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Calculus of Variations

Global existence and partial regularity for the *p*-harmonic flow

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Abstract

We show a global existence for the Cauchy problem with large initial data for the p-harmonic flow between two smooth, compact Riemannian manifolds. We devise new monotonicity type formulas of a local scaled energy and establish a partial regularity for the solution. The partial regularity obtained is almost optimal, comparing with that of the corresponding stationary case. The p-harmonic flow obtained also converges to a p-harmonic map along a certain time sequence tending to infinity.

Mathematics Subject Classification Primary: 35B45 · 35B65; Secondary: 35D30 · 35K59 · 35K65

1 Introduction

Let $\mathcal{M}$ and $\mathcal{N}$ be smooth compact Riemannian manifolds of dimension m and n with metric g and h, respectively. We assume that, by Nash's embedding theorem, $\mathcal{N}$ is isometrically embedded into $\mathbb{R}^l$ $(l > n)$. For a smooth map u from $\mathcal{M}$ to $\mathcal{N} \subset \mathbb{R}^l$, we consider the p-energy

$$E(u) := \int_{\mathcal{M}} \frac{1}{p}|Du|^p \, d\mathcal{M}, \qquad p \geq 2. \tag{1.1}$$

Here the unknown map $u = \left(u^i\right)$, $i = 1, \ldots, l$, is a vector-valued function, defined on $\mathcal{M}$ with values into $\mathcal{N} \subset \mathbb{R}^l$. In a local coordinate $x = (x_\alpha)$, $\alpha = 1, \ldots, m$, on $\mathcal{M}$, the usual notation is used : $g = \left(g_{\alpha\beta}\right)$, $\left(g_{\alpha\beta}\right)^{-1} = \left(g^{\alpha\beta}\right)$, $|g| = |\det\left(g_{\alpha\beta}\right)|$, and $d\mathcal{M} = \sqrt{|g|}dx$ is a volume element with m-dimensional Lebesgue measure dx, and $D_\alpha = \partial/\partial x_\alpha, \alpha = 1, \ldots, m$,

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$Du = \left(D_\alpha u^i\right)$ is the gradient of a map u, and $|Du|^2 = \sum_{\alpha,\beta=1}^m g^{\alpha\beta} D_\alpha u \cdot D_\beta u$ with an Euclidean inner product $\cdot$ in $\mathbb{R}^l$.

The p-harmonic map is a critical point of the p-energy and satisfies the Euler–Lagrange equation

$$\begin{cases} -\Delta_p u = |Du|^{p-2} A(u)(Du, Du) \\ u \in \mathcal{N} \end{cases} \tag{1.2}$$

where the p-Laplace operator is denoted by

$$\Delta_p u = \frac{1}{\sqrt{|g|}} \sum_{\alpha,\beta=1}^m D_\alpha \left(|Du|^{p-2} \sqrt{|g|} g^{\alpha\beta} D_\beta u \right) \tag{1.3}$$

and the second fundamental form $A(u)(Du, Du)$ of $\mathcal{N} \subset \mathbb{R}^l$ is a vector field along the map $u \in \mathcal{N}$ with values into the orthogonal complement of the tangent space of $\mathcal{N}$ at u (if necessary, the manifold $\mathcal{N}$ is assumed to be orientable).

An approach to look for p-harmonic maps is to exploit the gradient flow associated with the p-energy, called the p-harmonic flow, which are described by the evolutionary p-Laplacian system

$$\begin{cases} \partial_t u - \Delta_p u = |Du|^{p-2} A(u)(Du, Du) \\ u \in \mathcal{N} \end{cases} \tag{1.4}$$

where $u = u(t, x)$ is defined on $\mathcal{M}_\infty = (0, \infty) \times \mathcal{M}$ with values onto $\mathbb{R}^l$, $\partial_t u = \left(\partial_t u^i\right)$ is a partial derivative on time. In this paper we study a global existence and regularity of a solution to the Cauchy problem for the p-harmonic flow (1.4).

Let $\mathbb{R}^l = \mathcal{T}_u\mathcal{N} \oplus (\mathcal{T}_u\mathcal{N})^\perp$ be the orthogonal decomposition of $\mathbb{R}^l$ with respect to the tangent space $\mathcal{T}_u\mathcal{N}$ at each $u \in \mathcal{N}$. The corresponding orthonormal basis is $(e_1(u), \dots, e_n(u))$ of the tangent space $\mathcal{T}_u\mathcal{N}$ and $(e_{n+1}(u), \dots, e_l(u))$ of its orthogonal complement $(\mathcal{T}_u\mathcal{N})^\perp$. Then we find an equivalent representation for the p-harmonic flow

$$\partial_t u - \Delta_p u \perp \mathcal{T}_u\mathcal{N} \iff \partial_t u - \Delta_p u = |Du|^{p-2} A(u)(Du, Du). \tag{1.5}$$

In fact, there exists some vector-valued function $\lambda = \left(\lambda^j(u)\right)$, $j = n+1, \dots, l$, such that

$$\partial_t u - \Delta_p u \perp \mathcal{T}_u\mathcal{N} \iff \partial_t u - \Delta_p u = \sum_{j=n+1}^l \lambda^j(u) e^j(u)$$

and, simply multiplying each of the orthonormal basis $e_j(u)$, $j = n+1, \dots, l$, by the second equation above, we have

$$\lambda^j(u) = |Du|^{p-2} \sum_{\alpha,\beta=1}^m \sqrt{|g|} g^{\alpha\beta} D_\beta u \cdot \left(D_\alpha u \cdot D_u e_j(u) \right),$$

where $\partial_t u$, $Du \in \mathcal{T}_u\mathcal{N}$ because the map $u = u(t, x)$ moves on $\mathcal{N}$, and thus, the usual Euclidean innner product in $\mathbb{R}^l$ is taken, so that $\partial_t u \cdot e_j(u) = 0$ and $Du \cdot e_j(u) = 0$, for all $j = n+1, \dots, l$. Here the last summation term in the equation above is nothing but the second fundamental form of $\mathcal{N}$ along the map u. Furthermore, the Euclidean inner product in $\mathbb{R}^l$ of $\partial_t u$ with the p-harmonic flow Eq. (1.4) leads the energy identity

$$|\partial_t u|^2 - \frac{1}{\sqrt{|g|}} D_\alpha \left(|Du|^{p-2} \sqrt{|g|} g^{\alpha\beta} D_\beta u \cdot \partial_t u \right) + \partial_t \frac{1}{p} |Du|^p = 0,$$

integrated in $\mathcal{M}$ yielding, through integration by parts,

$$\frac{d}{dt}E(u(t)) = -\|\partial_t u(t)\|_2^2. \tag{1.6}$$

Thus, the p-energy $E(u(t))$ is decreasing along the solution $u(t)$ of the p-harmonic flow and, in fact, the solution $\{u(t)\} \subset C^\infty(\mathcal{M}, \mathcal{N})$, $0 < t < \infty$, is the trajectory of negative direction gradient vector field of the p-energy

$$\begin{aligned}\frac{du}{dt}(t) &= -\nabla E(u(t)) \\ &= \Delta_p u(t) + |Du(t)|^{p-2} A(u(t))(Du(t), Du(t)),\end{aligned}$$

by the Euler–Lagrange equation (1.2), where $\nabla E(u(t))$ is the Gâteaux derivative of $E(\cdot)$ at $u(t) \in C^\infty(\mathcal{M}, \mathcal{N})$. Therefore, a global in time solution to (1.4) for any initial data may converge to critical points of the p-energy, the p-harmonic maps, as time tends to ∞. This heat flow method was originally realized by J. Eells and J. H. Sampson for the harmonic flow in the case $p = 2$ under the condition that the sectional curvature of target manifold $\mathcal{N}$ is non-positive, in their pioneering work [15,23]. This fundamental result in the harmonic flow case $p = 2$ was also extends to hold similarly for the p-harmonic flow.

Theorem 1 [16,31] *Suppose that the sectional curvature of the target manifold $\mathcal{N}$ is non-positive, Sect$(\mathcal{N}) \leq 0$. Then, for any smooth initial map from $\mathcal{M}$ to $\mathcal{N}$, there exists a unique global in time weak solution of the Cauchy problem on $\mathcal{M}$ for p-harmonic flow* (1.4)*. The solution u and its gradient are Hölder continuous in time-space. The solution and its gradient uniformly converge to a weak solution and its gradient, respectively, of the p-harmonic map, as time tends to ∞, respectively, which are Hölder continuous.*

We call the weak solution which is locally continuous on time-space together with its gradient the *regular* solution. The curvature restriction on the target manifold in general is necessary for the global existence of regular solution of the p-harmonic flow. In fact, without any curvature restriction on the target manifold, we have some example of a blowing up solution at a finite time (see [5] in the case $p = m = 3$). But, a global in time weak solution may be exist.

Theorem 2 [24] *Let $p = m \geq 2$ and the initial data be in the set of Sobolev maps $W^{1,p}(\mathcal{M}, \mathcal{N})$ between two smooth, compact Riemannian manifolds $\mathcal{M}$ and $\mathcal{N}$. Then, there exists a global in time weak solution of Cauchy problem on $\mathcal{M}$ for the m-harmonic flow. The solution and its gradient are Hölder continuous on time-space, except for at most finitely many time slices.*

In the case $p = m = 2$, the global in time existence as above is also shown for the initial-boundary value problem of the two-dimensional harmonic flow. Moreover, the solution is smooth except for at most finitely many points [3,38]. In the case $p = m$, a nice Sobolev type inequality on time-space, referred as Ladyzhenskaya or Nash inequality, can be available and is crucial for regularity estimate in this case.

In the higher dimensional case $m \geq 3$, M. Struwe et al. established the following fundamental result for global existence and regularity of the harmonic flow in the case $p = 2$ in [8,9,39]

Theorem 3 [8,9,39] *Let $p = 2$. Let initial and boundary data u_0 be smooth map from $\mathcal{M}$ into $\mathcal{N}$. Then, there exists a global in time weak solution u of the harmonic flow* (1.4)*. The* *solution u satisfies the energy inequality: letting* $\mathcal{M}_\infty = (0, \infty) \times \mathcal{M}$,

$$\|\partial_t u\|^2_{L^2(\mathcal{M}_\infty)} + \sup_{0<t<\infty} E(u(t)) \leq E(u_0). \tag{1.7}$$

There exists a relatively closed subset $\Sigma \subset (0, \infty] \times \mathcal{M}$ *such that the solution u is smooth in the complement of* Σ, $\mathcal{M}_\infty \backslash \Sigma$; Σ *is of at most finite m-dimensional Hausdorff measure with respect to the usual parabolic metric in* $\mathcal{M}_\infty$, *and furthermore, for any time* $t_0 > 0$ *and some positive* $C_0 = C(\mathcal{M}, \mathcal{N}, t_0, E(u_0))$, $\mathcal{H}^{m-2}(\{t_0\} \times \Sigma) \leq C_0$; *As time suitably tends to* ∞, *the solution converges to a weakly harmonic map* u_∞ *weakly in Sobolev space* $W^{1,2}(\mathcal{M}, \mathbb{R}^l)$. *There exists a closed set* $\Sigma_\infty \subset \mathcal{M}$ *such that* u_∞ *is smooth on* $\mathcal{M} \backslash \Sigma_\infty$; Σ_∞ *is of at most finite* $(m-2)$*-dimensional Hausdorff measure: For some positive* $C'_0 = C'(\mathcal{M}, \mathcal{N}, t_0, E(u_0))$, $\mathcal{H}^{m-2}(\Sigma_\infty) \leq C'_0$.

There also exist blowing up solutions at a finite time (see [4,7,11,22]).

If the target manifold is the standard unit sphere, the global in time existence of weak solution to the p-harmonic flow is also shown by use of the special structure of the target standard unit sphere [6,25,27,32].

In differential geometry, the regularity has been studied under a smallness of image of a solution, instead of curvature condition [18], and the everywhere regularity of a small solution of harmonic flow is shown in [19,20,37]. Such regularity of a small solution of p-harmonic flow remained open (refer to [28]).

Theorem 3 implies the global in time existence of weak solution of the harmonic flow in the case $p = 2$, which is *partial regular* in the sense of regularity outside exceptional closed set. It has remained open whether or not the corresponding result holds for the p-harmonic flow, since the important result, Theorem 2, was obtained for the case $p = m$.

A compactness for regular p-harmonic flows with uniform boundedness of p-energy has been recently proved by the author in [33,34] (see [39, Theorem 6.1; its proof, pp. 494–497] for the harmonic flow). The compactness result will be the key ingredient for the global in time existence of p-harmonic flow (refer to [9] for the harmonic flow case).

Theorem 4 (A compactness of regular p-harmonic flows with uniformly bounded p-energy) *Let* $p > 2$. *Suppose that a family* $\{u_k\}$ *of regular p-harmonic flows on* $\mathbb{R}^m_\infty = (0, \infty) \times \mathbb{R}^m$ *satisfies the p-energy boundedness with uniform positive constant C*

$$p\,\|\partial_t u_k\|^2_{L^2(\mathbb{R}^m_\infty)} + \sup_{0<t<\infty} \|Du_k(t)\|^p_{L^p(\mathbb{R}^m)} \leq C \tag{1.8}$$

and converges to a limit map u in the sense

$$u_k \longrightarrow u \quad \text{weakly} * \text{ in } L^\infty\left(0, T;\ W^{1,p}(\mathbb{R}^m_\infty, \mathbb{R}^l)\right), \tag{1.9}$$

$$Du_k \longrightarrow Du \quad \text{weakly in } L^p\left(\mathbb{R}^m_\infty, \mathbb{R}^{ml}\right), \tag{1.10}$$

$$\partial_t u_k \longrightarrow \partial_t u \quad \text{weakly in } L^2\left(\mathbb{R}^m_\infty, \mathbb{R}^l\right). \tag{1.11}$$

Then, the limit map u is a global weak solution on $\mathbb{R}^m_\infty$ *of the p-harmonic flow such that* $u \in \mathcal{N}$ *almost everywhere in* $\mathbb{R}^m_\infty$, *and the p-energy boundedness is valid, replacing* u_k *by u in* (1.8). *Moreover, the limit map u is partial regular in the sense : There exists a relatively closed set* Σ *in* $\mathbb{R}^m_\infty$ *such that u and its gradient Du are locally in time-space continuous in the complement* $\mathbb{R}^m_\infty \backslash \Sigma$, *and the size of* Σ *is also estimated by the Hausdorff measure : For any positive number* γ_0, $2 < \gamma_0 < p$, *the set* Σ *is of at most locally zero m-dimensional Hausdorff measure with respect to the time-space metric* $|t|^{1/\gamma_0} + |x|$, *and, furthermore, for* *any positive time $\tau < \infty$, the $(m-\gamma_0)$-dimensional Hausdorff measure of $\{\tau\} \times \Sigma$ with respect to the usual Euclidean metric is locally zero.*

In this paper we show the global existence and regularity of a weak solution of the Cauchy problem for the p-harmonic flow (1.4) with an initial data u_0

$$\begin{cases} \partial_t u - \operatorname{div}\left(|Du|^{p-2} Du\right) = |Du|^{p-2} A(u)(Du, Du) & \text{in } \mathcal{M}_\infty \\ u(0) = u_0 & \text{on } \mathcal{M} \end{cases} \tag{1.12}$$

and a convergence of the solution of p-harmonic flow to a p-harmonic map along a time sequence tending to infinity. The Sobolev space on $\mathcal{M}$ is usually defined as

Definition 1

$$\mathrm{W}^{1,p}\left(\mathcal{M}, \mathbb{R}^l\right) := \left\{ v \in \mathrm{L}^p\left(\mathcal{M}, \mathbb{R}^l\right) \,\middle|\, \exists \text{ a weak derivative } Dv \in \mathrm{L}^p\left(\mathcal{M}, \mathbb{R}^{ml}\right) \right\};$$
$$\mathrm{W}^{1,p}(\mathcal{M}, \mathcal{N}) := \left\{ v \in \mathrm{W}^{1,p}\left(\mathcal{M}, \mathbb{R}^l\right) \mid v \in \mathcal{N} \text{ almost everywhere in } \mathcal{M} \right\};$$
$$\|v\|_{\mathrm{W}^{1,p}(\mathcal{M})} := \|v\|^p_{\mathrm{L}^p(\mathcal{M})} + \|Dv\|_{\mathrm{L}^p(\mathcal{M})} \tag{1.13}$$

Definition 2 Let $u_0 \in \mathrm{W}^{1,p}(\mathcal{M}, \mathcal{N})$. A map u is called a global weak solution of the Cauchy problem (1.12) if and only if u is a measurable vector-valued function defined on $\mathcal{M}_\infty := (0, \infty) \times \mathcal{M}$ with values into $\mathbb{R}^l$, satisfying the following four conditions :

(D1) $u \in \mathrm{L}^\infty(0, \infty; \mathrm{W}^{1,p}(\mathcal{M}, \mathbb{R}^l))$, $\partial_t u \in \mathrm{L}^2(\mathcal{M}_\infty, \mathbb{R}^l)$;
(D2) $u \in \mathcal{N}$ almost everywhere in $\mathcal{M}_\infty$;
(D3) u satisfies (1.4) in the sense of distributions, that is, for any smooth map $\phi \in \mathrm{C}^\infty_0(\mathcal{M}_\infty, \mathbb{R}^l)$,

$$\int_{\mathcal{M}_\infty} \{\partial_t u \cdot \phi + |Du|^{p-2} Du \cdot D\phi - |Du|^{p-2} \phi \cdot A(u)(Du, Du)\}\, dz = 0;$$

(D4) u attains the initial data continuously in the Sobolev space

$$|u(t) - u_0|_{\mathrm{W}^{1,p}(\mathcal{M}, \mathbb{R}^l)} \to 0 \quad \text{as } t \to 0.$$

Theorem 5 (A global existence and regularity for the p-harmonic flow) *Let $p > 2$. Let $u_0 \in \mathrm{W}^{1,p}(\mathcal{M}, \mathcal{N})$. Then, there exists a global weak solution u of (1.12), satisfying the energy inequality*

$$\|\partial_t u\|^2_{\mathrm{L}^2(\mathcal{M}_\infty)} + \sup_{0<t<\infty} E(u(t)) \le E(u_0). \tag{1.14}$$

Moreover, the solution u is partial regular in the following sense : There exists a relatively closed set Σ in $\mathcal{M}_\infty = (0, \infty) \times \mathcal{M}$ such that u and its gradient Du are locally in time-space continuous in the complement $\mathcal{M}_\infty \backslash \Sigma$, and the size of Σ is also estimated by the Hausdorff measure : For any positive number γ_0, $2 < \gamma_0 < p$, the set Σ is of at most zero m-dimensional Hausdorff measure with respect to the time-space metric $|t|^{1/\gamma_0} + |x|$, and, furthermore, for any positive time $\tau < \infty$, the $(m-\gamma_0)$-dimensional Hausdorff measure of $\{\tau\} \times \Sigma$ with respect to the usual Euclidean metric is zero. As time suitably tends to ∞, the solution converges to a weakly p-harmonic map u_∞ weakly in $W^{1,p}(\mathcal{M}, \mathbb{R}^l)$. There exists a closed set $\Sigma_\infty \subset \mathcal{M}$ such that u_∞ and its gradient Du_∞ are locally continuous on $\mathcal{M} \backslash \Sigma_\infty$; For any positive number γ_0, $2 < \gamma_0 < p$, Σ_∞ is of at most zero $(m-\gamma_0)$-dimensional Hausdorff measure.

Remark Measuring by use of the time-space metric $|t|^{1/p}+|x|$ on $\mathcal{M}_\infty$, the $(m+p-\gamma_0)$-dimensional Hausdorff size of Σ is zero. The scale order in the estimate of singular set Σ is almost optimal, since the exponent γ_0 can be as close to p as possible.

The contents of the paper are as follows :

In Sect. 2, we introduce the so-called penalty approximation for the p-harmonic flow. In Sect. 3, some preliminary estimates for the penalty approximating solutions are derived, those proofs are given in "Appendix", and then, the small energy regularity estimate is shown to hold uniformly for the penalty approximating solutions, and is applied for their convergence to a weak solution of the p-harmonic flow in Sect. 4, based on the compactness result, Theorem 4. The monotonicity estimate, Lemmata 12 and 13, is demonstrated in Sect. 5.

2 Penalty approximation

In this section we set the approximation scheme for the p-harmonic flow. We will approximate the p-harmonic flow by the solutions of the gradient flow for the so-called penalized functional, introduced in [9] for the harmonic flow case $p=2$ (also refer to [29,40]).

Since the manifold $\mathcal{N}$ is smooth and compact, there exists a tubular neighborhood $\mathcal{O}_{2\delta_\mathcal{N}}$ with width $2\delta_\mathcal{N}$ of $\mathcal{N}$ in $\mathbb{R}^l$ such that any point $u\in\mathcal{O}_{2\delta_\mathcal{N}}$ has a unique nearest point $\pi_\mathcal{N}(u)\in\mathcal{N}$ satisfying $\operatorname{dist}(u,\pi_\mathcal{N}(u))=\operatorname{dist}(u,\mathcal{N})$ for the Euclidean distance $\operatorname{dist}(\cdot,\cdot)$, where the projection $\pi_\mathcal{N}:\mathcal{O}_{2\delta_\mathcal{N}}\to\mathcal{N}$ is smooth, since the manifold $\mathcal{N}$ is smooth. The distance function $\operatorname{dist}(u,\mathcal{N})$ is Lipschitz continuous on $u\in\mathcal{O}_{2\delta_\mathcal{N}}$.

Let χ be a smooth, non-decreasing real-valued function defined on $[0,\infty)$ such that $\chi(s)=s$ for $s\le(\delta_\mathcal{N})^2$ and $\chi(s)=2(\delta_\mathcal{N})^2$ for $s\ge 4(\delta_\mathcal{N})^2$. Then, the function $\chi\left(\operatorname{dist}^2(u,\mathcal{N})\right)$ is smooth on $u\in\mathbb{R}^l$. Its gradient at $u\in\mathcal{O}_{2\delta_\mathcal{N}}$ is computed as

$$D_u\chi\left(\operatorname{dist}^2(u,\mathcal{N})\right)=2\chi'\left(\operatorname{dist}^2(u,\mathcal{N})\right)\operatorname{dist}(u,\mathcal{N})D_u\operatorname{dist}(u,\mathcal{N});$$
$$D_u\operatorname{dist}(u,\mathcal{N})=\frac{u-\pi_\mathcal{N}(u)}{|u-\pi_\mathcal{N}(u)|}$$

parallel to the vector field $u-\pi_\mathcal{N}(u)$ and orthogonal to $\mathcal{T}_{\pi_\mathcal{N}(u)}\mathcal{N}$. We also have that, for any $u\in\mathcal{N}$ and any tangent vector $\tau\in\mathcal{T}_u\mathcal{N}$,

$$\left|\tau^i\tau^jD_{u^i}D_{u^j}\operatorname{dist}(u,\mathcal{N})\right|\le C(\mathcal{N})|\tau|^2$$

(see [2, Theorem 3.1, pp. 704–705], [1, Theorem 2.1]).

For positive parameters $1\le K\nearrow\infty$ and $1>\epsilon\searrow 0$, we consider the Cauchy problem in $\mathcal{M}_\infty$ with initial data u_0 for the gradient flow, called the *penalized equation*,

$$\begin{cases}\partial_t u-\Delta_{p,\epsilon}u+C_0K\chi'\left(\operatorname{dist}^2(u,\mathcal{N})\right)\operatorname{dist}(u,\mathcal{N})D_u\operatorname{dist}(u,\mathcal{N})=0\\ u(0)=u_0\end{cases}\tag{2.1}$$

associated with the *penalized functional*, defined by

$$F_{K,\epsilon}(u) := E_\epsilon(u) + C_0 \frac{K}{2} \int_{\mathcal{M}} \chi\left(\mathrm{dist}^2(u, \mathcal{N})\right) d\mathcal{M}, \tag{2.2}$$

where the positive constant C_0 will be stipulated later, depending only on p, $\mathcal{M}$ and $\mathcal{N}$ (see Lemma 9 and its proof in "Appendix B"). The partial differential operator $\Delta_{p,\epsilon}$ and its corresponding energy, called the regularized p-Laplace operator and the regularized p-energy, respectively, are defined as

$$\Delta_{p,\epsilon} u := \frac{1}{\sqrt{|g|}} \sum_{\alpha,\beta=1}^{m} D_\alpha \left(\left(\epsilon + |Du|^2\right)^{\frac{p-2}{2}} \sqrt{|g|} g^{\alpha\beta} D_\beta u \right);$$

$$E_\epsilon(u) := \int_{\mathcal{M}} \frac{1}{p} \left(\epsilon + |Du|^2\right)^{\frac{p}{2}} d\mathcal{M} \tag{2.3}$$

We now state the global existence for (2.1). For the proof see "Appendix A".

Lemma 6 (Existence for the penalty approximation) *Let $p > 2$ and let $u_0 \in \mathrm{W}^{1,p}(\mathcal{M}, \mathcal{N})$. For each positive numbers K and ϵ, there exists a weak solution $u = u_{K,\epsilon}$ of the Cauchy problem for the penalized equation* (2.1) *such that $u = u_{K,\epsilon}$ satisfies the energy inequality*

$$\|\partial_t u\|^2_{\mathrm{L}^2(\mathcal{M}_\infty)} + \sup_{0<t<\infty} F_{K,\epsilon}(u) \le E_\epsilon(u_0) \tag{2.4}$$

and, that u, Du, $\partial_t u$ and $D^2 u$ are locally (Hölder) continuous on time and space (with some Hölder exponent) in $\mathcal{M}_\infty$ and u satisfies the penalized equation everywhere in $\mathcal{M}_\infty$.

3 Small energy regularity estimate

3.1 Preliminaries

In this section we show some regularity estimates for solutions $u = u_{K,\epsilon}$ of the penalized equations (2.1). Those proofs are given in "Appendix".

Lemma 7 (Energy inequality) *Let $u_0 \in \mathrm{W}^{1,p}\left(\mathcal{M}, \mathcal{N}\right)$ and $u = u_{K,\epsilon}$ be a regular solution of* (2.1)*. Then, it holds that*

$$\|\partial_t u\|^2_{\mathrm{L}^2(\mathcal{M}_\infty)} + \sup_{0<t<\infty} F_{K,\epsilon}(u) \le E_\epsilon(u_0). \tag{3.1}$$

A solution of the penalized equation is uniformly bounded, that is used in the regularity estimate.

Lemma 8 (Boundedness) *Let $u = u_{K,\epsilon}$ be a regular solution of* (2.1)*. Then it holds that $\sup_{\mathcal{M}_\infty} |u| \le H$, where the positive number H is so large that $B(H) \supset \mathcal{O}_{2\delta_{\mathcal{N}}}(\mathcal{N})$ in $\mathbb{R}^l$.*

We will put the setting for local estimates for the penalized Eq. (2.1). For this purpose we recall some standard geometrical settings. Let $R_{\mathcal{M}} > 0$ be a lower bound for the injective radius of the exponential map on $\mathcal{M}$. Thus, for any positive number $R < R_{\mathcal{M}}$ and any point $x_0 \in \mathcal{M}$, the geodesic ball $\mathcal{B}(R, x_0) \subset \mathcal{M}$ of radius R around x_0 is well-defined and diffeomorphic to the Euclidean ball $B(R, 0) \subset \mathbb{R}^m$, under the linear homeomorphism $\mathcal{T}_{x_0}\mathcal{M} \cong \mathbb{R}^m$, through the exponential map

$$\exp_{x_0} \cdot\, : \mathbb{R}^m \supset B(R,\, 0) \ni x \to \exp_{x_0} x \in \mathcal{B}(R,\, x_0) \subset \mathcal{M}.$$

For any $t \in (0,\, \infty)$, the map

$$u\left(t,\, \exp_{x_0} \cdot\right) \,:\, \mathbb{R}^m \supset B(R,\, 0) \ni x \to u\left(t,\, \exp_{x_0} x\right) \in \mathbb{R}^l \tag{3.2}$$

is well-defined. Hereafter let $x_0 \in \mathcal{M}$ be arbitrarily taken and fixed. We abbreviate as $B(R_{\mathcal{M}}) = B(R_{\mathcal{M}},\, 0)$. We denote $u\left(t,\, \exp_{x_0} x\right)$ by $u(t,\, x)$ for any $(t,\, x) \in (B(R_{\mathcal{M}}))_\infty := (0,\, \infty) \times B(R_{\mathcal{M}})$ and, furthermore, by translation, regard u as a map defined on $(B(R_{\mathcal{M}}))_\infty$ with values into $\mathbb{R}^l$.

Let us denote the penalized energy density for a map u by

$$e_{K,\,\epsilon}(u) := \frac{1}{p}\left(\epsilon + |Du|^2\right)^{\frac{p}{2}} + \frac{K}{2}\chi\left(\operatorname{dist}^2(u,\, \mathcal{N})\right). \tag{3.3}$$

We need the so-called Bochner type estimate for the penalized energy density. See "Appendix C" for the proof. Here the constant C_0 in (2.1) is appropriately chosen.

Lemma 9 (Bochner type estimate) *Let $p > 2$ and $u = u_{K,\,\epsilon}$ be a regular solution to* (2.1). *For brevity, put $e(u) = e_{K,\,\epsilon}(u)$. Then, it holds in $(B_{R_{\mathcal{M}}})_\infty$ that*

$$\begin{aligned}
&\partial_t e(u) - \frac{1}{\sqrt{|g|}} D_\alpha \left(\left(\epsilon + |Du|^2\right)^{\frac{p-2}{2}} \sqrt{|g|}\mathcal{A}^{\alpha\beta} D_\beta e(u)\right) \\
&\quad + C_1 \left(\epsilon + |Du|^2\right)^{\frac{p-2}{2}} \left|D^2 u\right|^2 + C_2 \left|2^{-1}\, K\, D_u \chi\left(\, dist^2\left(u,\, \mathcal{N}\right)\right)\right|^2 \\
&\le C_3 \left(1 + e(u)^{\frac{2}{p}}\right) e(u)^{2\left(1-\frac{1}{p}\right)},
\end{aligned} \tag{3.4}$$

where

$$\mathcal{A}^{\alpha\beta} := g^{\alpha\beta} + (p-2)\frac{g^{\alpha\gamma} g^{\beta\mu} D_\gamma u \cdot D_\mu u}{\epsilon + |Du|^2}, \quad \left|D^2 u\right|^2 = g^{\alpha\beta} g^{\gamma\mu} D_\alpha D_\gamma u \cdot D_\beta D_\mu u,$$

the summation convention over repeated indices is used and the positive constants C_i $(i = 1, 2, 3)$ depend on p, $\mathcal{M}$ and $\mathcal{N}$.

Let λ_0 be a positive number, R be a positive number such that $R < \min\{1,\, R_{\mathcal{M}}/2,\, T^{1/\lambda_0}\}$ and $(t_0,\, x_0)$ in the parabolic like envelope $\mathcal{P} := \left\{(t,\, x) \,:\, T - R^{\lambda_0} < t \le T,\, |x|^{\lambda_0} < t - (T - R^{\lambda_0})\right\}$. In the following we use time-space local cylinder. For $r,\, \tau > 0$, $Q\,(\tau,\, r)\,(t_0,\, x_0) = (t_0 - \tau,\, t_0) \times B(r,\, x_0)$, where $B(r,\, x_0)$ is an open ball in $B_{R_{\mathcal{M}}}$ with center x_0 and radius r. For brevity, we put $u = u_{K,\,\epsilon}$, $e(u) = e_{K,\,\epsilon}(u)$ in (3.3) and abbreviate the time-space Lebesgue measure $dt\, d\mathcal{M}$ as dz.

Lemma 10 (Gradient boundedness on a small region) *For some $(t_0,\, x_0) \in \mathcal{P}$, let $\rho_0 := \left((t_0 - (T - R^{\lambda_0}))^{1/\lambda_0} - |x_0|\right)/4$. Suppose that, for $\lambda_0 > 0$, $r_0 > 0$, $C_1 > 0$ and $L > 0$,*

$$r_0 \le \frac{\rho_0}{2}; \quad L^{2-p}\,(r_0)^2 \le (\rho_0)^{\lambda_0}; \quad r_0 \sup_{Q\left(L^{2-p}(r_0)^2,\, r_0\right)(t_0,\, x_0)} (e(u))^{\frac{1}{p}} \le C_1. \tag{3.5}$$

Let $q > 2$ be a positive number. Then there exists a positive number C depending only on q, p, $\mathcal{M}$ and $\mathcal{N}$, but, independent of L, such that

$$\sup_{Q(L^{2-p}(r_0/2)^2,\, r_0/2)(t_0,\, x_0)} e(u)\, \mathcal{C}^q$$
$$\le \frac{C\, L^{2-p}}{\left|Q\left(L^{2-p}(r_0)^2,\ r_0\right)\right|} \int_{Q\left(L^{2-p}(r_0)^2,\, r_0\right)(t_0,\, x_0)} (e(u))^{2-\frac{2}{p}}\, \mathcal{C}^q\, dz + C\, L^p;$$
$$\mathcal{C}(t,\ x) := \left((t - (T - R^{\lambda_0}))^{\frac{1}{\lambda_0}} - |x|\right)_+ . \tag{3.6}$$

The detail of proof is presented in "Appendix D" (refer to [10,12]).

3.2 Local regularity estimates

The partial regularity is based on the so-called *small energy regularity estimate* (refer to [39, Theorems 5.1, 5.3, 5.4; their proofs, pp. 491–494]). The small energy regularity estimate for the p-harmonic flow in the case $p > 2$ has been recently established in [33,34]. Our main task here is to demonstrate that the small energy regularity estimate holds uniformly for solutions of the penalized equations.

Theorem 11 (Small energy regularity) *Let $p > 2$. Let B_0 and a_0 be positive numbers satisfying the conditions*

$$\frac{6p-4}{p+2} < B_0 < p; \quad \frac{B_0 - 2}{p-2} < a_0 \le 1. \tag{3.7}$$

Let $u = u_{K,\epsilon}$ be a regular solution of (2.1) on $(B(R_{\mathcal{M}}))_T = (0,\ T) \times B(R_{\mathcal{M}},\ 0)$ for a positive $T < \infty$, satisfying the energy bound

$$\|\partial_t u\|^2_{L^2(\mathcal{M}_T)} + \sup_{0<t<T} F_{K,\epsilon}(u) \le C_1 \tag{3.8}$$

for a positive number C_1 depending only on $\mathcal{M}$, p and $\mathcal{N}$. Then, there exists a small positive numeber $R_0 < 1$, depending only on $\mathcal{M}$, $\mathcal{N}$, p, B_0, a_0 and C_1, and the following holds true : Let γ_0 be any positive number satisfying

$$2 < \gamma_0 < p.$$

If, for some small positive $R < \min\{R_{\mathcal{M}},\ R_0,\ T^{1/B_0}\}$,

$$\limsup_{r \searrow 0} r^{\gamma_0 - m} \int_{\{t = T - R^{B_0}\} \times B(r,\,0)} e_{K,\epsilon}(u(t,\ x))\, d\mathcal{M} \le 1, \tag{3.9}$$

then, there holds

$$\sup_{(T-(R/4)^{B_0},\, T) \times B(R/4,\, 0)} e_{K,\epsilon}(u(t,\ x)) \le C_2\, R^{-a_0 p}, \tag{3.10}$$

where the positive constant C_2 depends only on γ_0, B_0, a_0, p, $\mathcal{M}$, $\mathcal{N}$ and C_1.

The novelty here is a new *monotonicity* type estimate of a *localized* scaled energy, which may be of its own interest. Let us define our localized scaled energy in the following way: Let $T \ge 0$ be given, and $(t_0,\ x_0)$ in the parabolic like envelope

$$\left\{(t,\ x) \in (0, \infty) \times B(R_{\mathcal{M}})\ :\ \min\{(R_{\mathcal{M}})^{B_0},\ 1\} > t - T \ge |x|^{B_0}\right\}; \quad B_0 > 2.$$

The localized scaled energy is defined by

$$E_{\pm}(r) = \frac{1}{\Lambda^p} \int_{\{t=t_0 \pm \Lambda^{2-p} r^2\} \times B(R_{\mathcal{M}})} \bar{e}_{K,\epsilon}(u(t, x))\, \mathcal{B}_{\pm}(t_0, x_0; t, x)\, \mathcal{C}^q(t, x)\, d\mathcal{M};$$

$$\bar{e}_{K,\epsilon}(u) := \frac{1}{p}\left(\epsilon + |Du|^2\right)^{\frac{p}{2}} + C_0 \frac{K}{2} \chi\left(\operatorname{dist}^2(u, \mathcal{N}\right) \tag{3.11}$$

and $\Lambda = \Lambda(r)$ is a function of a scale radius r, defined as

$$\Lambda = \Lambda(r) = r^{\frac{B_0 - 2}{2-p}}; \quad p > B_0 > \frac{6p - 4}{p + 2} \tag{3.12}$$

for any r, $0 < r < R_{\mathcal{M}}/2$, where we note that

$$p > \frac{6p-4}{p+2} \Longleftrightarrow (p-2)^2 > 0.$$

The *forward* or *backward* in time Barenblatt like function, denoted by $\mathcal{B}_-$ and $\mathcal{B}_-$, respectively, are defined by

$$\mathcal{B}_{\pm}(t_0, x_0; t, x) = \frac{1}{(\mp t_0 \pm t)^{\frac{m}{B_0}}} \left(1 - \left(\frac{|x - x_0|}{2\,(\mp t_0 \pm t)^{\frac{1}{B_0}}}\right)^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}}, \quad \mp t < \mp t_0. \tag{3.13}$$

The localized function $\mathcal{C}$ is defined and used as

$$\mathcal{C}(t, x) := \left((t - T)^{1/B_0} - |x|\right)_+; \quad q > 2. \tag{3.14}$$

We call $E_+(r)$ and $E_-(r)$ the forward and backward localized scaled p-energy, respectively.

Our monotonicity type estimate of a scaled energy is the following. The proof is postponed by Sect. 5

Lemma 12 (Monotonicity estimate for the backward localized scaled p-energy) *Let $p > 2$ and $q > 2$. For any regular solution u to (2.1) the following estimate holds for all positive numbers r, ρ, $r^{B_0} = \Lambda(r)^{2-p} r^2 < \rho^{B_0} = \Lambda(\rho)^{2-p} \rho^2 \le \min\{1, (R_{\mathcal{M}})^{B_0}, (t_0 - T)/2\}$,*

$$E_-(r) \le E_-(\rho) + C\left(\rho^{\mu} - r^{\mu}\right) + C \int_{t_0 - \rho^{B_0}}^{t_0 - r^{B_0}} \|\mathcal{C}^{q-2}(t)\left(\bar{e}_{K,\epsilon}(u(t))\right)^{\theta_0}\|_{L^\infty(B((t_0 - t)^{1/B_0}, x_0))}\, dt, \tag{3.15}$$

where B_0 as in (3.12), and the positive exponents $\theta_0 \ge 2$ and μ depend only on B_0, p and $\mathcal{N}$, $\mathcal{M}$, p and B_0, respectively, and the positive constant C depends only on the same ones as μ and q.

Lemma 13 (Monotonicity estimate for the forward localized scaled p-energy) *Let $p > 2$ and $q > 2$. For any regular solution u to (2.1) the following estimate holds for all positive numbers r, ρ, $r^{B_0} = \Lambda(r)^{2-p} r^2 < \rho^{B_0} = \Lambda(\rho)^{2-p} \rho^2 \le \min\{1, T - t_0 + (R_{\mathcal{M}})^{B_0}\}$*

$$E_+(\rho) \le E_+(r) + C\left(\rho^{\mu} - r^{\mu}\right) + C \int_{t_0 + r^{B_0}}^{t_0 + \rho^{B_0}} \|\mathcal{C}^{q-2}(t)\left(\bar{e}_{K.\epsilon}(u(t))\right)^{\theta_0}\|_{L^\infty(B((t - t_0)^{1/B_0}, x_0))}\, dt, \tag{3.16}$$

where B_0 as in (3.12), and the positive constants $\theta_0 \geq 2$, μ and C have the same dependence as those in Lemma 12.

From now on we show the validity of Theorem 11.

First of all we make parallel translation $t' = t - T, x' = x$ of the Eq. (1.4) and its solutions u on $(0, T) \times B(R_{\mathcal{M}})$ to those on $(-T, 0) \times B(R_{\mathcal{M}})$ with the same notation. The Eq. (1.4) is invariant under parallel transformation.

Under this setting the statement of Theorem 11 is rewritten as

Lemma 14 *There exists a positive number $R_0 < 1$, depending only on B_0, p, $\mathcal{M}$ and $\mathcal{N}$, such that the following is valid : If*

$$\limsup_{r \searrow 0} r^{\gamma_0 - m} \int_{\{t=-R^{B_0}\} \times B(r,0)} e_{K,\epsilon}(u(t, x))\, d\mathcal{M} \leq 1, \tag{3.17}$$

is satisfied for some small positive $R \leq R_0$ with

$$\gamma_0 = \frac{p(B_0 - 2)}{p - 2}, \tag{3.18}$$

then, it holds that, for a positive constant C_2 depending only on p, $\mathcal{M}$, $\mathcal{N}$ and B_0,

$$\sup_{(-(R/4)^{B_0}, 0) \times B(R/4, 0)} e_{K,\epsilon}(u) \leq C_2 R^{-a_0 p}. \tag{3.19}$$

The proof of Lemma 14 consists of several steps, which are separately explained with those proofs. Our strategy of proof is based on a now classical argument similar to [9,39], originally introduced by Schoen for the partial regularity of harmonic maps [36]. Here we carefully make local estimates under an intrinsic scaling to the evolutionary p-Laplace operator.

Hereafter in this section we put, for brevity,

$$u = u_{K,\epsilon}; \quad e(u) = e_{K,\epsilon}(u).$$

Let positive numbers $\lambda_0 > 2$ and $a_0 < 1$ be determined later. According to λ_0 and a_0, we choose a positive number ϵ such that

$$0 < \epsilon < 2\left(a_0 - \frac{\lambda_0 - 2}{p - 2}\right), \tag{3.20}$$

where we should choose a_0 as

$$a_0 - \frac{\lambda_0 - 2}{p - 2} > 0 \Longleftrightarrow a_0 > \frac{\lambda_0 - 2}{p - 2}. \tag{3.21}$$

For t, $-R^{\lambda_0} \leq t \leq 0$, we define a function $f(t)$ as

$$f(t) := \left(\sup_{-R^{\lambda_0} < \tau < t}\left(\sup_{x \in B\left((\tau + R^{\lambda_0})^{\frac{1}{\lambda_0}}, 0\right)} \left((\tau + R^{\lambda_0})^{\frac{1}{\lambda_0}} - |x|\right)^{a_0} (e(u(\tau, x)))^{\frac{1}{p}}\right)\right)^{A_0};$$

$$A_0 := 2\left(1 - \frac{\lambda_0 - 2}{a_0(p - 2)}\right) - \frac{\epsilon}{a_0}, \tag{3.22}$$

where we notice by (3.20) that

$$A_0 = 2\left(1 - \frac{\lambda_0 - 2}{a_0(p - 2)}\right) - \frac{\epsilon}{a_0} > 0 \Longleftrightarrow \epsilon < 2\left(a_0 - \frac{\lambda_0 - 2}{p - 2}\right).$$

Now we also define a function $g(t)$ as

$$g(t) := \left(\sup_{x \in B((t+R^{\lambda_0})^{1/\lambda_0},\, 0)} \left((t + R^{\lambda_0})^{\frac{1}{\lambda_0}} - |x| \right)^{a_0} (e(u(t,\, x)))^{\frac{1}{p}} \right)^{A_0}, \quad -R^{\lambda_0} \le t \le 0. \tag{3.23}$$

It is readily seen that, for any t, $-R^{\lambda_0} \le t \le 0$,

$$\begin{aligned} (f(t))^{\frac{1}{A_0}} &= \sup_{-R^{\lambda_0} \le \tau \le t} \left(\sup_{x \in B\left((\tau + R^{\lambda_0})^{\frac{1}{\lambda_0}},\, 0 \right)} \left((\tau + R^{\lambda_0})^{\frac{1}{\lambda_0}} - |x| \right)^{a_0} (e(u(\tau,\, x)))^{\frac{1}{p}} \right) \\ &\ge \sup_{x \in B\left((t + R^{\lambda_0})^{1/\lambda_0},\, 0 \right)} \left((t + R^{\lambda_0})^{\frac{1}{\lambda_0}} - |x| \right)^{a_0} (e(u(t,\, x)))^{\frac{1}{p}} = (g(t))^{\frac{1}{A_0}}; \\ & 0 \le g(t) \le f(t). \end{aligned} \tag{3.24}$$

Let t, $-R^{\lambda_0} < t \le 0$, be arbitrarily taken and fixed. Then we can choose some time-space points $(t_0,\, x_0)$ such that $t_0 \in (-R^{\lambda_0},\, t]$ and $x_0 \in B\big((t_0 + R^{\lambda_0})^{1/\lambda_0},\, 0\big)$, and

$$\begin{aligned} (f(t))^{\frac{1}{A_0}} &= \left((t_0 + R^{\lambda_0})^{\frac{1}{\lambda_0}} - |x_0| \right)^{a_0} (e(u(t_0,\, x_0)))^{\frac{1}{p}} \\ &= 4^{a_0} (\rho_0)^{a_0} (e(u(t_0,\, x_0)))^{\frac{1}{p}} \end{aligned} \tag{3.25}$$

where we put

$$\rho_0 := \frac{(t_0 + R^{\lambda_0})^{\frac{1}{\lambda_0}} - |x_0|}{4}. \tag{3.26}$$

Here, if $t_0 = -R^{\lambda_0}$ or $|x_0| = (t_0 + R^{\lambda_0})^{1/\lambda_0}$, then $f(t) = 0$ and $g(t) = 0$.

Refined gradient boundedness on a small region By Lemma 10, we make the gradient bounded by a *local scaled energy* on a *small region*. We divide our consideration into two cases.

Case 1. First we treat the case that $(\rho_0)^{a_0} (e(u(t_0,\, x_0)))^{\frac{1}{p}} \le 1$.
Then we have that

$$\begin{aligned} & \left(\frac{(t_0 + R^{\lambda_0})^{\frac{1}{\lambda_0}} - |x_0|}{4} \right)^{a_0} (e(u(t_0,\, x_0)))^{\frac{1}{p}} \le 1 \\ \Longleftrightarrow\ & \left((t_0 + R^{\lambda_0})^{\frac{1}{\lambda_0}} - |x_0| \right)^{a_0} (e(u(t_0,\, x_0)))^{\frac{1}{p}} \le 4^{a_0} \\ \Longleftrightarrow\ & f(t) \le 4^{p\, a_0\, A_0}. \end{aligned} \tag{3.27}$$

By (3.24) and (3.27) we have

$$g(t) \le f(t) \le 4^{p\, a_0\, A_0}. \tag{3.28}$$

Case 2. Next we study the case that $(\rho_0)^{a_0}(e(u(t_0, x_0)))^{\frac{1}{p}} > 1$.
Then we have

$$r_1 := \left(\frac{1}{(e(u(t_0, x_0)))^{\frac{1}{p}}}\right)^{\frac{1}{a_0}} < \rho_0 \le 1. \tag{3.29}$$

Let L be

$$L := (r_1)^{\frac{\lambda_0 - 2}{2-p}}. \tag{3.30}$$

It holds that

$$L^{2-p}(r_1)^2 \le (\rho_0)^{\lambda_0}, \tag{3.31}$$

because

$$L^{2-p}(r_1)^2 = (r_1)^{\lambda_0} \le (\rho_0)^{\lambda_0} \Longleftrightarrow r_1 \le \rho_0.$$

Under (3.31) we have

$$r_1 \sup_{Q(L^{2-p}(r_1)^2, r_1)(t_0, x_0)} (e(u))^{\frac{1}{p}} \le C_1 := 2^{a_0}. \tag{3.32}$$

For the validity of (3.32), we observe from (3.22) and (3.25) that

$$\sup_{(t_0-(\rho_0)^{\lambda_0}, t_0)\times B(\rho_0, x_0)} (e(u))^{\frac{1}{p}} \le 2^{a_0}(e(u(t_0, x_0)))^{\frac{1}{p}}. \tag{3.33}$$

Then we find that, for L in (3.30),

$$\begin{aligned} r_1 \sup_{Q(L^{2-p}(r_1)^2, r_1)(t_0, x_0)} (e(u))^{\frac{1}{p}} &\le (r_1)^{a_0} \sup_{Q(L^{2-p}(r_1)^2, r_1)(t_0, x_0)} (e(u))^{\frac{1}{p}} \\ &\le \frac{1}{(e(u(t_0, x_0)))^{\frac{1}{p}}} \sup_{(t_0-(\rho_0)^{\lambda_0}, t_0)\times B(\rho_0, x_0)} (e(u))^{\frac{1}{p}} \\ &\le 2^{a_0}, \end{aligned} \tag{3.34}$$

where we choose a_0 as

$$0 < a_0 \le 1. \tag{3.35}$$

Here we show the validity of (3.33), through (3.22) and (3.25). For any τ, $t_0 - (\rho_0)^{\lambda_0} \le \tau \le t_0$, we find that

$$(2\rho_0)^{a_0} \sup_{x\in B(\rho_0, x_0)} (e(u(\tau, x)))^{\frac{1}{p}} \le \sup_{x\in B(\rho_0, x_0)} \left((\tau + R^{\lambda_0})^{1/\lambda_0} - |x|\right)^{a_0} (e(u(\tau, x)))^{\frac{1}{p}}, \tag{3.36}$$

because it holds that for any τ, $t_0 - (\rho_0)^{\lambda_0} \le \tau \le t_0$, and any $x \in B(\rho_0, x_0)$

$$\begin{aligned} &\left(t_0 + R^{\lambda_0} - (\rho_0)^{\lambda_0}\right)^{1/\lambda_0} \ge \left(t_0 + R^{\lambda_0}\right)^{1/\lambda_0} - \rho_0 \quad ; \\ &\left(\tau + R^{\lambda_0}\right)^{1/\lambda_0} - |x| \ge \left(t_0 - (\rho_0)^{\lambda_0} + R^{\lambda_0}\right)^{1/\lambda_0} - (|x_0| + \rho_0) \\ &\quad \ge \left(t_0 + R^{\lambda_0}\right)^{1/\lambda_0} - |x_0| - 2\rho_0 = 2\rho_0, \end{aligned} \tag{3.37}$$

where we note the definition ρ_0 in (3.26) and use the simple algebraic inequality for any positive number a and b

$$a^{1/\lambda_0} + b^{1/\lambda_0} \geq (a+b)^{1/\lambda_0}\,.$$

From (3.36) we obtain that

$$\begin{aligned}
&(2\rho_0)^{a_0} \sup_{(t_0-(\rho_0)^{\lambda_0},\, t_0)\times B(\rho_0,\, x_0)} (e(u))^{\frac{1}{p}} \\
&\leq \sup_{t_0-(\rho_0)^{\lambda_0}<\tau<t_0} \left(\sup_{x\in B(\rho_0,\, x_0)} \left\{ \left((\tau+R^{\lambda_0})^{1/\lambda_0} - |x|\right)^{a_0} (e(u(\tau,\, x)))^{\frac{1}{p}} \right\} \right) \\
&\leq \sup_{-R^{\lambda_0}<\tau<t} \left(\sup_{x\in B\left((\tau+R^{\lambda_0})^{1/\lambda_0},\, 0\right)} \left\{ \left((\tau+R^{\lambda_0})^{1/\lambda_0} - |x|\right)^{a_0} (e(u)(\tau,\, x))^{\frac{1}{p}} \right\} \right) \\
&= \left((t_0+R^{\lambda_0})^{1/\lambda_0} - |x_0|\right)^{a_0} (e(u(t_0,\, x_0)))^{\frac{1}{p}} = (4\rho_0)^{a_0} (e(u(t_0,\, x_0)))^{\frac{1}{p}},
\end{aligned}$$

where we use that for any τ, $t_0 - (\rho_0)^{\lambda_0} \leq \tau \leq t_0$

$$B(\rho_0,\, x_0) \subset B\left((\tau+R^{\lambda_0})^{1/\lambda_0},\, 0\right),$$

because by (3.37), for any τ, $t_0 - (\rho_0)^{\lambda_0} \leq \tau \leq t_0$,

$$\left(\tau + R^{\lambda_0}\right)^{1/\lambda_0} \geq \left(t_0 + R^{\lambda_0} - (\rho_0)^{\lambda_0}\right)^{1/\lambda_0} \geq |x_0| + \rho_0.$$

Thus, (3.33) is actually verified.

Under the choice of parameters $\lambda_0 > 2$ and a_0 in (3.21) and (3.35), we should have

$$\frac{\lambda_0-2}{p-2} < a_0 \leq 1 \Longleftarrow \frac{\lambda_0-2}{p-2} < 1 \tag{3.38}$$

$$\Longleftrightarrow \lambda_0 < p \tag{3.39}$$

and, (3.32) which verifies the condition (3.5) with letting $r_0 = r_1/2$. Thus, we can apply Lemma 10 and take the L^∞-estimate of gradient (3.6), yielding

$$\begin{aligned}
&L^{-p} e(u(t_0,\, x_0))\, \mathcal{C}^q(t_0,\, x_0) \\
&\leq L^{-p} \sup_{Q\left(L^{2-p}\,(r_1/4)^2,\, r_1/4\right)(t_0,\, x_0)} e(u)\, \mathcal{C}^q \\
&\leq \frac{C\, L^{-2p+2}}{\left|Q\left(L^{2-p}(r_1/2)^2,\, r_1/2\right)\right|} \int_{Q\left(L^{2-p}(r_1/2)^2,\, r_1/2\right)(t_0,\, x_0)} (e(u))^{2-\frac{2}{p}}\, \mathcal{C}^q\, dz + C,
\end{aligned} \tag{3.40}$$

where $C > 0$ depends only on a_0, p, $\mathcal{M}$ and $\mathcal{N}$.

Multiplying the both sides of (3.40) by $\left(L^{-p} e(u(t_0,\, x_0))\right)^{\frac{2}{p}-1}$, we have

$$\begin{aligned}
&\left(L^{-p} e(u(t_0,\, x_0))\right)^{\frac{2}{p}}\, \mathcal{C}^q(t_0,\, x_0) \\
&\leq \frac{C\, L^{-p}}{\left|Q\left(L^{2-p}\,(r_1/2)^2,\, r_1/2\right)\right|} \int_{Q\left(L^{2-p}\,(r_1/2)^2,\, r_1/2\right)(t_0,\, x_0)} e(u)\, \mathcal{C}^q\, dz + C,
\end{aligned} \tag{3.41}$$

where by r_1 in (3.29), L in (3.30) and (3.33) we compute as

$$\left(L^{-p}e(u(t_0,\ x_0))\right)^{\frac{2}{p}-1}=(r_1)^{(p-2)\left(a_0-\frac{\lambda_0-2}{p-2}\right)};\quad L^{-p+2}=(r_1)^{\lambda_0-2};$$

$$\sup_{Q\left(L^{2-p}(r_1/2)^2,\, r_1/2\right)}(e(u))^{1-\frac{2}{p}}\le\left(2^{a_0}(e(u(t_0,\ x_0)))^{\frac{1}{p}}\right)^{p-2}=2^{a_0(p-2)}(r_1)^{-a_0(p-2)};$$

$$\left(L^{-p}e(u(t_0,\ x_0))\right)^{\frac{2}{p}-1}L^{-p+2}\sup_{Q\left(L^{2-p}(r_1/2)^2,\, r_1/2\right)}(e(u))^{1-\frac{2}{p}}\le 2^{a_0(p-2)}.$$

Furthermore, we divide our estimations into two cases, depending on the size of r_1.

The positive number $q>2$ is selected later. Recall that the positive number ϵ is as in (3.20). Then $q/\epsilon>1$.

Case 2-1 : $0<r_1\le(\rho_0)^{\frac{q}{\epsilon}}$; *Case 2-2* : $(\rho_0)^{\frac{q}{\epsilon}}<r_1<\rho_0$.

Case 2-1 $0<r_1\le(\rho_0)^{\frac{q}{\epsilon}}$.

Lemma 15 *Suppose that*

$$0<r_1\le(\rho_0)^{\frac{q}{\epsilon}}. \tag{3.42}$$

Then there exists $t_0'\in\left[t_0-(r_1)^{\lambda_0}/4,\ t_0\right]$ *such that*

$$\left(e(u(t_0,\ x_0))\right)^{\frac{1}{p}\left(2\left(1-\frac{\lambda_0-2}{a_0(p-2)}\right)-\frac{\epsilon}{a_0}\right)}\le\frac{C\,(r_1)^{\frac{p(\lambda_0-2)}{p-2}}}{|B\,(r_1/2)|}\int_{\{t=t_0'\}\times B(r_1/2,\, x_0)}e(u)\,\mathcal{C}^q(t)\,d\mathcal{M}+C,$$

$$\mathcal{C}(t,\ x):=\left((t+R^{\lambda_0})^{1/\lambda_0}-|x|\right)_+;\quad q>1, \tag{3.43}$$

where the positive constant C *depends only on* a_0, m, p *and* $\mathcal{N}$.

Proof We will estimate both sides of (3.41).

By ρ_0 in (3.42), r_1 in (3.29) and L in (3.30), the left hand side of (3.41) is computed as

$$\begin{aligned}\mathcal{C}^q(t_0,\ x_0)\ \left(L^{-p}e(u(t_0,\ x_0))\right)^{\frac{2}{p}}&\ge 4^q\,(r_1)^{\epsilon}\ \left(L^{-p}e(u(t_0,\ x_0))\right)^{\frac{2}{p}}\\&=4^q\,\left(e(u(t_0,\ x_0))\right)^{\frac{1}{p}\left(2\left(1-\frac{\lambda_0-2}{a_0(p-2)}\right)-\frac{\epsilon}{a_0}\right)},\end{aligned}$$

where by (3.42),

$$\mathcal{C}^q(t_0,\ x_0)=(4\,\rho_0)^q\ge 4^q\,(r_1)^{\epsilon} \tag{3.44}$$

and the parameters a_0 and ϵ satisfy (3.38) and (3.20).

In the right hand side of (3.41), we take the supremum on time to have, by L and r_1 in (3.30),

$$\begin{aligned}&\frac{C\,(r_1)^{\frac{p(\lambda_0-2)}{p-2}}}{\left|Q\left(L^{2-p}(r_1/2)^2,\ r_1/2\right)\right|}\int_{Q\left(L^{2-p}(r_1/2)^2,\, r_1/2\right)(t_0,\, x_0)}e(u)\,\mathcal{C}^q\,dz\\&\le C\sup_{t_0-L^{2-p}(r_1/2)^2<s<t_0}\frac{(r_1)^{\frac{p(\lambda_0-2)}{p-2}}}{|B(r_1/2)|}\int_{B(r_1/2,\, x_0)}e(u(s))\,\mathcal{C}^q\,d\mathcal{M}\\&=\frac{C\,(r_1)^{\frac{p(\lambda_0-2)}{p-2}}}{|B(r_1/2)|}\int_{\{s=t_0'\}\times B(r_1/2,\, x_0)}e(u(s))\,\mathcal{C}^q\,d\mathcal{M},\end{aligned} \tag{3.45}$$

where by continuity of the gradient of solution, we choose some t_0' such that

$$t_0 - L^{2-p}(r_1/2)^2 \le t_0' \le t_0 \Longleftrightarrow t_0 - (r_1)^{\lambda_0}/4 \le t_0' \le t_0, \tag{3.46}$$

at which the supremum of the second line is attained. □

Case 2-2 $(\rho_0)^{\frac{q}{\epsilon}} < r_1 < \rho_0$.

Lemma 16 *Suppose that*

$$(\rho_0)^{\frac{q}{\epsilon}} < r_1 < \rho_0. \tag{3.47}$$

Then there exists $t_0' \in \left[t_0 - (\rho_0)^{q\,\lambda_0/\epsilon}/4,\ t_0\right]$ *such that*

$$\begin{aligned}&\left(e(u(t_0,\ x_0))\right)^{\frac{1}{p}\left(2\left(1-\frac{\lambda_0-2}{a_0(p-2)}\right)-\frac{\epsilon}{a_0}\right)}\\ &\le \frac{C\,(\rho_0)^{\frac{qp(B_0-2)}{\epsilon(p-2)}}}{\left|B\left((\rho_0)^{q/\epsilon}/2\right)\right|}\int_{\{t=t_0'\}\times B\left((\rho_0)^{q/\epsilon}/2,\,x_0\right)} e(u(t))\,\mathcal{C}^q\,d\mathcal{M} + C,\end{aligned} \tag{3.48}$$

where the positive constant C depends only on a_0, p, $\mathcal{M}$ *and* $\mathcal{N}$.

Proof First we take a look at the inequality (3.43) in *Case 2-1*. For r_1, $0 < r_1 \le (\rho_0)^{\frac{q}{\epsilon}}$ it holds that

$$\begin{aligned}&\left(e(u(t_0,\ x_0))\right)^{\frac{1}{p}\left(2\left(1-\frac{\lambda_0-2}{a_0(p-2)}\right)-\frac{\epsilon}{a_0}\right)} \le \frac{C\,(r_1)^{\frac{p(\lambda_0-2)}{p-2}}}{|B\,(r_1/2)|}\int_{\{t=t_0'\}\times B(r_1/2,\,x_0)} e(u(t))\,\mathcal{C}^q\,d\mathcal{M} + C\\ &\Longleftrightarrow (r_1)^{2\left(\frac{\lambda_0-2}{p-2}-a_0\right)+\epsilon} \le \frac{C\,(r_1)^{\frac{p(\lambda_0-2)}{p-2}}}{|B\,(r_1/2)|}\int_{\{t=t_0'\}\times B(r_1/2,\,x_0)} e(u(t))\,\mathcal{C}^q\,d\mathcal{M} + C,\end{aligned} \tag{3.49}$$

where we use the definition of r_1 in (3.29). In particular, (3.49) is valid for $r_1 = (\rho_0)^{\frac{q}{\epsilon}}$ and the corresponding t_0' as in (3.45) and (3.46)

$$\begin{aligned}&t_0 - (\rho_0)^{q\,\lambda_0/\epsilon}/4 \le t_0' \le t_0 \quad ;\\ &\left((\rho_0)^{\frac{q}{\epsilon}}\right)^{2\left(\frac{\lambda_0-2}{p-2}-a_0\right)+\epsilon} \le \frac{C\,(\rho_0)^{\frac{qp(\lambda_0-2)}{\epsilon(p-2)}}}{\left|B\left((\rho_0)^{q/\epsilon}/2\right)\right|}\int_{\{t=t_0'\}\times B\left((\rho_0)^{q/\epsilon}/2,\,x_0\right)} e(u(t))\,\mathcal{C}^q\,d\mathcal{M} + C.\end{aligned} \tag{3.50}$$

Thus, for r_1, $(\rho_0)^{\frac{q}{\epsilon}} < r_1 < \rho_0$, we simply have

$$(r_1)^{2\left(\frac{\lambda_0-2}{p-2}-a_0\right)+\epsilon} \le \frac{C\,(\rho_0)^{\frac{qp(\lambda_0-2)}{\epsilon(p-2)}}}{\left|B\left((\rho_0)^{q/\epsilon}/2\right)\right|}\int_{\{t=t_0'\}\times B\left((\rho_0)^{\lambda_0/B_0}/2,\,x_0\right)} e(u(t))\,\mathcal{C}^q\,d\mathcal{M} + C,$$

because of (3.20) and (3.38) again. □

Now we derive an ordinary differential inequality for $g(t)$, $-R^{\lambda_0} \le t \le 0$.

Lemma 17 *Let λ_0, B_0, a_0 and ϵ be positive parameters satisfying the conditions*

$$\frac{6p-4}{p+2} < \lambda_0 = B_0 < p; \tag{3.51}$$

$$\frac{\lambda_0 - 2}{p-2} < a_0 \le 1; \quad 0 < \epsilon < 2\left(a_0 - \frac{\lambda_0 - 2}{p-2}\right). \tag{3.52}$$

Then the differential inequality holds for any positive $R < 1$ and any t, $-R^{\lambda_0} \le t \le 0$

$$g(t) \le g_0 + C \int_{-R^{\lambda_0}}^{t} (g(\tau))^{\frac{p\theta_0}{A_0}} \, d\tau, \tag{3.53}$$

where the initial data g_0 is

$$g_0 := 4^{a_0 A_0} + C\, R^{a_0 A_0} + C\, R^{a_0 A_0} \limsup_{\rho \searrow 0} \frac{\rho^{\frac{p(B_0-2)}{p-2}}}{|B(\rho)|} \int_{\{t=-R^{\lambda_0}\}\times B(\rho, x_0)} e(u(t))\, d\mathcal{M} \tag{3.54}$$

and the positive constant C depends only on λ_0, p, $\mathcal{M}$ and $\mathcal{N}$.

Proof Simply saying, our desired inequality (3.53) in Lemma 17 is obtained from combining the gradient L^∞-estimate on a *small region* in Lemmata 15 and 16, and the *monotonicity estimate of local scaled energy* in Lemmata 12 and 13. Here we observe the admissible range of two parameters B_0 in Lemmata 12 and 13, and λ_0 in Lemmata 15 and 16, to choose as $\lambda_0 = B_0$. By (3.12) and (3.39) we have

$$\frac{6p-4}{p+2} < B_0 < p; \quad 2 < \lambda_0 < p$$

and thus, we can choose B_0 and λ_0 as in (3.51), because

$$\frac{6p-4}{p+2} < p \Longleftrightarrow (p-2)^2 > 0.$$

The choice of a_0 in (3.38) and ϵ in (3.20) are as in (3.52).

By use of the monotonicity estimate in Lemmata 12 and 13. we estimate the local in space scaled integral of gradient in the right hand side of (3.43) in Lemma 15 and (3.48) in Lemma 16

Backward monotonicity estimate, Lemma 12 First we apply the backward monotonicity estimate, Lemma 12, for the local scaled energy in the right hand side of (3.43) in Lemma 15 and (3.48) in Lemma 16.

Let us choose the time-component t_0 of the pole of Barenblatt function $\mathcal{B}_-$ in (5.3) as follows: For each

$$t_0 - (r_1)^{\lambda_0}/4 \le t_0' \le t_0 \text{ in Lemma 15,}$$
$$\text{or}$$
$$t_0 - (\rho_0)^{\frac{q\lambda_0}{\epsilon}}/4 \le t_0' \le t_0 \text{ in Lemma 16, with } r_1 \text{ replaced by } (\rho_0)^{\frac{q}{\epsilon}}, \tag{3.55}$$

let t_0 be as t_1

$$t_1 := t_0' + (r_1)^{B_0} = t_0' + (\Lambda(r_1))^{2-p}(r_1)^2. \tag{3.56}$$

Then, the local scaled integral in the right hand side of (3.43) and (3.48) is estimated as

$$\begin{aligned} &\frac{(r_1)^{\frac{p(B_0-2)}{p-2}}}{|B(r_1/2)|} \int_{\{t=t_0'\}\times B(r_1/2,\, x_0)} e(u(t))\, \mathcal{C}(t)^q \, d\mathcal{M} \\ &\le \frac{C}{(\Lambda(r_1))^p} \int_{\{t=t_1-(r_1)^{B_0}\}\times B\left(r_1,\, x_0\right)} e(u(t))\, \mathcal{B}_-\,(t_1,\ x_0;\ t)\ \mathcal{C}(t)^q d\mathcal{M}, \end{aligned} \tag{3.57}$$

because by (3.56) we have, for $t := t_0'$,

$$\begin{aligned} &t = t_1 - (\Lambda(r_1))^{2-p}\,(r_1)^2 \iff t_1 - t = (\Lambda(r_1))^{2-p}(r_1)^2 = (r_1)^{B_0}; \quad x \in B(r_1/2,\ x_0) \\ &\implies (r_1)^{-m}\left(1-2^{-\frac{p}{p-1}}\right)^{\frac{p-1}{p-2}} \le \mathcal{B}_-(t_1,\ x_0;\ t,\ x). \end{aligned}$$

Let ρ' be a positive number, chosen as

$$(\rho')^{B_0} = \frac{t_1 + R^{\lambda_0}}{2} \tag{3.58}$$

and then, the *backward* monotonicity estimate in Lemma 12 yields the upper-boundedness for (3.57) by

$$\begin{aligned} &\frac{C}{(\Lambda(\rho'))^p} \int_{\{t=t_1-(\rho')^{B_0}\}\times B\left(\rho',\, x_0\right)} e(u(t))\, \mathcal{B}_-\,(t_1,\ x_0;\ t)\ \mathcal{C}(t)^q\, d\mathcal{M} \\ &+ C\,\left((\rho')^{\mu} - r^{\mu}\right) \\ &+ C \int_{t_1-(\rho')^{B_0}}^{t_1-(r_1)^{B_0}} \|\mathcal{C}(\tau)^{q-2}\,(e(u(\tau)))^{\theta_0}\|_{L^\infty(B((t_1-\tau)^{1/B_0},\, x_0))}\, d\tau. \end{aligned} \tag{3.59}$$

Forward monotonicity estimate, Lemma 13 Next, we use the forward monotonicty estmate in Lemma 13 for estimating the first scaled energy in (3.59).

By use of $\mathcal{C}$, the first term of (3.59) is evaluated by the *forward* scaled energy

$$\frac{C}{(\Lambda(\rho))^p} \int_{\{t=t_1-\rho^{B_0}\}\times B\left(\rho,\, x_0\right)} e(u(t))\, \mathcal{B}_+\left(-R^{\lambda_0},\ 0;\ t\right)\, \mathcal{C}(t)^{q-\frac{p-1}{p-2}}\, d\mathcal{M}, \tag{3.60}$$

since by (3.58) we find that, for $t := t_1 - (\Lambda(\rho'))^{2-p}\,(\rho')^2$,

$$\begin{aligned} t_1 - t &= (\rho')^{B_0} = \frac{t_1 + R^{\lambda_0}}{2} = t - (-R^{\lambda_0}); \\ (t_1 - t)^{-\frac{m}{B_0}} &= \left(\frac{t_1 + R^{\lambda_0}}{2}\right)^{-\frac{m}{B_0}} = \left(t - (-R^{\lambda_0})\right)^{-\frac{m}{B_0}} \end{aligned}$$

and the function $\mathcal{C}$ can be evaluated above as

$$\begin{aligned} \mathcal{C}(t,\ x) = \left((t+R^{\lambda_0})^{1/\lambda_0} - |x|\right)_+ &= (t+R^{\lambda_0})^{1/\lambda_0}\left(1 - \frac{|x|}{(t+R^{\lambda_0})^{1/\lambda_0}}\right)_+ \\ &\le \left(1 - \left(\frac{|x|}{(t+R^{\lambda_0})^{1/\lambda_0}}\right)^{\frac{p}{p-1}}\right)_+; \end{aligned}$$

$$\mathcal{C}^q \le \mathcal{C}^{q-\frac{p-1}{p-2}} \left(1 - \left(\frac{|x|}{(t+R^{\lambda_0})^{1/\lambda_0}}\right)^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}},$$

because

$$\lambda_0 = B_0; \quad \frac{p}{p-1} > 1;$$
$$(t+R^{\lambda_0})^{1/\lambda_0} \le R \le 1; \quad \mathrm{supp}(\mathcal{C}(t)) = B\left((t+R^{\lambda_0})^{1/\lambda_0},\, 0\right),$$

and thus, for $t := t_1 - (\Lambda(\rho'))^{2-p}(\rho')^2$,

$$\mathcal{B}_-(t_1,\, x_0;\, t)\, \mathcal{C}(t)^q \le \mathcal{B}_+\left(-R^{\lambda_0},\, 0;\, t\right) \mathcal{C}(t)^{q-\frac{p-1}{p-2}}.$$

Also the third term of (3.59) is bounded above by

$$C \int\limits_{t_1-(\rho')^{B_0}}^{t_1-(r_1)^{B_0}} \|\mathcal{C}(\tau)^{q-2}\,(e(u(\tau)))^{\theta_0}\|_{L^\infty(B((\tau+R^{\lambda_0})^{1/B_0},\,0))}\, d\tau, \tag{3.61}$$

because by the support of $\mathcal{C}$ the region of L^∞ norm on space is actually

$$B((t_1-\tau)^{1/B_0},\, x_0) \cap B((\tau+R^{\lambda_0})^{1/B_0},\, 0) \subset B((\tau+R^{\lambda_0})^{1/B_0},\, 0).$$

Then, by the *forward* monotonicity estimate in Lemma 13 (3.60) is bounded by

$$\limsup_{\rho\searrow 0}\left(\frac{C}{(\Lambda(\rho))^p}\int\limits_{\{t=\rho^{B_0}-R^{\lambda_0}\}\times B\left(\rho,0\right)} e(u(t))\,\mathcal{B}_+\left(-R^{\lambda_0},\, 0;\, t\right) \mathcal{C}(t)^{q-\frac{p-1}{p-2}}\, d\mathcal{M}\right)$$
$$+ C\,(\rho')^{\mu} + C \int\limits_{-R^{\lambda_0}}^{(\rho')^{B_0}-R^{\lambda_0}} \|\mathcal{C}(\tau)^{q-\frac{p-1}{p-2}-2}\,(e(u(\tau)))^{\theta_0}\|_{L^\infty(B((\tau+R^{\lambda_0})^{1/B_0},\,0))}\, d\tau, \tag{3.62}$$

where again, we note that by (3.58)

$$(\rho')^{B_0} - R^{\lambda_0} = \frac{t_1 - R^{\lambda_0}}{2} = t_1 - (\rho')^{B_0}.$$

The first scaled energy term above is estimated above as

$$C\,\limsup_{\rho\searrow 0}\left(\frac{\rho^{\frac{p(B_0-2)}{p-2}}}{|B(\rho)|}\int\limits_{\{\tau=\rho^{\lambda_0}-R^{\lambda_0}\}\times B(\rho)(0)} e(u(\tau))\, d\mathcal{M}\right). \tag{3.63}$$

Now we combine the estimations above, (3.43), (3.48), (3.57), (3.59), (3.60), (3.61), (3.62) and (3.63) to have

$$\left(e(u(t_0,\, x_0))\right)^{\frac{1}{p}\left(2\left(1-\frac{\lambda_0-2}{a_0(p-2)}\right)-\frac{\epsilon}{a_0}\right)}$$
$$\le C\,\limsup_{\rho\searrow 0}\left(\frac{\rho^{\frac{p(B_0-2)}{p-2}}}{|B(\rho)|}\int\limits_{\{\tau=\rho^{B_0}-R^{\lambda_0}\}\times B(\rho)(0)} e(u(\tau))\, d\mathcal{M}\right)$$

$$+C\left((\rho')^{\mu}-r^{\mu}\right)+C(\rho')^{\mu}$$
$$+C\int_{-R^{\lambda_0}}^{t_1-(r_1)^{B_0}}\|\mathcal{C}(\tau)^{q-\frac{3p-5}{p-2}}\,(e(u(\tau)))^{\theta_0}\|_{L^\infty(B((\tau+R^{\lambda_0})^{1/B_0},0))}\,d\tau, \tag{3.64}$$

where the power exponent in the left hand side is positive by (3.52), and the second one in the right hand side is bounded as

$$C\left((\rho')^{\mu}-r^{\mu}\right)+C(\rho')^{\mu}\le C\,R^{\mu}, \tag{3.65}$$

where we recall t_1 in (3.56) and ρ' in (3.58)

$$t_1=t_0'+\Lambda(r_1)^{2-p}\,(r_1)^2;\quad (\Lambda(r_1))^{2-p}\,(r_1)^2=(r_1)^{B_0};\quad -R^{\lambda_0}<t_0'<0;\quad \lambda_0=B_0$$
$$\Longrightarrow t_1\le R^{\lambda_0};\quad \rho'=\left(\frac{t_1+R^{\lambda_0}}{2}\right)^{1/B_0}\le\left(R^{\lambda_0}\right)^{1/B_0}=R.$$

Differential inequality We gather (3.64) and (3.65) and then, multiply the resulting inequality by $\left(\left(t_0+R^{\lambda_0}\right)^{1/\lambda_0}-|x_0|\right)^{a_0\,A_0}$ to have

$$\left(\left(\left(t_0+R^{\lambda_0}\right)^{\frac{1}{\lambda_0}}-|x_0|\right)^{a_0}e(u(t_0,\,x_0))^{\frac{1}{p}}\right)^{A_0}$$
$$\le\,C\,R^{a_0A_0}\,\limsup_{\rho\searrow 0}\left(\frac{\rho^{\frac{p(\lambda_0-2)}{p-2}}}{|B(\rho)|}\int_{\{t=\rho^{B_0}-R^{\lambda_0}\}\times B(\rho,0)}e(u(t))\,d\mathcal{M}\right)$$
$$+C\,R^{a_0A_0}\left(1+R^{\mu}\right)$$
$$+C\,R^{a_0A_0}\int_{-R^{\lambda_0}}^{t}\|\mathcal{C}(\tau)^{q-\frac{3p-5}{p-2}}\,(e(u(\tau)))^{\theta_0}\|_{L^\infty(B((\tau+R^{\lambda_0})^{1/\lambda_0},0))}\,d\tau, \tag{3.66}$$

where we note by (3.51) that $B_0=\lambda_0$.

Moreover, we will modify some terms in (3.66) for our demand. By (3.24) the left hand side of (3.66) is estimated below by $g(t)$.

In the third term in the right hand side of (3.66) the integrand is bounded by

$$\|\left(\left(\tau+R^{\lambda_0}\right)^{\frac{1}{\lambda_0}}-|\cdot|\right)^{a_0p\theta_0}\,(e(u(\tau)))^{\theta_0}\,\|_{L^\infty B((\tau+R^{\lambda_0})^{1/\lambda_0},0)}, \tag{3.67}$$

since $q>2$ can be chosen to be large, comparing with $a_0p\theta_0$ and depending only on p and B_0, in fact,

$$0<a_0\le 1;\quad q-\frac{3p-5}{p-2}\ge a_0p\,\theta_0 \Longleftarrow q\ge p\,\theta_0+\frac{3p-5}{p-2},$$

where θ_0 depends only on p and B_0.

Finally, collecting (3.28) in *Case* 1, and (3.66) (3.67) in *Case* 2, we arrive at our desired estimation (3.53).

Here we observe that the principal integral quantity in (3.66) is rewritten as

$$\begin{aligned}
&\limsup_{\rho \searrow 0} \frac{\rho^{\frac{p(B_0-2)}{p-2}}}{|B(\rho)|} \int\limits_{\{t=\rho^{\lambda_0}-R^{\lambda_0}\}\times B(\rho,x_0)} e(u(t))\, d\mathcal{M} \\
&= \limsup_{\rho \searrow 0} \frac{\rho^{\frac{p(B_0-2)}{p-2}}}{|B(\rho)|} \int\limits_{\{t=-R^{\lambda_0}\}\times B(\rho,x_0)} e(u(t))\, d\mathcal{M}. \qquad (3.68)
\end{aligned}$$

In fact, by time-space continuity of Du, we have the estimation for sufficiently small positive ρ

$$\frac{\rho^{\frac{p(B_0-2)}{p-2}}}{|B(\rho)|} \int\limits_{B(\rho,x_0)} \left| e\left(u\left(\rho^{\lambda_0} - R^{\lambda_0}\right)\right) - e\left(\left(-R^{\lambda_0}\right)\right)\right| d\mathcal{M} \leq C\, \rho^{\frac{p(B_0-2)}{p-2}},$$

which converges to 0, by taking the lim sup on ρ tending to 0 in the both side. □

We are now in position to show the validity of Lemma 14. We solve the differential inequality (3.53) and (3.54), yielding the uniform gradient bound (3.19).

Proof of Lemma 14. The differential inequality (3.53) and (3.54) can be easily solved as

$$g(t) \leq g_0 / \left(1 - C\,(\beta-1)\left(R^{\lambda_0}+t\right)\,(g_0)^{\beta-1}\right)^{\frac{1}{\beta-1}}, \quad -R^{\lambda_0} \leq t \leq 0, \qquad (3.69)$$

with the exponent

$$\beta = \frac{p\,\theta_0}{A_0} > 1,$$

which is satisfied by $\theta_0 > 1$ and choice in (3.52).

We simply obtain from (3.69)

$$g(t) \leq 2^{\frac{1}{\beta-1}} g_0, \quad -R^{\lambda_0} \leq t \leq 0, \qquad (3.70)$$

under the choice of R such that

$$1 - C\,(\beta-1)\left(R^{\lambda_0}+t\right)\,(g_0)^{\beta-1} \geq 2^{-1} \iff \left(\frac{1}{g_0}\right)^{\beta-1} \frac{1}{2C(\beta-1)} \geq R^{\lambda_0}+t,$$

which is satisfied by

$$\left(\frac{1}{C}\right)^{\frac{\beta-1}{\lambda_0}} \left(\frac{1}{2C(\beta-1)}\right)^{\frac{1}{\lambda_0}} \geq R \Longleftarrow 0 < g_0 < C; \quad R^{\lambda_0}+t \leq R^{\lambda_0} \qquad (3.71)$$

and so, let R_0 be the positive number in the left hand side of the first inequality in (3.71). □

4 Passing to the limit

In this section we present the proof of Theorem 5, based on Theorem 11. As before we abbreviate the time-space Lebesgue measure $dtd\mathcal{M}$ as dz.

Let $\{\epsilon_k\}$ and $\{K_k\}$ be sequences such that $\epsilon_k \searrow 0$ and $K_k \nearrow \infty$ as $k \to \infty$. Let u_{K_k,ϵ_k}, $k = 1, 2, \ldots$, be a sequence of solutions of the Cauchy problem with initial data u_0 for the penalized equations (2.1) with approximating numbers $\epsilon = \epsilon_k$ and $K = K_k$, obtained in Lemma 6. Hereafter we put $u_k = u_{K_k,\epsilon_k}$ $e_k(u_k) = e_{K_k,\epsilon_k}(u_{K_k,\epsilon_k})$, for brevity.

By the energy inequality (2.4), there exist a subsequence of $\{u_k\}$, also denoted by the same notation, and the limit map u such that, as $k \to \infty$,

$$u_k \longrightarrow u \quad \text{weakly} * \text{ in } L^\infty\left(0, \infty; W^{1,p}(\mathcal{M}, \mathbb{R}^l)\right), \tag{4.1}$$

$$\partial_t u_k \longrightarrow \partial_t u \quad \text{weakly in } L^2\left(\mathcal{M}_\infty, \mathbb{R}^l\right), \tag{4.2}$$

$$Du_k \longrightarrow Du \quad \text{weakly in } L^p_{\mathrm{loc}}\left(\mathcal{M}_\infty, \mathbb{R}^{ml}\right), \tag{4.3}$$

$$\chi(\operatorname{dist}^2(u_k, \mathcal{N})) \longrightarrow 0 \quad \text{strongly in } L^2_{\mathrm{loc}}\left(\mathcal{M}_\infty, \mathbb{R}^l\right), \tag{4.4}$$

$$u_k \longrightarrow u \quad \text{strongly in } L^q_{\mathrm{loc}}\left(\mathcal{M}_\infty, \mathbb{R}^l\right) \text{ for any } q,\, 1 \le q < \frac{mp}{(m-p)_+}, \tag{4.5}$$

where the strong convergence in (4.5) follows from (4.1) and (4.2) (see [6, Lemma 1.4, p. 28]). Thus, furthermore, for a subsequence $\{u_k\}$ denoted by the same notation,

$$u_k \longrightarrow u, \quad \operatorname{dist}(u_k, \mathcal{N}) \longrightarrow 0 \quad \text{almost everywhere in } \mathcal{M}_\infty. \tag{4.6}$$

The use of convergence (4.3) and (4.2) in the energy inequality (3.1) for u_k also yields (1.14) for the limit map u.

We demonstrate that the limit map u is a *partial regular* weak solution of the p-harmonic flow, as in the statement of Theorem 5. The proof is divided to several steps and proceeded.

Let us define the *regular set* of the limit map u as

$$\operatorname{Reg}(u) := \{z_0 = (t_0, x_0) \in \mathcal{M}_\infty \mid u \text{ is regular in a neighborhood of } z_0\}$$

and thus, the *singular set* as the complement of $\operatorname{Reg}(u)$, $\Sigma := \operatorname{Sing}(u) = \mathcal{M}_\infty \setminus \operatorname{Reg}(u)$. By definition, $\operatorname{Reg}(u)$ is a relatively open set of $\mathcal{M}_\infty$ and $\operatorname{Sing}(u)$ is relatively closed in $\mathcal{M}_\infty$. Let R_0 be a sufficient small positive number, determined in Theorem 11. For τ, $0 < \tau < \infty$, and R, $0 < R < \min\{R_0, \tau^{1/B_0}\}$, we put two subsets in $\mathcal{M}$ as

$$\begin{aligned} \mathcal{S}(\tau, R) &:= \left\{x_0 \in \mathcal{M} \,:\, \limsup_{k\to\infty}\left(\limsup_{r\searrow 0} r^{\gamma_0 - m}\int_{\{t=\tau-R^{B_0}\}\times B(r,x_0)} e_k(u_k(t, x))\, d\mathcal{M}\right) \ge 1\right\}; \\ \mathcal{T}(\tau, R) &:= \bigcap_{l=1}^{\infty}\bigcup_{k=l}^{\infty}\left\{x_0 \in \mathcal{M} \,:\, \limsup_{r\searrow 0} r^{\gamma_0 - m}\int_{\{t=\tau-R^{B_0}\}\times B(r,x_0)} e_k(u_k(t, x))\, d\mathcal{M} > 1/2\right\}. \end{aligned} \tag{4.7}$$

Then, let us define as

$$\mathcal{S}(\tau) := \bigcap_{0<R<\min\{R_0,\, \tau^{1/B_0}\}} \mathcal{S}(\tau, R) \quad ; \quad \mathcal{S} := \bigotimes_{0<\tau<\infty} \mathcal{S}(\tau), \tag{4.8}$$

where $\bigotimes_{0<\tau<\infty}$ means the direct product of sets on positive time $\tau < \infty$.

Regularity of the limit map We will prove that $\Sigma = \operatorname{Sing}(u) \subset \mathcal{S}$. For this purpose, we now show the regularity of limit map u in the complement of $\mathcal{S}$. Let (t_0, x_0) be in the complement of $\mathcal{S}$. Thus, there exist a positive $R < \min\{R_0, (t_0)^{1/B_0}\}$ and an infinite family $\{u_k\}$ of regular solutions such that

$$\limsup_{r\searrow 0} r^{\gamma_0-m} \int_{\{t=t_0-R^{B_0}\}\times B(r,x_0)} e_k(u_k(t,x))\,dx < 1.$$

Then we can apply Theorem 11 for each u_k above to obtain

$$\sup_{(t_0-(R/4)^{B_0},\,t_0)\times B(R/4,\,x_0)} e_k(u_k) \le C\, R^{-pa_0}, \tag{4.9}$$

where the positive constant C depends only on B_0, p, $\mathcal{M}$ and $\mathcal{N}$.

Put $Q := \left(t_0 - (R/8)^{B_0},\ t_0\right) \times B(R/8,\ x_0)$. From (4.9), there exists a subsequence of $\{u_k\}$, denoted by the same notation, such that, as $k \to \infty$,

$$Du_k \longrightarrow Du \quad \text{weakly} * \text{ in } L^\infty(Q)\,; \quad \sup_Q |Du| \le C\, R^{-pa_0}. \tag{4.10}$$

Now we will show the uniform continuity of $\{u_k\}$ in Q. For this purpose we will derive a local L^2 estimate of derivative of the penalty term. For any smooth function ϕ of compact support in Q, we multiply the Bochner type estimate (3.4) in Lemma 9 by $\phi^2\sqrt{|g|}$ and integrate by parts in Q to have, letting $K = K_k$, $u = u_k$ and $e(u) = e_k(u_k)$,

$$\begin{aligned}
&\int_Q \phi^2 \left(\frac{C_1}{2}\left(\epsilon + |Du|^2\right)^{\frac{p-2}{2}} \left|D^2 u\right|^2 + \frac{C_2}{2}\left|\frac{K}{2} D_u \chi\left(\operatorname{dist}^2(u,\mathcal{N})\right)\right|^2 \right) dz \\
&\le \int_Q \left(\phi\, |\partial_t \phi|\, e(u) + |D\phi|^2 \left(\frac{2p}{C_1} e(u) + \frac{2}{C_2} e(u)^{\frac{2}{p}} \right) \right. \\
&\quad \left. + C_3\, \phi\, 2\left(1 + e(u)^{\frac{2}{p}}\right) e(u)^{2\left(1-\frac{1}{p}\right)} \right) dz,
\end{aligned} \tag{4.11}$$

where we use the Cauchy inequality in the first inequality.

Let $(t_0, x_0) \subset Q$ be any point and $r \le R/8$ be any positive number, and $Q(r) = (t_0 - r^q,\ t_0) \times B(r,\ x_0)$ with $q > 1$. In (4.11) we choose a smooth function ϕ such that $0 \le \phi \le 1$, $\phi = 1$ in $Q(r)$, $\phi = 0$ outside $Q(2r)$, and $|D\phi| \le C/r$ and $|\partial_t \phi| \le C/r^q$. Thus, from (4.9) and (4.11) we obtain

$$\begin{aligned}
&\int_{Q(r)} \left(\frac{C_1}{2}\left(\epsilon + |Du|^2\right)^{\frac{p-2}{2}} \left|D^2 u\right|^2 + \frac{C_2}{2}\left|\frac{K}{2} D_u \chi\left(\operatorname{dist}^2(u,\mathcal{N})\right)\right|^2 \right) dz \\
&\le C\left(r^m + r^{m+q-2} + r^{m+q}\right) \le C\, r^m
\end{aligned} \tag{4.12}$$

We also need the Poincaré inequality of parabolic type : Let $u = u_k$. There exists a positive constant C, depending only on $\mathcal{M}$ and p, such that, for any $Q(r) \subset Q$,

$$\begin{aligned}
\|u - \bar{u}_{Q(r)}\|^2_{L^2(Q(r))} \le C &\left(r^2\, \|Du\|^2_{L^2(Q(r))} + r^{-m+q-2}\, \|\left(\epsilon + |Du|^2\right)^{1/2}\|^{2(p-1)}_{L^{p-1}(Q(r))} \right. \\
&\left. + r^{2q}\, \|2^{-1} K\, D_u \chi(\operatorname{dist}^2(u,\mathcal{N}))\|^2_{L^2(Q(r))} \right),
\end{aligned} \tag{4.13}$$

where $\bar{u}_{Q(r)}$ is the integral mean of u in $Q(r)$. For the proof refer to [28].

Substituting (4.9) and (4.12) into (4.13), we have, for any $(t_0, x_0) \subset Q$, any positive $r \le R/8$, and $Q(r) = (t_0 - r^q,\ t_0) \times B(r,\ x_0)$,

$$\|u - \bar{u}_{Q(r)}\|^2_{L^2(Q(r))} \le C\left(r^{m+q+2} + r^{m+3q-2} + r^{m+2q}\right) \tag{4.14}$$

and thus, choosing $q > 1$ in (4.14), we obtain from Campanato's isomorphism theorem that $\{u_k\}$ is uniformly Hölder continuous in Q with exponent $\min\{1,\ q-1,\ \frac{q}{2}\}$ on the metric $|t|^{1/q}+|x|$, uniformly on u_k. Thus, we see that $\{u_k\}$ is equicontinuous, and uniformly bounded in Q by Lemma 8. Therefore, by Arzela-Ascoli theorem we find for a subsequence of $\{u_k\}$, denoted by the same notation, and the limit map u that, as $k \to \infty$,

$$u_k \longrightarrow u \quad \text{uniformly in } Q \tag{4.15}$$

and that the limit map u is uniformly continuous in Q. From (4.9) and (4.15), we see that, as $k \to \infty$,

$$\chi(\,\mathrm{dist}^2(u_k,\ \mathcal{N})) \le C/K_k \longrightarrow 0 \quad \text{uniformly in } Q \quad \Longrightarrow u \in \mathcal{N} \quad \text{in } Q \tag{4.16}$$

Now we will show that the limit map u satisfies the p-harmonic flow equation in Q. From (4.9) and (4.11) we also see that $\left\{(K_k/2)\ D_u\chi\big(\,\mathrm{dist}^2(u,\ \mathcal{N})\big|_{u=u_k}\right\}$ is bounded in $L^2(Q, \mathbb{R}^l)$ and then, there exists a vector-valued function $\nu \in L^2(Q, \mathbb{R}^l)$ such that, as $k \to \infty$,

$$(K_k/2)\ D_u\chi\big(\,\mathrm{dist}^2(u,\ \mathcal{N})\big)\big|_{u=u_k} \longrightarrow \nu \quad \text{weakly in } L^2(Q). \tag{4.17}$$

Let $\mathcal{P}_{\mathcal{N}}\big(u(Q)\big)$ be a neighborhood of $u(Q)$ in $\mathcal{N}$. Let $\tau(v)$ be any smooth tangent vector field of $\mathcal{N}$ on $\mathcal{P}_{\mathcal{N}}\big(u(Q)\big)$, $\tau(v) \in \mathcal{T}_v\mathcal{N}$ for any $v \in \mathcal{P}_{\mathcal{N}}\big(u(Q)\big)$. By (4.15), we can choose a sufficiently large k_0 such that, for any $k \ge k_0$, $u_k \in \mathcal{O}_{\delta_{\mathcal{N}}}$ in Q and $\pi_{\mathcal{N}}(u_k) \in \mathcal{P}_{\mathcal{N}}\big(u(Q)\big) \subset \mathcal{N}$ and $\tau(\pi_{\mathcal{N}}(u_k)) \in \mathcal{T}_{\pi_{\mathcal{N}}(u_k)}\mathcal{N}$ in Q, where $\mathcal{O}_{\delta_{\mathcal{N}}}$ is a tubular neighborhood in $\mathbb{R}^l$ of $\mathcal{N}$ with width $\delta_{\mathcal{N}}$, and $\pi_{\mathcal{N}}$ is the nearest point projection to $\mathcal{N}$ from the tubular neighborhood of $\mathcal{N}$. Thus, we have that

$$\begin{aligned} D_u\chi\big(\,\mathrm{dist}^2(u,\ \mathcal{N})\big)\big|_{u=u_k} \cdot \tau(\pi_{\mathcal{N}}(u_k)) &= 2\chi'\,\mathrm{dist}(u,\ \mathcal{N})\ D_u\,\mathrm{dist}(u,\ \mathcal{N})|_{u=u_k} \cdot \tau\big(\pi_{\mathcal{N}}(u_k)\big) \\ &= 0 \quad \text{in } Q, \end{aligned}$$

because $D_u\,\mathrm{dist}(u,\ \mathcal{N})|_{u=u_k(z)}$ is orthogonal to $\mathcal{T}_{\pi_{\mathcal{N}}(u_k(z))}\mathcal{N}$ for any $z \in Q$, and then,

$$\int_Q \frac{K_k}{2}\ D_u\chi\big(\,\mathrm{dist}^2(u,\ \mathcal{N})\big)\big|_{u=u_k} \cdot \tau(\pi_{\mathcal{N}}(u_k))\,dz = 0. \tag{4.18}$$

By (4.15) and (4.17), we can take the limit as $k \to \infty$ in (4.18) to have, for any smooth tangent vector field $\tau(v)$ of $\mathcal{N}$ on $\mathcal{P}_{\mathcal{N}}\big(u(Q)\big) \subset \mathcal{N}$, as $k \to \infty$,

$$\begin{aligned} 0 = \int_Q \frac{K_k}{2}\ D_u\chi\big(\,\mathrm{dist}^2(u,\ \mathcal{N})\big)\big|_{u=u_k} \cdot \tau(\pi_{\mathcal{N}}(u_k))\,dz &\longrightarrow \int_Q \nu \cdot \tau(u)\,dz \\ \Longrightarrow \int_Q \nu \cdot \tau(u)\,dz = 0 & \\ \Longleftrightarrow \nu(z) \perp \mathcal{T}_{u(z)}\mathcal{N} \quad \text{for any } z \in Q & \end{aligned} \tag{4.19}$$

and thus, $\nu(z)$ is a normal vector field along $u(z)$ for any $z \in Q$. In the weak form of (2.1), for any smooth map ϕ with compact support in Q,

$$\int_Q \left(\partial_t u_k \cdot \phi + \big(\epsilon_k + |Du_k|^2\big)^{\frac{p-2}{2}} g^{\alpha\beta} D_\beta u_k \cdot D_\alpha\phi + \frac{K_k}{2}\ D_u\chi\big(\,\mathrm{dist}^2(u,\ \mathcal{N})\big)\big|_{u=u_k} \cdot \phi\right) dz = 0,$$

we pass to the limit as $k \to \infty$ to find that the limit map u satisfies

$$\int_Q \big(\partial_t u \cdot \phi + |Du|^{p-2} g^{\alpha\beta} D_\beta u \cdot D_\alpha\phi + \nu \cdot \phi\big)\,dz = 0, \tag{4.20}$$

where we use the convergence in the first line of (4.19) and, the strong convergence of gradients $\{Du_k\}$, obtained from (2.1) with the convergence (4.1), (4.2) and (4.17) (see [6, Theorem 2.1 and its proof, pp. 31–33]). Therefore, we obtain that

$$\partial_t u - \Delta_p u + \nu = 0 \quad \text{almost everywhere in } Q \text{ as } L^2(Q) - \text{map}. \tag{4.21}$$

We now observe that

$$|\nu(z)| = -|Du(z)|^{p-2} g^{\alpha\beta}(x) D_\beta u(z) \cdot (D_\alpha u(z) \cdot D_u \gamma(u)|_{u=u(z)}) \\ \text{almost every } z = (t,\, x) \in Q. \tag{4.22}$$

Let $\bar{z} = (\bar{t},\, \bar{x}) \in Q$ be arbitrarily taken and fixed. Let $\gamma(v)$ be a smooth unit normal vector field of $\mathcal{N}$ in $u(Q) \subset \mathcal{N}$ such that $\gamma(v) \in (\mathcal{T}_v\mathcal{N})^\perp$, $|\gamma(v)| = 1$ for any $v \in u(Q)$ and $\gamma(u(\bar{z})) = \nu(\bar{z})/|\nu(\bar{z})|$. We take the composite map $\gamma(u)$ of $\gamma(\cdot)$ and the limit map u, and use a test function $\sqrt{|g|}\gamma(u)\,\eta$ for any smooth real-valued function η with compact support in Q to have

$$\int_Q \left(\partial_t u \cdot \gamma(u)\eta + |Du|^{p-2} g^{\alpha\beta} D_\beta u \cdot \left(D_\beta \gamma(u)\eta + \gamma(u) D_\beta \eta\right) + \nu \cdot \gamma(u)\,\eta\right) dz = 0;$$

$$\int_Q \left(|Du|^{p-2} g^{\alpha\beta} D_\beta u \cdot D_\beta \gamma(u) + \nu \cdot \gamma(u)\right) \eta\, dz = 0$$

$$\Longrightarrow \nu \cdot \gamma(u) = -|Du|^{p-2} g^{\alpha\beta} D_\beta u \cdot D_\beta \gamma(u) \quad \text{almost everywhere in } Q,$$

where in the second line, we use that $\partial_t u$, $D_\alpha u \in \mathcal{T}_u\mathcal{N}$, $\alpha = 1, \ldots, m$, and $\gamma(u) \in (\mathcal{T}_u\mathcal{N})^\perp$ in Q. The last line yields, at $z = \bar{z}$,

$$|\nu(\bar{z})| = -|Du(\bar{z})|^{p-2} g^{\alpha\beta}(\bar{x}) D_\beta u(\bar{z}) \cdot \left(D_\beta u(\bar{z}) \cdot D_u \gamma(u)|_{u=u(\bar{z})}\right).$$

Thus, (4.22) actually holds true.

Furthermore, there exists a positive constant C depending only on bounds of curvature of $\mathcal{N}$ and $(g^{\alpha\beta})$ such that

$$|\nu| \le C\,|Du|^p \quad \text{almost everywhere in } Q. \tag{4.23}$$

In fact, from (4.22) we obtain

$$|\nu(z)| \le C \max_{v \in u(Q)} |D_v \gamma(v)|\,|Du(z)|^p \quad \text{for almost every } z \in Q.$$

By (4.23) and (4.10) we have that

$$\partial_t u - \Delta_p u = -\nu \in L^\infty(Q) \quad \text{almost everywhere in } Q \\ \Longrightarrow Du \text{ is locally Hölder continuous in } Q, \tag{4.24}$$

where for the last statement of gradient continuity, we refer to [12, Theorem 1.1, p. 245; Sect. 4, p. 291; Sect. 1-(ii), pp. 217–218] (also see [26]).

As a consequence, we have that $(t_0,\, x_0)$ is a regular point and thus, $\Sigma \subset \mathcal{S}$. Furthermore, from (4.21) and (4.22) it follows that the limit map u satisfies the p-harmonic flow Eq. (1.4) almost everywhere in Q.

Size estimate of the singular set We recall again that $\Sigma = \mathrm{Sing}(u) \subset \mathcal{S}$. Let us estimate the size of $\mathcal{S}$.

From the definition of limit supremum on k and (4.7), we see that, for every $\tau, 0 < \tau < \infty$, and $R, 0 < R < \min\{R_0,\, \tau^{1/B_0}\}$,

$$\mathcal{S}(\tau,\, R) \subset \mathcal{T}(\tau,\, R). \tag{4.25}$$

Here we have the estimation of size (see [17, Theorem 2.2; its proof, pp. 101–103], [21] for the proof) : It holds that, for every τ, $0 < \tau < \infty$, and R, $0 < R < \min\{R_0,\ \tau^{1/B_0}\}$,

$$\mathcal{H}^{m-\gamma_0}(\mathcal{T}(\tau,\ R)) = 0$$

and so, by (4.25),

$$\mathcal{H}^{m-\gamma_0}(\mathcal{S}(\tau,\ R)) = 0; \quad \mathcal{H}^{m-\gamma_0}(\mathcal{S}(\tau)) = 0.$$

Thus, for any positive $\tau < \infty$,

$$\{\tau\} \times \Sigma \subset \mathcal{S}(\tau); \quad \mathcal{H}^{m-\gamma_0}\big(\{\tau\} \times \Sigma\big) = 0.$$

Then, the m-dimensional Hausdorff measure of $\mathcal{S} \cap \mathcal{M}_\infty$ with respect to the time-space metric $|t|^{1/\gamma_0} + |x|$ is locally zero : For any positive $T < \infty$, letting $\mathcal{M}_T = (0,\ T) \times \mathcal{M}$,

$$\mathcal{H}^m\Big(\mathcal{S} \bigcap \mathcal{M}_T\Big) = \int_0^T \mathcal{H}^{m-\gamma_0}\big(\mathcal{S}(\tau)\big)\, d\tau = 0.$$

Weak solution of the p-harmonic flow Now we set the two sequences of real-numbers as follows : Let Λ_0 be a positive number. Let ϵ be any small positive number and $R_0 < 1$ be a sufficient small positive number, which are sent to zero, later. For positive constants $M > 1$ and $\theta < 1$, we define two geometrical progressions as

$$\Lambda_l = \Lambda_0 M^l; \quad R_l = R_0 \theta^l, \quad l = 0, 1, 2, \ldots. \tag{4.26}$$

It is seen that $\Lambda_l \nearrow \infty$ and $R_l \searrow 0$ as $l \to \infty$.

Let K be any time-space domain, $K = (0, T) \times B(R_{\mathcal{M}},\ x_0)$ for $T > 0$ and a geodesic ball $B(R_{\mathcal{M}},\ x_0)$ in $\mathcal{M}$. We set a family of sets $\mathcal{S}_l$, $l = 0, 1, 2, \ldots$, as

$$\begin{aligned} \mathcal{S}_0 &= \Big\{z \in \mathcal{M}_\infty \ :\ |Du(z)| \le \Lambda_0\Big\} \bigcap \Big(K \bigcap \mathcal{S}\Big); \\ \mathcal{S}_l &= \Big\{z \in \mathcal{M}_\infty \ :\ \Lambda_{l-1} < |Du(z)| \le \Lambda_l\Big\} \bigcap \Big(K \bigcap \mathcal{S}\Big), \quad l = 1, 2, \ldots, \end{aligned} \tag{4.27}$$

where $\mathcal{S}$ is as in (4.8).

By the size of $\mathcal{S}$ shown as before and the compactness of $K \bigcap \mathcal{S}$, we can choose a covering of $K \bigcap \mathcal{S}$ in the following way : There exist sequences of positive numbers $\{r_{l\,i}\}$ and time-space points $\{z_{l\,i}\}$, $l = 0, 1, 2, \ldots$; $i = 1, 2, \ldots, I(l)$ with finite integer $I(l)$ depending on each l, such that, for each $l = 0, 1, 2, \ldots$,

$$\begin{aligned} &P(r_{l\,i})(z_{l\,i}) = (t_{l\,i} - (r_{l\,i})^{\gamma_0},\ t_{l\,i} + (r_{l\,i})^{\gamma_0}) \times B(r_{l,\,i})(x_{l\,i}), \\ &z_{l\,i} = (t_{l\,i},\ x_{l\,i}), \quad r_{l\,i} \le R_l, \quad i = 1, 2, \ldots I(l), \end{aligned} \tag{4.28}$$

are a family of time-space cylinders and a covering of $\mathcal{S}_l$ in the sense that

$$\begin{aligned} &P_{l\,i} := P(r_{l\,i})(z_{l\,i}) \quad : \quad \text{disjoint each other ;} \\ &P'_{l\,i} := P(5\,r_{l\,i})(z_{l\,i}), \quad \bigcup_{i=1}^{I(l)} P'_{l\,i} \supset \mathcal{S}_l; \quad \sum_{i=1}^{I(l)} (5\,r_{l\,i})^m \le \epsilon, \end{aligned} \tag{4.29}$$

where ϵ is firstly taken as small positive number.

Furthermore, by the compactness of $K \bigcap \mathcal{S}$, we can take a covering of $K \bigcap \mathcal{S}$ from $\{P'_{l\,i}\}$, obtained above, which consists of *finitely many* time-space cylinders $P'_{l\,i}$, $l = 0, 1, 2, \ldots, L$ with finite integer L; $i = 1, 2, \ldots, I(l)$, and has the properties

$$P_{l\,i} := P(r_{l\,i})(z_{l\,i}) \;:\; \text{disjoint each other}, \quad l = 0, 1, 2, \ldots, L;\; i = 1, 2, \ldots, I(l);$$
$$\bigcup_{l=0}^{L} \bigcup_{i=1}^{I(l)} P'_{l\,i} \supset K \bigcap \mathcal{S}; \quad \sum_{l=0}^{L} \sum_{i=1}^{I(l)} (5\,r_{l\,i})^m \le \epsilon. \tag{4.30}$$

Let η be a smooth function on $\mathcal{M}_\infty$ such that $0 \le \eta \le 1$, $\eta = 1$ in $P(1)(0) := (-1,\ 1) \times B(1)(0)$ and the support of η is contained in $P(2)(0) := \big(-2^{\gamma_0},\ 2^{\gamma_0}\big) \times B(2)(0)$, and $|\partial_t \eta| + |D\eta| \le C$ with a positive number C depending only on m and γ_0. For $l = 0, 1, 2, \ldots$; $i = 1, 2, \ldots, I(l)$, we denote by $\eta_{l\,i}$ the scaled function

$$\eta_{l\,i}(t,\ x) = \eta\Big((t - t_{l\,i})/(5\,r_{l\,i})^{\gamma_0},\ (x - x_i)/5\,r_{l\,i}\Big)$$

and then, the support of $\eta_{l\,i} \subset P''_{l\,i} := P(10\,r_{l\,i})(z_{l\,i})$.

Let $\mathcal{L} := \{0, 1, 2, \ldots, L\}$, $\mathcal{I}(l) := \{1, 2, \ldots, I(l)\}$. Let ϕ be any smooth map defined on $\mathcal{M}_\infty$ with values into $\mathbb{R}^l$ with compact support in K. From (4.21) we obtain

$$\begin{aligned} 0 &= \int_K \big(\partial_t u - \Delta_p u + \nu\big) \cdot \phi \inf_{l \in \mathcal{L};\, i \in \mathcal{I}(l)} (1 - \eta_{l\,i})\, dz \\ &= \int_K \big(\partial_t u \cdot \phi + |Du|^{p-2} g^{\alpha\beta} D_\beta u \cdot D_\alpha \phi + \nu \cdot \phi\big) \inf_{l \in \mathcal{L};\, i \in \mathcal{I}(l)} (1 - \eta_{l\,i})\, dz \\ &\quad - \int_K |Du|^{p-2} g^{\alpha\beta} D_\beta u \cdot \sup_{l \in \mathcal{L};\, i \in \mathcal{I}(l)} (\phi\, D_\alpha \eta_{l\,i})\, dz. \end{aligned} \tag{4.31}$$

We note that the number of overlaps of $\big\{P''_{l\,i}\big\}$, $l \in \mathcal{L};\ i \in \mathcal{I}(l)$, is at most finite and thus, there exists a subfamily $\{Q_{l\,i}\}$ of $\big\{P''_{l\,i}\big\}$ such that

$$|D\eta_{l\,i}(z)| = \sup_{l \in \mathcal{L},\, i \in \mathcal{I}(l)} |D\eta_{l\,i}(z)| \quad \text{for } Q_{l\,i} := \exists P''_{l\,i} \ni z$$

for any $z \in \bigcup_{l \in \mathcal{L}} \bigcup_{i \in \mathcal{I}(l)} \big(\operatorname{supp}(D\eta_{l\,i}) \bigcap P''_{l\,i} \bigcap \mathcal{S}_l\big)$. Thus, the last error term in (4.31) is estimated above by

$$\begin{aligned} \int_K |Du|^{p-1} |\phi| \sup_{l \in \mathcal{L};\, i \in \mathcal{I}(l)} |D\eta_{l\,i}|\, dz &= \int_{\bigcup_{l \in \mathcal{L}} \bigcup_{i \in \mathcal{I}(l)} (P''_{l\,i} \bigcap \mathcal{S}_l)} |Du|^{p-1} |\phi| \sup_{l \in \mathcal{L};\, i \in \mathcal{I}(l)} |D\eta_{l\,i}|\, dz \\ &= \int_{\bigcup_{l \in \mathcal{L}} \bigcup_{i \in \mathcal{I}(l)} (Q_{l\,i} \bigcap \mathcal{S}_l)} |Du|^{p-1} |\phi| |D\eta_{l\,i}|\, dz \\ &\le \sup_K |\phi| \sum_{l=0}^{L} \sum_{i=1}^{I(l)} \left(C\, (r_{l\,i})^{-1} \int_{Q_{l\,i} \bigcap \mathcal{S}_l} |Du|^{p-1}\, dz \right), \end{aligned}$$

of which the last integral is bounded by

$$
\begin{aligned}
(r_{l\,i})^{-1}\,|Q_{l\,i}|\,(\Lambda_l)^{p-1} &= (r_{l\,i})^{-1}\,\left|P''_{l\,i}\right|\,(\Lambda_l)^{p-1} \\
&\le C\,(\Lambda_l)^{p-1}\,(r_{l\,i})^{m+\gamma_0-1} \\
&\le C\,(\Lambda_l)^{p-1}\,(R_l)^{\gamma_0-1}\,(r_{l\,i})^{m},
\end{aligned}
$$

with a positive constant C depending only on m, where we use that $\gamma_0 > 1$ and that, by (4.27) and (4.28), for each $l = 0, 1, 2, \ldots,$

$$
|Du| \le \Lambda_l \quad \text{in } \mathcal{S}_l; \quad r_{l\,i} \le R_l, \quad i = 1, 2, \ldots, I(l).
$$

Thus, it holds that

$$
\begin{aligned}
\int_K |Du|^{p-1}|\phi| \sup_{l\in\mathcal{L};\, i\in\mathcal{I}(l)} |D\eta_{l\,i}|\,dz &\le C \sup_K |\phi| \sum_{l=0}^{L}(\Lambda_l)^{p-1}(R_l)^{\gamma_0-1}\sum_{i=1}^{I(l)}(r_{l\,i})^m \\
&\le C'\epsilon \sum_{l=0}^{\infty}(\Lambda_l)^{p-1}(R_l)^{\gamma_0-1},
\end{aligned}
\tag{4.32}
$$

where we use (4.29) and the positive constant C' depends only on m, γ_0 and $\sup_K |\phi|$. For summation on l, we choose the ratios $M > 1$ and $\theta < 1$ in (4.26) as

$$
0 < \theta < 1; \quad M = \theta^{-a} \quad \text{for some } a > 0 \text{ chosen later} \tag{4.33}
$$

and compute

$$
(\Lambda_l)^{p-1}(R_l)^{\gamma_0-1} = (\Lambda_0)^{p-1}\,(R_0)^{\gamma_0-1}\,\theta^{l(-a(p-1)+\gamma_0-1)}
$$

and thus,

$$
\begin{aligned}
\sum_{l=0}^{\infty}(\Lambda_l)^{p-1}(R_l)^{\gamma_0-1} &\le (\Lambda_0)^{p-1}\,(R_0)^{\gamma_0-1}\sum_{l=0}^{\infty}\theta^{l(-a(p-1)+\gamma_0-1)} \\
&= \frac{(R_0)^{\gamma_0-1}\,(\Lambda_0)^{p-1}}{1-\theta^{-a(p-1)+\gamma_0-1}},
\end{aligned}
\tag{4.34}
$$

provided that

$$
-a(p-1)+\gamma_0-1 > 0 \iff 0 < a < \frac{\gamma_0-1}{p-1}; \quad \gamma_0 > 1.
$$

Finally, we see from (4.29) and definition of $\eta_{l\,i}$ that

$$
\inf_{l\in\mathcal{L};\, i\in\mathcal{I}}(1-\eta_{l\,i}) \to 1 \quad \text{almost everywhere in } K \text{ as } R_0 \searrow 0 \tag{4.35}
$$

and thus, we can take the limit as $R_0 \searrow 0$ in (4.32) and (4.34), and use the Lebesgue convergence theorem with (4.35) in the second line of (4.31) to find that the limit map u satisfies the p-harmonic flow equation in the weak sense. □

Convergence to the p-harmonic map at a time-infinity We will present the convergence of u to a p-harmonic map as time tends to infinity. By (1.14) we choose a sequence of time $\{\tau_l\}$, $\tau_l \nearrow \infty$, and a limit map u_∞ such that, as $l \to \infty$,

$$
u(\tau_l) \longrightarrow u_\infty \quad \text{weakly in } W^{1,p}\big(\mathcal{M},\,\mathbb{R}^l\big) \tag{4.36}
$$

$$
Du(\tau_l) \longrightarrow Du_\infty \quad \text{weakly in } L^p\big(\mathcal{M},\,\mathbb{R}^{ml}\big) \tag{4.37}
$$

$$\partial_t u(\tau_l) \longrightarrow 0 \quad \text{strongly in } L^2\big(\mathcal{M},\ \mathbb{R}^l\big), \tag{4.38}$$

where from (1.14) we obtain that, for some time-sequence $\{t_l\}$, $t_l \nearrow \infty$ as $l \to \infty$,

$$\|\partial_t u(\tau_l)\|^2_{L^2(\mathcal{M})} = \|\partial_t u\|^2_{L^2((t_{l-1},\, t_l)\times\mathcal{M})} \longrightarrow 0.$$

Then, from the convergence (4.1), (4.2) and (4.3), there exists a subsequence of $\{u_k(\tau_k)\}$, satisfying the same convergence as in (4.36), (4.37) and (4.38) with $u(\tau_l)$ replaced by $u_k(\tau_k)$, as $k \nearrow \infty$.

Let us define the *regular set* of u_∞ as

$$\mathrm{Reg}(u_\infty) := \{x_0 \in \mathcal{M}\ :\ u_\infty \text{ is regular in a neghborhood of } x_0\} \tag{4.39}$$

and the *singular set* $\mathrm{Sing}(u_\infty)$ as the complement of $\mathrm{Reg}(u_\infty)$, $\Sigma_\infty := \mathrm{Sing}(u_\infty) = \mathcal{M}\setminus \mathrm{Reg}(u_\infty)$. By definition, $\mathrm{Reg}(u_\infty)$ is relatively open in $\mathcal{M}$ and $\mathrm{Sing}(u_\infty)$ is relatively closed in $\mathcal{M}$.

Let us put, for $0 < R < R_0$,

$$\mathcal{S}_\infty(R) := \left\{x_0 \in \mathcal{M}\ :\ \limsup_{k\to\infty}\left(\limsup_{r\searrow 0} r^{\gamma_0-m}\int_{\{t=\tau_k-R^{B_0}\}\times B(r,\,x_0)} e_k(u_k(t,\ x))\,d\mathcal{M}\right) \ge 1\right\};$$

$$\mathcal{S}_\infty := \bigcap_{0<R<R_0} \mathcal{S}_\infty(R). \tag{4.40}$$

Then, similarly as in *Size estimate of the singular set* before, we have that, for any positive $R < R_0$,

$$\mathcal{H}^{m-\gamma_0}\big(\mathcal{S}_\infty(R)\big) = 0; \quad \mathcal{H}^{m-\gamma_0}\big(\mathcal{S}_\infty\big) = 0. \tag{4.41}$$

We will show that $\Sigma_\infty \subset \mathcal{S}_\infty$. Let x_0 be in the complement of $\mathcal{S}_\infty$ and then, there exist a positive $R < R_0$, a subsequence of $\{u_k(\tau_k)\}$, denoted by the same notation as before, such that

$$\limsup_{r\searrow 0} r^{\gamma_0-m}\int_{\{t=\tau_k-R^{B_0}\}\times B(r,\,x_0)} e_k(u_k(t,\ x))\,d\mathcal{M} < 1.$$

Then, by Theorem 11, we have

$$\sup_{(\tau_k-(R/4)^{B_0},\ \tau_k)\times B(R/4,\,x_0)} e_k(u_k) \le C\,R^{-pa_0}, \tag{4.42}$$

where the positive constant C depends only on p, $\mathcal{M}$ and $\mathcal{N}$. Based on (4.42), we can proceed the same limit process as in (4.11)–(4.24) to find that u_∞ is regular in $B(R/8,\ x_0)$ and thus, $x_0 \in \mathrm{Reg}(u_\infty) = \mathcal{M}\setminus\Sigma_\infty$. Therefore, the complement of $\mathcal{S}_\infty$ is contained in that of Σ_∞, $\mathrm{Reg}(u_\infty)$, and thus, $\Sigma_\infty \subset \mathcal{S}_\infty$. By use of the size estimate of $\mathcal{S}_\infty$ in (4.41), we can adopt the similar argument as in (4.26)–(4.35), where time-space regions used are replaced by the corresponding space regions, and thus, find that u_∞ is a weak solution of the p-harmonic map. □

5 Monotonicity estimate of a local scaled energy

We now prove the monotonicity type estimate.

We make parallel translation on time of the Eq. (2.1) and its solutions u on $(T,\ \infty) \times B(R_\mathcal{M})$ to those on $(0,\ \infty) \times B(R_\mathcal{M})$ with the same notation.

Hereafter we assume that the metric $g = \left(g_{\alpha\beta}\right)$ is the identity matrix. In the general case with $\left(g_{\alpha\beta}\right)$, the lower order terms containing the derivatives of $g_{\alpha\beta}$ only appear and controlled well as in the following estimations.

Let (t_0, x_0) in the parabolic like envelope $\left\{(t, x) : \min\{1, (R_{\mathcal{M}})^{B_0}\} > t \geq |x|^{B_0}\right\}$, $B_0 > 2$.

First we prove the backward monotonicity estimate, Lemma 12. Our localized scaled penalized energy is defined as

$$E(r) = \frac{1}{\Lambda^p} \int_{\{t=t_0-\Lambda^{2-p} r^2\} \times B(R_{\mathcal{M}})} \bar{e}(u(t, x)) \mathcal{B}(t_0, x_0; t, x) \mathcal{C}^q(t, x)\, dx; \tag{5.1}$$

$$\bar{e}(u) := \bar{e}_{K,\epsilon}(u) = \frac{1}{p}\left(\epsilon + |Du|^2\right)^{\frac{p}{2}} + C_0 \frac{K}{2} \chi\left(\operatorname{dist}^2(u, \mathcal{N}\right);$$

$$\Lambda = \Lambda(r) = r^{\frac{B_0-2}{2-p}}; \quad p > B_0 > \frac{6p-4}{p+2}; \quad 0 < r \leq \min\{1, R_{\mathcal{M}}, (t_0)^{1/B_0}\} \tag{5.2}$$

with weight

$$\mathcal{B}(t_0, x_0; t, x) = \frac{1}{(t_0 - t)^{\frac{m}{B_0}}} \left(1 - \left(\frac{|x - x_0|}{(t_0 - t)^{\frac{1}{B_0}}}\right)^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}}, \qquad t < t_0;$$

$$\mathcal{C}(t, x) = \left(t^{1/B_0} - |x|\right)_+; \quad q > 2. \tag{5.3}$$

Hereafter, for brevity, we use the notation as above.

Lemma 18 *Let $p > 2$ and $q > 2$. For any regular solution u to (2.1) the following estimate is valid for any positive number $r < \rho \leq \min\{1, R_{\mathcal{M}}, (t_0/2)^{1/B_0}\}$*

$$\begin{aligned} E(r) \leq\ & E(\rho) + C\left(\rho^{\mu} - r^{\mu}\right) \\ & + C \int_{t_0-\rho^{B_0}}^{t_0-r^{B_0}} \|\mathcal{C}^{q-2}(t)\, (\bar{e}(u(t)))^{\theta_0}\|_{L^\infty(B((t_0-t)^{1/B_0}, x_0))}\, dt, \end{aligned} \tag{5.4}$$

where

$$\Lambda = \Lambda(r) = r^{\frac{B_0-2}{2-p}}; \quad (\Lambda(r))^{2-p} r^2 = r^{B_0}$$

and the positive exponents $\theta_0 \geq 2$ and μ depend only on B_0, p and $\mathcal{N}$, m, p and B_0, respectively, and the positive constant C depends only on the same ones as μ and q.

The proof is proceeded similarly as in [33, Lemmta 5 and 6]. Here we will study how to control well the approximating term, the derivative of penalty term.

Proof of Lemma 18. As before, let

$$\Lambda = r^{\frac{B_0-2}{2-p}}, \quad p > B_0 > \frac{6p-4}{p+2}$$

and let r be any positive number in the range $0 < r \leq \min\left\{1, R_{\mathcal{M}}, (t_0/2)^{1/B_0}\right\}$. First we make a scaling transformation intrinsic to the evolutionary p-Laplace operator

$$t = t_0 + \Lambda^{2-p} r^2 s; \quad x = x_0 + r\, y; \quad v(s, y) = \frac{u(t_0 + \Lambda^{2-p} r^2 s,\ x_0 + r\, y)}{\Lambda r} \tag{5.5}$$

and, under the scaling transformation

$$t = t_0 - \Lambda^{2-p} r^2 \iff s = -1.$$

Then the scaled solution v is a solution of the scaled equation on $\{s = -1\} \times \{y \in \mathbb{R}^m : x_0 + ry \in B(R_{\mathcal{M}})\}$

$$\partial_s v - \operatorname{div}\left(\left(\Lambda^{-2}\epsilon + |Dv|^2\right)^{\frac{p-2}{2}} Dv\right) = -C_0 \frac{K/\Lambda^p}{2} D_v \chi\left(\operatorname{dist}^2(\Lambda\, r\, v,\, \mathcal{N})\right) \tag{5.6}$$

and we put the notation

$$\begin{aligned}
&\bar{\epsilon} = \Lambda^{-2}\epsilon; \quad \bar{K} = \Lambda^{-p} K;\\
&f = f(v) := \frac{1}{p}\left(\bar{\epsilon} + |Dv|^2\right)^{\frac{p}{2}}; \quad g = g(v) := \frac{\bar{K}}{2}\chi\left(\operatorname{dist}^2(\Lambda r v,\, \mathcal{N})\right);\\
&\Delta_p v = \operatorname{div}\left(\left(p\, f\right)^{1-\frac{2}{p}} Dv\right); \quad \bar{e} = \bar{e}(v) = f(v) + C_0\, g(v).
\end{aligned}$$

The scaled penalized energy is rewritten as

$$\begin{aligned}
&E(r) = \int_{\{s=-1\}\times\mathbb{R}^m} \bar{e}(v(s,\, y))\, \mathcal{B}(s,\, y)\, \mathcal{C}^q(s,\, y)\, dy;\\
&\mathcal{B}(s,\, y) = \frac{1}{(-s)^{\frac{m}{B_0}}}\left(1 - \left(\frac{|y|}{(-s)^{\frac{1}{B_0}}}\right)^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}}; \ \mathcal{C}(s,\, y) = \left((t_0 + r^{B_0} s)^{1/B_0} - |x_0 + r\, y|\right)_+,
\end{aligned} \tag{5.7}$$

where the integral in (5.7) is well-defined by supp($\mathcal{C}$) and supp($\mathcal{B}$) and we simply compute as

$$\begin{aligned}
&Dv(s,\, y) = \frac{1}{\Lambda} D_x u(t,\, x); \quad \bar{e}(v) = \frac{1}{\Lambda^p}\bar{e}(u)\\
&\Lambda = r^{\frac{B_0-2}{2-p}} \iff \Lambda^{\frac{p-2}{B_0}}\, r^{\frac{B_0-2}{B_0}} = 1; \quad \mathcal{B}(s,\, y)\, dy = \mathcal{B}\left(t_0,\, x_0;\, t,\, x\right)\, dx.
\end{aligned}$$

Our main task in monotonicity estimate is to derive appropriate values of parameter such that

$$p > B_0 > \frac{6p-4}{p+2}. \tag{5.8}$$

Step 1 : differentiation of $E(r)$ on r. We compute differentiation of $E(r)$ on r.

$$\begin{aligned}
\frac{d}{dr}E(r) &= \int_{\{s=-1\}\times\mathbb{R}^m} \frac{d}{dr}\bar{e}(v)\, \mathcal{B}\,\mathcal{C}^q\, dy + \int_{\{s=-1\}\times\mathbb{R}^m} \bar{e}(v)\,\mathcal{B}\,\frac{d}{dr}\mathcal{C}^q\, dy\\
&= \int_{\{s=-1\}\times\mathbb{R}^m} \left((p\, f)^{1-\frac{2}{p}} Dv \cdot \frac{d}{dr}Dv + C_0\, \frac{dv}{dr}\cdot D_v g(v)\right)\mathcal{B}\,\mathcal{C}^q dy\\
&\quad + \frac{B-2}{r\,(p-2)}\int_{\{s=-1\}\times\mathbb{R}^m} \left(\bar{\epsilon}\,(p\, f)^{1-\frac{2}{p}} + p\, C_0\, g(v)\right)\mathcal{B}\,\mathcal{C}^q\, dy
\end{aligned}$$

$$+ \int_{\{s=-1\}\times\mathbb{R}^m} \bar{e}(v)\,\mathcal{B}\,\frac{d}{dr}\mathcal{C}^q\,dy$$
$$=: I + II + III, \tag{5.9}$$

since

$$\frac{d\bar{e}(v)}{dr} = (p\,f)^{1-\frac{2}{p}}\left(\frac{\bar{\epsilon}(B-2)}{r\,(p-2)} + Dv\cdot\frac{d}{dr}Dv\right) + \frac{C_0\,p(B-2)}{r\,(p-2)}g(v) + C_0\,\frac{dv}{dr}\cdot D_v g(v).$$

Estimations of II and III. The term II is nonnegative.

The term III is estimated by Young's inequality as

$$III = q\int_{\{s=-1\}\times\mathbb{R}^m} \bar{e}(v)\,\mathcal{B}\,\mathcal{C}^{q-1}\,\frac{d\mathcal{C}}{dr}\,dy$$
$$\geq -\frac{C}{r}\int_{\{s=-1\}\times\mathbb{R}^m} \bar{e}(v)\,\mathcal{C}^{q-1}\,\mathcal{B}\,dy$$
$$\geq -\frac{C}{r^{1+\delta}}\int_{\{s=-1\}\times\mathbb{R}^m} (\bar{e}(v))^{\frac{2(p-1)}{p}}\,\mathcal{C}^q\,\mathcal{B}\,dy - \frac{C\,r^{\frac{\delta p}{p-2}}}{r}\int_{\{s=-1\}\times\mathbb{R}^m} \mathcal{C}^{q-\frac{2(p-1)}{p-2}}\,\mathcal{B}\,dy, \tag{5.10}$$

where $\mathcal{C}(s,\ y) := \left((t_0 + r^{B_0}\,s)^{1/B_0} - |x_0 + r\,y|\right)_+$ is a Lipschitz function and the derivative of $\mathcal{C}$ on r is computed as

$$\left|\frac{d}{dr}\mathcal{C}(s,\ y)\right| = \left|\frac{d}{dr}\left((t_0 + r^{B_0}s)^{1/B_0} - |x_0 + ry|\right)_+\right|$$
$$= \chi_{\{|x_0+ry|\leq(t_0+r^{B_0}s)^{1/B_0}\}}\left|(t_0 + r^{B_0}s)^{1/B_0}\,\frac{r^{B_0-1}s}{(t_0 + r^{B_0}s)} - \frac{x_0+ry}{|x_0+ry|}\cdot y\right|$$

and thus, on the support $\{y\in\mathbb{R}^m\ :\ |y|<1\}$ of $\mathcal{B}(-1,\ y)$

$$\left|\frac{d}{dr}\mathcal{C}\right|\Big|_{s=-1} \leq 2\chi_{\{|x_0+ry|\leq(t_0+r^{B_0})^{1/B_0}\}}\,r^{-1}$$

because of the conditions

$$0 < t_0 \leq 1; \quad \frac{r^{B_0}}{t_0 - r^{B_0}} \leq 1 \Longleftrightarrow r^{B_0} \leq \frac{t_0}{2}.$$

The 1st term of (5.10) is scaled back and bounded below by

$$-\frac{C}{r^{1+\delta}}\frac{1}{\Lambda^{2(p-1)}}\ \|\,(\bar{e}(u(t)))^{\frac{2(p-1)}{p}}\ \mathcal{C}^q(t)\|_{L^\infty(\mathrm{supp}(\mathcal{B}(t)))}\Big|_{t=t_0-r^{B_0}}\int_{\{s=-1\}\times\mathbb{R}^m}\mathcal{B}\,dy,$$

where we use $\Lambda^{2-p}\,r^2 = r^{B_0}$,

$$\int_{\{s=-1\}\times\mathbb{R}^m}\mathcal{B}\,dy = \int_{\mathbb{R}^m}\left(1-|y|^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}}\,dy < \infty$$

and the notation

$$\bar{e}(u) := f(u) + C_0\,g(u) := \frac{1}{p}\left(\epsilon + |Du|^2\right)^{\frac{p}{2}} + C_0\,\frac{K}{2}\chi\left(\mathrm{dist}^2(u,\ \mathcal{N})\right).$$

Each term of I is separately estimated in the following.

Estimation of I. By integration by parts, we have

$$\begin{aligned}
I =& \int_{\{s=-1\}\times\mathbb{R}^m} \Big(-\operatorname{div}\big((p\,f)^{1-\frac{2}{p}} Dv\,\mathcal{B}\,\mathcal{C}^q\big) + C_0\, D_v g(v)\,\mathcal{B}\,\mathcal{C}^q\Big)\cdot \frac{dv}{dr}\,dy \\
=& \frac{1}{r}\left(1+\frac{r\,\Lambda'}{\Lambda}\right) \int_{\{s=-1\}\times\mathbb{R}^m} \Big\{ \operatorname{div}\big((p\,f)^{1-\frac{2}{p}} Dv\,\mathcal{B}\,\mathcal{C}^{q/2}\big)\cdot v\,\mathcal{C}^{q/2} - C_0\, v\cdot D_v g(v)\,\mathcal{B}\,\mathcal{C}^q \\
&\qquad + (p\,f)^{1-\frac{2}{p}} Dv\cdot\Big(v\,D\mathcal{C}^{q/2}\Big)\,\mathcal{B}\,\mathcal{C}^{q/2}\Big\}\,dy \\
&+\frac{1}{r}\int_{\{s=-1\}\times\mathbb{R}^m} \Big(-\operatorname{div}\big((p\,f)^{1-\frac{2}{p}} Dv\,\mathcal{B}\,\mathcal{C}^q\big) + C_0\, D_v g(v)\,\mathcal{B}\,\mathcal{C}^q\Big)\cdot \\
&\qquad \cdot\Big(\big((2-p)\,r\,\Lambda^{-1}\,\Lambda'+2\big)\,s\,\partial_s v + y\cdot Dv\Big)\,dy \\
=:& I_1+I_2+I_3,
\end{aligned} \tag{5.11}$$

where the generator of dilation is computed as

$$\frac{dv}{dr} = r^{-1}\,\Big(-\big(1+r\,\Lambda^{-1}\,\Lambda'\big)\,v + \big((2-p)\,r\,\Lambda^{-1}\,\Lambda'+2\big)\,s\,\partial_s v + y\cdot Dv\Big). \tag{5.12}$$

Estimation of I_1.

$$\begin{aligned}
I_1 =& \frac{1}{r}\left(1+\frac{r\,\Lambda'}{\Lambda}\right) \int_{\{s=-1\}\times\mathbb{R}^m} \Big\{\Big(\operatorname{div}\big((p\,f)^{1-\frac{2}{p}} Dv\,\mathcal{B}\,\mathcal{C}^{q/2}\big) - C_0\, D_v g(v)\,\mathcal{B}\,\mathcal{C}^{q/2}\Big)\cdot \\
&\qquad \cdot\big(v\,\mathcal{C}^{q/2}-\bar{v}\big) - C_0\,\bar{v}\cdot D_v g(v)\,\mathcal{B}\,\mathcal{C}^{q/2}\Big\}\,dy \\
=:& I_{11}+I_{12},
\end{aligned}$$

where $\bar{v}$ is a weighted integral mean as in (5.14) below, and

$$\begin{aligned}
&-\frac{1}{r}\left(1+r\,\frac{\Lambda'}{\Lambda}\right)\bar{v}\Big|_{s=-1}\cdot\int_{\{s=-1\}\times\mathbb{R}^m} \operatorname{div}\Big((p\,f)^{1-\frac{2}{p}} Dv\,\mathcal{B}\,\mathcal{C}^{q/2}\Big)\,dy \\
&\quad = 0,
\end{aligned}$$

because of Gauss's divergence theorem and the compactness of support of $\mathcal{B}$ and $\mathcal{C}$.

Each term I_{11} and I_{12} is separately estimated in the following.

Estimation of I_{11}. I_{11} is computed as

$$\begin{aligned}
I_{11} =& \frac{1}{r}\left(1+r\,\frac{\Lambda'}{\Lambda}\right) \int_{\{s=-1\}\times\mathbb{R}^m} \Big((v\,\mathcal{C}^{q/2})-\bar{v}\Big)\cdot\Big(\big(\Delta_p v - C_0\,D_v g(v)\big)\,\mathcal{B}\,\mathcal{C}^{q/2}+ \\
&\qquad +(p\,f)^{1-\frac{2}{p}} Dv\cdot D\mathcal{B}\,\mathcal{C}^{q/2} + (p\,f)^{1-\frac{2}{p}} Dv\cdot D\mathcal{C}^{q/2}\,\mathcal{B}\Big)\,dy \\
=& \frac{1}{r}\left(1+r\,\frac{\Lambda'}{\Lambda}\right) \int_{\{s=-1\}\times\mathbb{R}^m} \Big((v\,\mathcal{C}^{q/2})-\bar{v}\Big)\cdot
\end{aligned}$$

$$
\cdot\Big\{\partial_s v\,\mathcal{B}\,\mathcal{C}^{q/2}+\frac{p\,(p\,f)^{1-\frac{2}{p}}\,y\cdot Dv\,\Big(1-|y|^{\frac{p}{p-1}}\Big)_+^{\frac{1}{p-2}}\,\mathcal{C}^{q/2}}{(p-2)\,|y|^{\frac{p-2}{p-1}}}
+(p\,f)^{1-\frac{2}{p}}\,Dv\cdot D\mathcal{C}^{q/2}\,\mathcal{B}\Big\}\,dy
$$

$$
=: I_{111}+I_{112}+I_{113}. \tag{5.13}
$$

Now we will estimate each of three terms in (5.13).

For estimation of I_{111} we use the Poincaré type inequality with weight of Barenblatt like function [35, Theorem 5.3.4, p. 134]. Let $\bar{v}$ be a weighted integral mean

$$
\bar{v}=\int_{\{s=-1\}\times\mathbb{R}^m}(v\,\mathcal{C}^{q/2})\,\mathcal{B}\,dy\Big/\int_{\{s=-1\}\times\mathbb{R}^m}\mathcal{B}\,dy. \tag{5.14}
$$

Lemma 19 (Poincaré inequality)

$$
\int\limits_{\{s=-1\}\times\mathbb{R}^m}\left|(v\,\mathcal{C}^{q/2})-\bar{v}\right|^2\mathcal{B}\,dy\le C\int\limits_{\{s=-1\}\times\mathbb{R}^m}\left|D(v\,\mathcal{C}^{q/2})\right|^2\mathcal{B}\,dy. \tag{5.15}
$$

I_{111} of (5.13). I_{111} is estimated by Cauchy's inequality for small $c>0$ as

$$
I_{111}\ge-\frac{c}{2\,r}\int\limits_{\{s=-1\}\times\mathbb{R}^m}|\partial_s v|^2\,\mathcal{B}\,\mathcal{C}^q\,dy-\frac{1}{2c\,r}\int\limits_{\{s=-1\}\times\mathbb{R}^m}\left|(v\,\mathcal{C}^{q/2})-\bar{v}\right|^2\mathcal{B}\,dy, \tag{5.16}
$$

where by definition of Λ, $1+r\,\Lambda^{-1}\,\Lambda'=(p-B_0)(p-2)^{-1}$. The 1st time-derivative term is absorbed into that of (5.24) below, later. By the Poincaré inequality (5.15) and Young's inequality with $\delta>0$, the 2nd term is bounded below by

$$
\begin{aligned}
-\frac{C}{2c\,r}\int\limits_{\{s=-1\}\times\mathbb{R}^m}\left|D(v\,\mathcal{C}^{q/2})\right|^2\mathcal{B}\,dy\ge&-\frac{C}{r^{1+\delta}}\int\limits_{\{s=-1\}\times\mathbb{R}^m}(p\,f)^{2\left(1-\frac{1}{p}\right)}\,\mathcal{B}\,\mathcal{C}^q\,dy\\
&-\frac{C}{r^{1-\frac{\delta}{p-2}}}\int\limits_{\{s=-1\}\times\mathbb{R}^m}\mathcal{B}\,\mathcal{C}^q\,dy\\
&-\frac{C}{r\,\Lambda^2}\int\limits_{\{s=-1\}\times\mathbb{R}^m}\mathcal{C}^{q-2}\,\mathcal{B}\,dy,
\end{aligned} \tag{5.17}
$$

where the last term is obtained from the derivative of $\mathcal{C}$ on y, scaling back, a boundedness of the map u with a bound H depending only on $\mathcal{N}$ in Lemma 8

$$
\begin{aligned}
\left|D_y\mathcal{C}(t,\,x)\right|&=\chi_{\{|x_0+ry|\le(t_0+r^{B_0}s)^{1/B_0}\}}\left|-\frac{x_0+ry}{|x_0+ry|}\,r\right|\\
&\le r\,\chi_{\{|x_0+ry|\le(t_0+r^{B_0}s)^{1/B_0}\}};\\
|v|^2\,|D\mathcal{C}|^2&\le\frac{|u|^2}{\Lambda^2\,r^2}\,r^2=\Lambda^{-2}\,H^2.
\end{aligned} \tag{5.18}
$$

I_{112} *of* (5.13). By Cauchy's inequality,

$$
\begin{aligned}
I_{112} \geq & -\frac{C}{r} \int_{\{s=-1\}\times\mathbb{R}^m} \left(p\, f\right)^{1-\frac{1}{p}} \left|(v\, \mathcal{C}^{q/2}) - \bar{v}\right| |y|^{\frac{1}{p-1}} \left(1 - |y|^{\frac{p}{p-1}}\right)_+^{\frac{1}{p-2}} \mathcal{C}^{q/2}\, dy \\
\geq & -\frac{C}{r} \int_{\{s=-1\}\times\mathbb{R}^m} \left|(v\, \mathcal{C}^{q/2}) - \bar{v}\right|^2 \mathcal{B}\, dy \\
& -\frac{C}{r} \int_{\{s=-1\}\times\mathbb{R}^m} \left(p\, f\right)^{2-\frac{2}{p}} |y|^{\frac{2}{p-1}} \left(1 - |y|^{\frac{p}{p-1}}\right)_+^{\frac{3-p}{p-2}} \mathcal{C}^q\, dy,
\end{aligned}
\tag{5.19}
$$

where the 1st one is the same as the 2nd term in (5.16) and bounded below for $\delta > 0$ as in (5.17) and, the 2nd one of (5.19), together with the 1st one of (5.17), is estimated below by

$$
-\frac{C\,(r^{-\delta}+1)}{r\,\Lambda^{2(p-1)}} \, \|\mathcal{C}(t)^q \left(p\, f(u(t))\right)^{2\left(1-\frac{1}{p}\right)}\|_{L^\infty(\mathrm{supp}\,(\mathcal{B}(t)))}\Big|_{t=t_0-r^{B_0}}, \tag{5.20}
$$

where we make a scaling back and compute as

$$
\int_{\mathbb{R}^m} |y|^{\frac{2}{p-1}} \left(1 - |y|^{\frac{p}{p-1}}\right)_+^{\frac{3-p}{p-2}} dy < \infty; \quad \frac{3-p}{p-2} > -1 \Longleftrightarrow 3 > 2.
$$

I_{113} *of* (5.13). By the boundedness (5.18) of derivative of $\mathcal{C}$ and Cauchy's inequality,

$$
\begin{aligned}
I_{113} \geq & -\frac{q(p-B_0)}{2(p-2)\,r} \int_{\{s=-1\}\times\mathbb{R}^m} \left(p\, f\right)^{1-\frac{1}{p}} \left|v\, \mathcal{C}^{q/2} - \bar{v}\right| \mathcal{C}^{q/2-1}\, |D\mathcal{C}|\, \mathcal{B}\, dy \\
\geq & -C \int_{\{s=-1\}\times\mathbb{R}^m} \left(p\, f\right)^{2-\frac{2}{p}} \mathcal{C}^{q-2}\, \mathcal{B}\, dy - C \int_{\{s=-1\}\times\mathbb{R}^m} \left|v\, \mathcal{C}^{q/2} - \bar{v}\right|^2 \mathcal{B}\, dy,
\end{aligned}
$$

of which the 1st term is estimated, similarly as in (5.20), below by

$$
-\frac{C\,(r^{-\delta}+1)}{r\,\Lambda^{2(p-1)}} \, \|\mathcal{C}(t)^{q-2} \left(p\, f(u(t))\right)^{2\left(1-\frac{1}{p}\right)}\|_{L^\infty(\mathrm{supp}\,(\mathcal{B}(t)))}\Big|_{t=t_0-r^{B_0}} \tag{5.21}
$$

and the 2nd term is bounded below as in (5.17).

Estimation of I_{12}. By Cauchy's inequality with $\delta > 0$, we estimate as

$$
\begin{aligned}
r^{-1} \left| \bar{v} \cdot \int_{\{s=-1\}\times\mathbb{R}^m} D_v g\, \mathcal{B}\, \mathcal{C}^{q/2}\, dy \right| & \leq r^{-1}\, |\bar{v}| \int_{\{s=-1\}\times\mathbb{R}^m} |D_v g|\, \mathcal{B}\, \mathcal{C}^{q/2}\, dy \\
\leq & \frac{C(\|u_0\|_{L^\infty(\mathbb{R}^m)}, \mathcal{N})}{\Lambda\, r^2} \int_{\{s=-1\}\times\mathbb{R}^m} |D_v g|\, \mathcal{B}\, \mathcal{C}^{q/2}\, dy \\
\leq & C\, r^{\delta-1} \int_{\{s=-1\}\times\mathbb{R}^m} \mathcal{B}\, dy + \frac{C}{\Lambda^2\, r^{3+\delta}} \int_{\{s=-1\}\times\mathbb{R}^m} |D_v g|^2\, \mathcal{B}\, \mathcal{C}^q\, dy,
\end{aligned}
$$

where we use a boundedness of u with $H > 0$ depending only on $\mathcal{N}$ in Lemma 8 to have

$$
|v(s)| = \frac{|u|}{\Lambda\, r} \leq \frac{H}{\Lambda\, r}; \quad |\bar{v}| \leq \int_{\{s=-1\}\times\mathbb{R}^m} |v(s)| \mathcal{B}\, dy \Big/ \int_{\{s=-1\}\times\mathbb{R}^m} \mathcal{B}\, dy \leq \frac{H}{\Lambda\, r}.
$$

Estimation of I_2. As before by Cauchy's inequality

$$
\begin{aligned}
I_2 &\geq -\frac{q(p-B_0)}{2(p-2)\,r} \int_{\{s=-1\}\times\mathbb{R}^m} \big(p\,f\big)^{1-\frac{1}{p}}\,|v|\,\mathcal{C}^{q/2-1}|D\mathcal{C}|\,\mathcal{C}^{q/2}\,\mathcal{B}\,dy \\
&\geq -\frac{C}{r} \int_{\{s=-1\}\times\mathbb{R}^m} (p\,f)^{2\left(1-\frac{1}{p}\right)}\,\mathcal{C}^q\,\mathcal{B}\,dy - \frac{C}{r\Lambda^2} \int_{\{s=-1\}\times\mathbb{R}^m} \mathcal{C}^{q-2}\,\mathcal{B}\,dy, \quad (5.22)
\end{aligned}
$$

of which the 1st term is estimated below by (5.20).

Estimation of I_3. I_3 is treated as

$$
\begin{aligned}
I_3 = &\int_{\{s=-1\}\times\mathbb{R}^m} \Big(\big((2-p)\,r\,\Lambda^{-1}\,\Lambda'+2\big)\ s\,\partial_s v + y\cdot Dv\Big)\cdot \\
&\cdot\bigg(\Big(-\Delta_p v + C_0\,D_v g(v)\Big)\,\mathcal{B}\,\mathcal{C}^q - \big(p\,f\big)^{1-\frac{2}{p}}\,Dv\cdot\Big(D\mathcal{B}\,\mathcal{C}^q + D\mathcal{C}^q\,\mathcal{B}\Big)\bigg)\ dy \\
= &\frac{1}{r}\int_{\{s=-1\}\times\mathbb{R}^m} \Big(\big((2-p)\,r\,\Lambda^{-1}\,\Lambda'+2\big)\ s\,\partial_s v + y\cdot Dv\Big)\cdot \\
&\cdot\bigg\{-\partial_s v\,\mathcal{B}\,\mathcal{C}^q + \big(p\,f\big)^{1-\frac{2}{p}}\ y\cdot Dv\ \frac{p\left(1-|y|^{\frac{p}{p-1}}\right)_+^{\frac{1}{p-2}}}{(p-2)\,|y|^{\frac{p-2}{p-1}}}\,\mathcal{C}^q \\
&-\big(p\,f\big)^{1-\frac{2}{p}}\,Dv\cdot D\mathcal{C}^q\,\mathcal{B}\bigg\}\ dy. \qquad (5.23)
\end{aligned}
$$

Moreover each term of (5.23) is arranged as

$$
\begin{aligned}
&\frac{1}{r}\int_{\{s=-1\}\times\mathbb{R}^m} (-s)\ \big((2-p)\,r\,\Lambda^{-1}\,\Lambda'+2\big)\ |\partial_s v|^2\,\mathcal{B}\,\mathcal{C}^q\,dy \\
&-\frac{1}{r}\int_{\{s=-1\}\times\mathbb{R}^m} (y\cdot Dv)\cdot\partial_s v\,\mathcal{B}\,\mathcal{C}^q\,dy \\
&+\frac{p}{r\,(p-2)}\int_{\{s=-1\}\times\mathbb{R}^m} \bigg\{\big(p\,f\big)^{1-\frac{2}{p}}\ |y\cdot Dv|^2 \\
&\qquad + \big((2-p)\,r\,\Lambda^{-1}\,\Lambda'+2\big)\ \big(p\,f\big)^{1-\frac{2}{p}}\ (y\cdot Dv)\cdot(s\,\partial_s v)\bigg\}\ \mathcal{C}^q\times \\
&\qquad \times|y|^{-\frac{p-2}{p-1}}\ \left(1-|y|^{\frac{p}{p-1}}\right)_+^{\frac{1}{p-2}}\ dy \\
&-\frac{1}{r}\int_{\{s=-1\}\times\mathbb{R}^m} \Big(\big((2-p)\,r\,\Lambda^{-1}\,\Lambda'+2\big)\ s\,\partial_s v + y\cdot Dv\Big)\ \big(p\,f\big)^{1-\frac{2}{p}}\,Dv\cdot D\mathcal{C}^q\,\mathcal{B}\,dy. \\
&=: I_{31}+I_{32}+I_{33}+I_{34}+I_{35}. \qquad (5.24)
\end{aligned}
$$

Now each term in (5.24) is separately estimated.

I_{31} *of* (5.24). $I_{31}\geq 0$ by the positivity of the coefficient. In fact, by definition of Λ and $s=-1$

$$
(-s)\ \big((2-p)\,r\,\Lambda^{-1}\,\Lambda'+2\big) = B_0 > 0 \Longleftrightarrow \Lambda = r^{(B_0-2)/(2-p)}, \quad B_0 > 0.
$$

I_{32} of (5.24). By Cauchy's inequality for small $c > 0$,

$$I_{32} \geq \frac{c}{2r} \int_{\{s=-1\}\times\mathbb{R}^m} |\partial_s v|^2 \, \mathcal{B}\, \mathcal{C}^q \, dy - \frac{1}{2c\,r} \int_{\{s=-1\}\times\mathbb{R}^m} |y|^2 \, |Dv|^2 \, \mathcal{B}\, \mathcal{C}^q \, dy.$$

The time-derivative term is absorbed into I_{31}. By Young's inequality the 2nd term is estimated below for $\delta > 0$ by

$$-\frac{C}{r^{1+\delta}} \int_{\{s=-1\}\times\mathbb{R}^m} |Dv|^{2(p-1)} \, \mathcal{B}\, \mathcal{C}^q \, dy - \frac{C}{r^{1-\frac{\delta}{p-2}}} \int_{\{s=-1\}\times\mathbb{R}^m} \mathcal{B}\, \mathcal{C}^q \, dy, \tag{5.25}$$

of which the 1st term is bounded below by (5.20).

I_{33} of (5.24). Clearly, $I_{33} \geq 0$.

I_{34} of (5.24).

By Cauchy's inequality for small $c > 0$,

$$\begin{aligned} I_{34} &\geq -\frac{p\,B_0}{r\,(p-2)} \int_{\{s=-1\}\times\mathbb{R}^m} |\partial_s v| \left(p\, f\right)^{1-\frac{1}{p}} |y|^{\frac{1}{p-1}} \left(1 - |y|^{\frac{p}{p-1}}\right)_+^{\frac{1}{p-2}} \mathcal{C}^q \, dy \\ &\geq -\frac{c}{2r} \int_{\{s=-1\}\times\mathbb{R}^m} |\partial_s v|^2 \, \mathcal{B}\, \mathcal{C}^q \, dy \\ &\quad -\frac{C}{2c\,r} \int_{\{s=-1\}\times\mathbb{R}^m} (p\, f)^{2\left(1-\frac{1}{p}\right)} |y|^{\frac{2}{p-1}} \left(1 - |y|^{\frac{p}{p-1}}\right)_+^{\frac{3-p}{p-2}} \mathcal{C}^q \, dy, \end{aligned} \tag{5.26}$$

where in the 1st inequality $(2-p)\, r\, \Lambda^{-1} \Lambda' + 2 = B_0$ as before. The 1st term of (5.26) is absorbed into I_{31}. The 2nd term of (5.26) is estimated below by (5.20).

I_{35} of (5.24). By Young's inequality and the estimation (5.18) of derivative of $\mathcal{C}$,

$$\begin{aligned} I_{35} \geq &-\frac{c}{2} \int_{\{s=-1\}\times\mathbb{R}^m} |\partial_s v|^2 \, \mathcal{C}^q \, \mathcal{B}\, dy - \frac{1}{2c} \int_{\{s=-1\}\times\mathbb{R}^m} \left(p\, f\right)^{2\left(1-\frac{1}{p}\right)} \mathcal{C}^{q-2} \, \mathcal{B}\, dy \\ &- C \int_{\{s=-1\}\times\mathbb{R}^m} \left(p\, f\right)^{2\left(1-\frac{1}{p}\right)} \mathcal{C}^{\frac{2(q-1)(p-1)}{p}} \, \mathcal{B}\, dy - C \int_{\{s=-1\}\times\mathbb{R}^m} \mathcal{B}\, dy, \end{aligned} \tag{5.27}$$

where the 1st term can be absorbed into I_{31} and the 2nd and 3rd terms are bounded below by

$$-\frac{C\,(r^{-\delta}+1)}{r\,\Lambda^{2(p-1)}} \, \|\mathcal{C}(t)^{q-2} \left(p\, f(u(t))\right)^{2\left(1-\frac{1}{p}\right)} \|_{L^\infty(\mathrm{supp}\,(\mathcal{B}(t)))} \Big|_{t=t_0-r^{B_0}}, \tag{5.28}$$

because $2(q-1)(p-1)/p > q-2 \Longleftarrow q > 2$.

Resulting estimation. Combining all of the estimations above we have

$$\begin{aligned} \frac{d}{dr} E(r) &= I + II + III \\ &\geq J - C\left(r^{-1+\frac{\delta}{p-2}} + r^{-1+\delta}\right) - \frac{C}{r^{3+\delta}\,\Lambda^2} \int_{\{s=-1\}\times\mathbb{R}^m} |D_v g(v)|^2 \, \mathcal{B}\, \mathcal{C}^q \, dy \\ &\quad - C\, \frac{r^{-\delta}+1}{r\,\Lambda^{2(p-1)}} \, \|\mathcal{C}(t)^{q-2} \, \bar{e}(u(t))^{\frac{2(p-1)}{p}} \|_{L^\infty(\mathrm{supp}\,(\mathcal{B}(t)))} \Big|_{t=t_0-r^{B_0}} \end{aligned} \tag{5.29}$$

with $\Lambda = r^{(B_0-2)/(2-p)}$, according to (5.20), (5.21) and (5.28), and

$$J = \frac{B_0}{2\,r} \int\limits_{\{s=-1\}\times\mathbb{R}^m} (-s)\,|\partial_s v|^2\,\mathcal{B}\,\mathcal{C}^q\,dy + \frac{p}{r\,(p-2)} \int\limits_{\{s=-1\}\times\mathbb{R}^m} |Dv|^{p-2}\,|y\cdot Dv|^2\,|y|^{-\frac{p-2}{p-1}} \left(1-|y|^{\frac{p}{p-1}}\right)_+^{\frac{1}{p-2}} \mathcal{C}^q\,dy.$$

The term J is clearly nonnegative. From (5.29) integrated on the interval $(r,\ \rho)$ we derive

$$\begin{aligned} &E(\rho)-E(r)\\ &\geq -C\int_r^\rho \left(r^{-1+\frac{\delta}{p-2}}+r^{-1+\delta}\right)dr - \int_r^\rho \frac{C}{r^{3+\delta}\,\Lambda^2} \int\limits_{\{s=-1\}\times\mathbb{R}^m} |D_v g(v)|^2\,\mathcal{B}\,\mathcal{C}^q\,dy\,dr\\ &\quad -C\int_r^\rho \frac{r^{-\delta}+1}{r\,\Lambda^{2(p-1)}}\,\|\mathcal{C}(t)^{q-2}\bar{e}(u(t))^{\frac{2(p-1)}{p}}\|_{L^\infty(\mathrm{supp}(\mathcal{B}(t)))}\Big|_{t=t_0-\Lambda^{2-p}\,r^2}\,dr.\\ &=: C\left(U_1+U_2+U_3\right). \end{aligned} \tag{5.30}$$

Step 2 : a uniform bound. We will make a bound of each term U_i, $i=1,2,3$, in the right hand side of (5.30).

U_1 *of* (5.30). The 1st integrals on r in the 2nd line of (5.30) are computed as

$$\int_r^\rho r^{-1+\frac{\delta}{p-2}}\,dr = \frac{p-2}{\delta}\left(\rho^{\frac{\delta}{p-2}}-r^{\frac{\delta}{p-2}}\right);\quad \int_r^\rho r^{-1+\delta}\,dr = \frac{1}{\delta}\left(\rho^\delta - r^\delta\right), \tag{5.31}$$

U_3 *of* (5.30). $-U_3$ is computed as

$$\begin{aligned} &\int_r^\rho r^{-1}\left(-B_0\,\Lambda^{2-p}\,r\right)^{-1}\times\\ &\qquad\times\frac{1}{\Lambda^{2(p-1)}}\Lambda^\delta\|\mathcal{C}(t)^{q-2}\,\bar{e}(u(t))^{\frac{2(p-1)}{p}}\|_{L^\infty(\mathrm{supp}\,(\mathcal{B}(t)))}\left(-B_0\,\Lambda^{2-p}\,r\right)dt\\ &= \frac{1}{B_0}\int\limits_{t_0-(\Lambda(\rho))^{2-p}\,\rho^2}^{t_0-(\Lambda(r))^{2-p}\,r^2} (t_0-t)^{-1+\frac{2(p-1)(B_0-2)}{B_0(p-2)}-\frac{\delta(B_0-2)}{B_0(p-2)}}\times\\ &\qquad\qquad\times\|\mathcal{C}(t)^{q-2}\,\bar{e}(u(t))^{\frac{2(p-1)}{p}}\|_{L^\infty(\mathrm{supp}\,(\mathcal{B}(t)))}\,dt, \end{aligned} \tag{5.32}$$

where by definition of Λ

$$\Lambda = r^{(B_0-2)/(2-p)} \iff (\Lambda(r))^{2-p}\ r^2 = r^{B_0}$$

and, in the last term we make a changing of variable

$$\begin{aligned} &t = t_0-\Lambda^{2-p}\,r^2 \iff t_0-t = \Lambda^{2-p}\,r^2 = r^{B_0};\\ &\frac{dt}{dr} = -B_0\,\Lambda^{2-p}\,r \iff dt = -B_0\,\Lambda^{2-p}\,r\,dr. \end{aligned}$$

Here the exponent of power of (t_0-t) in (5.32) is estimated as

$$-1+\frac{2(p-1)(B_0-2)}{B_0(p-2)} > 0 \iff B_0 > \frac{4(p-1)}{p}$$

$$\Longleftarrow B_0 > \frac{6p-4}{p+2}; \quad \frac{6p-4}{p+2} > \frac{4(p-1)}{p} \iff (p-2)^2 > 0;$$
$$-1 + \frac{2(p-1)(B_0-2)}{B_0(p-2)} - \frac{\delta(B_0-2)}{B_0(p-2)} \geq 0 \iff 0 < \delta \leq 2(p-1) - \frac{B_0(p-2)}{B_0-2}$$

and then,

$$t_0 - (\Lambda(\rho))^{2-p}\,\rho^2 \leq t \leq t_0 - (\Lambda(r))^{2-p}\,r^2 \iff r^{B_0} \leq t_0 - t \leq \rho^{B_0},$$
$$(t_0-t)^{-1+\frac{2(p-1)(B_0-2)}{B_0(p-2)} - \frac{\delta(B_0-2)}{B_0(p-2)}} \leq \rho^{B_0\left(-1+\frac{2(p-1)(B_0-2)}{B_0(p-2)} - \frac{\delta(B_0-2)}{B_0(p-2)}\right)} \leq 1$$

and thus, the right hand side of (5.32) is bounded above by

$$\frac{1}{B_0}\int_{t_0-\rho^{B_0}}^{t_0-r^{B_0}} \|\mathcal{C}(t)^{q-2}\,\bar{e}(u(t))^{\frac{2(p-1)}{p}}\|_{L^\infty(\mathrm{supp}\,(\mathcal{B}(t)))}\,dt.$$

U_2 of (5.30). U_2 is given by the approximation term, the derivative of penalty term in (2.1) and our task is to control U_2 well in the appropriate way. U_2 is evaluated by use of the Bochner type estimate for the penalty term

$$\partial_s g - \mathrm{div}\big((p\,f)^{1-\frac{2}{p}}\,Dg\big) + C_0'\,|D_v g|^2 \leq C\,(\Lambda r)^2\,\bar{e}^2 \tag{5.33}$$

with positive constants C_0' and C depending only on p, $\mathcal{M}$ and $\mathcal{N}$. The derivation of (5.33) is done similarly as in "Appendix C", under the scaling settings (5.5) and (5.6), by using (6.8) below.

Let $\bar{r}$ be as $r \leq \bar{r} < \rho$ and chosen later. In the following we will replace r by $\bar{r}$, $r \leq \bar{r} < \rho$, and proceed to the similar estimations as for U_2.

Multiplying a test function $\mathcal{B}\,\mathcal{C}^q$ in (5.33) and then, integrating the resulting inequality on y in $\{s=-1\}\times\mathbb{R}^m$ and on r in a interval $(\bar{r},\ \rho)$, the estimation for U_2 is done as

$$\begin{aligned}
-\left(C_0' - \frac{c}{2}\right) U_2 &= \left(C_0' - \frac{c}{2}\right)\int_{\bar{r}}^{\rho}\frac{1}{r^{3+\delta}\,\Lambda^2}\int_{\{s=-1\}\times\mathbb{R}^m}|D_v g|^2\,\mathcal{B}\,\mathcal{C}^q\,dy\,dr\\
&\leq -\int_{\bar{r}}^{\rho}\frac{1}{r^{3+\delta}\,\Lambda^2}\int_{\{s=-1\}\times\mathbb{R}^m}\partial_s g\,\mathcal{B}\,\mathcal{C}^q\,dy\,dr\\
&\quad + \int_{\bar{r}}^{\rho}\frac{1}{r^{3+\delta}\,\Lambda^2}\int_{\{s=-1\}\times\mathbb{R}^m}\frac{1}{2c}\,\Big((p\,f)^{2-\frac{2}{p}}\,\big(\mathcal{B}'\,\mathcal{C}^q + \mathcal{B}\,q^2\mathcal{C}^{q-2}|D\mathcal{C}|^2\big)\\
&\qquad + C\,(\Lambda r)^2\,\bar{e}^2\,\mathcal{B}\,\mathcal{C}^q\Big)\,dy\,dr\\
&=: U_{21} + U_{22},
\end{aligned} \tag{5.34}$$

where by Cauchy's inequality with a small $c > 0$, the integrand term in the 3rd line is obtained from

$$\begin{aligned}
\left|(p\,f)^{1-\frac{2}{p}}\,Dg\cdot D\big(\mathcal{B}\,\mathcal{C}^q\big)\right| &= (p\,f)^{1-\frac{1}{p}}\,|D_v g|\,\big(|D\mathcal{B}|\mathcal{C}^q + \mathcal{B}|D\mathcal{C}^q|\big).\\
&\leq \frac{c}{2}\,|D_v g|^2\,\mathcal{B}\,\mathcal{C}^q + \frac{1}{2c}\,(p\,f)^{\frac{2(p-1)}{p}}\,\big(\mathcal{B}'\,\mathcal{C}^q + \mathcal{B}\,q^2\mathcal{C}^{q-2}|D\mathcal{C}|^2\big);\\
\mathcal{B}' &:= |y|^{\frac{2}{p-1}}\left(1 - |y|^{\frac{p}{p-1}}\right)_+^{\frac{3-p}{p-2}}.
\end{aligned}$$

Each term in the right hand side of (5.34) is separately treated in the following.

U_{22} *in* (5.34) The 3rd and 4th lines in the right hand side of (5.34) , U_{22}, is scaled back and

$$U_{22} \le C \int_{\bar r}^{\rho} \frac{1}{r^{3+\delta}\,\Lambda^{2p}} \, \|\mathcal{C}(t)^{q-2}\big(\epsilon + |Du(t)|^2\big)^{p-1} + \\ + \mathcal{C}(t)^q \, r^2 \, \bar e(u(t))^2 \|_{L^\infty(\mathrm{supp}(\mathcal{B}(t)))}\Big|_{t=t_0-r^{B_0}} \, dr \; \times \\ \times \int\limits_{\{s=-1\}\times\mathbb{R}^m} \big(\mathcal{B}'\,\mathcal{C}^2 + \mathcal{B}\,|D\mathcal{C}|^2 + \mathcal{B}\big)\, dy, \tag{5.35}$$

where in the 2nd line we compute as

$$(\Lambda r)^2\,\Lambda^{-2} = r^2$$

and, the integral on y in the 3rd line is bounded by a constant as before, since

$$\mathcal{C} + |D\mathcal{C}| \le 2; \quad \int\limits_{\mathbb{R}^m} (\mathcal{B} + \mathcal{B}')\, dy < \infty, \quad \frac{3-p}{p-2} > -1 \Longleftrightarrow 3 > 2.$$

The integral on r in the 1st and 2nd lines is transformed into that on time by changing a variable $t = t_0 - \Lambda(r)^{2-p} r^2 = t_0 - r^{B_0}$

$$C \int\limits_{t_0-\rho^{B_0}}^{t_0-\bar r^{B_0}} (t_0-t)^{\frac{1}{B_0}\left(-4-\delta+\frac{(p+2)(B_0-2)}{p-2}\right)} \|\mathcal{C}(t)^{q-2}\big(\bar e(u(t))^{\frac{2(p-1)}{p}} + \bar e(u(t))^2\big)\|_{L^\infty(\mathrm{supp}(\mathcal{B}(t))}\, dt \\ \le C \int\limits_{t_0-\rho^{B_0}}^{t_0-\bar r^{B_0}} \|\mathcal{C}(t)^{q-2}\big(\bar e(u(t))^{\frac{2(p-1)}{p}} + \bar e(u(t))^2\big)\|_{L^\infty(\mathrm{supp}(\mathcal{B}(t))}\, dt,$$

where the power exponents of scale radius are computed as

$$r^2 \le 1 \Longleftarrow 0 < r \le \rho \le 1; \\ \frac{\Lambda^{p-2}\, r^{-1}}{r^{3+\delta}\,\Lambda^{2p}} = r^{-4-\delta+\frac{(p+2)(B_0-2)}{p-2}}; \\ -4-\delta+\frac{(p+2)(B_0-2)}{p-2} \ge 0 \Longleftrightarrow 0 < \delta \le -4 + \frac{(p+2)(B_0-2)}{p-2} \\ \Longleftarrow -4 + \frac{(p+2)(B_0-2)}{p-2} > 0 \Longleftrightarrow B_0 > \frac{6p-4}{p+2}.$$

U_{21} *in* (5.34) $-U_{21}$ is computed as

$$-U_{21} = \int_{\bar r}^{\rho} \frac{1}{r^{3+\delta}\,\Lambda^2} \int\limits_{\{s=-1\}\times\mathbb{R}^m} \partial_s\big(g\,\mathcal{C}^q\big)\,\mathcal{B}\,dy\,dr \\ - \int_{\bar r}^{\rho} \frac{1}{r^{3+\delta}\,\Lambda^2} \int\limits_{\{s=-1\}\times\mathbb{R}^m} g\,\partial_s \mathcal{C}^q\,\mathcal{B}\,dy\,dr \\ =: U_{211} + U_{212} \tag{5.36}$$

Each term in (5.36) is separately estimated in the following.

U_{212} *in* (5.36) U_{212} is estimated by using

$$\partial_s \mathcal{C} = \chi_{\{|x_0+r\,y|^{B_0}\le (t_0+r^{B_0}\,s)\}}\,\frac{1}{B_0}(t_0+r^{B_0}\,s)^{\frac{1}{B_0}-1} r^{B_0} \le \frac{r}{B_0};$$
$$r^{B_0} \le t_0/2, \quad s=-1;$$
$$t_0 - r^{B_0} \ge t_0/2, \quad (t_0-r^{B_0})^{\frac{1}{B_0}-1} \le (t_0/2)^{\frac{1}{B_0}-1} \le r^{1-B_0};$$
$$\frac{1}{r^{3+\delta}\,\Lambda^2}\,\left|g\,\partial_s \mathcal{C}^q\right| \le \frac{q}{B_0}\,\frac{1}{r^{2+\delta}\,\Lambda^2}\,|g|\,\mathcal{C}^{q-1} \le \frac{q}{B_0}\,\frac{1}{r^{2+\delta}\,\Lambda^{2+p}}\,|g(u)|\,\mathcal{C}^{q-1}$$

and thus, by scaling back and a changing of variable $t = t_0 - \Lambda(r)^{2-p} r^2 = t_0 - r^{B_0}$,

$$g(u(t)) := \frac{K}{2}\chi\left(\operatorname{dist}^2\big(u(t,\,x_0+(t_0-t)^{1/B_0}y),\,\mathcal{N}\big)\right);$$
$$|U_{212}| \le \frac{q}{2B_0}\int_{\bar r}^{\rho}\frac{1}{r^{2+\delta}\,\Lambda^{2+p}}\int_{\mathbb{R}^m} g(u(t))\,\mathcal{C}^{q-1}(t)\,\mathcal{B}\,dy\Bigg|_{t=t_0-r^{B_0}}\,dr$$
$$\le \frac{q}{B_0}\int_{t_0-\rho^{B_0}}^{t_0-\bar r^{B_0}} (t_0-t)^{\frac{1}{B_0}\left(-2-\delta+\frac{(p+2)(B_0-2)}{p-2}\right)}\,\|e(u(t))\,\mathcal{C}^{q-1}(t)\|_{L^\infty\left(\operatorname{supp}(\mathcal{B}(t))\right)}\,dt\,\times$$
$$\times\int_{\mathbb{R}^m}\mathcal{B}\,dy$$
$$\le C\int_{t_0-\rho^{B_0}}^{t_0-\bar r^{B_0}} \|\mathcal{C}^{q-1}(t)\,\bar e(u(t))\|_{L^\infty\left(\operatorname{supp}(\mathcal{B}(t))\right)}\,dt, \tag{5.37}$$

where $\mathcal{B} = \mathcal{B}(s,\,y)|_{s=-1} = \left(1-|y|^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}}$ and the exponents are computed as

$$-2-\delta+\frac{(p+2)(B_0-2)}{p-2} \ge 0 \Longleftrightarrow 0<\delta\le -2+\frac{(p+2)(B_0-2)}{p-2}$$
$$\Longleftarrow -2+\frac{(p+2)(B_0-2)}{p-2} > 0 \Longleftrightarrow B_0 > \frac{4p}{p+2}$$
$$\Longleftarrow B_0 > \frac{6p-4}{p+2}\quad;\quad \frac{6p-4}{p+2} > \frac{4p}{p+2} \Longleftrightarrow p>2.$$

U_{211} *in* (5.36) U_{211} is transformed into an integral on time by scaling back.

$$U_{211} = \int_{\bar r}^{\rho}\frac{1}{r^{3+\delta}\,\Lambda^2}\int_{\{s=-1\}\times\mathbb{R}^m}\partial_s\big(g\,\mathcal{C}^q\big)\,\mathcal{B}\,dy\,dr$$
$$= \int_{\bar r}^{\rho}\frac{1}{r^{3+\delta}\,\Lambda^2}\int_{\{s=-1\}\times\mathbb{R}^m}\Lambda^{2-p}\,r^2\,\frac{\bar K}{2}\,\partial_t\widetilde{h}(t,\,y,\,r)\Big|_{t=t_0+\Lambda^{2-p}r^2 s}\,\mathcal{B}\,dy\,dr, \tag{5.38}$$

where we put

$$\widetilde{h}(t,\,y,\,r) := \frac{\bar K}{2}\chi\left(\operatorname{dist}^2\big(u(t,\,x_0+ry),\,\mathcal{N}\big)\right)\left(t^{\frac{1}{B_0}} - |x_0+ry|\right)_+^q.$$

By changing a variable $t = t_0 - \Lambda(r)^{2-p} r^2 = t_0 - r^{B_0}$, we have

$$
\begin{aligned}
& t = t_0 - r^{B_0} \Longleftrightarrow r = (t_0 - t)^{1/B_0}; \quad \bar{K} = K \Lambda^{-p} = K r^{\frac{p(B_0-2)}{p-2}} = K (t_0 - t)^{\frac{p(B_0-2)}{B_0(p-2)}}; \\
& dt = -B_0 r^{B_0 - 1} dr = -B_0 \Lambda^{2-p} r \, dr; \\
& \frac{1}{r^{2+\delta} \Lambda^2} = (t_0 - t)^{-c_0}; \quad c_0 := \frac{1}{B_0} \left(\frac{2(p - B_0)}{p - 2} + \delta \right)
\end{aligned}
$$

and an elementary computation

$$
\begin{aligned}
h(t, y) &:= \widetilde{h}(t, y, (t_0 - t)^{1/B_0}) \\
&= \frac{\bar{K}}{2} \chi \left(\operatorname{dist}^2\big(u(t, x_0 + (t_0 - t)^{1/B_0} y), \mathcal{N}\big) \right) \left(t^{\frac{1}{B_0}} - |x_0 + (t_0 - t)^{1/B_0} y| \right)_+^q ; \\
\partial_t h(t, y) &= -\frac{p(B_0 - 2)}{B_0(p-2)} (t_0 - t)^{-1} h(t, y) - \frac{1}{B_0} (t_0 - t)^{-1} y \cdot D_y h(t, y) \\
&\quad + \frac{\bar{K}}{2} \, \partial_\tau \widetilde{h}(\tau, y, (t_0 - t)^{1/B_0}) \Big|_{\tau = t} .
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
U_{211} &= \frac{1}{B_0} \int_{t_0 - \rho^{B_0}}^{t_0 - \bar{r}^{B_0}} (t_0 - t)^{-c_0} \frac{dP}{dt} \, dt + \frac{p(B_0 - 2)}{B_0^2 (p-2)} \int_{t_0 - \rho^{B_0}}^{t_0 - \bar{r}^{B_0}} (t_0 - t)^{-c_0 - 1} P(t) \, dt \\
&\quad + \frac{1}{B_0^2} \int_{t_0 - \rho^{B_0}}^{t_0 - \bar{r}^{B_0}} (t_0 - t)^{-c_0 - 1} \int_{\mathbb{R}^m} y \cdot D_y h(t, y) \, \mathcal{B}(y) \, dy \, dt \\
&=: U_{2111} + U_{2112} + U_{2113},
\end{aligned} \tag{5.39}
$$

where we put

$$
P(t) := \int_{\mathbb{R}^m} h(t, y) \, \mathcal{B}(y) \, dy, \quad \mathcal{B}(y) = \mathcal{B}(s, y)|_{s=-1} = \left(1 - |y|^{\frac{p}{p-1}} \right)_+^{\frac{p-1}{p-2}} .
$$

We will estimate each term in (5.39).

Now, we set $\bar{r}$ as

$$
\exists \, \bar{r}, \; r \le \bar{r} \le \rho \quad : \quad \max_{t_0 - \rho^{B_0} \le t \le t_0 - r^{B_0}} P(t) = P(t_0 - \bar{r}^{B_0}). \tag{5.40}
$$

Then, by integration by parts in the integral on t, we have

$$
\begin{aligned}
B_0 \times U_{2111} &= \int_{t_0 - \rho^{B_0}}^{t_0 - \bar{r}^{B_0}} (t_0 - t)^{-c_0} \frac{dP}{dt} \, dt \\
&= (t_0 - t)^{-c_0} P(t) \Big|_{t_0 - \rho^{B_0}}^{t_0 - \bar{r}^{B_0}} - \int_{t_0 - \rho^{B_0}}^{t_0 - \bar{r}^{B_0}} P(t) \, c_0 \, (t_0 - t)^{-c_0 - 1} \, dt \\
&\ge \bar{r}^{-c_0 B_0} P(t_0 - \bar{r}^{B_0}) - \rho^{-c_0 B_0} P(t_0 - \rho^{B_0})
\end{aligned}
$$

$$-P(t_0 - \bar{r}^{B_0})\left(\bar{r}^{-c_0 B_0} - \rho^{-c_0 B_0}\right)$$
$$= \rho^{-c_0 B_0}\left(P(t_0 - \bar{r}^{B_0}) - P(t_0 - \rho^{B_0})\right) \geq 0. \tag{5.41}$$

Clearly, $U_{2112} \geq 0$. By integration by parts in the integral on y, we also have

$$U_{2113} = \frac{1}{B_0^2} \int_{t_0-\rho^{B_0}}^{t_0-\bar{r}^{B_0}} (t_0 - t)^{-c_0-1} \int_{\mathbb{R}^m} h(t,\ y) \left(-m\,\mathcal{B} + \frac{p\big(1 - |y|^{\frac{p}{p-1}}\big)_+^{\frac{1}{p-2}} |y|^{\frac{p}{p-1}}}{p-2}\right) dy\, dt$$
$$\geq -\frac{m}{B_0^2} \int_{t_0-\rho^{B_0}}^{t_0-\bar{r}^{B_0}} (t_0 - t)^{-c_0-1} \int_{\mathbb{R}^m} h(t,\ y)\, \mathcal{C}^q(t)\, \mathcal{B}\, dy\, dt.$$

The last integral is estimated below as

$$\mathcal{C}^q(t) := \mathcal{C}^q(t,\ x_0 + (t_0 - t)^{1/B_0} y);$$
$$-\frac{m}{B_0^2} \int_{t_0-\rho^{B_0}}^{t_0-\bar{r}^{B_0}} (t_0 - t)^{b_0-1} \int_{\mathbb{R}^m} g(u(t))\mathcal{C}^q(t)\, \mathcal{B}\, dy\, dt$$
$$\geq -\frac{m}{B_0^2} \int_{\mathbb{R}^m} \mathcal{B}\, dy \times$$
$$\times \left(\int_{t_0-\rho^{B_0}}^{t_0-\bar{r}^{B_0}} (t_0 - t)^{\alpha_0(b_0-1)}\, dt \right.$$
$$\left. + \int_{t_0-\rho^{B_0}}^{t_0-\bar{r}^{B_0}} \left\| \frac{K}{2}\, \chi\big(\operatorname{dist}^2\big(u(t,\ \cdot),\ \mathcal{N}\big)\big)\, \mathcal{C}^q(t) \right\|_{L^\infty(\operatorname{supp}\mathcal{B}(t))}^{\frac{\alpha_0}{\alpha_0-1}} dt \right), \tag{5.42}$$

where we use Young's inequality with an exponent $\alpha_0 > 1$ and compute as

$$t = t_0 - r^{B_0} \Longleftrightarrow r = (t_0 - t)^{1/B_0};$$
$$\Lambda = r^{\frac{B_0-2}{2-p}}, \quad h(u(t)) = \Lambda^{-p} g(u(t));$$
$$(t_0 - t)^{-c_0-1}\, \Lambda^{-p} = (t_0 - t)^{b_0-1}; \quad b_0 := \frac{B_0(p+2) - 4p}{B_0(p-2)} - \frac{\delta}{B_0}$$

and we choose $\alpha_0 > 1$ as

$$1 < \alpha_0 < \frac{1}{1 - b_0} \Longleftarrow \alpha_0(b_0 - 1) > -1; \quad b_0 < 1 \Longleftarrow B_0 < p;$$
$$\frac{1}{1 - b_0} > 1 \Longleftrightarrow b_0 > 0 \Longleftrightarrow 0 < \delta < \frac{B_0(p+2) - 4p}{p-2};$$
$$\frac{B_0(p+2) - 4p}{p-2} > 0 \Longleftrightarrow B_0 > \frac{4p}{p+2}$$
$$\Longleftarrow B_0 > \frac{6p-4}{p+2}; \frac{6p-4}{p+2} > \frac{4p}{p+2} \Longleftrightarrow p > 2. \tag{5.43}$$

By (5.41) and (5.42) substituted into (5.39), we have

$$U_{211} \geq -\frac{m}{B_0^2} \int_{\mathbb{R}^m} \mathcal{B}\, dy \times \\ \times \left(\frac{\rho^{\alpha_0(b_0-1)+1} - \bar{r}^{\alpha_0(b_0-1)+1}}{\alpha_0(b_0-1)+1} + \int_{t_0-\rho^{B_0}}^{t_0-\bar{r}^{B_0}} \|\bar{e}(u(t))\, \mathcal{C}^q(t)\|_{L^\infty\left(\mathrm{supp}\mathcal{B}(t)\right)}^{\frac{\alpha_0}{\alpha_0-1}} \right), \tag{5.44}$$

yielding, with (5.36) and (5.37) for U_{212},

$$\begin{aligned} -U_{21} &\geq U_{211} + U_{212} \\ &\geq -C\left(\rho^{\alpha_0(b_0-1)+1} - \bar{r}^{\alpha_0(b_0-1)+1}\right) \\ &\quad - C \int_{t_0-\rho^{B_0}}^{t_0-\bar{r}^{B_0}} \|\bar{e}(u(t))\, \mathcal{C}^q(t)\|_{L^\infty\left(\mathrm{supp}\mathcal{B}(t)\right)}^{\frac{\alpha_0}{\alpha_0-1}} \\ &\quad - C \int_{t_0-\rho^{B_0}}^{t_0-\bar{r}^{B_0}} \|\mathcal{C}^{q-1}(t)\, \bar{e}(u(t))\|_{L^\infty\left(\mathrm{supp}(\mathcal{B}(t))\right)}\, dt. \end{aligned}$$

By definition of $P(t)$ in (5.39) and $\bar{r}$ in (5.40), $P(t)$ is the local scaled integral of the penalty term, because by changing a variable $x = x_0 + (t_0 - t)^{1/B_0} y$

$$h(t, y) = \frac{\bar{K}}{2} \chi\left(\mathrm{dist}^2\big(u(t, x_0 + (t_0-t)^{1/B_0} y), \mathcal{N}\big)\right)\left(t^{1/B_0} - |x_0 + (t_0-t)^{1/B_0} y|\right)_+^q;$$

$$P(t) = \int_{\mathbb{R}^m} h(t, y)\, \mathcal{C}^q(t)\, \mathcal{B}\, dy = \frac{1}{\Lambda^p} \int_{\mathbb{R}^m} \frac{K}{2} \chi\left(\mathrm{dist}^2(u(t, x), \mathcal{N})\right) \mathcal{C}^q(t, x)\, \mathcal{B}(t, x)\, dx$$

and it holds that

$$P(t_0 - r^{B_0}) \leq P(t_0 - \bar{r}^{B_0}) \quad \text{for } \bar{r} \text{ in (5.40)}. \tag{5.45}$$

Collecting the estimations for U_1, U_2 and U_3 above in (5.30), we have, for $\bar{r}$ in (5.40),

$$\begin{aligned} E(\rho) - E(\bar{r}) &\geq -C\left(\rho^\mu - \bar{r}^\mu\right) \\ &\quad - C \int_{t_0-\rho^{B_0}}^{t_0-\bar{r}^{B_0}} \|\mathcal{C}^{q-2}(t)\, \bar{e}(u(t))^{\frac{2(p-1)}{p}} \\ &\quad + \mathcal{C}^q(t)\left(\bar{e}(u(t))^2 + \bar{e}(u(t))^{\frac{\alpha_0}{\alpha_0-1}}\right)\|_{L^\infty(\mathrm{supp}(\mathcal{B}(t)))}\, dt. \end{aligned} \tag{5.46}$$

Let us put, for $\alpha_0 > 1$ in (5.43),

$$\theta_0 = \max\left\{2, \frac{\alpha_0}{\alpha_0 - 1}\right\}. \tag{5.47}$$

From (5.46), our desired monotonicity estimate is shown to hold true in the range of scale radius $[\bar{r}, \rho]$. Also (5.45) is the monotonicity estimate in the range $[r, \bar{r}]$ of the local scaled integral of the penalty term. Therefore, it remains to estimate the local scaled p-energy in the range of scale radius $[r, \bar{r}]$.

Step 3 : Monotonicity of the scaled p-energy. We now show a monotonicity estimate for the scaled p-energy without the penalty term. Under the same notation as before we denote the scaled p-energy by

$$F(r) = \int_{\{s=-1\}\times\mathbb{R}^m} f(v(s, y))\, \mathcal{B}(s, y)\, \mathcal{C}^q(s, y)\, dy; \quad f = f(v) := \frac{1}{p}\left(\bar{\epsilon} + |Dv|^2\right)^{\frac{p}{2}} \tag{5.48}$$

and compute the differentiation of $F(r)$ on a scale radius r

$$\begin{aligned} \frac{d}{dr}F(r) &= \int_{\{s=-1\}\times\mathbb{R}^m} \frac{d}{dr} f(v)\, \mathcal{B}\,\mathcal{C}^q\, dy + \int_{\{s=-1\}\times\mathbb{R}^m} f(v)\, \mathcal{B}\, \frac{d}{dr}\mathcal{C}^q\, dy \\ &= \int_{\{s=-1\}\times\mathbb{R}^m} (p\, f)^{1-\frac{2}{p}} Dv \cdot \frac{d}{dr} Dv\, \mathcal{B}\,\mathcal{C}^q\, dy \\ &\quad + \frac{B-2}{r\,(p-2)} \int_{\{s=-1\}\times\mathbb{R}^m} \bar{\epsilon}\, (p\, f)^{1-\frac{2}{p}}\, \mathcal{B}\,\mathcal{C}^q\, dy \\ &\quad + \int_{\{s=-1\}\times\mathbb{R}^m} f(v)\, \mathcal{B}\, \frac{d}{dr}\mathcal{C}^q\, dy \\ &=: H_1 + H_2 + H_3. \end{aligned} \tag{5.49}$$

Clearly, $H_2 \geq 0$. H_3 is similarly estimated as III of (5.9) in (5.10), and

$$\begin{aligned} H_1 &= \frac{1}{r}\left(1 + \frac{r\,\Lambda'}{\Lambda}\right) \int_{\{s=-1\}\times\mathbb{R}^m} \Big\{ \operatorname{div}\!\left((p\, f)^{1-\frac{2}{p}} Dv\, \mathcal{B}\,\mathcal{C}^{q/2}\right) \cdot \left(v\, \mathcal{C}^{q/2} - \bar{v}\right) \\ &\qquad + (p\, f)^{1-\frac{2}{p}} Dv \cdot \left(v\, D\mathcal{C}^{q/2}\right) \mathcal{B}\,\mathcal{C}^{q/2} \Big\}\, dy \\ &\quad - \frac{1}{r} \int_{\{s=-1\}\times\mathbb{R}^m} \operatorname{div}\!\left((p\, f)^{1-\frac{2}{p}} Dv\, \mathcal{B}\,\mathcal{C}^{q}\right) \cdot \left(\left(\frac{(2-p)\, r\, \Lambda'}{\Lambda} + 2\right) s\, \partial_s v + y \cdot Dv\right) dy \\ &=: H_{11} + H_{12} + H_{13}, \end{aligned} \tag{5.50}$$

where we use an integration by parts and the dilation derivative (5.12).

Estimation of H_{11} We have

$$\begin{aligned} H_{11} &= \frac{1}{r}\left(1 + \frac{r\,\Lambda'}{\Lambda}\right) \int_{\{s=-1\}\times\mathbb{R}^m} \operatorname{div}\!\left((p\, f)^{1-\frac{2}{p}} Dv\, \mathcal{B}\,\mathcal{C}^{q/2}\right) \cdot \left(v\, \mathcal{C}^{q/2} - \bar{v}\right) dy \\ &= \frac{1}{r}\left(1 + r\,\frac{\Lambda'}{\Lambda}\right) \int_{\{s=-1\}\times\mathbb{R}^m} \left((v\, \mathcal{C}^{q/2}) - \bar{v}\right) \cdot \Big\{ \left(\partial_s v + C_0\, D_v g(v)\right) \mathcal{B}\,\mathcal{C}^{q/2} \\ &\qquad + \frac{p\,(p\, f)^{1-\frac{2}{p}}\, y \cdot Dv\, \left(1 - |y|^{\frac{p}{p-1}}\right)_+^{\frac{1}{p-2}}}{(p-2)\,|y|^{\frac{p-2}{p-1}}}\, \mathcal{C}^{q/2} \\ &\qquad + (p\, f)^{1-\frac{2}{p}} Dv \cdot D\mathcal{C}^{q/2}\, \mathcal{B} \Big\}\, dy. \end{aligned}$$

In the bracket of the right hand side, the 1st time-derivative term, the 3rd term and the 4th term are the same ones as in I_{111}, I_{112} and I_{113} of I_{11} in (5.13), respectively. These terms are estimated as for I_{111}, I_{112} and I_{113}.

The 2nd term containing the derivative of penalty term is estimated in the following.

$$\frac{1}{r}\Big|\int_{\{s=-1\}\times\mathbb{R}^m} C_0\left((v\,\mathcal{C}^{q/2}-\bar{v}\right)\cdot D_v g\,\mathcal{B}\,\mathcal{C}^{q/2}\,dy\Big|$$
$$\leq \frac{(C_0)^2}{2r}\int_{\{s=-1\}\times\mathbb{R}^m}\left|(v\,\mathcal{C}^{q/2})-\bar{v}\right|^2\mathcal{B}\,dy+\frac{1}{2r}\int_{\{s=-1\}\times\mathbb{R}^m}\left|D_v g\right|^2\mathcal{B}\,\mathcal{C}^q\,dy.$$

The 1st term is the same as $(-1)\times$2nd one in (5.16) and thus, estimated above by $(-1)\times$the right hand side of (5.17). The 2nd term is estimated in the following. Multiplying a test function $\mathcal{B}\,\mathcal{C}^q$ by (5.33), we have, by Cauchy's inequality with a small $c>0$,

$$\begin{aligned}(C_0'-\frac{1}{c})\,|D_v g(v)|^2\,\mathcal{B}\,\mathcal{C}^q &\leq \frac{c}{2}\,|\partial_s v|^2\,\mathcal{B}\,\mathcal{C}^q+\ \mathrm{div}\big((p\,f)^{1-\frac{2}{p}}\,Dg\,\mathcal{B}\,\mathcal{C}^q\big)\\ &\quad+\frac{c}{2}\,(p\,f)^{2-\frac{2}{p}}\,\left(\mathcal{B}'\,\mathcal{C}^q+\mathcal{B}\,\mathcal{C}^{q-2}|D\mathcal{C}|^2\right)\\ &\quad+C\,(\Lambda r)^2\,\bar{e}(v)^2\,\mathcal{B}\,\mathcal{C}^q,\end{aligned}\tag{5.51}$$

where $\mathcal{B}'=|y|^{\frac{2}{p-1}}\left(1-\ |y|^{\frac{p}{p-1}}\right)_+^{\frac{3-p}{p-2}}$, and

$$\partial_s g(v)=\partial_s v\cdot D_v g(v);\quad Dg(v)=Dv\cdot D_v g(v).$$

The inequality (5.51) is integrated on y and then, estimated by integration by parts as

$$\begin{aligned}\frac{1}{r}\int_{\{s=-1\}\times\mathbb{R}^m}|D_v g|^2\,\mathcal{B}\,\mathcal{C}^q\,dy &\leq \frac{c}{2r}\int_{\{s=-1\}\times\mathbb{R}^m}|\partial_s v|^2\,\mathcal{B}\,\mathcal{C}^q\,dy\\ &+\frac{C}{r}\int_{\{s=-1\}\times\mathbb{R}^m}\left((\Lambda r)^2\,\bar{e}^2\,\mathcal{B}\,\mathcal{C}^q\ +\ (p\,f)^{2-\frac{2}{p}}\,\left(\mathcal{B}'\,\mathcal{C}^q+\mathcal{B}\,q^2\mathcal{C}^{q-2}|D\mathcal{C}|^2\right)\right)\,dy.\end{aligned}\tag{5.52}$$

Estimations of H_{12} *and* H_{13} H_{12} is the same as I_2 in (5.11) and thus, estimated as in (5.22) and (5.20).

H_{13} is the same as I_3 in (5.11) except the derivative term of the penalty term and thus, is estimated similarly as for I_{3i}, $i=1,\ldots,5$, and the estimation (5.52) for the derivative of the penalty term in H_{11}.

Gathering the estimations above and scaling back, we have, for $\delta>0$,

$$\begin{aligned}\frac{d}{dr}F(r)\geq J-C\,\left(r^{-1+\frac{\delta}{p-2}}+r^{-1+\delta}\right)\\ -\frac{C}{r\,\Lambda^{2(p-1)}}\,\|\bar{e}(u(t))^{\frac{2(p-1)}{p}}\,\mathcal{C}^{q-2}(t)+\bar{e}(u(t))^2\,\mathcal{C}^q(t)\|_{L^\infty(\mathrm{supp}\,(\mathcal{B}(t)))}\Big|_{t=t_0-r^{B_0}},\end{aligned}$$

where in the 2nd line we estimate as

$$(\Lambda r)^2\Lambda^{-2}=r^2\leq 1\Longleftarrow 0<r\leq\rho\leq 1;\qquad \int_{\{s=-1\}\times\mathbb{R}^m}\left(\mathcal{B}+\mathcal{B}'\,\mathcal{C}^2+\mathcal{B}\,|D\mathcal{C}|^2\right)\,dy<\infty,$$

and, integrated on r in $(r,\ \rho)$, yielding

$$
\begin{aligned}
F(\rho) - F(r) \geq & -C \int_r^\rho \left(r^{-1+\frac{\delta}{p-2}} + r^{-1+\delta} \right) dr \\
& - \int_r^\rho \frac{C}{r\,\Lambda^{2(p-1)}} \|\bar{e}(u(t))^{\frac{2(p-1)}{p}}\, \mathcal{C}^{q-2}(t) + \\
& + \bar{e}(u(t))^2\, \mathcal{C}^q(t) \|_{L^\infty(\mathrm{supp}\,(\mathcal{B}(t))}\Big|_{t=t_0-r^{B_0}} dr. \qquad (5.53)
\end{aligned}
$$

The 1st term in the right hand side is the same as U_1 in (5.30) and estimated as in (5.31). The term in the 2nd and 3rd lines is, by changing a variable $t = t_0 - \Lambda^{2-p} r^2 = t_0 - r^{B_0}$, computed as

$$
\begin{aligned}
& \int_r^\rho \frac{-1}{B_0\, r^{B_0}\, \Lambda^{2(p-1)}} \|\bar{e}(u(t))^{\frac{2(p-1)}{p}}\, \mathcal{C}^{q-2}(t) \\
& \qquad + \bar{e}(u(t))^2\, \mathcal{C}(t)^q \|_{L^\infty(\mathrm{supp}(\mathcal{B}(t))}\Big|_{t=t_0-r^{B_0}} (-B_0\, r^{B_0-1})\, dr \\
& = C \int_{t_0-\rho^{B_0}}^{t_0-r^{B_0}} (t_0-t)^{\frac{1}{B_0}\left(-2+\frac{p(B_0-2)}{p-2}\right)} \|\bar{e}(u(t))^{\frac{2(p-1)}{p}}\, \mathcal{C}(t)^{q-2} \\
& \qquad + \bar{e}(u(t))^2\, \mathcal{C}(t)^q \|_{L^\infty(\mathrm{supp}(\mathcal{B}(t)))}\, dt \\
& \leq C \int_{t_0-\rho^{B_0}}^{t_0-r^{B_0}} \|\bar{e}(u(t))^{\frac{2(p-1)}{p}}\, \mathcal{C}(t)^{q-2} + \bar{e}(u(t))^2\, \mathcal{C}(t)^q \|_{L^\infty(\mathrm{supp}(\mathcal{B}(t)))}\, dt, \qquad (5.54)
\end{aligned}
$$

where the power exponent of scale radius is evaluated as

$$
\begin{aligned}
& r^{-1} \frac{1}{\Lambda^{2(p-1)}} \Lambda^{p-2} r^{-1} = (t_0-t)^{\frac{1}{B_0}\left(-2+\frac{p(B_0-2)}{p-2}\right)} \Longleftarrow t = t_0 - r^{B_0}; \\
& -2 + \frac{p(B_0-2)}{p-2} \geq 0 \Longleftrightarrow B_0 > \frac{4(p-1)}{p} \\
& \Longleftarrow B_0 > \frac{6p-4}{p+2}; \quad \frac{6p-4}{p+2} > \frac{4(p-1)}{p} \Longleftrightarrow (p-2)^2 > 0.
\end{aligned}
$$

Finally, we collect the estimations (5.45), (5.46) in *Step 2*, and (5.53), (5.54) in *Step 3* to complete the proof of (5.4). □

Now we show the validity of the forward monotonicity estimate, Lemma 13.

As before by parallel transformation let the Eq. (2.1) and its solutions u be defined on $(0,\ \infty) \times \mathbb{R}^m$ with the same notation.

Let $(t_0,\ x_0)$ in the parabolic like envelope $\left\{(t,\ x)\ :\ \min\{1,\ (R_{\mathcal{M}})^{B_0}\} > t \geq |x|^{B_0}\right\}$, $B_0 > 2$.

The forward localized scaled penalized energy is

$$
E(r) = \frac{1}{\Lambda^p} \int_{\{t=t_0+\Lambda^{2-p} r^2\} \times B(R_{\mathcal{M}})} \frac{1}{p} \bar{e}(u(t,\ x))\, \mathcal{B}(t_0, x_0;\ t, x)\, \mathcal{C}^q(t,\ x)\, dx; \qquad (5.55)
$$

$$
\Lambda = \Lambda(r) = r^{\frac{B_0-2}{2-p}}; \quad p > B_0 > \frac{6p-4}{p+2}; \quad 0 < r \leq \min\{1,\ (R_{\mathcal{M}})^{1/B_0}\} \qquad (5.56)
$$

with weight

$$\mathcal{B}(t_0,\, x_0;\, t,\, x) = \frac{1}{(t-t_0)^{\frac{m}{B_0}}}\left(1-\left(\frac{|x-x_0|}{(t-t_0)^{\frac{1}{B_0}}}\right)^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}}, \qquad t > t_0;$$
$$\mathcal{C}(t,\, x) = \left(t^{1/B_0} - |x|\right)_+; \quad q > 2. \tag{5.57}$$

The notation as above is used.

Lemma 20 *Let $p > 2$ and $q > 2$. For any regular solution u to (2.1) the following estimate is valid for any positive number $r < \rho \le \min\{1,\, \left((R_{\mathcal{M}})^{B_0} - t_0\right)^{1/B_0}\}$*

$$E(\rho) \le\; E(r) + C\,\left(\rho^{\mu} - r^{\mu}\right)$$
$$+\, C \int_{t_0+r^{B_0}}^{t_0+\rho^{B_0}} \|\mathcal{C}^{q-2}(t)\,(\bar{e}(u(t)))^{\theta_0}\|_{L^\infty(B((t_0-t)^{1/B_0},\, x_0))}\, dt, \tag{5.58}$$

where

$$\Lambda = \Lambda(r) = r^{\frac{B_0-2}{2-p}}, \quad (\Lambda(r))^{2-p}\, r^2 = r^{B_0}$$

and the positive exponents $\theta_0 \ge 2$ and μ depend only on B_0, p and $\mathcal{N}$, m, p and B_0, respectively, and the positive constant C depends only on the same ones as μ and q.

Proof of Lemma 20. As before we put

$$\Lambda = r^{\frac{B_0-2}{2-p}}, \quad p > B_0 > \frac{6p-4}{p+2}$$

and let r any positive number in the range $0 < r \le \min\{1,\, \left((R_{\mathcal{M}})^{B_0} - t_0\right)^{1/B_0}\}$. We make a scaling transformation intrinsic to the evolutionary p-Laplace operator

$$t = t_0 + \Lambda^{2-p} r^2\, s; \quad x = x_0 + r\,y; \quad v(s,\, y) = \frac{u(t_0 + \Lambda^{2-p} r^2\, s,\; x_0 + r\,y)}{\Lambda\, r} \tag{5.59}$$

and, under the scaling transformation it holds that

$$t = t_0 + \Lambda^{2-p} r^2 \iff s = +1.$$

The scaled solution v is a solution of the scaled equation on $\{s = 1\} \times \mathbb{R}^m$

$$\partial_s v - \operatorname{div}\left(\left(\Lambda^{-2}\epsilon + |Dv|^2\right)^{\frac{p-2}{2}} Dv\right) = -C_0\, \frac{K/\Lambda^p}{2}\, \frac{d}{dv}\chi\left(\operatorname{dist}^2\left(\Lambda\, r\, v,\, \mathcal{N}\right)\right). \tag{5.60}$$

Hereafter we use the same notation as in (5.6).

Similarly as the backward case, the scaled energy is rewritten as

$$E(r) = \int_{\{s=1\}\times\mathbb{R}^m} \bar{e}((v(s,\, y))\, \mathcal{B}(s,\, y)\, \mathcal{C}^q(s,\, y)\, dy;$$

$$\mathcal{B}(s,\, y) = \frac{1}{s^{m/B_0}}\left(1-\left(\frac{|y|}{s^{1/B_0}}\right)^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}}; \; \mathcal{C}(s,\, y) = \left((t_0 + r^{B_0}\, s)^{1/B_0} - |x_0 + r\,y|\right)_+, \tag{5.61}$$

where the integral in (5.61) is well-defined by supp($\mathcal{C}$) and supp($\mathcal{B}$).

The computation and estimation are similar as in those for the backward monotonicity estimate. In the following we indicate only the part of estimations, different from the backward monotonicity. In the following the integral region on y is changed to $\{s = 1\} \times \mathbb{R}^m$.

Similarly as in (5.9) in the backward case, we make differentiation of $E(r)$ on r

$$\begin{aligned}\frac{d}{dr}E(r) &= \int_{\{s=1\}\times\mathbb{R}^m} \left((p\,f)^{1-\frac{2}{p}} Dv \cdot \frac{d}{dr}Dv + C_0\,\frac{dv}{dr}\cdot D_v g(v)\right) \mathcal{B}\,\mathcal{C}^q dy \\ &\quad + \frac{B-2}{r\,(p-2)} \int_{\{s=1\}\times\mathbb{R}^m} \left(\bar{\epsilon}\,(p\,f)^{1-\frac{2}{p}} + p\,C_0\,g(v)\right) \mathcal{B}\,\mathcal{C}^q\,dy \\ &\quad + \int_{\{s=1\}\times\mathbb{R}^m} \bar{e}(v)\,\mathcal{B}\,\frac{d}{dr}\mathcal{C}^q\,dy \\ &=: I + II + III. \end{aligned} \tag{5.62}$$

Estimation of II and III. By Young's inequality and $0 < \bar{\epsilon} = \Lambda^{-2}\epsilon \le 1$ the term II is bounded with $\delta > 0$ by

$$II \le \frac{C}{r^{1+\delta}} \int_{\{s=1\}\times\mathbb{R}^m} (\bar{e}(v))^{\frac{2(p-1)}{p}}\,\mathcal{C}^q\,\mathcal{B}\,dy + C\left(\frac{r^{\frac{\delta(p-2)}{p}}}{r} + \frac{r^{\frac{\delta p}{p-2}}}{r}\right)\int_{\{s=1\}\times\mathbb{R}^m} \mathcal{C}^q\,\mathcal{B}\,dy.$$

For estimation of III the derivative of $\mathcal{C}$ on r is computed as

$$\left|\frac{d}{dr}\mathcal{C}(t,\,x)\right| = \chi_{\{|x_0+ry|\le(t_0+r^{B_0}s)^{1/B_0}\}}\left|(t_0+r^{B_0}s)^{1/B_0}\,\frac{r^{B_0-1}s}{t_0+r^{B_0}s} - \frac{x_0+ry}{|x_0+ry|}\cdot y\right|$$

and thus, on the support $\{y \in \mathbb{R}^m \;:\; |y| < 1\}$ of $\mathcal{B}(1,\,y)$

$$\left|\frac{d}{dr}\mathcal{C}\right|\Bigg|_{s=1} \le 3\,\chi_{\{|x_0+ry|\le(t_0-r^{B_0})^{1/B_0}\}}\,r^{-1}$$

because of the conditions

$$0 < t_0 \le 1; \quad \frac{r^{B_0}}{t_0+r^{B_0}} \le 1.$$

Thus, exactly as (5.10) in the backward case, we have

$$III \le \frac{C}{r^{1+\delta}} \int_{\{s=1\}\times\mathbb{R}^m} (\bar{e}(v))^{\frac{2(p-1)}{p}}\,\mathcal{C}^q\,\mathcal{B}\,dy + \frac{C\,r^{\frac{\delta p}{p-2}}}{r} \int_{\{s=1\}\times\mathbb{R}^m} \mathcal{C}^{q-\frac{2(p-1)}{p-2}}\,\mathcal{B}\,dy.$$

The estimation of I is exactly same as (5.11) in the backward case. The terms corresponding to I_1 are bounded above by $(-1)\times$the terms (5.20), (5.21) and some controllable integral terms containing $\mathcal{B}$, $\mathcal{C}$ and their derivatives, where the integral region is replaced by $\{s = 1\} \times \mathbb{R}^m$. The term corresponding to I_2 is estimated above by $(-1)\times$the right hand side of (5.22) with the integral region replaced by $\{s = 1\} \times \mathbb{R}^m$.

I_3 is computed exactly as (5.23) and (5.24) with the integral region replaced by $\{s = 1\} \times \mathbb{R}^m$. In the term corresponding to I_{31} we note by $s = 1$ that

$$(-s)\,\big((2-p)\,r\,\Lambda^{-1}\,\Lambda' + 2\big) = -B_0 < 0 \Longleftrightarrow \Lambda = r^{(B_0-2)/(2-p)}, \quad B_0 > 0.$$

The term corresponding to I_{33} is estimated above by

$$\begin{aligned}
&\frac{p}{r\,(p-2)} \int_{\{s=1\}\times\mathbb{R}^m} (p\,f)^{1-\frac{2}{p}}\, |y\cdot Dv|^2\, \mathcal{C}^q\, |y|^{-\frac{p-2}{p-1}} \left(1-|y|^{\frac{p}{p-1}}\right)_+^{\frac{1}{p-2}} dy \\
&\quad \le \frac{p}{r\,(p-2)} \int_{\{s=1\}\times\mathbb{R}^m} p\,f\,\mathcal{C}^q\,|y|^{2-\frac{p-2}{p-1}} \left(1-|y|^{\frac{p}{p-1}}\right)_+^{\frac{1}{p-2}} dy \\
&\quad \le \frac{C\,r^{-\delta}}{r} \int_{\{s=1\}\times\mathbb{R}^m} f^{\frac{2(p-1)}{p}}\,\mathcal{C}^q\,|y|^{2-\frac{p-2}{p-1}} \left(1-|y|^{\frac{p}{p-1}}\right)_+^{\frac{1}{p-2}} dy \\
&\qquad + \frac{C\,r^{\frac{\delta p}{p-2}}}{r} \int_{\{s=1\}\times\mathbb{R}^m} \mathcal{C}^q\,|y|^{2-\frac{p-2}{p-1}} \left(1-|y|^{\frac{p}{p-1}}\right)_+^{\frac{1}{p-2}} dy.
\end{aligned}$$

The other terms corresponding to I_{3i}, $i = 2, 4, 5$, are bounded above by $(-1)\times$the terms of the right hand side of (5.25), (5.26), (5.27) and (5.28).

Combining all of the estimations above we have

$$\begin{aligned}
\frac{d}{dr}E(r) &\le J + C\left(r^{-1+\delta} + r^{-1+\frac{\delta}{p-2}} + r^{-1+\frac{\delta p}{p-2}}\right) \\
&\quad + \frac{C}{r^{3+\delta}\,\Lambda^2} \int_{\{s=1\}\times\mathbb{R}^m} |D_v g(v)|^2\,\mathcal{B}\,\mathcal{C}^q\,dy \\
&\quad + C\,\frac{r^{-\delta}+1}{r\,\Lambda^{2(p-1)}}\,\|\mathcal{C}^{q-2}(t)\,(\bar{e}(u(t)))^{\frac{2(p-1)}{p}}\|_{L^\infty(\mathrm{supp}\,(\mathcal{B}(t)))}\Big|_{t=t_0+r^{B_0}},
\end{aligned} \tag{5.63}$$

where $\Lambda = r^{(B_0-2)/(2-p)}$, but

$$J = -\frac{1}{2}\,B_0\,r^{-1} \int_{\{s=1\}\times\mathbb{R}^m} |\partial_s v|^2\,\mathcal{B}\,\mathcal{C}^q\,dy.$$

The term J is clearly nonpositive. From (5.63) integrated on the interval $(r,\ \rho)$ we derive

$$\begin{aligned}
&E(\rho) - E(r) \\
&\quad \le C\int_r^\rho \left(r^{-1+\delta} + r^{-1+\frac{\delta}{p-2}} + r^{-1+\frac{\delta p}{p-2}}\right) dr \\
&\qquad + \int_r^\rho \frac{C}{r^{3+\delta}\,\Lambda^2} \int_{\{s=1\}\times\mathbb{R}^m} |D_v g(v)|^2\,\mathcal{B}\,\mathcal{C}^q\,dy\,dr \\
&\qquad + C\int_r^\rho \frac{r^{-\delta}+1}{r\,\Lambda^{2(p-1)}}\,\|\mathcal{C}^{q-2}(t)\,(\bar{e}(u(t)))^{\frac{2(p-1)}{p}}\|_{L^\infty(\mathrm{supp}\mathcal{B}(t))}\Big|_{\tau=t_0+r^{B_0}}\,dr \\
&\quad =: C\,(U_1 + U_2 + U_3).
\end{aligned} \tag{5.64}$$

The terms in the right hand side of (5.64) correspond to those in (5.30). Note that U_1, U_2 and U_3 are just $(-1)\times$ the corresponding terms in (5.30). U_1 and U_3 can be estimated exactly similarly as the corresponding terms in (5.30).

We have to care the estimation of U_2. Under the scaling setting (5.59) and (5.60) in the forward case now, we also have the Bochner type estimate (5.33) for the penalty term. We can proceed to the estimations, similarly as in (5.34), to obtain

$$\begin{aligned}
\left(C_0' - \frac{c}{2}\right) U_2 &= \left(C_0' - \frac{c}{2}\right) \int_r^\rho \frac{1}{r^{3+\delta}\,\Lambda^2} \int\limits_{\{s=1\}\times\mathbb{R}^m} |D_v g|^2\, \mathcal{B}\,\mathcal{C}^q \,dy\,dr \\
&\le - \int_r^\rho \frac{1}{r^{3+\delta}\,\Lambda^2} \int\limits_{\{s=1\}\times\mathbb{R}^m} \partial_s g\, \mathcal{B}\,\mathcal{C}^q \,dy\,dr \\
&\quad + \int_r^\rho \frac{1}{r^{3+\delta}\,\Lambda^2} \int\limits_{\{s=1\}\times\mathbb{R}^m} \frac{1}{2c} \Big((p\,f)^{2-\frac{2}{p}} \left(\mathcal{B}'\,\mathcal{C}^q + \mathcal{B}\,q^2\mathcal{C}^{q-2}|D\mathcal{C}|^2\right) + \\
&\quad + C\,(\Lambda r)^2 \bar{e}^2\, \mathcal{B}\,\mathcal{C}^q \Big)\,dy\,dr \\
&=: U_{21} + U_{22},
\end{aligned}$$

The estimation for U_{22} is the same as in (5.35) in the backward case.

U_{21} is also computed as in (5.36) in the backward case

$$\begin{aligned}
-U_{21} &= \int_r^\rho \frac{1}{r^{3+\delta}\,\Lambda^2} \int\limits_{\{s=1\}\times\mathbb{R}^m} \partial_s\left(g\,\mathcal{C}^q\right) \mathcal{B}\,dy\,dr - \int_r^\rho \frac{1}{r^{3+\delta}\,\Lambda^2} \int\limits_{\{s=1\}\times\mathbb{R}^m} g\,\partial_s \mathcal{C}^q\,\mathcal{B}\,dy\,dr \\
&=: U_{211} + U_{212}.
\end{aligned}$$

The estimation for $|U_{212}|$ is done in the same way as in (5.37) in the backward case. The estimation for U_{211} is performed in the following. By changing a variable $t = t_0 + \Lambda(r)^{2-p} r^2 = t_0 + r^{B_0}$, we have

$$\begin{aligned}
&t = t_0 + r^{B_0} \Longleftrightarrow r = (t-t_0)^{1/B_0}; \quad \bar{K} = K\Lambda^{-p} = K r^{\frac{p(B_0-2)}{p-2}} = (t-t_0)^{\frac{p(B_0-2)}{B_0(p-2)}}; \\
&dt = B_0 r^{B_0-1} dr = B_0 \Lambda^{2-p} r\,dr; \\
&\frac{1}{r^{2+\delta}\,\Lambda^2} = (t-t_0)^{-c_0}; \quad c_0 := \frac{1}{B_0}\left(\frac{2(p-B_0)}{p-2} + \delta\right)
\end{aligned}$$

and a computation

$$\begin{aligned}
h(t,\,y) &:= \frac{\bar{K}}{2}\chi\left(\operatorname{dist}^2\!\left(u(t,\,x_0+(t-t_0)^{1/B_0}y),\,\mathcal{N}\right)\right)\left(t^{1/B_0} - |x_0 + (t-t_0)^{1/B_0}y|\right)_+^q; \\
\partial_t h(t,\,y) &= \frac{p(B_0-2)}{B_0(p-2)}(t-t_0)^{-1} h(t,\,y) + \frac{1}{B_0}(t-t_0)^{-1} y\cdot D_y h(t,\,y) \\
&\quad + \frac{\bar{K}}{2}\partial_\tau\left(\chi\left(\operatorname{dist}^2\!\left(u(\tau,\,x_0+(t-t_0)^{1/B_0}y),\,\mathcal{N}\right)\right)\times\right. \\
&\qquad\qquad \left.\left.\times\left(\tau^{1/B_0} - |x_0+(t-t_0)^{1/B_0}y|\right)_+^q\right)\right|_{\tau=t}.
\end{aligned}$$

Thus, we have

$$
\begin{aligned}
U_{211} = & \frac{1}{B_0} \int_{t_0+\bar{r}^{B_0}}^{t_0+\rho^{B_0}} (t-t_0)^{-c_0} \frac{dP}{dt}\, dt - \frac{p(B_0-2)}{B_0^2(p-2)} \int_{t_0+\bar{r}^{B_0}}^{t_0+\rho^{B_0}} (t-t_0)^{-c_0-1} P(t)\, dt \\
& - \frac{1}{B_0^2} \int_{t_0+\bar{r}^{B_0}}^{t_0+\rho^{B_0}} (t-t_0)^{-c_0-1} \int_{\mathbb{R}^m} y \cdot D_y h(t,\, y)\, \mathcal{B}(y)\, dy\, dt \\
=: & \, U_{2111} + U_{2112} + U_{2113},
\end{aligned} \tag{5.65}
$$

where we put

$$
P(t) := \int_{\mathbb{R}^m} h(t,\, y)\, \mathcal{B}(y)\, dy, \quad \mathcal{B}(y) = \mathcal{B}(s,\, y)|_{s=1} = \left(1 - |y|^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}}.
$$

We will estimate each term in (5.65).
Now, we set $\bar{r}$ as

$$
\exists\, \bar{r},\ r \le \bar{r} \le \rho \quad : \quad \min_{t_0+r^{B_0} \le t \le t_0+\rho^{B_0}} P(t) = P(t_0 + \bar{r}^{B_0}). \tag{5.66}
$$

Then, by integration by parts in the integral on t, we have

$$
\begin{aligned}
B_0 \times U_{2111} = & \int_{t_0+\bar{r}^{B_0}}^{t_0+\rho^{B_0}} (t-t_0)^{-c_0} \frac{dP}{dt}\, dt \\
= & \, (t-t_0)^{-c_0} P(t)\Big|_{t_0+\bar{r}^{B_0}}^{t_0+\rho^{B_0}} + \int_{t_0+\bar{r}^{B_0}}^{t_0+\rho^{B_0}} P(t)\, c_0\, (t-t_0)^{-c_0-1}\, dt \\
\ge & \, \rho^{-c_0 B_0} P(t_0+\rho^{B_0}) - \bar{r}^{-c_0 B_0} P(t_0+\bar{r}^{B_0}) \\
& + P(t_0+\bar{r}^{B_0}) \left(\bar{r}^{-c_0 B_0} - \rho^{-c_0 B_0}\right) \\
= & \, \rho^{-c_0 B_0} \left(P(t_0+\rho^{B_0}) - P(t_0+\bar{r}^{B_0})\right) \ge 0.
\end{aligned} \tag{5.67}
$$

By integration by parts in the integral on y, we also have

$$
\begin{aligned}
U_{2113} = & \frac{1}{B_0^2} \int_{t_0+\bar{r}^{B_0}}^{t_0+\rho^{B_0}} (t-t_0)^{-c_0-1} \times \\
& \times \int_{\mathbb{R}^m} h(t,\, y) \left(m\, \mathcal{B} - \frac{p}{p-2} \left(1 - |y|^{\frac{p}{p-1}}\right)_+^{\frac{1}{p-2}} |y|^{\frac{p}{p-1}} \right) dy\, dt;
\end{aligned}
$$

$$
U_{2112} + U_{2113} \ge \frac{1}{B_0^2} \left(m - \frac{p(B_0-2)}{p-2} \right) \int_{t_0+\bar{r}^{B_0}}^{t_0+\rho^{B_0}} (t-t_0)^{-c_0-1} P(t)\, dt
$$

$$- \frac{p}{B_0^2(p-2)} \int_{t_0+\bar{r}^{B_0}}^{t_0+\rho^{B_0}} (t-t_0)^{-c_0-1} \int_{\mathbb{R}^m} h(t,\, y)\, \mathcal{C}^q(t) \left(1-|y|^{\frac{p}{p-1}}\right)_+^{\frac{1}{p-2}} |y|^{\frac{p}{p-1}}\, dy\, dt.$$

The 1st term is nonnegative, since $m - \frac{p(B_0-2)}{p-2} > m - p \geq 0$, and the last integral is estimated below as

$$\begin{aligned}
&g(u(t)) := \frac{K}{2}\chi\left(\operatorname{dist}^2\left(u(t,\, x_0+(t-t_0)^{1/B_0}y),\, \mathcal{N}\right)\right),\\
&\mathcal{C}^q(t) := \mathcal{C}^q(t,\, x_0+(t-t_0)^{1/B_0}y);\\
&\quad - \frac{p}{B_0^2(p-2)} \int_{t_0+\bar{r}^{B_0}}^{t_0+\rho^{B_0}} (t-t_0)^{b_0-1} \int_{\mathbb{R}^m} g(u(t))\, \mathcal{C}^q(t) \left(1-|y|^{\frac{p}{p-1}}\right)_+^{\frac{1}{p-2}} |y|^{\frac{p}{p-1}}\, dy\, dt\\
&\geq - \frac{p}{B_0^2(p-2)} \int_{\mathbb{R}^m} \left(1-|y|^{\frac{p}{p-1}}\right)_+^{\frac{1}{p-2}} |y|^{\frac{p}{p-1}}\, dy \times\\
&\quad \times \left(\int_{t_0+\bar{r}^{B_0}}^{t_0+\rho^{B_0}} (t-t_0)^{\alpha_0(b_0-1)}\, dt + \int_{t_0+\bar{r}^{B_0}}^{t_0+\rho^{B_0}} \left\| \frac{K}{2}\chi\left(\operatorname{dist}^2\left(u(t,\,\cdot),\, \mathcal{N}\right)\right) \mathcal{C}^q(t) \right\|_{L^\infty\left(\operatorname{supp}\mathcal{B}(t)\right)}^{\frac{\alpha_0}{\alpha_0-1}} dt \right),
\end{aligned} \tag{5.68}$$

where $\alpha_0 > 1$ is as in (5.43) in the backward case. Therefore we have

$$\begin{aligned}
-U_{21} \geq & -C\left(\rho^{\alpha_0(b_0-1)+1} - \bar{r}^{\alpha_0(b_0-1)+1}\right)\\
& - C \int_{t_0+\bar{r}^{B_0}}^{t_0+\rho^{B_0}} \|\bar{e}(u(t))\, \mathcal{C}^q(t)\|_{L^\infty\left(\operatorname{supp}\mathcal{B}(t)\right)}^{\frac{\alpha_0}{\alpha_0-1}}\\
& - C \int_{t_0+\bar{r}^{B_0}}^{t_0+\rho^{B_0}} \|\mathcal{C}^{q-1}(t)\, \bar{e}(u(t))\|_{L^\infty\left(\operatorname{supp}(\mathcal{B}(t))\right)}\, dt.
\end{aligned}$$

By definition of $P(t)$ in (5.65) and $\bar{r}$ in (5.66), $P(t)$ is the local scaled integral of the penalty term, because by changing a variable $x = x_0 + (t-t_0)^{1/B_0} y$

$$\begin{aligned}
h(t,\, y) &= \frac{\bar{K}}{2}\chi\left(\operatorname{dist}^2\left(u(t,\, x_0+(t-t_0)^{1/B_0}y),\, \mathcal{N}\right)\right)\left(t^{1/B_0} - |x_0+(t-t_0)^{1/B_0}y|\right)_+^q;\\
P(t) &= \int_{\mathbb{R}^m} h(t,\, y)\, \mathcal{B}\, dy = \frac{1}{\Lambda^p}\int_{\mathbb{R}^m} \frac{K}{2}\chi\left(\operatorname{dist}^2(u(t,\, x),\, \mathcal{N})\right) \mathcal{C}^q(t,\, x)\, \mathcal{B}(t,\, x)\, dx
\end{aligned}$$

and it holds that

$$P(t_0+\bar{r}^{B_0}) \leq P(t_0+r^{B_0}) \quad \text{for } \bar{r} \text{ in (5.66).} \tag{5.69}$$

Collecting the estimations for U_1, U_2 and U_2 above in (5.64), we have, for $\bar{r}$ in (5.66),

$$\begin{aligned} E(\rho) - E(\bar{r}) \le\ & C\ \left(\rho^{\mu} - \bar{r}^{\mu}\right) \\ & + C \int_{t_0+\bar{r}^{B_0}}^{t_0+\rho^{B_0}} \|\mathcal{C}^{q-2}(t)\, \bar{e}(u(t))^{\frac{2(p-1)}{p}} + \\ & + \mathcal{C}^{q}(t) \left(\bar{e}(u(t))^2 + \bar{e}(u(t))^{\frac{\alpha_0}{\alpha_0-1}}\right) \|_{L^\infty(\mathrm{supp}(\mathcal{B}(t)))}\, dt. \end{aligned} \tag{5.70}$$

Let θ_0 be as in (5.43) and (5.47) in the backward case. By (5.70), our desired monotonicity estimate holds true in the range of scale radius $[\bar{r},\ \rho]$, and (5.69) is the monotonicity estimate in the range $[r,\ \bar{r}]$ of the local scaled integral of the penalty term. Therefore, it remains to estimate the local scaled p-energy in the range of scale radius $[r,\ \bar{r}]$. The monotonicity estimate of the local scaled p-energy in the range of scale radius $[r,\ \bar{r}]$ is estimated exactly as *Step* 3 in the backward case. In fact, letting as in (5.48)

$$F(r) = \int_{\{s=1\}\times\mathbb{R}^m} f(v(s,\ y))\, \mathcal{B}(s,\ y)\, \mathcal{C}^q(s,\ y)\, dy; \quad f = f(v) := \frac{1}{p}\left(\bar{\epsilon} + |Dv|^2\right)^{\frac{p}{2}},$$

we arrive at the estimate corresponding to (5.53)

$$\begin{aligned} F(\rho) - F(r) \le\ & C\ \left(\rho^{\mu} - r^{\mu}\right) \\ & + \int_r^{\rho} \frac{C}{r\, \Lambda^{2(p-1)}} \|\bar{e}(u(t))^{\frac{2(p-1)}{p}}\, \mathcal{C}^{q-2}(t) + \\ & + \bar{e}(u(t))^2\, \mathcal{C}^q(t) \|_{L^\infty(\mathrm{supp}\,(\mathcal{B}(t))}\Big|_{t=t_0+r^{B_0}}\, dr, \end{aligned}$$

where the last term is controlled as in (5.54). □

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Compliance with ethical standards

Conflicts of interest The author declares that it has no conflict of interest.

6. Appendix

Appendix A A global existence and regularity of a weak solution of (2.1).

Proof od Lemma 6. We use the Galerkin method and the monotonicity trick for p-Laplace operator to solve the Cauchy problem (2.1). The proof is standard and we can refer to [6, Theorem 1.5 and its proof, pp. 29–31].

Regularity of a weak solution. Let $u = u_{K,\epsilon}$ be a weak solution of (2.1). The lower-order term is bounded by the definition of χ as

$$\left|K\, \chi'\left(\mathrm{dist}^2(p,\ \mathcal{N})\right)\ \mathrm{dist}(p,\ \mathcal{N}) D_p\ \mathrm{dist}(p,\ \mathcal{N})\right| \le C\, K\, \delta_{\mathcal{N}} \sup_{s>0} \left|\chi'(s)\right|$$

and thus, we can apply the Hölder regularity result for the evolutionary p-Laplace operator in [12, Theorem 1.1′, p. 256] (also see [26]) to find that the solution u and its gradient are locally Hölder continuous on $\mathcal{M}_\infty$. We also have that the second derivative is integrable : $\left(\epsilon + |Du|^2\right)^{\frac{p-2}{2}} |D^2u|^2$ is locally integrable in $\mathcal{M}_\infty$ and that the gradient Du is locally bounded in $\mathcal{M}_\infty$ (see [12, Proposition 3.1, p. 223; Theorem 5.1 , p. 238]. Then, expanding the principal part of the p-Laplace operator, the solution u is also satisfies the linear parabolic systems with Hölder continuous coefficients and lower-order terms almost everywhere. Thus, it follows from the Schauder regularity theory that u, Du, D^2u and $\partial_t u$ are locally Hölder continuous in $\mathcal{M}_\infty$. □

Appendix B Energy inequality and maximum principle

We present the proof of Lemmata 7 and 8.

Proof of Lemma 7. The energy inequality (3.1) is shown to be valid in the proof of Lemma 6. However, as a priori estimates for regular solutions of (2.1), we naturally multiply (2.1) by $\partial_t u \sqrt{|g|}$ and integrate by parts on space variable in $\mathcal{M}_T$ for any $T > 0$. □

Proof of Lemma 8. We multiply (2.1) by $\sqrt{|g|} u\left(|u|^2 - H^2\right)_+$ and integrate in $\mathcal{M}_\infty$, where $\left(f\right)_+$ is the positive part of a function f. Since the support of χ' is in $\mathcal{O}_{2\delta_{\mathcal{N}}}(\mathcal{N}) \subset B(H)$, $\chi'(\mathrm{dist}^2(u, \mathcal{N}))$ is zero in $\mathbb{R}^l \setminus B(H, 0)$. Also $u_0 \in \mathcal{N} \subset B(H)$. Hence, we have

$$
\begin{aligned}
&\frac{1}{4}\int_{\mathcal{M}} \left(|u(t)|^2 - H^2\right)_+ d\mathcal{M} \\
&\quad + \int_{\mathcal{M}_t} \left(\epsilon + |Du|_g^2\right)^{\frac{p-2}{2}} \left(\frac{1}{2}\left|D\left(|u|^2 - H^2\right)_+\right|_g^2 + |Du|_g^2\left(|u|^2 - H^2\right)_+ \right) d\mathcal{M}dt = 0 \quad ; \\
&\frac{1}{4}\int_{\mathcal{M}} \left(|u(t)|^2 - H^2\right)_+^2 d\mathcal{M} \le 0
\end{aligned}
$$

and thus, $|u(t)| \le H$ in $\mathcal{M}$ and any $t \ge 0$. □

Appendix C Proof of the Bochner estimate.

Proof of Lemma 9. In the proof, for brevity, let the regularized p-energy density be

$$
f = f(u) := \frac{1}{p}\left(\epsilon + |Du|^2\right)^{\frac{p}{2}}.
$$

In the general case in $\mathcal{M}$, the terms containing the spatial derivative of $g_{\alpha\beta}$ only appear and are bounded by $C\,(\epsilon + |Du|^2)^{p-1}$. In fact, in (6.5) below, by a direct computation, we have the terms with derivatives of the metric

$$
\begin{aligned}
&-\frac{D_\gamma\sqrt{|g|}}{|g|} g^{\gamma\mu} D_\mu u \cdot D_\alpha\left((p\,f)^{1-\frac{2}{p}}\sqrt{|g|}g^{\alpha\beta}D_\beta u\right) \\
&+D_\alpha\left((pf)^{1-\frac{2}{p}} D_\gamma(\sqrt{|g|}g^{\alpha\beta})g^{\gamma\mu}D_\mu u \cdot D_\beta u\right) \\
&-\frac{1}{2}\, D_\alpha\left((pf)^{1-\frac{2}{p}}\sqrt{|g|}g^{\alpha\beta}D_\beta g^{\gamma\mu}D_\gamma u \cdot D_\mu u\right) \\
&-\frac{1}{\sqrt{|g|}}(pf)^{1-\frac{2}{p}}\Big\{D_\alpha g^{\gamma\mu}D_\gamma(\sqrt{|g|}g^{\alpha\beta})D_\mu u \cdot D_\beta u
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{p-2}{2}\sqrt{|g|}g^{\alpha\beta}\frac{D_\alpha g^{\gamma\mu}D_\mu u\cdot D_\beta u}{(pf)^{\frac{2}{p}}}\left(D_\gamma g^{\mu\nu}D_\mu u\cdot D_\nu u+2g^{\mu\nu}D_\mu D\gamma u\cdot D_\nu u\right)\\
&+\sqrt{|g|}g^{\alpha\beta}D_\alpha g^{\gamma\mu}D_\mu u\cdot D_\beta D_\gamma u+D_\gamma(\sqrt{|g|}g^{\alpha\beta}g^{\gamma\mu}D_\alpha D_\mu u\cdot D_\beta u\\
&+\frac{p-2}{2}\sqrt{|g|}g^{\gamma\mu}g^{\alpha\beta}D_\alpha D_\mu u\cdot D_\beta u\frac{D_\gamma g^{\mu\nu}D_\mu u\cdot D_\nu u}{(pf)^{\frac{2}{p}}}\Bigg\},
\end{aligned}
\tag{6.1}
$$

which are bounded by such terms as

$$
C\,|Dg^{\alpha\beta}|^2\,f;\quad C\,|Dg^{\alpha\beta}||g^{\alpha\beta}|\,f^{1-\frac{1}{p}}\,|D^2u| \tag{6.2}
$$

with a positive constant C depending only on $\mathcal{M}$, p and m. Here the 1st term is controllable lower-order one. By Cauchy's inequality with $c>0$, the 2nd term are estimated above as

$$
C\,|Dg^{\alpha\beta}||g^{\alpha\beta}|\,f^{1-\frac{1}{p}}\,|D^2u|\le cf^{1-\frac{2}{p}}\,|D^2u|^2+C(c^{-1})\,|Dg^{\alpha\beta}|^2|g^{\alpha\beta}|^2\,f,
$$

of which the 1st term with a small $c>0$ in the right hand side is abosrbed into the squared 2nd derivative term of the solution in (6.5) below, and the 2nd term is a controllable one. The controllable terms $C\,f$ above are multiplied by $(pf)^{1-\frac{2}{p}}$ in (6.6) below, and thus, becomes $C\,f^{2(1-\frac{1}{p})}$.

Hereafter, we assume that the metric $g=\left(g_{\alpha\beta}\right)$ is the identity matrix.

Since u, Du and D^2u are continuous in $\mathbb{R}^m_\infty$, it holds in the distribution sense that

$$
\begin{aligned}
Du\cdot D\Big(f^{1-\frac{2}{p}}Du\Big)=\;&D_\alpha\Big(\mathcal{A}^{\alpha\beta}D_\beta f\Big)-(p\,f)^{1-\frac{2}{p}}\big|D^2u\big|^2\\
&-(p-2)(p\,f)^{1-\frac{4}{p}}\Big|D\frac{1}{2}|Du|^2\Big|^2.
\end{aligned}
\tag{6.3}
$$

Hereafter the summation convention over repeated indices is used. Since $\chi\big(\operatorname{dist}^2(u,\mathcal{N})\big)=2(\delta_{\mathcal{N}})^2$ for $u\in\mathbb{R}_l\backslash\mathcal{O}_{2\delta_{\mathcal{N}}}$, $D\Big(D_u\chi\big(\operatorname{dist}^2(u,\mathcal{N})\big)\Big)=0$ if $\operatorname{dist}(u,\mathcal{N})>2\delta_{\mathcal{N}}$ and then, we have (3.4) by (2.1). We treat the case that $\operatorname{dist}(u,\mathcal{N})\le 2\delta_{\mathcal{N}}$. Noting that $\chi\big(\operatorname{dist}^2(u,\mathcal{N})\big)$ is smooth, by a direct calculation we have

$$
\begin{aligned}
&Du\cdot D\Big(\frac{K}{2}D_u\chi\big(\operatorname{dist}^2(u,\mathcal{N})\big)\Big)=\frac{K}{2}\Big(Du^i\cdot Du^j\Big)D_{u^i}D_{u^j}\chi\big(\operatorname{dist}^2(u,\mathcal{N})\big)\quad;\\
&D_{u^i}D_{u^j}\chi\big(\operatorname{dist}^2(u,\mathcal{N})\big)=2\,D_{u^i}\operatorname{dist}(u,\mathcal{N})D_{u^j}\operatorname{dist}(u,\mathcal{N})\left(\chi'+2\operatorname{dist}^2(u,\mathcal{N})\chi''\right)\\
&\qquad+2\chi'\operatorname{dist}(u,\mathcal{N})D_{u^i}D_{u^j}\operatorname{dist}(u,\mathcal{N}),
\end{aligned}
\tag{6.4}
$$

where the arguments in χ' are omitted. By (6.3) and (6.4) with (2.1), we have

$$
\begin{aligned}
&\partial_t\frac{1}{2}(p\,f)^{\frac{2}{p}}-D_\alpha\Big(\mathcal{A}^{\alpha\beta}D_\beta f\Big)+(p\,f)^{1-\frac{2}{p}}\big|D^2u\big|^2+(p-2)(p\,f)^{1-\frac{4}{p}}\Big|D\frac{1}{2}|Du|^2\Big|^2\\
&+C_0\,K|D\operatorname{dist}(u,\mathcal{N})|^2\left(\chi'+2\operatorname{dist}^2(u,\mathcal{N})\chi''\right)\\
&+C_0\,K\chi'\operatorname{dist}(u,\mathcal{N})Du^i\cdot Du^jD_{u^i}D_{u^j}\operatorname{dist}(u,\mathcal{N})=0.
\end{aligned}
\tag{6.5}
$$

Furthermore, multiplying (6.5) by $(p\,f)^{1-\frac{2}{p}}$, we obtain

$$\begin{aligned}
&\partial_t f - D_\alpha\Big((p\,f)^{1-\frac{2}{p}}\,\mathcal{A}^{\alpha\beta}D_\beta f\Big) + (p\,f)^{2-\frac{4}{p}}\big|D^2u\big|^2 + (p-2)(p\,f)^{2-\frac{6}{p}}\Big|D\frac{1}{2}|Du|^2\Big|^2\\
&+C_0\,(p\,f)^{1-\frac{2}{p}}\,K|D\,\mathrm{dist}(u,\mathcal{N})|^2\left(\chi'+2\,\mathrm{dist}^2(u,\mathcal{N})\chi''\right)\\
&+C_0\,(p\,f)^{1-\frac{2}{p}}\,K\chi'\ \mathrm{dist}(u,\mathcal{N})Du^i\cdot Du^j D_{u^i}D_{u^j}\ \mathrm{dist}(u,\mathcal{N})\le 0,
\end{aligned}\tag{6.6}$$

where we use the fact that

$$\mathcal{A}^{\alpha\beta}D_\beta f D_\alpha(p\,f)^{1-\frac{2}{p}} = p^{-\frac{2}{p}}(p-2)\,\mathcal{A}^{\alpha\beta}D_\beta f D_\alpha f\ge 0.$$

By differentiation of the penalty term

$$g=g(u):=\frac{K}{2}\chi\big(\,\mathrm{dist}^2(u,\mathcal{N})\big)$$

and (6.4), we have

$$\begin{aligned}
&\partial_t g - D_\alpha\Big((p\,f)^{1-\frac{2}{p}}\mathcal{A}^{\alpha\beta}D_\beta g\Big)\\
&=-C_0\,\big|D_u g\big|^2-(p\,f)^{1-\frac{2}{p}}K\,\mathcal{A}^{\alpha\beta}D_\alpha u\cdot D_u\ \mathrm{dist}(u,\mathcal{N})D_\beta u\\
&\qquad\qquad\cdot D_u\ \mathrm{dist}(u,\mathcal{N})\Big(\chi'+2\,\mathrm{dist}^2(u,\mathcal{N})\chi''\Big)\\
&\quad-(p\,f)^{1-\frac{2}{p}}\,K\,\chi'\ \mathrm{dist}(u,\mathcal{N})\mathcal{A}^{\alpha\beta}D_\alpha u^i D_\beta u^j D_{u^i}D_{u^j}\ \mathrm{dist}(u,\mathcal{N})\\
&\quad-(p-2)(p\,f)^{1-\frac{4}{p}}\left((D_\alpha u\cdot D_\beta u)D_\alpha D_\beta u\cdot D_u g - D\frac{1}{2}|Du|^2\cdot\big(Du\cdot D_u g\big)\right),
\end{aligned}\tag{6.7}$$

where in particular, multiplying (2.1) by the derivative of penalty term we compute

$$\begin{aligned}
&\partial_t g-\ \mathrm{div}\Big((p\,f)^{1-\frac{2}{p}}Dg\Big)\\
&=-C_0\,\big|D_u g\big|^2-(p\,f)^{1-\frac{2}{p}}\left(Du^i\cdot Du^j\right)D_{u^i}D_{u^j}\chi\\
&=-C_0\,\big|D_u g\big|^2-(p\,f)^{1-\frac{2}{p}}K\,\big|Du\cdot D_u\ \mathrm{dist}(u,\mathcal{N})\big|^2\Big(\chi'+2\,\mathrm{dist}^2(u,\mathcal{N})\chi''\Big)\\
&\qquad-(p\,f)^{1-\frac{2}{p}}\,K\,\chi'\ \mathrm{dist}(u,\mathcal{N})Du^i\cdot Du^j D_{u^i}D_{u^j}\ \mathrm{dist}(u,\mathcal{N}).
\end{aligned}\tag{6.8}$$

By the support of χ'', we have

$$\mathrm{dist}^2(u,\mathcal{N})\,\big|\chi''\big|\le 100\ \sup\big|\chi'\big|\,\chi$$

and thus, the 2nd terms in the 2nd line of (6.6) and the 3rd line of (6.7), and the 3rd term in the 3rd line of (6.8) are estimated above by

$$\begin{aligned}
&2\,K\,(p\,f)^{1-\frac{2}{p}}\ \mathrm{dist}^2(u,\mathcal{N})\ \big|\chi''\big|\ \Big((C_0+1)\big|D\ \mathrm{dist}(u,\mathcal{N})\big|^2\\
&\quad+\big|\mathcal{A}^{\alpha\beta}D_\alpha\ \mathrm{dist}(u,\mathcal{N})D_\beta\ \mathrm{dist}(u,\mathcal{N})\big|\Big)\\
&\le C\ (1+C_0)\,f\,g
\end{aligned}\tag{6.9}$$

with a positive constant C depending only on p and χ, because

$$|D_u \, \mathrm{dist}(u, \mathcal{N})| = 1; \quad |D \, \mathrm{dist}(u, \mathcal{N})| = |Du \cdot D_u \, \mathrm{dist}(u, \mathcal{N})| \\ \leq |Du| \leq (p\, f)^{\frac{2}{p}}.$$

By Schwarz's and Cauchy's inequalities, the terms in the 3rd lines of (6.6) and in the 4th line of (6.7), (6.8) are bounded by

$$\begin{aligned} & C(\mathcal{N})\ (C_0 + p - 1)\ (p\, f)^{1-\frac{2}{p}} |Du|^2\, K\chi' \ \mathrm{dist}(u, \mathcal{N}) \\ & = C(\mathcal{N})\ (C_0 + p - 1)\ (p\, f)^{1-\frac{2}{p}}\, |Du|^2\, |D_u g| \\ & \leq \left(\frac{C_0}{2} + 1\right) |D_u g|^2 + C^2(\mathcal{N}) \left(\frac{C_0}{2} + \frac{(p-1)^2}{4}\right) (p\, f)^{2-\frac{4}{p}} |Du|^4, \end{aligned} \tag{6.10}$$

where by a positive constant $C(\mathcal{N})$ depending on a bound for the curvature of $\mathcal{N}$, we have the boundedness for any $u \in \mathcal{N}$

$$\left| \left(2\delta^{\alpha\beta} + \mathcal{A}^{\alpha\beta}\right) D_\alpha u^i\, D_\beta u^j\ D_{u^i} D_{u^j}\ \mathrm{dist}(u, \mathcal{N}) \right| \leq C(\mathcal{N})\, |Du|^2, \tag{6.11}$$

of which the validity will be shown later.

The terms in the 5th line of (6.7) are bounded by

$$2(p-2)\,(p\, f)^{1-\frac{2}{p}} \left|D^2 u\right| |D_u g| \leq \frac{1}{2} (p\, f)^{2-\frac{4}{p}} \left|D^2 u\right|^2 + 2(p-2)^2\ |D_u g|^2. \tag{6.12}$$

Gathering (6.9), (6.10) and (6.12) in (6.6) and (6.7), respectively, we obtain

$$\begin{aligned} & \partial_t e(u) - D_\alpha \left((p\, f)^{1-\frac{2}{p}} \mathcal{A}^{\alpha\beta} D_\beta e(u) \right) \\ & \quad + (p\, f)^{2\left(1-\frac{2}{p}\right)} \left|D^2 u\right|^2 + (p-2)(p\, f)^{2\left(1-\frac{3}{p}\right)} \left| D \frac{1}{2} |Du|^2 \right|^2 + C_2 \left|D_u g\right|^2 \\ & \leq C^2(\mathcal{N}) \left(\frac{C_0}{2} + \frac{(p-1)^2}{4} \right) (p\, f)^{2\left(1-\frac{2}{p}\right)} |Du|^4 + C\ (1 + C_0)\, f\, g \end{aligned} \tag{6.13}$$

and thus, from (6.13), the desired inequality (3.4) is obtained, if the constant C_0 is so large that

$$C_2 := \frac{C_0}{2} - 1 - \frac{25(p-2)^2}{2} > 0. \tag{6.14}$$

We present the proof of (6.11). We follow the argument as in [2, Theorems 3.1 and 3.2, their proofs, pp. 704–707] (also refer to [1, Theorem 2.2]). □

Lemma 21 *There exists a positive constant C depending only on a bound of curvatures of $\mathcal{N}$ such that, for any $u \in \mathcal{O}_{2\delta_{\mathcal{N}}}$ and $q \in \mathbb{R}^l \cong \mathcal{T}_u \mathbb{R}^l$,*

$$\left| q^i\, q^j\ D_{u^i} D_{u^j}\ dist(u, \mathcal{N}) \right| \leq C\, |q|^2. \tag{6.15}$$

Proof For any $u \in \mathcal{O}_{2\delta_{\mathcal{N}}}$ such that $u \notin \mathcal{N}$, we make parallel transformation with the direction $\left(u - \pi_{\mathcal{N}}(u)\right) / \left|u - \pi_{\mathcal{N}}(u)\right|$ and follow the following argument. Therefore, we treat the case that $u \in \mathcal{N}$ and thus, $\pi_{\mathcal{N}}(u) = u$. For any $v \in \mathcal{O}_{2\delta_{\mathcal{N}}}$ let us put

$$d(v) := \ \mathrm{dist}(v, \mathcal{N}); \quad \eta(v) = \frac{1}{2} d(v)^2.$$

We know that the squared distance function $\eta(v)$ is smooth on $v \in \mathcal{O}_{2\delta_{\mathcal{N}}}$. Let $q \in \mathbb{R}^l$ be any vector in $\mathbb{R}^l$ and be fixed. Under the orthogonal decomposition of $\mathbb{R}^l$ with respect to the tangent space $\mathcal{T}_u\mathcal{N}$ at $u \in \mathcal{N}$, $\mathbb{R}^l = \mathcal{T}_u\mathcal{N} \oplus (\mathcal{T}_u\mathcal{N})^\perp$, we set as

$$q = q_\tau + q_\nu; \quad p := \frac{q_\nu}{|q_\nu|}; \quad f(t) := d(u+tp) \quad \text{for any } t \in (0,\ 2\delta_{\mathcal{N}}], \tag{6.16}$$

where p is the unit normal vector along the normal component of q and $f(t)$ is the distance to $\mathcal{N}$ measured along p. Then, we compute as

$$D_v d(u+tp) := D_v d(v)|_{v=u+tp} = p; \quad \frac{df}{dt}(t) = p \cdot D_u d(u+tp) = |p|^2 = 1 \tag{6.17}$$

and, also, for any $v \in \mathcal{O}_{2\delta_{\mathcal{N}}}$, $v \notin \mathcal{N}$,

$$\begin{aligned} &D_v\eta(v) = d(v)D_u d(v); \quad D_{v^i}D_{v^j}\eta(v) = d(v)D_{v^i}D_{v^j}d(v) + D_{v^i}d(v)D_{v^j}d(v) \\ &\iff D_{v^i}D_{v^j}d(v) = \frac{D_{v^i}D_{v^j}\eta(v) - D_{v^i}d(v)D_{v^j}d(v)}{d(v)}. \end{aligned} \tag{6.18}$$

Thus, letting, for any $t \in (0,\ 2\delta_{\mathcal{N}}]$,

$$D_{v^i}D_{v^j}d(u+tp) := D_{v^i}D_{v^j}d(v)\big|_{v=u+tp}; \quad D_{v^i}D_{v^j}\eta(u+tp) := D_{v^i}D_{v^j}\eta(v)\big|_{v=u+tp},$$

we have

$$q^i D_{v^i}D_{v^j}d(u+tp)q^j = \frac{q^i D_{v^i}D_{v^j}\eta(u+tp)q^j - \left|q \cdot D_v d(u+tp)\right|^2}{d(u+tp)},$$

where by (6.17) we have, as $t \searrow 0$,

$$\begin{aligned} &q^i D_{v^i}D_{v^j}\eta(u+tp)q^j \to q^i D_{v^i}D_{v^j}\eta(u)q^j = \left|q_\nu\right|^2; \\ &\left|q \cdot D_v d(u+tp)\right|^2 = (q \cdot p)^2 = \left|q_\nu\right|^2 \to \left|q_\nu\right|^2; \quad d(u+tp) \to 0 \end{aligned}$$

and the 1st convergence is valid because $D_{v^i}D_{v^j}\eta(u) = D_{v^i}D_{v^j}\eta(v)\big|_{v=u}$ is the orthogonal projection on $(\mathcal{T}_u\mathcal{N})^\perp$ (see [2, Theorem 3.1, p.704]). Therefore, from l'Hospital's theorem, we obtain

$$\begin{aligned} q^i D_{v^i}D_{v^j}d(u)q^j &:= \exists \lim_{t\searrow 0}\left(q^i D_{v^i}D_{v^j}d(u+tp)q^j\right) \\ &= \lim_{t\searrow 0}\frac{q^i q^j D_{v^i}D_{v^j}D_{v^k}\eta(u+tp)p^k - 2q \cdot D_v(p \cdot D_v d(u+tp))\, q \cdot D_v d(u+tp)}{p \cdot D_v d(u+tp)} \\ &= q^i q^j D_{v^i}D_{v^j}D_{v^k}\eta(u)p^k, \end{aligned}$$

where we use that $p \cdot D_v d(u+tp) = \left|p\right|^2 = 1$ and $D_v(p \cdot D_u d(u+tp)) = 0$. Thus, we have

$$\left|q^i D_{v^i}D_{v^j}d(u)q^j\right| = \left|q^i q^j D_{v^i}D_{v^j}D_{v^k}\eta(u)p^k\right| \le C\left|q\right|^2,$$

where in the last inequality the positive constant C depends only on a bound of curvatures of $\mathcal{N}$ (see [2, Remark 3.3; Theorem 3.5, its proof, pp. 707–709]). □

Appendix D Proof of the gradient boundedness.

Here we demonstrate the proof of Lemma 10, relying on Moser's iteration method as usual. Such estimate has been originally done for the evolutionary p-Laplacian system with controllable growth lower-order terms, by DiBenedetto, developing the intrinsic scaling transformation to the evolutionary p-Laplace operator (refer to [10,12]).

However, the emphasis here is to make localization by use of the cut-off function $\mathcal{C}$.

Proof of Lemma 10. In the following we use the same notation as in Lemma 10.

By use of a scaling transformation intrinsic to the evolutionary p-Laplace operator

$$t = t_0 + L^{2-p}(r_0)^2 s; \quad x = x_0 + r_0 y, \tag{6.19}$$

we now rewrite (3.4) in Lemma 9 by the scaled solution v on $Q(1,\ 1) := Q(1,\ 1)(0)$,

$$v(s,\ y) = \frac{u\left(t_0 + L^{2-p}(r_0)^2 s,\ x_0 + r_0 y\right)}{L\, r_0}$$

satisfying in $Q(1,\ 1)$

$$\begin{aligned} &\partial_s v - \frac{1}{\sqrt{|g|}} D_\alpha \left(\left(L^{-2}\epsilon + |Dv|^2 \right)^{\frac{p-2}{2}} \sqrt{|g|}\, g^{\alpha\beta} D_\beta v \right) \\ &= -C_0 \frac{K/L^p}{2} D_v \chi \left(\operatorname{dist}^2\left(L r_0 v,\ \mathcal{N} \right) \right). \end{aligned} \tag{6.20}$$

We put the notation

$$\begin{aligned} &\bar{\epsilon} = L^{-2}\epsilon; \quad \bar{K} = L^{-p} K; \\ &f = f(v) := \frac{1}{p}\left(\bar{\epsilon} + |Dv|^2\right)^{\frac{p}{2}}; \quad g = g(v) := \frac{\bar{K}}{2}\chi\left(\operatorname{dist}^2(L r v,\ \mathcal{N})\right); \\ &e(v) = f(v) + g(v); \quad \mathcal{B}^{\alpha\beta} = g^{\alpha\beta} + \frac{(p-2)\, g^{\alpha\gamma}\, g^{\beta\mu} D_\gamma v \cdot D_\mu v}{\bar{\epsilon} + |Dv|^2}. \end{aligned}$$

As in "Appendix C", we assume that the metric $(g_{\alpha\beta})$ is the identity matrix and, in the general case in $\mathcal{M}$, the terms with the derivative of $g_{\alpha\beta}(x_0 + ry)$ appear and are bounded by $C\, f(v)^{2(1-\frac{1}{p})}$, as in (6.1) and (6.2), where we note that $D_y g^{\alpha\beta}(x_0 + ry) = r D g^{\alpha\beta}(x)$ and $r \le 1$. We proceed to the same computation as (6.6) and (6.7), where the quantities appeared are transformed to the corresponding ones, respectively, defined by the scaled solution v as above. Now we will look at the transformed estimation for the scaled solution v.

By the support of χ'', we have

$$\operatorname{dist}^2(L r_0 v,\ \mathcal{N})\left|\chi''\right| \le 100 \sup\left|\chi'\right|\, \chi$$

and thus, the corresponding terms for the scaled solution v to 2nd terms in the 2nd line of (6.6) and the 3rd line of (6.7), and the 3rd term in the 3rd line of (6.8) are estimated above by

$$\begin{aligned} &2\,\bar{K}\,(p\,f)^{1-\frac{2}{p}}\ \operatorname{dist}^2(L r_0 v,\ \mathcal{N})\ \left|\chi''\right| \times \\ &\quad \times \left((C_0 + 1)\left| D \operatorname{dist}(L r_0 v,\ \mathcal{N})\right|^2 + \mathcal{B}^{\alpha\beta} D_\alpha \operatorname{dist}(L r_0 v,\ \mathcal{N}) D_\beta \operatorname{dist}(L r_0 v,\ \mathcal{N}) \right) \\ &\quad \le C\ (1 + C_0)\ (p\,f)^{1-\frac{2}{p}}\, g \end{aligned} \tag{6.21}$$

with a positive constant C depending only on p and χ, because, by the definition of r_0,

$$D_v \operatorname{dist}(Lr_0 v, \mathcal{N}) = Lr_0\ D_u \operatorname{dist}(u, \mathcal{N})|_{u=Lr_0 v}\,; \quad |D_v \operatorname{dist}(Lr_0 v, \mathcal{N})| = Lr_0;$$
$$|D \operatorname{dist}(Lr_0 v, \mathcal{N})| \le Lr_0\,|Dv| \le Lr_0\,\frac{1}{L}\|Du\|_{L^\infty(Q_0)} \le C.$$

By Schwarz's and Cauchy's inequality, the terms in the 3rd line of (6.6) and in the 4th line of (6.7), (6.8) are estimated as

$$\begin{aligned} &C(\mathcal{N})\ (C_0+p-1)\ L^2(r_0)^2\,(p\,f)^{1-\frac{2}{p}}|Dv|^2\,\bar{K}\,\chi'\ \operatorname{dist}(Lr_0 v, \mathcal{N}) \\ &\le C(\mathcal{N})\,C\ (C_0+p-1)\ (p\,f)^{1-\frac{2}{p}}|Dv|\ |D_v g| \\ &\le \left(\frac{C_0}{2}+1\right)|D_v g|^2 + (C(\mathcal{N})C)^2\left(\frac{C_0}{2}+\frac{(p-1)^2}{4}\right)(p\,f)^{2-\frac{4}{p}}|Dv|^2, \end{aligned} \tag{6.22}$$

where by the definition of r_0 as before, and a positive constant $C(\mathcal{N})$ depending on a bound for the curvature of $\mathcal{N}$, we compute as

$$\begin{aligned} &Lr_0\,|Dv| \le Lr_0\,\frac{1}{L}\|Du\|_{L^\infty(Q_0)} \le C; \\ &D_v g = Lr_0\,\bar{K}\,\chi'\ \operatorname{dist}(Lr_0 v, \mathcal{N})\ D_u \operatorname{dist}(u, \mathcal{N})|_{u=Lr_0 v}\,; \quad \left|D_u \operatorname{dist}(u, \mathcal{N})|_{u=Lr_0 v}\right| = 1; \\ &D_{v^i} D_{v^j} \operatorname{dist}(Lr_0 v, \mathcal{N}) = L^2(r_0)^2\ \ D_{u^i} D_{u^j} \operatorname{dist}(u, \mathcal{N})\big|_{u=Lr_0 v}\,; \\ &\left|\left(2\delta^{\alpha\beta}+\mathcal{B}^{\alpha\beta}\right)D_\alpha v^i D_\beta v^j\ \ D_{u^i} D_{u^j} \operatorname{dist}(u, \mathcal{N})\big|_{u=Lr_0 v}\right| \le C(\mathcal{N})|Dv|^2. \end{aligned}$$

The terms in 5th line of (6.7) are bounded by

$$2(p-2)\,(p\,f)^{1-\frac{2}{p}}\left|D^2 v\right||D_v g| \le \frac{1}{2}(p\,f)^{2-\frac{4}{p}}\left|D^2 v\right|^2 + 2(p-2)^2\ |D_v g|^2. \tag{6.23}$$

Gathering all of the estimations above yields

$$\begin{aligned} &\partial_t e(v) - \sum_{\alpha,\,\beta=1}^{m} D_\alpha\left((p\,f)^{1-\frac{2}{p}}\mathcal{B}^{\alpha\beta}D_\beta e(v)\right) + C_1\,(p\,f)^{2-\frac{4}{p}}\left|D^2 v\right|^2 \\ &\le C'(\mathcal{N})\left(\frac{C_0}{2}+\frac{(p-1)^2}{4}\right)(p\,f)^{2-\frac{4}{p}}|Dv|^2 + C\ (1+C_0)\,(p\,f)^{1-\frac{2}{p}}\,g. \end{aligned} \tag{6.24}$$

Finally we make Moser's iteration estimate by (6.24) and scaling back to have (3.6). Now, taking care of localization by the cut off function $\mathcal{C}$, we proceed to the estimations.

Let $B(\rho) = B(\rho, 0)$ be a ball in $\mathbb{R}^m$ with radius $\rho \le \min\{1, R_{\mathcal{M}}/2, T^{1/\lambda_0}\}$ and center of origin. Let $0 < r < \rho$. We use local parabolic cyllinders $Q(r^2, r) = (-r^2, 0) \times B(r)$ and $Q(\rho^2, \rho) = (-\rho^2, 0) \times B(\rho)$, Let η be a smooth real-valued function on $\mathbb{R}^m$ such that $0 \le \eta \le 1$, the support of η is contained in $B(\rho)$ and $\eta = 1$ on $B(r)$. Let $\sigma = \sigma(t)$ be a smooth real-valued function on $\mathbb{R}$ such that $0 \le \sigma \le 1$, $\sigma = 1$ on $[-r^2, \infty)$ and $\sigma = 0$ on $(-\infty, -\rho^2]$. We denote by the original notation the scaled function under (6.19). Put

$$\mathcal{C}(s,\ y) = \left((t_0 + L^{2-p}\,(r_0)^2\,s + R^{\lambda_0})^{1/\lambda_0} - |x_0 + r_0\,y|\right)_+, \quad (s,\ y) \in Q(1, 1),$$

and also write as $z = (s,\ y) \in Q(1, 1)$ and $dz = d\mathcal{M}ds$.

Put $w = e(v)$ in the Bochner type estimate (6.24). Let α be nonnegative number and use the test function $w^\alpha \eta^2 \sigma\,\mathcal{C}^q\sqrt{|g|}$ in the weak form of (6.24). After a routine computation we have the so-called reverse Poincaré inequality

$$\sup_{-r^2<\tau<0} \int\limits_{\{\tau\}\times B(\rho)} w^{\alpha+1}\eta^2\sigma\,\mathcal{C}^q\,d\mathcal{M} + \int\limits_{Q(\rho^2,\,\rho)} \left|Dw^{\frac{\alpha}{2}+1-\frac{1}{p}}\right|^2 \eta^2\sigma\,\mathcal{C}^q\,dz$$
$$\leq C\,(\alpha+p)^3 \int\limits_{Q(\rho^2,\,\rho)} \left\{w^{\alpha+1}\eta^2\,\partial_t\sigma + w^{\alpha+2-\frac{2}{p}}\left(\eta^2+|D\eta|^2\right)\sigma\right\}\mathcal{C}^q\,dz, \quad (6.25)$$

where we compute as

$$|Dw| \leq |Dv|\,\left((p\,f)^{1-\frac{2}{p}}\,\left|D^2v\right| + \left|D_v g\right|\right);$$
$$\left|Dw^{\frac{\alpha}{2}+1-\frac{1}{p}}\right|^2 \leq C\,(\alpha+p)^2\,w^\alpha\,\left((p\,f)^{2-\frac{4}{p}}\left|D^2v\right|^2 + \left|D_v g\right|^2\right).$$

Applying the Sobolev embedding $W_0^{1,2}(B(\rho)) \to L^{2m/(m-2)}(B(\rho))$ we have

$$\left(\int_{B(\rho)}\left(w^{\frac{\alpha}{2}+1-\frac{1}{p}}\,\eta\,\mathcal{C}^{\frac{q}{2}}\right)^{\frac{2m}{m-2}}\,d\mathcal{M}\right)^{\frac{m-2}{2m}} \leq C\,\left(\int_{B(\rho)}\left|D\left(w^{\frac{\alpha}{2}+1-\frac{1}{p}}\,\eta\,\mathcal{C}^{\frac{q}{2}}\right)\right|^2\,d\mathcal{M}\right)^{\frac{1}{2}},$$

which is combined with (6.25) and yields

$$\sup_{-r^2<\tau<0} \int\limits_{\{\tau\}\times B(r)} (w(\tau))^{\alpha+1}\,\mathcal{C}^q\,d\mathcal{M} + \int\limits_{-r^2}^{0}\left(\int\limits_{B(r)} w^{\frac{2m}{m+2}\,\frac{\alpha}{2}+1-\frac{1}{p}}\,\mathcal{C}^{\frac{mq}{m-2}}\,d\mathcal{M}\right)^{\frac{m-2}{m}}\,dt$$
$$\leq \frac{C\,(\alpha+p)^3}{(\rho-r)^2}\int\limits_{Q(\rho^2,\,\rho)}\left(w^{\alpha+1}+w^{\alpha+2-\frac{2}{p}}\right)\mathcal{C}^q\,dz. \quad (6.26)$$

By Hölder's inequality and (6.26) we compute as

$$\int\limits_{Q(r^2,\,r)} w^{\alpha+2-\frac{2}{p}+\frac{2(\alpha+1)}{m}}\,\mathcal{C}^{\frac{q(m+2)}{m}}\,dz$$
$$\leq \int\limits_{-r^2}^{0}\left(\int\limits_{B(r)} w^{\alpha+1}\,\mathcal{C}^q\,d\mathcal{M}\right)^{\frac{2}{m}}\left(\int\limits_{B(r)} w^{\frac{m(\alpha+2-2\,p^{-1})}{m-2}}\,\mathcal{C}^{\frac{qm}{m-2}}\,d\mathcal{M}\right)^{\frac{m-2}{m}}\,dt$$
$$\leq \left(\sup_{-r^2<\tau<0}\int\limits_{B(r)} w(\tau)^{\alpha+1}\,\mathcal{C}^q\,d\mathcal{M}\right)^{\frac{2}{m}}\int\limits_{-r^2}^{0}\left(\int\limits_{B(r)} w^{\frac{m(\alpha+2-2\,p^{-1})}{m-2}}\,\mathcal{C}^{\frac{qm}{m-2}}\,d\mathcal{M}\right)^{\frac{m-2}{m}}\,dt$$
$$\leq \left(\frac{C\,(\alpha+p)^3}{(\rho-r)^2}\int_{Q(\rho^2,\,\rho)} w^{\alpha+2-\frac{2}{p}}\,\mathcal{C}^q\,dz + \frac{C\,(\alpha+p)^3\,|Q(\rho^2,\,\rho)|}{(\rho-r)^2}\right)^{\frac{m+2}{m}},$$

where we use a simple inequality valid for $\alpha \geq 0$

$$w^{\alpha+1} = \left(\chi_{\{w\geq 1\}} + \chi_{\{w<1\}}\right)w^{\alpha+1}$$
$$\leq \chi_{\{w\geq 1\}}\,w^{\alpha+2-\frac{2}{p}} + \chi_{\{w<1\}}$$
$$\leq w^{\alpha+2-\frac{2}{p}} + 1$$

and also estimate the derivative of $\mathcal{C}$ as

$$
\begin{aligned}
|D\mathcal{C}(s,\ y)| &= \left| D\left((t_0 + L^{2-p}(r_0)^2\, s + R^{\lambda_0})^{1/\lambda_0} - |x_0 + r_0 y|\right)_+ \right| \\
&= \left| -\frac{x_0 + r_0 y}{|x_0 + r_1' y|}\, r_0 \right| \chi_{\{|x_0+r_0 y|<t_0;\, L^{2-p}(r_0)^2 s + R^{\lambda_0}\}}(s,\ y) \\
&\leq \left((t_0 + L^{2-p}(r_0)^2 s + R^{\lambda_0})^{1/\lambda_0} - |x_0 + r_0 y|\right)_+ \times \\
&\qquad \times \frac{r_0}{(t_0 + L^{2-p}(r_0)^2 s + R^{\lambda_0})^{1/\lambda_0} - |x_0 + r_0 y|} \\
&\leq \left((t_0 + L^{2-p}(r_0)^2 s + R^{\lambda_0})^{1/\lambda_0} - |x_0 + r_0 y|\right)_+ \\
&= \mathcal{C}(s,\ y); \\
|\partial_s \mathcal{C}(s,\ y)| &= \left| \partial_s \left((t_0 + L^{2-p}(r_0)^2 s + R^{\lambda_0})^{1/\lambda_0} - |x_0 + r_0 y|\right)_+ \right| \\
&= \left| \frac{1}{\lambda_0} (t_0 + L^{2-p}(r_0)^2 s + R^{\lambda_0})^{1/\lambda_0 - 1}\, L^{2-p}(r_0)^2 \right| \chi_{\{|x_0+r_0 y|<t_0+L^{2-p}(r_0)^2 s + R^{\lambda_0}\}}(s,\ y) \\
&\leq \frac{1}{\lambda_0} \chi_{\{|x_0+r_0 y|<t_0+L^{2-p}(r_0)^2 s + R^{\lambda_0}\}}(s,\ y)\ \left((\rho_0)^{\lambda_0}\right)^{1/\lambda_0 - 1}\ (\rho_0)^{\lambda_0} \\
&\leq \frac{\rho_0}{\rho_0/2} \frac{(\rho_0)^{\lambda_0}}{(\rho_0)^{\lambda_0}} \left((t_0 + L^{2-p}(r_0)^2 s + R^{\lambda_0})^{1/\lambda_0} - |x_0 + r_0 y|\right)_+ \\
&= 2\,\mathcal{C}(s,\ y),
\end{aligned}
$$

because, by the range $Q(1, 1)$ of $(s,\ y)$ and the condition (3.5) of r_0,

$$
\begin{aligned}
&-1 \leq s \leq 0; \quad |y| \leq 1; \\
&\rho_0 = \frac{(t_0 + R^{\lambda_0})^{1/\lambda_0} - |x_0|}{4}; \quad r_0 \leq \rho_0/2; \quad L^{2-p}(r_0)^2 \leq (\rho_0)^{\lambda_0}
\end{aligned}
$$

and so, we have the estimations

$$
\begin{aligned}
&r_0/(\rho_0/2) \leq 1; \\
&(t_0 + L^{2-p}(r_0)^2 s + R^{\lambda_0})^{1/\lambda_0} - |x_0 + r_0 y| \geq \left(t_0 + R^{\lambda_0} - (\rho_0)^{\lambda_0}\right)^{1/\lambda_0} - (|x_0| + r_0) \\
&\geq \frac{(t_0 + R^{\lambda_0})^{1/\lambda_0} + |x_0|}{2} - |x_0| - r_0 \geq \rho_0 - \frac{\rho_0}{2} = \frac{\rho_0}{2}.
\end{aligned}
$$

We arrange some terms in an appropriate way to have

$$
\begin{aligned}
&\frac{1}{|Q(r^2,\ r)|} \int_{Q(r^2, r)} w^{\alpha+2-\frac{2}{p}+\frac{2(\alpha+1)}{m}}\, \mathcal{C}^{\frac{q(m+2)}{m}}\, dz \\
&\leq \frac{C\,(\alpha+p)^{3(1+2/m)}\, |Q(\rho^2,\ \rho)|^{1+2/m}}{(\rho - r)^{2(1+2/m)}} \left(\frac{1}{|Q(\rho^2,\ \rho)|} \int_{Q(\rho^2, \rho)} w^{\alpha+2-\frac{2}{p}}\, \mathcal{C}^q\, dz + 1 \right)^{1+\frac{2}{m}}.
\end{aligned}
\tag{6.27}
$$

Here let $\{\rho_k\}$ be a sequence of radii, defined as

$$
\rho_k = 2^{-1}\left(1 + 2^{-k}\right); \quad 1 \geq \rho_k \searrow 1/2; \quad Q_k = Q((\rho_k)^2,\ \rho_k)(0) \tag{6.28}
$$

and $\{\alpha_k\}$ be a sequence of exponents

$$\theta = 1 + \frac{2}{m}; \quad q > 1; \quad q_k = q\,\theta^k; \quad 0 < q < q_k \nearrow \infty \quad ;$$
$$\alpha_k = \theta^k + 1 - \frac{2}{p}; \quad 2 - \frac{2}{p} =: \alpha_0 < \alpha_k \nearrow \infty; \quad \alpha_{k+1} = \alpha_k + \frac{2(\alpha_k - 1 + 2\,p^{-1})}{m}$$
$$= \alpha_k\,\theta - \frac{2(p-2)}{mp}. \tag{6.29}$$

We choose $r = \rho_{k+1}$, $\rho = \rho_k$ and $\alpha = \alpha_k$ in (6.27) and make routine computation to have

$$\frac{1}{|Q_{k+1}|}\int_{Q_{k+1}} w^{\alpha_{k+1}}\,\mathcal{C}^{q_{k+1}}\,dz \le C^k\,(\alpha_k + p)^{3\theta}\left(\frac{1}{|Q_k|}\int_{Q_k} w^{\alpha_k}\,\mathcal{C}^{q_k}\,dz + 1\right)^{\theta};$$
$$\frac{1}{|Q_{k+1}|}\int_{Q_{k+1}} w^{\alpha_{k+1}}\,\mathcal{C}^{q_{k+1}}\,dz + 1 \le 2\,C^k\,(\alpha_k + p)^{3\theta}\left(\frac{1}{|Q_k|}\int_{Q_k} w^{\alpha_k}\,\mathcal{C}^{q_k}\,dz + 1\right)^{\theta} \tag{6.30}$$

which is computed by sequences (6.28) and (6.29) as

$$\left(\frac{1}{|Q_{k+1}|}\int_{Q_{k+1}} w^{\alpha_{k+1}}\,\mathcal{C}^{q_{k+1}}\,dz + 1\right)^{\frac{1}{\theta^{k+1}}} \le C^{\frac{k}{\theta^k}}\left(\frac{1}{|Q_k|}\int_{Q_k} w^{\alpha_k}\,\mathcal{C}^{q_k}\,dz + 1\right)^{\frac{1}{\theta^k}}. \tag{6.31}$$

An iterative application of (6.31) yields, as $k \to \infty$,

$$\sup_{Q((\rho_0/2)^2,\,\rho_0/2)} w\,\mathcal{C}^{q_0} \longleftarrow \left(\frac{1}{|Q_{k+1}|}\int_{Q_{k+1}} w^{\alpha_{k+1}}\,\mathcal{C}^{q_0\alpha_{k+1}}\,dz\right)^{\frac{1}{\theta^{k+1}}}$$
$$\le \left(\frac{1}{|Q_{k+1}|}\int_{Q_{k+1}} w^{\alpha_{k+1}}\,\mathcal{C}^{q_{k+1}}\,dz + 1\right)^{\frac{1}{\theta^{k+1}}}$$
$$\le C^{\sum_{i=1}^{k}\frac{i}{\theta^i}}\left(\frac{1}{|Q_0|}\int_{Q_0} w^{\alpha_0}\,\mathcal{C}^{q_0}\,dz + 1\right)^{\frac{1}{\theta^0}}, \tag{6.32}$$

where we use the relation of exponents

$$q_{k+1} = q_0\,\theta^{k+1} < q_0\alpha_{k+1} \Longleftrightarrow \alpha_{k+1} = \theta^{k+1} + 1 - \frac{2}{p} > \theta^{k+1};$$
$$0 \le \mathcal{C}(s,\,y) \le 1, \quad (s,\,y) \in Q(1,1)(0)$$

and the limit as $k \to \infty$

$$\frac{\alpha_{k+1}}{\theta^{k+1}} = 1 + \frac{p-2}{p\,\theta^{k+1}} \to 1.$$

Finally, scaling back in (6.32) yields the desired estimate (3.6). □

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