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Calculus of Variations

Heat kernel and curvature bounds in Ricci flows with bounded scalar curvature—Part II

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Abstract
In this paper we analyze the behavior of the distance function under Ricci flows whose scalar curvature is uniformly bounded. We will show that on small time-intervals the distance function is $\frac{1}{2}$-Hölder continuous in a uniform sense. This implies that the distance function can be extended continuously up to the singular time.

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1 Introduction

In this paper, we extend the estimates of [1], to prove the following result:

Theorem 1.1 *For any* $0 < A < \infty$ *and* $n \in \mathbb{N}$ *there is a constant* $C = C(A, n) < \infty$ *such that the following holds:*

Let $(\mathbf{M}^n, (g_t)_{t\in[0,1]})$ *be a Ricci flow* ($\partial_t g_t = -2\,\mathrm{Ric}_{g_t}$) *on an* n*-dimensional compact manifold* $\mathbf{M}$ *with the property that* $\nu[g_0, 1 + A^{-1}] \geq -A$. *Assume that the scalar curvature satisfies* $|R| \leq R_0$ *on* $\mathbf{M} \times [0, 1]$ *for some constant* $0 \leq R_0 \leq A$.

Then for any $0 \leq t_1 \leq t_2 \leq 1$ *and* $x, y \in \mathbf{M}$ *we have the distance bound*

$$d_{t_1}(x, y) - C\sqrt{t_2 - t_1} \leq d_{t_2}(x, y) \leq \exp\left(C R_0^{1/2}\sqrt{t_2 - t_1}\right) d_{t_1}(x, y) + C\sqrt{t_2 - t_1}.$$

In particular, if $\min\{d_{t_1}(x, y), d_{t_2}(x, y)\} \leq D$ *for some* $D < \infty$*, then*

$$\left|d_{t_1}(x, y) - d_{t_2}(x, y)\right| \leq C'\sqrt{t_2 - t_1},$$

where C' *may depend on* A, D *and* n.

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By parabolic rescaling, we obtain distance bounds on larger time-intervals. Note that Theorem 1.1 is a generalization of [1, Theorem 1.1], which only provides a bound on the distance distortion that does not improve for t_2 close to t_1. The constant $\nu[g_0, 1 + A^{-1}]$ is defined as the infimum of Perelman's μ-functional (cf [4]) $\mu[g_0, \tau]$ over all $\tau \in (0, 1+A^{-1})$. For more details see [1, Sect. 2]. The condition $\nu[g_0, 1 + A^{-1}] \geq -A$, can be viewed as a non-collapsing condition. The exponential factor in the upper bound is necessary, as one can see for example in the case in which $(\mathbf{M}, (g_t)_{t\in[0,1]})$ is the Ricci flow on a hyperbolic manifold and the distance between x, y is very large. The proof of Theorem 1.1 will heavily use the results of [1], in particular the heat kernel bound, [1, Theorem 1.4].

As a consequence of Theorem 1.1, we obtain the following:

Corollary 1.2 *Let $(\mathbf{M}, (g_t)_{t\in[0,T)})$, $T < \infty$ be a Ricci flow on a compact manifold and assume that the scalar curvature satisfies $R < C < \infty$ on $\mathbf{M} \times [0, T)$. Then the distance function*

$$d : \mathbf{M} \times \mathbf{M} \times [0, T) \longrightarrow [0, \infty), \qquad (x, y, t) \longmapsto d_t(x, y)$$

can be extended continuously onto the domain $\mathbf{M} \times \mathbf{M} \times [0, T]$.

Note that the corollary does not state that $d_T : \mathbf{M} \times \mathbf{M} \to [0, \infty)$ is a metric on $\mathbf{M}$. It only follows that d_T is a pseudometric, which means that we may have $d_T(x, y) = 0$ for some $x \neq y$. After taking the metric identification, however, $(\mathbf{M}/\sim, d_T)$ is in fact the Gromov–Hausdorff limit of $(\mathbf{M}, g_t)$ as $t \nearrow T$. Here $x \sim y$ if and only if $d_T(x, y) = 0$. Moreover, since the volume measure converges as well, the space $(\mathbf{M}/\sim, d_T)$ becomes a metric measure space with doubling property and this space is the limit of $(\mathbf{M}, g_t)$ in the measured Gromov–Hausdorff sense.

More generally, we obtain the following consequence of Theorem 1.1.

Corollary 1.3 *Let $(\mathbf{M}^i, (g^i_t)_{t\in[0,1]})$ be a sequence of Ricci flows on n-dimensional compact manifolds $\mathbf{M}^i$ with the property that $\nu[g^i_0, 1 + A^{-1}] \geq -A$ and $|R| < A$ on $\mathbf{M} \times [0, 1]$ for some uniform $A < \infty$. Let $x_i \in \mathbf{M}^i$ be points. Then, after passing to a subsequence, we can find a pointed metric space $(\overline{\mathbf{M}}, \overline{d}, \overline{x})$, a continuous function*

$$d^\infty : \overline{\mathbf{M}} \times \overline{\mathbf{M}} \times [0, 1] \to [0, \infty), \qquad (x, y, t) \mapsto d^\infty_t(x, y)$$

and a continuous family of measures $(\mu_t)_{t\in[0,1]}$ such that for any $x, y \in \overline{\mathbf{M}}$, the function $t \mapsto d^\infty_t(x, y)$ is $\frac{1}{2}$-Hölder continuous and such that for any $t \in [0, 1]$, the metric identification $(\overline{\mathbf{M}}/\sim_t, d^\infty_t, \mu_t, \overline{x})$ is a metric measure space with doubling property for balls of radius less than $\sqrt{t}$. Here $x \sim_t y$ if and only if $d^\infty_t(x, y) = 0$. Moreover, for any $t \in [0, 1]$ the sequence $(\mathbf{M}^i, g^i_t, dg^i_t, x_i)$ converges to $(\overline{\mathbf{M}}/\sim_t, d^\infty_t, \mu_t, \overline{x})$ in the pointed, measured Gromov–Hausdorff sense.

For the proof of Corollary 1.3 see Sect. 5.

Note that if we impose the extra assumption that $|R| < R_i$ on $\mathbf{M}\times[0, 1]$ for some sequence R_i with $\lim_{i\to\infty} R_i = 0$, then the limiting family of measures $(\mu_t)_{t\in[0,1]}$ is constant in time. Unfortunately, however, our results do not imply that $(d^\infty_t)_{t\in[0,1]}$ is constant in time as well.

Finally, we mention a direct consequence of Theorem 1.1, which can be interpreted as an analogue of the main result of [2] in the parabolic case.

Corollary 1.4 *For any $0 < A < \infty$ and $n \in \mathbb{N}$ there is a constant $C = C(A, n) < \infty$ such that the following holds:*

Let $(\mathbf{M}^n, (g_t)_{t\in[0,1]})$ be a Ricci flow on an n-dimensional compact manifold $\mathbf{M}$ with the property that $\nu[g_0, 1 + A^{-1}] \geq -A$. Assume that the scalar curvature satisfies $|R| \leq A$ on $\mathbf{M} \times [0, 1]$.

Then for any $r > 0$ *and* $0 \le t_1 \le t_2 \le 1$ *and* $x \in \mathbf{M}$ *we have the following bound for Gromov–Hausdorff distance of* r*-balls*

$$d_{\mathrm{GH}}(B(x,t_1,r), B(x,t_2,r)) \le C\sqrt{|t_1-t_2|}.$$

For the rest of the paper, we will fix the dimension $n \ge 2$ of the manifold $\mathbf{M}$. Most of our constants will depend on n. For convenience we will not mention this dependence anymore.

2 Upper volume bound

We first generalize the upper volume bound from [5] or [3].

Lemma 2.1 *For any* $A < \infty$ *there is a uniform constant* $C_0 = C_0(A) < \infty$ *such that the following holds:*

Let $(\mathbf{M}^n, (g_t)_{t\in[-1,1]})$ *be a Ricci flow on a compact, n-dimensional manifold* $\mathbf{M}$ *with* $|R| \le 1$ *on* $\mathbf{M} \times [-1,1]$. *Assume that* $\nu[g_{-1}, 4] \ge -A$. *Then for any* $(x,t) \in \mathbf{M} \times [0,1]$ *and* $r > 0$ *we have*

$$|B(x,t,r)|_t < C_0 r^n e^{C_0 r}.$$

Here $|S|_t$ *denotes the volume of a set* $S \subset \mathbf{M}$ *with respect to the metric* g_t.

Proof It follows from [3–5] (see also [1, Sect. 2]), that for any $x \in \mathbf{M}$ and $0 \le r \le 1$, we have

$$cr^n \le |B(x,t_0,r)|_{t_0} \le Cr^n, \tag{2.1}$$

for some constants c, C, which only depend on A.

Fix some $x \in \mathbf{M}$ and let $N < \infty$ be maximal with the property that we can find points $x_1, \ldots, x_N \in B(x,t,\frac{1}{2})$ such that the balls $B(x_1,t,\frac{1}{8}), \ldots, B(x_N,t,\frac{1}{8})$ are pairwise disjoint. Note that then

$$B\left(x_1,t,\tfrac{1}{8}\right), \ldots, B\left(x_N,t,\tfrac{1}{8}\right) \subset B(x,t,1).$$

So, by (2.1), we have $N \le C_* := (c(\frac{1}{8})^n)^{-1}C$. Moreover, by the maximality of N, we have

$$B\left(x_1,t,\tfrac{1}{4}\right) \cup \ldots \cup B\left(x_N,t,\tfrac{1}{4}\right) \supset B\left(x,t,\tfrac{1}{2}\right). \tag{2.2}$$

We now argue that for all $r \ge \frac{1}{2}$

$$B(x_1,t,r) \cup \ldots \cup B(x_N,t,r) \supset B\left(x,t,r+\tfrac{1}{4}\right). \tag{2.3}$$

Let $y \in B(x,t,r+\frac{1}{4})$ and consider a time-t minimizing geodesic $\gamma : [0,l] \to \mathbf{M}$ between x and y that is parameterized by arclength. Then $l < r+\frac{1}{4}$. By (2.2) we may pick $i \in \{1,\ldots,N\}$ such that $\gamma(\frac{1}{2}) \in \overline{B(x_i,t,\frac{1}{4})}$. Then

$$\mathrm{dist}_t(x_i,y) \le \left(l-\tfrac{1}{2}\right) + \mathrm{dist}_t\left(\gamma\left(\tfrac{1}{2}\right),x_i\right) \le l-\tfrac{1}{4} < r.$$

So $y \in B(x_i,t_0,r)$, which confirms (2.3).

Let us now prove by induction on $k = 1,2,\ldots$ that for any $x \in \mathbf{M}$

$$\left|B\left(x,t,\tfrac{1}{4}k\right)\right|_t < C_*^k. \tag{2.4}$$

For $k = 1$, the inequality follows from (2.1) (assuming $c < 1$ and hence $C_* > C$). If the inequality is true for k, then we can use (2.3) to conclude

$$\begin{aligned}\left|B\left(x, t, \tfrac{1}{4}(k+1)\right)\right|_t &\leq \left|B\left(x_1, t, \tfrac{1}{4}k\right)\right|_t + \ldots + \left|B\left(x_N, t, \tfrac{1}{4}k\right)\right|_t \\ &\leq N \cdot C_*^k \leq C_* \cdot C_*^k = C_*^{k+1}.\end{aligned}$$

So (2.4) also holds for $k+1$. This finishes the proof of (2.4).

The assertion of the lemma now follows from (2.1) for $r < 1$. For $r \geq 1$ choose $k \in \mathbb{N}$ such that $\frac{1}{4}(k-1) \leq r < \frac{1}{4}k$. Then, by (2.4), we have

$$|B(x, t, r)|_t < \left|B\left(x, t, \tfrac{1}{4}k\right)\right|_t < C_*^k = C_* e^{(\log C_*)(k-1)} \leq C_* e^{4(\log C_*)r}.$$

This finishes the proof. □

3 Generalized maximum principle

Consider a Ricci flow $(g_t)_{t \in I}$ on a closed manifold $\mathbf{M}$. In the following we will consider the heat kernel $K(x, t; y, s)$ on a Ricci flow background. That is, for any $(y, s) \in \mathbf{M} \times I$ the kernel $K(\cdot, \cdot; y, s)$ is defined for $t > s$ and $x \in \mathbf{M}$ and satisfies

$$(\partial_t - \Delta_x) K(x, t; y, s) = 0 \qquad \text{and} \qquad \lim_{t \searrow s} K(\cdot, t; y, s) = \delta_y.$$

Then, for fixed $(x, t) \in \mathbf{M} \times I$, the function $K(x, t; \cdot, \cdot)$, which is defined for $s < t$, is a kernel for the conjugate heat equation

$$(-\partial_s - \Delta_y + R(y, s)) K(x, t; y, s) = 0 \qquad \text{and} \qquad \lim_{s \nearrow t} K(x, t; \cdot, s) = \delta_x.$$

Recall that for any $s < t$ and $x \in \mathbf{M}$ we have

$$\int_{\mathbf{M}} K(x, t; y, s) dg_s(y) = 1. \tag{3.1}$$

Lemma 3.1 *Let* $(\mathbf{M}, (g_t)_{t \in [0,1]})$ *be a Ricci flow on a compact manifold* $\mathbf{M}$ *with* $|R| \leq R_0$ *on* $\mathbf{M} \times [0, 1]$ *for some constant* $R_0 \geq 0$. *Then for any* $(x, t) \in \mathbf{M} \times (0, 1]$ *we have*

$$\int_0^t \int_{\mathbf{M}} K(x, t; y, s) |\mathrm{Ric}|^2(y, s) dg_s(y) ds \leq R_0.$$

Proof This follows from the identities

$$R(x, t) = \int_{\mathbf{M}} K(x, t; y, 0) R(y, 0) dg_0(y) + 2 \int_0^t \int_{\mathbf{M}} K(x, t; y, s) |\mathrm{Ric}|^2(y, s) dg_s(y) ds$$

and (3.1) as well as $R(x, t) \leq R_0$ and $R(\cdot, 0) \geq -R_0$ on $\mathbf{M}$. □

We will now use the Gaussian bounds from [1] to bound the forward heat kernel in terms of the backwards conjugate heat kernel based at a certain point and time. Note that in the following Lemma we only obtain estimates on the time-interval $[0, 1]$, but we need to assume that the flow exists on $[-1, 1]$. This is due to an extra condition in [1, Theorem 1.4].

Lemma 3.2 *For any $A < \infty$ there are uniform constants $C_1 = C_1(A), Y = Y(A) < \infty$ such that the following holds:*

Let $(\mathbf{M}^n, (g_t)_{t\in[-1,1]})$ be a Ricci flow on a compact, n-dimensional manifold $\mathbf{M}$ with the property that $\nu[g_{-1}, 4] \geq -A$. Assume that $|R| \leq 1$ on $\mathbf{M}\times[-1, 1]$. Let $0 \leq t_1 < t_2 < t_3 \leq 1$ such that

$$Y(t_2 - t_1) \leq t_3 - t_2 \leq 10Y(t_2 - t_1).$$

Then for all $x, y \in \mathbf{M}$

$$K(x, t_2; y, t_1) < C_1 K(y, t_3; x, t_2).$$

Proof Recall that, by [1, Theorem 1.4] and the remark afterwards, there are constants $C_1^* = C_1^*(A), C_2^* = C_2^*(A) < \infty$ such that for any $0 \leq s < t \leq 1$

$$\frac{1}{C_1^*(t-s)^{n/2}} \exp\Big(-\frac{C_2^* d_s^2(x, y)}{t-s}\Big) < K(x, t; y, s) < \frac{C_1^*}{(t-s)^{n/2}} \exp\Big(-\frac{d_t^2(x, y)}{C_2^*(t-s)}\Big). \tag{3.2}$$

Set now

$$Y := (C_2^*)^2 \qquad \text{and} \qquad C_1 := (C_1^*)^2(10Y)^{n/2}.$$

Then

$$\begin{aligned}
K(x, t_2; y, t_1) &< \frac{C_1^*}{(t_2-t_1)^{n/2}} \exp\Big(-\frac{d_{t_2}^2(x, y)}{C_2^*(t_2-t_1)}\Big)\\
&\leq \frac{C_1^*}{(10Y)^{-n/2}(t_3-t_2)^{n/2}} \exp\Big(-\frac{d_{t_2}^2(x, y)}{C_2^*(t_2-t_1)}\Big)\\
&\leq C_1 \frac{1}{C_1^*(t_3-t_2)^{n/2}} \exp\Big(-\frac{d_{t_2}^2(x, y)}{C_2^* Y^{-1}(t_3-t_2)}\Big)\\
&= C_1 \frac{1}{C_1^*(t_3-t_2)^{n/2}} \exp\Big(-\frac{C_2^* d_{t_2}^2(x, y)}{(t_3-t_2)}\Big) < C_1 K(y, t_3, x, t_2).
\end{aligned}$$

This finishes the proof. □

Next, we combine Lemmas 3.1 and 3.2 to obtain the following bound.

Lemma 3.3 *For any $A < \infty$ there are uniform constants $C_2 = C_2(A) < \infty, \theta_2 = \theta_2(A) > 0$ such that the following holds:*

Let $(\mathbf{M}^n, (g_t)_{t\in[-1,1]})$ be a Ricci flow on a compact, n-dimensional manifold $\mathbf{M}$ with the property that $\nu[g_{-1}, 4] \geq -A$. Assume that $|R| \leq R_0$ on $\mathbf{M} \times [-1, 1]$ for some constant $0 \leq R_0 \leq 1$. Then for any $0 \leq t < 1$ and $0 < a \leq \theta_2(1-t)$ and $x \in \mathbf{M}$ we have

$$\int_{t+a}^{t+2a} \int_{\mathbf{M}} K(y, s; x, t)|\mathrm{Ric}|(y, s) dg_s(y) ds < C_2 R_0^{1/2} \sqrt{a}.$$

Proof Choose $\theta_2 := \frac{1}{2}Y^{-1}$ and set

$$t_3 := t + 2Ya \leq 1.$$

So for any $s \in [t+a, t+2a]$ we have

$$Y(s-t) \leq Y \cdot 2a = t_3 - t \leq 10Ya \leq 10Y(s-t).$$

So by Lemma 3.2, we have for any $(y, s) \in \mathbf{M} \times [t+a, t+2a]$

$$K(y, s; x, t) < C_1 K(x, t_3; y, s).$$

We can then conclude, using Cauchy-Schwarz, (3.1) and Lemma 3.1, that

$$\begin{aligned}
&\int_{t+a}^{t+2a}\int_{\mathbf{M}} K(y, s; x, t)|\mathrm{Ric}|(y, s)dg_s(y)ds \\
&\quad \le C_1 \int_{t+a}^{t+2a}\int_{\mathbf{M}} K(x, t_3; y, s)|\mathrm{Ric}|(y, s)dg_s(y)ds \\
&\quad \le C_1 \Big(\int_{t+a}^{t+2a}\int_{\mathbf{M}} K(x, t_3; y, s)dg_s(y)ds \Big)^{1/2} \\
&\qquad \cdot \Big(\int_{t+a}^{t+2a}\int_{\mathbf{M}} K(x, t_3; y, s)|\mathrm{Ric}|^2(y, s)dg_s(y)ds \Big)^{1/2} \\
&\quad = C_1 \sqrt{a} \Big(\int_{t+a}^{t+2a}\int_{\mathbf{M}} K(x, t_3; y, s)|\mathrm{Ric}|^2(y, s)dg_s(y)ds \Big)^{1/2} \\
&\quad \le C_1 R_0^{1/2} \sqrt{a}.
\end{aligned}$$

This proves the desired result. □

Lemma 3.4 *For any $A < \infty$ there are constants $C_3 = C_3(A) < \infty$, $\theta_3 = \theta_3(A) > 0$ such that the following holds:*

Let $(\mathbf{M}^n, (g_t)_{t\in[0,1]})$ be a Ricci flow on a compact, n-dimensional manifold $\mathbf{M}$ with the property that $\nu[g_{-1}, 4] \ge -A$. Assume that $|R| \le R_0$ on $\mathbf{M} \times [-1, 1]$ for some constant $0 \le R_0 \le 1$. Then for any $0 \le s < t \le 1$ with $t - s \le \theta_3(1-s)$ and any $x \in \mathbf{M}$, we have

$$\int_s^t \int_{\mathbf{M}} K(y, s; x, t)|\mathrm{Ric}|(y, s)dg_s(y)ds < C_3 R_0^{1/2} \sqrt{t-s}.$$

Proof Choose $\theta_3(A) = \theta_2(A)$. Then, using Lemma 3.3,

$$\begin{aligned}
&\int_s^t \int_{\mathbf{M}} K(y, s; x, t)|\mathrm{Ric}|(y, s)dg_s(y) \\
&\quad = \sum_{k=1}^{\infty} \int_{s+(t-s)2^{-k}}^{s+2(t-s)2^{-k}} \int_{\mathbf{M}} K(y, s; x, t)|\mathrm{Ric}|(y, s)dg_s(y)ds \\
&\quad \le \sum_{k=1}^{\infty} C_2 R_0^{1/2} \sqrt{(t-s)2^{-k}} \\
&\quad = C_2 R_0^{1/2} \sqrt{t-s} \sum_{k=1}^{\infty} 2^{-k/2}
\end{aligned}$$

$$\leq C C_2 R_0^{1/2} \sqrt{t-s}.$$

This proves the desired estimate. □

Proposition 3.5 *For every $A < \infty$ there are constants $\theta_4 = \theta_4(A) > 0$ and $C_4 = C_4(A) < \infty$ such that the following holds:*

Let $(\mathbf{M}^n, (g_t)_{t\in[-1,1]})$ be a Ricci flow on a compact, n-dimensional manifold $\mathbf{M}$ with the property that $\nu[g_{-1}, 4] \geq -A$. Assume that $|R| \leq R_0$ on $\mathbf{M} \times [-1, 1]$ for some constant $0 \leq R_0 \leq 1$. Let $H > 1$ and $[t_1, t_2] \subset [0, 1)$ be a sub-interval with $t_2 - t_1 \leq \theta_4 \min\{(1-t_1), H^{-1}\}$ and consider a non-negative function $f \in C^\infty(\mathbf{M} \times [t_1, t_2])$ that satisfies the following evolution inequality in the barrier sense:

$$-\partial_t f \leq \Delta f + H|\mathrm{Ric}| f - Rf.$$

Then

$$\max_{\mathbf{M}} f(\cdot, t_1) \leq \big(1 + C_4 H R_0^{1/2} \sqrt{t_2 - t_1}\big) \max_{\mathbf{M}} f(\cdot, t_2).$$

Note that with similar techniques, we can analyze the evolution inequality $-\partial_t f \leq \Delta f + H|\mathrm{Ric}|^p f$ for any $p \in (0, 2)$.

Proof We first find that that for any $(x, t) \in \mathbf{M} \times [-1, 1)$ and $t < s \leq 1$

$$\frac{\mathrm{d}}{\mathrm{d}s} \int_{\mathbf{M}} K(y, s; x, t) dg_s(y) = \int_{\mathbf{M}} \big(\Delta_y K(y, s; x, t) - K(y, s; x, t) R(y, s)\big) dg_s(y)$$
$$\leq R_0 \int_{\mathbf{M}} K(y, s; x, t) dg_s(y),$$

which implies

$$\int_{\mathbf{M}} K(y, s; x, t) dg_s(y) \leq e^{R_0(s-t)}.$$

So for any $(x, t) \in \mathbf{M} \times [t_1, t_2]$ we have by Lemma 3.4, assuming $\theta_4 \leq \theta_3$ and $C_3 > 1$,

$$f(x, t) \leq \int_{\mathbf{M}} K(y, t_2; x, s) f(y, t_2) dg_{t_2}(y)$$
$$+ \int_t^{t_2} \int_{\mathbf{M}} K(y, s; x, t) \cdot H|\mathrm{Ric}|(y, s) \cdot f(y, s) dg_s(y) ds$$
$$\leq e^{R_0(t_2-t)} \max_{\mathbf{M}} f(\cdot, t_2) + H\Big(\max_{\mathbf{M}\times[t,t_2]} f\Big) \int_t^{t_2} \int_{\mathbf{M}} K(y, s; x, t) |\mathrm{Ric}|(y, s) dg_s(y) ds$$
$$\leq e^{R_0(t_2-t)} \max_{\mathbf{M}} f(\cdot, t_2) + H\Big(\max_{\mathbf{M}\times[t,t_2]} f\Big) \cdot C_3 R_0^{1/2} \sqrt{t_2 - t}.$$

It follows that

$$\max_{\mathbf{M}\times[t,t_2]} f \leq e^{R_0(t_2-t)} \max_{\mathbf{M}} f(\cdot, t_2) + \Big(\max_{\mathbf{M}\times[t,t_2]} f\Big) \cdot C_3 H R_0^{1/2} \sqrt{t_2 - t}.$$

So if $t_2 - t < (2C_3 H)^{-2}$, then

$$\max_{\mathbf{M}\times[t,t_2]} f \le \frac{e^{R_0(t_2-t)} \max_{\mathbf{M}} f(\cdot, t_2)}{1 - C_3 H R_0^{1/2}\sqrt{t_2 - t}} \le \left(1 + 10 C_3 H R_0^{1/2}\sqrt{t_2-t}\right) \max_{\mathbf{M}} f(\cdot, t_2).$$

This finishes the proof. □

4 Proof of theorem 1.1

We will first establish a lower bound on the distortion of the distance:

Lemma 4.1 *For every $A < \infty$ there is a constant $C_5 = C_5(A) < \infty$ such that the following holds:*

Let $(\mathbf{M}^n, (g_t)_{t\in[-1,1]})$ be a Ricci flow on a compact, n-dimensional manifold $\mathbf{M}$ with the property that $\nu[g_{-1}, 4] \ge -A$. Assume that $|R| \le 1$ on $\mathbf{M} \times [-1, 1]$. Let $[t_1, t_2] \subset [0, 1]$ be a sub-interval and consider two points $x_1, x_2 \in \mathbf{M}$. Then

$$d_{t_2}(x_1, x_2) \ge d_{t_1}(x_1, x_2) - C_5\sqrt{t_2 - t_1}.$$

Proof Set $d := d_{t_1}(x_1, x_2)$ and let $u \in C^0(\mathbf{M} \times [t_1, t_2]) \cap C^\infty(\mathbf{M} \times (t_1, t_2])$ be a solution to the heat equation

$$\partial_t u = \Delta u, \qquad u(\cdot, t_1) = d_{t_1}(x_1, \cdot).$$

Then for any $(x, t) \in \mathbf{M} \times [t_1, t_2]$

$$u(x,t) = \int_{\mathbf{M}} K(x, t; y, t_1) u(t_1) dg_{t_1}(y) = \int_{\mathbf{M}} K(x, t; y, t_1) d_{t_1}(x_1, y) dg_{t_1}(y).$$

Using [1, Theorem 1.4] (compare also with (3.2)), we find that by Lemma 2.1

$$\begin{aligned}
u(x_1, t_2) &\le \int_{\mathbf{M}} \frac{C_1^*}{(t_2 - t_1)^{n/2}} \exp\left(-\frac{d_{t_1}^2(x_1, y)}{C_2^*(t_2 - t_1)}\right) d_{t_1}(x_1, y) dg_{t_1}(y) \\
&= \sum_{k=-\infty}^{\infty} \int_{B(x_1,t_1,2^k)\setminus B(x_1,t_1,2^{k-1})} \frac{C_1^*}{(t_2-t_1)^{n/2}} \exp\left(-\frac{d_{t_1}^2(x_1,y)}{C_2^*(t_2-t_1)}\right) \\
&\qquad \cdot d_{t_1}(x_1, y) dg_{t_1}(y) \\
&\le \sum_{k=-\infty}^{\infty} |B(x_1, t_1, 2^k)|_{t_1} \frac{C_1^*}{(t_2 - t_1)^{n/2}} \exp\left(-\frac{2^{2k-2}}{C_2^*(t_2 - t_1)}\right) \cdot 2^k \\
&\le \sum_{k=-\infty}^{\infty} C_0 (2^k)^n e^{C_0 2^k} \frac{C_1^*}{(t_2 - t_1)^{n/2}} \exp\left(-\frac{2^{2k}}{4C_2^*(t_2 - t_1)}\right) \cdot 2^k \\
&\le \int_{\mathbb{R}^n} \frac{C C_0 C_1^*}{(t_2 - t_1)^{n/2}} \exp\left(2C_0|x| - \frac{|x|^2}{4C_2^*(t_2 - t_1)}\right) |x| dx \\
&= \sqrt{t_2 - t_1} \int_{\mathbb{R}^n} C C_0 C_1^* \exp\left(2C_0|x|\sqrt{t_2 - t} - \frac{|x|^2}{4C_2^*}\right) |x| dx \le C\sqrt{t_2 - t_1}
\end{aligned}$$

On the other hand, using (3.1),

$$
\begin{aligned}
|d - u(x_2, t_2)| &= \left| \int_{\mathbf{M}} K(x_2, t; y, t_1)(d - d_{t_1}(x_1, y)) dg_{t_1}(y) \right| \\
&\leq \int_{\mathbf{M}} K(x_2, t; y, t_1) |d_{t_1}(x_1, x_2) - d_{t_1}(x_1, y)| dg_{t_1}(y) \\
&\leq \int_{\mathbf{M}} K(x_2, t; y, t_1) d_{t_1}(x_2, y) dg_{t_1}(y).
\end{aligned}
$$

So similarly,

$$|d - u(x_2, t_2)| \leq C\sqrt{t_2 - t_1}.$$

It follows that

$$|u(x_1, t_2) - u(x_2, t_2)| \geq d - 2C\sqrt{t_2 - t_1}. \tag{4.1}$$

Next, consider the quantity $|\nabla u|$ on $\mathbf{M} \times [t_1, t_2]$. It is not hard to check that, in the barrier sense,

$$\partial_t |\nabla u| \leq \Delta |\nabla u|. \tag{4.2}$$

Since $|\nabla u|(\cdot, t_1) \leq 1$, we have by the maximum principle that $|\nabla u| \leq 1$ on $\mathbf{M} \times [t_1, t_2]$. So

$$|u(x_1, t_2) - u(x_2, t_2)| \leq d_{t_2}(x_1, x_2).$$

Together with (4.1) this gives us

$$d_{t_2}(x_1, x_2) \geq d - 2C\sqrt{t_2 - t_1} = d_{t_1}(x_1, x_2) - 2C\sqrt{t_2 - t_1}.$$

This finishes the proof. □

For the upper bound on the distance distortion, we will argue similarly, by reversing time. The derivation of the bound on $|\nabla u|$ will now be more complicated, since the equation (4.2) will have an extra $4|\mathrm{Ric}||\nabla u|$ term. We will overcome this difficulty by applying the generalized maximum principle from Proposition 3.5.

Lemma 4.2 *For every $A < \infty$ there are constants $\theta_6 = \theta_6(A) > 0$ and $C_6 = C_6(A) < \infty$ such that the following holds:*

Let $(\mathbf{M}^n, (g_t)_{t\in[-1,1]})$ be a Ricci flow on a compact, n-dimensional manifold $\mathbf{M}$ with the property that $\nu[g_{-1}, 4] \geq -A$. Assume that $|R| \leq R_0$ on $\mathbf{M} \times [-1, 1]$ for some constant $0 \leq R_0 \leq 1$. Let $[t_1, t_2] \subset [0, 1)$ be a sub-interval with $t_2 - t_1 \leq \theta_6(1 - t_1)$ and consider two points $x_1, x_2 \in \mathbf{M}$. Then

$$d_{t_2}(x_1, x_2) \leq \exp\left(C_6 R_0^{1/2}\sqrt{t_2 - t_1}\right) d_{t_1}(x_1, x_2) + C_6\sqrt{t_2 - t_1}.$$

Proof Set $d := d_{t_2}(x_1, x_2)$. For $i = 1, 2$ let $u_i \in C^0(\mathbf{M} \times [t_1, t_2]) \cap C^\infty(\mathbf{M} \times [t_1, t_2))$ be a solution to the backwards (not the conjugate!) heat equation

$$-\partial_t u_i = \Delta u_i, \qquad u_i(\cdot, t_2) = d_{t_2}(x_i, \cdot) \tag{4.3}$$

and let $v_i \in C^0(\mathbf{M} \times [t_1, t_2]) \cap C^\infty(\mathbf{M} \times [t_1, t_2))$ be a solution to the conjugate heat equation

$$-\partial_t v_i = \Delta v_i - R v_i, \qquad v_i(\cdot, t_2) = d_{t_2}(x_i, \cdot).$$

Note that by the maximum principle, we have on $\mathbf{M} \times [t_1, t_2]$

$$u_1 + u_2 \geq \min_{\mathbf{M}} \big(u_1(\cdot, t_2) + u_2(\cdot, t_2)\big) \geq \min_{\mathbf{M}} \big(d_{t_2}(x_1, \cdot) + d_{t_2}(x_2, \cdot)\big) \geq d. \tag{4.4}$$

We also claim that we have for all $t \in [t_1, t_2]$

$$u_i(\cdot, t) \leq e^{R_0(t_2 - t)} v_i(\cdot, t). \tag{4.5}$$

This inequality follows by the maximum principle and by the fact that whenever $v_i \geq 0$, we have

$$(-\partial_t - \Delta)\big(e^{R_0(t_2-t)} v_i(\cdot, t)\big) = e^{R_0(t_2-t)} R_0 v_i(\cdot, t) - e^{R_0(t_2-t)} R(\cdot, t) v_i(\cdot, t) \geq 0.$$

We now make use of the fact that for any $x \in \mathbf{M}$,

$$v_i(x, t_1) = \int_{\mathbf{M}} K(y, t_2; x, t_1) v_i(y, t_2) dg_{t_2}(y) = \int_{\mathbf{M}} K(y, t_2; x, t_1) d_{t_2}(x_i, y) dg_{t_2}(y)$$

and

$$K(y, t_2; x, t_1) < \frac{C_1^*}{(t_2 - t_1)^{n/2}} \exp\left(-\frac{d_{t_2}^2(x, y)}{C_2^*(t_2 - t_1)}\right),$$

for some constants C_1^*, C_2^*, which depend only on A. Note that the latter inequality is similar to (3.2) except that the distance between x, y is taken at time t_2. This inequality follows from [1, Theorem 1.4] and the subsequent comment in that paper. We can hence estimate, similarly as in the proof of Lemma 4.1,

$$v_i(x_i, t_1) \leq \int_{\mathbf{M}} \frac{C_1^*}{(t_2 - t_1)^{n/2}} \exp\left(-\frac{d_{t_2}^2(x_i, y)}{C_2^*(t_2 - t_1)}\right) d_{t_2}(x_i, y) dg_{t_2}(y) \leq C\sqrt{t_2 - t_1}.$$

So, using (4.5), we have

$$u_i(x_i, t_1) \leq C e^{R_0(t_2 - t_1)} \sqrt{t_2 - t_1} \leq 10C\sqrt{t_2 - t_1}.$$

So by (4.4) we have

$$u_1(x_2, t_1) \geq d - u_2(x_2, t_1) \geq d - 10C\sqrt{t_2 - t_1}.$$

This implies

$$|u_1(x_1, t_1) - u_1(x_2, t_2)| \geq d - 20C\sqrt{t_2 - t_1}. \tag{4.6}$$

Taking derivatives of (4.3), we obtain the evolution inequality

$$-\partial_t |\nabla u_1| \leq \Delta |\nabla u_1| + 4|\mathrm{Ric}| \cdot |\nabla u_1| \leq \Delta |\nabla u_1| + (4 + \sqrt{n})|\mathrm{Ric}| \cdot |\nabla u_1| - R|\nabla u_1|,$$

which holds in the barrier sense. Note that by definition $|\nabla u_1(\cdot, t_2)| \leq 1$. So, by Proposition 3.5, we have for sufficiently small θ_6

$$|\nabla u_1(\cdot, t_1)| \leq 1 + C R_0^{1/2} \sqrt{t_2 - t_1}.$$

So, using (4.6), we obtain

$$\begin{aligned} d_{t_2}(x_1, x_2) - 10C\sqrt{t_2 - t_1} &\leq |u(x_1, t_1) - u(x_2, t_2)| \\ &\leq \big(1 + C R_0^{1/2} \sqrt{t_2 - t_1}\big) d_{t_1}(x_1, x_2) \leq \exp\big(C R_0^{1/2} \sqrt{t_2 - t_1}\big) d_{t_1}(x_1, x_2). \end{aligned}$$

This finishes the proof. □

Next, we remove the assumption $t_2 - t_1 \leq \theta_6(1-t_1)$ from Lemma 4.2.

Lemma 4.3 *For every $A < \infty$ there is a constant $C_7 = C_7(A) < \infty$ such that the following holds:*

Let $(\mathbf{M}^n, (g_t)_{t\in[-1,1]})$ be a Ricci flow on a compact, n-dimensional manifold $\mathbf{M}$ with the property that $\nu[g_{-1}, 4] \geq -A$. Assume that $|R| \leq R_0$ on $\mathbf{M} \times [-1,1]$ for some constant $0 \leq R_0 \leq 1$. Let $0 \leq t_1 \leq t_2 \leq 1$ and consider two points $x, y \in \mathbf{M}$. Then

$$d_{t_2}(x,y) \leq \exp\big(C_7 R_0^{1/2}\sqrt{t_2-t_1}\big) d_{t_1}(x,y) + C_7\sqrt{t_2-t_1}.$$

Proof In the case in which $t_2 - t_1 \leq \theta_6(1-t_1)$, the bound follows immediately from Lemma 4.2. Let us now assume that $t_2 - t_1 > \theta_6(1-t_1)$. By continuity we may also assume without loss of generality that $t_2 < 1$.

Choose times

$$t'_k := 1 - (1-\theta_6)^k(1-t_1)$$

and observe that $t'_0 = t_1$ and

$$t'_{k+1} - t'_k = \theta_6(1-\theta_6)^k(1-t_1) = \theta_6(1-t'_k).$$

So by Lemma 4.2

$$d_{t'_k}(x,y) \leq \exp\Big(C_6 R_0^{1/2}\sum_{l=1}^k \sqrt{t'_l - t'_{l-1}}\Big) d_{t_1}(x,y)$$
$$+ C_6 \sum_{l=1}^k \exp\Big(C_6 R_0^{1/2} \sum_{j=l+1}^k \sqrt{t'_j - t'_{j-1}}\Big)\sqrt{t'_l - t'_{l-1}}.$$

Since

$$\sum_{l=1}^k \sqrt{t'_l - t'_{l-1}} = \sum_{l=1}^k \sqrt{\theta_6}(1-\theta_6)^{l/2}\sqrt{1-t_1} \leq C'\sqrt{1-t_1}$$

and

$$\sum_{l=1}^k \exp\Big(C_6 R_0^{1/2}\sum_{j=l+1}^k \sqrt{t'_j - t'_{j-1}}\Big)\sqrt{t'_l - t'_{l-1}}$$
$$\leq \sum_{l=1}^k \exp\Big(C_6 C' R_0^{1/2}\sqrt{1-t_1}\Big)\sqrt{t'_l - t'_{l-1}} \leq C''\sqrt{1-t_1},$$

we find that for a generic constant $C < \infty$

$$d_{t'_k}(x,y) \leq \exp\big(C R_0^{1/2}\sqrt{1-t_1}\big) d_{t_1}(x,y) + C\sqrt{1-t_1}.$$

Choose now k such that $t'_k \leq t_2 < t'_{k+1}$. Then $t_2 - t'_k \leq t'_{k+1} - t'_k \leq \theta_6(1-t'_1)$, so again by Lemma 4.2, we have

$$d_{t_2}(x,y) \leq \exp\big(C_6 R_0^{1/2}\sqrt{t_2 - t'_k}\big) d_{t'_k}(x,y) + C_6\sqrt{t_2 - t'_k}$$
$$\leq \exp\big((C+C_6) R_0^{1/2}\sqrt{1-t_1}\big) d_{t_1}(x,y) + C\exp(1+C_6)\sqrt{1-t_1} + C_6\sqrt{1-t_1}.$$

The claim now follows using $\sqrt{1-t_1} < \theta_6^{-1/2}\sqrt{t_2-t_1}$. □

We can finally prove Theorem 1.1.

Proof of Theorem 1.1 Consider the Ricci flow $(\mathbf{M}^n, (g_t)_{t\in[0,1]})$ with $\nu[g_0, 1+A^{-1}] \geq -A$ and $|R| \leq R_0$ for $0 \leq R_0 \leq A$. After replacing A by $4A+2$, we may assume without loss of generality that $A > 2$ and that we even have $\nu[g_0, 1+4A^{-1}] \geq -A$.

We will first prove the distance bounds for the case in which $t_1 > 0$ and $t_2 \leq (1+A^{-1})t_1$. By monotonicity of ν (compare with [1, Sect. 2]), we find that for any $t \in [0,1]$ we have

$$\nu[g_t, 4A^{-1}] \geq \nu[g_0, 1+4A^{-1}] \geq -A.$$

Restrict the flow to the time-interval $[(1-A^{-1})t_1, (1+A^{-1})t_1]$ and parabolically rescale by $A^{1/2}t_1^{-1/2}$ to obtain a flow $(\widetilde{g}_t)_{t\in[A-1,A+1]}$. Then $\nu[\widetilde{g}_{A-1}, 4] \geq -A$ and $|\widetilde{R}| \leq \widetilde{R}_0 := A^{-1}t_1R_0 \leq 1$. Then t_1, t_2 correspond to times $\widetilde{t}_1 := A$, $\widetilde{t}_2 := At_1^{-1}t_2$ and we have

$$\widetilde{R}_0^{1/2}\sqrt{\widetilde{t}_2 - \widetilde{t}_1} = R_0^{1/2}\sqrt{t_2 - t_1}.$$

So the distance bounds follow from Lemmas 4.1 and 4.3.

Consider now the case in which $t_2 > (1+A^{-1})t_1$. So $t_1 < \lambda t_2$, where $\lambda := (1+A^{-1})^{-1} < 1$. By continuity we may assume without loss of generality that $t_1 > 0$. Then we can find $1 \leq k_2 < k_1$ such that $t_1 \in [\lambda^{k_1}, \lambda^{k_1-1}]$ and $t_2 \in [\lambda^{k_2}, \lambda^{k_2-1}]$. Using our previous conclusions, we find

$$d_{t_2}(x,y) \geq d_{\lambda^{k_2}}(x,y) - C\sqrt{\lambda^{k_2}} \geq \ldots \geq d_{t_1}(x,y) - C\sum_{l=k_1}^{k_2}\sqrt{\lambda^l} \geq d_{t_1}(x,y) - C'C\lambda^{k_2/2}.$$

Since $t_1 < \lambda t_2$, we have $\sqrt{t_2-t_1} > \sqrt{(1-\lambda)t_2} > \sqrt{1-\lambda}\sqrt{\lambda^{k_2}}$. So

$$d_{t_2}(x,y) \geq d_{t_1}(x,y) - C'C(1-\lambda)^{-1/2}\sqrt{t_2-t_1}.$$

This establishes the lower bound.

For the upper bound, set $t'_0 := t_1$, $t'_1 := \lambda^{k_1-1}, \ldots, t'_{k_1-k_2} := \lambda^{k_2}$, $t'_{k_1-k_2+1} := t_2$. Then we have by our previous conclusions

$$d_{t_2}(x,y) \leq \exp\left(CR_0^{1/2}\sum_{l=1}^{k_1-k_2+1}\sqrt{t'_l - t'_{l-1}}\right)d_{t_1}(x,y)$$
$$+C\sum_{l=1}^{k_2-k_1+1}\exp\left(CR_0^{1/2}\sum_{j=l+1}^{k_1-k_2+1}\sqrt{t'_j - t'_{j-1}}\right)\sqrt{t'_l - t'_{l-1}}$$

Similarly as in the proof of Lemma 4.3, we conclude

$$d_{t_2}(x,y) \leq \exp\left(CR_0^{1/2}\sqrt{\lambda^{k_2}}\right)d_{t_1}(x,y) + C\sqrt{\lambda^{k_2}}.$$

Again, using $\sqrt{t_2-t_1} > \sqrt{1-\lambda}\sqrt{\lambda^{k_2}}$, we get the desired bound. □

5 Proof of corollary 1.3

Proof of Corollary 1.3 For each i consider the metric $\overline{d}^i$ on $\mathbf{M}^i$ with

$$\overline{d}^i(x,y) := \int_0^1 d_t^i(x,y)dt.$$

Note that by the Hölder bound in Theorem 1.1 there is a uniform constant $c' > 0$ such that for all $t, t' \in [0,1]$ we have $d^i_{t'}(x,y) > \frac{1}{2} d^i_t(x,y)$ whenever $|t-t'| \leq c'(d^i_t(x,y))^2$. So there is a uniform constant $c > 0$ such that for all $t \in [0,1]$

$$\overline{d}^i(x,y) \geq c\big(\min\{d^i_t(x,y), 1\}\big)^3. \tag{5.1}$$

So by the triangle inequality and Theorem 1.1, for any $A < \infty$ there is a constant $C < \infty$ such that for any $x, y, x', y' \in \mathbf{M}$ and $t, t' \in [0,1]$ with $\overline{d}^i(x,y) + \overline{d}^i(x,x') + \overline{d}^i(y,y') < A$ we have

$$\big|d^i_t(x,y) - d^i_{t'}(x',y')\big| \leq C\big(\overline{d}^i(x,x')\big)^{1/3} + C\big(\overline{d}^i(y,y')\big)^{1/3} + C|t-t'|^{1/2}. \tag{5.2}$$

We first argue that the sequence $(\mathbf{M}^i, \overline{d}^i)$ is uniformly totally bounded in the following sense: For any $0 < a < b$ there is a number $N = N(a,b) < \infty$ such that for any i and any $x \in \mathbf{M}^i$, the ball $\overline{B}^i(x,b) := \{x \in \mathbf{M}^i \;:\; \overline{d}^i(x,z) < b\}$ contains at most N pairwise disjoint balls $\overline{B}^i(y_j, a)$, $j = 1, \ldots, m$. Fix $0 < a < b$ and assume without loss of generality that $a < 1$. By (5.1) there is a constant $b' = b'(b) < \infty$ such that $\overline{B}^i(x,b) \subset B^i(x,t,b')$ for all $t \in [0,1]$.

Assume that $y_1, \ldots, y_m \in \overline{B}^i(x,b)$ such that the balls $\overline{B}^i(y_j, a)$ are pairwise disjoint. This implies $\overline{d}^i(y_{j_1}, y_{j_2}) \geq 2a$ for all $j_1 \neq j_2$. By the Hölder bound in Theorem 1.1, we may find a large integer $L = L(a) < \infty$ such that whenever $\overline{d}^i(y,y') \geq 2a$ for some points $y, y' \in \mathbf{M}^i$, then $d^i_{\frac{l}{L}}(y,y') > a$ for some $l \in \{1, \ldots, L\}$. So for any $j_1 \neq j_2$, there is an $l_{j_1,j_2} \in \{1, \ldots, L\}$ such that

$$d^i_{\frac{l_{j_1,j_2}}{L}}(y_{j_1}, y_{j_2}) > a.$$

This implies the following statement: If we form the L-fold Cartesian product $\mathbf{M}^{i,L} := (\mathbf{M}^i)^L = \mathbf{M} \times \ldots \times \mathbf{M}$ equipped with the metric $g^i_{\frac{1}{L}} \oplus \ldots \oplus g^i_{\frac{L-1}{L}}$ and if we define $y^L_j := (y_j, \ldots, y_j) \in \mathbf{M}^{i,L}$, then $d^{\mathbf{M}^{i,L}}(y^L_{j_1}, y^L_{j_2}) > a$ for any $j_1 \neq j_2$. So the $\frac{1}{2}a$-balls around $y^L_{j_1}$ are pairwise disjoint and contained in $B^i(x, \frac{1}{L}, b'+a) \times \ldots \times B^i(x, \frac{L-1}{L}, b'+a)$. Using (2.1) and Lemma 2.1, we conclude that

$$\left(c\left(\frac{a}{\sqrt{L}}\right)^n\right)^L \cdot m \leq \left(C_0 (b')^n e^{C_0 b'}\right)^L,$$

which yields an upper bound on m. So the sequence $(\mathbf{M}^i, \overline{d}^i)$ is in fact uniformly totally bounded.

We may now pass to a subsequence and assume that $(\mathbf{M}^i, \overline{d}^i, x_i)$ converges to some metric space $(\overline{\mathbf{M}}, \overline{d}, \overline{x})$ in the pointed Gromov–Hausdorff sense. By (5.2) and Arzelá–Ascoli and after passing to another subsequence, the sequence of time-dependent metrics $(d^i)_{t\in[0,1]}$ converges locally uniformly to a time-dependent, continuous family of pseudometrics $(d^\infty_t)_{t\in[0,1]}$ on $\overline{\mathbf{M}}$. So for any $t \in [0,1]$, the pointed metric spaces $(\mathbf{M}^i, d^i_t, x_i)$ converge to $(\overline{\mathbf{M}}/\sim_t, d^\infty_t, \overline{x})$ in the pointed Gromov–Hausdorff sense. Passing to another subsequence once again, and using (2.1), we can ensure that also the volume forms dg^i_t converge uniformly for every rational $t \in [0,1]$. Since $e^{-A|t_2-t_1|} dg^i_{t_1} \leq dg^i_{t_2} \leq e^{A|t_2-t_1|} dg^i_{t_1}$, the convergence holds for any $t \in [0,1]$. The doubling property for balls of radius less than $\sqrt{t}$ follows from (2.1) after parabolic rescaling by $(\frac{1}{2}t)^{-1/2}$. □

References

1. Bamler, R., Zhang, Q.: Heat kernel and curvature bounds in Ricci flows with bounded scalar curvature (2015). arXiv:1501.01291
2. Colding, T.H., Naber, A.: Sharp Hölder continuity of tangent cones for spaces with a lower Ricci curvature bound and applications. Ann. Math. **176**(2), 1173–1229 (2012). https://doi.org/10.4007/annals.2012.176.2.10
3. Chen, X., Wang, B.: On the conditions to extend Ricci flow (III). Int. Math. Res. Not. **10**, 2349–2367 (2013)
4. Perelman, G.: The entropy formula for the Ricci flow and its geometric applications (2002). arXiv:math/0211159
5. Zhang, Q.S.: Bounds on volume growth of geodesic balls under Ricci flow. Math. Res. Lett. **19**(1), 245–253 (2012). https://doi.org/10.4310/MRL.2012.v19.n1.a19