Calculus of Variations

A universal thin film model for Ginzburg–Landau energy with dipolar interaction

Cyrill B. Muratov[1]

Abstract
We present an analytical treatment of a three-dimensional variational model of a system that exhibits a second-order phase transition in the presence of dipolar interactions. Within the framework of Ginzburg–Landau theory, we concentrate on the case in which the domain occupied by the sample has the shape of a flat thin film and obtain a reduced two-dimensional, non-local variational model that describes the energetics of the system in terms of the order parameter averages across the film thickness. Namely, we show that the reduced two-dimensional model is in a certain sense asymptotically equivalent to the original three-dimensional model for small film thicknesses. Using this asymptotic equivalence, we analyze two different thin film limits for the full three-dimensional model via the methods of Γ-convergence applied to the reduced two-dimensional model. In the first regime, in which the film thickness vanishes while all other parameters remain fixed, we recover the local two-dimensional Ginzburg–Landau model. On the other hand, when the film thickness vanishes while the sample's lateral dimensions diverge at the right rate, we show that the system exhibits a transition from homogeneous to spatially modulated global energy minimizers. We identify a sharp threshold for this transition.

Mathematics Subject Classification 35B36 · 35B40 · 49S05

1 Introduction

This paper is concerned with the behavior of ground states in systems exhibiting a second-order phase transition which gives rise to the emergence of dipolar order. A prototypical example may be found in strongly uniaxial ferromagnets, such as magnetic garnet films with perpendicular easy axis [5,17,23,26]. In such films, spontaneous magnetization appears below the Curie temperature due to ferromagnetic exchange, with the magnetic moments of

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the electrons aligning in the direction normal to the film plane. However, this local ordering is frustrated by the weak dipole-dipole coupling, which instead favors anti-parallel alignment of distant magnetic moments. Under appropriate conditions, this competition between the short-range attractive and long-range repulsive interactions is well known to produce various types of inhomogeneous spatial patterns of magnetization, often referred to as "modulated phases" [2,17,37]. Other physical systems with similar behavior include uniaxial ferroelectrics [23, 38], ferrofluids [36] and Langmuir layers [1,37].

Within the mean-field approximation, these types of systems are usually modeled by an appropriate free energy functional that contains non-local terms coming from the dipolar interaction. Spatially modulated phases are interpreted as either local or global minimizers of the respective energy functional. A phase diagram is then established by comparing the energies of the candidate "phases" and selecting those corresponding to the global minimum of the energy. Mathematically, this leads to a formidable variational problem, which has been well known to exhibit intricate dependence on the model parameters and geometry because of its non-convex and non-local character. In the context of micromagnetics, a whole zoo of different behaviors have been recently established (see, e.g., [6,7,10,20,22,34]; this list is certainly very far from exclusive).

The complexity of the problem may be somewhat reduced near a phase transition point, where the energy functional attains an asymptotically universal form coming from the Landau expansion (still within the mean-field approximation). This is the approach taken by [2,13,18, 29,30,35,37], which is also adopted by us here. We start by formulating the three-dimensional Ginzburg–Landau theory of a system undergoing a second-order phase transition, in which the order parameter is associated with dipolar ordering (for a recent review of the general Ginzburg–Landau formalism, see [16]; for a stochastic perspective, see also [8,28]). We then derive a reduced two-dimensional Ginzburg–Landau theory with a modified non-local term which becomes asymptotically equivalent to the full three-dimensional theory as the film thickness vanishes. This reduction is done in the spirit of Γ-development [4] and is the main result of the paper.

Consider a region $\Omega \subset \mathbb{R}^3$ occupied by the material and assume that this region is in the shape of a film of thickness $\delta > 0$, cross-section $D \subset \mathbb{R}^2$ and rounded edges (a pancake-shaped domain). Namely, we assume that[1]

$$D \times (0, \delta) \subset \Omega \subset (D + B_\delta) \times (0, \delta), \tag{1.1}$$

and both D and Ω have boundaries of class C^2. Note that we do not necessarily assume that D is connected. We are particularly interested in the case when δ is sufficiently small, corresponding to a *thin film* (how small the value of δ should be in order for a film to be considered as thin will be discussed later). Inside Ω, the state of the material is described by a scalar order parameter $\phi = \phi(\mathbf{r})$, where $\mathbf{r} = (x, y, z) \in \mathbb{R}^3$ stands for the spatial coordinate. The order parameter represents the magnitude of the magnetization or polarization vector in the z-direction. In the following, we extend ϕ by zero outside Ω. Then the Ginzburg–Landau free energy plus the dipolar interaction energy can be written in the following form [23]:

$$\frac{\mathcal{F}(\phi)}{k_B T_c} = \int_\Omega \left(\frac{g}{2} |\nabla \phi|^2 + \frac{a}{2}(T - T_c)\phi^2 + \frac{b}{4}\phi^4 - h\phi \right) d^3r + \frac{c}{2} \int_{\mathbb{R}^3} \partial_z \phi (-\Delta)^{-1} \partial_z \phi \, d^3r. \tag{1.2}$$

Here, k_B is the Boltzmann constant, T is temperature, T_c is the transition temperature in the absence of the dipolar interaction, $h = h(x, y)$ is the applied field normal to the film

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[1] Recall that $D + B_\delta = \{\mathbf{r} \in \mathbb{R}^2 : \text{dist}(\mathbf{r}, D) < \delta\}$.

plane, and a, b, c, g are positive material constants. Also, the symbol $(-\Delta)^{-1}$ stands for the convolution with the Newtonian potential $1/(4\pi|\mathbf{r}|)$ in three space dimensions, and the derivative $\partial_z \phi$ in $\mathbb{R}^3$ is understood distributionally.

When δ is small, the gradient term is expected to strongly penalize the variations of ϕ in the z-direction. Furthermore, it is easy to see that to the leading order the dipolar term should become local. Indeed, since for small δ we have $\Delta \approx \partial^2 z$ in a certain sense, the energy in (1.2) may be equivalently rewritten as

$$\frac{\mathcal{F}(\phi)}{k_B T_c} = \int_\Omega \left(\frac{g}{2}|\nabla\phi|^2 + \frac{a}{2}(T - T_c^*)\phi^2 + \frac{b}{4}\phi^4 - h\phi \right) d^3r + \frac{c}{2} \int_{\mathbb{R}^3} \left(\partial_z \phi (-\Delta)^{-1} \partial_z \phi - \phi^2 \right) d^3r, \tag{1.3}$$

where we introduced the renormalized critical temperature $T_c^* = T_c - \frac{c}{a}$ that contains the contribution of the dipolar interaction and rewrote the last term so that it is expected to be $o(\delta)$ as $\delta \to 0$. Note that in the context of micromagnetics, such an argument was made rigorous by Gioia and James [15] (see also the following sections). Furthermore, plugging in a z-independent ansatz $\phi(x, y, z) = \bar{\phi}(x, y)$, where $\bar{\phi} : D \to \mathbb{R}$ is sufficiently smooth (extended by zero outside D), one straightforwardly obtains (here and everywhere below we use $\mathbf{r}$ to denote either a point in $\mathbb{R}^3$ or $\mathbb{R}^2$, depending on the context)

$$\frac{\mathcal{F}(\phi)}{k_B T_c} = \delta \int_D \left(\frac{g}{2}|\nabla\bar{\phi}|^2 + \frac{a}{2}(T - T_c^*)\bar{\phi}^2 + \frac{b}{4}\bar{\phi}^4 - h\bar{\phi} \right) d^2r + O(\delta^2)$$
$$+ \frac{c}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{\sqrt{|\mathbf{r} - \mathbf{r}'|^2 + \delta^2}} - 2\pi\delta^{(2)}(\mathbf{r} - \mathbf{r}')\delta \right) \bar{\phi}(\mathbf{r})\bar{\phi}(\mathbf{r}') d^2r \, d^2r', \tag{1.4}$$

where $\delta^{(2)}(\mathbf{r})$ is the two-dimensional Dirac delta-function. Formally expanding the integrand in the last term in (1.4) in the powers of δ, one can then see that to the leading order the kernel becomes $\delta^2/(8\pi|\mathbf{r} - \mathbf{r}'|^3)$. In the physics literature, this approximation is often adopted to arrive at a leading order asymptotic theory for thin films with dipolar interactions, with the $1/r^3$ kernel representing the dipole-dipole repulsion (as is done, e.g., in the review [2]). This, however, is incorrect, since the $1/r^3$ kernel is too singular in two dimensions, and thus the resulting double integral does not make sense. A more sound approach mathematically is to go to Fourier space, perform an expansion there and then invert the transform. This leads to the following formula:

$$\frac{\mathcal{F}(\phi)}{k_B T_c \delta} \approx \int_D \left(\frac{g}{2}|\nabla\bar{\phi}|^2 + \frac{a}{2}(T - T_c^*)\bar{\phi}^2 + \frac{b}{4}\bar{\phi}^4 - h\bar{\phi} \right) d^2r$$
$$- \frac{c\delta}{16\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(\bar{\phi}(\mathbf{r}) - \bar{\phi}(\mathbf{r}'))^2}{|\mathbf{r} - \mathbf{r}'|^3} d^2r \, d^2r'. \tag{1.5}$$

Contrary to the previous case, the last integral in the right-hand side of (1.5) is well defined, at least for smooth functions vanishing on ∂D. Moreover, since this term can be interpreted, up to a constant factor, as the homogeneous $H^{1/2}$-norm squared of $\bar{\phi}$ (see, e.g., [11]):

$$\|\bar{\phi}\|_{\dot{H}^{1/2}(\mathbb{R}^2)}^2 = \frac{1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(\bar{\phi}(\mathbf{r}) - \bar{\phi}(\mathbf{r}'))^2}{|\mathbf{r} - \mathbf{r}'|^3} d^2r \, d^2r' = \int_{\mathbb{R}^2} \bar{\phi}(-\Delta)^{1/2}\bar{\phi} \, d^2r, \tag{1.6}$$

one can write (1.5) as [1,18]

$$\frac{\mathcal{F}(\phi)}{k_B T_c \delta} \approx \int_D \left(\frac{g}{2} |\nabla \bar{\phi}|^2 + \frac{a}{2}(T - T_c^*)\bar{\phi}^2 + \frac{b}{4}\bar{\phi}^4 - h\bar{\phi} \right) d^2r - \frac{c\delta}{4} \int_{\mathbb{R}^2} \bar{\phi}(-\Delta)^{1/2}\bar{\phi}\, d^2r,$$
(1.7)

where the half-Laplacian operator $(-\Delta)^{1/2}$ is understood as a map whose Fourier symbol is $|k|$, or, equivalently, as an integral operator whose action on smooth functions with compact support is given by [11]

$$(-\Delta)^{1/2}\bar{\phi}(\mathbf{r}) = \frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{2\bar{\phi}(\mathbf{r}) - \bar{\phi}(\mathbf{r} - \mathbf{z}) - \bar{\phi}(\mathbf{r} + \mathbf{z})}{|\mathbf{z}|^3}\, d^2z \qquad \mathbf{r} \in \mathbb{R}^2. \qquad (1.8)$$

In particular, since D is assumed to be bounded, we must necessarily have $\bar{\phi} \in H^1(D)$ in order for the right-hand side of (1.5) to be less than $+\infty$. If also $\bar{\phi} \in H_0^1(D)$, then by interpolation the energy is bounded below and is thus well defined [24]. Yet, there is still an issue with the expression for the energy in (1.5), which becomes negative infinity as soon as $\bar{\phi}$ does not vanish at the boundary of D. This issue is quite severe and exists even for $\bar{\phi} = const$ in D. The reason for the latter is that the energy in (1.5) fails to capture a reduced local contribution of the dipoles near the boundary, since only half of the neighbors are present at ∂D. In the following, we fix this issue by introducing a smooth cutoff near the boundary of D in computing the last term in the right-hand side of (1.5). This allows us to estimate, under appropriate assumptions, the original energy from (1.2) from below by a reduced energy similar to the one in (1.5) evaluated on the average of the order parameter in the z-direction, with the relative error controlled only by δ (for precise statements, see the following section). Since the latter energy is also a good approximation to the original energy for z-independent configurations, this then allows us to make a number of conclusions regarding the energy minimizers of the full energy in (1.2) defined on three-dimensional configurations. Thus, understanding the behavior of the energy minimizers for (1.2) can be achieved by looking at a somewhat simpler energy of the type in (1.5), which, nevertheless, retains most of the complexity of the former.

To summarize, in this paper we show that the energy in (1.5) is in a certain sense asymptotically equivalent to the energy in (1.2) *without* assuming that the order parameter does not vary in the z-direction. Instead, we show that the energy in (1.5) correctly describes the energetics of the low-energy three-dimensional order parameter configurations in terms of their z-averages. More precisely, under some technical assumptions the energy in (1.5) evaluated on the z-average of the order parameter gives an asymptotically accurate lower bound for the full energy in (1.2) evaluated on the three-dimensional order parameter configuration. On the other hand, extending a two-dimensional order parameter configuration to a three-dimensional z-independent configuration, one gets a value of the full energy in (1.2) that is asymptotically bounded above by the value of the reduced energy in (1.5) evaluated on the two-dimensional configuration. We note that the first result in that direction was obtained by Kohn and Slastikov in the context of micromagnetics, see [22, Lemma 3]. Our analysis differs from that in [22] in that it identifies the first two non-trivial leading order terms in the expansion of the dipolar energy in δ and provides sharp universal estimates for the remainder.

The main result of this paper on the asymptotic equivalence of the two energies is presented in Theorem 2.1. This theorem relies on key Lemma 4.1, which establishes matching upper and lower bounds for the dipolar energy of three-dimensional order parameter configurations in terms of a non-local energy functional evaluated on the z-averages in the plane, with the error controlled by the Dirichlet energy with a vanishingly small coefficient as the film thickness becomes small. This produces errors that can be controlled by the L^∞ norm of the order parameter, apart from some possible additional contributions near the film edge in the upper

bounds. Notice that boundedness of the L^∞ norm of both the three- and two-dimensional energy minimizing order parameter configurations is a reasonable assumption in view of the regularity of minimizers established in Propositions 3.3 and 3.6. We also note that a uniform L^∞ bound by the equilibrium value of the order parameter is a fairly standard assumption for the ansatz-based computations in the physics literature and is a property which is also observed in some numerical simulations (see, e.g., [13,18,19,33,35]).

With the reduced energy identified, we proceed to analyze two thin film regimes. In the first regime, only the film thickness is sent to zero, with all the other parameters as well as the film cross-section fixed. In the context of micromagnetics, such a result was first obtained by Gioia and James in [15]. Here under a uniform L^∞ bound this type of result follows immediately from Theorem 2.1. Still, we are able to relax the L^∞ constraint and prove the result in the full generality by establishing Γ-convergence of the full energy to the *local* energy evaluated on the *z*-averages, see Theorem 2.2. Here the proof requires a different treatment of the non-local contributions to the energy near the film edge.

Finally, we consider a regime in which simultaneously the film thickness goes to zero, while the film's lateral dimension goes to infinity with a suitable rate that is exponential in the film thickness. We note that these types of scalings were previously discussed in the physics literature [19,33] and have been recently treated by Knüpfer, Muratov and Nolte within the framework of micromagnetics [21]. In this regime, after a rescaling that fixes the domain in the plane we prove a Γ-convergence result for the reduced energy in Theorem 2.4. Together with Theorem 2.1, this result then gives asymptotic non-existence of non-trivial minimizers of the full energy, under a uniform L^∞ bound and a technical assumption that the sample is maintained in a single phase near the edge. We further identify a critical value of the rescaled film thickness above which pattern formation occurs, see Corollaries 2.6 and 2.7. The proof relies on the standard Modica-Mortola trick [27] and an interpolation Lemma 6.1 similar to the one obtained in the context of thin film micromagnetics [9], and follows closely the arguments that lead to Theorem 3.5 in our companion paper [21]. Note that combining Theorem 2.4 with Theorem 2.1 yields an analog of Theorem 3.1 in [21]. A novel aspect of Theorem 2.4 is the consideration of the energy contribution from the non-local term near the sample edge.

Our paper is organized as follows. In Sect. 2, we present the main results of the paper. In Sect. 3, a number of auxiliary results is obtained that are used throughout the proofs. Here we also derive the Euler-Lagrange equations associated with minimizers of the full and the reduced energies, see Propositions 3.3 and 3.6. Then, in Sect. 4 we give the proof of Theorem 2.1 and in Sect. 5, we give the proof of Theorem 2.2. Finally, in Sect. 6 we present the proof of Theorem 2.4 and Corollary 2.6.

2 Main results

We now turn to our main results. We start by carrying out a suitable non-dimensionalization for the energy in (1.2). To that end, we use instead the representation in (1.3) and choose the units of length, ϕ and the energy in such a way that $k_B T_c = a(T_c^* - T) = b = g = 1$, treating the most interesting case $T < T_c^*$. Also, to simplify the presentation we set $h = 0$ throughout the rest of the paper. The external field h can be trivially added back in all the results below.

Denoting the dimensionless dipolar strength by $\gamma > 0$, we write the rescaled version of the energy in (1.3), up to an additive constant, as

$$\mathcal{E}(\phi) := \int_\Omega \left(\frac{1}{2} |\nabla \phi|^2 + \frac{1}{4} \left(1 - \phi^2 \right)^2 \right) d^3 r + \frac{\gamma}{2} \int_{\mathbb{R}^3} \left(\partial_z \phi (-\Delta)^{-1} \partial_z \phi - \phi^2 \right) d^3 r, \quad (2.1)$$

where $\phi \in H^1(\Omega)$, extended by zero to $\mathbb{R}^3 \backslash \Omega$. The energy $\mathcal{E}$ in (2.1) thus depends on only two dimensionless parameters, δ and γ, as well as on the domain D, whose diameter may have a relationship with these two parameters when considering various asymptotic regimes. The unit of length above is chosen so that the characteristic length scale of variation of ϕ in the absence of the dipolar interaction is of order unity. Therefore, the thin film regime that we are interested in should correspond to $\delta \lesssim 1$. Note that in terms of the original, dimensional variables, we have

$$\gamma = \frac{c}{a(T_c^* - T)}. \quad (2.2)$$

In the context of ferromagnetism, the parameter γ may be both small and large, depending on how close the value of T is to T_c^*. Indeed, since the stray field interaction is a relativistic effect in comparison with the exchange interaction driving the phase transition, it should be considerably weaker than the latter away from the critical temperature [5]. At the same time, as T approaches T_c^*, the value of γ diverges.

We next introduce a cutoff function $\chi_\delta \in C_c^\infty(\mathbb{R}^2)$. Namely, we define $\eta : \mathbb{R} \to [0, 1]$ such that $\eta \in C^\infty(\mathbb{R})$, $\eta(t) = 0$ for all $t \leq 1$, $\eta(t) = 1$ for all $t \geq 2$ and $0 \leq \eta'(t) \leq 2$ for all $t \in \mathbb{R}$. We then define $\chi_\delta(\mathbf{r}) = \eta(\delta^{-1} \text{dist}(\mathbf{r}, \mathbb{R}^2 \backslash D))$. We also define

$$D_\delta := \{ \mathbf{r} \in D : \text{dist}(\mathbf{r}, \partial D) > \delta \} \quad \text{and} \quad \Omega_\delta := D_\delta \times (0, \delta), \quad (2.3)$$

and note that $\overline{D}_\delta = \text{supp}(\chi_\delta)$. Finally, with a slight abuse of notation we will also treat χ_δ as a z-independent function of all three coordinates, depending on the context.

We now define the following *reduced energy* for $\bar\phi \in H^1(D)$ and $\alpha > 0$:

$$E(\bar\phi) := \int_D \left(\frac{1}{2} \left(1 - \alpha \delta^2 \right) |\nabla \bar\phi|^2 + \frac{1}{4} \left(1 - \bar\phi^2 \right)^2 \right) d^2 r$$
$$- \frac{\gamma \delta}{16\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(\chi_\delta(\mathbf{r}) \bar\phi(\mathbf{r}) - \chi_\delta(\mathbf{r}') \bar\phi(\mathbf{r}'))^2}{|\mathbf{r} - \mathbf{r}'|^3} \, d^2 r \, d^2 r'. \quad (2.4)$$

This definition makes sense, because we have $\chi_\delta \bar\phi \in H^1(\mathbb{R}^2)$ and, hence, the last term in (2.4) is well-defined [24, Section 7.11]. What we will show below is that if

$$\bar\phi(x, y) = \frac{1}{\delta} \int_0^\delta \phi(x, y, z) \, dz \quad (x, y) \in D, \quad (2.5)$$

then with a suitable explicit choice of α the value of $E(\bar\phi)\delta$ may be used to bound from below the value of $\mathcal{E}(\phi)$, up to a small error in δ. Conversely, the value of $E(\bar\phi)\delta$ provides a good approximation for the value of $\mathcal{E}(\phi)$, with a small relative error, when ϕ is chosen to be independent of z. We make this statement precise in the following theorem.

Theorem 2.1 *There exist universal constants $\alpha_1 > 0$, $\alpha_2 > 0$ and $\beta > 0$ such that for every $\delta > 0$ sufficiently small there holds:*

(i) *If $\phi \in H^1(\Omega) \cap L^\infty(\Omega)$ and $\bar\phi$ is defined by (2.5), then*

$$\mathcal{E}(\phi) \geq E(\bar\phi)\delta - \beta \gamma \delta^2 \|\phi\|_{L^\infty(\Omega)}^2 |\partial D|, \quad (2.6)$$

with $\alpha = \alpha_1 + \gamma \alpha_2$.

(ii) *For every $\bar\phi \in H^1(D) \cap L^\infty(D)$ there exists $\phi \in H^1(\Omega) \cap L^\infty(\Omega)$ such that $\|\phi\|_{L^\infty(\Omega)} \le \|\bar\phi\|_{L^\infty(D)}$, $\phi(x, y, z) = \bar\phi(x, y)$ for all $(x, y) \in D$, and*

$$\mathcal{E}(\phi) \le (1 - 2\alpha\delta^2)^{-1} E(\bar\phi)\delta$$
$$+ \beta\delta^2(1 + \gamma^2)\left(1 + \|\bar\phi\|_{L^\infty(D)}^4\right)|\partial D| + \beta\delta\|\nabla\bar\phi\|_{L^2(D\setminus D_\delta)}^2. \qquad (2.7)$$

Note that for $\gamma \lesssim 1$ and $\|\phi\|_{L^\infty(\Omega)} \lesssim 1$ the additive error term appearing in both the upper and the lower bound in Theorem 2.1 is of the order of the dipolar self-interaction energy of ϕ at the sample edge $\Omega\setminus\Omega_\delta$. Thus, the asymptotic equivalence of $\mathcal{E}$ and E established in Theorem 2.1 holds when $|\mathcal{E}(\phi)| \gg \delta^2$, when the bulk contribution to the energy dominates that of the edge. Note that in this case the non-local term in E is expected to capture the leading $O(\delta^2|\log\delta|)$ contribution to $\mathcal{E}$ from the film edge. Hence, the additive error term appearing in Theorem 2.1 should still be negligible even when the edge effects are prominent. We point out that a smooth cutoff near the sample edge was recently used to model boundary effects in computational micromagnetic studies of ultrathin ferromagnetic films, a closely related problem [32].

We now show how Theorem 2.1 may be used to establish some of the asymptotic properties of the energy minimizing configurations for the original energy $\mathcal{E}$ as $\delta \to 0$ by studying the reduced energy E. We begin by establishing a result similar to that of Gioia and James for a closely related vectorial model of micromagnetics in the thin film limit [15]. Namely, we consider the simplest thin film regime, in which $\delta \to 0$ with both γ and D fixed. In this regime, we show that the energetics of the low energy configurations in the original three-dimensional model can be asymptotically described via the local two-dimensional energy. The proof for uniformly bounded sequences follows by combining the result in Theorem 2.1 with the $\delta \to 0$ limit behavior of E established in Proposition 5.2. A slight modification of the proof of Theorem 2.1 in this regime allows to remove the assumption of boundedness, so below we state the result in its full generality.

For fixed D, consider a family of bounded open sets $\Omega^\delta \subset \mathbb{R}^3$ such that $D \times (0, \delta) \subset \Omega^\delta \subset (D + B_\delta) \times (0, \delta)$. Given $\phi_\delta \in H^1(\Omega^\delta)$, we define $\bar\phi_\delta$ to be its z-average on D, i.e., $\bar\phi_\delta \in H^1(D)$ is defined by (2.5) with ϕ replaced by ϕ_δ. We next define $\mathcal{E}_\delta$ to be the family of functionals given by (2.1) with $\Omega = \Omega^\delta$. We also define E_0 to be given by (2.4) with δ formally set to zero, i.e., we define

$$E_0(\bar\phi) := \begin{cases} \int_D \left(\frac{1}{2}|\nabla\bar\phi|^2 + \frac{1}{4}\left(1 - \bar\phi^2\right)^2\right) d^2r & \bar\phi \in H^1(D), \\ +\infty & \text{otherwise.} \end{cases} \qquad (2.8)$$

Then the following Γ-convergence result holds true (for a general introduction to Γ-convergence, see, e.g., [3]).

Theorem 2.2 *As $\delta \to 0$, we have*

$$\delta^{-1}\mathcal{E}_\delta \xrightarrow{\Gamma} E_0, \qquad (2.9)$$

with respect to the L^2 convergence of the z-averages, in the following sense:

(i) *For any sequence of $\delta \to 0$ and $\phi_\delta \in H^1(\Omega^\delta)$ such that $\|\nabla\phi_\delta\|_{L^2(\Omega)}^2 \le C\delta$ for some $C > 0$ independent of δ, $\bar\phi_\delta \rightharpoonup \bar\phi$ in $H^1(D)$ and $\bar\phi_\delta \to \bar\phi$ in $L^2(D)$, we have*

$$\liminf_{\delta \to 0} \delta^{-1}\mathcal{E}_\delta(\phi_\delta) \ge E_0(\bar\phi). \qquad (2.10)$$

(ii) *For any $\bar{\phi} \in H^1(D)$ and every sequence of $\delta \to 0$, there exists $\phi_\delta \in H^1(\Omega^\delta)$ such that $\|\nabla\phi_\delta\|^2_{L^2(\Omega)} \leq C\delta$ for some $C > 0$ independent of δ, $\bar{\phi}_\delta \to \bar{\phi}$ in $L^2(D)$ and*

$$\limsup_{\delta \to 0} \delta^{-1}\mathcal{E}_\delta(\phi_\delta) \leq E_0(\bar{\phi}). \tag{2.11}$$

The assumption on the gradient in Theorem 2.2 is a natural assumption consistent with the scaling of the minimum energy for $\mathcal{E}_\delta$. In particular, the theorem above applies, upon extraction of subsequences, to $\phi_\delta \in H^1(\Omega^\delta)$ satisfying

$$\limsup_{\delta \to 0} \delta^{-1}\mathcal{E}_\delta(\phi_\delta) < +\infty, \tag{2.12}$$

in view of the compactness of their z-averages in $H^1(D)$, see Proposition 5.1. Therefore, by Corollary 3.2 we have the following immediate consequence of Theorem 2.2 concerning global minimizers of $\mathcal{E}_\delta$. Note that the latter exist for each $\delta > 0$ by Proposition 3.3.

Corollary 2.3 *Let $\phi_\delta \in H^1(\Omega^\delta)$ by a minimizer of $\mathcal{E}_\delta$. Then for any sequence of $\delta \to 0$ we have $\bar{\phi}_\delta \to \bar{\phi}$ in $L^2(D)$, where $\bar{\phi}$ takes a constant value ± 1 in every connected component of D.*

Let us point out that the addition to $\mathcal{E}_\delta(\phi)$ of an applied field term $-\int_{\Omega^\delta} h\phi \, d^3r$ with $h = h(x, y) \in L^2(\Omega^\delta)$ does not change the Γ-convergence result in Theorem 2.2, provided that the term $-\int_D h\bar{\phi} \, d^2r$ is added to the definition of E_0 in (2.8). Thus, as expected, in the thin film limit with D and γ fixed one recovers the local Ginzburg–Landau energy functional. We note, however, that physically the effect of the dipolar interaction is still present in the renormalization of the transition temperature from T_c to T_c^*.

We finally turn our attention to a regime of practical interest in which modulated patterns spontaneously emerge. In view of the previous result, this requires simultaneous vanishing of the film thickness and blowup of the film's lateral dimensions. To this end, we introduce a small parameter $\varepsilon > 0$ and consider domain $D^\varepsilon = \varepsilon^{-1}D$, with a fixed bounded open set $D \subset \mathbb{R}^2$ with C^2 boundary describing the shape of the film in the plane and lateral length scale $\varepsilon^{-1} \gg 1$. Next, we rescale all lengths with ε^{-1} and define the rescaled domain $\Omega^\varepsilon \subset \mathbb{R}^3$ occupied by the material. Thus, for a film of thickness $\delta = \delta_\varepsilon$ we have $D \times (0, \varepsilon\delta_\varepsilon) \subset \Omega^\varepsilon \subset (D + B_{\varepsilon\delta_\varepsilon}) \times (0, \varepsilon\delta_\varepsilon)$.

In the rescaled variables, the energy in (2.1) takes the following form:

$$\mathcal{E}_\varepsilon(\phi) := \int_{\Omega^\varepsilon} \left(\frac{1}{2}|\nabla\phi|^2 + \frac{1}{4\varepsilon^2}\left(1 - \phi^2\right)^2\right) d^3r + \frac{\gamma}{2\varepsilon^2} \int_{\mathbb{R}^3} \left(\partial_z\phi(-\Delta)^{-1}\partial_z\phi - \phi^2\right) d^3r, \tag{2.13}$$

where $\phi \in H^1(\Omega^\varepsilon)$ and the energy has been rescaled with an overall factor ε. Similarly, rescaling the energy in (2.4) with ε as well, for $\bar{\phi} \in H^1(D)$ we define

$$E_\varepsilon(\bar{\phi}) := \int_D \left(\frac{\varepsilon}{2}(1 - \alpha\delta_\varepsilon^2)|\nabla\bar{\phi}|^2 + \frac{1}{4\varepsilon}\left(1 - \bar{\phi}^2\right)^2\right) d^2r$$
$$- \frac{\gamma\delta_\varepsilon}{16\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(\chi_{\varepsilon\delta_\varepsilon}(\mathbf{r})\bar{\phi}(\mathbf{r}) - \chi_{\varepsilon\delta_\varepsilon}(\mathbf{r}')\bar{\phi}(\mathbf{r}'))^2}{|\mathbf{r} - \mathbf{r}'|^3} d^2r \, d^2r'. \tag{2.14}$$

Notice that the overall factor of ε in the energy scale for both energies above is chosen to obtain the Modica-Mortola scaling [27] in the reduced two-dimensional energy E_ε, in anticipation of its limit behavior as $\varepsilon \to 0$.

With these notations, the lower bound in Theorem 2.1 for $\phi_\varepsilon \in H^1(\Omega^\varepsilon) \cap L^\infty(\Omega^\varepsilon)$ satisfying $\|\phi_\varepsilon\|_{L^\infty(\Omega^\varepsilon)} \leq M$ for some $M \geq 1$ fixed and for all δ_ε sufficiently small becomes

$$\mathcal{E}_\varepsilon(\phi_\varepsilon) \geq E_\varepsilon(\bar{\phi}_\varepsilon)\delta_\varepsilon - C\delta_\varepsilon^2, \tag{2.15}$$

where

$$\bar{\phi}_\varepsilon(x, y) = \frac{1}{\varepsilon\delta_\varepsilon} \int_0^{\varepsilon\delta_\varepsilon} \phi_\varepsilon(x, y, z)\, dz \qquad (x, y) \in D, \tag{2.16}$$

and $C > 0$ depends only on γ, D and M, for a suitable choice of α depending only on γ. Concentrating on the bulk properties of the configurations, we further assume that the order parameter is equal to its bulk equilibrium value near the film edge and does not exceed it in magnitude throughout the film (a more thorough analysis of the behavior of global minimizers as $\varepsilon \to 0$ goes far beyond the scope of the present paper and will be treated elsewhere). Hence, we set $M = 1$ and for $\rho > 0$ sufficiently small fixed we assume that $\bar{\phi}_\varepsilon = 1$ in $D \backslash D_\rho$, where D_ρ is as in (2.3). In this case the upper bound from Theorem 2.1 reads for all $\bar{\phi}_\varepsilon \in H^1(D) \cap L^\infty(D)$ such that $\|\bar{\phi}_\varepsilon\|_{L^\infty(D)} = 1$ and $\bar{\phi}_\varepsilon = 1$ in $D \backslash D_\rho$, for all δ_ε sufficiently small:

$$\mathcal{E}_\varepsilon(\phi_\varepsilon) \leq (1 - 2\alpha\delta_\varepsilon^2)^{-1} E_\varepsilon(\bar{\phi}_\varepsilon)\delta_\varepsilon + C\delta_\varepsilon^2, \tag{2.17}$$

where $C > 0$ is as before and $\phi_\varepsilon \in H^1(\Omega^\varepsilon) \cap L^\infty(\Omega^\varepsilon)$ satisfies $\phi_\varepsilon(x, y, z) = \bar{\phi}_\varepsilon(x, y)$ for all $(x, y) \in D$, and $\|\phi_\varepsilon\|_{L^\infty(\Omega^\varepsilon)} = 1$. We note that related ideas were used in [31] in the asymptotic analysis of the two-dimensional Ohta-Kawasaki energy.

We now specify the scaling of δ_ε with ε for which modulated patterns emerge. This scaling has been recently identified in [21] in the studies of a closely related model from micromagnetism. For $\lambda > 0$ fixed, we set

$$\delta_\varepsilon = \frac{\lambda}{\gamma|\ln\varepsilon|}, \tag{2.18}$$

and consider the limit behavior of the energies in (2.13) and (2.14) as $\varepsilon \to 0$. In [21], a critical value of $\lambda = \lambda_c$ has been identified, below which no modulated patterns emerge as energy minimizers in this limit, while above this value pattern formation occurs. A similar phenomenon takes place in our problem, too. In the subcritical regime, the conclusion above is a consequence of the following Γ-convergence result. In our case, the threshold value of λ is

$$\lambda_c := \frac{2\pi\sqrt{2}}{3}. \tag{2.19}$$

We also define the constants

$$\sigma_0 = \frac{2\sqrt{2}}{3}, \qquad \sigma_1 = \frac{1}{\pi}, \tag{2.20}$$

and notice that $\lambda_c = \sigma_0/\sigma_1$. The following theorem is a close analog of Theorem 3.5 in [21] obtained in a periodic setting.

Theorem 2.4 *Let $\rho > 0$, $0 < \lambda < \lambda_c$ and let E_ε be defined by (2.14) with δ_ε given by (2.18). Then as $\varepsilon \to 0$ we have*

$$E_\varepsilon \xrightarrow{\Gamma} E_*, \qquad E_*(\bar{\phi}) := -\frac{1}{4}\sigma_1\lambda|\partial D| + \frac{1}{2}(\sigma_0 - \sigma_1\lambda)\int_D |\nabla\bar{\phi}|\, d^2r, \tag{2.21}$$

where $\bar{\phi} \in BV(D; \{-1, 1\})$, with respect to the $L^1(D)$ convergence, in the following sense:

(i) *For every sequence of $\bar{\phi}_\varepsilon \in H^1(D) \cap L^\infty(D)$ such that $\bar{\phi}_\varepsilon = 1$ in $D \backslash D_\rho$, $\|\bar{\phi}_\varepsilon\|_{L^\infty(D)} = 1$, and*

$$\limsup_{\varepsilon \to 0} E_\varepsilon(\bar{\phi}_\varepsilon) < +\infty, \tag{2.22}$$

there exists a subsequence (not relabelled) such that $\bar{\phi}_\varepsilon \to \bar{\phi}$ in $L^1(D)$ and

$$\liminf_{\varepsilon \to 0} E_\varepsilon(\bar{\phi}_\varepsilon) \geq E_*(\bar{\phi}), \tag{2.23}$$

for some $\bar{\phi} \in BV(D; \{-1, 1\})$ such that $\bar{\phi} = 1$ in $D \backslash D_\rho$.

(ii) *For any $\bar{\phi} \in BV(D; \{-1, 1\})$ such that $\bar{\phi} = 1$ in $D \backslash D_\rho$ there exists a sequence of $\bar{\phi}_\varepsilon \in H^1(D) \cap L^\infty(D)$ such that $\bar{\phi}_\varepsilon = 1$ in $D \backslash D_\rho$, $\|\bar{\phi}_\varepsilon\|_{L^\infty(D)} = 1$, $\bar{\phi}_\varepsilon \to \bar{\phi}$ in $L^1(D)$ and*

$$\limsup_{\varepsilon \to 0} E_\varepsilon(\bar{\phi}_\varepsilon) \leq E_*(\bar{\phi}). \tag{2.24}$$

Remark 2.5 The inequalities in (2.23) and (2.24) remain true for $\lambda \geq \lambda_c$ as well, if one assumes that $\bar{\phi}_\varepsilon \to \bar{\phi}$ in $BV(D)$ in addition to $\bar{\phi}_\varepsilon = 1$ in $D \backslash D_\rho$ and $\|\bar{\phi}_\varepsilon\|_{L^\infty(D)} = 1$. However, the compactness statement of Theorem 2.4 no longer holds for $\lambda > \lambda_c$ (for more details in a periodic setting, see [21]).

Theorem 2.4 implies, in particular, that for $\lambda < \lambda_c$ all minimizers of E_ε among functions $\bar{\phi}_\varepsilon \in H^1(D) \cap L^\infty(D)$ satisfying $\|\bar{\phi}_\varepsilon\|_{L^\infty(D)} = 1$ and $\bar{\phi}_\varepsilon = 1$ in $D \backslash D_\rho$ for some $\rho > 0$ converge a.e. to $\bar{\phi} = 1$ in D as $\varepsilon \to 0$, implying that minimizers within this class approach a monodomain state for all ε sufficiently small. This is consistent with the result in Corollary 2.3 in the other scaling regime considered earlier. As was already noted, relaxing the assumption of boundedness and the behavior near the edge to make the same conclusion about the unconstrained minimizers of E_ε would require a rather delicate analysis of the energy minimizing configurations near the film edge, which goes beyond the scope of the present paper. Still, within the considered restricted class we may conclude, by (2.15) and (2.17), that the same result is true for the z-averages $\bar{\phi}_\varepsilon$ of the minimizers ϕ_ε of $\mathcal{E}_\varepsilon$ in the respective class. The precise statement is in the following corollary.

Corollary 2.6 *Let $\rho > 0$, $0 < \lambda < \lambda_c$ and let $\mathcal{E}_\varepsilon$ be defined by (2.13) with δ_ε given by (2.18). Let $\phi_\varepsilon \in H^1(\Omega^\varepsilon) \cap L^\infty(\Omega^\varepsilon)$ be a minimizer of $\mathcal{E}_\varepsilon$ among all functions satisfying $\phi_\varepsilon = 1$ in $\Omega^\varepsilon \backslash (D_\rho \times (0, \varepsilon\delta_\varepsilon))$ and $\|\phi_\varepsilon\|_{L^\infty(\Omega^\varepsilon)} = 1$. Then if $\bar{\phi}_\varepsilon$ is defined by (2.16), we have $\bar{\phi}_\varepsilon \to 1$ in $BV(D)$ as $\varepsilon \to 0$.*

We also point out that, despite asymptotic non-existence of non-trivial minimizers of E_ε for $\lambda < \lambda_c$ the effect of the dipolar interaction can still be seen in the energetics via a renormalized line tension $\sigma = \sigma_0 - \lambda\sigma_1$ for the domain patterns in the plane. At the same time, Remark 2.5 also allows us to conclude that for $\lambda > \lambda_c$ the minimizers in Corollary 2.6 must develop spatial oscillations as $\varepsilon \to 0$.

Corollary 2.7 *Let ρ, ϕ_ε and $\bar{\phi}_\varepsilon$ be as in Corollary 2.6, and let $\lambda > \lambda_c$. Then $\bar{\phi}_\varepsilon \not\to 1$ in $BV(D)$, as $\varepsilon \to 0$.*

In fact, it is possible to show that for $\lambda > \lambda_c$ minimizers of E_ε or the z-averages of minimizers of $\mathcal{E}_\varepsilon$ cannot converge in $BV(D)$. Instead, they develop fine oscillations throughout D (for an analogous result in micromagnetics, see [21, Theorem 3.6]).

3 Preliminaries

In this section, we collect a few basic facts for various terms appearing both in the original energy in (2.1) and the reduced energy in (2.4). In particular, we establish existence and regularity of the minimizers of both energies. We remind the reader that, except in the following lemma, we always consider a function $\phi \in H^1(\Omega)$ to be extended by zero to the whole space whenever we view ϕ as a function defined on $\mathbb{R}^3$. Similarly, a function $\bar{\phi} \in H^1(D)$ is assumed to be extended by zero to the rest of $\mathbb{R}^2$ whenever it is treated as a function defined on $\mathbb{R}^2$.

We begin by a characterization of the non-local term appearing in (2.1). Recall that the derivative $\partial_z \phi$ in (2.1) is understood in the distributional sense in the whole of $\mathbb{R}^3$.

Lemma 3.1 *Let $\phi, \psi \in H^1(\Omega)$ and let $\tilde{\phi}, \tilde{\psi}$ be their extensions by zero to $\mathbb{R}^3 \setminus \Omega$, respectively. Then*

$$\int_{\mathbb{R}^3} \partial_z \tilde{\phi} (-\Delta)^{-1} \partial_z \tilde{\psi} \, d^3 r := \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\partial_z \tilde{\phi}(\mathbf{r}) \, \partial_z \tilde{\psi}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d^3 r \, d^3 r' \tag{3.1}$$

defines an inner product on $H^1(\Omega)$. Furthermore, $\int_{\mathbb{R}^3} \partial_z \tilde{\phi} (-\Delta)^{-1} \partial_z \tilde{\psi} \, d^3 r \leq \|\phi\|_{L^2(\Omega)} \|\psi\|_{L^2(\Omega)}$, and we have

$$\int_{\mathbb{R}^3} \partial_z \tilde{\phi} (-\Delta)^{-1} \partial_z \tilde{\psi} \, d^3 r = -\int_{\Omega} \phi \, \partial_z^2 (-\Delta)^{-1} \psi \, d^3 r, \tag{3.2}$$

where $(-\Delta)^{-1} \psi \in W_{loc}^{2,2}(\mathbb{R}^3)$ is the Newtonian potential of $\tilde{\psi}$:

$$(-\Delta)^{-1} \psi(\mathbf{r}) := \frac{1}{4\pi} \int_{\Omega} \frac{\psi(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d^3 r \qquad \mathbf{r} \in \mathbb{R}^3. \tag{3.3}$$

Proof First of all, observe that since $\phi, \psi \in H^1(\Omega)$ and Ω is a bounded set with boundary of class C^2, we have $\tilde{\phi}, \tilde{\psi} \in BV(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$, with $\partial_z \tilde{\phi} = \partial_z \tilde{\phi}^a + \partial_z \tilde{\phi}^j$, where $\partial_z \tilde{\phi}^a = \mathcal{L}^3(\Omega) \llcorner \partial_z \phi$ is the absolutely continuous part and $\partial_z \tilde{\phi}^j = \mathcal{H}^2(\partial\Omega) \llcorner (-\mathbf{e}_z \cdot \nu) T(\phi)$ is the jump part [12]. Here, $T(\phi)$ denotes the trace of ϕ on $\partial\Omega$, ν is the outward unit normal vector to $\partial\Omega$ and $\mathbf{e}_z$ is the unit vector in the positive z direction. Furthermore, since $T(\phi) \in L^2(\partial\Omega)$ by the trace embedding theorem [12], it is easy to see that the right-hand side of (3.1) defines an absolutely convergent integral. Then, arguing by approximation, we can write

$$\int_{\mathbb{R}^3} \partial_z \tilde{\phi} (-\Delta)^{-1} \partial_z \tilde{\psi} \, d^3 r = \int_{\mathbb{R}^3} \frac{(\mathbf{k} \cdot \mathbf{e}_z)^2}{|\mathbf{k}|^2} \, \widehat{\phi}_{\mathbf{k}}^* \widehat{\psi}_{\mathbf{k}} \, \frac{d^3 k}{(2\pi)^3}, \tag{3.4}$$

where $\widehat{\phi}_{\mathbf{k}}$ and $\widehat{\psi}_{\mathbf{k}}$ are the Fourier transforms of $\tilde{\phi}$ and $\tilde{\psi}$, respectively, with the convention

$$\widehat{\phi}_{\mathbf{k}} := \int_{\mathbb{R}^3} e^{i\mathbf{k} \cdot \mathbf{r}} \tilde{\phi}(\mathbf{r}) \, d^3 r. \tag{3.5}$$

Thus, by Cauchy–Schwarz inequality and Parseval's identity, the first part of the statement follows. To complete the proof of the second part, we note that by standard elliptic regularity [14], we have $(-\Delta)^{-1} \psi \in W_{loc}^{2,2}(\mathbb{R}^3)$ and, therefore, $\partial_z^2 (-\Delta)^{-1} \psi \in L_{loc}^2(\mathbb{R}^3)$. The claim then follows by passing again to the Fourier space. □

As can be seen from the proof of Lemma 3.1, the Fourier representation in (3.4) of the integral in the right-hand side of (3.1) justifies our choice of notation for the left-hand side of (3.1). Throughout the rest of the paper, we drop the tildes from all the formulas involving the

extensions. An immediate corollary to Lemma 3.1 is the following, with the last statement obtained by testing the energy against $\phi \equiv 1$.

Corollary 3.2 *We have for all $\phi \in H^1(\Omega)$*

$$0 \leq \int_{\mathbb{R}^3} \partial_z \phi (-\Delta)^{-1} \partial_z \phi \, d^3 r \leq \int_{\Omega} \phi^2 \, d^3 r. \tag{3.6}$$

In particular, $\inf_{\phi \in H^1(\Omega)} \mathcal{E}(\phi) \leq 0$.

We next turn to existence and some basic properties of the minimizers of $\mathcal{E}$. The arguments of the proof are fairly standard, based on the direct method of calculus of variations and standard elliptic regularity theory, with the exception of a separate treatment of the contributions to the non-local term coming from the boundary trace of ϕ.

Proposition 3.3 *There exists a minimizer ϕ of $\mathcal{E}$ in (2.1) among all functions in $H^1(\Omega)$. Furthermore, we have $\phi \in C^\infty(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ for all $\alpha \in (0, 1)$, and ϕ satisfies*

$$0 = \Delta \phi(\mathbf{r}) + (1 + \gamma)\phi(\mathbf{r}) - \phi^3(\mathbf{r}) - \frac{\gamma}{4\pi} \int_{\Omega} \frac{\mathbf{e}_z \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \partial_z \phi(\mathbf{r}') d^3 r'$$

$$+ \frac{\gamma}{4\pi} \int_{\partial\Omega} \frac{\mathbf{e}_z \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} (\mathbf{e}_z \cdot \nu(\mathbf{r}')) \phi(\mathbf{r}') d\mathcal{H}^2(\mathbf{r}') \quad \forall \mathbf{r} \in \Omega, \tag{3.7}$$

where $\mathbf{e}_z$ is the unit vector in the positive z direction and ν is the outward unit normal to $\partial\Omega$, with $\nu \cdot \nabla \phi(\mathbf{r}) = 0$ for all $\mathbf{r} \in \partial\Omega$.

Proof By Lemma 3.1 and Sobolev embedding [12], the energy in (2.1) is well-defined and bounded below for all $\phi \in H^1(\Omega)$. Let $\phi_n \in H^1(\Omega)$ be a minimizing sequence. Then by Corollary 3.2 and Cauchy–Schwarz inequality we have

$$\frac{1}{2} \|\nabla \phi_n\|_{L^2(\Omega)}^2 - \frac{1}{2} (1 + \gamma) |\Omega|^{1/2} \|\phi_n\|_{L^4(\Omega)}^2 + \frac{1}{4} \|\phi_n\|_{L^4(\Omega)}^4 \leq C, \tag{3.8}$$

for some $C > 0$ independent of n. Therefore, upon extraction of a subsequence we may assume that $\phi_n \rightharpoonup \phi$ in $H^1(\Omega)$ as $n \to \infty$, and upon further extraction we also have $\phi_n \to \phi$ in $L^p(\Omega)$ for all $1 \leq p < 6$ [12]. In particular, up to a subsequence (not relabeled) we have $\phi_n \to \phi$ in $L^2(\Omega)$, and by Lemma 3.1 we also have $\int_{\mathbb{R}^3} \partial_z \phi_n (-\Delta)^{-1} \partial_z \phi_n \, d^3 r \to \int_{\mathbb{R}^3} \partial_z \phi (-\Delta)^{-1} \partial_z \phi \, d^3 r$ as $n \to \infty$. Therefore, by lower semicontinuity of the gradient squared term in the energy, we have $\liminf_{n \to \infty} \mathcal{E}(\phi_n) \geq \mathcal{E}(\phi)$, and so ϕ is a minimizer.

By Lemma 3.1 and an explicit calculation, the energy in (2.1) is Fréchet differentiable, and the minimizer ϕ satisfies

$$\int_{\Omega} \left(\nabla \phi \cdot \nabla \psi - (1 + \gamma)\phi\psi + \phi^3 \psi \right) d^3 r + \gamma \int_{\mathbb{R}^3} \partial_z (\psi \chi_\Omega)(-\Delta)^{-1} \partial_z \phi \, d^3 r = 0, \tag{3.9}$$

where χ_Ω is the characteristic function of Ω, for every $\psi \in C_c^\infty(\mathbb{R}^3)$. Therefore, by Lemma 3.1 we have

$$0 = \int_{\Omega} \nabla \phi \cdot \nabla \psi \, d^3 r - \int_{\Omega} \left((1 + \gamma)\phi - \phi^3 + \gamma \partial_z^2 (-\Delta)^{-1} \phi \right) \psi \, d^3 r. \tag{3.10}$$

Since the bracket in the last integral in (3.10) is in $L^2(\Omega)$ by Sobolev embedding and Lemma 3.1, by standard elliptic estimates [14] we have $\phi \in W^{2,2}(\Omega)$ and, therefore, $\phi \in L^p(\Omega)$ for any $1 \leq p < \infty$, again, by Sobolev embedding (recall that $\Omega \subset \mathbb{R}^3$ is a bounded open set with boundary of class C^2). Then, again by standard elliptic regularity we also

have $(-\Delta)^{-1}\phi \in W^{2,p}(\Omega)$ and, hence, $\partial_z^2(-\Delta)^{-1}\phi \in L^p(\Omega)$. Thus, we conclude that $\phi \in W^{2,p}(\Omega)$ as well, and by Sobolev embedding $\phi \in C^{1,\alpha}(\overline{\Omega})$, for any $\alpha \in (0,1)$. In particular, ϕ satisfies Neumann boundary condition.

Finally, to arrive at (3.7) we note that with the above regularity of ϕ we can write

$$\partial_z^2(-\Delta)^{-1}\phi(\mathbf{r}) = -\frac{1}{4\pi}\int_\Omega \frac{\mathbf{e}_z \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}\partial_z\phi(\mathbf{r}')\,d^3r'$$
$$+ \frac{1}{4\pi}\int_{\partial\Omega} \frac{\mathbf{e}_z \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}(\mathbf{e}_z \cdot \nu(\mathbf{r}'))\phi(\mathbf{r}')\,d\mathcal{H}^2(\mathbf{r}'), \tag{3.11}$$

in $\mathcal{D}'(\mathbb{R}^3)$. The last term in the right-hand side of (3.11) defines a smooth function of $\mathbf{r} \in \Omega$, while the first term has derivatives belonging to $L^p(\Omega)$, in view of the fact that $\phi \in W^{1,p}(\Omega)$ and using standard elliptic regularity. Thus, we can apply a bootstrap argument to establish interior C^∞ regularity of ϕ in Ω. This then allows us to obtain (3.7) from (3.10). $\qquad\square$

Remark 3.4 Observe that by Proposition 3.3 every minimizer ϕ of $\mathcal{E}$ over $H^1(\Omega)$ is bounded. However, it is not a priori clear under which conditions the L^∞ norm of ϕ remains bounded as $\delta \to 0$, with other parameters such as γ or the diameter of Ω possibly going to infinity. It is natural to expect that in some thin film regimes the minimizers may develop a boundary layer near the edge, i.e., in the vicinity of $\Omega\backslash(D \times (0,\delta))$, and blow up at $\partial D \times (0,\delta)$ as $\delta \to 0$.

Before turning to the discussion of the reduced energy E in (2.4), we consider the contribution of the film's edge to the non-local term in the energy in (2.1). We have the following estimate for the contribution of the edge to the non-local term in the energy in the following lemma. Note that this estimate is expected to be optimal for small δ, since for $\phi = 1$, for example, the self-interaction energy associated with the edge can be easily seen to be of order δ^2.

Lemma 3.5 Let $\phi \in H^1(\Omega)$ and $\delta > 0$. Then

$$\left|\int_{\mathbb{R}^3}\partial_z\phi(-\Delta)^{-1}\partial_z\phi\,d^3r - \int_{\mathbb{R}^3}\partial_z(\chi_\delta\phi)(-\Delta)^{-1}\partial_z(\chi_\delta\phi)\,d^3r\right| \le 3\|\phi\|_{L^2(\Omega)}\|\phi\|_{L^2(\Omega\backslash\Omega_{2\delta})}. \tag{3.12}$$

Furthermore, there exists $\delta_0 > 0$ depending only on D such that for all $0 < \delta \le \delta_0$ we have for all $\phi \in H^1(\Omega) \cap L^\infty(\Omega)$:

$$\left|\int_{\mathbb{R}^3}\partial_z\phi(-\Delta)^{-1}\partial_z\phi\,d^3r - \int_{\mathbb{R}^3}\partial_z(\chi_\delta\phi)(-\Delta)^{-1}\partial_z(\chi_\delta\phi)\,d^3r\right|$$
$$\le 98|\partial D|\delta^2\|\phi\|_{L^\infty(\Omega)}^2. \tag{3.13}$$

Proof Denoting the left-hand side of (3.13) by R, writing $\phi = \chi_\delta\phi + (1-\chi_\delta)\phi$ and expanding the difference, we have $R \le 2R_1 + R_2$, where

$$R_1 := \left|\int_{\mathbb{R}^3}\partial_z((1-\chi_\delta)\phi)(-\Delta)^{-1}\partial_z(\chi_\delta\phi)\,d^3r\right|, \tag{3.14}$$

$$R_2 := \int_{\mathbb{R}^3}\partial_z((1-\chi_\delta)\phi)(-\Delta)^{-1}\partial_z((1-\chi_\delta)\phi)\,d^3r. \tag{3.15}$$

The rough bound in (3.12) is then an immediate consequence of Lemma 3.1.

To proceed towards the proof of (3.13), we still estimate R_2 roughly:

$$R_2 \leq \|\phi\|^2_{L^2(\Omega \setminus \Omega_{6\delta})} \leq 14 |\partial D| \delta^2 \|\phi\|^2_{L^\infty(\Omega)}, \tag{3.16}$$

where we chose δ so small depending only on D that $|(D + B_\delta) \setminus D_{6\delta}| \leq 14 |\partial D| \delta$ and, hence, $|\Omega \setminus \Omega_{6\delta}| \leq 14 |\partial D| \delta^2$. Focusing on R_1, we write, using again Lemma 3.1 to estimate the first term:

$$
\begin{aligned}
R_1 &\leq \left| \int_{\mathbb{R}^3} \partial_z ((1 - \chi_\delta)\phi)(-\Delta)^{-1} \partial_z ((\chi_\delta - \chi_{3\delta})\phi) \, d^3 r \right| \\
&\quad + \left| \int_{\mathbb{R}^3} \partial_z ((1 - \chi_\delta)\phi)(-\Delta)^{-1} \partial_z (\chi_{3\delta}\phi) \, d^3 r \right| \\
&\leq 14 |\partial D| \delta^2 \|\phi\|^2_{L^\infty(\Omega)} \\
&\quad + \frac{1}{4\pi} \left| \int_{\Omega \setminus \Omega_{2\delta}} \int_{\Omega_{3\delta}} \frac{3(\mathbf{e}_z \cdot (\mathbf{r} - \mathbf{r}'))^2 - |\mathbf{r} - \mathbf{r}'|^2}{|\mathbf{r} - \mathbf{r}'|^5} (1 - \chi_\delta(\mathbf{r})) \chi_{3\delta}(\mathbf{r}') \phi(\mathbf{r}) \phi(\mathbf{r}') \, d^3 r' \, d^3 r \right| \\
&\leq 14 |\partial D| \delta^2 \|\phi\|^2_{L^\infty(\Omega)} + \frac{1}{\pi} \|\phi\|^2_{L^\infty(\Omega)} \int_{\Omega \setminus \Omega_{2\delta}} \left(\int_{\Omega_{3\delta}} \frac{1}{|\mathbf{r} - \mathbf{r}'|^3} \, d^3 r' \right) d^3 r \\
&\leq 14 |\partial D| \delta^2 \|\phi\|^2_{L^\infty(\Omega)} + \frac{1}{\pi} \delta^2 \|\phi\|^2_{L^\infty(\Omega)} \int_{(D + B_\delta) \setminus D_{2\delta}} \left(\int_{\mathbb{R}^2 \setminus B_\delta(\mathbf{r})} \frac{1}{|\mathbf{r} - \mathbf{r}'|^3} \, d^2 r' \right) d^2 r \\
&\leq 42 |\partial D| \delta^2 \|\phi\|^2_{L^\infty(\Omega)}. \tag{3.17}
\end{aligned}
$$

Combining this estimate with (3.16) yields (3.13). □

Let us note that the universal constants appearing in Lemma 3.5 are not intended to be optimal.

We now proceed to establishing existence and regularity of the minimizers of E from (2.4) among all $\bar{\phi} \in H^1(D)$.

Proposition 3.6 *For every $\alpha > 0$ and every $\delta > 0$ such that $\alpha \delta^2 < 1$ there exists a minimizer $\bar{\phi}$ of E in (2.4) among all functions in $H^1(D)$. Furthermore, we have $\bar{\phi} \in C^\infty(D) \cap C^{1,\alpha}(\overline{D})$ for all $\alpha \in (0, 1)$, and $\bar{\phi}$ satisfies for every $\mathbf{r} \in D$:*

$$
\begin{aligned}
0 &= (1 - \alpha \delta^2) \Delta \bar{\phi}(\mathbf{r}) + \bar{\phi}(\mathbf{r}) - \bar{\phi}^3(\mathbf{r}) \\
&\quad + \frac{\gamma \delta}{8\pi} \chi_\delta(\mathbf{r}) \int_{\mathbb{R}^2} \frac{2\chi_\delta(\mathbf{r}) \bar{\phi}(\mathbf{r}) - \chi_\delta(\mathbf{r} - \mathbf{z}) \bar{\phi}(\mathbf{r} - \mathbf{z}) - \chi_\delta(\mathbf{r} + \mathbf{z}) \bar{\phi}(\mathbf{r} + \mathbf{z})}{|\mathbf{z}|^3} \, d^2 z, \tag{3.18}
\end{aligned}
$$

with $\nu \cdot \nabla \bar{\phi}(\mathbf{r}) = 0$ for all $\mathbf{r} \in \partial D$, where ν is the outward unit normal.

Proof The proof is analogous to that of Proposition 3.3 and is simpler, because now $\chi_\delta \bar{\phi} \in H^1(\mathbb{R}^2)$. This means that the non-local term in the energy may be controlled by the $H^1(\mathbb{R}^2)$ norm of $\chi_\delta \bar{\phi}$ [24, Section 7.11], which, in turn, can be controlled by the $H^1(D)$ norm of ϕ. Thus, if $\bar{\phi}_n \in H^1(D)$ is a minimizing sequence, recalling (1.6) we may write

$$
\begin{aligned}
C &\geq E(\bar{\phi}_n) - \frac{1}{4} |D| \\
&= \frac{1}{2}(1 - \alpha \delta^2) \|\nabla \bar{\phi}_n\|^2_{L^2(D)} - \frac{1}{2} \|\bar{\phi}_n\|^2_{L^2(D)} + \frac{1}{4} \|\bar{\phi}_n\|^4_{L^4(D)} - \frac{\gamma \delta}{4} \|\chi_\delta \bar{\phi}_n\|^2_{\mathring{H}^{1/2}(\mathbb{R}^2)} \\
&\geq \frac{1}{2}(1 - \alpha \delta^2) \|\nabla \bar{\phi}_n\|^2_{L^2(D)} - \frac{1}{2} \|\bar{\phi}_n\|^2_{L^2(D)} + \frac{1}{4} \|\bar{\phi}_n\|^4_{L^4(D)} \\
&\quad - \frac{\gamma \delta}{4} \|\chi_\delta \bar{\phi}_n\|_{L^2(\mathbb{R}^2)} \|\nabla (\chi_\delta \bar{\phi}_n)\|_{L^2(\mathbb{R}^2)}
\end{aligned}
$$

$$\geq \frac{1}{2}(1 - \alpha\delta^2)\|\nabla\bar{\phi}_n\|_{L^2(D)}^2 - \frac{1}{2}\|\bar{\phi}_n\|_{L^2(D)}^2 + \frac{1}{4}\|\bar{\phi}_n\|_{L^4(D)}^4$$
$$- \frac{\gamma\delta}{4}\|\bar{\phi}_n\|_{L^2(D)}\left(\|\nabla\bar{\phi}_n\|_{L^2(D)} + \|\nabla\chi_\delta\|_{L^\infty(\mathbb{R}^2)}\|\bar{\phi}_n\|_{L^2(D\setminus D_{2\delta})}\right)$$
$$\geq \frac{1}{2}(1 - \alpha\delta^2)\|\nabla\bar{\phi}_n\|_{L^2(D)}^2 - \frac{1}{2}\|\bar{\phi}_n\|_{L^2(D)}^2 + \frac{1}{4}\|\bar{\phi}_n\|_{L^4(D)}^4$$
$$- \frac{\gamma}{4}\|\bar{\phi}_n\|_{L^2(D)}\left(\delta\|\nabla\bar{\phi}_n\|_{L^2(D)} + 2\|\bar{\phi}_n\|_{L^2(D\setminus D_{2\delta})}\right), \tag{3.19}$$

for some $C > 0$ independent of n and all δ sufficiently small. Therefore, by Cauchy–Schwarz and Young's inequalities we obtain

$$\|\nabla\bar{\phi}_n\|_{L^2(D)}^2 - C_1\|\bar{\phi}_n\|_{L^4(D)}^2 + C_2\|\bar{\phi}_n\|_{L^4(D)}^4 \leq C_3, \tag{3.20}$$

for some $C_1, C_2, C_3 > 0$ independent of n. This yields compactness in $H^1(D)$ which, upon extraction of a subsequence, produces $\bar{\phi} \in H^1(D)$ such that $\bar{\phi}_n \rightharpoonup \bar{\phi}$ in $H^1(D)$, $\bar{\phi}_n \to \bar{\phi}$ in $L^p(D)$ for any $1 \leq p < \infty$ (recall that $D \subset \mathbb{R}^2$ and is bounded), and again by interpolation we have $\bar{\phi}_n \to \bar{\phi}$ in $H^{1/2}(\mathbb{R}^2)$ [24]. Finally, by lower semicontinuity of the gradient squared term, we obtain that $\bar{\phi}$ is a minimizer.

Once existence of a minimizer $\bar{\phi}$ is established, the weak form of (3.18) is obtained by an explicit computation:

$$0 = (1 - \alpha\delta^2)\int_D \nabla\bar{\phi} \cdot \nabla\bar{\psi}\, d^2r + \int_D (\bar{\phi}^3 - \bar{\phi})\bar{\psi}\, d^2r$$
$$- \frac{\gamma\delta}{8\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{(\chi_\delta(\mathbf{r})\bar{\phi}(\mathbf{r}) - \chi_\delta(\mathbf{r}')\bar{\phi}(\mathbf{r}'))(\chi_\delta(\mathbf{r})\bar{\psi}(\mathbf{r}) - \chi_\delta(\mathbf{r}')\bar{\psi}(\mathbf{r}'))}{|\mathbf{r} - \mathbf{r}'|^3}\, d^2r\, d^2r', \tag{3.21}$$

for any $\bar{\psi} \in H^1(D)$. Passing to Fourier space in the last term, we can then interpret this equation distributionally in D:

$$0 = (1 - \alpha\delta^2)\Delta\bar{\phi} + \bar{\phi} - \bar{\phi}^3 + \frac{\gamma\delta}{2}\chi_\delta(-\Delta)^{1/2}(\chi_\delta\bar{\phi}), \tag{3.22}$$

where for test functions the operator $(-\Delta)^{1/2}$ is defined by (1.8) (for a more detailed discussion of various representations of half-Laplacian in $\mathbb{R}^2$, see [25]). Moreover, since $\chi_\delta\bar{\phi} \in H^1(D)$, the last term in (3.22) belongs to $L^2(\mathbb{R}^2)$, and by standard elliptic regularity $\bar{\phi} \in H^2(D)$, with Neumann boundary condition. Applying bootstrap then yields the remaining claims. $\qquad\square$

We finish this section with an estimate for the energy E on a fixed domain D and small δ that will be useful in establishing the asymptotic behavior of the energy for $\delta \to 0$.

Lemma 3.7 *There exists $\delta_0 > 0$ depending only on α and D such that for all $0 < \delta \leq \delta_0$ and all $\bar{\phi} \in H^1(D)$ there holds*

$$E(\bar{\phi}) \geq \frac{1}{8}\|\nabla\bar{\phi}\|_{L^2(D)}^2 - \frac{1}{2}\left(1 + \frac{\gamma^2\delta^2}{4}\right)\|\bar{\phi}\|_{L^2(D)}^2 + \frac{1}{4}\|\bar{\phi}\|_{L^4(D)}^4 + \frac{1}{4}|D|$$
$$- \gamma|\partial D|^{1/4}\delta^{1/4}\|\bar{\phi}\|_{L^2(D)}\|\bar{\phi}\|_{L^4(D)}. \tag{3.23}$$

Proof We argue as in the proof of Proposition 3.6. Taking $\alpha\delta^2 \leq \frac{1}{2}$ and using the estimate in (3.19), with the help of Young's inequality we obtain

$$E(\bar{\phi}) \geq \frac{1}{4}\|\nabla\bar{\phi}\|^2_{L^2(D)} - \frac{1}{2}\|\bar{\phi}\|^2_{L^2(D)} + \frac{1}{4}\|\bar{\phi}\|^4_{L^4(D)} + \frac{1}{4}|D|$$

$$- \frac{\gamma}{4}\|\bar{\phi}\|_{L^2(D)}\left(\delta\|\nabla\bar{\phi}\|_{L^2(D)} + 2\|\bar{\phi}\|_{L^2(D\setminus D_{2\delta})}\right)$$

$$\geq \frac{1}{8}\|\nabla\bar{\phi}\|^2_{L^2(D)} - \frac{1}{2}\|\bar{\phi}\|^2_{L^2(D)} + \frac{1}{4}\|\bar{\phi}\|^4_{L^4(D)} + \frac{1}{4}|D|$$

$$- \frac{\gamma^2\delta^2}{8}\|\bar{\phi}\|^2_{L^2(D)} - \frac{\gamma}{2}\|\bar{\phi}\|_{L^2(D)}\|\bar{\phi}\|_{L^2(D\setminus D_{2\delta})}. \tag{3.24}$$

On the other hand, choosing δ so small that $|D\setminus D_{2\delta}| \leq 16|\partial D|\delta$, by Cauchy–Schwarz inequality we have

$$\|\bar{\phi}\|_{L^2(D\setminus D_{2\delta})} \leq 2|\partial D|^{1/4}\delta^{1/4}\|\bar{\phi}\|_{L^4(D)}. \tag{3.25}$$

Combining (3.25) with (3.24) then yields the result.   □

4 Proof of Theorem 2.1

The main ingredient in the proof of Theorem 2.1 is a careful estimate of the non-local part of the three-dimensional energy $\mathcal{E}$ evaluated on $\chi_\delta\phi$ (to exclude the effect of the edge) in terms of the non-local part of the two-dimensional energy E evaluated on $\chi_\delta\bar{\phi}$, where $\bar{\phi}$ is given by (2.5). The key point is that the difference between the two can be controlled by the gradient squared term in $\mathcal{E}(\phi)$. Note that a similar argument in the periodic setting was recently introduced in [21]. We establish the estimate in the following lemma.

Lemma 4.1 *Let $\phi \in H^1(\Omega)$ be extended by zero to the whole of $\mathbb{R}^3$ and let $\bar{\phi}$ be defined by (2.5). Then*

$$\left|\int_{\mathbb{R}^3}\left(\partial_z(\chi_\delta\phi)(-\Delta)^{-1}\partial_z(\chi_\delta\phi) - \chi_\delta^2\phi^2\right)d^3r + \frac{\delta^2}{2}\int_{\mathbb{R}^2}\chi_\delta\bar{\phi}(-\Delta)^{1/2}\chi_\delta\bar{\phi}\,d^2r\right|$$

$$\leq \frac{\delta^2}{2}\int_\Omega|\nabla(\chi_\delta\phi)|^2\,d^3r. \tag{4.1}$$

Proof To simplify the notations, let us introduce $\psi := \chi_\delta\phi$ and $\bar{\psi} := \chi_\delta\bar{\phi}$. Notice that in view of Lemma 3.1, we can argue by approximation and assume that $\psi \in C_c^\infty(\mathbb{R}^3)$ and $\bar{\psi} \in C_c^\infty(\mathbb{R}^2)$. Next, for each $z \in \mathbb{R}$ define the Fourier transform of $\psi = \psi(x, y, z)$ in the first two variables $(x, y) = \mathbf{r} \in \mathbb{R}^2$:

$$\widehat{\psi}_\mathbf{k}(z) := \int_{\mathbb{R}^2}e^{i\mathbf{k}\cdot\mathbf{r}}\psi(\mathbf{r}, z)\,d^2r \qquad \mathbf{k} \in \mathbb{R}^2. \tag{4.2}$$

We write the three-dimensional dipolar interaction energy (up to a factor) in terms of the associated potential $\varphi \in C^\infty(\mathbb{R}^3)$:

$$\mathcal{E}_d(\psi) := \int_{\mathbb{R}^3}\partial_z\psi(-\Delta)^{-1}\partial_z\psi\,d^3r = -\int_{\mathbb{R}^3}\varphi\,\partial_z\psi\,d^3r, \qquad \Delta\varphi = \partial_z\psi \ \text{ in } \ \mathbb{R}^3. \tag{4.3}$$

Passing to the Fourier space, with the help of Parseval's identity we get

$$\mathcal{E}_d(\psi) = -\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\int_0^\delta\widehat{\varphi}_k^*\partial_z\widehat{\psi}_\mathbf{k}\,dz\,d^2k, \tag{4.4}$$

where we introduced the Fourier transform $\widehat{\varphi}_{\mathbf{k}} = \widehat{\varphi}_{\mathbf{k}}(z)$ of φ, which solves

$$\frac{d^2\widehat{\varphi}_{\mathbf{k}}}{dz^2} - |\mathbf{k}|^2\widehat{\varphi}_{\mathbf{k}} = \partial_z\widehat{\psi}_{\mathbf{k}} \quad z \in \mathbb{R}. \tag{4.5}$$

Introducing the fundamental solution

$$H_{\mathbf{k}}(z) := \frac{e^{-|\mathbf{k}||z|}}{|\mathbf{k}|} \quad \mathbf{k} \in \mathbb{R}^2\backslash\{0\}, \ z \in \mathbb{R}, \tag{4.6}$$

of the ordinary differential equation

$$-\frac{d^2 H_{\mathbf{k}}(z)}{dz^2} + |\mathbf{k}|^2 H_{\mathbf{k}}(z) = 2\delta^{(1)}(z) \quad z \in \mathbb{R}, \tag{4.7}$$

where $\delta^{(1)}(z)$ is the one-dimensional Dirac delta-function, we can write the solution of (4.5) in terms of $H_{\mathbf{k}}(z)$ as

$$\widehat{\varphi}_{\mathbf{k}}(z) = -\frac{1}{2}\int_{\mathbb{R}} H_{\mathbf{k}}(z-z')\partial_z\widehat{\psi}_{\mathbf{k}}(z')dz' \quad z \in \mathbb{R}. \tag{4.8}$$

Thus, we have

$$\mathcal{E}_d(\psi) = \frac{1}{8\pi^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}} \partial_z\widehat{\psi}_{\mathbf{k}}^*(z)H_{\mathbf{k}}(z-z')\partial_z\widehat{\psi}_{\mathbf{k}}(z')dz\,dz'\,d^2k. \tag{4.9}$$

Introduce now $H_{\mathbf{k}}^{(0)}(z-z') := |\mathbf{k}|^{-1}$ and observe that

$$\begin{aligned}\mathcal{E}_d^{(0)}(\psi) &:= \frac{1}{8\pi^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}} \partial_z\widehat{\psi}_{\mathbf{k}}^*(z)H_{\mathbf{k}}^{(0)}(z-z')\partial_z\widehat{\psi}_{\mathbf{k}}(z')dz\,dz'\,d^2k\\ &= \frac{1}{8\pi^2}\int_{\mathbb{R}^2}\frac{1}{|\mathbf{k}|}\int_{\mathbb{R}}\int_{\mathbb{R}} \partial_z\widehat{\psi}_{\mathbf{k}}^*(z)\partial_z\widehat{\psi}_{\mathbf{k}}(z')dz\,dz'\,d^2k = 0.\end{aligned} \tag{4.10}$$

Similarly, with $H_{\mathbf{k}}^{(1)}(z-z') := -|z-z'|$ we have by Parseval's identity

$$\begin{aligned}\mathcal{E}_d^{(1)}(\psi) &= \frac{1}{8\pi^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}} \partial_z\widehat{\psi}_{\mathbf{k}}^*(z)H_{\mathbf{k}}^{(1)}(z-z')\partial_z\widehat{\psi}_{\mathbf{k}}(z')dz\,dz'\,d^2k\\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}} \partial_z\widehat{\psi}_{\mathbf{k}}^*(z)(-\partial_z^2)^{-1}\partial_z\widehat{\psi}_{\mathbf{k}}\,dz\,d^2k\\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}} |\widehat{\psi}_{\mathbf{k}}|^2\,dz\,d^2k = \int_{\mathbb{R}^3}\psi^2 d^3r.\end{aligned} \tag{4.11}$$

In turn, with $H_{\mathbf{k}}^{(2)}(z-z') = \frac{1}{2}|\mathbf{k}|(z-z')^2$ we have

$$\begin{aligned}\mathcal{E}_d^{(2)}(\psi) &= \frac{1}{8\pi^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}} \partial_z\widehat{\psi}_{\mathbf{k}}^*(z)H_{\mathbf{k}}^{(2)}(z-z')\partial_z\widehat{\psi}_{\mathbf{k}}(z')dz\,dz'\,d^2k\\ &= -\frac{1}{8\pi^2}\int_{\mathbb{R}^2}|\mathbf{k}|\int_0^\delta\int_0^\delta \widehat{\psi}_{\mathbf{k}}^*(z)\widehat{\psi}_{\mathbf{k}}(z')\,dz\,dz'\,d^2k\\ &= -\frac{\delta^2}{8\pi^2}\int_{\mathbb{R}^2}|\mathbf{k}|\left|\widehat{\overline{\psi}}_{\mathbf{k}}\right|^2 d^2k = -\frac{\delta^2}{2}\int_{\mathbb{R}^2}\bar{\psi}\,(-\Delta)^{1/2}\,\bar{\psi}\,d^2r.\end{aligned} \tag{4.12}$$

We now estimate the energy difference $\Delta\mathcal{E}_d(\psi) = \mathcal{E}_d(\psi) - \mathcal{E}_d^{(0)}(\psi) - \mathcal{E}_d^{(1)}(\psi) - \mathcal{E}_d^{(2)}(\psi)$. Introduce $I_{\mathbf{k}}(z) := H_{\mathbf{k}}(z) - H_{\mathbf{k}}^{(0)}(z) - H_{\mathbf{k}}^{(1)}(z) - H_{\mathbf{k}}^{(2)}(z)$, and observe that $I_{\mathbf{k}} \in C^2(\mathbb{R})$ and

$$I_{\mathbf{k}}''(z) = \frac{d^2 I_{\mathbf{k}}(z)}{dz^2} = -|\mathbf{k}|\left(1 - e^{-|\mathbf{k}||z|}\right), \tag{4.13}$$

In particular, we have

$$|I_{\mathbf{k}}''(z)| \leq |\mathbf{k}|^2 \delta \qquad \forall |z| \leq \delta. \tag{4.14}$$

Integrating by parts, we express the excess energy in terms of $I_{\mathbf{k}}''(z)$:

$$\Delta \mathcal{E}_d(\psi) = \frac{1}{8\pi^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \partial_z \widehat{\psi}_{\mathbf{k}}^*(z) I_{\mathbf{k}}(z-z') \partial_z \widehat{\psi}_{\mathbf{k}}(z') dz\, dz'\, d^2 k$$
$$= -\frac{1}{8\pi^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{\psi}_{\mathbf{k}}^*(z) I_{\mathbf{k}}''(z-z') \widehat{\psi}_{\mathbf{k}}(z')\, dz\, dz'\, d^2 k. \tag{4.15}$$

Therefore, applying Cauchy–Schwarz inequality and using (4.14), we obtain

$$|\Delta \mathcal{E}_d(\psi)| \leq \frac{1}{8\pi^2} \int_{\mathbb{R}^2} \int_0^\delta \int_0^\delta |\widehat{\psi}_{\mathbf{k}}(z)|\, |I_{\mathbf{k}}''(z-z')|\, |\widehat{\psi}_{\mathbf{k}}(z')|\, dz\, dz'\, d^2 k$$
$$\leq \frac{\delta^2}{8\pi^2} \int_{\mathbb{R}^2} \int_0^\delta |\mathbf{k}|^2 |\widehat{\psi}_{\mathbf{k}}(z)|^2\, dz\, d^2 k \leq \frac{\delta^2}{2} \int_\Omega |\nabla \psi|^2 d^3 r, \tag{4.16}$$

which yields the claim. $\qquad\square$

Proof of Theorem 2.1 We begin with a lower bound and split the energy into the local and the dipolar parts:

$$\mathcal{E}(\phi) = \mathcal{E}_l(\phi) + \frac{\gamma}{2} \left(\mathcal{E}_d(\phi) - \int_{\mathbb{R}^3} \phi^2\, d^3 r \right), \tag{4.17}$$

where $\mathcal{E}_d$ is defined in (4.3). Applying Jensen's inequality to the positive terms, for any $\alpha_1 > 0$, $\alpha_2 > 0$ and $\delta > 0$ such that $(\alpha_1 + \alpha_2 \gamma)\delta^2 < 1$ we have

$$\mathcal{E}_l(\phi) = \int_\Omega \left(\frac{1}{2} |\nabla \phi|^2 - \frac{1}{2} \phi^2 + \frac{1}{4} \phi^4 - \frac{1}{4} \right) d^3 r$$
$$\geq \int_{D \times (0,\delta)} \left(\frac{1}{2}(1 - \alpha_1 \delta^2 - \alpha_2 \gamma \delta^2) |\nabla \bar{\phi}|^2 - \frac{1}{2}\phi^2 + \frac{1}{4}\bar{\phi}^4 - \frac{1}{4} \right) d^3 r$$
$$+ \frac{\delta^2}{2}(\alpha_1 + \alpha_2 \gamma) \int_\Omega |\nabla \phi|^2 d^3 r$$
$$\geq \int_{D \times (0,\delta)} \left(\frac{1}{2}(1 - \alpha_1 \delta^2 - \alpha_2 \gamma \delta^2) |\nabla \bar{\phi}|^2 + \frac{1}{4}\left(1 - \bar{\phi}^2\right)^2 \right) d^3 r$$
$$- \frac{1}{2} \int_{D \times (0,\delta)} (\phi - \bar{\phi})^2 d^3 r + \frac{\delta^2}{2}(\alpha_1 + \alpha_2 \gamma) \int_\Omega |\nabla \phi|^2 d^3 r. \tag{4.18}$$

Therefore, by Poincaré's inequality we obtain

$$\mathcal{E}_l(\phi) \geq \int_{D \times (0,\delta)} \left(\frac{1}{2}(1 - \alpha_1 \delta^2 - \alpha_2 \gamma \delta^2) |\nabla \bar{\phi}|^2 + \frac{1}{4}\left(1 - \bar{\phi}^2\right)^2 \right) d^3 r + \frac{\alpha_2 \gamma \delta^2}{2} \int_\Omega |\nabla \phi|^2 d^3 r, \tag{4.19}$$

with $\alpha_1 = \pi^{-2}$.

Turning now to the dipolar part, we observe that by Lemma 3.5 we have for all δ sufficiently small:

$$\mathcal{E}_d(\phi) \geq \mathcal{E}_d(\chi_\delta \phi) - 98\delta^2 \|\phi\|_{L^\infty(\Omega)}^2 |\partial D|. \tag{4.20}$$

At the same time, by Lemma 4.1 we may write

$$\mathcal{E}_d(\chi_\delta\phi) \geq \int_\Omega \chi_\delta^2\phi^2\,d^3r - \frac{\delta^2}{2}\int_{\mathbb{R}^2}\chi_\delta\bar\phi(-\Delta)^{1/2}\chi_\delta\bar\phi\,d^2r - \frac{\delta^2}{2}\int_\Omega |\nabla(\chi_\delta\phi)|^2 d^3r. \quad (4.21)$$

By Young's inequality, the last term in (4.21) may be estimated as

$$\frac{\delta^2}{2}\int_\Omega |\nabla(\chi_\delta\phi)|^2 d^3r \leq \delta^2\int_\Omega \left(|\nabla\chi_\delta|^2\phi^2 + \chi_\delta^2|\nabla\phi|^2\right)d^3r$$
$$\leq 4\int_{\Omega\backslash\Omega_{2\delta}} \phi^2 d^3r + \delta^2\int_\Omega |\nabla\phi|^2 d^3r. \quad (4.22)$$

Therefore, we have

$$\mathcal{E}_d(\chi_\delta\phi) - \int_\Omega \phi^2 d^3r + \frac{\delta^2}{2}\int_{\mathbb{R}^2}\chi_\delta\bar\phi(-\Delta)^{1/2}\chi_\delta\bar\phi\,d^2r$$
$$\geq -5\|\phi\|^2_{L^\infty(\Omega)}|\Omega\backslash\Omega_{2\delta}| - \delta^2\int_\Omega |\nabla\phi|^2 d^3r. \quad (4.23)$$

Noting that $|\Omega\backslash\Omega_{2\delta}| \leq 6|\partial D|\delta^2$ for all $\delta > 0$ sufficiently small depending only on D and combining (4.23) with (4.20), we finally arrive at

$$\mathcal{E}_d(\phi) - \int_\Omega \phi^2 d^3r + \frac{\delta^2}{2}\int_{\mathbb{R}^2}\chi_\delta\bar\phi(-\Delta)^{1/2}\chi_\delta\bar\phi\,d^2r$$
$$\geq -\beta\delta^2\|\phi\|^2_{L^\infty(\Omega)}|\partial D| - \delta^2\int_\Omega |\nabla\phi|^2 d^3r, \quad (4.24)$$

for some universal $\beta > 0$. The lower bound in (2.6) then follows by combining the above estimate with (4.19) and choosing $\alpha_2 = 1$.

We now proceed to proving (2.7). To begin, we define ϕ in Ω to be a z-independent function, thus, satisfying (2.5) in $D \times (0, \delta)$. Namely, for $(x, y, z) \in D \times (0, \delta)$, we define $\phi(x, y, z) := \bar\phi(x, y)$. Next, we extend ϕ to the rest of Ω by a reflection about $\partial D \times (0, \delta)$. More precisely, for $\mathbf{r} \in \mathbb{R}^2$ define $\rho(\mathbf{r}) := \text{dist}(\mathbf{r}, \mathbb{R}^2\backslash D) - \text{dist}(\mathbf{r}, D)$ to be the signed distance function to ∂D in the plane. Then, for all δ sufficiently small depending only on D there is a tubular neighborhood of ∂D in which we can define a continuous unit outward normal vector ν to the projection on ∂D, i.e., we have $\mathbf{r} + \rho(\mathbf{r})\nu(\mathbf{r}) \in \partial D$ for all $\mathbf{r} \in D$ such that $|\rho(\mathbf{r})| \leq \delta$ and all $0 < \delta \leq \delta_0$ for some $\delta_0 > 0$ depending only on D. We then define for $\mathbf{r} = (x, y) \in \mathbb{R}^2\backslash D$ and all $z \in (0, \delta)$ the extension of $\bar\phi$ as $\phi(x, y, z) := \bar\phi(\mathbf{r} + 2\nu(\mathbf{r})\rho(\mathbf{r}))$. In view of the regularity of ∂D, we then have

$$\int_{\Omega\backslash(D\times(0,\delta))} |\nabla\phi|^2 d^3r \leq 2\delta\int_{D\backslash D_\delta} |\nabla\bar\phi|^2 d^2r, \quad (4.25)$$

for δ_0 sufficiently small depending only on D.

We now use positivity of different terms in the energy and Lemma 3.1 to estimate

$$(1 - 2\alpha\delta^2)\mathcal{E}(\phi) \leq \delta\int_D \left(\frac{1}{2}(1-\alpha\delta^2)|\nabla\bar\phi|^2 + \frac{1}{4}\left(1-\bar\phi^2\right)^2\right)d^2r$$
$$+ \int_{\Omega\backslash(D\times(0,\delta))} \left(\frac{1}{2}|\nabla\phi|^2 + \frac{1}{4}\left(1-\phi^2\right)^2\right)d^3r + \frac{\gamma}{2}\int_{\mathbb{R}^3}\left(\partial_z\phi(-\Delta)^{-1}\partial_z\phi - \phi^2\right)d^3r$$
$$+ \gamma\alpha\delta^2\int_\Omega \phi^2 d^3r - \frac{\alpha\delta^2}{2}\int_\Omega |\nabla\phi|^2 d^3r. \quad (4.26)$$

Accordingly, possibly increasing the values of α_1, α_2 and β, by Lemmas 3.5 and 4.1, Young's inequality and using (4.25), we may write

$$
\begin{aligned}
(1 - 2\alpha\delta^2)\mathcal{E}(\phi) \leq & E(\bar{\phi})\delta + \delta \int_{D \setminus D_\delta} |\nabla \bar{\phi}|^2 d^2 r \\
& + \beta\delta^2(1 + \gamma^2)\left(1 + \|\phi\|_{L^\infty(\Omega)}^4\right)(|\partial D| + |D|\delta).
\end{aligned} \tag{4.27}
$$

The result then follows by possibly further decreasing the value of δ_0 and increasing the value of β.

5 Proof of Theorem 2.2

We begin by establishing compactness of sequences satisfying (2.12). As in the statement of Theorem 2.2, for $\phi_\delta \in H^1(\Omega^\delta)$ we define $\bar{\phi}_\delta \in H^1(D)$ to be given by (2.5) with ϕ replaced by ϕ_δ. We also define Ω_δ^δ, etc., to be given by (2.3) with Ω replaced by Ω^δ.

Proposition 5.1 *For a sequence of $\delta \to 0$, assume $\phi_\delta \in H^1(\Omega^\delta)$ satisfies (2.12). Then, we have $\|\nabla\phi_\delta\|_{L^2(\Omega^\delta)}^2 \leq C\delta$ for some $C > 0$ independent of δ, and upon extraction of a subsequence $\bar{\phi}_\delta \rightharpoonup \bar{\phi}$ in $H^1(D)$ and $\bar{\phi}_\delta \to \bar{\phi}$ in $L^p(D)$ for any $1 \leq p < \infty$.*

Proof By Corollary 3.2 and Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
C\delta \geq \mathcal{E}_\delta(\phi_\delta) \geq & \int_{\Omega^\delta} \left(\frac{1}{2}|\nabla\phi_\delta|^2 - \frac{1}{2}(1+\gamma)\phi_\delta^2 + \frac{1}{4}\phi_\delta^4\right) d^3 r \\
\geq & \int_{\Omega^\delta} \left(\frac{1}{2}|\nabla\phi_\delta|^2 + \frac{1}{4}\phi_\delta^4\right) d^3 r - \frac{1}{2}(1+\gamma)|\Omega^\delta|^{1/2}\left(\int_{\Omega^\delta}\phi_\delta^4 d^3 r\right)^{1/2} \\
\geq & \frac{1}{4}\int_{\Omega^\delta}|\nabla\phi_\delta|^2 d^3 r + \frac{1}{4}\int_D \int_0^\delta |\nabla'\phi_\delta|^2 dz\, d^2 r + \frac{1}{8}\int_{\Omega^\delta}\phi_\delta^4 d^3 r - \frac{1}{2}(1+\gamma)^2|\Omega^\delta|,
\end{aligned} \tag{5.1}
$$

where $\nabla' = (\partial_x, \partial_y, 0)$, for some $C > 0$ independent of δ. On the other hand, for all δ sufficiently small we have $|\Omega^\delta| \leq 2|D|\delta$. Hence, by Jensen's inequality and arguing by approximation we have

$$
\begin{aligned}
C + (1+\gamma)^2|D| \geq & \frac{1}{4\delta}\int_{\Omega^\delta}|\nabla\phi_\delta|^2 d^3 r + \frac{1}{4\delta^2}\int_D \left|\int_0^\delta \nabla'\phi_\delta\, dz\right|^2 d^2 r + \frac{1}{8}\int_D \bar{\phi}_\delta^4\, d^2 r \\
\geq & \frac{1}{4\delta}\int_{\Omega^\delta}|\nabla\phi_\delta|^2 d^3 r + \frac{1}{4}\int_D |\nabla\bar{\phi}_\delta|^2 d^2 r + \frac{1}{8|D|}\left(\int_D \bar{\phi}_\delta^2 d^2 r\right)^2, \tag{5.2}
\end{aligned}
$$

where we again used Cauchy–Schwarz inequality in the last step. Thus, the sequence of $\delta^{-1/2}|\nabla\phi_\delta|$ is bounded in $L^2(\Omega^\delta)$, the sequence of $\bar{\phi}_\delta$ is bounded in $H^1(D)$, and by compact embedding there exists a subsequence with the desired properties. □

We now turn to the proof of Theorem 2.2. We note that if we also assume that $\|\phi_\delta\|_{L^\infty(\Omega^\delta)} \leq M$ for some $M > 0$ independent of δ, we could immediately combine the result of Theorem 2.1 with the result of the following proposition to obtain the claim (however, see Remark 3.4). Below E_δ denotes the energy E from (2.4) with the dependence on δ highlighted.

Proposition 5.2 *Let $\bar{\phi}_\delta \in H^1(D)$, and assume that for a sequence of $\delta \to 0$ we have $\bar{\phi}_\delta \to \bar{\phi}$ in $L^2(D)$. Then*

$$\liminf_{\delta \to 0} E_\delta(\bar{\phi}_\delta) \geq E_0(\bar{\phi}). \tag{5.3}$$

Conversely, for any $\bar{\phi} \in H^1(D)$ we have

$$\limsup_{\delta \to 0} E_\delta(\bar{\phi}) \leq E_0(\bar{\phi}). \tag{5.4}$$

Proof Without loss of generality, we may assume that

$$\limsup_{\delta \to 0} E_\delta(\bar{\phi}_\delta) < +\infty. \tag{5.5}$$

Then, by Lemma 3.7 and Young's inequality the sequence of $\bar{\phi}_\delta$ is bounded in $H^1(D)$ and $L^4(D)$. Therefore, upon extraction of a subsequence, we also have $\bar{\phi}_\delta \rightharpoonup \bar{\phi}$ in $H^1(D)$. Arguing as in the proof of Lemma 3.7, by lower semicontinuity of the $H^1(D)$ and $L^4(D)$ norms as well as strong convergence in $L^2(D)$ we then obtain (5.3). Lastly, to obtain (5.4) we simply note that $E_\delta(\bar{\phi}) \leq E_0(\bar{\phi})$. □

Proof of Theorem 2.2 The proof follows closely the arguments of the proof of Theorem 2.1, except we only apply the rough bound in (3.12). The local part of the energy may be estimated exactly as in (4.19). For the non-local part, we apply the first part of Lemma 3.5. This leads to a lower bound

$$\mathcal{E}_\delta(\phi_\delta) \geq E_\delta(\bar{\phi}_\delta)\delta - 2\gamma \|\phi_\delta\|_{L^2(\Omega^\delta)} \|\phi_\delta\|_{L^2(\Omega^\delta \setminus \Omega_{2\delta}^\delta)} + \frac{1}{4} \int_{\Omega^\delta \setminus (D \times (0,\delta))} \left(1 - \phi_\delta^2\right)^2 d^3 r. \tag{5.6}$$

To proceed, we note that by Poincaré's inequality we have

$$\|\phi_\delta\|_{L^2(D \times (0,\delta))} \leq \delta^{1/2} \|\bar{\phi}_\delta\|_{L^2(D)} + \frac{\delta}{\pi} \|\nabla \phi_\delta\|_{L^2(D \times (0,\delta))}, \tag{5.7}$$

and a similar estimate holds for $\|\phi_\delta\|_{L^2((D \setminus D_{2\delta}) \times (0,\delta))}$. Therefore, from our assumption on the gradient of ϕ_δ we obtain

$$\|\phi_\delta\|_{L^2((D \times (0,\delta))} \leq \delta^{1/2} \|\bar{\phi}_\delta\|_{L^2(D)} + C\delta^{3/2}, \tag{5.8}$$

$$\|\phi_\delta\|_{L^2((D \setminus D_{2\delta}) \times (0,\delta))} \leq \delta^{1/2} \|\bar{\phi}_\delta\|_{L^2(D \setminus D_{2\delta})} + C\delta^{3/2}, \tag{5.9}$$

for some $C > 0$ independent of δ. Using these estimates, we get

$$\begin{aligned}
\|\phi_\delta\|_{L^2(\Omega^\delta)} \|\phi_\delta\|_{L^2(\Omega^\delta \setminus \Omega_{2\delta}^\delta)} \leq & \|\phi_\delta\|_{L^2(\Omega^\delta \setminus (D \times (0,\delta)))}^2 \\
& + 2\delta^{1/2} \|\phi_\delta\|_{L^2(\Omega^\delta \setminus (D \times (0,\delta)))} (\|\bar{\phi}_\delta\|_{L^2(D)} + C\delta) \\
& + \delta(\|\bar{\phi}_\delta\|_{L^2(D)} + C\delta)(\|\bar{\phi}_\delta\|_{L^2(D \setminus D_{2\delta})} + C\delta).
\end{aligned} \tag{5.10}$$

Therefore, since $\bar{\phi}_\delta \to \bar{\phi}$ in $L^2(D)$, there is $C > 0$ such that

$$\begin{aligned}
\|\phi_\delta\|_{L^2(\Omega^\delta)} \|\phi_\delta\|_{L^2(\Omega^\delta \setminus \Omega_{2\delta}^\delta)} \leq & \|\phi_\delta\|_{L^2(\Omega^\delta \setminus (D \times (0,\delta)))}^2 + C\delta^{1/2} \|\phi_\delta\|_{L^2(\Omega^\delta \setminus (D \times (0,\delta)))} \\
& + C\delta(\|\bar{\phi}_\delta\|_{L^2(D \setminus D_{2\delta})} + \delta),
\end{aligned} \tag{5.11}$$

for all δ sufficiently small. Thus, by Cauchy–Schwarz inequality we have

$$\|\phi_\delta\|_{L^2(\Omega^\delta)}\|\phi_\delta\|_{L^2(\Omega^\delta\setminus\Omega_{2\delta}^\delta)} \leq C\delta(\|\phi_\delta\|_{L^4(\Omega^\delta\setminus(D\times(0,\delta)))}^2$$
$$+ \|\phi_\delta\|_{L^4(\Omega^\delta\setminus(D\times(0,\delta)))} + \|\bar\phi_\delta\|_{L^2(D\setminus D_{2\delta})} + \delta), \tag{5.12}$$

for some $C > 0$ and all δ small enough.

On the other hand, for δ sufficiently small depending only on D we have

$$\frac{1}{4}\int_{\Omega^\delta\setminus(D\times(0,\delta))}\left(1-\phi_\delta^2\right)^2 d^3r \geq \int_{\Omega^\delta\setminus(D\times(0,\delta))}\left(\frac{1}{8}\phi_\delta^4 - 1\right)d^3r$$
$$\geq -2|\partial D|\delta^2 + \frac{1}{8}\|\phi_\delta\|_{L^4(\Omega^\delta\setminus(D\times(0,\delta)))}^4. \tag{5.13}$$

Combining this estimate with (5.12) and (5.6), we then get

$$\mathcal{E}_\delta(\phi_\delta) - E_\delta(\bar\phi_\delta)\delta \geq \frac{1}{8}\|\phi_\delta\|_{L^4(\Omega^\delta\setminus(D\times(0,\delta)))}^4 - C\delta(\|\phi_\delta\|_{L^4(\Omega^\delta\setminus(D\times(0,\delta)))}^2$$
$$+ \|\phi_\delta\|_{L^4(\Omega^\delta\setminus(D\times(0,\delta)))} + \|\bar\phi_\delta\|_{L^2(D\setminus D_{2\delta})} + \delta)$$
$$\geq -C'\delta^{4/3} - C\delta\|\bar\phi_\delta\|_{L^2(D\setminus D_{2\delta})}, \tag{5.14}$$

for some $C, C' > 0$ and all δ small enough. The lower bound in (2.10) then follows from Proposition 5.2 and the fact that $\|\bar\phi_\delta\|_{L^2(D\setminus D_{2\delta})} \to 0$ as $\delta \to 0$. The latter is an immediate consequence of the strong convergence of $\bar\phi_\delta$ to $\bar\phi$ in $L^2(D)$.

For the upper bound, we use the same construction as in the proof of Theorem 2.1. Let $\phi_\delta \in H^1(\Omega^\delta)$ be the function obtained from a given $\bar\phi \in H^1(D)$ in this way. Note that by construction we have

$$\|\phi_\delta\|_{L^2(\Omega^\delta\setminus(D\times(0,\delta)))}^2 \leq 2\delta\|\bar\phi\|_{L^2(D\setminus D_\delta)}^2, \tag{5.15}$$

$$\|\phi_\delta\|_{L^4(\Omega^\delta\setminus(D\times(0,\delta)))}^4 \leq 2\delta\|\bar\phi\|_{L^4(D\setminus D_\delta)}^4, \tag{5.16}$$

$$\|\nabla\phi_\delta\|_{L^2(\Omega^\delta\setminus(D\times(0,\delta)))}^2 \leq 2\delta\|\nabla\bar\phi\|_{L^2(D\setminus D_\delta)}^2, \tag{5.17}$$

for all δ sufficiently small depending only on D. In particular, we have $\|\nabla\phi_\delta\|_{L^2(\Omega^\delta)}^2 \leq C\delta$ for some $C > 0$ independent of δ. By Lemmas 3.5 and 4.1, we get

$$(1 - 2\alpha\delta^2)\mathcal{E}_\delta(\phi_\delta) \leq E_\delta(\bar\phi)\delta + 2\gamma\|\phi_\delta\|_{L^2(\Omega^\delta)}\|\phi_\delta\|_{L^2(\Omega^\delta\setminus\Omega_{2\delta}^\delta)} + \gamma\alpha\delta^2\|\phi\|_{L^2(\Omega_\delta^\delta)}^2$$
$$+ \int_{\Omega^\delta\setminus(D\times(0,\delta))}\left(\frac{1}{2}|\nabla\phi_\delta|^2 + \frac{1}{4}(1-\phi_\delta^2)^2\right)d^3r. \tag{5.18}$$

Therefore, for δ sufficiently small depending only on D we obtain

$$(1 - 2\alpha\delta^2)\mathcal{E}_\delta(\phi_\delta) \leq E_\delta(\bar\phi)\delta + 6\gamma\delta\|\bar\phi\|_{L^2(D)}\|\bar\phi\|_{L^2(D\setminus D_{2\delta})} + \gamma\alpha\delta^3\|\bar\phi\|_{L^2(D)}^2$$
$$+ \delta\int_{D\setminus D_\delta}\left(|\nabla\bar\phi|^2 + 1 + \bar\phi^4\right)d^2r. \tag{5.19}$$

Note that the integral in the right-hand side of (5.19) vanishes as $\delta \to 0$, since $\bar\phi \in H^1(D) \subset L^4(D)$ by Sobolev embedding. Similarly, $\|\bar\phi\|_{L^2(D\setminus D_{2\delta})} \to 0$ as $\delta \to 0$. Thus, the estimate in (2.11) follows by Proposition 5.2. $\qquad\square$

6 Rest of the proofs

We begin this section by presenting a brief demonstration of Corollary 2.6. Assume Theorem 2.4 holds true. We use $\bar{\phi} \equiv 1$ as an admissible test function for E_* to estimate the minimum energy from above. Then, if ϕ_ε is a minimizer of $\mathcal{E}_\varepsilon$ and $\bar{\phi}_\varepsilon$ is its z-average given by (2.16), by (2.15) and (2.17) we have

$$E_\varepsilon(\bar{\phi}_\varepsilon)\delta_\varepsilon \leq \mathcal{E}_\varepsilon(\phi_\varepsilon) + O(\delta_\varepsilon^2) \leq (1 - 2\alpha\delta_\varepsilon^2)^{-1} E_*(1)\delta_\varepsilon + o(\delta_\varepsilon), \tag{6.1}$$

as $\varepsilon \to 0$. Thus, by the Γ-convergence of E_ε to E_* we get

$$\frac{1}{2}(\sigma_0 - \sigma_1\lambda) \int_D |\nabla\bar{\phi}_\varepsilon|\, d^2r \to 0 \quad \text{as} \quad \varepsilon \to 0, \tag{6.2}$$

and in view of the fact that $\bar{\phi}_\varepsilon = 1$ in $D\backslash D_\rho$, we have $\bar{\phi}_\varepsilon \to 1$ in $BV(D)$.

The proof of Theorem 2.4 relies on a key interpolation lemma that goes back to [9] and is generalized in [21], all in the periodic setting, to estimate the homogeneous $H^{1/2}$ norm of $\bar{\phi}$ from above by the L^∞ and the BV norms of $\bar{\phi}$. As was already pointed out in [9], this is impossible without an additional penalty term due to the "logarithmic failure" of the corresponding embedding [9]. Here we use the approach of [21] to extend a version of the estimate in [21, Lemma 4.1] to our setting, noting that we need a nonlinear version of [21, Lemma 4.1] in order to combine it with the Modica-Mortola lower bound for the local part of the energy.

Lemma 6.1 *Let* $\bar{\phi} \in H^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ *be such that* $\|\bar{\phi}\|_{L^\infty(\mathbb{R}^2)} \leq 1$ *and* $\mathrm{supp}(\bar{\phi}) \in B_R$. *Then*

$$\frac{1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(\bar{\phi}(\mathbf{r}) - \bar{\phi}(\mathbf{r}'))^2}{|\mathbf{r} - \mathbf{r}'|^3}\, d^2r\, d^2r'$$
$$\leq \frac{3}{\pi} \ln\left(\frac{R}{r}\right) \|\nabla\left(\bar{\phi} - \tfrac{1}{3}\bar{\phi}^3\right)\|_{L^1(\mathbb{R}^2)} + r\|\nabla\bar{\phi}\|_{L^2(\mathbb{R}^2)}^2 + \pi R, \tag{6.3}$$

for any $r \in (0, R)$.

Proof The proof is a close adaptation of the proof of [21, Lemma 4.1]. Write the integral in (6.3) as

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{(\bar{\phi}(\mathbf{r}) - \bar{\phi}(\mathbf{r}'))^2}{|\mathbf{r} - \mathbf{r}'|^3}\, d^2r\, d^2r' = \int_{B_{2R}}\int_{B_{2R}} \frac{(\bar{\phi}(\mathbf{r}) - \bar{\phi}(\mathbf{r}'))^2}{|\mathbf{r} - \mathbf{r}'|^3}\, d^2r\, d^2r'$$
$$+ 2\int_{B_{2R}}\int_{\mathbb{R}^2\backslash B_{2R}} \frac{\bar{\phi}^2(\mathbf{r})}{|\mathbf{r} - \mathbf{r}'|^3}\, d^2r'\, d^2r$$
$$\leq \int_{B_{2R}}\int_{B_{4R}} \frac{(\bar{\phi}(\mathbf{r} + \mathbf{z}) - \bar{\phi}(\mathbf{r}))^2}{|\mathbf{z}|^3}\, d^2z\, d^2r$$
$$+ 2\int_{B_R}\int_{\mathbb{R}^2\backslash B_R} \frac{\bar{\phi}^2(\mathbf{r})}{|\mathbf{z}|^3}\, d^2z\, d^2r$$
$$= \int_{B_{2R}}\int_{B_{4R}} \frac{(\bar{\phi}(\mathbf{r} + \mathbf{z}) - \bar{\phi}(\mathbf{r}))^2}{|\mathbf{z}|^3}\, d^2z\, d^2r + \frac{4\pi}{R}\|\bar{\phi}\|_{L^2(\mathbb{R}^2)}^2. \tag{6.4}$$

Focusing now on the first term above, we observe that by Jensen's inequality we have for all $\mathbf{z} \in B_{4R}$:

$$\int_{B_{2R}} (\bar{\phi}(\mathbf{r} + \mathbf{z}) - \bar{\phi}(\mathbf{r}))^2 d^2r \leq \int_{B_{2R}} \int_0^1 |\mathbf{z} \cdot \nabla \bar{\phi}(\mathbf{r} + t\mathbf{z})|^2 \, dt \, d^2r \leq \int_{B_{6R}} |\mathbf{z} \cdot \nabla \bar{\phi}(\mathbf{r})|^2 \, d^2r.$$
(6.5)

Similarly, introducing $\bar{\psi} := \bar{\phi} - \frac{1}{3}\bar{\phi}^3$, we have

$$\int_{B_{2R}} (\bar{\phi}(\mathbf{r} + \mathbf{z}) - \bar{\phi}(\mathbf{r}))^2 d^2r$$

$$\leq \int_{B_{2R} \cap \{\bar{\phi}(\mathbf{r}+\mathbf{z}) \neq \bar{\phi}(\mathbf{r})\}} \frac{(\bar{\phi}(\mathbf{r} + \mathbf{z}) - \bar{\phi}(\mathbf{r}))^2}{|\bar{\phi}(\mathbf{r} + \mathbf{z}) - \frac{1}{3}\bar{\phi}^3(\mathbf{r} + \mathbf{z}) - \bar{\phi}(\mathbf{r}) + \frac{1}{3}\bar{\phi}^3(\mathbf{r})|} \int_0^1 |\mathbf{z} \cdot \nabla \bar{\psi}(\mathbf{r} + t\mathbf{z})| \, dt \, d^2r$$

$$\leq 3 \int_{B_{6R}} |\mathbf{z} \cdot \nabla \bar{\psi}(\mathbf{r})| \, d^2r,$$
(6.6)

where we used the fact that

$$\left| \frac{(s - t)^2}{s - \frac{1}{3}s^3 - t + \frac{1}{3}t^3} \right| \leq 3 \quad \forall (s, t) \in (-1, 1)^2, \ s \neq t,$$
(6.7)

which can be readily verified by means of elementary calculus. Indeed, for every $-1 < s < t < 1$ we have

$$F(s, t) := \frac{(s - t)^2}{s - \frac{1}{3}s^3 - t + \frac{1}{3}t^3} = \frac{3(s - t)}{3 - t^2 - ts - s^2},$$
(6.8)

and taking partial derivatives, we obtain

$$\frac{\partial F}{\partial t} = -\frac{9(1 - s^2) + 3(s - t)^2}{(3 - s^2 - st - t^2)^2} < 0, \quad \frac{\partial F}{\partial s} = \frac{9(1 - t^2) + 3(s - t)^2}{(3 - s^2 - st - t^2)^2} > 0.$$
(6.9)

Hence $0 > F(s, t) > F(-1, 1) = -3$ for all $-1 < s < t < 1$. Since $F(s, t) = -F(t, s)$, we conclude that $|F(s, t)| \leq 3$ for all $(s, t) \in (-1, 1)^2$ with $s \neq t$.

Now, splitting the integral over $\mathbf{z}$ in (6.4) into a near-field part and a far-field part and using (6.5) and (6.6) to estimate the respective pieces, we get for any $0 < r < R$:

$$\int_{B_{2R}} \int_{B_{4R}} \frac{(\bar{\phi}(\mathbf{r} + \mathbf{z}) - \bar{\phi}(\mathbf{r}))^2}{|\mathbf{z}|^3} \, d^2z \, d^2r$$

$$\leq \int_{B_{4r}} \int_{B_{6R}} \frac{|\mathbf{z} \cdot \nabla \bar{\phi}(\mathbf{r})|^2}{|\mathbf{z}|^3} \, d^2r \, d^2z + 3 \int_{B_{4R} \setminus B_{4r}} \int_{B_{6R}} \frac{|\mathbf{z} \cdot \nabla \bar{\psi}(\mathbf{r})|}{|\mathbf{z}|^3} \, d^2r \, d^2z$$

$$\leq \int_0^{2\pi} \int_0^{4r} \int_{\mathbb{R}^2} \cos^2 \theta \, |\nabla \bar{\phi}(\mathbf{r})|^2 \, d^2r \, ds \, d\theta + 3 \int_0^{2\pi} \int_{4r}^{4R} \int_{\mathbb{R}^2} \frac{|\cos \theta|}{s} |\nabla \bar{\psi}(\mathbf{r})| \, d^2r \, ds \, d\theta$$

$$= 4\pi r \|\nabla \bar{\phi}\|_{L^2(\mathbb{R}^2)}^2 + 12 \ln\left(\frac{R}{r}\right) \|\nabla \bar{\psi}\|_{L^1(\mathbb{R}^2)},$$
(6.10)

where in the next-to-last line we passed to polar coordinates (s, θ) for $\mathbf{z}$. Then, combining this estimate with (6.4), we obtain the result. □

We point out that, importantly, the constant in front of the logarithm in Lemma 6.1 is the best possible one (as was already observed in [9,21] in a slightly different setting), which can be easily seen by considering the characteristic function of $B_{R/2}$ mollified at scale r as a test function, provided that r is small enough.

We will also need a slightly modified version of Lemma 6.1.

Lemma 6.2 *Let $\bar{\phi} \in L^\infty(\mathbb{R}^2) \cap H^1_{loc}(\mathbb{R}^2)$ be such that $\bar{\phi} = 1$ in $\mathbb{R}^2 \setminus D$ and $\|\bar{\phi}\|_{L^\infty(\mathbb{R}^2)} = 1$. Then*

$$\frac{1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(\bar{\phi}(\mathbf{r}) - \bar{\phi}(\mathbf{r}'))^2}{|\mathbf{r} - \mathbf{r}'|^3} \, d^2r \, d^2r'$$

$$\leq \frac{3}{\pi} \ln\left(\frac{R}{r}\right) \|\nabla\left(\bar{\phi} - \tfrac{1}{3}\bar{\phi}^3\right)\|_{L^1(\mathbb{R}^2)} + r\|\nabla\bar{\phi}\|^2_{L^2(\mathbb{R}^2)} + 4\pi R, \qquad (6.11)$$

for some $R > 0$ and all $r \in (0, R)$.

The proof of Lemma 6.2 is identical to that of Lemma 6.1. We note that the left-hand side in (6.11) makes sense because $\bar{\phi} - 1 \in H^1(\mathbb{R}^2)$ and can be interpreted as the homogeneous $H^{1/2}$ norm squared of $\bar{\phi} - 1$.

Proof of Theorem 2.4 We begin with the proof of compactness. Let $\bar{\phi}_\varepsilon$ be as in part (i) of Theorem 2.4 and define $\bar{\psi}_\varepsilon := \bar{\phi}_\varepsilon - \tfrac{1}{3}\bar{\phi}^3_\varepsilon$. Using the Modica-Mortola trick [27] and weak chain rule [12], we write for all ε sufficiently small

$$E_\varepsilon(\bar{\phi}_\varepsilon) \geq \frac{(\lambda_c + \lambda)\sqrt{1 - \alpha\delta^2_\varepsilon}}{2\lambda_c\sqrt{2}} \int_D |\nabla\bar{\psi}_\varepsilon| \, d^2r$$

$$+ \frac{\varepsilon(\lambda_c - \lambda)(1 - \alpha\delta^2_\varepsilon)}{4\lambda_c} \int_D |\nabla\bar{\phi}_\varepsilon|^2 \, d^2r + \frac{\lambda_c - \lambda}{8\varepsilon\lambda_c} \int_D (1 - \bar{\phi}^2_\varepsilon)^2 \, d^2r$$

$$- \frac{\lambda}{16\pi|\ln\varepsilon|} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(\chi_{\varepsilon\delta_\varepsilon}(\mathbf{r})\bar{\phi}_\varepsilon(\mathbf{r}) - \chi_{\varepsilon\delta_\varepsilon}(\mathbf{r}')\bar{\phi}_\varepsilon(\mathbf{r}'))^2}{|\mathbf{r} - \mathbf{r}'|^3} \, d^2r \, d^2r'. \qquad (6.12)$$

Applying Lemma 6.1 with $r = \varepsilon\delta_\varepsilon$ and R sufficiently large independent of ε, we get

$$E_\varepsilon(\bar{\phi}_\varepsilon) + C\|\nabla\chi_{\varepsilon\delta_\varepsilon}\|_{L^1(D)} + C\varepsilon\delta^2_\varepsilon\|\nabla\chi_{\varepsilon\delta_\varepsilon}\|^2_{L^2(D)} + C\delta_\varepsilon$$

$$\geq \frac{1}{4}(\sigma_0 - \lambda\sigma_1) \int_D |\nabla\bar{\psi}_\varepsilon| \, d^2r + \frac{\lambda_c - \lambda}{8\varepsilon\lambda_c} \int_D (1 - \bar{\phi}^2_\varepsilon)^2 \, d^2r, \qquad (6.13)$$

for some $C > 0$ and all ε small enough. Here we used the fact that $\bar{\phi} = 1$ on $D \setminus D_\rho$ and that to the leading order the coefficient of $\int_D |\nabla\bar{\psi}_\varepsilon| \, d^2r$ in the right-hand side equals $\frac{\lambda_c + \lambda}{2\lambda_c\sqrt{2}} - \frac{3\lambda}{4\pi} = \frac{3}{8}(\sigma_0 - \lambda\sigma_1) > \frac{1}{4}(\sigma_0 - \lambda\sigma_1)$, allowing to absorb the additional logarithmic terms coming from Lemma 6.1 for $\lambda < \lambda_c$. Since the left-hand side of the above expression is bounded as $\varepsilon \to 0$, we obtain, upon extraction of a subsequence, that $|\bar{\phi}_\varepsilon| \to 1$ in $L^1(D)$ and a.e. in D. Furthermore, by compactness in BV [12], we have, upon extraction of another subsequence, that $\bar{\psi}_\varepsilon \rightharpoonup \bar{\psi}$ in $BV(D)$, and $\bar{\psi}_\varepsilon \to \bar{\psi}$ in $L^1(D)$ and a.e. in D, with $|\bar{\psi}| = \frac{2}{3}$ a.e. in D. Thus, we get that $\bar{\psi} \in BV(D; \{-\frac{2}{3}, \frac{2}{3}\})$, which, in turn, implies that $\bar{\phi}_\varepsilon \to \bar{\phi}$ in $L^1(D)$ with $\bar{\phi} = \frac{3}{2}\bar{\psi} \in BV(D; \{-1, 1\})$. Also, clearly $\bar{\phi} = \bar{\phi}_\varepsilon = 1$ in $D \setminus D_\rho$.

We now prove the lower bound in (2.23). To that end, we make the estimate in (6.13) quantitative by isolating the contribution of the edge to the non-local energy. We redefine $\bar{\phi}(x) := 1$ for all $x \in \mathbb{R}^2 \setminus D$ and introduce

$$E^0_\varepsilon(\bar{\phi}) := \int_D \left(\frac{\varepsilon}{2}\left(1 - \alpha\delta^2\right)|\nabla\bar{\phi}|^2 + \frac{1}{4\varepsilon}\left(1 - \bar{\phi}^2\right)^2\right) d^2r$$

$$- \frac{\lambda}{16\pi|\ln\varepsilon|} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(\bar{\phi}(\mathbf{r}) - \bar{\phi}(\mathbf{r}'))^2}{|\mathbf{r} - \mathbf{r}'|^3} \, d^2r \, d^2r', \qquad (6.14)$$

which represents the energy E_ε without the contribution of the edges. Then, since by our assumption $\bar{\phi}_\varepsilon = 1$ in $\mathbb{R}^2 \setminus D_\rho$, we have $\chi_{\varepsilon\delta_\varepsilon} \bar{\phi}_\varepsilon = \bar{\phi}_\varepsilon - 1 + \chi_{\varepsilon\delta_\varepsilon}$ for all ε sufficiently small and, therefore,

$$E_\varepsilon(\bar{\phi}_\varepsilon) = E_\varepsilon^0(\bar{\phi}_\varepsilon) - \frac{\lambda}{16\pi \, |\ln \varepsilon|} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(\chi_{\varepsilon\delta_\varepsilon}(\mathbf{r}) - \chi_{\varepsilon\delta_\varepsilon}(\mathbf{r}'))^2}{|\mathbf{r} - \mathbf{r}'|^3} \, d^2r \, d^2r'$$
$$- \frac{\lambda}{8\pi \, |\ln \varepsilon|} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(\bar{\phi}_\varepsilon(\mathbf{r}) - \bar{\phi}_\varepsilon(\mathbf{r}'))(\chi_{\varepsilon\delta_\varepsilon}(\mathbf{r}) - \chi_{\varepsilon\delta_\varepsilon}(\mathbf{r}'))}{|\mathbf{r} - \mathbf{r}'|^3} \, d^2r \, d^2r'. \quad (6.15)$$

Using Lemma 6.2 and arguing as in (6.13), we can estimate

$$E_\varepsilon^0(\bar{\phi}_\varepsilon) \geq \frac{3}{4}(\sigma_0 - \lambda\sigma_1) \int_D |\nabla \bar{\psi}_\varepsilon| \, d^2r - \frac{C}{|\ln \varepsilon|}, \quad (6.16)$$

for some $C > 0$ and all ε small enough. At the same time, by a direct computation as in the proof of [21, Lemma 5.3] we have

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(\chi_{\varepsilon\delta_\varepsilon}(\mathbf{r}) - \chi_{\varepsilon\delta_\varepsilon}(\mathbf{r}'))^2}{|\mathbf{r} - \mathbf{r}'|^3} \, d^2r \, d^2r' \leq 4 \, |\partial D| \, |\ln \varepsilon| + C \ln |\ln \varepsilon|, \quad (6.17)$$

for some $C > 0$ and ε small enough. Finally, we estimate the integral involving the mixed term in (6.15) for all ε so small that $\chi_{\varepsilon\delta_\varepsilon} = 1$ in $D_{\rho/2}$:

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(\bar{\phi}_\varepsilon(\mathbf{r}) - \bar{\phi}_\varepsilon(\mathbf{r}'))(\chi_{\varepsilon\delta_\varepsilon}(\mathbf{r}) - \chi_{\varepsilon\delta_\varepsilon}(\mathbf{r}'))}{|\mathbf{r} - \mathbf{r}'|^3} \, d^2r \, d^2r'$$
$$= 2 \int_{D_\rho} \int_{\mathbb{R}^2 \setminus D_{\rho/2}} \frac{(\bar{\phi}_\varepsilon(\mathbf{r}) - 1)(1 - \chi_{\varepsilon\delta_\varepsilon}(\mathbf{r}'))}{|\mathbf{r} - \mathbf{r}'|^3} \, d^2r' \, d^2r$$
$$\leq 8 \int_{D_\rho} \left(\int_{\mathbb{R}^2 \setminus B_{\rho/2}(\mathbf{r})} \frac{1}{|\mathbf{r} - \mathbf{r}'|^3} \, d^2r' \right) d^2r \leq \frac{32\pi |D|}{\rho}. \quad (6.18)$$

Putting all these estimates together, we then obtain

$$E_\varepsilon(\bar{\phi}_\varepsilon) \geq \frac{3}{4}(\sigma_0 - \lambda\sigma_1) \int_D |\nabla \bar{\psi}_\varepsilon| \, d^2r - \frac{\lambda\sigma_1}{4}|\partial D| - C \frac{\ln |\ln \varepsilon|}{|\ln \varepsilon|}, \quad (6.19)$$

for some $C > 0$ and all ε small enough. The proof is concluded from the lower semicontinuity of the total variation [12] and the fact that $\bar{\phi} = \frac{3}{2}\bar{\psi}$.

Finally, the upper bound in (2.24) follows from the standard construction of the recovery sequence for the Ginzburg–Landau energy exactly as in [21, Lemma 5.3] □

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