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Calculus of Variations

Critical exponent for the global existence of solutions to a semilinear heat equation with degenerate coefficients

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Abstract

We investigate a non-homogeneous semilinear heat equation which involves degenerate coefficients. More precisely, in order to give a rather complete theory, we focus on two types of weights $w(x) = |x_1|^a$ or $w(x) = |x|^b$ where $a, b > 0$ in a suitable range. We prove the existence of a Fujita exponent and describe the dichotomy existence/non-existence of global in time solutions. The coefficients under consideration admit either a singularity at the origin or a line of singularities. In this latter case, the problem is related to the fractional Laplacian, through the Caffarelli–Silvestre extension and is a first attempt to develop a parabolic theory in this setting.

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Contents

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1 Introduction

We consider the problem

$$\begin{cases} \partial_t u - \operatorname{div}(w(x)\nabla u) = u^p, & x \in \mathbb{R}^N, \quad t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where the coefficient w is either $w(x) = |x_1|^a$ with $a \in [0,1)$, or $w(x) = |x|^b$ with $b \in [0,N)$. Here one has $N \geq 1$, $\partial_t := \partial/\partial t$ and $p > 1$.

The aim of the present work is to develop a global-in-time existence theory of mild solutions for the problem (1.1). We prove that there is a critical exponent for the global existence of positive solutions of problem (1.1), the so-called Fujita exponent.

We give first the definition of a solution to (1.1).

Let $\Gamma = \Gamma(x,y,t)$ be the fundamental solution of

$$\partial_t v - \operatorname{div}(w(x)\nabla v) = 0, \quad x \in \mathbb{R}^N, \quad t > 0,$$

with a pole at $(y,0)$. Under the condition either $w(x) = |x_1|^a$ with $a \in [0,1)$, or $w(x) = |x|^b$ with $b \in [0,1)$, this fundamental solution Γ satisfies the mass conservation property, the semigroup one and suitable Gaussian estimates [see (K1)–(K3) in Sect. 2]. Using Γ, we define the solution of (1.1) as follows.

Definition 1.1 Let u_0 be a nonnegative measurable function in $\mathbb{R}^N$. Let $T \in (0,\infty]$ and u be a nonnegative measurable function in $\mathbb{R}^N \times (0,T)$ such that $u \in L^\infty(0,T\colon L^\infty(\mathbb{R}^N))$. Then we call u a solution of (1.1) in $\mathbb{R}^N \times (0,T)$ if u satisfies

$$u(x,t) = \int_{\mathbb{R}^N} \Gamma(x,y,t)u_0(y)\,dy + \int_0^t \int_{\mathbb{R}^N} \Gamma(x,y,t-s)u(y,s)^p\,dy\,ds < \infty \tag{1.2}$$

for almost all $x \in \mathbb{R}^N$ and $t \in (0,T)$. In particular, we call u a global-in-time solution of (1.1) if u is a solution of (1.1) in $\mathbb{R}^N \times (0,\infty)$.

The previous definition is the well-known class of *mild solutions* and is natural to tackle parabolic problems. A main point of the previous definition is that it involves the fundamental solution of the operator under consideration. It is important to notice that in our context, due to the non-homogeneity of the operator, the fundamental solution is not translation-invariant. Furthermore, there is no explicit expression of it, though bounds are known. This makes the theory harder.

Remark 1.1 (i) For the case of the semilinear heat equation, namely (1.1) with either $a = 0$ or $b = 0$, if $u_0 \in L^\infty(\mathbb{R}^N)$, then there exists a local-in-time solution u of (1.1) in $\mathbb{R}^N \times (0,T)$ for some $T > 0$ satisfying (1.2). See e.g. [22,40].

(ii) In order to prove the regularity of the solution satisfying (1.2), even for the case of the semilinear heat equation, we need suitable bounds for the derivatives and the translation-invariant property of the fundamental solution. (See e.g. [15,25].) However, unfortunately, it seems that they have been still left open. (See also Remark 1.3 (iii).) On the other hand, under our definition, in order to prove the existence/nonexistence of global-in-time solutions of (1.1), we only need properties (K1)–(K3) and decay estimates, which are given in Lemma 2.2.

We discuss now the features of the weight $w(x)$. In both cases under consideration, the weights belong to the class A_2 of Muckenhoupt functions [33]. This class of functions is very important in harmonic analysis for the boundedness of Maximal Functions. From the PDE point of view, elliptic equations and potential theory involving these weights have been studied in [10–12]. See also [7,23] for the parabolic counterpart. In the present work, we do not consider general weights since it is very complicated in this case to give precise results as our aim is. We will consider two types of weights. The first one is $|x_1|^a$ which is A_2 if and only if $a \in (-1, 1)$ and exhibits singularities along the line $x_1 = 0$. The other weight under consideration is $|x|^b$ which is A_2 for $b \in (-N, N)$ and exhibits a singularity at the origin $x = 0$.

Motivation

We now explain what motivated this work, in order to put it in perspective. Our original project was to develop a blow-up theory in the spirit of Giga and Kohn (see [18]) for the nonlocal parabolic equation

$$\partial_t u + (-\Delta)^s u = u^p, \quad x \in \mathbb{R}^N, \quad t > 0, \quad u(x, 0) = u_0(x) \geq 0, \quad x \in \mathbb{R}^N \tag{1.3}$$

As far as the Fujita problem for the previous equation is concerned, the theory is now more or less well-understood. Among others, Sugitani [38] showed the Fujita exponent for this problem (see also [25]). In the problem (1.3), the operator is non-local, but we can construct the fundamental solution by using the Fourier transform as $\mathcal{F}^{-1}(e^{-t|\xi|^s})$. In particular, for the case $s = 1/2$, we have the explicit representation of $\mathcal{F}^{-1}(e^{-t|\xi|^{1/2}})$, which is given by the Poisson kernel. (See e.g. [2,3,5,27,39] and, for some regularity estimates for the derivatives of the heat kernel associated to the fractional Laplacian on $\mathbb{R}^N$ can be found in [4].) For this case, even if we don't have the explicit representation of the kernel, applying the Hörmander–Mikhlin type multiplier theorem (see [37]), we can obtain point-wise estimates for the fundamental solution and its derivative with respect to x and t. (See e.g. [27, Lemma 2.1]).

The blow-up theory, however, is much more involved. Indeed, several attempts on the Eq. (1.3) to prove a monotonicity formula have failed and the very first beginning of Giga–Kohn theory is at the moment out of reach. On the other hand, if one considers the modified nonlocal equation

$$(\partial_t - \Delta)^s u = u^p, \quad x \in \mathbb{R}^N, \quad t \in \mathbb{R}, \tag{1.4}$$

then one can prove a monotonicity formula (see [1]). We postpone to future work the Fujita dichotomy for this problem.

Considering the problem (1.3) from another point of view, invoking Caffarelli–Silvestre extension [6], one gets a degenerate/singular elliptic equation in the half-space with a nonlinear dynamical boundary condition. To analyze this problem, one needs to compare this problem with the degenerate/singular parabolic equation with nonlinear boundary condition in the half space. From this point of view, before treating this half space problem, we consider the similar problem in the whole space. This is one of our motivations. Another one relies on expanding the theory of degenerate semilinear parabolic equations, which is at an early stage, particularly as far as blow-up theory is concerned. The case of degenerate weights along a line as $w(x) = |x_1|^a$ is particularly relevant.

We now describe our results. We first introduce some notations. For any $x \in \mathbb{R}^N$ and $R > 0$, we put $B_R(x) := \{y \in \mathbb{R}^N : |x - y| < R\}$. For any $1 \leq r \leq \infty$, we denote by $\|\cdot\|_r$ the usual norm of $L^r := L^r(\mathbb{R}^N)$. For any measurable function f in $\mathbb{R}^N$,

$$\mu_f(\lambda) := |\{x : |f(x)| > \lambda\}|, \quad \lambda \geq 0, \tag{1.5}$$

is the distribution function of f, and we define the non-increasing rearrangement of f by

$$f^*(s) := \inf\{\lambda > 0 : \mu_f(\lambda) \leq s\}. \tag{1.6}$$

The spherical rearrangement of f is defined by

$$f^{\sharp}(x) := f^*(c_N|x|^N),$$

where c_N is the volume of the unit ball in $\mathbb{R}^N$.

Then, for any $1 \le r \le \infty$ and $1 \le \sigma \le \infty$, we define the Lorentz space $L^{r,\sigma} := L^{r,\sigma}(\mathbb{R}^N)$ by

$$L^{r,\sigma} := \{f\colon f \text{ is measurable in } \mathbb{R}^N,\ \ \|f\|_{r,\sigma} < \infty\},$$

where

$$\|f\|_{r,\sigma} := \begin{cases} \left(\displaystyle\int_0^\infty \left[s^{\frac{1}{r}} f^*(s)\right]^\sigma \frac{ds}{s}\right)^{\frac{1}{\sigma}} & \text{if } 1 \le \sigma < \infty, \\ \displaystyle\sup_{s>0} s^{\frac{1}{r}} f^*(s) & \text{if } \sigma = \infty. \end{cases} \tag{1.7}$$

The Lorentz $L^{r,\sigma}$ is a Banach space and the following holds (see e.g. [19,43]):

- Let $1 < r < \infty$. Then $f \in L^{r,\infty}$ if and only if
$$0 \le f^{\sharp}(x) \le C_1|x|^{-N/r}, \quad x \in \mathbb{R}^N, \tag{1.8}$$
for some constant C_1;
- For $1 < r < \infty$, it follows
$$\|f\|_{r,\sigma} = \begin{cases} r^{\frac{1}{\sigma}} \left(\displaystyle\int_0^\infty \left[s\mu_f(s)^{\frac{1}{r}}\right]^\sigma \frac{ds}{s}\right)^{\frac{1}{\sigma}} & \text{if } 1 \le \sigma < \infty, \\ \displaystyle\sup_{s>0} s\mu_f(s)^{\frac{1}{r}} & \text{if } \sigma = \infty. \end{cases} \tag{1.9}$$
- $L^{r,r} = L^r$ if $1 < r \le \infty$ and $L^{r,\sigma_1} \subset L^{r,\sigma_1}$ if $1 \le r \le \infty$ and $1 \le \sigma_1 \le \sigma_2 \le \infty$;
- Let $1 \le r_0 \le r \le r_1 \le \infty$ be such that
$$\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1} \quad \text{for} \quad \theta \in [0, 1].$$
Then it holds that
$$\|f\|_{r,\infty} \le \|f\|_{r_0,\infty}^{1-\theta} \|f\|_{r_1,\infty}^{\theta}, \quad f \in L^{r_0,\infty} \cap L^{r_1\infty}; \tag{1.10}$$
- Let $1 \le r_1 \le \infty$ and r_2 be the Hölder conjugate number of r_1, namely $1/r_1 + 1/r_2 = 1$. Then it holds that
$$\|fg\|_1 \le \|f\|_{r_1,1}\|g\|_{r_2,\infty}, \quad f \in L^{r_1,1}, \quad g \in L^{r_2,\infty}; \tag{1.11}$$

Now we state the main results of this paper but several explanations are in order. In most of the parabolic problem dealing with homogeneous equations, a crucial role is played by the fundamental solution. It happens that one can deduce several strong results as soon as one has an explicit form of the fundamental solution, allowing to get estimates for the function and its derivatives (see e.g. [25–27,38]). In our problems, even if the coefficients are rather simple, such an explicit form is unavailable. On the other hand, bounds on the solution are known (see e.g. [8,20,21]). In order to apply known bounds one has to impose additional properties on the weights under consideration. More precisely, the weights have to belong to the $A_{1+\frac{2}{N}}$ class additionally to being A_2 and $w^{-N/2}$ has to satisfy a reverse doubling condition. We refer the reader to Sect. 2 for a discussion of these fact. In what follows, we put

$$p_*(\alpha) := 1 + \frac{2-\alpha}{N} \quad \text{for} \quad \alpha \in \{a, b\}.$$

Furthermore, we assume either

(A) $w(x) = |x_1|^a$ with $a \in [0, 1)$ if $N = 1, 2$ and $a \in [0, 2/N)$ if $N \geq 3$,

or

(B) $w(x) = |x|^b$ with $b \in [0, 1)$.

The first theorem is concerned with the nonexistence of global-in-time solutions of (1.1).

Theorem 1.1 *Assume either* (A) *or* (B). *Let* α *be such that* $\alpha = a$ *for the case* (A) *and* $\alpha = b$ *for the case* (B). *Assume* $1 < p \leq p_*(\alpha)$. *Then problem* (1.1) *has no nontrivial global-in-time solutions.*

In second theorem we give a sufficient condition for the existence of nontrivial global-in-time solutions of (1.1).

Theorem 1.2 *Assume either* (A) *or* (B). *Let* α *be such that* $\alpha = a$ *for the case* (A) *and* $\alpha = b$ *for the case* (B). *Assume* $p > p_*(\alpha)$. *Put*

$$r_* := \frac{N}{2-\alpha}(p-1) > 1. \tag{1.12}$$

Then the following holds:

(i) *There exists a positive constant* δ *such that, for any* $u_0 \in L^\infty \cap L^{r_*,\infty}$ *with*

$$\|u_0\|_{r_*,\infty} < \delta, \tag{1.13}$$

a unique global-in-time solution u *of* (1.1) *exists and it satisfies*

$$\sup_{t>0}\,(1+t)^{\frac{N}{2-\alpha}\left(\frac{1}{r_*}-\frac{1}{q}\right)}\|u(t)\|_{q,\infty} < \infty, \quad r_* \leq q \leq \infty. \tag{1.14}$$

(ii) *Let* $1 \leq r \leq r_*$. *Then there exists a positive constant* δ *such that, for any* $u_0 \in L^\infty \cap L^r$ *with*

$$\|u_0\|_r^{\frac{r}{r_*}}\|\varphi\|_\infty^{1-\frac{r}{r_*}} < \delta, \tag{1.15}$$

a unique global-in-time solution u *of* (1.1) *exists and it satisfies*

$$\sup_{t>0}\,(1+t)^{\frac{N}{2-\alpha}\left(\frac{1}{r}-\frac{1}{q}\right)}\|u(t)\|_q < \infty, \quad r \leq q \leq \infty. \tag{1.16}$$

Remark 1.2 As far as the regularity of the mild solutions constructed in this paper is concerned, Chiarenza and Serapioni [7] considered degenerate parabolic equations with A_2-weights. However, their starting point are weak solutions. To upgrade our mild solutions to weak solutions one needs gradient bounds on the fundamental solution Γ, which are not available.

As a direct consequence of Theorem 1.2, we have:

Corollary 1.1 *Let* $\alpha \in \{a, b\}$. *Assume* $p > p_*(\alpha)$. *Then there exists a positive constant* δ *such that, if*

$$|u_0(x)| \leq \frac{\delta}{1+|x|^{(2-\alpha)/(p-1)}}, \quad x \in \mathbb{R}^N, \tag{1.17}$$

then a unique global-in-time solution u *of* (1.1) *exists and it satisfies* (1.14).

Remark 1.3 (i) For the case $\alpha = 0$, it is well known that the decay rate for initial data, which is given by (1.17), at spatial infinity is optimal to obtain the global existence of solutions for (1.1) (see e.g. [30]). If $u_0(x)$ satisfies (1.17), then it follows from (1.8) that $u_0 \in L^{r_*,\infty}$. On the other hand, if $u_0(x) = O(|x|^{-(2-\alpha)/(p-1)})$ as $|x| \to \infty$, then $u_0 \notin L^{r_*}$. This is a clear advantage in using $L^{r,\infty}$ spaces instead of the classical L^r spaces.

(ii) Beginning with the classical paper by Fujita [16], critical exponents for the global existence of solutions (not only positive ones but also sign-changing ones) were established for many classes of evolution problems. It seems almost impossible to make complete list of this topics. So we only refer a part of them for instance [9,17,24,25,28,31,32,34,41,42] and references therein. (See also [36], which includes a nice survey for the semilinear parabolic equation.) By Theorems 1.1 and 1.2 we see that $p_*(\alpha)$ is the Fujita exponent for problem (1.1). In fact, if $\alpha = 0$, then $p_*(0) = 1 + 2/N$, which is the Fujita exponent for (1.1) with $w(x) \equiv 1$.

(iii) If we have suitable bounds for the derivatives of the fundamental solution, then, applying the arguments in [25], we can obtain the asymptotic behavior of solutions for (1.1). However, unfortunately, it seems that they have been still left open.

(iv) By Theorem 1.2, for suitable small initial data, the solution exists globally in time and it is bounded. On the other hand, in general, it seems difficult to prove the boundedness of solutions even if the solution exists globally in time. In fact, for the case $\alpha = 0$, if $p < p_S := (N+2)/(N-2)$ and u_0 belongs to a weighted H^1 space, then global-in-time solutions are bounded (see e.g. [29]), and if p is critical or supercritical in the sense of Joseph-Lundgren, then there exists a continuous function u_0 such that solution u is global and $\lim_{t\to\infty} \|u(t)\|_\infty = \infty$ (see [35, Theorem 1.3]).

A digression on another problem with weighted degenerate diffusion.
We would like to mention here another problem related to ours, but substantially different at the linear level. Consider the degenerate equation

$$u_t - L_w u = u^p \tag{1.18}$$

where $L_w = w^{-1}\operatorname{div}(w\nabla)$ is a *self-adjoint* operator on $L^2(w)$ and w is an A_2 weight. In this case, Cruz-Uribe and Rios proved the following Gaussian bounds (see Corrigendum of [8]) for the fundamental solution

$$|\Gamma(x, y, t)| \le \frac{C_1}{\sqrt{w_t(x)}\sqrt{w_t(y)}} e^{-C_2 \frac{|x-y|^2}{t}} \tag{1.19}$$

where $w_t(x) = \int_{B_{\sqrt{t}}(x)} w(z)\,dz$. Notice that such a bound is possible because the operator L_w is self-adjoint in $L^2(w)$. This is not the case of the operator in (1.1). Seeing the homogeneous space $(\mathbb{R}^N, w(x)\,dx)$ as Ahlfors-regular, i.e. there exists $s > 0$ such that $w(B_R(x)) = \int_{B_R(x)} w(z)\,dz \sim R^s$ uniformly in x, one can run the same estimates as in the present paper and exhibit a Fujita exponent.

2 Preliminaries

A crucial tool in our arguments is based on the use of the fundamental solution of the operator $\partial_t - \operatorname{div}(w(x)\nabla\cdot)$. As already mentioned due to the inhomogeneity of the operator, an explicit formula is not known but bounds are available (see below). In order to check these bounds, following [20], one has to check that the coefficient $w(x)$ is a $A_{1+\frac{2}{N}}$ weight in the sense of

Muckenhoupt class and that the function $w^{-N/2}$ satisfies a doubling and reverse doubling condition of order μ with $\mu > 1/2$. Here we say that the function $w^{-N/2}$ satisfies doubling and a reverse doubling conditions if there exist positive constants C_1 and C_2 such that

$$\int_{B_{sR}(x)} w(y)^{-\frac{N}{2}} dy \le C_1 s^{\mu N} \int_{B_R(x)} w(y)^{-\frac{N}{2}} dy$$

and

$$\int_{B_{sR}(x)} w(y)^{-\frac{N}{2}} dy \ge C_2 s^{\mu N} \int_{B_R(x)} w(y)^{-\frac{N}{2}} dy$$

for all $x \in \mathbb{R}^N$, $s \ge 1$ and $R > 0$, respectively.

It is a direct computation to check that under condition (A) or (B) depending on the case, the weight $w(x)$ is an $A_{1+\frac{2}{N}}$ weight in the sense of Muckenhoupt class. Furthermore, the function $w^{-N/2}$ satisfies a doubling condition and reverse doubling condition of order μ with $\mu > 1/2$.

Under condition either (A) or (B), the fundamental solution $\Gamma = \Gamma(x, y, t)$ has the following properties (see [20]):

(K1) $\displaystyle\int_{\mathbb{R}^N} \Gamma(x, y, t)\, dx = \int_{\mathbb{R}^N} \Gamma(x, y, t)\, dy = 1$ for $x, y \in \mathbb{R}^N$ and $t > 0$;

(K2) $\displaystyle\Gamma(x, y, t) = \int_{\mathbb{R}^N} \Gamma(x, \xi, t - s)\, \Gamma(\xi, y, s)\, d\xi$ for $x, y \in \mathbb{R}^N$ and $t > s > 0$;

(K3) Put

$$c_0 := \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)\, dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1}\, dx \right) < \infty,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^N$. Then there exist positive constants c_* and C_* depending only on N and c_0 such that

$$\begin{aligned} c_*^{-1} \left(\frac{1}{[h_x^{-1}(t)]^N} + \frac{1}{[h_y^{-1}(t)]^N} \right) e^{-c_* \left(\frac{h_x(|x-y|)}{t} \right)^{\frac{1}{1-\alpha}}} &\le \Gamma(x, y, t) \\ &\le C_*^{-1} \left(\frac{1}{[h_x^{-1}(t)]^N} + \frac{1}{[h_y^{-1}(t)]^N} \right) e^{-C_* \left(\frac{h_x(|x-y|)}{t} \right)^{\frac{1}{1-\alpha}}} \end{aligned}$$

for $x, y \in \mathbb{R}^N$ and $t > 0$, where $\alpha \in \{a, b\}$. Here

$$h_x(r) = \left(\int_{B_r(x)} w(y)^{-\frac{N}{2}}\, dy \right)^{\frac{2}{N}} \tag{2.1}$$

and h_x^{-1} denotes the inverse function of h_x.

See [20]. (See also [8,21].) By (A), (B) and (2.1) we state a lemma on upper and lower estimates of $h_x(r)$. In what follows, by the letters C and C' we denote generic positive constants (independent of x and t) and they may have different values also within the same line.

Lemma 2.1 *The following hold.*

(i) *Let* $w(x) = |x_1|^a$ *and assume condition* (A). *There exist positive constants* C *and* C' *depending only on* N *and* a *such that*

$$h_x(r) \le Cr^{2-a} \tag{2.2}$$

and

$$h_x(r) \ge C' \begin{cases} r^2|x_1|^{-a} & \text{if } 0 < r \le |x_1|, \\ r^{2-a} & \text{if } r \ge |x_1|, \end{cases} \tag{2.3}$$

for all $x \in \mathbb{R}^N$ *and* $r > 0$.

(ii) *Let* $w(x) = |x|^b$ *and assume condition* (B). *There exist positive constants* C *and* C' *depending only on* N *and* b *such that*

$$h_x(r) \le Cr^{2-b} \tag{2.4}$$

and

$$h_x(r) \ge C' \begin{cases} r^2|x|^{-b} & \text{if } 0 < r \le |x|, \\ r^{2-b} & \text{if } r \ge |x|, \end{cases} \tag{2.5}$$

for all $x \in \mathbb{R}^N$ *and* $r > 0$, *where* C_2 *is a constant given in* $(W2)$.

Proof We first prove assertion (i).

Note that since the constant c_0 in (K3) depends only on N and a, the constants appearing in (K3) depend only on N and a.

Since $w(y)^{-N/2}$ is monotonically decreasing function with respect to the distance from the origin, by (2.1) we have

$$\begin{aligned} h_x(r) &= \left(\int_{B_r(x)} |y_1|^{-\frac{aN}{2}}\, dy \right)^{\frac{2}{N}} \le \left(\int_{B_r(0)} |y_1|^{-\frac{aN}{2}}\, dy \right)^{\frac{2}{N}} \\ &= \left[\int_{-r}^{r} |y_1|^{-\frac{aN}{2}} \left(\int_{|y'|_{N-1} < \sqrt{r^2 - y_1^2}} dy' \right) dy_1 \right]^{\frac{2}{N}} = \left(2\omega_{N-1} \int_0^r y_1^{-\frac{aN}{2}} (r^2 - y_1^2)^{\frac{N-1}{2}}\, dy_1 \right)^{\frac{2}{N}} \\ &= \left(\omega_{N-1} \cdot r^{N-\frac{aN}{2}} \int_0^1 \zeta^{-\frac{aN}{4}-\frac{1}{2}} (1-\zeta)^{\frac{N-1}{2}}\, d\zeta \right)^{\frac{2}{N}} \\ &= \left[\omega_{N-1} B\left(-\frac{aN}{4} + \frac{1}{2}, \frac{N+1}{2} \right) \right]^{\frac{2}{N}} r^{2-a} \end{aligned}$$

for all $x \in \mathbb{R}^N$ and $r > 0$, where $y = (y_1, y') \in \mathbb{R}^N$, $|\cdot|_{N-1}$ denotes the usual Euclidean norm in $\mathbb{R}^{N-1}$, ω_{N-1} denotes the volume of the unit ball in $\mathbb{R}^{N-1}$ and $B(\cdot,\cdot)$ denotes the beta function. Note that $-aN/4 + 1/2 > 0$ since $a < 2/N$.

This implies (2.2). On the other hand, since $w(y)$ depends only on y_1 variable, for any $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1}$, we can choose a point $x_* = (x_1, 0)$ such that

$$\int_{B_r(x)} w(y)^{-\frac{N}{2}}\, dy = \int_{B_r(x_*)} w(y)^{-\frac{N}{2}}\, dy, \quad r > 0. \tag{2.6}$$

Furthermore, for any $r > 0$, we see that

$$|y_1| \le |x_1| + r, \quad y \in B_r(x_*). \tag{2.7}$$

Since $a \geq 0$, by (2.6) and (2.7), for any $x \in \mathbb{R}^N$ and $r > 0$, we have

$$\int_{B_r(x)} w(y)^{-\frac{N}{2}}\,dy = \int_{B_r(x_*)} |y_1|^{-\frac{aN}{2}}\,dy$$
$$\geq (|x_1|+r)^{-\frac{aN}{2}} \int_{B_r(x_*)} dy = \omega_N r^N (|x_1|+r)^{-\frac{aN}{2}}.$$

This together with (2.1) yields (2.3). Thus assertion (i) holds.

Next we prove assertion (ii). Since $w(y)^{-N/2}$ is monotonically decreasing function with respect to the distance from the origin, by (2.1) we have

$$h_x(r) = \left(\int_{B_r(x)} |y|^{-\frac{bN}{2}}\,dy\right)^{\frac{2}{N}} \leq \left(\int_{B_r(0)} |y|^{-\frac{bN}{2}}\,dy\right)^{\frac{2}{N}}$$
$$= \left(\int_0^r \rho^{N-\frac{bN}{2}-1}\,d\rho \cdot \int_{\mathbb{S}^{N-1}} d\omega\right)^{\frac{2}{N}} = \left(\frac{\omega_N}{1-\frac{b}{2}}\right)^{\frac{2}{N}} r^{2-b}$$

for all $x \in \mathbb{R}^N$ and $r > 0$. This implies (2.4). On the other hand, for any $r > 0$, we see that

$$|y| \leq |x| + r, \quad y \in B_r(x). \tag{2.8}$$

Since $b \geq 0$, by (2.6) and (2.8), for any $x \in \mathbb{R}^N$ and $r > 0$, we have

$$\int_{B_r(x)} w(y)^{-\frac{N}{2}}\,dy = \int_{B_r(x)} |y|^{-\frac{bN}{2}}\,dy$$
$$\geq (|x|+r)^{-\frac{bN}{2}} \int_{B_r(x)} dy = \omega_N r^N (|x|+r)^{-\frac{bN}{2}}.$$

This together with (2.1) yields (2.5). Thus assertion (ii) holds, and Lemma 2.1 follows. □

For any $x \in \mathbb{R}^N$, since $w(x) \geq 0$, by (2.1) we can easily obtain that

$$\frac{d}{dr} h_x(r) > 0 \tag{2.9}$$

for all $r > 0$. Then, in the case (A), by (2.2) and (2.9) we have

$$h_x^{-1}(Cr^{2-a}) \geq r$$

for all $x \in \mathbb{R}^N$ and $r > 0$, and we see that

$$h_x^{-1}(t) \geq Ct^{\frac{1}{2-a}} \tag{2.10}$$

for all $x \in \mathbb{R}^N$ and $t > 0$. Similarly, by (2.3) we see that

$$h_x^{-1}(t) \leq C \begin{cases} |x_1|^{\frac{a}{2}} t^{\frac{1}{2}} & \text{if } 0 < t \leq |x_1|^{2-a}, \\ t^{\frac{1}{2-a}} & \text{if } t \geq |x_1|^{2-a}, \end{cases}$$

for all $x \in \mathbb{R}^N$ and $t > 0$. This together with Lemma 2.1, (K3) and (2.10) implies that

$$\begin{aligned} & d^{-1}\left(\min\left\{|x_1|^{-\frac{aN}{2}}, t^{-\frac{aN}{2}\frac{1}{2-a}}\right\} + \min\left\{|y_1|^{-\frac{aN}{2}}, t^{-\frac{aN}{2}\frac{1}{2-a}}\right\}\right) t^{-\frac{N}{2}} e^{-d\left(\frac{|x-y|^{2-a}}{t}\right)^{\frac{1}{1-a}}} \\ & \le \Gamma(x,y,t) \le D^{-1} t^{-\frac{N}{2-a}} \begin{cases} e^{-D\left(\frac{|x-y|^2}{t|x_1|^a}\right)^{\frac{1}{1-a}}} & \text{if } |x-y| \le |x_1|, \\ e^{-D\left(\frac{|x-y|^{2-a}}{t}\right)^{\frac{1}{1-a}}} & \text{if } |x-y| > |x_1|, \end{cases} \end{aligned} \tag{2.11}$$

for $x, y \in \mathbb{R}^N$ and $t > 0$. Here D and d are positive constant depending only on N and a. Similarly to (2.11), in the case (B), we see that

$$\begin{aligned} & d^{-1}\left(\min\left\{|x|^{-\frac{bN}{2}}, t^{-\frac{bN}{2}\frac{1}{2-b}}\right\} + \min\left\{|y|^{-\frac{bN}{2}}, t^{-\frac{bN}{2}\frac{1}{2-b}}\right\}\right) t^{-\frac{N}{2}} e^{-d\left(\frac{|x-y|^{2-b}}{t}\right)^{\frac{1}{1-b}}} \\ & \le \Gamma(x,y,t) \le D^{-1} t^{-\frac{N}{2-b}} \begin{cases} e^{-D\left(\frac{|x-y|^2}{t|x|^b}\right)^{\frac{1}{1-b}}} & \text{if } |x-y| \le |x|, \\ e^{-D\left(\frac{|x-y|^{2-b}}{t}\right)^{\frac{1}{1-b}}} & \text{if } |x-y| > |x|, \end{cases} \end{aligned} \tag{2.12}$$

for $x, y \in \mathbb{R}^N$ and $t > 0$. Here D and d are positive constant depending only on N and b. By (2.11) and (2.12) we obtain

$$\Gamma(x,y,t) \le D^{-1} t^{-\frac{N}{2-\alpha}}$$

for $x, y \in \mathbb{R}^N$ and $t > 0$. This together with (K1) implies that

$$\|\Gamma(\cdot,y,t)\|_r \le C t^{-\frac{N}{2-\alpha}(1-\frac{1}{r})}, \quad \|\Gamma(x,\cdot,t)\|_r \le C t^{-\frac{N}{2-\alpha}\left(1-\frac{1}{r}\right)}, \tag{2.13}$$

for any $1 \le r \le \infty$, where we can take the constant C so that it depends only on N and $\alpha \in \{a, b\}$. Furthermore, we have the following.

Remark 2.1 It has to be noticed that the previous computations, and in particular Lemma 2.1 are the cornerstone of our results since they provide the desired estimates to run the existence/nonexistence proof.

Lemma 2.2 *Assume either* (A) *or* (B). *Let α be such that $\alpha = a$ for the case* (A) *and $\alpha = b$ for the case* (B). *Then, for any $1 \le r < \infty$, there exists a positive constant C depending only on α, r and N such that*

$$\|\Gamma(x,\cdot,t)\|_{r,1} \le C t^{-\frac{N}{2-\alpha}\left(1-\frac{1}{r}\right)}, \tag{2.14}$$

for $x \in \mathbb{R}^N$ and $t > 0$.

Proof By (2.11) and (2.12), if we can prove (2.14) for the case (A), then, replacing $|x_1|$ and a with $|x|$ and b, respectively, we can obtain (2.14) for the case (B). So it suffices to prove (2.14) for the case (A).

Assume (A). For any $x \in \mathbb{R}^N$ and $t > 0$, put

$$f(y) := D^{-1} t^{-\frac{N}{2-a}} \begin{cases} e^{-D\left(\frac{|x-y|^2}{t|x_1|^a}\right)^{\frac{1}{1-a}}} & \text{if } |x-y| \le |x_1|, \\ e^{-D\left(\frac{|x-y|^{2-a}}{t}\right)^{\frac{1}{1-a}}} & \text{if } |x-y| > |x_1|, \end{cases} \tag{2.15}$$

for $y \in \mathbb{R}^N$, where D is the constant given in (2.11). Let $\lambda > 0$. If

$$D^{-1} t^{-\frac{N}{2-a}} e^{-D\left(\frac{|x_1|^{2-a}}{t}\right)^{\frac{1}{1-a}}} \le \lambda \le D^{-1} t^{-\frac{N}{2-a}},$$

then, by (1.5) we obtain

$$\mu_f(\lambda) = \left|\{y : |f(y)| > \lambda\}\right| = (t|x_1|^a)^{\frac{N}{2}} \left[-\frac{1}{D} \log\left(D\lambda t^{\frac{N}{2-a}}\right)\right]^{\frac{(1-a)N}{2}}, \tag{2.16}$$

and if

$$\lambda < D^{-1} t^{-\frac{N}{2-a}} e^{-D\left(\frac{|x_1|^{2-a}}{t}\right)^{\frac{1}{1-a}}},$$

then, by (1.5) and (2.15) we have

$$\mu_f(\lambda) = \left|\{y : |f(y)| > \lambda\}\right| = t^{\frac{N}{2-a}} \left[-\frac{1}{D} \log\left(D\lambda t^{\frac{N}{2-a}}\right)\right]^{\frac{(1-a)N}{2-a}}. \tag{2.17}$$

By (1.6), (2.16) and (2.17) we see that

$$f^*(s) = \begin{cases} D^{-1} t^{-\frac{N}{2-a}} e^{-D\left(s(t|x_1|^a)^{-\frac{N}{2}}\right)^{\frac{2}{(1-a)N}}} & \text{if } s \ge |x_1|^N, \\ D^{-1} t^{-\frac{N}{2-a}} e^{-D\left(s t^{-\frac{N}{2-a}}\right)^{\frac{2-a}{(1-a)N}}} & \text{if } s < |x_1|^N. \end{cases} \tag{2.18}$$

Then, by (1.7) and (2.18) we have

$$\begin{aligned}
\|f\|_{r,1} &= \int_0^\infty s^{\frac{1}{r}-1} f^*(s)\, ds \\
&= D^{-1} t^{-\frac{N}{2-a}} \int_0^{|x_1|^N} s^{\frac{1}{r}-1} e^{-D\left(s t^{-\frac{N}{2-a}}\right)^{\frac{2-a}{(1-a)N}}}\, ds \\
&\quad + D^{-1} t^{-\frac{N}{2-a}} \int_{|x_1|^N}^\infty s^{\frac{1}{r}-1} e^{-D\left(s(t|x_1|^a)^{-\frac{N}{2}}\right)^{\frac{2}{(1-a)N}}}\, ds \\
&\le D^{-1} t^{-\frac{N}{2-a}\left(1-\frac{1}{r}\right)} \int_0^\infty \zeta^{\frac{1}{r}-1} e^{-D\zeta^{\frac{2-a}{(1-a)N}}}\, d\zeta \\
&\quad + D^{-1} t^{-\frac{N}{2-a}} (t|x_1|^a)^{\frac{N}{2r}} \int_{(t^{-1}|x_1|^{2-a})^{N/2}}^\infty \zeta^{\frac{1}{r}-1} e^{-D\zeta^{\frac{2}{(1-a)N}}}\, d\zeta \\
&\le C t^{-\frac{N}{2-a}\left(1-\frac{1}{r}\right)} + D^{-1} t^{-\frac{N}{2-a}} (t|x_1|^a)^{\frac{N}{2r}} e^{-\frac{D}{2}\left(\frac{|x_1|^{2-a}}{t}\right)^{\frac{1}{1-a}}} \int_0^\infty \zeta^{\frac{1}{r}-1} e^{-\frac{D}{2}\zeta^{\frac{2}{(1-a)N}}}\, d\zeta \\
&\le C t^{-\frac{N}{2-a}\left(1-\frac{1}{r}\right)} + C t^{-\frac{N}{2-a}} (t|x_1|^a)^{\frac{N}{2r}} e^{-\frac{D}{2}\left(\frac{|x_1|^{2-a}}{t}\right)^{\frac{1}{1-a}}}
\end{aligned} \tag{2.19}$$

for any $1 \le r < \infty$, where C depends only on a, r and N. Since $s^{\frac{N}{2r}\frac{a}{2-a}} e^{-\frac{D}{2}s^{\frac{1}{1-a}}} \le C$ for all $s \ge 0$, we see that

$$\begin{aligned} & t^{-\frac{N}{2-a}} (t|x_1|^a)^{\frac{N}{2r}} e^{-\frac{D}{2}\left(\frac{|x_1|^{2-a}}{t}\right)^{\frac{1}{1-a}}} \\ & \le t^{-\frac{N}{2-a}+\frac{N}{2r}\left(1+\frac{a}{2-a}\right)} \left(\frac{|x_1|^{2-a}}{t}\right)^{\frac{N}{2r}\frac{a}{2-a}} e^{-\frac{D}{2}\left(\frac{|x_1|^{2-a}}{t}\right)^{\frac{1}{1-a}}} \le Ct^{-\frac{N}{2-a}(1-\frac{1}{r})}. \end{aligned}$$

This together with (2.19) implies that

$$\|f\|_{r,1} \le Ct^{-\frac{N}{2-a}(1-\frac{1}{r})}, \quad r \in [1,\infty). \tag{2.20}$$

By (2.11) and (2.15) we see that $\Gamma(x, y, t) \le f(y)$ for $x, y \in \mathbb{R}^N$ and $t > 0$, and it follows from (2.20) that

$$\|\Gamma(x,\cdot,t)\|_{r,1} \le \|f\|_{r,1} \le Ct^{-\frac{N}{2-a}(1-\frac{1}{r})}, \quad r \in [1,\infty),$$

for $x \in \mathbb{R}^N$ and $t > 0$. Thus the proof of Lemma 2.2 is complete. □

For any measurable function φ, we put

$$[S(t)\varphi](x) := \int_{\mathbb{R}^N} \Gamma(x, y, t)\varphi(y)\, dy \tag{2.21}$$

for all $x \in \mathbb{R}^N$ and $t > 0$.

We will prove L^q–L^r estimate and $L^{q,\infty}$–$L^{r,\infty}$ estimate for $S(t)\varphi$. Since the fundamental solution Γ is not translation-invariant, we can not apply the usual Young inequality and weak Young inequality to get these estimates. We can, however, prove the following estimates with a slight modification of the proof of L^q–L^r and $L^{q,\infty}$–$L^{r,\infty}$ estimates for the solution of the heat equation.

(G1) For any $\varphi \in L^q$ and $1 \le q \le r \le \infty$, it holds that

$$\|S(t)\varphi\|_r \le c_1 t^{-\frac{N}{2-a}(\frac{1}{q}-\frac{1}{r})}\|\varphi\|_q, \quad t > 0.$$

Here c_1 can be taken so that it depends only on N and $\alpha \in \{a, b\}$.

(G2) For any $\varphi \in L^{q,\infty}$ with $1 < q \le \infty$ and $q \le r \le \infty$, it holds that

$$\|S(t)\varphi\|_{r,\infty} \le c_2 t^{-\frac{N}{2-a}(\frac{1}{q}-\frac{1}{r})}\|\varphi\|_{q,\infty}, \quad t > 0.$$

Here c_2 can be taken so that it depends only on q, N and $\alpha \in \{a, b\}$. In particular, the constant c_2 is bounded in $q \in (1+\epsilon, \infty)$ for any fixed $\epsilon > 0$ and $c_2 \to \infty$ as $q \to 1$.

Proof of (G1) Fix $t > 0$. Then, by the Hölder inequality and (2.13) we have

$$\|S(t)\varphi\|_\infty \le Ct^{-\frac{N}{2-\alpha}\cdot\frac{1}{q}}\|\varphi\|_q$$

for any $1 \le q \le \infty$, where C depends only on N and α. Furthermore, by (K1) we apply the Jensen inequality and the Fubini theorem to obtain

$$\begin{aligned} \|S(t)\varphi\|_q^q &= \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \Gamma(x, y, t)\varphi(y)\, dy\right)^q dx \le \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \Gamma(x, y, t)\varphi(y)^q\, dy\right) dx \\ &= \int_{\mathbb{R}^N} \varphi(y)^q \left(\int_{\mathbb{R}^N} \Gamma(x, y, t)\, dx\right) dy = \|\varphi\|_q^q. \end{aligned}$$

These imply that

$$\|S(t)\varphi\|_r \le \|S(t)\varphi\|_\infty^{\frac{r-q}{r}} \|S(t)\varphi\|_q^{\frac{q}{r}} \le C^{\frac{r-q}{r}} t^{-\frac{N}{2-\alpha}\cdot(\frac{1}{q}-\frac{1}{r})}\|\varphi\|_q.$$

The constant $C^{\frac{r-q}{r}}$ is bounded by the constant depending only on N and α for all $1 \le q \le r \le \infty$, so we can prove property (G1). □

Before proving property (G2), we prepare the following lemma.

Lemma 2.3 *Let $1 < r \le \infty$. Assume $\varphi \in L^{r,\infty}$. Then there exists a positive constant C_r depending only on r such that*

$$\|S(t)\varphi\|_{r,\infty} \le C_r\|\varphi\|_{r,\infty}, \quad t > 0. \tag{2.22}$$

The constant C_r is bounded in $1+\epsilon < r < \infty$ for any fixed $\epsilon > 0$ and $C_r \to \infty$ as $r \to 1$.

Proof The proof of this lemma is almost same as in the proof of [19, Theorem 1.2.13], which gives the Young inequality for weak type spaces. For the completeness of this paper, we give it here.

In case of $r = \infty$, since $L^{\infty,\infty} = L^\infty$, (2.22) holds true with the constant $C_r = 1$. We consider the case where $1 < r < \infty$. Let M be a positive real number to be chosen later. Put $M_- := \{x : |\varphi| \le M\}$ and $M_+ := \{x : |\varphi| > M\}$. Then we can define

$$\varphi_1 := \varphi\chi_{M_-}, \quad \varphi_2 := \varphi\chi_{M_+},$$

where χ_E is the characteristic function of E. By the fundamental properties of $\mu_f(\lambda)$ we have

$$\mu_{\varphi_1}(\lambda) = \begin{cases} 0 & \text{if } \lambda \ge M, \\ \mu_\varphi(\lambda) - \mu_\varphi(M) & \text{if } \lambda < M, \end{cases} \quad \mu_{\varphi_2}(\lambda) = \begin{cases} \mu_\varphi(\lambda) & \text{if } \lambda \ge M, \\ \mu_\varphi(M) & \text{if } \lambda < M, \end{cases} \tag{2.23}$$

and since $\{x : S(t)\varphi > \lambda\} \subset \{x : S(t)\varphi_1 > \lambda/2\} \cup \{x : S(t)\varphi_2 > \lambda/2\}$ for $\lambda > 0$, we have

$$\mu_{S(t)\varphi}(\lambda) \le \mu_{S(t)\varphi_1}(\lambda/2) + \mu_{S(t)\varphi_2}(\lambda/2) \tag{2.24}$$

for all $t > 0$. Then, by (1.9) and (2.23) we obtain

$$\begin{aligned} \|\varphi_2\|_1 &= \int_0^\infty \mu_{\varphi_2}(\lambda)\,d\lambda \\ &= \int_0^M \mu_\varphi(M)\,d\lambda + \int_M^\infty \mu_\varphi(\lambda)\,d\lambda \\ &\le M\mu_\varphi(M) + \int_M^\infty \lambda^{-r}\|\varphi\|_{r,\infty}^r\,d\lambda \\ &\le M^{1-r}\|\varphi\|_{r,\infty}^r + \frac{1}{r-1}M^{1-r}\|\varphi\|_{r,\infty}^r = \frac{r}{r-1}M^{1-r}\|\varphi\|_{r,\infty}^r. \end{aligned}$$

This together with (K1) implies that

$$\|S(t)\varphi_2\|_1 \le \|\varphi_2\|_1 \le \frac{r}{r-1}M^{1-r}\|\varphi\|_{r,\infty}^r \tag{2.25}$$

for all $t > 0$. On the other hand, applying the Hölder inequality with (K1), we see that

$$|[S(t)\varphi_1](x)| \le \|\Gamma(x,\cdot,t)\|_{L^1}\|\varphi_1\|_\infty \le M \tag{2.26}$$

for all $x \in \mathbb{R}^N$ and $t > 0$.

Let $\nu > 0$ and fix it. Taking the constant M as $M = \nu/2$, by (2.26) we have

$$\mu_{S(t)\varphi_1}(\nu/2) = 0, \quad t > 0.$$

Then, applying the Chebyshev inequality with (2.24) and (2.25)

$$\mu_{S(t)\varphi}(\nu) \le \mu_{S(t)\varphi_2}(\nu/2) \le \frac{2}{\nu}\|S(t)\varphi_2\|_1 \le \frac{2^r r}{r-1}\nu^{-r}\|\varphi\|_{r,\infty}^r$$

for all $t > 0$. Since the above inequality holds for all $\nu > 0$, by (1.9) we obtain (2.22) with $C_r = 2(r/(r-1))^{\frac{1}{r}}$, and the constant C_r is bounded as $r \to \infty$. Thus the proof of Lemma 2.3 is complete. □

Proof of (G2) For any $1 < q \le \infty$, it follows from (1.11) and (2.14) that

$$\|S(t)\varphi\|_\infty \le \sup_{x\in\mathbb{R}^N} \|\Gamma(x,\cdot,t)\varphi\|_1 \le \|\Gamma(x,\cdot,t)\|_{\frac{q}{q-1},1}\|\varphi\|_{q,\infty} \le Ct^{-\frac{N}{2-\alpha}\cdot\frac{1}{q}}\|\varphi\|_{q,\infty} \quad (2.27)$$

for all $t > 0$, where C depends only on α, q and N. Therefore, combining (1.10), (2.22) and (2.27), we have

$$\|S(t)\varphi\|_{r,\infty} \le \|S(t)\varphi\|_{q,\infty}^{1-\theta}\|S(t)\varphi\|_\infty^\theta \le C_r^{1-\theta}C^\theta t^{-\frac{N}{2-\alpha}\cdot\frac{\theta}{q}}\|\varphi\|_{q,\infty}$$

where $\theta \in [0,1]$ satisfies $1/r = (1-\theta)/q$. Since $\theta/q = 1/q - 1/r$, we obtain property (G2). □

Furthermore, by (2.11) and (2.12) we have the following lemmas.

Lemma 2.4 *Assume same conditions as in Lemma 2.2. Let $\varphi \in L^\infty$ be a non-trivial measurable function such that $\varphi \ge 0$ in $\mathbb{R}^N$. Then there exists a positive constant C depending only on α and N such that*

$$[S(t)\varphi](x) \ge C^{-1}t^{-\frac{N}{2-\alpha}}\int_{|y|\le t^{\frac{1}{2-\alpha}}}\varphi(y)\,dy$$

for $|x| \le t^{\frac{1}{2-\alpha}}$ and $t > 0$.

Proof Since it follows from $\alpha < 1$ that

$$|x-y|^{2-\alpha} \le 2^{1-\alpha}(|x|^{2-\alpha}+|y|^{2-\alpha}),$$

by (2.11) and (2.12) we can find positive constant C depending only on α and N such that

$$\Gamma(x,y,t) \ge Ct^{-\frac{N}{2-\alpha}}$$

for $|x|, |y| \le t^{\frac{1}{2-\alpha}}$ and $t > 0$.

Then, by (2.21) we have

$$[S(t)\varphi](x) \ge \int_{|y|\le t^{\frac{1}{2-\alpha}}}\Gamma(x,y,t)\varphi(y)\,dy \ge Ct^{-\frac{N}{2-\alpha}}\int_{|y|\le t^{\frac{1}{2-\alpha}}}\varphi(y)\,dy$$

for all $|x| \le t^{\frac{1}{2-\alpha}}$ and $t > 0$.

Then Lemma 2.4 follows from (2.21). □

3 Proof of Theorem 1.1

In what follows, we assume either (A) or (B). Furthermore, let α be such that $\alpha = a$ for the case (A) and $\alpha = b$ for the case (B).

In this section we prove Theorem 1.1, which means that problem (1.1) has no nontrivial global solutions in the case $1 < p \le p_*(\alpha)$. The proof of Theorem 1.1 is based on the arguments of [41, Theorem 5] and [42, Theorem 1] (see also [14, Theorem 1.1]).

We first prove the following lemma.

Lemma 3.1 *Let u be a solution of (1.1) in $\mathbb{R}^N \times (0, T)$ with $0 < T \le \infty$. Then there exists a constant C^* depending only on p such that*

$$t^{\frac{1}{p-1}} \|S(t)u_0\|_\infty \le C^* \tag{3.1}$$

for any $t \in [0, T)$.

Proof This lemma follows from the proof of [41, Theorem 5]. For completeness of this paper, we will add the proof of it.

Since it follows from (2.11) and (2.12) that the fundamental solution Γ is positive for $x, y \in \mathbb{R}^N$ and $t > 0$, by (1.2) and (2.21) we have

$$[S(t)u_0](x) \le u(x, t) < \infty \tag{3.2}$$

for almost all $x \in \mathbb{R}^N$ all $t \in (0, T)$. This together with (1.2) again implies

$$u(x, t) \ge \int_0^t \int_{\mathbb{R}^N} \Gamma(x, y, t-s) \left([S(s)u_0](y)\right)^p \, dy \, ds \tag{3.3}$$

for almost all $x \in \mathbb{R}^N$ and all $t \in (0, T)$. Then, applying the Jensen inequality with the aid of (K1) and (K2) to (3.3), we obtain

$$\begin{aligned} u(x, t) &\ge \int_0^t \left(\int_{\mathbb{R}^N} \Gamma(x, y, t-s)[S(s)u_0](y) \, dy \right)^p ds \\ &= \int_0^t \left(\int_{\mathbb{R}^N} \Gamma(x, y, t-s) \int_{\mathbb{R}^N} \Gamma(y, z, s) u_0(z) \, dz \, dy \right)^p ds \\ &= \int_0^t \left[\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \Gamma(x, y, t-s)\Gamma(y, z, s) \, dy \right) u_0(z) \, dz \right]^p ds \\ &= \int_0^t \left(\int_{\mathbb{R}^N} \Gamma(x, z, t) u_0(z) \, dz \right)^p ds = t([S(t)u_0](x))^p \end{aligned} \tag{3.4}$$

for almost all $x \in \mathbb{R}^N$ and all $t \in (0, T)$. We repeat the above argument with (3.2) replaced by (3.4), and have

$$\begin{aligned} u(x, t) &\ge \int_0^t \int_{\mathbb{R}^N} \Gamma(x, y, t-s) \Big(s([S(s)u_0](y))^p \Big)^p dy \, ds \\ &\ge \int_0^t s^p \left(\int_{\mathbb{R}^N} \Gamma(x, y, t-s)[S(s)u_0](y) dy \right)^{p^2} ds = \frac{1}{p+1} t^{p+1} ([S(t)u_0](x))^{p^2} \end{aligned}$$

for almost all $x \in \mathbb{R}^N$ and all $t \in (0, T)$. Repeating the above argument, for any $k = 2, 3, \ldots,$ it holds that

$$u(x, t) \ge A_k t^{\frac{p^k - 1}{p-1}} ([S(t)u_0](x))^{p^k} \tag{3.5}$$

for almost all $x \in \mathbb{R}^N$ and all $t \in (0, T)$, where

$$A_k := \left(\frac{1}{p+1}\right)^{p^{k-2}} \left(\frac{1}{(p+1)p+1}\right)^{p^{k-3}} \cdots \left(\frac{1}{(1+p+\cdots+p^{k-2})p+1}\right)$$
$$= \prod_{j=1}^{k-1} \left(\frac{p-1}{p^{j+1}-1}\right)^{p^{k-j-1}}.$$

Therefore, by (3.5) we have

$$t^{\frac{1}{p-1}(1-\frac{1}{p^k})}[S(t)u_0](x) \le u(x,t)^{p^{-k}} \left(\prod_{j=1}^{k-1} \left(\frac{p-1}{p^{j+1}-1}\right)^{p^{k-j-1}}\right)^{-p^{-k}} \tag{3.6}$$

for almost all $x \in \mathbb{R}^N$ and all $t \in (0, T)$. On the other hand, we have

$$\begin{aligned} \log\left(\prod_{j=1}^{\infty}\left(\frac{p^{j+1}-1}{p-1}\right)^{p^{-j-1}}\right) &= \sum_{j=1}^{\infty} p^{-j-1} \log\left(\frac{p^{j+1}-1}{p-1}\right) \\ &\le \sum_{j=1}^{\infty} p^{-j-1} \log\left((j+1)p^j\right) < \infty. \end{aligned} \tag{3.7}$$

Then, by (3.6) and (3.7) we can find a constant C^* depending only on p such that

$$t^{\frac{1}{p-1}}[S(t)u_0](x) \le C^* < \infty$$

for almost all $x \in \mathbb{R}^N$ and all $t \in (0, T)$. This implies (3.1), and Lemma 3.1 follows. □

We prove Theorem 1.1 by using Lemma 3.1.

Proof of Theorem 1.1 The proof is by contradiction. Let u be a global-in-time solution of (1.1). Since $u(\cdot, 1)$ is a positive measurable function in $\mathbb{R}^N$, we can find a non-trivial measurable function $U_1 \in L^\infty$ such that $\operatorname{supp} U_1 \subset B_1(0)$ and $0 \le U_1(x) \le u(x, 1)$ for almost all $x \in \mathbb{R}^N$. Then it follows from Lemma 2.4 that

$$[S(t)U_1](x) \ge CMt^{-\frac{N}{2-\alpha}}, \quad M := \int_{B_1(0)} U_1(x)\,dx, \tag{3.8}$$

for all $|x| \le t^{\frac{1}{2-\alpha}}$ and $t \ge 1$. Furthermore, by (1.2), (2.21) and (K2) we see that

$$u(x, t+1) \ge [S(t)u(1)](x) \ge [S(t)U_1](x) \tag{3.9}$$

for almost all $x \in \mathbb{R}^N$ and all $t > 0$.

We first consider the case $1 < p < p_*(\alpha)$. By (3.8) and (3.9) we have

$$[S(t)u(1)](x) \ge CMt^{-\frac{N}{2-\alpha}} \tag{3.10}$$

for all $|x| \le t^{\frac{1}{2-\alpha}}$ and $t \ge T$. It follows from $1 < p < p_*(\alpha)$ with (3.10) that

$$t^{\frac{1}{p-1}}\|S(t)u(1)\|_\infty \to \infty \quad \text{as} \quad t \to \infty,$$

which contradicts (3.1). This means that problem (1.1) possesses no global-in-time positive solutions.

Next we consider the case $p = p_*(\alpha)$. Since $t + 1 - s \le t$ and $s \le t + 1 - s$ for $1 \le s \le t/2$ and it follows from (2.11) that

$$\int_{|x| \le t^{\frac{1}{2-\alpha}}} \Gamma(x, y, t)\, dx \ge C \int_{|x| \le t^{\frac{1}{2-\alpha}}} t^{-\frac{N}{2} \cdot \frac{\alpha}{2-\alpha}} \cdot t^{-\frac{N}{2}}\, dx \ge C$$

for all $|y| \le t^{\frac{1}{2-\alpha}}$, by (1.2), (2.11), (2.12), (3.8) and (3.9) we have

$$\begin{aligned}
&\int_{|x| \le t^{\frac{1}{2-\alpha}}} u(x, t+1)\, dx \\
&\ge \int_{|x| \le t^{\frac{1}{2-\alpha}}} \int_1^{\frac{t}{2}} \int_{|y| \le (t+1-s)^{\frac{1}{2-\alpha}}} \Gamma(x, y, t+1-s) u(y, s)^p\, dy\, ds\, dx \\
&\ge \int_1^{\frac{t}{2}} \int_{|y| \le (t+1-s)^{\frac{1}{2-\alpha}}} \left(\int_{|x| \le (t+1-s)^{\frac{1}{2-\alpha}}} \Gamma(x, y, t+1-s)\, dx \right) u(y, s)^p dy\, ds \\
&\ge C \int_1^{\frac{t}{2}} \int_{|y| \le (t+1-s)^{\frac{1}{2-\alpha}}} u(y, s)^{p-1} \cdot u(y, s)\, dy\, ds \\
&\ge CM^p \int_1^{\frac{t}{2}} \left(s^{-\frac{N}{2-\alpha}} \right)^{p-1} \left(\int_{|y| \le s^{\frac{1}{2-\alpha}}} s^{-\frac{N}{2-\alpha}}\, dy \right) ds \\
&\ge CM^p \int_1^{\frac{t}{2}} s^{-\frac{N}{2-\alpha}(p-1)}\, ds \ge CM^p \log t, \quad t > 3.
\end{aligned} \tag{3.11}$$

Let m be a sufficiently large positive constant. By (3.11) we can find $T > 0$ such that the function U_2 defined by $U_2 := u(\cdot, T) \in L^\infty$ satisfies

$$\int_{|x| \le T^{\frac{1}{2-\alpha}}} U_2(x)\, dx \ge m.$$

Similarly to (3.8) and (3.9), we have

$$u(x, t+T) \ge [S(t)U_2](x) \ge Cmt^{-\frac{N}{2-\alpha}}$$

for almost all $x \in \mathbb{R}^N$ and all $t > 0$. This implies that

$$t^{\frac{N}{2-\alpha}} \|S(t)U_2\|_\infty \ge Cm, \quad t > 1. \tag{3.12}$$

Let v be a solution of (1.1) with initial data U_2. Then, since u is a global-in-time solution of (1.1), v is also a global-in-time solution of (1.1). Therefore we can apply Lemma 3.1 to the solution v, and obtain (3.1) replacing u_0 with U_2. By the arbitrariness of m, this contradicts (3.12) and we see that problem (1.1) possesses no global-in-time positive solutions for the case $p = p_*(\alpha)$. Therefore the proof of Theorem 1.1 is complete. □

4 Proof of Theorem 1.2

In this section we prove Theorem 1.2. We first prove the uniqueness of solutions of (1.1). (See also [13, Lemma 3.1].)

Lemma 4.1 *Let $i = 1, 2$, $\tau > 0$, and u_i be a solution of (1.1) in $\mathbb{R}^N \times (0, \tau)$ with $u_0 = u_{0,i} \in L^\infty$. Then, for any $\sigma \in (0, \tau)$, there exists a constant C such that*

$$\sup_{0 < t \le \sigma} \|u_1(t) - u_2(t)\|_\infty \le C \|u_{0,1} - u_{0,2}\|_\infty. \tag{4.1}$$

Here the constant C depends on $\|u_1\|_{L^\infty(0,\sigma:L^\infty)}$ *and* $\|u_2\|_{L^\infty(0,\sigma:L^\infty)}$.

Remark 4.1 Let $\tau > 0$ and u be a solution of (1.1) in $\mathbb{R}^N \times (0,\tau]$. If $\|u\|_{L^\infty(0,\tau:L^\infty)}$ is bounded, then we can take a constant in (4.1) uniformly with respect to σ. Therefore, if u is a global-in-time bounded solution of (1.1), then, applying this lemma, we see that u is a unique solution of (1.1).

Proof Let $\sigma \in (0,\tau)$. Put $v = u_1 - u_2$. Then we have

$$\|v\|_{L^\infty(0,\sigma:L^\infty)} \le \|u_1\|_{L^\infty(0,\sigma:L^\infty)} + \|u_2\|_{L^\infty(0,\sigma:L^\infty)} < \infty.$$

This together with (1.2) and (K2) yields

$$\begin{aligned} |v(x,\tilde{t})| &\le \|v(t)\|_\infty + \int_t^{\tilde{t}} \int_{\mathbb{R}^N} \Gamma(x,y,\tilde{t}-s)|u_1(y,s)^p - u_2(y,s)^p|\,dy\,ds \\ &\le \|v(t)\|_\infty + C_1 \int_t^{\tilde{t}} \int_{\mathbb{R}^N} \Gamma(x,y,\tilde{t}-s)|v(y,s)|\,dy\,ds \\ &\le \|v(t)\|_\infty + C_1 \sup_{t<\tau\le\tilde{t}} \|v(\tau)\|_\infty(\tilde{t}-t) \end{aligned}$$

for almost all $x \in \mathbb{R}^N$ and all $0 \le t < \tilde{t} \le \sigma$, where C_1 is a positive constant depending only on p, $\|u_1\|_{L^\infty(0,\sigma:L^\infty)}$ and $\|u_2\|_{L^\infty(0,\sigma:L^\infty)}$. This implies that

$$\sup_{t<\tau\le\tilde{t}} \|v(t)\|_\infty \le \|v(t)\|_\infty + C_1 \sup_{t<\tau\le\tilde{t}} \|v(\tau)\|_\infty(\tilde{t}-t)$$

for all $0 \le t < \tilde{t} \le \sigma$.

Let ε be a sufficiently small positive constant such that $C_1\varepsilon \le 1/2$ and $\varepsilon < \sigma$. Then, by (3.2) we have

$$\sup_{t<\tau\le t+\varepsilon} \|v(\tau)\|_\infty \le 2\|v(t)\|_\infty$$

for all $t \in [0, \sigma-\varepsilon]$. Therefore there exists a constant C_2 such that

$$\sup_{0<\tau\le\sigma} \|v(t)\|_\infty \le C_2\|v(0)\|_\infty,$$

and we have inequality (3.1). Thus the proof of Lemma 4.1 is complete. □

Next we prove local existence of solutions of (1.2). For any nonnegative function $u_0 \in L^\infty$, we define $\{u_n\}$ inductively by

$$\begin{aligned} u_1(x,t) &:= \int_{\mathbb{R}^N} \Gamma(x,y,t)u_0(y)\,dy, \\ u_{n+1}(x,t) &:= u_1(x,t) + \int_0^t \int_{\mathbb{R}^N} \Gamma(x,y,t-s)u_n(y,s)^p\,dy\,ds, \quad n = 1,2,\dots, \end{aligned} \tag{4.2}$$

for almost all $x \in \mathbb{R}^N$ and all $t > 0$. Then we can prove inductively that

$$0 \le u_n(x,t) \le u_{n+1}(x,t) \tag{4.3}$$

for almost all $x \in \mathbb{R}^N$, $t > 0$ and all $n \in \mathbb{N}$. In fact, we clearly obtain $u_2 \ge u_1$ since Γ and u_1 are nonnegative functions, and if there exists a number $k \in \mathbb{N}$ such that $u_k(x,t) \le u_{k+1}(x,t)$

for almost all $x \in \mathbb{R}^N$ and all $t > 0$, then

$$\begin{aligned} u_{k+2}(x,t) &= u_1(x,t) + \int_0^t \int_{\mathbb{R}^N} \Gamma(x,y,t-s) u_{k+1}(y,s)^p \, dy \, ds \\ &\geq u_1(x,t) + \int_0^t \int_{\mathbb{R}^N} \Gamma(x,y,t-s) u_k(y,s)^p \, dy \, ds = u_{k+1}(x,t) \end{aligned}$$

for $x \in \mathbb{R}^N$ and $t > 0$. This implies that (4.3) holds true for all $n \in \mathbb{N}$. Therefore we see that the limit function

$$u_*(x,t) := \lim_{n \to \infty} u_n(x,t) \in [0, \infty] \tag{4.4}$$

can be defined for almost all $x \in \mathbb{R}^N$ and all $t > 0$. Furthermore, by properties (G1) and (G2) we can put a constant $c_{**} = \max\{c_1, c_2\}$ such that

$$\begin{aligned} &\sup_{0<t<\infty} \|u_1(t)\|_\infty \leq c_{**} \|u_0\|_\infty, \\ &\sup_{0<t<\infty} t^{\frac{N}{2-a}\left(\frac{1}{r_*} - \frac{1}{q}\right)} \|u_1(t)\|_{q,\infty} \leq c_{**} \|u_0\|_{r_*,\infty}, \end{aligned} \tag{4.5}$$

for any $q \in [r_*, \infty]$ if $u_0 \in L^{r_*,\infty} \cap L^\infty$, where c_1 and c_2 are constants given in (G1) and (G2), respectively. Then we have the following lemma, which implies the local existence of solutions of (1.1). (See also [13, Lemma 3.2] and [25, Lemma 3.1].)

Lemma 4.2 *Let $u_0 \in L^\infty$. Then there exists a positive constant T such that the problem (1.1) possesses a unique solution u of (1.1) in $\mathbb{R}^N \times (0,T)$ satisfying*

$$\sup_{0<t<T} \|u(t)\|_\infty \leq 2c_{**} \|u_0\|_\infty. \tag{4.6}$$

Here c_{**} is the constant given in (4.5).

Proof Let T be a sufficiently small positive constant to be chosen later. By induction we prove that

$$\sup_{0<t<T} \|u_n(t)\|_\infty \leq 2c_{**} \|u_0\|_\infty \tag{4.7}$$

for all $n = 1, 2, \ldots$. By (4.5) we have (4.7) for $n = 1$. Assume that (4.7) holds for some $n = n_* \in \{1, 2, \ldots\}$, that is,

$$\sup_{0<t<T} \|u_{n_*}(t)\|_\infty \leq 2c_{**} \|u_0\|_\infty.$$

Then, by (4.2) and (G1) we have

$$\begin{aligned} \|u_{n_*+1}(t)\|_\infty &\leq \|u_1(t)\|_\infty + \int_0^t \|S(t-s) u_{n_*}(s)^p\|_\infty \, ds \\ &\leq c_{**} \|u_0|_\infty + C_1 \int_0^t \|u_{n_*}(s)\|_\infty^p \, ds \\ &\leq c_{**} \|u_0\|_\infty + C_1 T (2c_{**} \|u_0\|_\infty)^p \end{aligned} \tag{4.8}$$

for all $t \in (0,T)$, where C_1 is a constant independent of n_* and T. Let T be a sufficiently small constant such that

$$C_1 T 2^p (c_{**} \|u_0\|_\infty)^{p-1} \leq 1.$$

Then, by (4.8) we have (4.7) for $n = n_* + 1$. Therefore (4.7) holds for all $n = 1, 2, \ldots$. By (4.3), (4.4) and (4.7) we see that the limit function u_* satisfies (1.2) and

$$\sup_{0<t<T} \|u_*(t)\|_\infty \le 2c_{**}\|u_0\|_\infty.$$

This together with Lemma 4.1 implies that the function $u = u_*$ is a solution of (1.1) in $\mathbb{R}^N \times (0, T)$. Thus Lemma 4.2 follows. □

Now we are ready to prove Theorem 1.2.

Proof of the assertion (i) of Theorem 1.2 Assume (1.12). Let δ be a sufficiently small positive constant. Assume (1.13). By induction we prove

$$\begin{aligned} \|u_n(t)\|_{r_*,\infty} &\le 2c_{**}\delta, \\ \|u_n(t)\|_\infty &\le 2c_{**}\delta t^{-\frac{N}{(2-\alpha)r_*}}, \end{aligned} \tag{4.9}$$

for all $t > 0$. By (4.5) we have (4.9) for $n = 1$. Assume that (4.9) holds for some $n = n_* \in \{1, 2, \ldots\}$, that is,

$$\begin{aligned} \|u_{n_*}(t)\|_{r_*,\infty} &\le 2c_{**}\delta, \\ \|u_{n_*}(t)\|_\infty &\le 2c_{**}\delta t^{-\frac{N}{(2-\alpha)r_*}}, \end{aligned} \tag{4.10}$$

for all $t > 0$. These together with (1.10) imply that

$$\|u_{n_*}(t)\|_{q,\infty} \le \|u_{n_*}(t)\|_{r_*,\infty}^{\frac{r_*}{q}} \|u_{n_*}(t)\|_\infty^{1-\frac{r_*}{q}} \le 2c_{**}\delta t^{-\frac{N}{2-\alpha}\left(\frac{1}{r_*}-\frac{1}{q}\right)} \tag{4.11}$$

for all $t > 0$ and $r_* < q < \infty$. Since $r_* = N(p-1)/(2-\alpha)$, by (4.10) we have

$$\|u_{n_*}(t)^p\|_\infty = \|u_{n_*}(t)\|_\infty^p \le \left(2c_{**}\delta t^{-\frac{N}{(2-\alpha)r_*}}\right)^p = (2c_{**}\delta)^p t^{-\frac{N}{(2-\alpha)r_*}-1} \tag{4.12}$$

for all $t > 0$. Similarly, for any $\eta > 1$ with $\eta \le r_* < \eta p$, by (4.11) we obtain

$$\|u_{n_*}(t)^p\|_{\eta,\infty} = \|u_{n_*}(t)\|_{\eta p,\infty}^p \le \left(2c_{**}\delta t^{-\frac{N}{2-\alpha}\left(\frac{1}{r_*}-\frac{1}{\eta p}\right)}\right)^p \le C\delta^p t^{\frac{N}{(2-a)\eta}-\frac{N}{(2-\alpha)r_*}-1} \tag{4.13}$$

for all $t > 0$. Therefore, by (G1), (G2), (4.12) and (4.13) we have

$$\begin{aligned} \left\| \int_{t/2}^t S(t-s)u_{n_*}(s)^p\, ds \right\|_\infty &\le \int_{t/2}^t \|S(t-s)u_{n_*}(s)^p\|_\infty\, ds \\ &\le \int_{t/2}^t \|u_{n_*}(s)^p\|_\infty\, ds \\ &\le C\delta^p \int_{t/2}^t s^{-\frac{N}{(2-\alpha)r_*}-1}\, ds \le C\delta^p t^{-\frac{N}{(2-\alpha)r_*}} \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} \left\| \int_{t/2}^t S(t-s)u_{n_*}(s)^p\, ds \right\|_{r_*,\infty} &\le \int_{t/2}^t \|S(t-s)u_{n_*}(s)^p\|_{r_*,\infty}\, ds \\ &\le \int_{t/2}^t \|u_{n_*}(s)^p\|_{r_*,\infty}\, ds \le C\delta^p \int_{t/2}^t s^{-1}\, ds \le C\delta^p \end{aligned} \tag{4.15}$$

for all $t > 0$. On the other hand, by (G2), (4.12) and (4.13) with $\eta < r_*$ we have

$$\begin{aligned}
&\left\| \int_0^{t/2} S(t-s)u_{n_*}(s)^p\, ds \right\|_\infty \\
&\le \int_0^{t/2} \|S(t-s)u_{n_*}(s)^p\|_\infty\, ds \le C\int_0^{t/2} (t-s)^{-\frac{N}{(2-\alpha)\eta}} \|u_{n_*}(s)^p\|_{\eta,\infty} ds \\
&\le C\delta^p t^{-\frac{N}{(2-\alpha)\eta}} \int_0^{t/2} s^{\frac{N}{(2-\alpha)\eta}-\frac{N}{(2-\alpha)r_*}-1}\, ds \le C\delta^p t^{-\frac{N}{(2-\alpha)r_*}}
\end{aligned} \tag{4.16}$$

and

$$\begin{aligned}
&\left\| \int_0^{t/2} S(t-s)u_{n_*}(s)^p\, ds \right\|_{r_*,\infty} \\
&\le \int_0^{t/2} \|S(t-s)u_{n_*}(s)^p\|_{r_*,\infty}\, ds \le C\int_0^{t/2} (t-s)^{-\frac{N}{2-\alpha}(\frac{1}{\eta}-\frac{1}{r_*})} \|u_{n_*}(s)^p\|_{\eta,\infty}\, ds \\
&\le C\delta^p t^{-\frac{N}{2-\alpha}(\frac{1}{\eta}-\frac{1}{r_*})} \int_0^{t/2} s^{\frac{N}{(2-\alpha)\eta}-\frac{N}{(2-\alpha)r_*}-1}\, ds \le C\delta^p
\end{aligned} \tag{4.17}$$

for all $t > 0$. Then, taking a sufficiently small δ if necessary, by (4.5) and (4.14), (4.15), (4.16) and (4.17) we see that

$$\left. \begin{array}{r} t^{\frac{N}{(2-\alpha)r_*}} \|u_{n_*+1}(t)\|_\infty \\ \|u_{n_*+1}(t)\|_{r_*,\infty} \end{array} \right\} \le c_{**}\delta + C_1\delta^p \le 2c_{**}\delta$$

for all $t > 0$, where C_1 is a constant independent of n_* and δ. Hence we obtain (4.9) for $n = n_* + 1$. Thus (4.9) holds for all $n = 1, 2, \dots$. Therefore, applying a similar argument as in the proof of Lemma 4.2, by (4.9) we see that there exists a unique global-in-time solution u of (1.1) such that

$$\|u(t)\|_{r_*,\infty} \le 2c_{**}\delta, \quad \|u(t)\|_\infty \le 2c_{**}\delta t^{-\frac{N}{(2-\alpha)r_*}},$$

for all $t > 0$. This together with (4.6) implies that

$$\|u(t)\|_\infty \le C(1+t)^{-\frac{N}{(2-\alpha)r_*}}$$

for all $t > 0$. Furthermore, we apply the interpolation inequality (1.10) to obtain

$$\|u(t)\|_{q,\infty} \le C(1+t)^{-\frac{N}{2-\alpha}(\frac{1}{r_*}-\frac{1}{q})}, \quad r_* \le q \le \infty,$$

for all $t > 0$. Thus we have (1.14), and the proof of the assertion of Theorem 1.2 is complete. □

Proof of the assertion (ii) of Theorem 1.2 Assume (1.12). Let δ be a sufficiently small constant and assume (1.15). Then, by the assertion (i) of Theorem 1.2 we see that there exists a unique global-in-time solution u of (1.1) satisfying (1.14).

We prove the existence of a global-in-time solution of (1.1) satisfying (1.16). For $r = r_*$, it follows from a similar argument as in the proof of the assertion (i) of Theorem 1.2. So we assume $1 \le r < r_*$. Put

$$u_{0,\lambda}(x) := \lambda^\beta u_0(\lambda x), \quad u_{n,\lambda}(x,t) := \lambda^\beta u_n(\lambda x, \lambda^{2-\alpha}t),$$

where $\beta = N/r_*$ and λ is a positive constant such that

$$\|u_{0,\lambda}\|_r = \|u_{0,\lambda}\|_\infty.$$

Since

$$\|u_{0,\lambda}\|_r^{\frac{r}{r_*}} \|u_{0,\lambda}\|_\infty^{1-\frac{r}{r_*}} = \|u_0\|_r^{\frac{r}{r_*}} \|u_0\|_\infty^{1-\frac{r}{r_*}},$$

it follows from (1.15) that

$$\|u_{0,\lambda}\|_r = \|u_{0,\lambda}\|_\infty < \delta. \tag{4.18}$$

Furthermore, $u_{n,\lambda}$ satisfies

$$u_{n,\lambda}(t) = S(t-\tau)u_{n,\lambda}(\tau) + \int_\tau^t S(t-s)u_{n-1,\lambda}(s)^p\, ds, \tag{4.19}$$

for all $t > \tau \geq 0$. On the other hand, by (G1), (4.5) and (4.18) we can find a constant C_{**} independent of δ, q and r, such that

$$\|S(t)u_{0,\lambda}\|_q \leq C_{**}\delta(1+t)^{-\frac{N}{2-\alpha}(\frac{1}{r}-\frac{1}{q})}, \quad t > 0, \tag{4.20}$$

for any $q \in [r, \infty]$.

By induction we prove that

$$\|u_{n,\lambda}(t)\|_q \leq 2C_{**}\delta, \quad 0 < t \leq 2, \tag{4.21}$$

for any $q \in [r, \infty]$ and $n = 1, 2, \ldots$. By (4.20) we have (4.21) for $n = 1$. Assume that (4.21) holds for some $n = n_*$, that is,

$$\|u_{n_*,\lambda}(t)\|_q \leq 2C_{**}\delta, \quad 0 < t \leq 2, \tag{4.22}$$

for any $q \in [r, \infty]$. Then, by (4.22), for any $q \in [r, \infty]$, we have

$$\|u_{n_*,\lambda}(t)^p\|_q = \|u_{n_*,\lambda}(t)\|_{pq}^p \leq (2C_{**}\delta)^p \tag{4.23}$$

for all $0 < t \leq 2$. Taking a sufficiently small δ if necessary, by (G1), (4.19), (4.20) and (4.23) we obtain

$$\begin{aligned} \|u_{n_*+1,\lambda}(t)\|_q &\leq \|S(t)u_{0,\lambda}\|_q + \int_0^t \|S(t-s)u_{n_*,\lambda}(s)^p\|_q\, ds \\ &\leq \|S(t)u_{0,\lambda}\|_q + C_1\int_0^t \|u_{n_*,\lambda}(s)^p\|_q\, ds \\ &\leq C_{**}\delta + C_2\delta^p \leq 2C_{**}\delta, \quad 0 < t \leq 2, \end{aligned} \tag{4.24}$$

for any $q \in [r, \infty]$, where C_1 and C_2 are constants independent of n_* and δ. Thus we have (4.21) for $n = n_* + 1$, and (4.21) holds for all $n = 1, 2, \ldots$.

Let $C_*^{'}$ be a constant to be chosen later such that $C_*^{'} \geq 2C_{**}$. By induction we prove that

$$\|u_{n,\lambda}(t)\|_q \leq C_*^{'}\delta t^{-\frac{N}{2-\alpha}(\frac{1}{r}-\frac{1}{q})}, \quad t > 1/2, \tag{4.25}$$

for any $q \in [r, \infty]$ and $n = 1, 2, \ldots$. By (4.20) we have (4.25) for $n = 1$. Assume that (4.23) holds for some $n = n_*$. Then, similarly to (4.24), since $r_* = \frac{N}{2-\alpha}(p-1) > r$, taking a sufficiently small δ if necessary, by (G1), (4.19) and (4.21) we have

$$
\begin{aligned}
\|u_{n_*+1,\lambda}(t)\|_q &\le C_3(t-1/2)^{-\frac{N}{2-\alpha}(\frac{1}{r}-\frac{1}{q})}\|u_{n_*+1,\lambda}(1/2)\|_r \\
&\quad + C_3\int_{1/2}^{t/2}(t-s)^{-\frac{N}{2-\alpha}(\frac{1}{r}-\frac{1}{q})}\|u_{n_*,\lambda}(s)^p\|_r\,ds \\
&\quad + C_3\int_{t/2}^{t}\|u_{n_*,\lambda}(s)^p\|_q\,ds \\
&\le C_4C_{**}\delta t^{-\frac{N}{2-\alpha}(\frac{1}{r}-\frac{1}{q})} + C_4(C_*^{'}\delta)^p t^{-\frac{N}{2-\alpha}(\frac{1}{r}-\frac{1}{q})}\int_{1/2}^{t/2}s^{-\frac{r_*}{r}}\,ds \\
&\quad + C_4(C_*^{'}\delta)^p\int_{t/2}^{t}s^{-\frac{N}{2-\alpha}(\frac{p}{r}-\frac{1}{q})}\,ds \\
&\le C_5C_{**}\delta t^{-\frac{N}{2-\alpha}(\frac{1}{r}-\frac{1}{q})} + C_5(C_*^{'}\delta)^p t^{-\frac{N}{2-\alpha}(\frac{1}{r}-\frac{1}{q})}
\end{aligned}
$$

for all $t>1$, where C_3, C_4 and C_5 are constants independent of n_* and δ. Let $C_*^{'} \ge 2C_5C_{**}$. Then, taking a sufficiently small δ if necessary, we have

$$\|u_{n_*+1,\lambda}(t)\|_q \le C_*^{'}\delta t^{-\frac{N}{2-\alpha}(\frac{1}{r}-\frac{1}{q})}, \quad t>1.$$

This together with (4.22) implies (4.25) with $n=n_*+1$. Thus (4.25) holds for all $n=1,2,\ldots$.

By (4.22) and (4.25) we can find a constant C such that

$$\|u_{n,\lambda}(t)\|_q \le C\delta(1+t)^{-\frac{N}{2-\alpha}(\frac{1}{r}-\frac{1}{q})}, \quad t>0,$$

for all $q\in[r,\infty]$ and $n=1,2,\ldots$. This implies that

$$\|u_n(t)\|_q \le C(1+t)^{-\frac{N}{2-\alpha}(\frac{1}{r}-\frac{1}{q})}, \quad t>0,$$

for any $q\in[r,\infty]$ and $n=1,2,\ldots$. Then, by the same argument as in the proof of the assertion (i) of Theorem 1.2, we see that there exists a solution u of (1.1) satisfying (1.16). Thus the assertion (ii) of Theorem 1.2 follows, and the proof of Theorem 1.2 is complete. □

Proof of Corollary 1.1 Since $r_*=N(p-1)/(2-\alpha)$, by (1.17) we can find a constant C_1 independent of δ such that

$$\|\varphi\|_{r_*,\infty} \le C_1\delta.$$

Therefore, by the assertion (i) of Theorem 1.2 we see that, if δ is sufficiently small, then a global-in-time solution of (1.1) exists and it satisfies (1.14) for $\alpha=a$ and for $\alpha=b$. Thus Corollary 1.1 follows. □

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