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Calculus of Variations

New affine inequalities and projection mean ellipsoids

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Abstract
A variational formula for the Lutwak affine surface areas Λ_j of convex bodies in $\mathbb{R}^n$ is established when $1 \leq j \leq n-1$. By using introduced new ellipsoids associated with projection functions of convex bodies, we prove a sharp isoperimetric inequality for Λ_j, which opens up a new passage to attack the longstanding Lutwak conjecture in convex geometry.

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1 Introduction

Among all compact domains of given surface area S in the Euclidean space $\mathbb{R}^n$, the volume V_n of a domain D is maximized only by the ball. This isoperimetric property of ball is usually formulated as the following classical *isoperimetric inequality*

$$S(D)^n \geq n^n \omega_n V_n(D)^{n-1}, \tag{1.1}$$

with equality if and only if the compact domain D is a ball, where $\omega_n = \pi^{n/2}/\Gamma(1+\frac{n}{2})$ is the volume of unit ball B in $\mathbb{R}^n$. The literature on isoperimetric inequality, as well as its various generalizations and applications, is abundant. See, e.g., the excellent survey articles by Osserman [20] and Gardner [4].

Let K be a convex body in $\mathbb{R}^n$. Write $V_j(K|\xi)$ for the j-dimensional volume of projection of K onto a j-dimensional subspace $\xi \subseteq \mathbb{R}^n$, and call it the *jth projection function*. The

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important geometric quantities related to $V_j(K|\xi)$ are the *jth surface area*, defined by

$$S_j(K) = \frac{n\omega_n}{\omega_j} \int_{G_{n,j}} V_j(K|\xi)\, d\mu_j(\xi), \quad j = 1, 2, \ldots, n-1, n, \tag{1.2}$$

where the Grassmann manifold $G_{n,j}$ is endowed with the normalized Haar measure μ_j. The jth surface area is a generalization of the surface area and volume. Indeed, $\frac{1}{n}S_n(K)$ is the volume of K. Let $j = n-1$. We have the celebrated *Cauchy surface area formula*

$$S_{n-1}(K) = S(K) = \frac{1}{\omega_{n-1}} \int_{\mathbb{S}^{n-1}} V_{n-1}(K|u^\perp)\, d\mathcal{H}^{n-1}(u),$$

where $u^\perp$ denotes the $(n-1)-$dimensional subspace orthogonal to u, $\mathcal{H}^{n-1}$ denotes the Lebesgue measure on unit sphere $\mathbb{S}^{n-1}$. This formula says that the surface area of a convex body is, up to a factor depending only on n, the average volume of its shadows.

Note that in accordance with the conventional terminology in convex geometry, $\frac{1}{n}S_j(K)$ is precisely the so-called *$(n-j)$th quermassintegral* $W_{n-j}(K)$ of convex body K. Here, we prefer to calling it the jth surface area and denoting it by S_j because the integral in (1.2) shows the true nature of "surface area".

For jth surface area $S_j(K)$, there holds the extended isoperimetric inequality

$$S_j(K)^n \geq n^n \omega_n^{n-j} V_n(K)^j, \tag{1.3}$$

with equality if and only if K is a ball.

Without doubt, the Euclidean ball, uniquely characterized by isoperimetric inequalities, such as (1.1) and (1.3), is one of the most important geometric objects. However, to study isoperimetric features of other important geometric objects, such as ellipsoids, simplices and parallelotopes, a fruitful and natural approach is from *affine geometry*. First of all, we need to study geometric quantities which are affine invariant. As an aside, the jth surface area is *not* affine invariant. In some sense, to establish sharp affine isoperimetric inequalities, is a central problem in isoperimetric theory, as well as in affine geometry.

In 1970s, Petty [22] proved the following celebrated affine isoperimetric inequality, which is now known as the *Petty projection inequality*

$$V_n(\Pi^* K) V_n(K)^{n-1} \leq \left(\frac{\omega_n}{\omega_{n-1}}\right)^n, \tag{1.4}$$

with equality if and only if K is an ellipsoid. Here, ΠK is the *projection body* of a convex body K with its support function $h_{\Pi K}(u) = V_{n-1}(K|u^\perp)$, for $u \in \mathbb{S}^{n-1}$. $\Pi^* K$ denotes the polar body of ΠK. It is noted that by monotonicity of power means, the Petty projection inequality (1.4) is *far* stronger than the Euclidean isoperimetric inequality (1.1). The reverse form of (1.4) is known as the *Zhang projection inequality*, which was conjectured by Ball [1] and was first proved by Zhang [26].

Since

$$[V_n(\Pi^* K)]^{-\frac{1}{n}} = \left(\frac{1}{n}\int_{\mathbb{S}^{n-1}} V_{n-1}(K|u^\perp)^{-n}\, du\right)^{-\frac{1}{n}},$$

it indicates the functional $[V_n(\Pi^* K)]^{-\frac{1}{n}}$ has the true nature of "surface area". Later, analogous quantities were considered by Lutwak and Grinberg in the setting of convex bodies. In [11], Lutwak proposed to define *affine quermassintegrals* $\Phi_0(K)$, $\Phi_1(K)$, ..., $\Phi_n(K)$ for each convex body K in $\mathbb{R}^n$, by taking $\Phi_0(K) = V_n(K)$, $\Phi_n(K) = \omega_n$, and for $1 \leq j \leq n-1$, by

$$\Phi_j(K) = \frac{\omega_n}{\omega_{n-j}} \left(\int_{G_{n,n-j}} V_{n-j}(K|\xi)^{-n} \, d\mu_{n-j}(\xi) \right)^{-1/n}. \tag{1.5}$$

Grinberg [7] proved that these geometric quantities, as their names suggest, are invariant under volume-preserving affine transformations. Concerning the Lutwak *dual affine quermassintegral* and its related affine isoperimetric inequality extended to the bounded integrable functions, one can refer to the excellent article [2] by S. Dann, G. Paouris and P. Pivovarov.

In light of the integral in (1.5) has the character of surface area, we slightly modify these quantities $\Phi_j(K)$ and write them by

$$\Lambda_j(K) = n\Phi_{n-j}(K), \quad j = 0, 1, \ldots, n-1, n. \tag{1.6}$$

We call $\Lambda_j(K)$ the *jth integral affine surface area* of convex body K. Note that $\Lambda_{n-1}(K)$ is a constant multiple (depending only on n) of $[V_n(\Pi^* K)]^{-\frac{1}{n}}$. Thus, the Petty projection inequality (1.4) can be reformulated as the following

$$\Lambda_{n-1}(K)^n \geq n^n \omega_n V_n(K)^{n-1},$$

with equality if and only if K is an ellipsoid.

In contrast to the classical isoperimetric inequality for surface area functional and the Petty projection inequality for $(n-1)$th integral affine surface area, Lutwak [12] proposed the following insightful conjecture for general jth integral affine surface areas.

The Lutwak conjecture *Suppose K is a convex body in $\mathbb{R}^n$. Then*

$$\Lambda_j(K)^n \geq n^n \omega_n^{n-j} V_n(K)^j, \quad j = 1, 2, \ldots, n-1,$$

with equality if and only if K is an ellipsoid.

Unfortunately, the Lutwak conjecture has not made any essential progress during the last 3 decades. It has not even received the attention it deserves, because only two nontrivial cases follow from the classical results: when $j = n-1$, it is the above mentioned Petty projection inequality; when $j = 1$ and K is symmetric, it is exactly the celebrated Blaschke–Santaló inequality. In each case equality holds precisely when K is an ellipsoid. For $j = 2, 3, \ldots, n-1$, the Lutwak conjecture still remains open.

In this article, we focus on the Lutwak integral affine surface areas. In Sect. 2, a variational formula for the affine surface area $\Lambda_j(K)$ of convex body K in $\mathbb{R}^n$ is established when $j = 1, 2, \ldots, n-1$. From the established variational formula, we define a new measure, called *affine projection measure*, and show this measure is indeed affine invariant. In Sect. 3, we introduce a new ellipsoid $\mathrm{P}_j K$, which is associated with the jth projection function $V_j(K|\cdot)$ of convex body K, and call it the *jth projection mean ellipsoid* of K. It is with this projection mean ellipsoid that we prove the following main results in Sects. 4 and 5, respectively.

Theorem 1.1 *Suppose K is a convex body in $\mathbb{R}^n$. Then,*

$$\Lambda_j(K)^n \geq n^n \omega_n^{n-j} V_n(\mathrm{P}_j K)^j, \quad j = 1, 2, \ldots, n-1. \tag{1.7}$$

If $j = 2, 3, \ldots, n-1$, or $j = 1$ and K is centrally symmetric, the equality holds if and only if K is an ellipsoid. If $j = 1$, the equality holds if and only if K has an $\mathrm{SL}(n)$ *image with constant width.*

Theorem 1.2 *Suppose K is an origin-symmetric convex body in $\mathbb{R}^n$. Then,*

$$V_n(K^*)V_n(\mathrm{P}_1K) \leq \omega_n^2, \tag{1.8}$$

with equality if and only if K is an ellipsoid.

The sharp affine isoperimetric inequality (1.7) within Theorem 1.1, including its equality condition, can be viewed as a modified version of the Lutwak conjecture. In this new geometric inequality, as well as inequality (1.8), projection mean ellipsoid plays a crucial and indispensable role.

It is worth mentioning that *projection* and *intersection*, are two most fundamental geometric means to study structures of convex bodies in convex geometry. Meanwhile, *ellipsoid*, especially the classical John ellipsoid and its various generalizations, such as L_p John ellipsoids [16], mixed L_p John ellipsoids [9], Orlicz–John ellipsoids [28], Orlicz–Legendre ellipsoids [29], are both powerful to attack *reverse* isoperimetric problems and effective to establish *reverse* isoperimetric inequalities. See, e.g., [1,9,13,15,17,18,24,25,27,28], etc. In this article, for the first time we take into account these two important characters: *projection* and *ellipsoid*, and introduce a new ellipsoid by using projection function. It is wonderful that this new ellipsoid is tailor-made to do extremum problem for the Lutwak integral affine surface area, which opens up a entirely distinctive passage to tackle the longstanding Lutwak conjecture in convex geometry.

As for the ellipsoid associated with intersection function and its applications to affine isoperimetric problem, one can refer to [10]. In Sect. 6, we provide an example to compare the volumes of convex body itself and the projection mean ellipsoid.

2 A variational formula for the integral affine surface area

The setting for this paper is Euclidean n-dimensional space $\mathbb{R}^n$. As usual, write B and $\mathbb{S}^{n-1}$ for standard Euclidean unit ball and unit sphere in $\mathbb{R}^n$, respectively. Write $G_{n,j}$ for the Grassmannian manifold of all j-dimensional linear subspaces in $\mathbb{R}^n$. For $\xi \in G_{n,j}$, let $|\xi$ denote the orthogonal projection from $\mathbb{R}^n$ onto ξ.

2.1 Basics on convex bodies

Write $\mathcal{K}^n$ for the class of convex bodies in $\mathbb{R}^n$. A compact convex set K in $\mathbb{R}^n$ is uniquely determined by its *support function* $h_K : \mathbb{R}^n \to \mathbb{R}$, defined for $x \in \mathbb{R}^n$ by

$$h_K(x) = \max\{x \cdot y : y \in K\}. \tag{2.1}$$

It is clear that the support function is positively homogeneous with degree 1.

Suppose K is a convex body in $\mathbb{R}^n$ with the origin in its interior. Its *radial function* $\rho_K : \mathbb{S}^{n-1} \to (0, \infty)$ is defined for $u \in \mathbb{S}^{n-1}$ by $\rho_K(u) = \max\{\lambda > 0 : \lambda u \in K\}$. The *polar body* K^* of K is still a convex body with the origin in its interior, and $\rho_{K^*}(u) = h_K(u)^{-1}$.

For compact convex sets K and L, their *Hausdorff distance* is defined by

$$\delta(K, L) = \|h_K - h_L\|_\infty, \tag{2.2}$$

where $\|\cdot\|_\infty$ denotes the L_∞ norm on $\mathbb{S}^{n-1}$.

For compact convex sets K, L in $\mathbb{R}^n$, the volume of $K + \varepsilon L$, $\varepsilon \geq 0$, can be represented as the following Steiner–Minkowski polynomial

$$V_n(K + \varepsilon L) = \sum_{j=0}^{n} \binom{n}{j} V_{n,j}(K, L)\varepsilon^j, \tag{2.3}$$

where $V_{n,j}(K, L)$ is called the *jth mixed volume* of (K, L). Note that the notation $V_{n,j}(K, L)$ is slightly different from common use, but it is convenient for our purpose. When $L = B^n$, $nV_{n,j}(K, B) = S_j(K)$.

From (2.3), it follows that

$$V_{n,1}(K, L) = \lim_{\varepsilon \to 0^+} \frac{V_n(K + \varepsilon L) - V_n(K)}{n\varepsilon}. \tag{2.4}$$

If in addition K is a convex body, then there is the following integral representation

$$V_{n,1}(K, L) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_L(u)\, dS(K, u). \tag{2.5}$$

Here, $S(K, \cdot)$ denotes the surface area measure of K. For more information on surface area measure, see, e.g., Gardner [5], Gruber [8] and Schneider [23].

For $\xi \in G_{n,j}$ and $1 \leq j \leq n - 1$, write $V_{j,1}(K|\xi, L|\xi)$ for the first mixed volume of $(K|\xi, L|\xi)$ defined in the subspace ξ. It is convenient to use the normalization of $V_{j,1}(K|\xi, L|\xi)$. That is,

$$\bar{V}_{j,1}(K|\xi, L|\xi) = \frac{V_{j,1}(K|\xi, L|\xi)}{V_j(K|\xi)}. \tag{2.6}$$

2.2 Affine projection measures

Let $K \in \mathcal{K}^n$ and $1 \leq j \leq n - 1$. It is useful to introduce a Borel measure $\mu_j(K, \cdot)$ of convex body K, which is defined over $G_{n,j}$ and called the *jth affine projection measure* of K. $\mu_j(K, \cdot)$ is absolutely continuous to Haar measure μ_j with Radon–Nikodym derivative

$$\frac{d\mu_j(K, \xi)}{d\mu_j(\xi)} = V_j(K|\xi)^{-n}. \tag{2.7}$$

Obviously, $\mu_j(K + x, \cdot) = \mu_j(K, \cdot)$ for $x \in \mathbb{R}^n$, and $\mu_j(\alpha K, \cdot) = \alpha^{-nj}\mu_j(K, \cdot)$ for $\alpha > 0$.

Note that the total mass $\mu_j(K, G_{n,j})$ of $\mu_j(K, \cdot)$ and the jth integral affine surface area $\Lambda_j(K)$ have the equality

$$\mu_j(K, G_{n,j}) = \left(\frac{n\omega_n}{\omega_j \Lambda_j(K)}\right)^{-n}.$$

So, $\mu_j(K, \cdot)$ can be viewed as the differential of the jth integral affine surface area Λ_j.

For convenience, write $\bar{\mu}_j(K, \cdot)$ for the normalization of $\mu_j(K, \cdot)$, that is,

$$\bar{\mu}_j(K, \cdot) = \mu_j(K, \cdot)/\mu_j(K, G_{n,j}), \tag{2.8}$$

which will be appeared in the variational formula in Theorem 2.3.

The following theorem shows $\mu_j(K, \cdot)$ is indeed affine invariant.

Theorem 2.1 *Suppose $K \in \mathcal{K}^n$ and $1 \leq j \leq n - 1$. Then for $g \in \mathrm{SL}(n)$,*

$$d\mu_j(gK, \xi) = d\mu_j(K, g^T\xi), \quad \forall \xi \in G_{n,j}.$$

Proof Since g induces a linear transformation from ξ to $g\xi$, for any Lebesgue measurable $A \subset \xi$ with positive Lebesgue measure, the volume ratio $V_j(gA)/V_j(A)$ depends only on g (and is independent of the choice of A). Thus, it is reasonable to define

$$\sigma_j(g, \xi) = V_j(gA)/V_j(A). \tag{2.9}$$

Let $g\mu_j$ be the image measure of μ_j under the map $g : G_{n,j} \to G_{n,j}, \xi \mapsto g\xi$. Since the Grassmannian $G_{n,j}$ is of class C^∞, through local coordinates, its Riemannian volume element $d\mu_j(\xi)$ is always represented as the differential form $f(x_1, \ldots, x_l)dx_1 \cdots dx_l$, where f is of class C^∞ and $l = \dim(G_{n,j})$. So, $g\mu_j$ is absolutely continuous with respect to μ_j, with a positive Radon–Nikodym derivative everywhere.

Hence, we can take the Radon–Nikodym derivative, $\sigma_{G_{n,j}}(g, \xi)$, of $g^{-1}\mu_j$ with respect to μ_j. Then, $\sigma_{G_{n,j}}(g, \xi) = d\mu_j(g^{-1}\xi)/d\mu_j(\xi)$. Using the fact that $\sigma_{G_{n,j}}(g, \xi) = \sigma_j(g, \xi)^{-n}$, proved by Furstenberg and Tzkoni [3], we have

$$d\mu_j(g^{-1}\xi) = \sigma_j(g, \xi)^{-n} d\mu_j(\xi). \tag{2.10}$$

Recall that in [7], Grinberg proved the following identity

$$V_j((gK)|\xi) = \sigma_j(g^T, \xi)V_j(K|g^T\xi). \tag{2.11}$$

Now, from (2.7), (2.11), the fact $\xi = g^{-T}(g^T\xi)$, (2.10), (2.9) and finally (2.7) again, it follows that

$$\begin{aligned}
d\mu_j(gK, \xi) &= V_j((gK)|\xi)^{-n}\, d\mu_j(\xi)\\
&= V_j(K|g^T\xi)^{-n}\sigma_j(g^T, \xi)^{-n}\, d\mu_j(\xi)\\
&= V_j(K|g^T\xi)^{-n}\sigma_j(g^T, g^{-T}(g^T\xi))^{-n}\, d\mu_j(g^{-T}(g^T\xi))\\
&= V_j(K|g^T\xi)^{-n}\sigma_j(g^T, g^{-T}(g^T\xi))^{-n}\sigma_j(g^{-T}, g^T\xi)^{-n}\, d\mu_j(g^T\xi)\\
&= V_j(K|g^T\xi)^{-n}\sigma_j(g^T g^{-T}, g^T\xi)^{-n}\, d\mu_j(g^T\xi)\\
&= V_j(K|g^T\xi)^{-n} d\mu_j(g^T\xi)\\
&= d\mu_j(K, g^T\xi),
\end{aligned}$$

as desired. □

The following lemma shows the weak convergence of affine projection measure.

Lemma 2.2 *Suppose $K, K_i \in \mathcal{K}^n$, $i \in \mathbb{N}$ and $1 \le j \le n-1$. If $K_i \to K$ in the Hausdorff metric as $i \to \infty$, then $\mu_j(K_i, \cdot) \to \mu_j(K, \cdot)$ weakly.*

Proof Let f be a continuous function on $G_{n,j}$. We aim to prove the convergence

$$\int_{G_{n,j}} f(\xi)\, d\mu_j(K_i, \xi) \to \int_{G_{n,j}} f(\xi)\, d\mu_j(K, \xi).$$

For each $\xi \in G_{n,j}$, since $K_i \to K$, it follows that $K_i|\xi \to K|\xi$. Since the volume functional V_j is continuous in the Hausdorff metric, this implies that $V_j(K_i|\xi) \to V_j(K|\xi)$. So,

$$f(\xi)V_j(K_i|\xi)^{-n} \to f(\xi)V_j(K|\xi)^{-n}.$$

To make use of the Lebesgue dominated theorem to obtain the desired limit, we need to show

$$\max_{(i,\xi)\in\mathbb{N}\times G_{n,j}} |f(\xi)|V_j(K_i|\xi)^{-n} < \infty.$$

Since $G_{n,j}$ is compact, the continuity of f implies that $\max_{G_{n,j}} |f| < \infty$. So, it suffices to prove

$$0 < c_1 := \min_{(i,\xi)\in\mathbb{N}\times G_{n,j}} V_j(K_i|\xi).$$

In fact, by the convergence $K_i \to K$, it yields that there exists a constant $c_2 > 0$, a point $x \in K$ and an index $i_0 \in \mathbb{N}$, such that $c_2 B + x \subset \mathrm{int} K$ and $c_2 B + x \subseteq K_i$, for $i \geq i_0 + 1$. Note that

$$0 < \min\left\{\min_{\xi\in G_{n,j}} V_j(K_1|\xi), \ldots, \min_{\xi\in G_{n,j}} V_j(K_{i_0}|\xi), c_2^j \omega_j\right\} \leq c_1,$$

which completes the proof. □

2.3 Integral affine surface area

The starting point of this article is to calculate the first variational of Λ_j.

Theorem 2.3 *Suppose $K, L \in \mathcal{K}^n$ and $1 \leq j \leq n-1$. Then,*

$$\lim_{\varepsilon\to 0^+} \frac{\Lambda_j(K+\varepsilon L) - \Lambda_j(K)}{j\Lambda_j(K)\varepsilon} = \int_{G_{n,j}} \bar{V}_{j,1}(K|\xi, L|\xi)\, d\bar{\mu}_j(K,\xi)$$

Proof From compactness of convex bodies, there are positive constant numbers R_K and R_L such that $K \subseteq R_K B^n$ and $L \subseteq R_L B^n$. Let $0 < \varepsilon \leq \varepsilon_0 < \infty$ and $\xi \in G_{n,j}$. From monotonicity of mixed volumes with respect to set inclusion and homogeneity of mixed volumes, for $1 \leq l \leq j$, we have

$$\begin{aligned} V_{j,l}(K|\xi, L|\xi) &\leq V_{j,l}((R_K B^n)|\xi, (R_L B^n)|\xi) \\ &= {R_K}^{j-l} {R_L}^{l} V_{j,l}(B^n|\xi, B^n|\xi) \\ &= {R_K}^{j-l} {R_L}^{l} V_j(B^n|\xi) \\ &= {R_K}^{j-l} {R_L}^{l} \omega_j. \end{aligned}$$

By using Steiner–Minkowski (2.3) to $V_j((K|\xi) + \varepsilon L|\xi)$, it yields

$$\frac{V_j((K+\varepsilon L)|\xi) - V_j(K|\xi)}{\varepsilon} \leq c := \omega_j \sum_{l=1}^{j} \binom{l}{j} {R_K}^{j-l} {R_L}^{l} {\varepsilon_0}^{l-1}.$$

Observe that the constant c is positive and finite, and is independent of $\xi \in G_{n,j}$. Hence, the following family of positive integrable functions

$$\left\{\frac{V_j((K+\varepsilon L)|\cdot) - V_j(K|\cdot)}{\varepsilon} : 0 < \varepsilon \leq \varepsilon_0\right\}$$

is uniformly bounded on the Grassmannian $G_{n,j}$.

Moreover,

$$\begin{aligned} \frac{\left|V_j((K+\varepsilon L)|\cdot)^{-n} - V_j(K|\cdot)^{-n}\right|}{\varepsilon} &\leq \frac{n}{V_j(K|\cdot)^{n+1}} \cdot \frac{V_j((K+\varepsilon L)|\cdot) - V_j(K|\cdot)}{\varepsilon} \\ &\leq \frac{nc}{\min_{G_{n,j}} V_j(K|\cdot)^{n+1}} \\ &< \infty. \end{aligned}$$

Thus, the set

$$\left\{\frac{V_j((K+\varepsilon L)|\cdot)^{-n}-V_j(K|\cdot)^{-n}}{\varepsilon}:0<\varepsilon\leq\varepsilon_0\right\}$$

is also uniformly bounded on the Grassmannian $G_{n,j}$.

Meanwhile, by (2.4) and (2.6), for each ε, the function $\varepsilon^{-1}\left(V_j((K+\varepsilon L)|\cdot)^{-n}-V_j(K|\cdot)^{-n}\right)$ is μ_j-integrable on $G_{n,j}$, and for each $\xi\in G_{n,j}$, there holds the limit

$$\lim_{\varepsilon\to 0^+}\frac{V_j((K+\varepsilon L)|\xi)^{-n}-V_j(K|\xi)^{-n}}{\varepsilon}=-njV_j(K|\xi)^{-n}\bar{V}_{j,1}(K|\xi,L|\xi).$$

By the Lebesgue dominated theorem, the functional $V_j(K|\cdot)^{-n}\bar{V}_{j,1}(K|\cdot,L|\cdot)$ is integrable with respect to μ_j. From (2.7), we have

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0^+}\int_{G_{n,j}}V_j((K+\varepsilon L)|\xi)^{-n}\,d\mu_j(\xi)=-nj\int_{G_{n,j}}\bar{V}_{j,1}(K|\xi,L|\xi)\,d\mu_j(K,\xi).$$

This shows $\Lambda_j(K+\varepsilon L)^{-n}$ has right derivative at 0 with respect to ε. By direct calculations, we obtain the desired formula. □

For $K,L\in\mathcal{K}^n$ and $1\leq j\leq n-1$, the previous theorem suggests us to define the following geometric quantity

$$\bar{\Lambda}_j(K,L)=\int_{G_{n,j}}\bar{V}_{j,1}(K|\xi,L|\xi)\,d\bar{\mu}_j(K,\xi). \tag{2.12}$$

Then, $\bar{\Lambda}_j(K,K)=1$. If we set $\Lambda_n(K)=V_n(K)$, then $\bar{\Lambda}_n(K,L)=\bar{V}_{n,1}(K,L)$.

What follows provides some fundamental properties for $\bar{\Lambda}_j(K,L)$.

Lemma 2.4 *Suppose $K\in\mathcal{K}^n$ and $1\leq j\leq n-1$. Then the following claims hold.*

(1) $\Lambda_j(gK)=\Lambda_j(K)$, *for* $g\in\mathrm{SL}(n)$.
(2) $\Lambda_j(\alpha K)=\alpha^j\Lambda_j(K)$, *for* $\alpha>0$.
(3) $\Lambda_j(K+x)=\Lambda_j(K)$, *for* $x\in\mathbb{R}^n$.

Proof (1) was shown by Grinberg [7]. Also, it is an immediate consequence of Theorem 2.1. From the definition of Λ_j and the fact that $V_j((\lambda K+x)|\xi)=\lambda^jV_j(K|\xi)$, for $\lambda>0$ and $x\in\mathbb{R}^n$, (2) and (3) are obtained. □

Lemma 2.5 *Suppose $K,L\in\mathcal{K}^n$ and $1\leq j\leq n-1$. Then the following claims hold.*

(1) $\bar{\Lambda}_j(gK,L)=\bar{\Lambda}_j(K,g^{-1}L)$, *for* $g\in\mathrm{SL}(n)$.
(2) $\bar{\Lambda}_j(\alpha_1K,\alpha_2L)=\alpha_1^{-1}\alpha_2\bar{\Lambda}_j(K,L)$, *for* $\alpha_1,\alpha_2>0$.
(3) $\bar{\Lambda}_j(K+x,L+y)=\bar{\Lambda}_j(K,L)$, *for* $x,y\in\mathbb{R}^n$.

Proof From (2.12) together with Theorem 2.3, Lemma 2.4 (1), and Theorem 2.3 together with (2.12) again, we have

$$\begin{aligned}\bar{\Lambda}_j(gK,L)&=\lim_{\varepsilon\to 0^+}\frac{\Lambda_j(gK+\varepsilon L)-\Lambda_j(gK)}{j\Lambda_j(gK)\varepsilon}\\&=\lim_{\varepsilon\to 0^+}\frac{\Lambda_j(K+\varepsilon g^{-1}L)-\Lambda_j(K)}{j\Lambda_j(K)\varepsilon}\\&=\bar{\Lambda}_j(K,g^{-1}L),\end{aligned}$$

as desired. Combining (2.12) with Theorem 2.3 and Lemma 2.4, (2) and (3) can be obtained similarly. □

Lemma 2.6 *Suppose $K, L_1, L_2 \in \mathcal{K}^n$ and $1 \leq j \leq n-1$. If $L_1 \subseteq L_2$, then*

$$\bar{\Lambda}_j(K, L_1) \leq \bar{\Lambda}_j(K, L_2).$$

Proof Let $L_1 \subseteq L_2$. By the monotonicity of mixed volumes with respect to set inclusions, it implies that $V_{j,1}(K|\xi, L_1|\xi) \leq V_{j,1}(K|\xi, L_2|\xi)$, for any $\xi \in G_{n,j}$. From this fact together with the definition of $\bar{\Lambda}_j(K, \cdot)$, the desired inequality is obtained. □

From the definition of $\bar{\Lambda}_j(K, \cdot)$ together with the fact that for $\xi \in G_{n,j}$, we have

$$V_{j,1}(K|\xi, (L_1 + L_2)|\xi) = V_{j,1}(K|\xi, L_1|\xi) + V_{j,1}(K|\xi, L_2|\xi),$$

the following lemma is obtained.

Lemma 2.7 *Suppose $K, L_1, L_2 \in \mathcal{K}^n$ and $1 \leq j \leq n-1$. Then,*

$$\bar{\Lambda}_j(K, L_1 + L_2) = \bar{\Lambda}_j(K, L_1) + \bar{\Lambda}_j(K, L_2).$$

3 Projection mean ellipsoids

In this section, a new kind of ellipsoid operators P_j associated with projection functions, $j = 1, \ldots, n-1$, for convex bodies are introduced. It is remarkable that these ellipsoid operators are closely connected with the Lutwak conjecture. For $K \in \mathcal{K}^n$, these ellipsoids $\mathrm{P}_j K$ are well defined by solving an optimization problem.

Theorem 3.1 *Suppose K is a convex body in $\mathbb{R}^n$ and $j = 1, \ldots, n-1$. Among all origin-symmetric ellipsoids E, there exists a unique ellipsoid $\mathrm{P}_j K$ which solves the constrained maximization problem*

$$\max_E \; V_n(E) \quad \textit{subject to} \quad \bar{\Lambda}_j(K, E) \leq 1.$$

Proof Give an ellipsoid E, let d_E denote its maximal principal radius and u_E be the maximal principal direction. Write $[-d_E u_E, d_E u_E]$ for the line segment with endpoints $\pm d_E u_E$. Then, $[-d_E u_E, d_E u_E]|\xi \subset E|\xi$.

By compactness of convex body K, there exist finite positive numbers r, R and a point $x \in K$ such that $rB + x \subseteq K \subseteq RB$. From monotonicity of mixed volume with respect to set inclusion together with the fact $(rB + x)|\xi \subseteq K|\xi$, the homogeneity and translation invariance of mixed volume, and (2.5), it follows that for any $\xi \in G_{n,j}$,

$$\begin{aligned} V_{j,1}(K|\xi, E|\xi) &\geq V_{j,1}((rB^n + x)|\xi, [-d_E u_E, d_E u_E]|\xi) \\ &\geq r^{j-1} d_E V_{j,1}(B^n|\xi, [u_E, u_E]|\xi) \\ &= \frac{r^{j-1} d_E}{j} \int_{\xi \cap \mathbb{S}^{n-1}} |u_E \cdot v| d\mathcal{H}^{j-1}(v). \end{aligned}$$

Thus, from (2.12) together with (2.7) and (2.8), the fact that $r^j \omega_j \leq V_j(K|\xi) \leq R^j \omega_j$ for all $\xi \in G_{n,j}$, Fubini's theorem, and the fact that $\int_{\mathbb{S}^{n-1}} |u_E \cdot v| d\mathcal{H}^{n-1}(v) = 2V_{n-1}(B|u_E^{\perp})$, we have

$$
\begin{aligned}
\bar{\Lambda}_j(K,E) &= \frac{\int_{G_{n,j}} V_{j,1}(K|\xi, E|\xi) V_j(K|\xi)^{-(n+1)} d\mu_j(\xi)}{\int_{G_{n,j}} V_j(K|\xi)^{-n} d\mu_j(\xi)} \\
&\geq \left(\frac{r}{R}\right)^{j(n+1)} \frac{d_E}{r} \frac{1}{j\omega_j} \int_{G_{n,j}} \int_{\xi\cap\mathbb{S}^{n-1}} |u_E \cdot v| d\mathcal{H}^{j-1}(v) d\mu_j(\xi) \\
&= \left(\frac{r}{R}\right)^{j(n+1)} \frac{d_E}{r} \frac{1}{n\omega_n} \int_{\mathbb{S}^{-1}} |u_E \cdot v| d\mathcal{H}^{n-1}(v) \\
&= \left(\frac{r}{R}\right)^{j(n+1)} \frac{d_E}{r} \frac{2\omega_{n-1}}{n\omega_n}.
\end{aligned}
$$

Hence, an origin-symmetric ellipsoid E satisfying the constraint satisfies the condition

$$
d_E \leq \frac{n\omega_n}{2\omega_{n-1}} \left(\frac{R}{r}\right)^{j(n+1)} r < \infty.
$$

Consequently, any maximizing ellipsoid sequence $\{E_i\}_{i\in\mathbb{N}}$ for the extremum problem is bounded from above. By Blaschke selection theorem, there exists a convergent subsequence $\{E_{i_k}\}_{k\in\mathbb{N}}$ converging to an origin-symmetric ellipsoid E_0. It remains to prove that E_0 is not degenerated. Note that $0 < \bar{\Lambda}_j(K,B) < \infty$. Then, $\bar{\Lambda}_j\left(K, \frac{B}{\bar{\Lambda}_j(K,B)}\right) = 1$. This implies that the ball $\bar{\Lambda}_j(K,B)^{-1}B$ satisfies the constraint. Therefore,

$$
0 < \bar{\Lambda}_j(K,B)^{-n}\omega_n \leq V_n(E_0),
$$

which ensures $\dim(E_0) = n$.

Now, we show the uniqueness. Assume two positive definite symmetric transformations $g_1, g_2 \in \mathrm{GL}(n)$ are such that the ellipsoids $E_i = g_i B$, $i = 1, 2$, solve the maximization problem. We aim to prove that $g_1 = g_2$. From the definition of support function of ellipsoid and triangle inequality, we obtain that $\frac{g_1+g_2}{2} B \subseteq \frac{g_1 B + g_2 B}{2}$. So, from Lemmas 2.6, 2.7 and that $\bar{\Lambda}_j(K, E_i) \leq 1$ for $i = 1, 2$, it follows that

$$
\bar{\Lambda}_j\left(K, \frac{g_1+g_2}{2} B\right) \leq 1.
$$

This means that the ellipsoid $\frac{g_1+g_2}{2} B$ also satisfies the constraint of extremum problem. So, $V_n\left(\frac{g_1+g_2}{2} B\right) \leq V_n(g_1 B) = V_n(g_2 B)$. Consequently,

$$
\det\left(\frac{g_1+g_2}{2}\right)^{1/n} \leq \frac{\det(g_1)^{1/n} + \det(g_2)^{1/n}}{2}.
$$

On the other hand, the Minkowski inequality for positive definite matrices asserts that the reverse of the above inequality always holds. Thus, equality has to occur in the above inequality. By equality condition of the Minkowski inequality, it follows that $g_1 = \lambda g_2$ for some $\lambda > 0$. Since $\det(g_1) = \det(g_2)$, it follows that $g_1 = g_2$. □

Therefore, for $K \in \mathcal{K}^n$, by Theorem 3.1, a family of ellipsoids $\mathrm{P}_j K$, $j = 1, \ldots, n-1$, are produced. We call $\mathrm{P}_j K$ the *jth projection mean ellipsoid* of K.

Recall that for convex body $K \in \mathcal{K}^n$, the John ellipsoid $\mathrm{J}K$ is the unique ellipsoid of maximal volume contained in K. For each $\xi \in G_{n,j}$, $1 \leq j \leq n-1$, we have $\mathrm{J}K|\xi \subseteq K|\xi$. By Theorem 3.1, $V_n(\mathrm{P}_j K) \geq V_n(\mathrm{J}K)$.

In additional, $\bar{\Lambda}_n(K, L)$ is just the normalized mixed volume $\bar{V}_{n,1}(K, L)$. When $j = n$, it is interesting that the nth projection mean ellipsoid $\mathrm{P}_n K$ is precisely the classical Petty ellipsoid $\mathrm{P}K$. The volume-normalized Petty ellipsoid [21] is obtained by minimizing the surface area of K under $\mathrm{SL}(n)$ transformations of K. See also Giannopoulos [6].

From Theorem 3.1 and Lemma 2.5, we obtain the following result.

Lemma 3.2 *Suppose $K \in \mathcal{K}^n$ and $1 \le j \le n-1$. Then for any $g \in \mathrm{GL}(n)$ and $x \in \mathbb{R}^n$,*

$$\mathrm{P}_j(gK + x) = g\mathrm{P}_j K.$$

4 A new affine isoperimetric inequality for the integral affine surface area

For convex body K in $\mathbb{R}^n$, Lutwak [11] conjectured that

$$\Lambda_j(K)^n \ge n^n \omega_n^{n-j} V_n(K)^j, \quad j = 2, \ldots, n-2,$$

with equality if and only if K is an ellipsoid. In this section, we present a variant of the Lutwak conjecture.

Theorem 4.1 *Suppose K is a convex body in $\mathbb{R}^n$. Then,*

$$\Lambda_j(K)^n \ge n^n \omega_n^{n-j} V_n(\mathrm{P}_j K)^j, \quad j = 1, \ldots, n-1.$$

If $j = 2, 3, \ldots, n-1$, or $j = 1$ and K is centrally symmetric, the equality holds if and only if K is an ellipsoid. If $j = 1$, the equality holds if and only if K has an $\mathrm{SL}(n)$ image with constant width.

To prove this theorem, we need to prove several lemmas.

Lemma 4.2 *Suppose $K, L \in \mathcal{K}^n$. Then,*

$$\bar{\Lambda}_j(K, L) \ge \left(\frac{\Lambda_j(L)}{\Lambda_j(K)} \right)^{1/j}, \quad j = 1, \ldots, n-1. \tag{4.1}$$

If $j = 2, 3, \ldots, n-1$, the equality holds if and only if K and L are homothetic. If $j = 1$, the equality holds if and only if $w_K = \lambda w_L$ for some $\lambda > 0$. If $j = 1$ and K, L are centrally symmetric, the equality holds if and only if K and L are homothetic.

Proof By the Minkowski first inequality, for each $\xi \in G_{n,j}$, there holds

$$\bar{V}_{j,1}(K|\xi, L|\xi) \ge \left(\frac{V_j(L|\xi)}{V_j(K|\xi)} \right)^{1/j},$$

with equality if and only if $K|\xi$ and $L|\xi$ are homothetic. If $j = 1$, the equality always holds.

From (2.12), Minkowski's first inequality, the definition of $\mu_j(K,\cdot)$, Hölder's inequality, and finally the definition of Λ_j, it follows that

$$\begin{aligned}
\bar{\Lambda}_j(K,L) &= \int_{G_{n,j}} \bar{V}_{j,1}(K|\xi, L|\xi)\, d\mu_j(K,\xi) \\
&\geq \int_{G_{n,j}} \left(\frac{V_j(L|\xi)}{V_j(K|\xi)}\right)^{\frac{1}{j}} d\mu_j(K,\xi) \\
&= \frac{\int_{G_{n,j}} V_j(L|\xi)^{(-n)\cdot\frac{-1}{jn}} V_j(K|\xi)^{(-n)\cdot\frac{jn+1}{jn}}\, d\mu_j(\xi)}{\int_{G_{n,j}} V_j(K|\xi)^{-n}\, d\mu_j(\xi)} \\
&\geq \frac{\left(\int_{G_{n,j}} V_j(L|\xi)^{-n}\, d\mu_j(\xi)\right)^{\frac{-1}{jn}} \left(\int_{G_{n,j}} V_j(L|\xi)^{-n}\, d\mu_j(\xi)\right)^{\frac{jn+1}{jn}}}{\int_{G_{n,j}} V_j(K|\xi)^{-n}\, d\mu_j(\xi)} \\
&= \left(\frac{\Lambda_j(L)}{\Lambda_j(K)}\right)^{\frac{1}{j}},
\end{aligned}$$

which establishes inequality (4.1).

Assume the equality holds in (4.1). Then equalities in the second line and the fourth line both hold. If $2 \leq j \leq n-1$, by the equality condition of the Minkowski inequality, $K|\xi$ and $L|\xi$ are homothetic for all $\xi \in G_{n-1}$, and therefore K and L are homothetic (see, e.g., Theorem 3.1.3 in [5]). If $j=1$, the equality condition of the Holder inequality implies that $w_K = \lambda w_L$ for some constant $\lambda > 0$. If in addition K and L are centrally symmetric, then they are homothetic.

On the contrary, if K and L are homothetic, by Lemma 2.5 (1) and (2), it follows that the equality holds in (4.1). □

From Lemma 2.5 together with the definition of $\mathrm{P}_j K$, we obtain the following result.

Lemma 4.3 *Suppose $K \in \mathcal{K}^n$. Then,*

$$\bar{\Lambda}_j(K, \mathrm{P}_j K) = 1, \quad j = 1, \ldots, n-1.$$

Lemma 4.4 *Suppose E is an ellipsoid in $\mathbb{R}^n$. Then,*

$$\Lambda_j(E) = n\omega_n^{(n-j)/n} V_n(E)^{j/n}, \quad j = 1, 2, \ldots, n-1.$$

Proof From the jth positive homogeneity of Λ_j, the $\mathrm{SL}(n)$ invariance of $\Lambda_j(K)$, and the fact $\Lambda_j(B) = n\omega_n$, it follows that

$$\begin{aligned}
\Lambda_j(E) &= \Lambda_j\left(V_n(E)^{1/n}\frac{E}{V_n(E)^{1/n}}\right) \\
&= V_n(E)^{j/n}\Lambda_j\left(\frac{E}{V_n(E)^{1/n}}\right) \\
&= V_n(E)^{j/n}\Lambda_j\left(\frac{B}{\omega_n^{1/n}}\right) \\
&= V_n(E)^{j/n}\omega_n^{-j/n}\Lambda_j(B) \\
&= n\omega_n^{(n-j)/n} V_n(E)^{j/n},
\end{aligned}$$

as desired. □

Lemma 4.5 *Suppose E is an ellipsoid with center c_E. Then,*

$$\mathrm{P}_j E = E - c_E, \quad j = 1, 2, \ldots, n-1, n.$$

Proof By Lemma 3.2, it suffices to prove $\mathrm{P}_j B = B$. From Lemmas 4.3, 4.2 and 4.4, it follows that

$$1 = \bar{\Lambda}_j(B, \mathrm{P}_j B) \geq \left(\frac{\Lambda_j(\mathrm{P}_j B)}{\Lambda_j(B)}\right)^{1/j} = \left(\frac{V_n(\mathrm{P}_j B)}{V_n(B)}\right)^{1/n}.$$

So, $V_n(\mathrm{P}_j B) \geq V_n(B)$.

On the other hand, since $\bar{\Lambda}_j(B, B) = 1$, i.e., unit ball B satisfies the constraint of the extremum problem in Theorem 3.1 for (B, j), it follows that $V_n(\mathrm{P}_j B) \leq V_n(B)$.

Thus, $V_n(\mathrm{P}_j B) = V_n(B)$. By uniqueness of projection mean ellipsoid, $\mathrm{P}_j B = B$ is obtained. □

Lemma 4.6 *Suppose $K \in \mathcal{K}^n$. Then,*

$$\Lambda_j(K) \geq \Lambda_j(\mathrm{P}_j K), \quad j = 1, 2, \ldots, n-1. \tag{4.2}$$

If $j = 2, 3, \ldots, n-1$, or $j = 1$ and K is centrally symmetric, the equality holds if and only if K is an ellipsoid. If $j = 1$, the equality holds if and only if K has an SL(n) *image with constant width.*

Proof From Lemmas 4.2 and 4.3, it follows that for $j = 1, 2, \ldots, n-1$,

$$1 = \bar{\Lambda}_j(K, \mathrm{P}_j K) \geq \left(\frac{\Lambda_j(\mathrm{P}_j K)}{\Lambda_j(K)}\right)^{\frac{1}{j}}.$$

That is, $\Lambda_j(K) \geq \Lambda_j(\mathrm{P}_j K)$.

Assume the equality holds. By Lemma 4.6, if $j = 2, 3, \ldots, n-1$, then the bodies $\mathrm{P}_j K$ and K are homothetic. Therefore, K is an ellipsoid. Let $j = 1$. Since $w_K = \alpha w_{\mathrm{P}_1 K}$ for some $\alpha > 0$, from the affine nature of $\mathrm{P}_1 K$, there exists an SL(n) transformation g such that $g\mathrm{P}_1 K$ is an origin-symmetric ball. Thus, $w_{gK} = \alpha w_{\mathrm{P}_1(gK)}$. That is, the body gK is of constant width. Moreover, if in addition K is centrally symmetric, then gK is a ball, and therefore, K is an ellipsoid.

Assume that K is an ellipsoid. By Lemma 4.5, $\mathrm{P}_j K = K - c_K$, here c_K is the center of K. By Lemma 2.4 (3), $\Lambda_j(\mathrm{P}_j K) = \Lambda_j(K)$. □

We are now in the position to finish the proof of Theorem 4.1.

Proof of Theorem 4.1 From Lemma 4.4, it follows that

$$\Lambda_j(\mathrm{P}_j K) = n\omega_n^{(n-j)/n} V_n(\mathrm{P}_j K)^{j/n}.$$

Combining this fact with Lemma 4.6, it follows that

$$\Lambda_j(K) \geq \Lambda_j(\mathrm{P}_j K) = n\omega_n^{(n-j)/n} V_n(\mathrm{P}_j K)^{j/n},$$

as desired. The equality condition are derived from Lemma 4.6 immediately. □

5 A sharp affine isoperimetric inequality for 1st projection mean ellipsoid

Lemma 5.1 *Suppose K and L are origin-symmetric convex bodies in $\mathbb{R}^n$ with the origin in their interior. Then,*

$$\bar{\Lambda}_1(K, L) = \frac{\int_{S^{n-1}} \rho_{L^*}(u)^{-1} \rho_{K^*}(u)^{n+1}\, d\mathcal{H}^{n-1}(u)}{n V_n(K^*)}. \tag{5.1}$$

Proof Since

$$\begin{aligned}
&\int_{G_{n,1}} V_{1,1}(K|\xi, L|\xi) V_1(K|\xi)^{-(n+1)}\, d\mu_1(\xi) \\
&= (n\omega_n)^{-1} \int_{\mathbb{S}^{n-1}} w_L(u) w_K(u)^{-(n+1)}\, d\mathcal{H}^{n-1}(u) \\
&= 2^{-n}(n\omega_n)^{-1} \int_{\mathbb{S}^{n-1}} h_L(u) h_K(u)^{-(n+1)}\, d\mathcal{H}^{n-1}(u) \\
&= 2^{-n}(n\omega_n)^{-1} \int_{\mathbb{S}^{n-1}} \rho_{L^*}(u)^{-1} \rho_{K^*}(u)^{n+1}\, d\mathcal{H}^{n-1}(u),
\end{aligned}$$

and

$$\int_{G_{n,1}} V_1(K|\xi)^{-n} d\mu_1(\xi) = 2^{-n}(n\omega_n)^{-1} \int_{\mathbb{S}^{n-1}} \rho_{K^*}(u)^n d\mathcal{H}^{n-1}(u).$$

By the definition of $\bar{\Lambda}_1(K, L)$, (5.1) is obtained. □

Theorem 5.2 *Suppose K is an origin-symmetric convex body in $\mathbb{R}^n$. Then,*

$$V_n(K^*) V_n(\mathrm{P}_1 K) \le \omega_n^2,$$

with equality if and only if K is an ellipsoid.

Proof From the Hölder inequality and the polar formula for volume, it yields that

$$n^{-1} \int_{S^{n-1}} \rho_{L^*}(u)^{-1} \rho_{K^*}(u)^{n+1} d\mathcal{H}^{n-1}(u) \ge V_n(K^*)^{(n+1)/n} V_n(L^*)^{-1/n},$$

with equality if and only if K^* and L^* are dilates. From Lemma 5.1, it follows that

$$\bar{\Lambda}_1(K, L) \ge \left(\frac{V_n(K^*)}{V_n(L^*)} \right)^{\frac{1}{n}},$$

with equality if and only if K and L are dilates.

Let $L = \mathrm{P}_1 K$. Using Lemma 4.3, we obtain

$$1 \ge \frac{V_n(K^*)}{V_n((\mathrm{P}_1 K)^*)},$$

with equality if and only if K is an ellipsoid. By the Blaschke–Santaló inequality, we have

$$\frac{{\omega_n}^2}{V_n((\mathrm{P}_1 K)^*)} = V_n(\mathrm{P}_1 K).$$

Thus,

$$\omega_n{}^2 \geq \frac{\omega_n{}^2 V_n(K^*)}{V_n((\mathrm{P}_1 K)^*)} = V_n(K^*)V_n(\mathrm{P}_1 K),$$

as desired. □

6 Which one is bigger, $V_n(K)$ or $V_n(\mathrm{P}_j K)$?

In light of the longstanding Lutwak conjecture, a natural question is posed as follows: For a convex body K in $\mathbb{R}^n$, which geometric quantity is bigger, $V_n(K)$ or $V_n(\mathrm{P}_j K)$?

Recall that when $j = n$, $\mathrm{P}_n K$ is just the classical Petty ellipsoid of K. It is known that $V_n(K) \geq V_n(\mathrm{P}_n K)$. As a result, one may tempt to conjecture that $V_n(K) \geq V_n(\mathrm{P}_j K)$, for $j = 1, 2, \ldots, n-1$?

In this section, we provide an example to show that it is *not* always true. So, projection mean ellipsoid not only owns strong geometric intuition, but also is of great value to attack the Lutwak conjecture for affine surface area.

Lemma 6.1 *Suppose B_p is the unit ball in $\mathbb{R}^n$ with l_p norm, $1 \leq p \leq \infty$. Then the mean projection ellipsoid $\mathrm{P}_j B_p$, $j = 1, 2, \ldots, n$, is an origin-symmetric Euclidean ball.*

Proof We argue by contradiction. Assume $\mathrm{P}_j B_p$ is not an Euclidean ball. We prove that there exists an orthogonal transformation g, such that

$$gB_p = B_p \quad \text{but} \quad g\mathrm{P}_j B_p \neq \mathrm{P}_j B_p.$$

However, this is impossible, since by Lemma 3.2 and $gB_p = B_p$, it necessarily yields that $g\mathrm{P}_j B_p = \mathrm{P}_j B_p$.

By the above assumption, among all principal radii of the ellipsoid $\mathrm{P}_j B_p$ there exists a principal radius, say λ_0, which differs from the others. Suppose $\pm u_0$ are the principal directions corresponding to λ_0. Say, $u_0 = (u_1^0, \ldots, u_n^0)$.

We first handle the case where $u_{i_0}^0 = 1$ or -1, for some index i_0. W.l.o.g., assume that $i_0 = 1$. Then, $u_i^0 = 0$ for $i \neq 1$. Take the orthogonal transformation $g : \mathbb{R}^n \to \mathbb{R}^n$,

$$(x_1, x_2, x_3, \ldots, x_n) \mapsto (x_2, x_1, x_3, \ldots, x_n).$$

Clearly, $gB_p = B_p$. Observe that the principal radii of $g\mathrm{P}_j B_p$ are identical to those of $\mathrm{P}_j B_p$, and $\pm gu_0$ are the unit principal directions corresponding to principal radius λ_0 of $g\mathrm{P}_j B_p$. The choice of g implies that $\{\pm gu_0\} \neq \{\pm u_0\}$. Moreover, $g\mathrm{P}_j B_p \neq \mathrm{P}_j B_p$, since if $g\mathrm{P}_j B_p = \mathrm{P}_j B_p$, then it yields that $\{\pm gu_0\} = \{\pm u_0\}$.

To complete the proof, it remains to consider the case where vector u_0 has two nonzero components, say $u_{i_1}^0$ and $u_{i_2}^0$. W.l.o.g., assume that $i_1 = 1$ and $i_2 = 2$. Take the orthogonal transformation $g : \mathbb{R}^n \to \mathbb{R}^n$,

$$(x_1, x_2, x_3, \ldots, x_n) \mapsto (-x_2, x_1, x_3, \ldots, x_n).$$

Clearly, $gB_p = B_p$. An argument similar to the above yields that $g\mathrm{P}_j B_p \neq \mathrm{P}_j B_p$. □

For convex body K with its centroid at the origin, its *isotropic constant* L_K is given by

$$L_K{}^2 = \frac{1}{n} \min \left\{ \frac{1}{V(gK)^{1+\frac{2}{n}}} \int_{gK} |x|^2 \, dx : g \in \mathrm{GL}(n) \right\}.$$

In particular, modulo orthogonal transformations, there is a unique $\mathrm{SL}(n)$ transformation g such that

$$\frac{1}{nV(gK)^{1+\frac{2}{n}}}\int_{gK}|x|^2dx = {L_K}^2,$$

i.e.,

$$\frac{1}{nV(gK)}\int_{\mathbb{S}^{n-1}}\rho_{gK}^{n+2}(u)\,d\mathcal{H}^{n-1}(u) = (n+2)V(gK)^{\frac{2}{n}}{L_K}^2.$$

If in addition g_K is orthogonal, then K is said to be *isotropic*. One of the main remained open problems in asymptotic theory of convex bodies is the *hyperplane conjecture*, which is equivalently asks whether there exists an absolute upper bound for isotropic constant. For more information, see, e.g., the classical paper by Milman and Pajor [19].

Recall that a known fact concerning B_p, $1 \le p \le \infty$, is that it is isotropic. Meanwhile, note that $B_p^* = B_{p^*}$, with p^* denoting the conjugate of p. Thus, we have

$$\frac{1}{nV(B_p^*)}\int_{\mathbb{S}^{n-1}}\rho_{B_p^*}^{n+2}(u)\,d\mathcal{H}^{n-1}(u) = (n+2)V(B_p^*)^{\frac{2}{n}}{L_{B_p^*}}^2. \tag{6.1}$$

Theorem 6.2 *Let* $1 \le p \le \infty$. *Then,*

$$\left(\frac{V_n(\mathrm{P}_1 B_p)}{V_n(B_p)}\right)^{1/n} \ge \frac{\omega_n^{-1/n}}{\sqrt{n+2}}L_{B_p^*}^{-1}, \tag{6.2}$$

with equality if and only if $p = 2$.

Proof By Lemma 6.1, $\mathrm{P}_1 B_p$ is an origin-symmetric Euclidean ball. Let r_p be its radius. From Lemmas 4.3 and 5.1, it follows that

$$\frac{r_p}{nV_n(B_p^*)}\int_{\mathbb{S}^{n-1}}\rho_{B_p^*}(u)^{n+1}\,d\mathcal{H}^{n-1}(u) = 1. \tag{6.3}$$

Then,

$$\frac{V_n(\mathrm{P}_1 B_p)}{V_n(B_p)} = \frac{n^n\omega_n V_n(B_p^*)^n}{V_n(B_p)\Big(\int_{S^{n-1}}\rho_{B_p^*}(u)^{n+1}\,d\mathcal{H}^{n-1}(u)\Big)^n}.$$

Meanwhile, from the Jensen inequality and (6.1), it follows that

$$\begin{aligned}\frac{1}{nV_n(B_p^*)}\int_{S^{n-1}}\rho_{B_p^*}(u)^{n+1}d\mathcal{H}^{n-1}(u) &\le \left(\frac{1}{nV_n(B_p^*)}\int_{S^{n-1}}\rho_{B_p^*}(u)^{n+2}d\mathcal{H}^{n-1}(u)\right)^{1/2}\\ &= (n+2)^{1/2}V_n(B_p^*)^{1/n}L_{B_p^*},\end{aligned}$$

with equality in the first line if and only if $p = 2$. Thus,

$$\frac{V_n(\mathrm{P}_1 B_p)}{V_n(B_p)} \ge \frac{\omega_n}{V_n(B_p)V_n(B_p^*)(n+2)^{n/2}{L_{B_p^*}}^n},$$

with equality if and only if $p = 2$. Finally, with the Blaschke–Santaló inequality, the desired inequality is obtained. □

An important fact goes back to Milman and Pajor [19] (also see LYZ [14]) states that for a convex body K with centroid at the origin,

$$L_K \geq \frac{\omega_n^{-1/n}}{\sqrt{n+2}}, \tag{6.4}$$

with equality if and only if K is an origin-symmetric ellipsoid.

Corollary 6.3 *Let* $1 \leq p \leq \infty$. *Then,*

$$\max_{1\leq p\leq\infty} \frac{V_n(\mathrm{P}_1 B_p)}{V_n(B_p)} \geq 1.$$

Proof Since B_p^* is continuous in $p \in [1, \infty]$, then r_p is also continuous in $p \in [1, \infty]$ by Eq. (6.3). From (6.2) and (6.4), it follows that

$$\max_{1\leq p\leq\infty} \frac{V_n(\mathrm{P}_1 B_p)}{V_n(B_p)} \geq \frac{1}{\min_{1\leq p\leq\infty} \omega_n (n+2)^{n/2} {L_{B_p^*}}^n} \geq 1,$$

as desired. □

Specifically, let $n = 2$ and $p = \infty$. We show that $V_2(\mathrm{P}_1 B_\infty) > V_2(B_\infty)$.

For this aim, we use the polar coordinate $\{(\rho, \theta) : 0 \leq \rho \leq \infty, 0 \leq \theta \leq 2\pi\}$. Since $\rho_{B_\infty^*}(\theta) = (|\cos\theta| + |\sin\theta|)^{-1}$, it yields that

$$\begin{aligned}
\frac{V_2(\mathrm{P}_1 B_\infty)}{V_2(B_\infty)} &= \frac{4\pi\, V_2(B_\infty^*)^2}{V_2(B_\infty)} \left(\int_0^{2\pi} \rho_{B_\infty^*}(\theta)^3 d\theta \right)^{-2} \\
&= \frac{\pi}{4} \left(\int_0^{\pi/2} (\cos\theta + \sin\theta)^{-3} d\theta \right)^{-2} \\
&= 2\pi \left(2 \int_{\pi/4}^{\pi/2} \sin^{-3}\theta d\theta \right)^{-2} \\
&= \frac{2\pi}{\left(\sqrt{2} + \log(\sqrt{2}+1)\right)^2} = 1.1923\cdots > 1.
\end{aligned}$$

Recall that $\mathrm{P}_1 B_p = r_p B$ is continuous in $p \in [1, \infty]$. So, there exists a $p_0 \in (2, \infty)$, so that for $p_0 < p \leq \infty$, $V_2(\mathrm{P}_1 B_p) > V_2(B_p)$.

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