



(n, ρ) -harmonic mappings and energy minimal deformations between annuli

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Abstract

We extend the main results obtained by Iwaniec and Onninen in *Memoirs of the AMS* (2012). In this paper, we solve the (ρ, n) -energy minimization problem for Sobolev homeomorphisms between two concentric annuli in the Euclidean space \mathbf{R}^n . Here ρ is a radial metric defined in the image annulus. The key element in the proofs is the solution to the Euler–Lagrange equation for a radial harmonic mapping. This is a new contribution on the topic related to the famous J. C. C. Nitsche conjecture on harmonic mappings between annuli on the complex plane. Namely we prove that the minimum of (ρ, n) -energy of diffeomorphisms between annuli is attained by a certain (ρ, n) -harmonic diffeomorphisms if and only if the original annulus can be mapped onto the image annulus by a radial (ρ, n) -harmonic diffeomorphisms and the last fact is equivalent with a certain inequality for annuli which we call a generalized J. C. C. Nitsche type inequality.

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1 Introduction

Let $0 < r < R$, $0 < r_* < R_*$ and let $\mathbb{A} = A(r, R) \stackrel{\text{def}}{=} \{x : r < |x| < R\}$ and $\mathbb{A}_* = A(r_*, R_*) \stackrel{\text{def}}{=} \{x : r_* < |x| < R_*\}$ be two annuli in the Euclidean space \mathbb{R}^n equipped with the Euclidean norm $|\cdot|$. Here $n \geq 2$. Let ρ be a continuous function on the closure of \mathbb{A}_* . The (ρ, n) -energy integral of a mapping $h \in \mathcal{W}^{1,n}(\mathbb{A}, \mathbb{A}_*)$ is defined by

$$\mathcal{E}_\rho[h] = \int_{A(r, R)} \rho(h(x)) \|Dh(x)\|^n dx. \quad (1.1)$$

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The central aim of this paper is to minimize the (ρ, n) -energy integral between \mathbb{A} and \mathbb{A}_* throughout the class of homomorphisms from the Sobolev class $\mathcal{W}^{1,n}(\mathbb{A}, \mathbb{A}_*)$. We will assume that ρ is C^1 radial metric that is $\rho(y) = \rho(|y|)$ for $y \in \mathbb{A}_*$. We will also assume that

$$\min_{r_* \leq s \leq R_*} \rho(s)s^n = \rho(r_*)r_*^n \tag{1.2}$$

and refer to such metrics as *regular metrics*. The modulus of $A(r, R)$ is defined by the formula $\text{Mod } A(r, R) = \omega_{n-1} \log \frac{R}{r}$, where ω_{n-1} is the area of the unit sphere \mathbb{S}^{n-1} .

For a homomorphism f of Sobolev class $\mathcal{W}^{1,1}$ we say that f has finite outer distortion if

$$\|Df\|^n \leq n^{n/2} K_O[f, x] J_f(x),$$

where $1 \leq K_O[f, x]$ is measurable and the smallest function with the above property. Then $K_O[f, x]$ is called the outer distortion of f . Here

$$\|Df\| = \sqrt{\langle Df, Df \rangle} = \sqrt{\sum_{k=1}^n |Df(x)e_k|^2},$$

and J_f is the determinant of the Jacobian matrix. A concept dual to the outer distortion is the inner distortion defined by the so-called co-factor matrix of $Df(x)$. Namely we define $\mathbf{adj}(Df(x)) = J_f(D^*f(x))^{-1}$. Then

$$\mathbb{K}_I[f, x] = \frac{\|\mathbf{adj}(Df(x))\|^n}{n^{n/2} \det(\mathbf{adj}(Df(x)))} = \frac{\|\mathbf{adj}(Df(x))\|^n}{n^{n/2} J_f^{n-1}(x)},$$

for $J_f(x) \neq 0$ and $K_I[f, x] = 1$ for $J_f(x) = 0$.

An important fact to be noticed in this introduction is that if ρ is a continuous function on the closure of \mathbb{A}_* and if $f \in \mathcal{W}^{1,n-1}(\mathbb{A}_*, \mathbb{A})$ is a homeomorphism, then its inverse mapping h belongs to the Sobolev class $\mathcal{W}^{1,n}(\mathbb{A}, \mathbb{A}_*)$ and we have the following formula

$$\int_{\mathbb{A}} \rho(h(x)) \|Dh(x)\|^n dx = n^{n/2} \int_{\mathbb{A}_*} \rho(y) \mathbb{K}_I[f, y] dy. \tag{1.3}$$

Concerning the criteria of integrability of inverse mapping and related problems we refer to the papers [2,3].

In this paper we extend the main result and simplify the proofs in [6]. We made a unified approach to the minimizing problem of (ρ, n) -energy for the class of all C^1 radial metrics ρ satisfying the condition $\rho(s)s^n$ is non-decreasing. This condition is fulfilled by two metrics $\rho(s) \equiv 1$ and $\rho(s) \equiv s^{-n}$ considered by Iwaniec and Onninen in [6]. The paper generalizes also the main results by Astala, Iwaniec and Martin in [1], and also by the author in [9] where it is treated the similar problem but only for the case $n = 2$. The case of non-circular annuli and non radial metrics has been treated in the papers [5] and [8] respectively, but also for the case $n = 2$. In this paper we assume that $n \geq 3$. The paper is a continuation of study of the so-called Nitsche phenomenon, invented by J. C. C. Nitsche in [14] where he stated his famous conjecture. Further the conjecture has been proved by Iwaniec, Kovalev and Onninen in [4], after some partial results obtained by Weitsman [15], Lyzzaik [12] and Kalaj [10]. For its counterpart to general annuli on Riemannian surfaces we refer to the recent paper [11]. The Nitsche conjecture in the context of (ρ, n) -harmonic mappings is given in (2.6). We prove in the first result (Theorem 2.2) that we can find a radial (ρ, n) -harmonic harmonic mapping between two annuli \mathbb{A} and \mathbb{A}_* if and only if the generalized Nitsche bound (2.6), is satisfied. This bound said roughly speaking that if we have a (ρ, n) harmonic diffeomorphism

between annuli \mathbb{A} and \mathbb{A}_* , then the image annulus cannot be too thin, but can be arbitrary thick. On the other hand this bound is equivalent with the fact that the (ρ, n) -energy integral is minimized for a certain radial (ρ, n) -harmonic diffeomorphism if $n = 3$; if $n \geq 4$, then we have some obstruction, and in this case the image annulus cannot be too thick (Theorem 3.1), provided that there exists a radial mapping which is a minimizer. The precise estimate how thick the image annulus could be, remains an open interesting problem.

1.1 (ρ, n) -harmonic equation

Assume that \mathbb{X} is domain in \mathbf{R}^n (for example \mathbb{X} is homeomorphic to an circular annulus $\{x \in \mathbf{R}^n | 1 < |x| < R\}$). The classical Dirichlet problem concerns the energy minimal mapping $h: \mathbb{X} \rightarrow \mathbf{R}^n$ of the Sobolev class $h \in h_o + \mathcal{W}_o^{1,n}(\mathbb{X}, \mathbf{R}^n)$ whose boundary values are explicitly prescribed by means of a given mapping $h_o \in \mathcal{W}^{1,n}(\mathbb{A}, \mathbf{R}^n)$. Let us consider the variation $h \rightsquigarrow h + \epsilon\eta$, in which $\eta \in \mathcal{C}_o^\infty(\mathbb{X}, \mathbf{R}^n)$ and $\epsilon \rightarrow 0$, leads to the integral form of the familiar n -harmonic system of equations

$$\int_{\mathbb{X}} (\langle \nabla \rho, \eta \rangle \| Dh \|^n + \langle \rho(h) \| Dh \|^{n-2} Dh, D\eta \rangle) = 0, \text{ for every } \eta \in \mathcal{C}_o^\infty(\mathbb{X}, \mathbf{R}^n). \tag{1.4}$$

Equivalently

$$\Delta_n h = \text{Div}(\rho(h) \| Dh \|^{n-2} Dh) - \frac{1}{n} \| Dh \|^n \nabla \rho = 0, \tag{1.5}$$

in the sense of distributions.

Similarly to [6], one may derive the general (ρ, n) -harmonic equation which by using a different variation as the following.

The situation is different if we allow h to slip freely along the boundaries. The *inner variation* come to stage in this case. This is simply a change of the variable; $h_\epsilon = h \circ \eta_\epsilon$, where $\eta_\epsilon: \mathbb{X} \xrightarrow{\text{onto}} \mathbb{X}$ is a \mathcal{C}^∞ -smooth diffeomorphism of \mathbb{X} onto itself, depending smoothly on a parameter $\epsilon \approx 0$ where $\eta_o = id: \mathbb{X} \xrightarrow{\text{onto}} \mathbb{X}$. Let us take on the inner variation of the form

$$\eta_\epsilon(x) = x + \epsilon \eta(x), \quad \eta \in \mathcal{C}_o^\infty(\mathbb{X}, \mathbf{R}^n). \tag{1.6}$$

By using the notation $y = x + \epsilon \eta(x) \in \mathbb{X}$, we obtain

$$\rho(h_\epsilon) Dh_\epsilon(x) = \rho(h(y)) Dh(y) (I + \epsilon D\eta(x)).$$

Hence

$$\begin{aligned} \rho(h_\epsilon(x)) \| Dh_\epsilon(x) \|^n &= \rho(h(y)) \| Dh(y) \|^n \\ &+ n\epsilon \rho(h(y)) \langle \| Dh(y) \|^{n-2} D^*h(y) \cdot Dh(y), D\eta \rangle + o(\epsilon). \end{aligned}$$

By integrating with respect to $x \in \mathbb{X}$ we obtain

$$\begin{aligned} \mathcal{E}_\rho[h_\epsilon] &= \int_{\mathbb{X}} \rho(h_\epsilon(x)) \| Dh_\epsilon(x) \|^n dx \\ &= \int_{\mathbb{X}} \left[\rho(h(y)) \| Dh(y) \|^n \right. \\ &\quad \left. + n\epsilon \rho(h(y)) \langle \| Dh(y) \|^{n-2} D^*h(y) \cdot Dh(y), D\eta(x) \rangle \right] dx + o(\epsilon). \end{aligned}$$

We now make the substitution $y = x + \epsilon \eta(x)$, which is a diffeomorphism for small ϵ , for which we have: $x = y - \epsilon \eta(y) + o(\epsilon)$, $D\eta(x) = D\eta(y) + o(1)$, when $\epsilon \rightarrow 0$, and the change of volume element $dx = [1 - \epsilon \operatorname{Tr} D\eta(y)] dy + o(\epsilon)$. Further

$$\int_{\mathbb{X}} \rho(h(y)) \| Dh(y) \|^n dx = \int_{\mathbb{X}} \rho(h(y)) \| Dh(y) \|^n [1 - \epsilon \operatorname{Tr} D\eta(y)] dy + o(\epsilon).$$

The so called equilibrium equation for the inner variation is obtained from $\frac{d}{d\epsilon} \mathcal{E}_{h_\epsilon} = 0$ at $\epsilon = 0$,

$$\int_{\mathbb{X}} \langle \rho(h) \| Dh \|^n D^*h \cdot Dh - \frac{\rho(h)}{n} \| Dh \|^n I, D\eta \rangle dy = 0 \tag{1.7}$$

or, by using distributions

$$\operatorname{Div} \left(\rho(h) \| Dh \|^n D^*h \cdot Dh - \frac{\rho(h)}{n} \| Dh \|^n I \right) = 0. \tag{1.8}$$

The name *generalized n-harmonic equation* is given to (1.8) in [6, Chapter 3] for the Euclidean metric $\rho \equiv 1$ because of the following:

Lemma 1.1 [6] *Assume that ρ is the Euclidean metric. Every n-harmonic mapping $h \in \mathcal{W}_{loc}^{1,n}(\mathbb{X}, \mathbb{R}^n)$ is a solution of the generalized n-harmonic equation (1.8).*

We believe that a similar statement related to Lemma 1.1 holds true for the general ρ metric, but we didn't consider this question, since we do not need in this paper.

In dimension $n = 2$, the generalized harmonic equation reduces to

$$\operatorname{Div} \left(\rho(h) D^*h Dh - \frac{\rho(h)}{2} \| Dh \|^2 I \right) = 0. \tag{1.9}$$

This equation is known as the Hopf equation, and the corresponding differential is called the Hopf differential. Since for $h(z) = (a(z), b(z))$, we have

$$\rho(h) D^*h Dh - \frac{\rho(h)}{2} \| Dh \|^2 I = \begin{pmatrix} U & V \\ V & -U \end{pmatrix},$$

where

$$U = \frac{\rho(h)}{2} (a_x^2 + b_x^2 - a_y^2 - b_y^2)$$

and

$$V = \rho(h)(a_x a_y + b_x b_y),$$

then (1.9) in complex notation takes the form

$$(U_x + U_y) - i(V_x + V_y) = 0$$

or what is the same

$$\frac{\partial}{\partial \bar{z}} (\rho(h(z)) h_z \bar{h}_{\bar{z}}) = 0, \quad z = x + iy. \tag{1.10}$$

In [5] and [8], it is used the fact that Hopf's differential of a minimizer has special form namely

$$\rho(h(z)) h_z \bar{h}_{\bar{z}} = \frac{c}{z^2}$$

for a certain constant c that depends on the ration of modulus of annuli. In this paper, this constant c will be also crucial for proving the minimization result.

If, in addition $h \in \mathcal{C}^2$ then (1.10) is equivalent with

$$h_{z\bar{z}} + (\log \rho)_w \circ h \cdot h_z h_{\bar{z}} = 0, \tag{1.11}$$

which is known as the harmonic mapping equation. In particular, if $\rho(w) = (1 - |w|^2)^{-2}$, then the equation leads to hyperbolic harmonic mappings. The class is particularly interesting, due to recent discover that every quasimetric map of the unit circle onto itself can be extended to a quasiconformal hyperbolic harmonic mapping of the unit disk onto itself. This problem is known as the Schoen conjecture and it was proved by Marković in [13].

2 Radial solutions to the generalized n -harmonic equation

We assume that $R > 1$, and $R_* > 1$ and $\mathbb{A} = A(1, R)$, $\mathbb{A}_* = A(1, R_*)$. Recall that ρ is a radial C^1 function in $\mathbb{A}_* = A(1, R_*)$ so that $\rho(s)s^n$ attains its minimum for $s = 1$. Let us consider a radial mapping

$$h(x) = H(|x|) \frac{x}{|x|}, \quad \text{where } H = H(t) \text{ is } C^2$$

We find that

$$\begin{aligned} \Lambda &= \rho(h) \| Dh \|^n \left(D^* h \cdot Dh - \frac{1}{n} \| Dh \|^2 I \right) \\ &= \rho(h)(n-1) \frac{n-2}{n} \left(H^2 + \frac{|x|^2 \dot{H}^2}{n-1} \right)^{\frac{n-2}{2}} \cdot (H^2 - |x|^2 \dot{H}^2) \frac{1}{|x|^n} \left(\frac{x \otimes x}{|x|^2} - \frac{1}{n} I \right). \end{aligned} \tag{2.1}$$

Thus (1.9) reduces to

$$\text{Div} \Lambda \equiv 0. \tag{2.2}$$

We show that if h is a \mathcal{C}^2 -smooth n -harmonic mapping then $H = H(t)$ must satisfy the characteristic equation

$$\rho(h) \left(H^2 + \frac{|x|^2 \dot{H}^2}{n-1} \right)^{\frac{n-2}{2}} \cdot (H^2 - |x|^2 \dot{H}^2) \equiv \text{const}. \tag{2.3}$$

Assume that $h = H(t) \frac{x}{|x|}$, $t = |x|$, is a radial function, where H is a real diffeomorphism between intervals $[1, R]$ and $[1, R_*]$. Then by a direct calculation we obtain

$$\| Dh \|^2 = \dot{H}^2(t) + (n-1) \frac{H^2(t)}{t^2},$$

and

$$J_h = \frac{\dot{H} H^{n-1}}{t^{n-1}}.$$

Thus

$$\mathcal{E}_\rho[h] = \mathcal{E}[H] \stackrel{\text{def}}{=} \omega_{n-1} \int_1^R \rho(H(t)) t^{n-1} \left(\dot{H}^2(t) + (n-1) \frac{H^2(t)}{t^2} \right)^{n/2} dt.$$

If

$$L[t, H, \dot{H}] \stackrel{\text{def}}{=} \rho(H(t))t^{n-1} \left(\dot{H}(t)^2 + (n-1) \frac{H^2(t)}{t^2} \right)^{n/2}$$

then the Euler–Lagrange equation is

$$L_H[t, H, \dot{H}] = \frac{\partial}{\partial t} L_{\dot{H}}[t, H, \dot{H}]. \tag{2.4}$$

Then (2.4) is equivalent to (2.2), because $H \in C^2$. The Eq. (2.4) further reduces to

$$\begin{aligned} M \stackrel{\text{def}}{=} & \frac{1}{(n-1)H(t)^2 + t^2\dot{H}^2} (n-1)t^{n-1} \left(\frac{(n-1)H(t)^2}{t^2} + \dot{H}^2 \right)^{n/2} \\ & \times \left(-n\rho[H(t)](H(t) - t\dot{H})((n-1)H(t)^2 + (-2+n)tH(t)\dot{H} + t^2\dot{H}^2) \right. \\ & - (H(t)^2 - t^2\dot{H}^2)((n-1)H(t)^2 + t^2\dot{H}^2)\rho'[H(t)] \\ & \left. + nt^2\rho[H(t)](H(t)^2 + t^2\dot{H}^2)\ddot{H} \right) = 0. \end{aligned}$$

Now we gave the following formula, which is the key of our approach

$$\begin{aligned} M = & \frac{(n-1)t^{-1-n}((n-1)H(t)^2 + t^2\dot{H}^2)}{\dot{H}} \\ & \times \frac{\partial}{\partial t} \left(\rho[H(t)](H(t)^2 - t^2\dot{H}^2)((n-1)H(t)^2 + t^2\dot{H}^2)^{\frac{1}{2}(n-2)} \right) = 0. \end{aligned}$$

Thus we obtain

$$\mathcal{L}[H] = \rho[H(t)](H(t)^2 - t^2\dot{H}^2) \left(H(t)^2 + \frac{t^2\dot{H}^2}{n-1} \right)^{\frac{1}{2}(n-2)} \equiv c. \tag{2.5}$$

Further we look at increasing diffeomorphisms H between two intervals $[1, R]$ and $[1, R_*]$ that are solutions of the previous equation. Then

$$c = \rho[H(t)](H(t)^2 - t^2\dot{H}^2) \left(H(t)^2 + \frac{t^2\dot{H}^2}{n-1} \right)^{\frac{1}{2}(n-2)}$$

Since the function

$$\psi(b) \stackrel{\text{def}}{=} (a^2 - b^2) \left(a^2 + \frac{b^2}{n-1} \right)^{\frac{1}{2}(n-2)}$$

is decreasing, because

$$\psi'(b) = - \frac{b(a^2 + b^2) \left(a^2 + \frac{b^2}{n-1} \right)^{n/2} (n-1)n}{(b^2 + a^2(n-1))^2}$$

we obtain that

$$c \leq \rho[H(t)]H(t)^n,$$

and thus

$$c \leq \min\{\rho[H(t)]H(t)^n, 1 \leq |t| \leq R\} = \rho[H(1)]H(1)^n = \rho(1).$$

Thus we conclude that if the equation has a solution then

$$c \leq c_\diamond \stackrel{\text{def}}{=} \rho(1), \tag{2.6}$$

which we call the generalized J. C. C. Nitsche inequality.

Let us demonstrate the connection of (2.6) with the standard Nitsche inequality.

In this special case $\rho \equiv 1$ and $n = 2$. So the inequality (2.6) is equivalent with the inequality

$$H(t)^2 - t^2 \dot{H}^2 \leq 1.$$

Assuming that $\dot{H} \geq 0$, $H(1) = 1$ and $H(R) = R_*$ then the last inequality is equivalent with

$$\int_1^R \frac{dr}{r} \leq \int_1^{R_*} \frac{dH}{\sqrt{H^2 - 1}}$$

or what is the same as

$$\log \left[R_* + \sqrt{R_*^2 - 1} \right] \geq \log R.$$

Thus we obtain

$$R_* \geq \frac{1 + R^2}{2R}, \tag{2.7}$$

which is the standard Nitsche inequality. Recall that the condition (2.7) is sufficient and necessary for the existence of a planar harmonic diffeomorphism between annuli $A(1, R)$ and $A(1, R_*)$ ([4]).

We will prove that the condition (2.6) is equivalent with the fact that there exists a radial (ρ, n) -harmonic diffeomorphism between given annuli and conjecture the following

Conjecture 2.1 *There is a (ρ, n) harmonic mappings between annuli $A(1, R)$ and $A(1, R_*)$ if and only (2.6) holds.*

The conjecture will be verified on the class of minimizers of (ρ, n) energy (Theorem 3.1).
Let

$$\eta_H(t) = \eta(t) \stackrel{\text{def}}{=} \frac{t^2(\dot{H}(t))^2}{H^2(t)},$$

and let

$$\zeta(t) \stackrel{\text{def}}{=} \eta^2(t).$$

Assume also that the constant c satisfies (2.6). Then the equation $\mathcal{L}[H] = c$ is equivalent with the equation

$$(1 - \eta^2(t)) \left(1 + \frac{\eta^2(t)}{n - 1} \right)^{(n-2)/2} = v_c(H(t)),$$

or

$$\Phi(\zeta) \stackrel{\text{def}}{=} (1 - \zeta(t)) \left(1 + \frac{\zeta(t)}{n - 1} \right)^{(n-2)/2} = v_c(H(t)),$$

where

$$v_c(H) \stackrel{\text{def}}{=} \frac{c}{H^n \rho(H)} \leq 1.$$

Since

$$\Phi'(\zeta) = -\frac{1}{2}(n-1)^{1-\frac{n}{2}}n(1+\zeta)(n-1+\zeta)^{-2+\frac{n}{2}},$$

we conclude that $\Phi : [0, \infty) \rightarrow (-\infty, 1]$ is strictly decreasing and smooth function and thus a diffeomorphism. Moreover $\Phi(0) = 1, \Phi(1) = 0$ and $\Phi(\infty) = -\infty$. Let $\Psi = \Phi^{-1} : (-\infty, 1] \rightarrow [0, \infty)$. Then Ψ is strictly decreasing as well with $\Psi(0) = 1$ and thus

$$\Psi(\zeta) \geq 1, \text{ if } \zeta \leq 0. \tag{2.8}$$

Then

$$\zeta(t) = \Psi(v_c(t)).$$

Further

$$\eta_H(t) = \frac{tH'(t)}{H(t)} = \sqrt{\Psi(v_c(H(t)))}. \tag{2.9}$$

Now by taking the initial condition $H(1) = 1$, we arrive to the implicit solution

$$\log t = \int_1^s \frac{1}{y\sqrt{\Psi(v_c(y))}} dy,$$

with $s = H(t)$.

Thus for

$$T_c(s) \stackrel{\text{def}}{=} \exp \left[\int_1^s \frac{1}{y\sqrt{\Psi(v_c(y))}} dy \right],$$

the diffeomorphism

$$H_c \stackrel{\text{def}}{=} T_c^{-1} \tag{2.10}$$

is a solution of the equation $\mathcal{L}[H](t) = c$ with the initial conditions $H(1) = 1$ and $H'(1) \geq 0$. Further $T_c(R_*) = R$, where

$$R = \exp \left[\int_1^{R_*} \frac{1}{y\sqrt{\Psi(v_c(y))}} dy \right]. \tag{2.11}$$

Let us emphasize the following important fact. Every parameter from the set $\{R, R_*, c\}$ is uniquely determined by two others. More precisely, we have

$$c = c(R, R_*), \quad R = R(c, R_*), \quad \text{and} \quad R_* = R_*(c, R).$$

Since $H_c(t)$ is increasing, then $v_c(H_c(t))$ is decreasing for $c > 0$ and increasing for $c < 0$, and thus $\sqrt{\Psi(v_c(H(t)))}$ increases for $c \geq 0$ and decreases for $c \leq 0$. Thus we obtain that

$$\begin{cases} \eta_H(t) \geq 1, & \text{and } \eta_H(t) \text{ increases on } [1, R] \text{ if } c < 0; \\ \eta_H(t) \leq 1, & \text{and } \eta_H(t) \text{ decreases on } [1, R] \text{ if } c > 0. \\ \eta_H(t) \equiv 1, & \text{on } [1, R] \text{ if } c = 0. \end{cases} \tag{2.12}$$

Let $n \geq 4$ and let κ_n be the solution of the equation

$$(n-1 + \eta^2)^{\frac{n-2}{2}}(\eta^2 - 1) = \eta^n$$

on the interval $[1, \frac{\sqrt{n-1}}{\sqrt{n-3}}]$. Then for $n \geq 4$ we put $c_\diamond := -\rho(1)\kappa_n^n$. If $n = 3$ we put $c^\diamond := -\infty$.

Furthermore, if $c^\diamond \leq c_1 < c_2 \leq c_\diamond$, then $T_{c_1}(R_*) < T_{c_2}(R_*)$. Now if $H_{c_1}(R) = R_*$, then we have $R < T_{c_2}(H_{c_1}(R))$ and thus $H_{c_2}(R) \leq H_{c_1}(R)$. If we use the convention $H_{c^\diamond} \equiv +\infty$ for $n = 3$, then we infer that for every $n \geq 3$

$$H_{c_\diamond}(R) < H_{c^\diamond}(R), \quad R > 1. \tag{2.13}$$

We conclude this section by proving the following theorem.

Theorem 2.2 *Let $R > 1$ be fixed. Let ρ be a regular metric on $\mathbb{A} = A(1, R)$. If $R_* > 1$, then there is a radial (ρ, n) -harmonic diffeomorphism $h = h_c$ between annuli $\mathbb{A} = A(1, R)$ and $\mathbb{A}_* = A(1, R_*)$ if and only if*

$$H_{c_\diamond}(R) \leq R_*, \tag{2.14}$$

or equivalently if

$$c(R, R_*) \leq \rho(1). \tag{2.15}$$

Proof Let $R_* \geq H_{c_\diamond}(R)$. We prove that there is $c \leq c_\diamond$ so that $H_c(R) = R_*$. Then

$$h_c(x) \stackrel{\text{def}}{=} H_c(|x|) \frac{x}{|x|} \tag{2.16}$$

is a n -harmonic diffeomorphism between $A(1, R)$ and $A(1, R_*)$. In order to do so define the function

$$\Lambda(c) = \exp \left[\int_1^{R_*} \frac{1}{h\sqrt{\Psi(v_c(h))}} dh \right].$$

Then $\Lambda(c)$ is continuous for $c \in (-\infty, c_\diamond]$. Moreover the function $c \rightarrow \Psi(v_c(y))$ is increasing for fixed y and so $c \rightarrow y\sqrt{\Psi(v_c(y))}$ is strictly decreasing. So $c \rightarrow \frac{1}{y\sqrt{\Psi(v_c(y))}}$ is increasing and thus Λ is increasing. As $\Lambda(-\infty) = 1$, by Mean value theorem there is a unique c so that $\Lambda(c) = R_*$.

To prove the converse part, assume that $h = H(|x|) \frac{x}{|x|}$ is a harmonic diffeomorphism between annuli $A(1, R)$ and $A(1, R_*)$. Then, because of Lemma 1.1, for a constant c , $\mathcal{L}[H] = c$. Further, H is a diffeomorphism, and so $c \leq c_\diamond$. It follows that (2.14). \square

3 The main result

Theorem 3.1 *Assume that $n \geq 3$ and ρ is a regular metric in $\mathbb{A}_* = A(1, R_*)$, $R_* > 1$.*

a) *Let $R > 1$ be fixed. We have the sharp inequality*

$$\int_{A(1,R)} \rho(|h|) \|Dh\|^n \geq \int_{A(1,R)} \rho(|h_c|) \|Dh_c\|^n, \tag{3.1}$$

for orientation preserving homeomorphisms h of the class $\mathcal{W}^{1,n}$ between $A(1, R)$ and $A(1, R_*)$ mapping the inner boundary onto the inner boundary if

$$H_{c_\diamond}(R) \leq R_* \leq H_{c^\diamond}(R), \tag{3.2}$$

or in its equivalent form if

$$-\rho(1)\kappa_n^n \leq c \leq \rho(1). \tag{3.3}$$

Here h_c is defined in (2.16). The equality is attained if and only if $h = \mathcal{T}h_c$ where \mathcal{T} is a linear isometry of \mathbf{R}^n .

b) If $R_* > 1, n > 3$ and if

$$c < -\frac{2\rho(R_*)R_*^n}{n-2} \left(\frac{n-2}{n-3}\right)^{n/2}, \tag{3.4}$$

then the (ρ, n) -harmonic diffeomorphism $h_c = H_c(|x|) \frac{x}{|x|} : A(1, R) \rightarrow A(1, R_*)$ is not the minimizer of the functional of energy. Here $R = R(c, R_*)$ is defined in (2.11).

Remark 3.2 Theorem 3.1 extends the corresponding result by Iwaniec and Onninen in [6], where the authors have considered only the cases $\rho(x) \equiv 1$ and $\rho(x) = 1/|x|^n$. Moreover the item b) contains a more explicit result even for the case $\rho(x) \equiv 1$.

By using (1.3) and Theorem 3.1 we obtain

Corollary 3.3 Assume that $\mathbb{A}_* = A(1, R_*)$ is an annulus and assume that ρ is a regular metric on \mathbb{A}_* . For $-\rho(1)\kappa_n^n \leq c \leq \rho(1)$ let $R = R(c, R_*)$ and let $f^c = (h_c)^{-1}$. Then we have the following sharp inequality

$$\int_{\mathbb{A}_*} \rho(y) \mathbb{K}_I[f, y] \geq \int_{\mathbb{A}_*} \rho(y) \mathbb{K}_I[f^c, y], \tag{3.5}$$

for every homeomorphism $f : \mathbb{A}_* \rightarrow \mathbb{A}$ preserving the inner boundary and the orientation and belonging to the Sobolev space $\mathcal{W}^{1, n-1}$.

Remark 3.4 The question arises how general can be two doubly connected domains, in order to have similar result.

- Instead of $A(1, R)$ and $A(1, R_*)$, we could take the annuli $A(r, R)$ and $A(r_*, R_*)$. The last case reduces to the previous one because the (ρ, n) -harmonic mappings are invariant under homothety of domain and of image domain. Namely if h is (ρ, n) harmonic mapping between annuli $A(r, R)$ and $A(r_*, R_*)$, then $\lambda h(\mu x)$ is harmonic as well w.r.t. the metric $\rho(\mu|x|)$ between annuli $A(\mu r, \mu R)$ and $A(\lambda r_*, \lambda R_*)$.
- If $h : A(1, R) \rightarrow A(1, R_*)$ is a harmonic homeomorphism that map the inner boundary onto the outer boundary, then

$$h_1(x) = h \left(R \frac{x}{|x|^2} \right) : A(1, R) \rightarrow A(1, R_*)$$

that map the inner boundary onto the inner boundary. This follows from the fact that the class of n -harmonic mappings is invariant under precomposing by conformal mappings of the space, exactly as in the planar case. More precisely, if $h : D \rightarrow \mathbf{R}^n$ is (ρ, n) -harmonic, then $h \circ T$ is (ρ, n) -harmonic in $D' = T^{-1}(D)$, for every metric ρ and every Möbius transformation T on the space \mathbf{R}^n . Here $D \subset \mathbf{R}^n$ is an open subset. This follows from the following formulas

$$\begin{aligned} \mathcal{E}_\rho[h] &= \int_D \rho(h(x)) \|Dh(x)\|^n dx \\ &= \int_D \rho(h(x)) \|Dh(x)\|^n dx \\ &= \int_{D'} \rho(h(T(y))) \|Dh(T(y))\|^n J_T(y) dy \\ &= \int_{D'} \rho(h \circ T(y)) \|Dh(T(y))\|^n |DT(y)|^n dy \\ &= \int_{D'} \rho(h \circ T(y)) \|D(h \circ T)(y)\|^n dy = \mathcal{E}_\rho[h \circ T] \end{aligned}$$

Here $|DT(y)| \stackrel{\text{def}}{=} \max_{|h|=1} |DT(y)h|$. Thus if h is a stationary point of energy integral, then so is $h \circ T$.

- If f is a n -harmonic mapping between annuli $A(r, R)$ and $A(r_*, R_*)$ w.r.t. the metric $\rho(|w|)$, then $\tilde{f}(x) = \frac{f(x)}{|f(x)|^2}$ is a n -harmonic mapping between annuli $A(r, R)$ and $A(1/R_*, 1/r_*)$ w.r.t. the metric

$$\tilde{\rho}(|\omega|) = |\omega|^n \rho\left(\frac{1}{|\omega|}\right).$$

Namely, if $g(x) = \frac{x}{|x|^2}$, then g is conformal and thus

$$\langle Dg(x)h, Dg(x)k \rangle = |Dg(x)|^2 \langle h, k \rangle,$$

where

$$|Dg(x)| = \max_{|h|=1} |Dg(x)h|.$$

Here

$$Dg(x)h = \frac{h}{|x|^2} - \frac{x \langle x, h \rangle}{|x|^4},$$

and thus

$$\|Dg(x)\|^2 = n|Dg(x)|^2 = \frac{n}{|x|^2}.$$

Further we obtain

$$\begin{aligned} \|D\tilde{f}\|^2 &= \text{Tr}((D\tilde{f})^* D\tilde{f}) = \sum_{k=1}^n \langle D\tilde{f}e_k, D\tilde{f}e_k \rangle \\ &= \sum_{k=1}^n \langle g'(f(x))Dfe_k, g'(f(x))Dfe_k \rangle = \frac{\|Df\|^2}{|f|^2}, \end{aligned}$$

and so

$$\mathcal{E}_{\tilde{\rho}}[\tilde{f}] = \mathcal{E}_{\rho}[f].$$

So if f is the minimizer of \mathcal{E}_{ρ} then \tilde{f} is the minimizer of $\mathcal{E}_{\tilde{\rho}}$.

- The main result can be formulated in a slightly more general case, namely for two double connected domains whose boundry components are two spheres (which are not concentric). Namely for annuli $\mathcal{A} = T_1(\mathbb{A})$ and $\mathcal{A}_* = T_1(\mathbb{A}_*)$, where T_1 and T_2 are certain Möbius transformations of the space \mathbf{R}^n . The class of conformal mappings on the space is very rigid, indeed it coincides with the class of Möbius transformations. The planar case is far more interesting but also more difficult in this context (cf. [5,8]).

Remark 3.5 If we take the substitution

$$\eta(t) = \frac{tH'(t)}{H(t)}$$

in (2.5) we obtain

$$n + \frac{nt(1 + \eta[t]^2)\eta'[t]}{(-1 + \eta[t]^2)(n - 1 + \eta[t]^2)} + \frac{H(t)R'[H(t)]}{R[H(t)]} = 0.$$

In particular if $\rho(s) = s^\nu$, we have

$$\frac{(1 + \eta[t^2]) \eta'[t]}{(1 - \eta[t^2]) (n - 1 + \eta[t^2])} = \frac{n + \nu}{nt}. \tag{3.6}$$

By following the approach as in [6], where are considered the special cases $\nu = 0$ and $\nu = -n$, we can find that the solution $H = H_c$ can be expressed by mean of the so called elasticity function $\eta = \eta_H$, and has the similar features as in the case $\nu = 0$ (see [6, p. 35-42]). However we do not need those properties in order to prove our main result. Instead, we use only some general results regarding the modulus of annuli obtained in [6] (Corollary 3.6).

For $x \in \mathbb{A}$ let $N = \frac{x}{|x|}$. Further let T_2, \dots, T_n be $n - 1$ unit vectors mutually orthogonal and orthogonal to N . Denote by $h_N, h_{T_2}, \dots, h_{T_n}$ the corresponding directional derivatives. Use the notation

$$|h_T| = \sqrt{\frac{|h_{T_2}|^2 + \dots + |h_{T_n}|^2}{n - 1}}.$$

Then we have

Corollary 3.6 [6]. *Let h be a homeomorphism between spherical rings \mathbb{A} and \mathbb{A}^* in the Sobolev class $\mathcal{W}^{1,n}(\mathbb{A}, \mathbb{A}^*)$. Then*

$$\int_{\mathbb{A}} \Phi(|h|) |h_N| |h_T|^{n-1} \geq \omega_{n-1} \int_1^{R_*} \tau^{n-1} \Phi(\tau) d\tau \tag{3.7}$$

whenever Φ is integrable in $[1, R_*]$. We have the equality in (3.7) if and only if

$$|h_N| |h_T|^{n-1} = J_h(x). \tag{3.8}$$

Furthermore,

$$\int_{\mathbb{A}} \frac{|h_N|}{|h| |x|^{n-1}} \geq \text{Mod } \mathbb{A}^* \tag{3.9}$$

$$\int_{\mathbb{A}} \frac{|h_T|^{n-1}}{|h|^{n-1} |x|} \geq \text{Mod } \mathbb{A} \tag{3.10}$$

Note that we have equalities if h is a radial mapping.

We also need the following simple lemmas.

Lemma 3.7 [6] *Let $u, v \geq 0$ and $0 \leq \sigma \leq 1$. Then*

$$[u^2 + (n - 1)v^2]^{\frac{n}{2}} \geq a(\sigma) v^n + b(\sigma) uv^{n-1} \tag{3.11}$$

where

$$a(\sigma) = (n - 1) (\sigma^2 + n - 1)^{\frac{n-2}{2}} (1 - \sigma^2) \tag{3.12}$$

and

$$b(\sigma) = n\sigma (\sigma^2 + n - 1)^{\frac{n-2}{2}} \tag{3.13}$$

Equality holds if and only if $u = \sigma v$.

Lemma 3.8 [6] *Let $u, v \geq 0$ and $1 \leq \sigma < \sigma_n$. Then*

$$a = a(\sigma) \stackrel{\text{def}}{=} \frac{(\sigma^2 + n - 1)^{\frac{n-2}{2}} (\sigma^2 - 1)}{\sigma^n} < 1 \tag{3.14}$$

and, we have

$$[u^2 + (n - 1)v^2]^{\frac{n}{2}} \geq a u^n + b uv^{n-1} \tag{3.15}$$

where

$$b = \frac{n (\sigma^2 + n - 1)^{\frac{n-2}{2}}}{\sigma} \tag{3.16}$$

Equality holds if and only if $u = \sigma v$.

4 Proof of Theorem 3.1

♣ Proof of a).

Here we assume (3.2). This bound means that there is a radial (ρ, n) -harmonic homeomorphism

$$h_c : \mathbb{A} \rightarrow \mathbb{A}^*, \quad h_c(x) = H_c(|x|) \frac{x}{|x|} \tag{4.1}$$

Recall the characteristic equation for $H = H_c(t)$ is

$$\left[H^2 + \frac{t^2 \dot{H}^2}{n - 1} \right]^{\frac{n-2}{2}} (H^2 - t^2 \dot{H}^2) \equiv \frac{c}{\rho(H)} \tag{4.2}$$

where $c = C(R, R_*)$ is a constant determined by (R, R_*) .

- The case $0 \leq c \leq c^\circ$.

Let

$$\eta_H = \frac{t \dot{H}}{\dot{H}}.$$

Then (4.2) is equivalent with

$$\left(1 + \frac{\eta_H^2}{n - 1} \right)^{\frac{n-2}{2}} (1 - \eta_H^2) = \frac{c}{\rho(H)H^n}. \tag{4.3}$$

Here c satisfies the condition $c \leq \rho(1)$ and so

$$\frac{c}{\rho(H)H^n} \leq 1.$$

Now, let $h : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$, $h \in \mathcal{H}^{1,n}(\mathbb{A}, \mathbb{A}^*)$, be arbitrary orientation preserving homeomorphism of annuli mapping the inner boundary onto the inner boundary. For $x \in \mathbb{A}$, let $u = |h_N(x)|$, $v = |h_T(x)|$. The equation (4.3) suggests that we should consider the nonnegative solution $\eta = \eta(t)$ to the equation

$$\left(1 + \frac{\eta^2}{n - 1} \right)^{\frac{n-2}{2}} (1 - \eta^2) = \frac{c}{\rho(t)t^n}, \quad 1 < t < R_*. \tag{4.4}$$

There is exactly one such η and it lies in the interval $[0, 1]$ because

$$\frac{c}{\rho(t)t^n} \leq \frac{c}{\rho(1)} \leq 1.$$

Let $\sigma = \eta(t)$ be the solution of (4.4), where $t = |h(x)|$. Then $0 \leq \sigma \leq 1$. We apply Lemma 3.7 to obtain the point-wise inequality

$$\begin{aligned} \rho(h) \| Dh \|^n &= \rho(h) [|h_N|^2 + (n - 1) |h_T|^2]^{\frac{n}{2}} \\ &\geq \rho(h)a(\sigma) |h_T|^n + \rho(h)b(\sigma) |h_N| |h_T|^{n-1} \end{aligned} \tag{4.5}$$

Now we find

$$\begin{aligned} a(\sigma) &= (n - 1) (\sigma^2 + n - 1)^{\frac{n-2}{2}} (1 - \sigma^2) \\ &= (n - 1)^{\frac{n}{2}} \frac{c}{|h|^n \rho(|h|)} \end{aligned}$$

and so

$$\begin{aligned} \rho(h) \| Dh \|^n &= \rho(h) [|h_N|^2 + (n - 1) |h_T|^2]^{\frac{n}{2}} \\ &\geq (n - 1)^{\frac{n}{2}} c \frac{|h_T|^n}{|h|^n} + B(|h|) |h_N| |h_T|^{n-1} \end{aligned} \tag{4.6}$$

Here

$$B(|h|) = n\rho(|h|) (\eta^2(|h|) + n - 1)^{\frac{n-2}{2}}$$

comes from (3.13). An important fact about $B(|h|)$ is that we have equality at (4.6) if $|h_N| = \eta(|h|) |h_T|$. This is true for the radial (ρ, n) -harmonic map at (4.1), by the definition of the constant c . Let us integrate (4.6) over the annulus \mathbb{A} . For the last term we apply the lower bound at (3.7). To estimate the first term in the right hand side of (4.6) we use Hölder’s inequality and we have

$$\left(\int_{\mathbb{A}} \frac{|h_T|^{n-1}}{|x| |h|^{n-1}} \right)^{\frac{n}{n-1}} \leq \int_{\mathbb{A}} \frac{|h_T|^n}{|h|^n} \left(\int_{\mathbb{A}} \frac{dx}{|x|^n} \right)^{\frac{1}{n-1}},$$

and then use (3.10). Thus we have

$$\begin{aligned} \int_{\mathbb{A}} \rho(h) \| Dh \|^n &\geq (n - 1)^{\frac{n}{2}} c \left(\int_{\mathbb{A}} \frac{|h_T|^{n-1}}{|x| |h|^{n-1}} \right)^{\frac{n}{n-1}} \left(\int_{\mathbb{A}} \frac{dx}{|x|^n} \right)^{\frac{-1}{n-1}} \\ &\quad + \omega_{n-1} \int_{r_*}^{R_*} \tau^{n-1} B(\tau) d\tau \\ &\geq (n - 1)^{\frac{n}{2}} c \text{Mod } \mathbb{A} + \omega_{n-1} \int_{r_*}^{R_*} \tau^{n-1} B(\tau) d\tau \end{aligned} \tag{4.7}$$

Finally, observe that we have equalities in all estimates for the radial stretchings. Thus

$$\int_{\mathbb{A}} \rho(h) \| Dh \|^n \geq \int_{\mathbb{A}} \rho(h_c) \| Dh_c \|^n \tag{4.8}$$

as stated.

- **The case $c_{\diamond} \leq c \leq 0$.** Then \mathbb{A}^* is thinner than \mathbb{A} . Let $H = \bar{H}_c$. Then

$$\mathcal{L}[H] = c \leq 0.$$

Thus

$$c_1 \equiv -c = (n - 1)^{\frac{2-n}{2}} \rho(H) [t^2(\dot{H})^2 + (n - 1)H^2]^{\frac{n-2}{2}} (t^2(\dot{H})^2 - H^2) \geq 0.$$

or

$$(n - 1 + \eta^2)^{\frac{n-2}{2}} (\eta^2 - 1) = \frac{c_1(n - 1)^{\frac{n-2}{2}}}{|H|^n p(H)}$$

Now we consider the general mapping h . There is exactly one solution $\sigma = \eta(|h(x)|)$ of the equation

$$(n - 1 + \eta^2)^{\frac{n-2}{2}} (\eta^2 - 1) = \frac{c_1(n - 1)^{\frac{n-2}{2}}}{|h(x)|^n p(|h(x)|)}.$$

Since $1 \leq |h(x)| \leq R_*$, we conclude that

$$a(\sigma) := \frac{(n - 1 + \sigma^2)^{\frac{n-2}{2}} (\sigma^2 - 1)}{\sigma^n} \leq 1.$$

From Lemma 3.8 we obtain

$$\begin{aligned} \rho(|h|)(\|Dh\|^n) &= \rho(|h|)(|h_N|^2 + (n - 1)|h_T|^2)^{n/2} \\ &\geq \rho(|h|) (a(\sigma)|h_N|^n + b(\sigma)|h_N| \cdot |h_T|^{n-1}) \\ &= c_1(n - 1)^{\frac{n-2}{2}} \left[\frac{|h_N|}{|h|\eta(|h|)} \right]^n + \Phi(|h|)|h_N| \cdot |h_T|^{n-1} \end{aligned} \tag{4.9}$$

where

$$\Phi(|h|) = \rho(|h|)b(\eta(|h|)) = \rho(|h|) \frac{n(\eta^2(|h|) + n - 1)^{\frac{n-2}{2}}}{\eta(|h|)}.$$

According to Lemma 3.8, equality holds at a given point x if and only if $|h_N(x)| = \eta(|h(x)|) |h_T(x)|$. In particular, it holds almost everywhere for $h = h_c(x)$, because $|(h_c)_N| = \eta_H |(h_c)_T|$. We now integrate over the annulus \mathbb{A} . The last term at (4.9) is estimated by using (3.7),

$$\begin{aligned} \int_{\mathbb{A}} \Phi(|h|) |h_N| |h_T|^{n-1} &\geq \omega_{n-1} \int_{r_*}^{R_*} \tau^{n-1} \Phi(\tau) d\tau \\ &= \int_{\mathbb{A}} \Phi(|h_c|) |(h_c)_N| |(h_c)_T|^{n-1} \end{aligned} \tag{4.10}$$

To estimate the first term in the right hand side of (4.9) we make use of the identities

$$\frac{|(h_c)_N|}{|h_c| \eta(|h_c|)} = \frac{\dot{H}}{H \eta_H} = \frac{1}{|x|}. \tag{4.11}$$

Having in mind the simple inequality $|h|_N \leq |h_N|$, by using Hölder’s inequality we obtain

$$\left(\int_{\mathbb{A}} \frac{|h|_N dx}{|h| \eta(|h|) |x|^{n-1}} \right)^n \leq \int_{\mathbb{A}} \left[\frac{|h_N|}{|h| \eta(|h|)} \right]^n \left(\int_{\mathbb{A}} \frac{dx}{|x|^n} \right)^{n-1}. \tag{4.12}$$

Further, as in the proof of [6, Proposition 12.1], we obtain

$$\int_{\mathbb{A}} \frac{|h|_N dx}{|h| \eta(|h|) |x|^{n-1}} = \int_{\mathbb{A}} \frac{|h_c|_N dx}{|h_c| \eta(|h_c|) |x|^{n-1}} = \int_{\mathbb{A}} \frac{\dot{H}(|x|) dx}{H \eta_H |x|^{n-1}} = \int_{\mathbb{A}} \frac{dx}{|x|^n}$$

Hence

$$\int_{\mathbb{A}} \left[\frac{|h_N|}{|h| \eta(|h|)} \right]^n \geq \int_{\mathbb{A}} \frac{dx}{|x|^n} = \text{Mod } \mathbb{A}. \tag{4.13}$$

Thus

$$\int_{\mathbb{A}} \rho(h) \| Dh \|^n \geq c_1(n-1)^{\frac{n-2}{2}} \text{Mod } \mathbb{A} + \omega_{n-1} \int_{r_*}^{R_*} \tau^{n-1} \Phi(\tau) d\tau, \tag{4.14}$$

with equality attained for h_c , as stated. This finishes the proof of the fact that if the condition (3.2) is satisfied, then we have the sharp inequality (3.1). In order to prove the opposite statement, assume that $R_* > H_{c^\diamond}(R)$. Then by Theorem 2.2 there is $c = c(R, R_*) < c^\diamond$ and a diffeomorphism $H = H_c : [1, R] \rightarrow [1, R_*]$, so that $h(x) = H(|x|) \frac{x}{|x|}$ is a (ρ, n) -harmonic diffeomorphism between \mathbb{A} and \mathbb{A}_* .

This finishes the proof of Theorem 3.1 a), up to the uniqueness part. The uniqueness part follows by repetition the approach of the similar statement from [6], and we will not write the details here. It is important to emphasize that in some key places where we used the sharp inequalities, the equality statement is attained if and only if

$$J_h(x) = |h_T|^{n-1} |h_N|$$

and so the matrix

$$C(x, h) \stackrel{\text{def}}{=} D^* h \cdot Dh = \begin{bmatrix} |h_N|^2 & 0 & \dots & 0 \\ 0 & |h_T|^2 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & |h_T|^2 \end{bmatrix}$$

arises, in order to prove that h is radial.

♣ **Proof of b).** Let $\mathcal{R} = \mathcal{R}(\mathbb{A}, \mathbb{A}^*)$ be the class of orientation preserving radial (ρ, n) -harmonic diffeomorphisms mapping the inner boundary onto itself and let $\mathcal{D} = \mathcal{D}(R, R_*)$ be the class of orientation preserving C^2 diffeomorphisms of $[1, R]$ onto $[1, R_*]$. Now, we find the infimum in the left hand side of (3.1) for $n > 3$ and obtain

$$\inf_{h \in \mathcal{R}} \int_{\mathbb{A}} \rho(h) \| Dh \|^n = \omega_{n-1} \inf_{H \in \mathcal{D}} \int_1^R \rho(H) [\dot{H}^2 + (n-1)t^{-2}H^2]^{\frac{n}{2}} t^{n-1} dt. \tag{4.15}$$

Here

$$L(t, H, \dot{H}) \stackrel{\text{def}}{=} \rho(H) [\dot{H}^2 + (n-1)t^{-2}H^2]^{\frac{n}{2}} t^{n-1}$$

is strictly convex in $K = \dot{H}$ and coercive and thus the minimum is attained for a smooth function H_o satisfying the Euler–Lagrange equation and boundary conditions $H_o(1) = 1$ and $H_o(R) = R_*$. Then $H_o = H_c$. In order to prove this fact notice that, in view of (2.8) and (2.11) we obtain $R_* > R$. Thus

$$\begin{aligned} \int_1^R \frac{|\dot{H}_o|}{H_o} dt &= \int_1^R |d \log H_o(t)| dt \\ &\geq \left\| \int_1^R d \log H_o(t) dt \right\| \\ &= \log R_* > \log R = \int_1^R \frac{1}{t} dt. \end{aligned}$$

By (2.5) the expression

$$\frac{|\dot{H}_o|}{H_o} - \frac{1}{t}$$

has a constant sign, and thus $\mathcal{L}[H_o] = c_1 < 0$.

So by (2.5) we infer that $H'_o(t) > 0$, and thus H_o is an increasing diffeomorphism. But then it coincides with H_c , because of uniqueness of the solution under this constraint. We obtain that

$$\begin{aligned} \inf_{h \in \mathbb{A}} \int_{\mathbb{A}} \rho(h) \| Dh \|^n &= \omega_{n-1} \int_1^R \rho(H_c) \left[t^2 \dot{H}_c^2 + (n-1)H_c^2 \right]^{\frac{n}{2}} \frac{dt}{t} \\ &= \omega_{n-1} \int_1^R \rho[H_c(t)] [H_c(t)]^n \left[\eta_{H_c}^2(t) + n-1 \right]^{\frac{n}{2}} \frac{dt}{t}. \end{aligned} \tag{4.16}$$

Let $\Phi^\lambda : S^{n-1} \rightarrow S^{n-1}$ be the so called spherical homothety constructed in [6], where $\lambda > 0$ is a real parameter, so that $\Phi^1 = \mathbf{Id}$. More precisely, if $(\theta, \varphi_1, \dots, \varphi_{n-2})$ are spherical coordinates of x , then $(\varphi(\theta), \varphi_1, \dots, \varphi_{n-2})$ are spherical coordinates of $\Phi^\lambda(x)$, where $\varphi(\theta) = 2 \tan^{-1}(\lambda \tan \frac{\theta}{2})$. Then φ is a diffeomorphism of $[0, \pi]$ onto itself. Furthermore Φ^λ is a conformal self-mapping of the unit sphere. Thus if $\zeta = \mathcal{S}(\theta, \varphi_1, \dots, \varphi_{n-2})$ are spherical coordinates, and $\Phi^\lambda(\zeta) = \mathcal{S}(\varphi(\theta), \varphi_1, \dots, \varphi_{n-2})$, by using conformality of Φ^λ and the formula

$$\Phi^\lambda(\mathcal{S}(\theta, \varphi_1, \dots, \varphi_{n-2})) = \mathcal{S}(\varphi(\theta), \varphi_1, \dots, \varphi_{n-2}),$$

we obtain that the ratio between Gram determinants of

$$DS(\varphi(\theta), \varphi_1, \dots, \varphi_{n-2})$$

and of

$$DS(\theta, \varphi_1, \dots, \varphi_{n-2})$$

is equal to $\varphi'(\theta)^{n-1}$. Thus, having in mind the conformality of Φ^λ we define

$$|D\Phi^\lambda(\zeta)| = \varphi'(\theta) = \frac{\sin \varphi(\theta)}{\sin \theta} = \frac{2\lambda}{1 + \lambda^2 + (1 - \lambda^2) \cos \theta}, \tag{4.17}$$

where $\theta \in [0, \pi]$ is the meridian of ζ .

Notice that φ is the only diffeomorphism that produces a conformal mapping on S^{n-1} . Indeed it is only solution of the differential equation with respect to φ in (4.17).

By [6, Eq. 14.50] we have

$$\phi(\lambda) := \frac{1}{\omega_{n-1}} \int_{S^{n-1}} [\sigma^2 + (n-1)|D\Phi^\lambda|^2]^{\frac{n}{2}} < [\sigma^2 + n-1]^{\frac{n}{2}} \tag{4.18}$$

for every parameter $1 < \lambda \sqrt{\frac{n-3}{n-1}} \sigma$, where $\sigma > \sqrt{\frac{n-1}{n-3}}$.

This mean that $\lambda = 1$ is a local maximum of ϕ . We prove here more, $\lambda = 1$ is local maximum of ϕ if and only if $\sigma > \sqrt{\frac{n-1}{n-3}}$.

Then, by direct computation, in view of (4.17) we find that

$$\phi'(1) = 0$$

and

$$\phi''(1) = \frac{2(-1 - \sigma^2(-3 + n) + n)\sqrt{\pi}\Gamma\left(\frac{1+n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}.$$

So $\phi''(1) < 0$ if and only if $\sigma > \sqrt{\frac{n-1}{n-3}}$.

Then we test the infimum in the right hand side of (3.1) with the mapping

$$h_\lambda(x) = H(|x|)\Phi^\lambda\left(\frac{x}{|x|}\right) \tag{4.19}$$

where, as in the previous case, $\Phi^\lambda : \mathbf{S}^{n-1} \rightarrow \mathbf{S}^{n-1}$ is the spherical homothety and $H = H_c$. An important facts concerning Φ^λ , which follows from (4.17), is the following

$$\int_{\mathbf{S}^{n-1}} |D\Phi^\lambda|^{n-1} = \omega_{n-1}.$$

From the equation

$$(\eta^2(t) - 1)\left(1 + \frac{\eta^2(t)}{n-1}\right)^{(n-2)/2} = -v_c(H(t)) = \frac{-c}{\rho[H(t)]H(t)^n}$$

in view of (2.12) we infer that

$$(\eta^2(t) - 1)\left(1 + \frac{\eta^2(t)}{n-1}\right)^{(n-2)/2} \geq \frac{-c}{\rho(R_*)R_*^n}.$$

From (3.4) we obtain

$$(\eta^2(t) - 1)\left(1 + \frac{\eta^2(t)}{n-1}\right)^{(n-2)/2} > \left(\left(\sqrt{\frac{n-1}{n-3}}\right)^2 - 1\right)\left(1 + \frac{\left(\sqrt{\frac{n-1}{n-3}}\right)^2}{n-1}\right)^{(n-2)/2},$$

and thus

$$\eta = \eta_H(t) > \sqrt{\frac{n-1}{n-3}}. \tag{4.20}$$

From (4.16) and (4.18) we find that

$$\begin{aligned} \inf_{h \in \mathcal{P}(\mathbb{A}, \mathbb{A}^*)} \int_{\mathbb{A}} \rho(h) \| Dh \| ^n &\leq \int_{\mathbb{A}} \rho(h_\lambda) \| Dh_\lambda \| ^n \\ &= \int_r^R \rho[H(t)] [H(t)]^n \int_{\mathbf{S}^{n-1}} [\eta_H^2(t) + (n-1)|D\Phi^\lambda|^2]^{\frac{n}{2}} \frac{dt}{t} \\ &< \omega_{n-1} \int_r^R \rho[H(t)] [H(t)]^n [\eta_H^2(t) + n-1]^{\frac{n}{2}} \frac{dt}{t} \\ &= \inf_{h \in \mathcal{R}(\mathbb{A}, \mathbb{A}^*)} \int_{\mathbb{A}} \rho(h) \| Dh \| ^n. \end{aligned}$$

Here we have chosen $\lambda > 1$ sufficiently close to 1.

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