in W^{1,\infty}(\Omega) \text{ with } c_0 \geq 0 \text{ in } \bar{\Omega}, \\ u_0 \in D(A_r^\gamma) \text{ for some } \gamma \in (3/4, 1) \text{ and any } r \in (1, \infty), \end{cases} \tag{1.8}$$

where A_r denotes the Stokes operator with domain $D(A_r) := W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega) \cap L_\sigma^r(\Omega)$ and $L_\sigma^r(\Omega) := \{\varphi \in L^r(\Omega) | \nabla \cdot \varphi = 0\}$ for $r \in (1, \infty)$ (similar to that in [19]).

Our main result assert the existence of the global weak solution for system (1.3).

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary.* (1.7) *and* (1.8) *hold, and suppose that S satisfies* (1.4) *and* (1.5) *with some*

$$\alpha > \frac{1}{3}.$$

Then problem (1.3) *possesses at least one global weak solution (n, c, u, P) in the sense of Definition* 2.1.

Remark 1.1 (i) From Theorem 1.1, we conclude that $\alpha > \frac{1}{3}$ is sufficient to guarantee the existence of global (weak) solutions. Compared with the results (1.2), we know such a restriction on α seems to be optimal.

(ii) Obviously, $\frac{3}{7} > \frac{1}{3}$, so Theorem 1.1 improves the results of Liu and Wang [14], which showed the global weak existence of solutions in cases $S(x, n, c)$ satisfying (1.4) and (1.5) with $\alpha > \frac{3}{7}$.

(iii) If $S := S(n) = C_S(1 + n)^{-\alpha}$ is a scalar function which satisfies that $\alpha > \frac{1}{3}$, the boundedness of solution to Keller–Segel–Stokes [$\kappa = 0$ in the first equation of (1.3)] system (1.3) is obtained by Winkler (see [40]). Recalling the condition (1.2) for global existence in the fluid-free setting, as implied by the previously mentioned studied (see Horstmann and Winkler [10]), this result appears to be optimal with respect to α.

This paper is organized as followed. In Sect. 2, we give the definition of weak solutions to (1.3), the regularized problems of (1.3), and some preliminary properties. Sections 3 and 4 will be devoted to an analysis of regularized problems of (1.3). Next, on the basis of the compactness properties thereby implied, in Sects. 5 and 6, we can pass to the limit along with an adequate sequence of numbers $\varepsilon = \varepsilon_j \searrow 0$ and thereby verify Theorem 1.1.

2 Preliminaries

In light of the strong nonlinear term $(u \cdot \nabla)u$, problem (1.3) has no classical solutions in general, thus we consider its weak solutions.

Definition 2.1 Let $T > 0$ and assume that (n_0, c_0, u_0) fulfills (1.8). Then a triple of functions (n, c, u) is called a weak solution of (1.3) if the following conditions are satisfied:

$$\begin{cases} n \in L^1_{loc}(\bar{\Omega} \times [0, T)), \\ c \in L^1_{loc}([0, T); W^{1,1}(\Omega)), \\ u \in L^1_{loc}([0, T); W^{1,1}(\Omega); \mathbb{R}^3), \end{cases} \tag{2.1}$$

where $n \geq 0$ and $c \geq 0$ in $\Omega \times (0, T)$ as well as $\nabla \cdot u = 0$ in the distributional sense in $\Omega \times (0, T)$. Moreover,

$$\begin{aligned} &u \otimes u \in L^1_{loc}(\bar{\Omega} \times [0, \infty); \mathbb{R}^{3\times 3}) \text{ and } n \text{ belongs to } L^1_{loc}(\bar{\Omega} \times [0, \infty)), \\ &cu,\ nu, \text{ and } nS(x, n, c)\nabla c \text{ belong to } L^1_{loc}(\bar{\Omega} \times [0, \infty); \mathbb{R}^3) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} &-\int_0^T \int_\Omega n\varphi_t - \int_\Omega n_0 \varphi(\cdot, 0) \\ &= -\int_0^T \int_\Omega \nabla n \cdot \nabla \varphi + \int_0^T \int_\Omega nS(x, n, c)\nabla c \cdot \nabla \varphi + \int_0^T \int_\Omega nu \cdot \nabla \varphi \end{aligned} \tag{2.3}$$

for any $\varphi \in C_0^\infty(\bar{\Omega} \times [0, T))$ satisfying $\frac{\partial \varphi}{\partial \nu} = 0$ on $\partial\Omega \times (0, T)$, as well as

$$\begin{aligned} &-\int_0^T \int_\Omega c\varphi_t - \int_\Omega c_0 \varphi(\cdot, 0) \\ &= -\int_0^T \int_\Omega \nabla c \cdot \nabla \varphi - \int_0^T \int_\Omega c\varphi + \int_0^T \int_\Omega n\varphi + \int_0^T \int_\Omega cu \cdot \nabla \varphi \end{aligned} \tag{2.4}$$

for any $\varphi \in C_0^\infty(\bar{\Omega} \times [0, T))$ and

$$\begin{aligned} &-\int_0^T \int_\Omega u\varphi_t - \int_\Omega u_0 \varphi(\cdot, 0) - \kappa \int_0^T \int_\Omega u \otimes u \cdot \nabla \varphi \\ &= -\int_0^T \int_\Omega \nabla u \cdot \nabla \varphi - \int_0^T \int_\Omega n \nabla \phi \cdot \varphi \end{aligned} \tag{2.5}$$

for any $\varphi \in C_0^\infty(\bar{\Omega} \times [0, T); \mathbb{R}^3)$ fulfilling $\nabla \varphi \equiv 0$ in $\Omega \times (0, T)$.

If $(n, c, u) : \Omega \times (0, \infty) \longrightarrow \mathbb{R}^5$ is a weak solution of (1.3) in $\Omega \times (0, T)$ for all $T > 0$, then (n, c, u) is called a global weak solution of (1.3).

To obtain the solution of system (1.3), we first consider the following approximate system of (1.3):

$$\begin{cases} n_{\varepsilon t} + u_\varepsilon \cdot \nabla n_\varepsilon = \Delta n_\varepsilon - \nabla \cdot (n_\varepsilon F'_\varepsilon(n_\varepsilon) S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \nabla c_\varepsilon), & x \in \Omega, \quad t > 0, \\ c_{\varepsilon t} + u_\varepsilon \cdot \nabla c_\varepsilon = \Delta c_\varepsilon - c_\varepsilon + F_\varepsilon(n_\varepsilon), & x \in \Omega, \quad t > 0, \\ u_{\varepsilon t} + \nabla P_\varepsilon = \Delta u_\varepsilon - \kappa (Y_\varepsilon u_\varepsilon \cdot \nabla) u_\varepsilon + n_\varepsilon \nabla \phi, & x \in \Omega, \quad t > 0, \\ \nabla \cdot u_\varepsilon = 0, & x \in \Omega, \quad t > 0, \\ \nabla n_\varepsilon \cdot \nu = \nabla c_\varepsilon \cdot \nu = 0, u_\varepsilon = 0, & x \in \partial\Omega, \quad t > 0, \\ n_\varepsilon(x, 0) = n_0(x), c_\varepsilon(x, 0) = c_0(x),\ u_\varepsilon(x, 0) = u_0(x), & x \in \Omega, \end{cases} \tag{2.6}$$

where

$$F_\varepsilon(s) := \frac{1}{\varepsilon} \ln(1 + \varepsilon s) \quad \text{for all } s \geq 0 \text{ and } \varepsilon > 0, \tag{2.7}$$

as well as

$$S_\varepsilon(x,n,c) := \rho_\varepsilon(x)S(x,n,c), \quad x\in\bar{\Omega}, \quad n\geq 0, \quad c\geq 0 \tag{2.8}$$

and

$$Y_\varepsilon w := (1+\varepsilon A)^{-1}w \quad \text{for all} \ \ w\in L^2_\sigma(\Omega)$$

is a standard Yosida approximation and A is the realization of the Stokes operator (see [19]). Here, $(\rho_\varepsilon)_{\varepsilon\in(0,1)}\in C_0^\infty(\Omega)$ is a family of standard cutoff functions satisfying $0\leq\rho_\varepsilon\leq 1$ in Ω and $\rho_\varepsilon\nearrow 1$ in Ω as $\varepsilon\searrow 0$.

The local solvability of (2.6) can be derived by a suitable extensibility criterion and a slight modification of the well-established fixed-point arguments in Lemma 2.1 of [37] (see also [36] and Lemma 2.1 of [16]), so here we omit the proof.

Lemma 2.1 *Assume that* $\varepsilon\in(0,1)$. *Then there exist* $T_{max,\varepsilon}\in(0,\infty]$ *and a classical solution* $(n_\varepsilon,c_\varepsilon,u_\varepsilon,P_\varepsilon)$ *of* (2.6) *in* $\Omega\times(0,T_{max,\varepsilon})$ *such that*

$$\begin{cases} n_\varepsilon\in C^0(\bar{\Omega}\times[0,T_{max,\varepsilon}))\cap C^{2,1}(\bar{\Omega}\times(0,T_{max,\varepsilon})),\\ c_\varepsilon\in C^0(\bar{\Omega}\times[0,T_{max,\varepsilon}))\cap C^{2,1}(\bar{\Omega}\times(0,T_{max,\varepsilon})),\\ u_\varepsilon\in C^0(\bar{\Omega}\times[0,T_{max,\varepsilon});\mathbb{R}^3)\cap C^{2,1}(\bar{\Omega}\times(0,T_{max,\varepsilon});\mathbb{R}^3),\\ P_\varepsilon\in C^{1,0}(\bar{\Omega}\times(0,T_{max,\varepsilon})),\end{cases}$$

classically solving (2.6) *in* $\Omega\times[0,T_{max,\varepsilon})$. *Moreover,* n_ε *and* c_ε *are nonnegative in* $\Omega\times(0,T_{max,\varepsilon})$, *and*

$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}+\|c_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)}+\|A^\gamma u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}\to\infty \ \ as \ \ t\to T_{max,\varepsilon},$$

where γ *is given by* (1.8).

Lemma 2.2 [32,47] *Let* $(e^{\tau\Delta})_{\tau\geq 0}$ *be the Neumann heat semigroup in* Ω *and* $p>3$. *Then there exist positive constants* $c_1:=c_1(\Omega)$, $c_2:=c_2(\Omega)$, *and* $c_3:=c_3(\Omega)$ *such that for all* $\tau>0$ *and any* $\varphi\in W^{1,p}(\Omega)$,

$$\|\nabla e^{\tau\Delta}\varphi\|_{L^p(\Omega)}\leq c_1(\Omega)\|\nabla\varphi\|_{L^p(\Omega)},$$

and for all $\tau>0$ *and each* $\varphi\in L^\infty(\Omega)$

$$\|\nabla e^{\tau\Delta}\varphi\|_{L^p(\Omega)}\leq c_2(1+\tau^{-\frac{1}{2}})\|\varphi\|_{L^\infty(\Omega)},$$

as well as for all $\tau>0$ *and all* $\varphi\in C^1(\bar{\Omega};\mathbb{R}^3)$ *fulfilling* $\varphi\cdot\nu=0$ *on* $\partial\Omega$

$$\|e^{\tau\Delta}\nabla\cdot\varphi\|_{L^\infty(\Omega)}\leq c_3(1+\tau^{-\frac{1}{2}-\frac{3}{2p}})\|\varphi\|_{L^p(\Omega)}.$$

3 Some a priori estimates for the regularized problem (2.6) that is independent of ε

In this section, we are going to establish an iteration step to develop the main ingredients of our result. The iteration depends on a series of a priori estimates. To proceed, first, we recall some properties of F_ε and F'_ε, which play an important role in demonstrating Theorem 1.1.

Lemma 3.1 *Assume* F_ε *is given by* (2.7). *Then*

$$0\leq F'_\varepsilon(s)=\frac{1}{1+\varepsilon s}\leq 1 \ \ for\ all \ \ s\geq 0 \ \ and \ \ \varepsilon>0 \tag{3.1}$$

as well as

$$\lim_{\varepsilon\to 0^+} F_\varepsilon(s) = s, \quad \lim_{\varepsilon\to 0^+} F'_\varepsilon(s) = 1 \ \ \textit{for all} \ \ s \geq 0 \tag{3.2}$$

and

$$0 \leq F_\varepsilon(s) \leq s \ \ \textit{for all} \ \ s \geq 0. \tag{3.3}$$

Proof Recalling (2.7), by tedious and simple calculations, we can derive (3.1)–(3.3). □

The proof of this lemma is very similar to that of Lemmas 2.2 and 2.6 of [23] (see also Lemma 3.2 of [25]), so we omit it here.

Lemma 3.2 *There exists $\lambda > 0$ independent of ε such that the solution of* (2.6) *satisfies*

$$\int_\Omega n_\varepsilon + \int_\Omega c_\varepsilon \leq \lambda \ \ \textit{for all} \ \ t \in (0, T_{max,\varepsilon}). \tag{3.4}$$

Lemma 3.3 *Let $\alpha > \frac{1}{3}$. Then there exists $C > 0$ independent of ε such that the solution of* (2.6) *satisfies*

$$\int_\Omega n_\varepsilon^{2\alpha} + \int_\Omega c_\varepsilon^2 + \int_\Omega |u_\varepsilon|^2 \leq C \ \ \textit{for all} \ \ t \in (0, T_{max,\varepsilon}). \tag{3.5}$$

Moreover, for $T \in (0, T_{max,\varepsilon})$, one can find a constant $C > 0$ independent of ε such that

$$\int_0^T \int_\Omega \left[n_\varepsilon^{2\alpha-2} |\nabla n_\varepsilon|^2 + |\nabla c_\varepsilon|^2 + |\nabla u_\varepsilon|^2 \right] \leq C. \tag{3.6}$$

Proof The proof consists of two cases.

Case $2\alpha \neq 1$: We first obtain from $\nabla \cdot u_\varepsilon = 0$ in $\Omega \times (0, T_{max,\varepsilon})$ and straightforward calculations that

$$\begin{aligned}
&\mathrm{sign}(2\alpha-1)\frac{1}{2\alpha}\frac{d}{dt}\|n_\varepsilon\|^{2\alpha}_{L^{2\alpha}(\Omega)} + \mathrm{sign}(2\alpha-1)(2\alpha-1)\int_\Omega n_\varepsilon^{2\alpha-2}|\nabla n_\varepsilon|^2 \\
&= -\int_\Omega \mathrm{sign}(2\alpha-1) n_\varepsilon^{2\alpha-1} \nabla \cdot (n_\varepsilon F'_\varepsilon(n_\varepsilon) S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \cdot \nabla c_\varepsilon) \\
&\leq \mathrm{sign}(2\alpha-1)(2\alpha-1)\int_\Omega n_\varepsilon^{2\alpha-2} n_\varepsilon F'_\varepsilon(n_\varepsilon)|S_\varepsilon(x, n_\varepsilon, c_\varepsilon)||\nabla n_\varepsilon||\nabla c_\varepsilon|
\end{aligned} \tag{3.7}$$

for all $t \in (0, T_{max,\varepsilon})$. Therefore, from (3.1), in light of (1.5) and (2.7), we can estimate the right-hand side of (3.7) as follows:

$$\begin{aligned}
&\mathrm{sign}(2\alpha-1)(2\alpha-1)\int_\Omega n_\varepsilon^{2\alpha-2} n_\varepsilon F'_\varepsilon(n_\varepsilon)|S_\varepsilon(x, n_\varepsilon, c_\varepsilon)||\nabla n_\varepsilon||\nabla c_\varepsilon| \\
&\leq \mathrm{sign}(2\alpha-1)(2\alpha-1)\int_\Omega n_\varepsilon^{2\alpha-2} n_\varepsilon C_S (1+n_\varepsilon)^{-\alpha}|\nabla n_\varepsilon||\nabla c_\varepsilon| \\
&\leq \mathrm{sign}(2\alpha-1)\frac{2\alpha-1}{2}\int_\Omega n_\varepsilon^{2\alpha-2}|\nabla n_\varepsilon|^2 \\
&\quad + \frac{|2\alpha-1|}{2} C_S^2 \int_\Omega n_\varepsilon^{2\alpha-2} n_\varepsilon^2 (1+n_\varepsilon)^{-2\alpha}|\nabla c_\varepsilon|^2 \\
&\leq \mathrm{sign}(2\alpha-1)\frac{2\alpha-1}{2}\int_\Omega n_\varepsilon^{2\alpha-2}|\nabla n_\varepsilon|^2 \\
&\quad + \frac{|2\alpha-1|}{2} C_S^2 \int_\Omega |\nabla c_\varepsilon|^2 \text{ for all } t \in (0, T_{max,\varepsilon})
\end{aligned} \tag{3.8}$$

by using Young's inequality, where in the last inequality we have used the fact that $n_\varepsilon^{2\alpha-2}n_\varepsilon^2(1+n_\varepsilon)^{-2\alpha} \le 1$ for all $\varepsilon \ge 0$ and $(x,t) \in \Omega \times (0, T_{max,\varepsilon})$. Inserting (3.8) into (3.7), we conclude that

$$\begin{aligned}&\operatorname{sign}(2\alpha-1)\frac{1}{2\alpha}\frac{d}{dt}\|n_\varepsilon\|^{2\alpha}_{L^{2\alpha}(\Omega)} + \operatorname{sign}(2\alpha-1)\frac{2\alpha-1}{2}\int_\Omega n_\varepsilon^{2\alpha-2}|\nabla n_\varepsilon|^2\\ &\le \frac{|2\alpha-1|}{2}C_S^2\int_\Omega |\nabla c_\varepsilon|^2 \quad \text{for all} \quad t \in (0, T_{max,\varepsilon}).\end{aligned} \tag{3.9}$$

To track the time evolution of c_ε, taking c_ε as the test function for the second equation of (2.6) and using $\nabla \cdot u_\varepsilon = 0$ and (3.3) together with Hölder's inequality yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|c_\varepsilon\|^2_{L^2(\Omega)} + \int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega |c_\varepsilon|^2 = \int_\Omega F_\varepsilon(n_\varepsilon)c_\varepsilon\\ &\le \int_\Omega n_\varepsilon c_\varepsilon \le \|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)}\|c_\varepsilon\|_{L^6(\Omega)} \text{ for all } t \in (0, T_{max,\varepsilon}).\end{aligned} \tag{3.10}$$

By applying Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ in the three-dimensional setting, in view of (3.4), there exist positive constants C_1 and C_2 such that

$$\begin{aligned}\|c_\varepsilon\|^2_{L^6(\Omega)} &\le C_1\|\nabla c_\varepsilon\|^2_{L^2(\Omega)} + C_1\|c_\varepsilon\|^2_{L^1(\Omega)}\\ &\le C_1\|\nabla c_\varepsilon\|^2_{L^2(\Omega)} + C_2 \text{ for all } t \in (0, T_{max,\varepsilon}).\end{aligned} \tag{3.11}$$

Thus, by means of Young's inequality and (3.11), we proceed to estimate

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|c_\varepsilon\|^2_{L^2(\Omega)} + \int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega |c_\varepsilon|^2 \le \frac{1}{2C_1}\|c_\varepsilon\|^2_{L^6(\Omega)} + \frac{C_1}{2}\|n_\varepsilon\|^2_{L^{\frac{6}{5}}(\Omega)}\\ &\le \frac{1}{2}\|\nabla c_\varepsilon\|^2_{L^2(\Omega)} + \frac{C_1}{2}\|n_\varepsilon\|^2_{L^{\frac{6}{5}}(\Omega)} + C_3 \text{ for all } t \in (0, T_{max,\varepsilon})\end{aligned} \tag{3.12}$$

and some positive constant C_3 independent of ε. Therefore,

$$\frac{1}{2}\frac{d}{dt}\|c_\varepsilon\|^2_{L^2(\Omega)} + \frac{1}{2}\int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega |c_\varepsilon|^2 \le \frac{C_1}{2}\|n_\varepsilon\|^2_{L^{\frac{6}{5}}(\Omega)} + C_3 \quad \text{for all} \quad t \in (0, T_{max,\varepsilon}). \tag{3.13}$$

To estimate $\|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)}$ for all $t \in (0, T_{max,\varepsilon})$, we should notice that $\alpha > \frac{1}{3}$ ensures that $\frac{2}{6\alpha-1} < 2$, so that, in light of (3.4), the Gagliardo–Nirenberg inequality and Young's inequality allow us to estimate that

$$\begin{aligned}\|n_\varepsilon\|^2_{L^{\frac{6}{5}}(\Omega)} &= \|n_\varepsilon^\alpha\|^{\frac{2}{\alpha}}_{L^{\frac{6}{5\alpha}}(\Omega)} \le C_4\left(\|\nabla n_\varepsilon^\alpha\|^{\frac{2}{6\alpha-1}}_{L^2(\Omega)}\|n_\varepsilon^\alpha\|^{\frac{2}{\alpha}-\frac{2}{6\alpha-1}}_{L^{\frac{1}{\alpha}}(\Omega)} + \|n_\varepsilon^\alpha\|^{\frac{2}{\alpha}}_{L^{\frac{1}{\alpha}}(\Omega)}\right)\\ &\le \frac{1}{4}\frac{1}{C_1\alpha^2C_S^2}\|\nabla n_\varepsilon^\alpha\|^2_{L^2(\Omega)} + C_5 \text{ for all } t \in (0, T_{max,\varepsilon})\end{aligned} \tag{3.14}$$

with some positive constants C_4 and C_5 independent of ε. This together with (3.13) contributes to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|c_\varepsilon\|^2_{L^2(\Omega)} + \frac{1}{2}\int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega |c_\varepsilon|^2\\ &\le \frac{1}{8}\frac{1}{\alpha^2C_S^2}\|\nabla n_\varepsilon^\alpha\|^2_{L^2(\Omega)} + C_6 \quad \text{for all} \quad t \in (0, T_{max,\varepsilon})\end{aligned} \tag{3.15}$$

and some positive constant C_6. Taking an evident linear combination of the inequalities provided by (3.9) and (3.15), one can obtain

$$\begin{aligned} &\operatorname{sign}(2\alpha-1)\frac{1}{2\alpha}\frac{d}{dt}\|n_\varepsilon\|^{2\alpha}_{L^{2\alpha}(\Omega)}+|2\alpha-1|C_S^2\frac{d}{dt}\|c_\varepsilon\|^2_{L^2(\Omega)}\\ &\quad+\frac{|2\alpha-1|}{2}C_S^2\int_\Omega|\nabla c_\varepsilon|^2+2|2\alpha-1|C_S^2\int_\Omega|c_\varepsilon|^2\\ &\quad+\left(\operatorname{sign}(2\alpha-1)\frac{2\alpha-1}{2}-\frac{1}{4}|2\alpha-1|\right)\int_\Omega n_\varepsilon^{2\alpha-2}|\nabla n_\varepsilon|^2\\ &\le C_7 \text{ for all } t\in(0,T_{max,\varepsilon}) \end{aligned} \tag{3.16}$$

and some positive constant C_7. Since $\operatorname{sign}(2\alpha-1)\dfrac{2\alpha-1}{2}=\dfrac{|2\alpha-1|}{2}$, (3.16) implies that

$$\begin{aligned} &\operatorname{sign}(2\alpha-1)\frac{1}{2\alpha}\frac{d}{dt}\|n_\varepsilon\|^{2\alpha}_{L^{2\alpha}(\Omega)}+|2\alpha-1|C_S^2\frac{d}{dt}\|c_\varepsilon\|^2_{L^2(\Omega)}\\ &\quad+\frac{|2\alpha-1|}{2}C_S^2\int_\Omega|\nabla c_\varepsilon|^2+2|2\alpha-1|C_S^2\int_\Omega|c_\varepsilon|^2\\ &\quad+\frac{|2\alpha-1|}{4}\int_\Omega n_\varepsilon^{2\alpha-2}|\nabla n_\varepsilon|^2\\ &\le C_7 \text{ for all } t\in(0,T_{max,\varepsilon}). \end{aligned} \tag{3.17}$$

If $2\alpha>1$, then $\operatorname{sign}(2\alpha-1)=1>0$, thus, integrating (3.17) over time, we can obtain

$$\int_\Omega n_\varepsilon^{2\alpha}+\int_\Omega c_\varepsilon^2\le C_8 \quad \text{for all} \quad t\in(0,T_{max,\varepsilon}) \tag{3.18}$$

and

$$\int_0^T\int_\Omega\left[n_\varepsilon^{2\alpha-2}|\nabla n_\varepsilon|^2+|\nabla c_\varepsilon|^2\right]\le C_8(T+1) \quad \text{for all} \quad T\in(0,T_{max,\varepsilon}) \tag{3.19}$$

and some positive constant C_8. If $2\alpha<1$, then $\operatorname{sign}(2\alpha-1)=-1<0$; hence, in view of (3.4), integrating (3.17) over time and employing Hölder's inequality, we also conclude that there exists a positive constant C_9 such that

$$\int_\Omega n_\varepsilon^{2\alpha}+\int_\Omega c_\varepsilon^2\le C_9 \quad \text{for all} \quad t\in(0,T_{max,\varepsilon}) \tag{3.20}$$

and

$$\int_0^T\int_\Omega\left[n_\varepsilon^{2\alpha-2}|\nabla n_\varepsilon|^2+|\nabla c_\varepsilon|^2\right]\le C_9(T+1) \quad \text{for all} \quad T\in(0,T_{max,\varepsilon}). \tag{3.21}$$

Case $2\alpha=1$: Using the first equation of (2.6) and (2.7), integrating by parts, and applying (1.5) and (3.1), we obtain

$$\begin{aligned} \tfrac{d}{dt}\textstyle\int_\Omega n_\varepsilon\ln n_\varepsilon &=\int_\Omega n_{\varepsilon t}\ln n_\varepsilon+\int_\Omega n_{\varepsilon t}\\ &=\int_\Omega\Delta n_\varepsilon\ln n_\varepsilon-\int_\Omega\ln n_\varepsilon\nabla\cdot(n_\varepsilon F'_\varepsilon(n_\varepsilon)S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon)\\ &\le-\textstyle\int_\Omega\frac{|\nabla n_\varepsilon|^2}{n_\varepsilon}+\int_\Omega C_S(1+n_\varepsilon)^{-\alpha}\frac{n_\varepsilon}{n_\varepsilon}|\nabla n_\varepsilon||\nabla c_\varepsilon| \quad \text{for all} \quad t\in(0,T_{max,\varepsilon}), \end{aligned}$$

which combined with Young's inequality and $2\alpha=1$ implies that

$$\frac{d}{dt}\int_\Omega n_\varepsilon\ln n_\varepsilon+\frac{1}{2}\int_\Omega\frac{|\nabla n_\varepsilon|^2}{n_\varepsilon}\le\frac{1}{2}C_S^2\int_\Omega|\nabla c_\varepsilon|^2 \quad \text{for all} \quad t\in(0,T_{max,\varepsilon}).$$

However, since $2\alpha = 1$ yields $\alpha > \frac{1}{3}$, by employing almost exactly the same arguments as in the proof of (3.10)–(3.16) (with the minor necessary changes being left as an easy exercise to the reader), we conclude an estimate of

$$\int_\Omega n_\varepsilon \ln n_\varepsilon + \int_\Omega c_\varepsilon^2 \leq C_{10} \quad \text{for all} \quad t \in (0, T_{max,\varepsilon}) \tag{3.22}$$

and

$$\int_0^T \int_\Omega \left[\frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + |\nabla c_\varepsilon|^2 \right] \leq C_{10}(T+1) \quad \text{for all} \quad T \in (0, T_{max,\varepsilon}). \tag{3.23}$$

Now, multiplying the third equation of (2.6) by u_ε, integrating by parts, and using $\nabla \cdot u_\varepsilon = 0$ give

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 = \int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla \phi \quad \text{for all} \quad t \in (0, T_{max,\varepsilon}). \tag{3.24}$$

Here we use Hölder's inequality, Young's inequality, (1.7), and the continuity of the embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ to find C_{11} and $C_{12} > 0$ such that

$$\begin{aligned} \int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla \phi &\leq \|\nabla \phi\|_{L^\infty(\Omega)} \|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)} \|u_\varepsilon\|_{L^6(\Omega)} \\ &\leq C_{11} \|\nabla \phi\|_{L^\infty(\Omega)} \|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)} \|\nabla u_\varepsilon\|_{L^2(\Omega)} \\ &\leq \frac{1}{2}\|\nabla u_\varepsilon\|^2_{L^2(\Omega)} + C_{12}\|n_\varepsilon\|^2_{L^{\frac{6}{5}}(\Omega)} \quad \text{for all} \quad t \in (0, T_{max,\varepsilon}). \end{aligned} \tag{3.25}$$

Next, in view of (3.4) and $\alpha > \frac{1}{3}$, (3.14) and Young's inequality along with the Gagliardo–Nirenberg inequality yields

$$\begin{aligned} \int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla \phi &\leq \frac{1}{2}\|\nabla u_\varepsilon\|^2_{L^2(\Omega)} + C_8 \|\nabla n_\varepsilon^\alpha\|^{\frac{2}{6\alpha-1}}_{L^2(\Omega)} \|n_\varepsilon^\alpha\|^{\frac{2}{\alpha}-\frac{2}{6\alpha-1}}_{L^{\frac{1}{\alpha}}(\Omega)} \\ &\leq \frac{1}{2}\|\nabla u_\varepsilon\|^2_{L^2(\Omega)} + \|\nabla n_\varepsilon^\alpha\|^2_{L^2(\Omega)} + C_{13} \quad \text{for all} \quad t \in (0, T_{max,\varepsilon}) \end{aligned} \tag{3.26}$$

and some positive constant C_{13}. Now, inserting (3.25) and (3.26) into (3.24) and using (3.21) and (3.23), one has

$$\int_\Omega |u_\varepsilon|^2 \leq C_{14} \quad \text{for all} \quad t \in (0, T_{max,\varepsilon}) \tag{3.27}$$

and

$$\int_0^T \int_\Omega |\nabla u_\varepsilon|^2 \leq C_{14}(T+1) \quad \text{for all} \quad T \in (0, T_{max,\varepsilon}) \tag{3.28}$$

and some positive constant C_{14}. Finally, collecting (3.20)–(3.21), (3.22)–(3.23), and (3.27)–(3.28), we can get (3.5) and (3.6). □

With the help of Lemma 3.3, based on the Gagliardo–Nirenberg inequality and an application of well-known arguments from parabolic regularity theory, we can derive the following lemmas:

Lemma 3.4 *Let $\alpha > \frac{1}{3}$. Then there exists $C > 0$ independent of ε such that, for each $T \in (0, T_{max,\varepsilon})$, the solution of* (2.6) *satisfies*

$$\int_0^T \int_\Omega \left[|\nabla n_\varepsilon|^{\frac{3\alpha+1}{2}} + n_\varepsilon^{\frac{6\alpha+2}{3}} \right] \le C(T+1) \quad \text{if } \frac{1}{3} < \alpha \le \frac{1}{2}, \tag{3.29}$$

$$\int_0^T \int_\Omega \left[|\nabla n_\varepsilon|^{\frac{10\alpha}{3+2\alpha}} + n_\varepsilon^{\frac{10\alpha}{3}} \right] \le C(T+1) \quad \text{if } \frac{1}{2} < \alpha < 1, \tag{3.30}$$

as well as

$$\int_0^T \int_\Omega \left[|\nabla n_\varepsilon|^2 + n_\varepsilon^{\frac{10}{3}} \right] \le C(T+1) \quad \text{if } \alpha \ge 1 \tag{3.31}$$

and

$$\int_0^T \left\{ \int_\Omega [c_\varepsilon^{\frac{10}{3}} + |u_\varepsilon|^{\frac{10}{3}}] + \|u_\varepsilon\|^2_{L^6(\Omega)} \right\} \le C(T+1). \tag{3.32}$$

Proof Case $\frac{1}{3} < \alpha \le \frac{1}{2}$: From (3.4), (3.5), and (3.6), in light of the Gagliardo–Nirenberg inequality, for some C_1 and $C_2 > 0$ that are independent of ε, one may verify that

$$\begin{aligned} \int_0^T \int_\Omega n_\varepsilon^{\frac{6\alpha+2}{3}} &= \int_0^T \|n_\varepsilon^\alpha\|^{\frac{6\alpha+2}{3\alpha}}_{L^{\frac{6\alpha+2}{3\alpha}}(\Omega)} \\ &\le C_1 \int_0^T \left(\|\nabla n_\varepsilon^\alpha\|^2_{L^2(\Omega)} \|n_\varepsilon^\alpha\|^{\frac{2}{3\alpha}}_{L^{\frac{1}{\alpha}}(\Omega)} + \|n_\varepsilon^\alpha\|^{\frac{6\alpha+2}{3\alpha}}_{L^{\frac{1}{\alpha}}(\Omega)} \right) \\ &\le C_2(T+1) \quad \text{for all } T > 0. \end{aligned} \tag{3.33}$$

Therefore, employing Hölder's inequality (with two exponents $\frac{4}{3\alpha+1}$ and $\frac{4}{3-3\alpha}$), we conclude that there exists a positive constant C_3 such that

$$\begin{aligned} \int_0^T \int_\Omega |\nabla n_\varepsilon|^{\frac{3\alpha+1}{2}} &\le \left[\int_0^T \int_\Omega n_\varepsilon^{2\alpha-2} |\nabla n_\varepsilon|^2 \right]^{\frac{3\alpha+1}{4}} \left[\int_0^T \int_\Omega n_\varepsilon^{\frac{6\alpha+2}{3}} \right]^{\frac{3-3\alpha}{4}} \\ &\le C_3(T+1) \quad \text{for all } T > 0. \end{aligned} \tag{3.34}$$

Case $\frac{1}{2} < \alpha < 1$: Again by (3.4), (3.5), and (3.6) and the Gagliardo–Nirenberg inequality and Hölder's inequality (with two exponents $\frac{3+2\alpha}{5\alpha}$ and $\frac{3+2\alpha}{3-3\alpha}$), we derive that there exist positive constants C_4, C_5, and C_6 such that

$$\begin{aligned} \int_0^T \int_\Omega n_\varepsilon^{\frac{10\alpha}{3}} &= \int_0^T \|n_\varepsilon^\alpha\|^{\frac{10}{3}}_{L^{\frac{10}{3}}(\Omega)} \\ &\le C_4 \int_0^T \left(\|\nabla n_\varepsilon^\alpha\|^2_{L^2(\Omega)} \|n_\varepsilon^\alpha\|^{\frac{4}{3}}_{L^2(\Omega)} + \|n_\varepsilon^\alpha\|^{\frac{10}{3}}_{L^2(\Omega)} \right) \\ &\le C_5(T+1) \quad \text{for all } T > 0 \end{aligned} \tag{3.35}$$

and

$$\begin{aligned} \int_0^T \int_\Omega |\nabla n_\varepsilon|^{\frac{10\alpha}{3+2\alpha}} &\le \left[\int_0^T \int_\Omega n_\varepsilon^{2\alpha-2} |\nabla n_\varepsilon|^2 \right]^{\frac{5\alpha}{3+2\alpha}} \left[\int_0^T \int_\Omega n_\varepsilon^{\frac{10\alpha}{3}} \right]^{\frac{3-3\alpha}{3+2\alpha}} \\ &\le C_6(T+1) \quad \text{for all } T > 0. \end{aligned} \tag{3.36}$$

Case $\alpha \geq 1$: Multiplying the first equation in (2.6) by n_ε, in view of (2.7) and using $\nabla \cdot u_\varepsilon = 0$, we derive

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|n_\varepsilon\|^2_{L^2(\Omega)} + \int_\Omega |\nabla n_\varepsilon|^2 &= -\int_\Omega n_\varepsilon \nabla \cdot (n_\varepsilon F'_\varepsilon(n_\varepsilon) S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \cdot \nabla c_\varepsilon) \\ &\leq \int_\Omega n_\varepsilon F'_\varepsilon(n_\varepsilon) |S_\varepsilon(x, n_\varepsilon, c_\varepsilon)| |\nabla n_\varepsilon| |\nabla c_\varepsilon| \quad \text{for all } t \in (0, T_{max,\varepsilon}).\end{aligned} \tag{3.37}$$

Recalling (1.5) and (2.7) and using $\alpha \geq 1$, via Young's inequality, we derive

$$\begin{aligned}\int_\Omega n_\varepsilon F'_\varepsilon(n_\varepsilon) |S_\varepsilon(x, n_\varepsilon, c_\varepsilon)| |\nabla n_\varepsilon| |\nabla c_\varepsilon| &\leq C_S \int_\Omega |\nabla n_\varepsilon| |\nabla c_\varepsilon| \\ &\leq \frac{1}{2}\int_\Omega |\nabla n_\varepsilon|^2 + \frac{C_S^2}{2}\int_\Omega |\nabla c_\varepsilon|^2 \quad \text{for all } t \in (0, T_{max,\varepsilon}).\end{aligned} \tag{3.38}$$

Here we have used the fact that

$$n_\varepsilon F'_\varepsilon(n_\varepsilon) |S_\varepsilon(x, n_\varepsilon, c_\varepsilon)| \leq C_S n_\varepsilon (1 + n_\varepsilon)^{-1} \leq C_S$$

by using (1.5). Therefore, collecting (3.37) and (3.38) and using (3.6), we conclude that

$$\int_\Omega n_\varepsilon^2 \leq C_7 \quad \text{for all } t \in (0, T_{max,\varepsilon}) \tag{3.39}$$

and

$$\int_0^T \int_\Omega |\nabla n_\varepsilon|^2 \leq C_7(T+1). \tag{3.40}$$

Hence, from (3.39)–(3.40) and (3.5)–(3.6), in light of the Gagliardo–Nirenberg inequality, we derive that there exist positive constants C_8, C_9, C_{10}, C_{11}, C_{12}, C_{13}, C_{14}, C_{15}, C_{16} and C_{17} such that

$$\begin{aligned}\int_0^T \int_\Omega n_\varepsilon^{\frac{10}{3}} &\leq C_8 \int_0^T \left(\|\nabla n_\varepsilon\|^2_{L^2(\Omega)} \|n_\varepsilon\|^{\frac{4}{3}}_{L^2(\Omega)} + \|n_\varepsilon\|^{\frac{10}{3}}_{L^2(\Omega)} \right) \\ &\leq C_9(T+1) \quad \text{for all } T > 0,\end{aligned} \tag{3.41}$$

$$\begin{aligned}\int_0^T \int_\Omega c_\varepsilon^{\frac{10}{3}} &\leq C_{10} \int_0^T \left(\|\nabla c_\varepsilon\|^2_{L^2(\Omega)} \|c_\varepsilon\|^{\frac{4}{3}}_{L^2(\Omega)} + \|c_\varepsilon\|^{\frac{10}{3}}_{L^2(\Omega)} \right) \\ &\leq C_{11}(T+1) \quad \text{for all } T > 0\end{aligned} \tag{3.42}$$

as well as

$$\begin{aligned}\int_0^T \int_\Omega |u_\varepsilon|^{\frac{10}{3}} &\leq C_{14} \int_0^T \left(\|\nabla u_\varepsilon\|^2_{L^2(\Omega)} \|u_\varepsilon\|^{\frac{4}{3}}_{L^2(\Omega)} + \|u_\varepsilon\|^{\frac{10}{3}}_{L^2(\Omega)} \right) \\ &\leq C_{15}(T+1) \quad \text{for all } T > 0\end{aligned} \tag{3.43}$$

and

$$\int_0^T \|u_\varepsilon\|^2_{L^6(\Omega)} \leq C_{16} \int_0^T \|\nabla u_\varepsilon\|^2_{L^2(\Omega)} \leq C_{17}(T+1) \text{ for all } T > 0, \tag{3.44}$$

where the last inequality we have used the embedding $W^{1,2}_{0,\sigma}(\Omega) \hookrightarrow L^6(\Omega)$ and the Poincaré inequality. Finally, combining (3.33)–(3.36) with (3.40)–(3.44), we can obtain the results. □

Lemma 3.5 *Let* $\frac{1}{3} < \alpha \le \frac{8}{21}$. *Then there exist* $\gamma = \frac{2\alpha+\frac{2}{3}}{\alpha+1} \in (1,2)$ *and* $C > 0$ *independent of* ε *such that, for each* $T \in (0, T_{max,\varepsilon})$, *the solution of* (2.6) *satisfies*

$$\int_0^T \|n_\varepsilon\|_{L^{\frac{6\gamma}{6-\gamma}}(\Omega)}^{\frac{2\gamma}{2-\gamma}} \le C(T+1). \tag{3.45}$$

Proof To this end, we first prove that for all $p \in (1, 6\alpha)$, then there exists a positive constant C_1 independent of ε such that, for each $T \in (0, T_{max,\varepsilon})$, the solution of (2.6) satisfies

$$\int_0^T \|n_\varepsilon\|_{L^p(\Omega)}^{\frac{2p(\alpha-\frac{1}{6})}{p-1}} \le C_1(T+1). \tag{3.46}$$

In fact, by (3.4) and (3.6), we derive that for some positive constants C_2 and C_3 independent of ε such that

$$\begin{aligned}\int_0^T \|n_\varepsilon\|_{L^p(\Omega)}^{\frac{2p(\alpha-\frac{1}{6})}{p-1}} &= \int_0^T \|n_\varepsilon^\alpha\|_{L^{\frac{p}{\alpha}}(\Omega)}^{\frac{2p}{p-1}\cdot\frac{6\alpha-1}{6\alpha}} \\ &\le C_2 \int_0^T \left(\|\nabla n_\varepsilon^\alpha\|_{L^2(\Omega)}^2 \|n_\varepsilon^\alpha\|_{L^{\frac{1}{\alpha}}(\Omega)}^{\frac{2p}{p-1}\cdot\frac{6\alpha-1}{6\alpha}-2} + \|n_\varepsilon^\alpha\|_{L^{\frac{1}{\alpha}}(\Omega)}^{\frac{2p}{p-1}\cdot\frac{6\alpha-1}{6\alpha}} \right) \\ &\le C_3(T+1) \quad \text{for all} \quad T > 0.\end{aligned}$$

Therefore, (3.46) holds. Next, by $\alpha \in (\frac{1}{3}, \frac{8}{21}]$, we may choose $\gamma = \frac{2\alpha+\frac{2}{3}}{\alpha+1}$ such that

$$1 < \gamma < \min\left\{\frac{6\alpha}{\alpha+1}, 2\right\} \tag{3.47}$$

as well as

$$p := \frac{6\gamma}{6-\gamma} \in (1, 6\alpha) \tag{3.48}$$

and

$$\frac{2p(\alpha-\frac{1}{6})}{p-1} = \frac{12\gamma(\alpha-\frac{1}{6})}{7\gamma-6} > \frac{2\gamma}{2-\gamma}. \tag{3.49}$$

Collecting (3.46)–(3.49), one can derive (3.45) by using the Young inequality. □

4 Global solvability of the regularized problem (2.6)

The main task of this section is to prove the global solvability of the regularized problem (2.6). To this end, first, we need to establish some ε-dependent estimates for n_ε, c_ε, and u_ε.

4.1 A priori estimates for the regularized problem (2.6) that depend on ε

In this subsection, we obtain some regularity properties for n_ε, c_ε, and u_ε in the following form on the basis of Lemma 3.3.

Lemma 4.1 *Let* $\alpha > \frac{1}{3}$. *Then there exists* $C = C(\varepsilon) > 0$ *depending on* ε *such that the solution of* (2.6) *satisfies*

$$\int_\Omega n_\varepsilon^{2\alpha+2} + \int_\Omega |\nabla u_\varepsilon|^2 \le C \quad \textit{for all} \quad t \in (0, T_{max,\varepsilon}). \tag{4.1}$$

In addition, for each $T \in (0, T_{max,\varepsilon}]$ with $T < \infty$, one can find a constant $C > 0$ depending on ε such that

$$\int_0^T \int_\Omega \left[n_\varepsilon^{2\alpha}|\nabla n_\varepsilon|^2 + |\Delta u_\varepsilon|^2\right] \le C. \tag{4.2}$$

Proof In view of (2.7), we derive

$$F_\varepsilon'(n_\varepsilon) \le \frac{1}{\varepsilon n_\varepsilon},$$

so that, by multiplying the first equation in (2.6) by $n_\varepsilon^{1+2\alpha}$, using $\nabla \cdot u_\varepsilon = 0$, and applying the same argument as in the proof of (3.7)–(3.20), one can obtain that there exist positive constants C_1 and C_2 depending on ε such that

$$\int_\Omega n_\varepsilon^{2\alpha+2} \le C_1 \quad \text{for all } \ t \in (0, T_{max,\varepsilon}) \tag{4.3}$$

and

$$\int_0^T \int_\Omega n_\varepsilon^{2\alpha}|\nabla n_\varepsilon|^2 \le C_2 \quad \text{for all } \ T \in (0, T_{max,\varepsilon}] \ \text{ with } \ T < \infty.$$

Now, from $D(1+\varepsilon A) := W^{2,2}(\Omega) \cap W^{1,2}_{0,\sigma}(\Omega) \hookrightarrow L^\infty(\Omega)$ and (3.5), it follows that, for some $C_3 > 0$ and $C_4 > 0$,

$$\begin{aligned}\|Y_\varepsilon u_\varepsilon\|_{L^\infty(\Omega)} &= \|(I+\varepsilon A)^{-1} u_\varepsilon\|_{L^\infty(\Omega)} \\ &\le C_3\|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C_4 \quad \text{for all } \ t \in (0, T_{max,\varepsilon}).\end{aligned} \tag{4.4}$$

Next, testing the projected Stokes equation $u_{\varepsilon t} + Au_\varepsilon = \mathcal{P}[-\kappa(Y_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon + n_\varepsilon \nabla\phi]$ by Au_ε, we derive

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{2}}u_\varepsilon\|^2_{L^2(\Omega)} + \int_\Omega |Au_\varepsilon|^2 &= \int_\Omega Au_\varepsilon \mathcal{P}(-\kappa(Y_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon) + \int_\Omega \mathcal{P}(n_\varepsilon\nabla\phi)Au_\varepsilon \\ &\le \frac{1}{2}\int_\Omega |Au_\varepsilon|^2 + \kappa^2\int_\Omega |(Y_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon|^2 + \|\nabla\phi\|^2_{L^\infty(\Omega)}\int_\Omega n_\varepsilon^2 \quad \text{for all } \ t \in (0, T_{max,\varepsilon}).\end{aligned} \tag{4.5}$$

However, in light of the Gagliardo–Nirenberg inequality, Young's inequality, and (4.4), there exists a positive constant C_5 such that

$$\begin{aligned}\kappa^2\int_\Omega |(Y_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon|^2 &\le \kappa^2\|Y_\varepsilon u_\varepsilon\|^2_{L^\infty(\Omega)}\int_\Omega |\nabla u_\varepsilon|^2 \\ &\le \kappa^2\|Y_\varepsilon u_\varepsilon\|^2_{L^\infty(\Omega)}\int_\Omega |\nabla u_\varepsilon|^2 \\ &\le C_5\int_\Omega |\nabla u_\varepsilon|^2 \quad \text{for all } \ t \in (0, T_{max,\varepsilon}).\end{aligned} \tag{4.6}$$

Here we have used the well-known fact that $\|A(\cdot)\|_{L^2(\Omega)}$ defines a norm equivalent to $\|\cdot\|_{W^{2,2}(\Omega)}$ on $D(A)$ (see Theorem 2.1.1 of [19]). Now, recall that $\|A^{\frac{1}{2}}u_\varepsilon\|^2_{L^2(\Omega)} = \|\nabla u_\varepsilon\|^2_{L^2(\Omega)}$. Substituting (4.6) into (4.5) yields

$$\frac{1}{2}\frac{d}{dt}\|\nabla u_\varepsilon\|^2_{L^2(\Omega)} + \int_\Omega |\Delta u_\varepsilon|^2 \le C_6\int_\Omega |\nabla u_\varepsilon|^2 + \|\nabla\phi\|^2_{L^\infty(\Omega)}\int_\Omega n_\varepsilon^2 \quad \text{for all } \ t \in (0, T_{max,\varepsilon}). \tag{4.7}$$

Since $\alpha > \frac{1}{3}$ yields $2\alpha + 2 > \frac{8}{3} > 2$, by collecting (4.3) and (4.7) and performing some basic calculations, we can get the results. □

Lemma 4.2 *Under the assumptions of Theorem 1.1, one can find that there exists $C = C(\varepsilon) > 0$ depending on ε such that*

$$\int_\Omega |\nabla c_\varepsilon(\cdot, t)|^2 \le C \quad \textit{for all} \quad t \in (0, T_{max,\varepsilon}) \tag{4.8}$$

and

$$\int_0^T \int_\Omega |\Delta c_\varepsilon|^2 \le C \quad \textit{for all} \quad T \in (0, T_{max,\varepsilon}] \quad \textit{with} \quad T < \infty. \tag{4.9}$$

Proof First, testing the second equation in (2.6) against $-\Delta c_\varepsilon$, employing Young's inequality, and using (3.3) yields

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla c_\varepsilon\|^2_{L^2(\Omega)} &= \int_\Omega -\Delta c_\varepsilon(\Delta c_\varepsilon - c_\varepsilon + F_\varepsilon(n_\varepsilon) - u_\varepsilon \cdot \nabla c_\varepsilon) \\ &= -\int_\Omega |\Delta c_\varepsilon|^2 - \int_\Omega |\nabla c_\varepsilon|^2 - \int_\Omega F_\varepsilon(n_\varepsilon)\Delta c_\varepsilon - \int_\Omega (u_\varepsilon \cdot \nabla c_\varepsilon)\Delta c_\varepsilon \\ &\le -\frac{1}{4}\int_\Omega |\Delta c_\varepsilon|^2 - \int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega n_\varepsilon^2 + \int_\Omega |u_\varepsilon \cdot \nabla c_\varepsilon||\Delta c_\varepsilon| \end{aligned} \tag{4.10}$$

for all $t \in (0, T_{max,\varepsilon})$. Next, one needs to estimate the last term on the right-hand side of (4.10). Indeed, in view of Sobolev's embedding theorem ($W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$) and applying (4.1) and (3.5), we derive from Hölder's inequality, the Gagliardo–Nirenberg inequality, and Young's inequality that there exist positive constants C_1, C_2, C_3, and C_4 such that

$$\begin{aligned} \int_\Omega |u_\varepsilon \cdot \nabla c_\varepsilon||\Delta c_\varepsilon| &\le \|u_\varepsilon\|_{L^6(\Omega)}\|\nabla c_\varepsilon\|_{L^3(\Omega)}\|\Delta c_\varepsilon\|_{L^2(\Omega)} \\ &\le C_1\|\nabla c_\varepsilon\|_{L^3(\Omega)}\|\Delta c_\varepsilon\|_{L^2(\Omega)} \\ &\le C_2(\|\Delta c_\varepsilon\|^{\frac{3}{4}}_{L^2(\Omega)}\|c_\varepsilon\|^{\frac{1}{4}}_{L^2(\Omega)} + \|c_\varepsilon\|^2_{L^2(\Omega)})\|\Delta c_\varepsilon\|_{L^2(\Omega)} \\ &\le C_3(\|\Delta c_\varepsilon\|^{\frac{7}{4}}_{L^2(\Omega)} + \|\Delta c_\varepsilon\|_{L^2(\Omega)}) \\ &\le \frac{1}{4}\|\Delta c_\varepsilon\|^2_{L^2(\Omega)} + C_4 \text{ for all } t \in (0, T_{max,\varepsilon}). \end{aligned} \tag{4.11}$$

Inserting (4.11) into (4.10) and using (4.1), one obtains (4.8) and (4.9). This completes the proof of Lemma 4.2. □

Lemma 4.3 *Let $\alpha > \frac{1}{3}$. Assume that the hypothesis of Theorem 1.1 holds. Then there exists a positive constant $C = C(\varepsilon)$ depending on ε such that, for any $3 < q < 6$, the solution of (2.6) from Lemma 2.1 satisfies*

$$\|A^\gamma u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \le C \quad \textit{for all} \quad t \in (0, T_{max,\varepsilon}) \tag{4.12}$$

as well as

$$\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \le C \quad \textit{for all} \quad t \in (0, T_{max,\varepsilon}) \tag{4.13}$$

and

$$\|\nabla c_\varepsilon(\cdot, t)\|_{L^q(\Omega)} \le C \quad \textit{for all} \quad t \in (0, T_{max,\varepsilon}), \tag{4.14}$$

where γ is the same as in (1.8).

Proof Let $h_\varepsilon(x,t)=\mathcal{P}[n_\varepsilon\nabla\phi-\kappa(Y_\varepsilon u_\varepsilon\cdot\nabla)u_\varepsilon]$. Because $\alpha>\frac{1}{3}$, then , along with (4.1), (1.7), and (4.4), there exist positive constants $q_0>\frac{3}{2}$ and C_1 such that

$$\|n_\varepsilon(\cdot,t)\|_{L^{q_0}(\Omega)}\le C_1 \quad \text{for all} \ \ t\in(0,T_{max,\varepsilon}) \tag{4.15}$$

and

$$\|h_\varepsilon(\cdot,t)\|_{L^{q_0}(\Omega)}\le C_1 \quad \text{for all} \ \ t\in(0,T_{max,\varepsilon}). \tag{4.16}$$

Hence, because $q_0>\frac{3}{2}$, we pick an arbitrary $\gamma\in(\frac{3}{4},1)$ and, then, $-\gamma-\frac{3}{2}(\frac{1}{q_0}-\frac{1}{2})>-1$. Therefore, in view of the smoothing properties of the Stokes semigroup [7], we derive that, for some λ, $C_2>0$, and $C_3>0$,

$$\begin{aligned}\|A^\gamma u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} &\le \|A^\gamma e^{-tA}u_0\|_{L^2(\Omega)}+\int_0^t\|A^\gamma e^{-(t-\tau)A}h_\varepsilon(\cdot,\tau)d\tau\|_{L^2(\Omega)}d\tau\\ &\le \|A^\gamma u_0\|_{L^2(\Omega)}+C_2\int_0^t(t-\tau)^{-\gamma-\frac{3}{2}(\frac{1}{q_0}-\frac{1}{2})}e^{-\lambda(t-\tau)}\|h_\varepsilon(\cdot,\tau)\|_{L^{q_0}(\Omega)}d\tau\\ &\le C_3 \quad \text{for all} \ \ t\in(0,T_{max,\varepsilon}).\end{aligned} \tag{4.17}$$

Observe that $\gamma>\frac{3}{4}$, $D(A^\gamma)$ is continuously embedded into $L^\infty(\Omega)$. Therefore, we derive that there exists a positive constant C_4 such that

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}\le C_4 \quad \text{for all} \ \ t\in(0,T_{max,\varepsilon}) \tag{4.18}$$

from (4.17). However, from (4.8), with the help of Sobolev's imbedding theorem, it follows that, for any fixed $\tilde{q}\in(3,6)$,

$$\|c_\varepsilon(\cdot,t)\|_{L^{\tilde{q}}(\Omega)}\le C_5 \quad \text{for all} \ \ t\in(0,T_{max,\varepsilon}). \tag{4.19}$$

Now, involving the variation-of-constants formula for c_ε and applying $\nabla\cdot u_\varepsilon=0$ in $x\in\Omega, t>0$, we have

$$c_\varepsilon(t)=e^{t(\Delta-1)}c_0+\int_0^t e^{(t-s)(\Delta-1)}(F_\varepsilon(n_\varepsilon(s))+\nabla\cdot(u_\varepsilon(s)c_\varepsilon(s))ds,\ \ t\in(0,T_{max,\varepsilon}), \tag{4.20}$$

so that, for any $3<q<\min\{\frac{3q_0}{(3-q_0)_+},\tilde{q}\}$, we have

$$\begin{aligned}\|\nabla c_\varepsilon(\cdot,t)\|_{L^q(\Omega)} &\le \|\nabla e^{t(\Delta-1)}c_0\|_{L^q(\Omega)}+\int_0^t\|\nabla e^{(t-s)(\Delta-1)}F_\varepsilon(n_\varepsilon(s))\|_{L^q(\Omega)}ds\\ &\quad+\int_0^t\|\nabla e^{(t-s)(\Delta-1)}\nabla\cdot(u_\varepsilon(s)c_\varepsilon(s))\|_{L^q(\Omega)}ds.\end{aligned} \tag{4.21}$$

To address the right-hand side of (4.21), in view of (1.8), we first use Lemma 2.2 to get

$$\|\nabla e^{t(\Delta-1)}c_0\|_{L^q(\Omega)}\le C_6 \quad \text{for all} \ \ t\in(0,T_{max,\varepsilon}). \tag{4.22}$$

Since (4.15) and (4.19) yields

$$-\frac{1}{2}-\frac{3}{2}\left(\frac{1}{q_0}-\frac{1}{q}\right)>-1,$$

together with this and (3.3), by using Lemma 2.2 again, the second term of the right-hand side is estimated as

$$\begin{aligned}&\int_0^t \|\nabla e^{(t-s)(\Delta-1)}F_\varepsilon(n_\varepsilon(s))\|_{L^q(\Omega)}ds\\&\le C_7\int_0^t[1+(t-s)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q_0}-\frac{1}{q})}]e^{-(t-s)}\|n_\varepsilon(s)\|_{L^{q_0}(\Omega)}ds\\&\le C_8 \text{ for all } t\in(0,T_{max,\varepsilon}).\end{aligned}\tag{4.23}$$

Finally, we will address the third term on the right-hand side of (4.21). To this end, we choose $0<\iota<\frac{1}{2}$ satisfying $\frac{1}{2}+\frac{3}{2}(\frac{1}{\tilde{q}}-\frac{1}{q})<\iota$ and $\tilde{\kappa}\in(0,\frac{1}{2}-\iota)$. In view of Hölder's inequality, we derive from Lemma 2.2, (4.19), and (4.18) that there exist constants C_9, C_{10}, C_{11}, and C_{12} such that

$$\begin{aligned}&\int_0^t \|\nabla e^{(t-s)(\Delta-1)}\nabla\cdot(u_\varepsilon(s)c_\varepsilon(s))\|_{L^{\tilde{q}}(\Omega)}ds\\&\le C_9\int_0^t\|(-\Delta+1)^{\iota}e^{(t-s)(\Delta-1)}\nabla\cdot(u_\varepsilon(s)c_\varepsilon(s))\|_{L^q(\Omega)}ds\\&\le C_{10}\int_0^t(t-s)^{-\iota-\frac{1}{2}-\tilde{\kappa}}e^{-\lambda(t-s)}\|u_\varepsilon(s)c_\varepsilon(s)\|_{L^{\tilde{q}}(\Omega)}ds\\&\le C_{11}\int_0^t(t-s)^{-\iota-\frac{1}{2}-\tilde{\kappa}}e^{-\lambda(t-s)}\|u_\varepsilon(s)\|_{L^\infty(\Omega)}\|c_\varepsilon(s)\|_{L^{\tilde{q}}(\Omega)}ds\\&\le C_{12}\quad\text{for all}\quad t\in(0,T_{max,\varepsilon}).\end{aligned}\tag{4.24}$$

Here we have used the fact that

$$\int_0^t(t-s)^{-\iota-\frac{1}{2}-\tilde{\kappa}}e^{-\lambda(t-s)}ds\le\int_0^\infty\sigma^{-\iota-\frac{1}{2}-\tilde{\kappa}}e^{-\lambda\sigma}d\sigma<+\infty.$$

Finally, collecting (4.21)–(4.24), we can obtain that there exists a positive constant C_{13} such that

$$\int_\Omega|\nabla c_\varepsilon(t)|^q\le C_{13}\quad\text{for all}\quad t\in(0,T_{max,\varepsilon})\quad\text{and some}\quad q\in\left(3,\min\left\{\frac{3q_0}{(3-q_0)_+},\tilde{q}\right\}\right).\tag{4.25}$$

The proof of Lemma 4.3 is complete. □

Then we can establish global existence in the approximate problem (2.6) by using Lemmas 4.1 and 4.2 .

Lemma 4.4 *Let $\alpha>\frac{1}{3}$. Then, for all $\varepsilon\in(0,1)$, the solution of (2.6) is global in time.*

Proof Assume that $T_{max,\varepsilon}$ is finite for some $\varepsilon\in(0,1)$. Fix $T\in(0,T_{max,\varepsilon})$, and let $M(T):=\sup_{t\in(0,T)}\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}$ and $\tilde{h}_\varepsilon:=F'_\varepsilon(n_\varepsilon)S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon+u_\varepsilon$. Then, by Lemma 4.3, (1.5), and (3.1), there exists $C_1>0$ such that

$$\|\tilde{h}_\varepsilon(\cdot,t)\|_{L^q(\Omega)}\le C_1\quad\text{for all}\quad t\in(0,T_{max,\varepsilon})\quad\text{and some}\quad 3<q<6.\tag{4.26}$$

Hence, because $\nabla\cdot u_\varepsilon=0$, we can derive

$$n_\varepsilon(t)=e^{(t-t_0)\Delta}n_\varepsilon(\cdot,t_0)-\int_{t_0}^t e^{(t-s)\Delta}\nabla\cdot(n_\varepsilon(\cdot,s)\tilde{h}_\varepsilon(\cdot,s))ds,\quad t\in(t_0,T)\tag{4.27}$$

by means of an associate variation-of-constants formula for n, where $t_0 := (t-1)_+$. If $t \in (0, 1]$, by virtue of the maximum principle, we can derive

$$\|e^{(t-t_0)\Delta} n_\varepsilon(\cdot, t_0)\|_{L^\infty(\Omega)} \le \|n_0\|_{L^\infty(\Omega)}, \tag{4.28}$$

while if $t > 1$ then, with the help of the L^p–L^q estimates for the Neumann heat semigroup and Lemma 3.2, we conclude that

$$\|e^{(t-t_0)\Delta} n_\varepsilon(\cdot, t_0)\|_{L^\infty(\Omega)} \le C_2(t-t_0)^{-\frac{3}{2}} \|n_\varepsilon(\cdot, t_0)\|_{L^1(\Omega)} \le C_3. \tag{4.29}$$

Finally, we fix an arbitrary $p \in (3, q)$ and then once more invoke known smoothing properties of the Stokes semigroup (see Page 201 of [7]) and Hölder's inequality to find $C_4 > 0$ such that

$$\begin{aligned}
&\int_{t_0}^t \|e^{(t-s)\Delta} \nabla \cdot (n_\varepsilon(\cdot, s)\tilde{h}_\varepsilon(\cdot, s)\|_{L^\infty(\Omega)} ds \\
&\le C_4 \int_{t_0}^t (t-s)^{-\frac{1}{2}-\frac{3}{2p}} \|n_\varepsilon(\cdot, s)\tilde{h}_\varepsilon(\cdot, s)\|_{L^p(\Omega)} ds \\
&\le C_4 \int_{t_0}^t (t-s)^{-\frac{1}{2}-\frac{3}{2p}} \|n_\varepsilon(\cdot, s)\|_{L^{\frac{pq}{q-p}}(\Omega)} \|\tilde{h}_\varepsilon(\cdot, s)\|_{L^q(\Omega)} ds \\
&\le C_4 \int_{t_0}^t (t-s)^{-\frac{1}{2}-\frac{3}{2p}} \|u_\varepsilon(\cdot, s)\|_{L^\infty(\Omega)}^b \|u_\varepsilon(\cdot, s)\||_{L^1(\Omega)}^{1-b} \|\tilde{h}_\varepsilon(\cdot, s)\|_{L^q(\Omega)} ds \\
&\le C_5 M^b(T) \quad \text{for all } t \in (0, T),
\end{aligned} \tag{4.30}$$

where $b := \frac{pq-q+p}{pq} \in (0, 1)$ and

$$C_5 := C_4 C_1^{2-b} \int_0^1 \sigma^{-\frac{1}{2}-\frac{3}{2p}} d\sigma.$$

Since $p > 3$, we conclude that $-\frac{1}{2} - \frac{3}{2p} > -1$. In combination with (4.27)–(4.30) and using the definition of $M(T)$, we obtain $C_6 > 0$ such that

$$M(T) \le C_6 + C_6 M^b(T) \quad \text{for all } T \in (0, T_{max,\varepsilon}). \tag{4.31}$$

Hence, in view of $b < 1$, with some basic calculation, since $T \in (0, T_{max,\varepsilon})$ was arbitrary, we can obtain there exists a positive constant C_7 such that

$$\|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \le C_7 \quad \text{for all } t \in (0, T_{max,\varepsilon}). \tag{4.32}$$

To prove the boundedness of $\|\nabla c_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)}$, we rewrite the variation-of-constants formula for c_ε in the form

$$c_\varepsilon(\cdot, t) = e^{t(\Delta-1)} c_0 + \int_0^t e^{(t-s)(\Delta-1)} [F_\varepsilon(n_\varepsilon)(s) - u_\varepsilon(s) \cdot \nabla c_\varepsilon(s)] ds \quad \text{for all } t \in (0, T_{max,\varepsilon}).$$

Now, we choose $\theta \in (\frac{1}{2} + \frac{3}{2q}, 1)$, where $3 < q < 6$ [see (4.25)], then the domain of the fractional power $D((-\Delta+1)^\theta) \hookrightarrow W^{1,\infty}(\Omega)$ (see [10]). Hence, in view of L^p–L^q estimates associated with the heat semigroup, (4.13), (4.14), and (3.3), we derive that there exist positive constants λ, C_8, C_9, C_{10}, and C_{11} such that

$$\begin{aligned}\|c_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} &\le C_8\|(-\Delta+1)^\theta c_\varepsilon(\cdot,t)\|_{L^q(\Omega)}\\ &\le C_9 t^{-\theta}e^{-\lambda t}\|c_0\|_{L^q(\Omega)} + C_9\int_0^t (t-s)^{-\theta}e^{-\lambda(t-s)}\|(F_\varepsilon(n_\varepsilon)-u_\varepsilon\cdot\nabla c_\varepsilon)(s)\|_{L^q(\Omega)}ds\\ &\le C_{10}+C_{10}\int_0^t (t-s)^{-\theta}e^{-\lambda(t-s)}[\|n_\varepsilon(s)\|_{L^q(\Omega)}+\|u_\varepsilon(s)\|_{L^\infty(\Omega)}\|\nabla c_\varepsilon(s)\|_{L^q(\Omega)}]ds\\ &\le C_{11} \text{ for all } t\in(0,T_{max,\varepsilon}).\end{aligned} \tag{4.33}$$

Here we have used Hölder's inequality as well as

$$\int_0^t (t-s)^{-\theta}e^{-\lambda(t-s)} \le \int_0^\infty \sigma^{-\theta}e^{-\lambda\sigma}d\sigma < +\infty.$$

In view of (4.12), (4.33), and (4.32), we apply Lemma 2.1 to reach a contradiction. □

5 Regularity properties of time derivatives

In preparation of an Aubin–Lions type compactness argument, we will rely on an additional regularity estimate for $n_\varepsilon F_\varepsilon'(n_\varepsilon)S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon$, $u_\varepsilon\cdot\nabla c_\varepsilon$, $n_\varepsilon u_\varepsilon$, and $c_\varepsilon u_\varepsilon$.

Lemma 5.1 *Let $\alpha > \frac{1}{3}$, and assume that (1.7) and (1.8) hold. Then one can find $C > 0$ independent of ε such that, for all $T \in (0,\infty)$,*

$$\int_0^T\int_\Omega\left[|n_\varepsilon F_\varepsilon'(n_\varepsilon)S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon|^{\frac{3\alpha+1}{2}}+|n_\varepsilon u_\varepsilon|^{\frac{2\alpha+\frac{2}{3}}{\alpha+1}}\right]\le C(T+1),\quad \textit{if}\ \frac{1}{3}<\alpha\le\frac{8}{21}, \tag{5.1}$$

$$\int_0^T\int_\Omega\left[|n_\varepsilon F_\varepsilon'(n_\varepsilon)S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon|^{\frac{3\alpha+1}{2}}+|n_\varepsilon u_\varepsilon|^{\frac{10(3\alpha+1)}{9(\alpha+2)}}\right]\le C(T+1),\quad \textit{if}\ \frac{8}{21}<\alpha\le\frac{1}{2}, \tag{5.2}$$

$$\int_0^T\int_\Omega\left[|n_\varepsilon F_\varepsilon'(n_\varepsilon)S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon|^{\frac{10\alpha}{3+2\alpha}}+|n_\varepsilon u_\varepsilon|^{\frac{10\alpha}{3(\alpha+1)}}\right]\le C(T+1),\quad \textit{if}\ \frac{1}{2}<\alpha<1 \tag{5.3}$$

as well as

$$\int_0^T\int_\Omega\left[|n_\varepsilon F_\varepsilon'(n_\varepsilon)S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon|^2+|n_\varepsilon u_\varepsilon|^{\frac{5}{3}}\right]\le C(T+1),\quad \textit{if}\ \alpha\ge 1 \tag{5.4}$$

and

$$\int_0^T\int_\Omega\left[|u_\varepsilon\cdot\nabla c_\varepsilon|^{\frac{5}{4}}+|c_\varepsilon u_\varepsilon|^{\frac{5}{3}}\right]\le C(T+1). \tag{5.5}$$

Proof First, by (1.5), (3.1), and (2.8), we derive

$$n_\varepsilon F_\varepsilon'(n_\varepsilon)S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\le C_S n_\varepsilon^{(1-\alpha)_+}$$

with $(1-\alpha)_+ = \max\{0,1-\alpha\}$. Case $\frac{8}{21}<\alpha\le\frac{1}{2}$: It is not difficult to verify that

$$\frac{2}{3\alpha+1}=\frac{1}{2}+\frac{3}{6\alpha+2}(1-\alpha)$$

and

$$\frac{9(\alpha+2)}{10(3\alpha+1)} = \frac{3}{10} + \frac{3}{6\alpha+2},$$

so that, recalling (3.29), (3.44), and Hölder's inequality, we can obtain (5.2). While if $\frac{1}{3} < \alpha \le \frac{8}{21}$, in light of (3.6), (3.29), (3.32), (3.45), an employment of the Hölder and Young inequalities to shows that

$$\begin{aligned}
&\int_0^T \int_\Omega \left[|n_\varepsilon F'_\varepsilon(n_\varepsilon) S_\varepsilon(x, n_\varepsilon, c_\varepsilon)\nabla c_\varepsilon|^{\frac{3\alpha+1}{2}} + |n_\varepsilon u_\varepsilon|^\gamma \right] \\
&\quad \le C_1 \left[\int_0^T \int_\Omega n_\varepsilon^{\frac{6\alpha+2}{3}} \right]^{\frac{3-3\alpha}{4}} \left[\int_0^T \int_\Omega |\nabla c_\varepsilon|^2 \right]^{\frac{3\alpha+1}{4}} \\
&\qquad + C_1 \int_0^T \|n_\varepsilon\|^\gamma_{L^{\frac{6\gamma}{6-\gamma}}(\Omega)} \|u_\varepsilon\|^\gamma_{L^6(\Omega)} \le C_2(T+1),
\end{aligned}$$

where $\gamma = \frac{2\alpha+\frac{2}{3}}{\alpha+1}$ is given by Lemma 3.5.

Other cases can be proved very similarly. Therefore, we omit their proofs. □

To prepare our subsequent compactness properties of $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ by means of the Aubin–Lions lemma (see Simon [18]), we use Lemmas 3.2–3.4 to obtain the following regularity property with respect to the time variable.

Lemma 5.2 *Let $\alpha > \frac{1}{3}$, and assume that (1.7) and (1.8) hold. Then there exists $C > 0$ independent of ε such that*

$$\int_0^T \|\partial_t n_\varepsilon(\cdot, t)\|_{(W^{2,4}(\Omega))^*} dt \le C(T+1) \quad \textit{for all} \quad T \in (0, \infty) \tag{5.6}$$

as well as

$$\int_0^T \|\partial_t c_\varepsilon(\cdot, t)\|^{\frac{5}{4}}_{(W^{1,5}(\Omega))^*} dt \le C(T+1) \quad \textit{for all} \quad T \in (0, \infty) \tag{5.7}$$

and

$$\int_0^T \|\partial_t u_\varepsilon(\cdot, t)\|^{\frac{5}{4}}_{(W^{1,5}_{0,\sigma}(\Omega))^*} dt \le C(T+1) \quad \textit{for all} \quad T \in (0, \infty). \tag{5.8}$$

Proof Firstly, testing the first equation of (2.6) by certain $\varphi \in C^\infty(\bar{\Omega})$, we have

$$\begin{aligned}
\left| \int_\Omega (n_{\varepsilon,t})\varphi \right| &= \left| \int_\Omega \left[\Delta n_\varepsilon - \nabla\cdot(n_\varepsilon F'_\varepsilon(n_\varepsilon) S_\varepsilon(x, n_\varepsilon, c_\varepsilon)\nabla c_\varepsilon) - u_\varepsilon \cdot \nabla n_\varepsilon \right] \varphi \right| \\
&= \left| \int_\Omega \left[-\nabla n_\varepsilon \cdot \nabla\varphi + n_\varepsilon F'_\varepsilon(n_\varepsilon) S_\varepsilon(x, n_\varepsilon, c_\varepsilon)\nabla c_\varepsilon \cdot \nabla\varphi + n_\varepsilon u_\varepsilon \cdot \nabla\varphi \right] \right| \\
&\le \left| \int_\Omega \left[|\nabla n_\varepsilon| + |n_\varepsilon F'_\varepsilon(n_\varepsilon) S_\varepsilon(x, n_\varepsilon, c_\varepsilon)\nabla c_\varepsilon| + |n_\varepsilon u_\varepsilon| \right] \right| \|\varphi\|_{W^{1,\infty}(\Omega)}
\end{aligned}$$

for all $t > 0$.

Observe that the embedding $W^{2,4}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$, so that, in view of $\alpha > \frac{1}{3}$, Lemmas 3.4 and 5.1, we deduce from the Young inequality that for some C_1 and C_2 such that

$$\begin{aligned}
&\int_0^T \|\partial_t n_\varepsilon(\cdot,t)\|_{(W^{2,4}(\Omega))^*} dt \\
&\le C_1 \left\{ \int_0^T \int_\Omega |\nabla n_\varepsilon|^{r_1} + \int_0^T \int_\Omega |n_\varepsilon F_\varepsilon'(n_\varepsilon) S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \nabla c_\varepsilon|^{r_1} + \int_0^T \int_\Omega |n_\varepsilon u_\varepsilon|^{r_2} + T \right\} \\
&\le C_2(T+1) \ \text{ for all } \ T > 0,
\end{aligned} \tag{5.9}$$

where

$$r_1 = \begin{cases} \frac{3\alpha+1}{2} & \text{if } \frac{1}{3} < \alpha \le \frac{1}{2}, \\ \frac{10\alpha}{3+2\alpha} & \text{if } \frac{1}{2} < \alpha < 1, \\ 2 & \text{if } \alpha \ge 1 \end{cases}$$

and

$$r_2 = \begin{cases} \frac{2\alpha+\frac{2}{3}}{\alpha+1} & \text{if } \frac{1}{3} < \alpha \le \frac{8}{21}, \\ \frac{10(3\alpha+1)}{9(\alpha+2)} & \text{if } \frac{8}{21} < \alpha \le \frac{1}{2}, \\ \frac{10\alpha}{3(\alpha+1)} & \text{if } \frac{1}{2} < \alpha < 1, \\ \frac{5}{3} & \text{if } \alpha \ge 1, \end{cases}$$

Likewise, given any $\varphi \in C^\infty(\bar{\Omega})$, we may test the second equation in (2.6) against φ to conclude that

$$\begin{aligned}
\left| \int_\Omega \partial_t c_\varepsilon(\cdot,t)\varphi \right| &= \left| \int_\Omega [\Delta c_\varepsilon - c_\varepsilon + n_\varepsilon - u_\varepsilon \cdot \nabla c_\varepsilon] \cdot \varphi \right| \\
&= \left| -\int_\Omega \nabla c_\varepsilon \cdot \nabla\varphi - \int_\Omega c_\varepsilon \varphi + \int_\Omega n_\varepsilon \varphi + \int_\Omega c_\varepsilon u_\varepsilon \cdot \nabla\varphi \right| \\
&\le \left\{ \|\nabla c_\varepsilon\|_{L^{\frac{5}{4}}(\Omega)} + \|c_\varepsilon\|_{L^{\frac{5}{4}}(\Omega)} + \|n_\varepsilon\|_{L^{\frac{5}{4}}(\Omega)} + \|c_\varepsilon u_\varepsilon\|_{L^{\frac{5}{4}}(\Omega)} \right\} \|\varphi\|_{W^{1,5}(\Omega)}
\end{aligned}$$

for all $t > 0$. Thus, from Lemmas 3.4 and 5.1 again, in light of $\alpha > \frac{1}{3}$, we invoke the Young inequality again and obtain that there exist positive constant C_3 and C_4 such that

$$\begin{aligned}
&\int_0^T \|\partial_t c_\varepsilon(\cdot,t)\|^{\frac{5}{4}}_{(W^{1,5}(\Omega))^*} dt \\
&\le C_3 \left(\int_0^T \int_\Omega |\nabla c_\varepsilon|^2 + \int_0^T \int_\Omega n_\varepsilon^{r_3} + \int_0^T \int_\Omega c_\varepsilon^{\frac{10}{3}} + \int_0^T \int_\Omega |u_\varepsilon|^{\frac{10}{3}} + T \right) \\
&\le C_4(T+1) \quad \text{for all} \quad T > 0
\end{aligned}$$

with

$$r_3 = \begin{cases} \frac{6\alpha+2}{3} & \text{if } \frac{1}{3} < \alpha \le \frac{1}{2}, \\ \frac{10\alpha}{3} & \text{if } \frac{1}{2} < \alpha < 1, \\ \frac{10}{3} & \text{if } \alpha \ge 1. \end{cases} \tag{5.10}$$

Finally, for any given $\varphi \in C^\infty_{0,\sigma}(\Omega; \mathbb{R}^3)$, we infer from the third equation in (2.6) that

$$\left| \int_\Omega \partial_t u_\varepsilon(\cdot,t)\varphi \right| = \left| -\int_\Omega \nabla u_\varepsilon \cdot \nabla\varphi - \kappa \int_\Omega (Y_\varepsilon u_\varepsilon \otimes u_\varepsilon) \cdot \nabla\varphi + \int_\Omega n_\varepsilon \nabla\phi \cdot \varphi \right| \ \text{ for all } \ t > 0.$$

Now, by virtue of (3.6), Lemmas 3.4 and 5.1 , we also get that there exist positive constants C_5, C_6 and C_7 such that

$$\begin{aligned}&\int_0^T \|\partial_t u_\varepsilon(\cdot,t)\|^{\frac{5}{4}}_{(W^{1,5}_{0,\sigma}(\Omega))^*}dt\\&\le C_5\left(\int_0^T\int_\Omega|\nabla u_\varepsilon|^{\frac{5}{4}}+\int_0^T\int_\Omega|Y_\varepsilon u_\varepsilon\otimes u_\varepsilon|^{\frac{5}{4}}+\int_0^T\int_\Omega n_\varepsilon^{\frac{5}{4}}\right)\\&\le C_6\left(\int_0^T\int_\Omega|\nabla u_\varepsilon|^2+\int_0^T\int_\Omega|Y_\varepsilon u_\varepsilon|^2+\int_0^T\int_\Omega|u_\varepsilon|^{\frac{10}{3}}+\int_0^T\int_\Omega n_\varepsilon^{r_3}+T\right)\\&\le C_7(T+1)\quad\text{for all}\quad T>0,\end{aligned}$$

which implies (5.8). Here r_3 is the same as (5.10). □

6 Passing to the limit: Proof of Theorem 1.1

Based on the above lemmas and by extracting suitable subsequences in a standard way, we can prove Theorem 1.1.

Lemma 6.1 *Let (1.4), (1.5), (1.7) and (1.8) hold, and suppose that $\alpha > \frac{1}{3}$. There exists $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ such that $\varepsilon_j\searrow 0$ as $j\to\infty$ and such that as $\varepsilon=\varepsilon_j\searrow 0$ we have*

$$n_\varepsilon\to n \text{ a.e. in } \Omega\times(0,\infty) \text{ and in } L^r_{loc}(\bar\Omega\times[0,\infty)) \text{ with } r=\begin{cases}\frac{3\alpha+1}{2} & \text{if } \frac{1}{3}<\alpha\le\frac{1}{2},\\ \frac{10\alpha}{3+2\alpha} & \text{if } \frac{1}{2}<\alpha<1,\\ 2 & \text{if } \alpha\ge 1,\end{cases}\tag{6.1}$$

$$\nabla n_\varepsilon\rightharpoonup\nabla n \text{ in } L^r_{loc}(\bar\Omega\times[0,\infty)) \text{ with } r=\begin{cases}\frac{3\alpha+1}{2} & \text{if } \frac{1}{3}<\alpha\le\frac{1}{2},\\ \frac{10\alpha}{3+2\alpha} & \text{if } \frac{1}{2}<\alpha<1,\\ 2 & \text{if } \alpha\ge 1,\end{cases}\tag{6.2}$$

$$c_\varepsilon\to c \text{ in } L^2_{loc}(\bar\Omega\times[0,\infty)) \text{ and a.e. in } \Omega\times(0,\infty),\tag{6.3}$$

$$\nabla c_\varepsilon\to\nabla c \text{ a.e. in } \Omega\times(0,\infty),\tag{6.4}$$

$$u_\varepsilon\to u \text{ in } L^2_{loc}(\bar\Omega\times[0,\infty)) \text{ and a.e. in } \Omega\times(0,\infty)\tag{6.5}$$

as well as

$$\nabla c_\varepsilon\rightharpoonup\nabla c \ \text{ in } L^2_{loc}(\bar\Omega\times[0,\infty))\tag{6.6}$$

and

$$\nabla u_\varepsilon\rightharpoonup\nabla u \ \text{ in } L^2_{loc}(\bar\Omega\times[0,\infty))\tag{6.7}$$

and

$$u_\varepsilon\rightharpoonup u \ \text{ in } L^{\frac{10}{3}}_{loc}(\bar\Omega\times[0,\infty))\tag{6.8}$$

with some triple (n,c,u) that is a global weak solution of (1.3) in the sense of Definition 2.1.

Proof First, from Lemma 3.4 and (5.6), we derive that there exists a positive constant C_0 such that

$$\|n_\varepsilon\|_{L^r_{loc}([0,\infty);W^{1,r}(\Omega))}\le C_0(T+1) \text{ and } \|\partial_t n_\varepsilon\|_{L^1_{loc}([0,\infty);(W^{2,4}(\Omega))^*)}\le C_0(T+1),\tag{6.9}$$

where r is given by (6.1). Hence, from (6.9) and the Aubin–Lions lemma (see, e.g., [18]), we conclude that

$$(n_\varepsilon)_{\varepsilon\in(0,1)} \text{ is strongly precompact in } L^r_{loc}(\bar{\Omega}\times[0,\infty)), \tag{6.10}$$

so that, there exists a sequence $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ such that $\varepsilon=\varepsilon_j\searrow 0$ as $j\to\infty$ and

$$n_\varepsilon\to n \text{ a.e. in } \Omega\times(0,\infty) \text{ and in } L^r_{loc}(\bar{\Omega}\times[0,\infty)) \quad \text{as} \quad \varepsilon=\varepsilon_j\searrow 0, \tag{6.11}$$

where r is the same as (6.1). Now, in view of Lemmas 3.3, 3.4, 5.1, and 5.2, employing the same arguments as in the proof of (6.9)–(6.11), we can derive (6.1)–(6.3) and (6.5)–(6.8) holds. Next, let $g_\varepsilon(x,t):=-c_\varepsilon+F_\varepsilon(n_\varepsilon)-u_\varepsilon\cdot\nabla c_\varepsilon$. With this notation, the second equation of (2.6) can be rewritten in component form as

$$c_{\varepsilon t}-\Delta c_\varepsilon=g_\varepsilon. \tag{6.12}$$

Case $\frac{1}{3}<\alpha\le\frac{1}{2}$: Observe that

$$\frac{5}{4}<\frac{4}{3}<\min\left\{\frac{6\alpha+2}{3},\frac{10}{3}\right\} \text{ for } \frac{1}{3}<\alpha\le\frac{1}{2}.$$

Thus, recalling (3.29), (3.32), and (5.5) and applying Hölder's inequality, we conclude that, for any $\varepsilon\in(0,1)$, g_ε is bounded in $L^{\frac{5}{4}}(\Omega\times(0,T))$, and we may invoke the standard parabolic regularity theory to (6.12) and infer that $(c_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^{\frac{5}{4}}((0,T);W^{2,\frac{5}{4}}(\Omega))$. Hence, by virtue of (5.7) and the Aubin–Lions lemma, we derive the relative compactness of $(c_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^{\frac{5}{4}}((0,T);W^{1,\frac{5}{4}}(\Omega))$. We can pick an appropriate subsequence that is still written as $(\varepsilon_j)_{j\in\mathbb{N}}$ such that $\nabla c_{\varepsilon_j}\to z_1$ in $L^{\frac{5}{4}}(\Omega\times(0,T))$ for all $T\in(0,\infty)$ and some $z_1\in L^{\frac{5}{4}}(\Omega\times(0,T))$ as $j\to\infty$. Therefore, by (5.7), we can also derive that $\nabla c_{\varepsilon_j}\to z_1$ a.e. in $\Omega\times(0,\infty)$ as $j\to\infty$. In view of (6.6) and Egorov's theorem, we conclude that $z_1=\nabla c$ and hence (6.4) holds. Next, we pay attention to the case $\frac{1}{2}<\alpha<1$: By straightforward calculations, and using relation $\frac{1}{2}<\alpha<1$, one has

$$\frac{5}{4}<\frac{5}{3}<\min\left\{\frac{10\alpha}{3},\frac{10}{3}\right\}.$$

Consequently, based on (3.30), (3.32), and (5.5), it follows from Hölder's inequality that

$$c_{\varepsilon t}-\Delta c_\varepsilon=g_\varepsilon \text{ is bounded in } L^{\frac{5}{4}}(\Omega\times(0,T)) \text{ for any } \varepsilon\in(0,1). \tag{6.13}$$

Employing almost exactly the same arguments as in the proof of the case $\frac{1}{3}<\alpha\le\frac{1}{2}$, and taking advantage of (6.13), we conclude the estimate (6.6). The proof of case $\alpha\ge 1$ is similar to that of case $\frac{1}{3}<\alpha\le\frac{1}{2}$, so we omit it.

In the following proof, we shall prove that (n,c,u) is a weak solution of problem (1.3) in Definition 2.1. In fact, by $\alpha>\frac{1}{3}$, we conclude that

$$r>1,$$

where r is given by (6.1). Therefore, with the help of (6.1)–(6.3) and (6.5)–(6.7), we can derive (2.1). Now, by the nonnegativity of n_ε and c_ε, we obtain $n\ge 0$ and $c\ge 0$. Next, from (6.7) and $\nabla\cdot u_\varepsilon=0$, we conclude that $\nabla\cdot u=0$ a.e. in $\Omega\times(0,\infty)$. However, in view of (5.2), (5.3), and (5.4), we conclude that

$$n_\varepsilon F'_\varepsilon(n_\varepsilon)S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\nabla c_\varepsilon\rightharpoonup z_2 \text{ in } L^r(\Omega\times(0,T)) \text{ as } \varepsilon=\varepsilon_j\searrow 0 \text{ for each } T\in(0,\infty), \tag{6.14}$$

where r is given by (6.1). However, it follows from (1.4), (2.8), (3.2), (6.1), (6.3), and (6.4) that

$$n_\varepsilon F'_\varepsilon(n_\varepsilon)S_\varepsilon(x, n_\varepsilon, c_\varepsilon)\nabla c_\varepsilon \to nS(x, n, c)\nabla c \text{ a.e. in } \Omega \times (0, \infty) \text{ as } \varepsilon = \varepsilon_j \searrow 0. \tag{6.15}$$

Again by Egorov's theorem, we gain $z_2 = nS(x, n, c)\nabla c$, and therefore (6.14) can be rewritten as

$$\begin{aligned} &n_\varepsilon F'_\varepsilon(n_\varepsilon)S_\varepsilon(x, n_\varepsilon, c_\varepsilon)\nabla c_\varepsilon \rightharpoonup nS(x, n, c)\nabla c \text{ in } L^r(\Omega \times (0, T)) \text{ as } \varepsilon \\ &\quad = \varepsilon_j \searrow 0 \text{ for each } T \in (0, \infty), \end{aligned} \tag{6.16}$$

which together with $r > 1$ implies the integrability of $nS(x, n, c)\nabla c$ in (2.2) as well. It is not difficult to check that

$$\begin{aligned} &\frac{2\alpha + \frac{2}{3}}{\alpha + 1} > 1 \text{ if } \frac{1}{3} < \alpha \le \frac{8}{21}, \ \frac{10(3\alpha + 1)}{9(\alpha + 2)} > 1 \text{ if } \frac{8}{21} \\ &< \alpha \le \frac{1}{2} \text{ and } \frac{10\alpha}{3(\alpha + 1)} > 1 \text{ if } \frac{1}{2} < \alpha < 1. \end{aligned}$$

Thereupon, recalling (5.2), (5.3), and (5.4), we infer that, for each $T \in (0, \infty)$,

$$n_\varepsilon u_\varepsilon \rightharpoonup z_3 \text{ in } L^{\tilde{r}}(\Omega \times (0, T)) \text{ with } \tilde{r} = \begin{cases} \frac{2\alpha+\frac{2}{3}}{\alpha+1} & \text{if } \frac{1}{3} < \alpha \le \frac{8}{21}, \\ \frac{10(3\alpha+1)}{9(\alpha+2)} & \text{if } \frac{8}{21} < \alpha \le \frac{1}{2}, \\ \frac{10\alpha}{3(\alpha+1)} & \text{if } \frac{1}{2} < \alpha < 1, \\ \frac{5}{3} & \text{if } \alpha \ge 1. \end{cases} \text{ as } \varepsilon = \varepsilon_j \searrow 0, \tag{6.17}$$

(6.17) together with (6.1) and (6.5) implies

$$n_\varepsilon u_\varepsilon \to nu \text{ a.e. in } \Omega \times (0, \infty) \text{ as } \varepsilon = \varepsilon_j \searrow 0. \tag{6.18}$$

(6.17) along with (6.18) and Egorov's theorem guarantees that $z_3 = nu$, whereupon we derive from (6.17) that

$$n_\varepsilon u_\varepsilon \rightharpoonup nu \text{ in } L^{\tilde{r}}(\Omega \times (0, T)) \text{ with } \tilde{r} = \begin{cases} \frac{2\alpha+\frac{2}{3}}{\alpha+1} & \text{if } \frac{1}{3} < \alpha \le \frac{8}{21}, \\ \frac{10(3\alpha+1)}{9(\alpha+2)} & \text{if } \frac{8}{21} < \alpha \le \frac{1}{2}, \\ \frac{10\alpha}{3(\alpha+1)} & \text{if } \frac{1}{2} < \alpha < 1, \\ \frac{5}{3} & \text{if } \alpha \ge 1 \end{cases} \text{ as } \varepsilon = \varepsilon_j \searrow 0, \tag{6.19}$$

for each $T \in (0, \infty)$.

As a straightforward consequence of (6.3) and (6.5), it holds that

$$c_\varepsilon u_\varepsilon \to cu \ \text{ in } L^1_{loc}(\bar{\Omega} \times (0, \infty)) \text{ as } \varepsilon = \varepsilon_j \searrow 0. \tag{6.20}$$

Thus, the integrability of nu and cu in (2.2) is verified by (6.3) and (6.5).

Next, by (6.5) and the fact that $\|Y_\varepsilon \varphi\|_{L^2(\Omega)} \le \|\varphi\|_{L^2(\Omega)} (\varphi \in L^2_\sigma(\Omega))$ and $Y_\varepsilon \varphi \to \varphi$ in $L^2(\Omega)$ as $\varepsilon \searrow 0$, we can get that there exists a positive constant C_1 such that, for any $\varepsilon \in (0, 1)$,

$$\begin{aligned} \|Y_\varepsilon u_\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)} &\le \|Y_\varepsilon[u_\varepsilon(\cdot, t) - u(\cdot, t)]\|_{L^2(\Omega)} + \|Y_\varepsilon u(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)} \\ &\le \|u_\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)} + \|Y_\varepsilon u(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)} \\ &\to 0 \text{ as } \varepsilon = \varepsilon_j \searrow 0 \end{aligned}$$

and

$$\begin{aligned}\|Y_\varepsilon u_\varepsilon(\cdot,t)-u(\cdot,t)\|^2_{L^2(\Omega)} &\le \left(\|Y_\varepsilon u_\varepsilon(\cdot,t)|\|_{L^2(\Omega)}+\|u(\cdot,t)|\|_{L^2(\Omega)}\right)^2\\ &\le \left(\|u_\varepsilon(\cdot,t)|\|_{L^2(\Omega)}+\|u(\cdot,t)|\|_{L^2(\Omega)}\right)^2\\ &\le C_1 \text{ for all } t\in(0,\infty)/N \text{ with some null set } N\subset(0,\infty),\end{aligned}$$

so that, by the dominated convergence theorem, we also find that

$$\int_0^T \|Y_\varepsilon u_\varepsilon(\cdot,t)-u(\cdot,t)\|^2_{L^2(\Omega)}dt \to 0 \text{ as } \varepsilon=\varepsilon_j\searrow 0 \quad \text{for all} \quad T>0.$$

Therefore,

$$Y_\varepsilon u_\varepsilon \to u \text{ in } L^2_{loc}([0,\infty);L^2(\Omega)). \tag{6.21}$$

Now, combining (6.5) with (6.21), we derive

$$Y_\varepsilon u_\varepsilon\otimes u_\varepsilon \to u\otimes u \text{ in } L^1_{loc}(\bar\Omega\times[0,\infty)) \text{ as } \varepsilon=\varepsilon_j\searrow 0. \tag{6.22}$$

Therefore, the integrability of $nS(x,n,c)\nabla c$, nu, cu, and $u\otimes u$ in (2.2) is verified by (6.16), (6.19), (6.20) and (6.22). Finally, for any fixed $T\in(0,\infty)$, applying (6.1), one can get

$$\begin{aligned}&\int_0^T \|F_\varepsilon(n_\varepsilon(\cdot,t))-n(\cdot,t)\|^r_{L^r(\Omega)}\,dt\\ &\le \int_0^T \|F_\varepsilon(n_\varepsilon(\cdot,t))-F_\varepsilon(n(\cdot,t))\|^r_{L^r(\Omega)}\,dt+\int_0^T \|F_\varepsilon(n(\cdot,t))-n(\cdot,t)\|^r_{L^r(\Omega)}\,dt\\ &\le \|F'_\varepsilon\|_{L^\infty(\Omega\times(0,\infty))}\int_0^T \|n_\varepsilon(\cdot,t)-n(\cdot,t)\|^r_{L^r(\Omega)}\,dt+\int_0^T \|F_\varepsilon(n(\cdot,t))-n(\cdot,t)\|^r_{L^r(\Omega)}\,dt,\end{aligned} \tag{6.23}$$

where r is the same as in (6.1). Besides that, we also deduce from (3.3) and $r>1$ that

$$\|F_\varepsilon(n(\cdot,t))-n(\cdot,t)\|^r_{L^r(\Omega\times(0,T))}\le 2^r\|n(\cdot,t)\|$$

for each $t\in(0,T)$, which together with (6.1) shows the integrability of $\|F_\varepsilon(n(\cdot,t)) -n(\cdot,t)\|^r_{L^r(\Omega)}$ on $(0,T)$. Thereupon, by virtue of (3.2), we infer from the dominated convergence theorem that

$$\int_0^T \|F_\varepsilon(n)-n\|^r_{L^r(\Omega)}\,dt \to 0 \text{ as } \varepsilon=\varepsilon_j\searrow 0 \tag{6.24}$$

for each $T\in(0,\infty)$. Inserting (6.24) into (6.23) and using (6.1) and (3.1), we can see clearly that

$$F_\varepsilon(n)\to n \text{ in } L^r_{loc}(\bar\Omega\times[0,\infty)) \text{ as } \varepsilon=\varepsilon_j\searrow 0. \tag{6.25}$$

Finally, according to (6.1)–(6.3), (6.5)–(6.7), (6.16), (6.19), (6.20), (6.21), (6.22), and (6.25), we may pass to the limit in the respective weak formulations associated with the regularized system (2.6) and obtain the integral identities (2.3)–(2.5). □

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