Calc. Var. (2018) 57:163
https://doi.org/10.1007/s00526-018-1442-7

Calculus of Variations

Liouville type equation with exponential Neumann boundary condition and with singular data

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Received: 8 February 2018 / Accepted: 7 September 2018 / Published online: 1 October 2018

Abstract

In this paper we will analyze the blow-up behaviors of solutions to the singular Liouville type equation with exponential Neumann boundary condition. We generalize the Brezis–Merle type concentration-compactness theorem to this Neumann problem. Then along the line of the Li–Shafrir type quantization property we show that the blow-up value $m(0) \in 2\pi\mathbb{N} \cup \{2\pi(1+\alpha) + 2\pi(\mathbb{N} \cup \{0\})\}$ if the singular point 0 is a blow-up point. In the end, when the boundary value of solutions has an additional condition, we can obtain the precise blow-up value $m(0) = 2\pi(1+\alpha)$.

Mathematics Subject Classification 35B40 · 35J65

1 Introduction

Let Ω be a bounded smooth domain in $\mathbb{R}^2$. As is well known, topological degree and variational methods can be used to obtain existence results for many Liouville type equations. And this requires the compactness property for the solution set. So it is important to obtain the blow-up analysis for the equations. The asymptotic blow-up analysis for Liouville type equations has already a lot of progresses. In 1991, Brezis and Merle [2] showed a concentration-compactness phenomena of solutions to the following Liouville equation:

$$-\Delta u = V(x)e^u \quad \text{in } \Omega.$$

And then Li and Shafrir [10] initiated to evaluate the blow-up value at the blow-up point. They showed at the each blow-up point the blow-up value is quantized, i.e., there is no

Communicated by J. Jost.

The authors are supported partially by NSFC of China (No. 11771285).

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contribution of mass outside the m disjoint balls which contain a contribution of $8\pi m$ mass for some positive integer m.

In recent years, the Liouville type equation with singular data attracts much attention due to their many applications in Mathematics and Physics, such as cosmic string equation, Chern–Simons and Electroweak self-dual vortices, etc, see [12,14–17]. This type equation can be reduced to the following equation:

$$-\Delta u = |x|^{2\alpha} V(x) e^u \quad \text{in } \Omega, \alpha > -1.$$

The Brezis–Merle type concentration-compactness type result has been established in [4] and [3]. Furthermore, Tarantello [13] generalized Li–Shafrir type quantization property to show that the blow-up value $m(0) \in 8\pi\mathbb{N} \cup \{8\pi(1+\alpha) + 8\pi(\mathbb{N} \cup \{0\})\}$ if the singular point 0 is a blow-up point.

In addition, there have been some progresses in the blow-up analysis of the Liouville type equation under the Neumann boundary condition. Guo and Liu [7] have analyzed the following equation:

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \beta = h(x) e^u & \text{on } \partial\Omega. \end{cases}$$

Here and in the sequel, ν is the out unit normal vector on the boundary. They also obtained Brezis–Merle type concentration-compactness phenomena and Li–Shafrir type quantization property. Later, Bao et al. [5] have studied the following geometric equations on compact Riemnn surface (M, g):

$$\begin{cases} -\Delta u_g = 2e^{2u} - K_g & \text{in } M^o, \\ \dfrac{\partial u}{\partial \nu} = ce^u - h_g & \text{on } \partial M. \end{cases}$$

They obtained Brezis–Merle type concentration-compactness phenomena. Recently, Zhang et al. [18] have proved the quantization property of blowing-up solutions for the local equations:

$$\begin{cases} -\Delta u = V(x) e^{2u} & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = h(x) e^u & \text{on } L. \end{cases}$$

Here L is a proper subset of $\partial\Omega$, $V(x)$ and $h(x)$ are nonnegative bounded functions.

In this paper we will consider the local singular Liouville type equation with Neumann boundary condition. Without loss of generality, we consider the following boundary value problem in $B_R^+(0)$:

$$\begin{cases} -\Delta u_n = V_n(x)|x|^{2\alpha} e^{2u_n} & \text{in } B_R^+(0), \\ \dfrac{\partial u_n}{\partial \nu} = h_n(x)|x|^{\alpha} e^{u_n} & \text{on } \partial B_R^+(0) \cap \partial\mathbb{R}_+^2, \end{cases} \tag{1}$$

where $\alpha \in (-1, +\infty)$ and the coefficient functions V_n and h_n satisfy

$$\begin{aligned} & V_n \to V, h_n \to h \text{ uniformly in } \bar{B}_R^+; \\ & 0 < a \le V_n \le C, |\nabla V_n| \le A; \quad 0 < b \le h_n \le C, |\nabla h_n| \le B. \end{aligned} \tag{2}$$

In the sequel, we always assume that $V_n(x)$ and $h_n(x)$ satisfy the above assumptions. We set $B_R^+(x_0) = \{x = (s,t) \in \mathbb{R}^2 | |x - x_0| < R, t > 0\}$, $L_R(x_0) = \partial B_R^+(x_0) \cap \partial\mathbb{R}_+^2$ and $S_R^+(x_0) = \partial B_R^+(x_0) \cap \mathbb{R}_+^2$. We also use the notations B_R^+, L_R, S_R^+ for $B_R^+(0)$, $L_R(0)$, $S_R^+(0)$ respectively.

Our first main result is about "Brezis–Merle type concentration-compactness phenomena Theorem".

Theorem 1.1 *Assume that $\{u_n\}$ is a sequence of solutions of (1) with $\alpha \in (-1, +\infty)$. If $\{u_n\}$ satisfies the energy conditions*

$$\int_{B_R^+} V_n|x|^{2\alpha} e^{2u_n} \le C \quad \text{and} \quad \int_{L_R} h_n|x|^{\alpha} e^{u_n} \le C \tag{3}$$

for the constant C which is independent of n, then there exists a subsequence, denoted still by $\{u_n\}$, satisfying one of the following alternatives:

(i) *$\{u_n\}$ is bounded in $L^\infty_{loc}(B_R^+ \cup L_R)$,*
(ii) *$\{u_n\} \to -\infty$ uniformly on compact subsets of $B_R^+ \cup L_R$,*
(iii) *We can define a finite and nonempty blow-up set of u_n*

$$S = \{x \in B_R^+ \cup L_R, \text{ there is a sequence } y_n \to x \text{ such that } u_n(y_n) \to +\infty\}.$$

such that

$$\{u_n\} \to -\infty \text{ uniformly on compact subsets of } (B_R^+ \cup L_R)\backslash S.$$

Our second main result is about "Li–Shafrir type quantization property".

Theorem 1.2 *Assume that $\{u_n\}$ is a sequence of solutions of (1) with $R = 1$ and $\alpha \in (-1, +\infty)\backslash\{2k+1\}$, $k = 0, 1, 2, \ldots$. If $\{u_n\}$ satisfies in addition that*

$$\int_{B_1^+} V_n|x|^{2\alpha} e^{2u_n}\phi + \int_{L_1} h_n|x|^{\alpha} e^{u_n}\phi \to m(0)\phi(0), \text{ for every } \phi \in C_c^\infty(B_1^+ \cup L_1), \tag{4}$$

i.e. zero is the only blow-up point of u_n in $\bar{B}_1^+$, then $m(0) \in 2\pi\mathbb{N} \cup \{2\pi(1+\alpha) + 2\pi(\mathbb{N} \cup \{0\})\}$.

From Theorem 1.2 it is natural to ask what is the precise value of the "mass" $m(0)$. We give an affirmative answer under an extra boundary condition:

$$\max_{S_1^+} u_n - \min_{S_1^+} u_n \le C \tag{5}$$

with C a suitable positive constant.

Theorem 1.3 *Under the assumptions of Theorem 1.2, if we suppose in addition that u_n satisfies (5), then we have $m(0) = 2\pi(1+\alpha)$.*

The proof of our main results follow closely the ideas in [4,10,13]. Since the problems involve Neumann boundary condition and the singular data, the steps of the blow-up analysis become more delicate. When we prove Theorem 1.1, we need to use the local Green representation formula and the Pohozaev type identity of Neumann problem. For the proof of Theorem 1.2, we will use the approach in [10,13], which is based on a classification result of bubbling equation

$$-\Delta u = e^{2u} \quad \text{in } \mathbb{R}^2$$

with $\int_{\mathbb{R}^2} e^{2u} < \infty$ and a "sup + inf" type inequality

$$u(0) + C_1 \inf_{B_1} u \le C_2$$

for equation $-\Delta u = Ve^{2u}$ in B_1. For our problem, we need the corresponding results. On one hand, besides of the above bubbling equation, there exist the other two kinds of bubbling equation, i.e.

$$\begin{cases} -\Delta u = V(0)e^{2u} & \text{in } \mathbb{R}^2 \cap \{t > -\Lambda\}, \\ \dfrac{\partial u}{\partial \nu} = h(0)e^u & \text{on } \mathbb{R}^2 \cap \{t = -\Lambda\}, \end{cases}$$

with the energy conditions

$$\int_{\mathbb{R}^2 \cap \{t > -\Lambda\}} V(0)e^{2u} < +\infty, \quad \int_{\mathbb{R}^2 \cap \{t = -\Lambda\}} h(0)e^u < +\infty;$$

and

$$\begin{cases} -\Delta u = V(0)|x|^{2\alpha}e^{2u} & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial u}{\partial \nu} = h(0)|x|^\alpha e^u & \text{on } \partial\mathbb{R}^2_+. \end{cases}$$

with the energy condition

$$\int_{\mathbb{R}^2_+} |x|^{2\alpha}e^{2u}dx < +\infty, \quad \int_{\partial\mathbb{R}^2_+} |x|^\alpha e^u ds < \infty.$$

We will use the classification results shown in [8,11] to handle our problem. On the other hand, we need to prove a "sup + inf" type inequality for this Neumann problem by using the moving plan method.

This paper is organized as follows. In this introduction, we state our main theorems. In Sect. 2, we study the blow-up behaviors for the considered Neumann boundary value problem, and give the proof of corresponding concentration-compactness Theorem 1.1. In Sects. 3 and 4, we give the version of Tarantello's decomposition Proposition and "sup + inf" type inequality under the Neumann boundary conditions separately. In Sect. 5, we will prove Theorem 1.2. In Sect. 6, we will consider the case u_n satisfy (5) and then we give the the proof of Theorem 1.3.

2 Blow-up analysis

In this section, we will study the blow-up behaviors for the considered Liouville type equation with Neumann boundary value condition and with singular data. We shall analyze the regularity of solutions to (1), (2) and (3). Consequently, we can prove Theorem 1.1. In the sequel, we will handle the problem with $\alpha \geq 0$ and $-1 < \alpha < 0$ separately.

Proposition 2.1 *Let $\alpha \geq 0$, $\epsilon_1 < \frac{\pi}{2}$ and $\epsilon_2 < \pi$. Assume that $\{u_n\}$ is a sequence of solutions which satisfies that*

$$\begin{cases} -\Delta u_n = V_n|x|^{2\alpha}e^{2u_n}, & \text{in } B_r^+, \\ \dfrac{\partial u_n}{\partial \nu} = h_n|x|^\alpha e^{u_n}, & \text{on } L_r, \end{cases}$$

and

$$\int_{B_r^+} V_n|x|^{2\alpha}e^{2u_n}dx < \epsilon_1, \quad \int_{L_r} h_n|x|^\alpha e^{u_n}dx < \epsilon_2. \tag{6}$$

Then u_n^+ is bounded in $L^\infty(\bar{B}^+_{\frac{r}{4}})$.

Proof Define $u_{1,n}$, $u_{2,n}$ by

$$\begin{cases} -\Delta u_{1,n} = V_n|x|^{2\alpha}e^{2u_n}, & \text{in } B_r^+, \\ \dfrac{\partial u_{1,n}}{\partial \nu} = 0, & \text{on } L_r, \\ u_{1,n} = 0, & \text{on } S_r^+. \end{cases}$$

$$\begin{cases} -\Delta u_{2,n} = 0, & \text{in } B_r^+, \\ \dfrac{\partial u_{2,n}}{\partial \nu} = h_n|x|^{\alpha}e^{u_n}, & \text{on } L_r, \\ u_{2,n} = 0, & \text{on } S_r^+. \end{cases}$$

Extending u_n, $u_{1,n}$ and V_n evenly we have

$$\begin{cases} -\Delta u_{1,n} = V_n|x|^{2\alpha}e^{2u_n}, & \text{in } B_r, \\ u_{1,n} = 0, & \text{on } \partial B_r. \end{cases}$$

Due to $\epsilon_1 < \frac{\pi}{2}$ we obtain that

$$\int_{B_r} V_n|x|^{2\alpha}e^{2u_n} < 2\epsilon_1 < \pi.$$

Now by Theorem 1 in [2] we can choose δ_1 such that

$$\frac{4\pi - \delta}{2\epsilon_1} = 4 + \delta_1,$$

with $\delta_1 > 0$. Then we have

$$\int_{B_r^+} e^{(4+\delta_1)|u_{1,n}|} = \frac{1}{2}\int_{B_r} e^{(4+\delta_1)|u_{1,n}|} \le C.$$

For $u_{2,n}$, since $\epsilon_2 < \pi$, by Lemma 3.2 in [8] we also can choose $\delta_2 > 0$, $\delta_3 > 0$ such that

$$\int_{B_r^+} e^{(4+\delta_2)|u_{2,n}|} \le C, \quad \int_{L_r} e^{(2+\delta_3)|u_{2,n}|} \le C.$$

Let $u_{3,n} = u_n - u_{1,n} - u_{2,n}$. Then we have

$$\begin{cases} -\Delta u_{3,n} = 0, & \text{in } B_r^+, \\ \dfrac{\partial u_{3,n}}{\partial \nu} = 0, & \text{on } L_r. \end{cases}$$

Extending $u_{3,n}$ evenly, $u_{3,n}$ becomes a harmonic function in B_r. Then the mean value theorem for harmonic functions implies that

$$\|u_{3,n}^+\|_{L^\infty(\bar{B}^+_{\frac{r}{2}})} \le C\|u_{3,n}^+\|_{L^1(B_r^+)}.$$

Notice that

$$u_{3,n}^+ \le u_n^+ + |u_{1,n}| + |u_{2,n}|.$$

Now we choose $t > 0$ such that $\int_{B_r^+} \frac{1}{|x|^{2\alpha t}} dx \le C$. Set $s = \frac{t}{t+1} < 1$ when $\alpha > 0$ and $s = 1$ when $\alpha = 0$. Then it follows from Holder's inequality to get

$$\int_{B_r^+} e^{2su_n} dx \le \left(\int_{B_r^+} |x|^{2\alpha} e^{2u_n} dx\right)^s \left(\int_{B_r^+} \frac{1}{|x|^{2\alpha t}} dx\right)^{1-s} \le C.$$

Therefore we have

$$\int_{B_r^+} u_n^+ dx \le \frac{1}{2s}\int_{B_r^+} e^{2su_n} dx \le C,$$

and consequently we have

$$\|u_{3,n}^+\|_{L^\infty(\bar{B}^+_{\frac{r}{2}})} \le C.$$

Finally, we rewrite the equations as

$$\begin{cases} -\Delta u_n = V_n|x|^{2\alpha}e^{2u_n} = f_n, & \text{in } \ B_r^+, \\ \dfrac{\partial u_n}{\partial \nu} = h_n|x|^\alpha e^{u_n} = g_n, & \text{on } \ L_r. \end{cases}$$

Since

$$f_n = V_n|x|^{2\alpha}e^{2u_{3,n}+2u_{1,n}+2u_{2,n}}, \quad g_n = h_n|x|^\alpha e^{u_{3,n}+u_{1,n}+u_{2,n}},$$

we know that $\|f_n\|_{L^q(B^+_{\frac{r}{2}})} \le C$ and $\|g_n\|_{L^q(L_{\frac{r}{2}})} \le C$ for some $q > 1$. Then the standard elliptic estimates imply that

$$\|u_n^+\|_{L^\infty(\bar{B}^+_{\frac{r}{4}})} \le C.$$

□

Next we consider the case $-1 < \alpha < 0$. There have subtle differences between the case $-1 < \alpha < 0$ and the case $\alpha \ge 0$.

Proposition 2.2 *Let $-1 < \alpha < 0$, and choose suitable constants $1 < p < \frac{2}{1-\alpha}$, $p_1 = \frac{\alpha-1}{2\alpha}$ and $p_2 = \frac{1-\alpha}{1+\alpha}$ such that $\frac{1}{p_1} + \frac{1}{p_2} = 1$. Let $\epsilon_1 < \frac{\pi}{2pp_2}$, and $\epsilon_2 < \frac{\pi}{pp_2}$. Assume that $\{u_n\}$ is a sequence of solutions which satisfies that*

$$\begin{cases} -\Delta u_n = V_n|x|^{2\alpha}e^{2u_n}, & \text{in } \ B_r^+, \\ \dfrac{\partial u_n}{\partial \nu} = h_n|x|^\alpha e^{u_n}, & \text{on } \ L_r, \end{cases}$$

and

$$\int_{B_r^+} V_n|x|^{2\alpha}e^{2u_n}dx < \epsilon_1, \quad \int_{L_r} h_n|x|^\alpha e^{u_n}dx < \epsilon_2. \tag{7}$$

Then $\|u_n^+\|_{L^\infty(\bar{B}^+_{\frac{r}{4}})}$ is bounded.

Proof Define $u_{1,n}, u_{2,n}$ by

$$\begin{cases} -\Delta u_{1,n} = V_n|x|^{2\alpha}e^{2u_n}, & \text{in } \ B_r^+, \\ \dfrac{\partial u_{1,n}}{\partial \nu} = 0, & \text{on } \ L_r, \\ u_{1,n} = 0, & \text{on } \ S_r^+. \end{cases}$$

$$\begin{cases} -\Delta u_{2,n} = 0, & \text{in } \ B_r^+, \\ \dfrac{\partial u_{2,n}}{\partial \nu} = h_n|x|^\alpha e^{u_n}, & \text{on } \ L_r, \\ u_{2,n} = 0, & \text{on } \ S_r^+. \end{cases}$$

Extending u_n, $u_{1,n}$ and $V_n(x)$ evenly we have

$$\begin{cases} -\Delta u_{1,n} = V_n|x|^{2\alpha}e^{2u_n}, & \text{in} \quad B_r, \\ u_{1,n} = 0, & \text{on} \quad \partial B_r. \end{cases}$$

As the similar arguments in Proposition 2.1 we can obtain for some $\delta > 0$ that

$$\int_{B_r^+} e^{(4+\delta)pp_2|u_{1,n}|} = \frac{1}{2}\int_{B_r} e^{(4+\delta)pp_2|u_{1,n}|} \leq C,$$

and

$$\int_{B_r^+} e^{(4+\delta)pp_2|u_{2,n}|} \leq C, \quad \int_{L_r} e^{(2+\delta)pp_2|u_{2,n}|} \leq C.$$

Let $u_{3,n} = u_n - u_{1,n} - u_{2,n}$. Then we have

$$\begin{cases} -\Delta u_{3,n} = 0, & \text{in} \quad B_r^+, \\ \dfrac{\partial u_{3,n}}{\partial \nu} = 0, & \text{on} \quad L_r. \end{cases}$$

Extending $u_{3,n}$ evenly, $u_{3,n}$ becomes a harmonic function in B_r. Then the mean value theorem for harmonic function implies that

$$\|u_{3,n}^+\|_{L^\infty(\bar{B}^+_{\frac{r}{2}})} \leq C\|u_{3,n}^+\|_{L^1(B_r^+)}.$$

Notice that

$$u_{3,n}^+ \leq u_n^+ + |u_{1,n}| + |u_{2,n}|.$$

Since $\alpha < 0$, (7) implies

$$\int_{B_r^+} e^{u_n} dx \leq C.$$

So we get

$$\int_{B_r^+} u_n^+ dx \leq \int_{B_r^+} e^{u_n} dx \leq C.$$

And we have

$$\|u_{3,n}^+\|_{L^\infty(\bar{B}^+_{\frac{r}{2}})} \leq C.$$

Thus, by Holder' inequality and $pp_1 < -\frac{1}{\alpha}$,

$$\int_{B^+_{\frac{r}{2}}} |x|^{2\alpha p} e^{2pu_n} dx \leq \int_{B^+_{\frac{r}{2}}} |x|^{2\alpha pp_1} dx \cdot \int_{B^+_{\frac{r}{2}}} e^{2pp_2 u_n} dx \leq C.$$

Hence we have $u_{1,n}$ is uniformly bounded in $B_r^+ \cup L_r$ and consequently

$$\int_{L_{\frac{r}{2}}} |x|^{\alpha p} e^{pu_n} dx \leq \int_{L_{\frac{r}{2}}} |x|^{\alpha pp_1} dx \cdot \int_{L_{\frac{r}{2}}} e^{pp_2 u_n} dx \leq C.$$

The standard elliptic estimates imply that

$$\|u_n^+\|_{L^\infty(\bar{B}^+_{\frac{r}{4}})} \leq C.$$

□

Next we present an inequality which has been established in [7].

Lemma 2.3 *[7] Let l be an imbedded C^1 curve in $\mathbb{R}^2$. $f \in L^1(l)$. Set $\|f\|_1 = \int_l |f(x)|dx$, and $\rho = diam\, l$. If we define*

$$\omega(x) = \frac{1}{\pi}\int_l \log\frac{\rho}{|x-y|}f(y)dy,$$

then for every $\delta \in (0, \pi)$ we have

$$\int_l \exp[(\pi-\delta)|\omega(x)|/\|f\|_1]dx \le \frac{C}{\delta}. \tag{8}$$

By using Lemma 2.3, we can get the following Lemma.

Lemma 2.4 *Set $f(x) \in L^1(L_r)$. If we define*

$$\omega(x) = \frac{1}{\pi}\int_{L_r} \log\frac{2r}{|x-y|}f(y)dy,$$

then for every $k > 0$ we have $e^{k|\omega|} \in L^1(L_r)$ and $e^{k|\omega|} \in L^1(B_r^+)$.

Proof Let $0 < \epsilon < \frac{1}{k}$. Since $f(x) \in L^1(L_r)$, we can split $f(x)$ as $f(x) = f_1(x) + f_2(x)$ with $\|f_1\|_1 < \epsilon$ and $f_2 \in L^\infty(L_r)$. Write $\omega(x) = \omega_1(x) + \omega_2(x)$ where

$$\omega_i(x) = \frac{1}{\pi}\int_{L_r} \log\frac{2r}{|x-y|}f_i(y)dy.$$

Choosing $\delta = \pi - 1$ in Lemma 2.3 we find $\int_{L_r} \exp[|\omega_1(x)|/\|f_1\|_1]dx \le C$. This implies that $e^{k|\omega_1|} \in L^1(L_r)$ for every $k > 0$. Thus the conclusion follows the fact $|\omega| \le |\omega_1| + |\omega_2|$ and $\omega_2 \in L^\infty(L_r)$. Using the same method of Lemma 2.3, we can get $\int_{B_r^+} \exp[(2\pi - \delta)|\omega(x)|/\|f\|_1]dx \le C$. Further more we can also obtain $e^{k|\omega|} \in L^1(B_r^+)$ for every $k > 0$.
□

Remark 2.5 If we set $f(x) \in L^1(B_r^+)$ and

$$\omega(x) = \frac{1}{\pi}\int_{B_r^+} \log\frac{2r}{|x-y|}f(y)dy,$$

by using the arguments in Lemma 2.4 again, then we can also obtain $e^{k|\omega|} \in L^1(L_r)$ and $e^{k|\omega|} \in L^1(B_r^+)$ for every $k > 0$.

In addition we need a Harnack inequality for a non-homogenous Neumann-type boundary problem for second-order elliptic equations, which has been established in [9].

Proposition 2.6 *Let $f \in L^p(B_r^+)$ for some $1 < p \le +\infty$, $g \in L^q(B_r^+ \cap \partial\mathbb{R}_+^2)$ for some $1 < q \le +\infty$, and u satisfy*

$$\begin{cases} -\Delta u = f, & in\ \ B_r^+, \\ \dfrac{\partial u}{\partial \nu} = g, & on\ \ B_r^+ \cap \partial\mathbb{R}_+^2, \\ u \le 0, & on\ \ \partial B_r^+ \cap \mathbb{R}_+^2. \end{cases}$$

Then for any $0 < \theta < 1$, there exist a constant $\beta \in (0, 1)$ depending on r, θ only, and a constant $\gamma > 0$ depending on r, p, q only such that

$$\sup_{\bar{B}_{\theta r}^+} u \le \beta \inf_{\bar{B}_{\theta r}^+} u + \gamma(\|f\|_{L^p(B_r^+)} + \|g\|_{L^q(\partial B_r^+ \cap \partial\mathbb{R}_+^2)}).$$

When the energy $\int_{B_R^+} V_n|x|^{2\alpha}e^{2u_n}$ and $\int_{L_R} h_n|x|^{\alpha}e^{u_n}$ are large, the blow-up phenomenon may occur, which is declared in Theorem 1.1. Next we give the proof of Theorem 1.1.

Proof of Theorem 1.1 Firstly we treat the case $\alpha \geq 0$. Since $V_n|x|^{2\alpha}e^{2u_n}$ is bounded in $L^1(B_R^+)$ and $h_n|x|^{\alpha}e^{u_n}$ is bounded in $L^1(L_R)$, along a subsequence (still denoted by u_n), such that

$$\int_{B_R^+} V_n|x|^{2\alpha}e^{2u_n}\varphi \to \int_{B_R^+} \varphi d\mu,$$
$$\int_{L_R} h_n|x|^{\alpha}e^{u_n}\phi \to \int_{L_R} \phi d\vartheta,$$

for every $\varphi \in C_c(B_R^+ \cup L_R)$ and $\phi \in C_c(L_R)$. Here μ and ϑ are two nonnegative bounded measures. A point $x \in B_R^+ \cup L_R$ is called an $\epsilon-$ regular point with respect to μ and ϑ if there is a function $\varphi \in C(B_R^+ \cup L_R)$, $supp\varphi \in B_r(x) \cap (B_R^+ \cup L_R)$ with $0 \leq \varphi \leq 1$ and $\varphi = 1$ in a neighborhood of x such that

$$\int_{B_R^+} \varphi d\mu < \epsilon, \text{ if } x \in B_R^+;$$
$$\int_{B_R^+} \varphi d\mu < \epsilon \text{ and } \int_{L_R} \varphi d\vartheta < \epsilon, \text{ if } x \in L_R.$$

□

We define the

$$\Sigma(\epsilon) = \{x \in B_R^+ \cup L_R : x \text{ is not an } \epsilon - \text{ regular point with respect to } \mu \text{ and } \vartheta\}.$$

By $\int_{B_R^+} V_n|x|^{2\alpha}e^{2u_n} \leq C$ and $\int_{L_R} h_n|x|^{\alpha}e^{u_n} \leq C$, we have $\Sigma(\epsilon)$ is finite. Furthermore we have $S = \Sigma(\epsilon_0)$ by using the similar arguments in [2,5], where $\epsilon_0 = \min\{\epsilon_1, \epsilon_2\}$ as in Proposition 2.1.

When $S = \emptyset$, it follows that (i) or (ii) holds. $S = \emptyset$ means that u_n^+ is uniformly bounded in $L^\infty(B_R^+ \cup L_R)$. Thus $f_n = V_n|x|^{2\alpha}e^{2u_n}$ is bounded in $L^p(B_R^+)$ for any $p > 1$, and $g_n = h_n|x|^{\alpha}e^{u_n}$ is bounded in $L^p(L_R)$ for any $p > 1$. Apply Harnack inequality in Proposition 2.6, we know that (i) or (ii) holds.

For the case $-1 < \alpha < 0$, we will use Proposition 2.2 instead of Proposition 2.1. Then similar with the case $\alpha \geq 0$, we can show (i) or (ii) holds when $S = \emptyset$.

When $S \neq \emptyset$, we can show that (iii) holds. Actually in this case, we know that u_n^+ is uniformly bounded in $L^\infty_{loc}(B_R^+ \cup L_R \backslash S)$ and therefore f_n is bounded in $L^p_{loc}(B_R^+ \backslash S)$ for some $p > 1$ and g_n is bounded in $L^p_{loc}(L_R \backslash S)$ for some $p > 1$. Then we have that either

$$u_n \text{ is bounded in } L^\infty_{loc}(B_R^+ \cup L_R \backslash S), \tag{9}$$

or

$$u_n \to -\infty \text{ uniformly on compact subsets of } (B_R^+ \cup L_R) \backslash S. \tag{10}$$

We should show that (9) does not happen when $S \neq \emptyset$. To this purpose, we can take a point $p \in S$ and choose a small $r_0 > 0$ such that p is the only blow-up point in $\bar{B}_{r_0}^+$. Then it is suffice to prove that

$$u_n \to -\infty \text{ uniformly on compact subsets of } \bar{B}_{r_0}^+ \backslash \{p\}. \tag{11}$$

If $p \neq 0$, this is a smooth case and (11) has been shown in [5]. So next we suppose $p = 0$. Since u_n is uniformly bounded in $L^\infty_{loc}(\bar{B}_{r_0}^+ \backslash \{0\})$, then we use elliptic estimates, and along a subsequence, we may assume that

$$u_n \to \xi \text{ pointwise a.e. and in } C^{1,\delta}_{loc}(\bar{B}^+_{r_0}\backslash\{0\}), \text{ for some } \delta \in (0,1), \tag{12}$$

Noticing that, by Fatou's lemma, $V(x)|x|^{2\alpha}e^{2\xi} \in L^1(B^+_{r_0})$ and $h(x)|x|^{\alpha}e^{\xi} \in L^1(L_{r_0})$, we have for any $0 < r \leq r_0$

$$\int_{B_r^+} V_n|x|^{2\alpha}e^{2u_n} + \int_{L_r} h_n|x|^{\alpha}e^{u_n} \to \int_{B_r^+} V|x|^{2\alpha}e^{2\xi} + \int_{L_r} h|x|^{\alpha}e^{\xi} + \beta, \tag{13}$$

here β is the blow-up value for the blow-up point $p = 0$, which is defined by

$$\beta = \lim_{r\to 0}\lim_{n\to\infty}\left\{\int_{B_r^+} V_n|x|^{2\alpha}e^{2u_n} + \int_{L_r} h_n|x|^{\alpha}e^{u_n}\right\}.$$

Set

$$\varphi_1(x) = V(x)|x|^{2\alpha}e^{2\xi} \quad \text{and } \varphi_2(x) = h(x)|x|^{\alpha}e^{\xi}.$$

By Green's representation formula for u_n in $\bar{B}^+_{r_0}$ and (13) we derive that

$$\xi(x) = \frac{\beta}{\pi}\ln\frac{1}{|x|} + \phi(x) + \gamma(x),$$

with

$$\phi(x) = \frac{1}{\pi}\int_{B^+_{r_0}} \ln\frac{1}{|x-y|}\varphi_1(y)dy + \frac{1}{\pi}\int_{L_{r_0}} \ln\frac{1}{|x-y|}\varphi_2(y)dy,$$

and

$$\gamma(x) = \frac{1}{\pi}\int_{S^+_{r_0}} \log\frac{1}{|x-y|}\frac{\partial\xi}{\partial\nu}dy + \frac{1}{\pi}\int_{S^+_{r_0}} \frac{(x-y)\cdot\nu}{|x-y|^2}\xi(y)dy.$$

Clearly,

$$\gamma(x) \in C^1(\bar{B}^+_r), \quad \text{for every } r \in (0, r_0). \tag{14}$$

For $\phi(x)$, we want to estimate the decay of ϕ near the zero. we observe first that $\phi(x)$ is bounded from below on $\bar{B}^+_{r_0}$, as we have,

$$\phi(x) \geq \frac{1}{\pi}\ln\frac{1}{2r_0}\left(\int_{B^+_{r_0}} V(y)|y|^{2\alpha}e^{2\xi}dy + \int_{L_{r_0}} h(y)|y|^{\alpha}e^{\xi}dy\right), \quad \forall x \in \bar{B}^+_{r_0}.$$

By (2) we find

$$\varphi_1(x) = V(x)|x|^{2\alpha}e^{2\xi} = V(x)\frac{|x|^{2\alpha}}{|x|^{\frac{2\beta}{\pi}}}e^{2\phi(x)+2\gamma(x)} \geq \frac{C}{|x|^{2(\frac{\beta}{\pi}-\alpha)}},$$

and

$$\varphi_2(x) = h(x)|x|^{\alpha}e^{\xi} = h(x)\frac{|x|^{\alpha}}{|x|^{\frac{\beta}{\pi}}}e^{\phi(x)+\gamma(x)} \geq \frac{C}{|x|^{\frac{\beta}{\pi}-\alpha}}.$$

Thus by the integrability of φ_1 and φ_2, we see that necessarily

$$\beta < \pi(1+\alpha). \tag{15}$$

On the other hand, let us set $s = \frac{\beta}{\pi} - \alpha$ and split $\phi = \phi_1 + \phi_2$, where

$$\phi_1(x) = \frac{1}{\pi}\int_{B^+_{r_0}} \ln\frac{1}{|x-y|}\varphi_1(y)dy, \quad \text{and} \quad \phi_2(x) = \frac{1}{\pi}\int_{L_{r_0}} \ln\frac{1}{|x-y|}\varphi_2(y)dy.$$

Noticing that, in view of (15), $s < 1$, it follows that

$$\varphi_1(x) = V(x)|x|^{2\alpha}e^{2\xi} \leq \frac{C}{|x|^{2s}}e^{2\phi_1(x)+2\phi_2(x)}, \quad \text{in } \bar{B}^+_{r_0},$$

and

$$\varphi_2(x) = h(x)|x|^{\alpha}e^{\xi} \leq \frac{C}{|x|^{s}}e^{\phi_1(x)+\phi_2(x)}, \quad \text{in } \bar{B}^+_{r_0}.$$

By Lemma 2.4 and Remark 2.5, for every $k > 0$ we have $e^{k|\phi_1|} \in L^1(L_{r_0})$, $e^{k|\phi_2|} \in L^1(L_{r_0})$, $e^{k|\phi_1|} \in L^1(B^+_{r_0})$ and $e^{k|\phi_2|} \in L^1(B^+_{r_0})$. By Holder's inequality it follows that $\varphi_1(x) \in L^t(B^+_{r_0})$ for any $t \in (1, \frac{1}{s})$ if $s > 0$, and $V(x)|x|^{2\alpha}e^{2\xi} \in L^t(B^+_{r_0})$ for any $t > 1$ if $s \leq 0$. We also have $\varphi_2(x) \in L^t(L_{r_0})$ for any $t \in (1, \frac{1}{s})$ if $s > 0$, and $\varphi_2(x) \in L^t(L_{r_0})$ for any $t > 1$ if $s \leq 0$. But if $-1 < \alpha < 0$, we have $0 < s < 1$. Since ϕ satisfies that

$$\begin{cases} -\Delta\phi = \varphi_1, & \text{in } B^+_{r_0}, \\ \dfrac{\partial\phi}{\partial\nu} = \varphi_2, & \text{on } L^+_{r_0}, \end{cases}$$

we get that ϕ is in $L^\infty(B^+_{r_0} \cap L_{r_0})$. Furthermore, if $s \leq 0$, then ϕ is in $C^1(B^+_{r_0} \cap L_{r_0})$. If $s > 0$, $\nabla\phi(x)$ will have a decay when $x \to 0$. Without loss of generality, we assume that $0 < s < 1$ in the sequel. We estimate $\nabla\phi(x)$ for $x \in B^+_{r_0}(0)$.

$$\begin{aligned} |\nabla\phi(x)| &\leq \frac{1}{\pi}\int_{B^+_{r_0}} \frac{1}{|x-y|}\varphi_1(y)dy + \frac{1}{\pi}\int_{L_{r_0}} \frac{1}{|x-y|}\varphi_2(y)dy \\ &= \frac{1}{\pi}\int_{\{|x-y|\geq\frac{|x|}{2}\}\cap B^+_{r_0}} \frac{1}{|x-y|}\varphi_1(y)dy + \frac{1}{\pi}\int_{\{|x-y|\leq\frac{|x|}{2}\}\cap B^+_{r_0}} \frac{1}{|x-y|}\varphi_1(y)dy \\ &\quad + \frac{1}{\pi}\int_{\{|x-y|\geq\frac{|x|}{2}\}\cap L_{r_0}} \frac{1}{|x-y|}\varphi_2(y)dy + \frac{1}{\pi}\int_{\{|x-y|\leq\frac{|x|}{2}\}\cap L_{r_0}} \frac{1}{|x-y|}\varphi_2(y)dy \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

For I_1, we fix $t \in (1, \frac{1}{s})$ and choose $\tau_1 > 0$ such that $\frac{\tau_1 t}{t-1} < 2$, and hence we have $0 < \tau_1 < 2 - 2s$. By Holder's inequality we obtain,

$$\begin{aligned} |I_1| \leq \frac{1}{\pi}&\left(\int_{\{|x-y|\geq\frac{|x|}{2}\}\cap B^+_{r_0}} \frac{1}{|x-y|^{\frac{\tau_1 t}{t-1}}}dy\right)^{\frac{t-1}{t}} \\ &\cdot\left(\int_{\{|x-y|\geq\frac{|x|}{2}\}\cap B^+_{r_0}} \frac{1}{|x-y|^{t(1-\tau_1)}}|\varphi_1(y)|^t dy\right)^{\frac{1}{t}} \leq \frac{C}{|x|^{1-\tau_1}}. \end{aligned}$$

For I_2, since $|x-y| \leq \frac{|x|}{2}$ implies that $|y| \geq \frac{|x|}{2}$, we have

$$\begin{aligned} |I_2| &\leq C\int_{\{|x-y|\leq\frac{|x|}{2}\}\cap B^+_{r_0}} \frac{1}{|x-y|}\frac{1}{|y|^s}dy \\ &\leq \frac{C}{|x|^s}\int_{\{|x-y|\leq\frac{|x|}{2}\}\cap B^+_{r_0}} \frac{1}{|x-y|}dy \leq C|x|^{1-s}. \end{aligned}$$

Similarly, for I_3, we fix $t \in (1, \frac{1}{s})$ and choose $\tau_2 > 0$ such that $\frac{\tau_2 t}{t-1} < 1$. and hence we have $0 < \tau_2 < 1 - s$. By Holder's inequality we obtain,

$$|I_3| \leq \frac{1}{\pi}\left(\int_{\{|x-y|\geq\frac{|x|}{2}\}\cap L_{r_0}} \frac{1}{|x-y|^{\frac{\tau_2 t}{t-1}}}dy\right)^{\frac{t-1}{t}} \cdot\left(\int_{\{|x-y|\geq\frac{|x|}{2}\}\cap L_{r_0}} \frac{1}{|x-y|^{t(1-\tau_2)}}|\varphi_2(y)|^t dy\right)^{\frac{1}{t}} \leq \frac{C}{|x|^{1-\tau_2}}.$$

For I_4 we have

$$|I_4| \leq C\int_{\{|x-y|\leq\frac{|x|}{2}\}\cap L_{r_0}} \frac{1}{|x-y|}\frac{1}{|y|^s}dy \leq \frac{C}{|x|^s}\int_{\{|x-y|\leq\frac{|x|}{2}\}\cap L_{r_0}} \frac{1}{|x-y|}dy \leq \frac{C}{|x|^{\tau_3}},$$

for some τ_3 with $0 < \tau_3 < 1$.

In conclusion, for all $x \in B^+_{r_0}(0)$ we have

$$|\nabla\phi(x)| \leq \frac{C}{|x|^{1-\tau_1}} + \frac{C}{|x|^{1-\tau_2}} + \frac{C}{|x|^{\tau_3}}, \tag{16}$$

for suitable constants $0 < \tau_1 < 2 - s$, $0 < \tau_2 < 1 - s$ and $0 < \tau_3 < 1$.

At this point we are ready to derive our contradiction by means of a Pohozaev type identity. We multiply all terms in (1) by $x \cdot \nabla u_n$ and integrate over $B^+_r(0)$ for any $r \in (0, r_0)$ to get

$$\begin{aligned} r\int_{S_r^+}(\frac{1}{2}|\nabla u_n|^2 - |\frac{\partial u_n}{\partial \nu}|^2)d\sigma &= -(1+\alpha)\int_{B_r^+} V_n|x|^{2\alpha}e^{2u_n}dx - (1+\alpha)\int_{L_r} h_n|x|^{\alpha}e^{u_n}d\sigma \\ &+\frac{r}{2}\int_{S_r^+} V_n|x|^{2\alpha}e^{2u_n}d\sigma - \frac{1}{2}\int_{B_r^+}|x|^{2\alpha}e^{2u_n}(x\cdot\nabla V_n)dx \\ &+h_n(x_1,0)|x_1|^{\alpha}x_1e^{u_n(x_1,0)}\,|_{x_1=-r}^{x_1=r} - \int_{-r}^{r}\frac{\partial h(x_1,0)}{\partial x_1}|x_1|^{\alpha}x_1e^{u_n(x_1,0)}dx_1. \end{aligned} \tag{17}$$

Passing to the limit in (17) to derive the following identity

$$\begin{aligned} r\int_{S_r^+}(\frac{1}{2}|\nabla \xi|^2 - |\frac{\partial \xi}{\partial \nu}|^2)d\sigma &= -(1+\alpha)\int_{B_r^+} V|x|^{2\alpha}e^{2\xi}dx - (1+\alpha)\int_{L_r} h|x|^{\alpha}e^{\xi}d\sigma + \frac{r}{2}\int_{S_r^+} V|x|^{2\alpha}e^{2\xi}d\sigma \\ &-\frac{1}{2}\int_{B_r^+}|x|^{2\alpha}e^{2\xi}(x\cdot\nabla V)dx - \int_{-r}^{r}\frac{\partial h(x_1,0)}{\partial x_1}|x_1|^{\alpha}x_1e^{\xi(x_1,0)}dx_1 \\ &+h(x_1,0)|x_1|^{\alpha}x_1e^{\xi(x_1,0)}\,|_{x_1=-r}^{x_1=r} - \beta(1+\alpha). \end{aligned} \tag{18}$$

Set $\eta=\phi+\gamma$. Since $\nabla\xi(x)=-\frac{\beta}{\pi}\frac{x}{|x|^2}+\nabla\eta(x)$, we have

$$\begin{aligned}\Phi_r &:= r\int_{S_r^+}(\frac{1}{2}|\nabla\xi|^2-|\frac{\partial\xi}{\partial\nu}|^2)ds\\ &= r\int_{S_r^+}\frac{1}{2}[(\frac{\beta}{\pi})^2\frac{1}{|x|^2}-2\frac{\beta}{\pi}\frac{x\cdot\nabla\eta}{|x|^2}+|\nabla\eta|^2]ds-r\int_{S_r^+}(-\frac{\beta}{\pi}\frac{1}{|x|}+\frac{x\cdot\nabla\eta}{|x|})^2ds\\ &= r\int_{S_r^+}[-\frac{1}{2}(\frac{\beta}{\pi})^2\frac{1}{|x|^2}+\frac{\beta}{\pi}\frac{x\cdot\nabla\eta}{|x|^2}+\frac{1}{2}|\nabla\eta|^2-(\frac{x\cdot\nabla\eta}{|x|})^2]d\sigma\\ &= -\frac{1}{2}(\frac{\beta}{\pi})^2\pi+\frac{\beta}{\pi}r\int_{S_r^+}\frac{x\cdot\nabla\eta}{|x|^2}+\frac{r}{2}\int_{S_r^+}|\nabla\eta|^2-r\int_{S_r^+}(\frac{x\cdot\nabla\eta}{|x|})^2.\end{aligned}$$

Since $\gamma\in C^1(B_r^+)$, by (16) we have

$$|\nabla\eta(x)|\le\frac{C}{|x|^{1-\tau_1}}+\frac{C}{|x|^{1-\tau_2}}+\frac{C}{|x|^{\tau_3}}+C,$$

with $0<\tau_1<2-s$, $0<\tau_2<1-s$ and $0<\tau_3<1$. So,

$$\Phi_r=-\frac{\beta^2}{2\pi}+o(1),\ \text{as } r\to 0. \tag{19}$$

Similarly, letting $r\to 0$ on the right side of (18) we also can obtain that

$$\Phi_r=-\beta(1+\alpha)+o(1),\ \text{as } r\to 0. \tag{20}$$

Comparing (19) and (20), we see that necessarily $\beta=2\pi(1+\alpha)$, in contradiction with (15). Therefore, the proof of Theorem 1.1 is finished.

3 A version of Tarantello's decomposition proposition

In this section, we would like to show the new version of Tarantello' decomposition Proposition for Liouville equation under the Neumann boundary condition. Firstly we give the "Minimal-Mass Lemma", which is frequently used in the following Proposition.

Lemma 3.1 *Assume that $\{u_n\}$ is a sequence of solutions to (1) for $R=1$ and u_n satisfies (2) and (4). If there exists a sequence $\{x_n\}\subset\bar{B}_1^+\backslash\{0\}$ such that*

$$x_n\to x_0\subset\bar{B}_1^+ \text{ and } u_n(x_n)+(\alpha+1)\log|x_n|\to+\infty. \tag{21}$$

Then we must have $x_0=0$ and

$$\limsup_{n\to+\infty}\left(\int_{B^+_{\delta|x_n|}(x_n)}V_n|x|^{2\alpha}e^{2u_n}+\int_{L_{\delta|x_n|}(x_n)}h_n|x|^{\alpha}e^{u_n}\right)\ge 2\pi, \tag{22}$$

for every small $\delta>0$.

Proof Noticing that 0 is the only blow-up point for u_n in $\bar{B}_1^+$ and $u_n(x_n)\to+\infty$, we have $x_0=0$. Next we consider the new function

$$v_n(x)=u_n(|x_n|x)+(\alpha+1)\log|x_n|.$$

Then $v_n(x)$ satisfies

$$\begin{cases} -\Delta v_n = V_n(|x_n|x)|x|^{2\alpha}e^{2v_n} & \text{in } B^+_{\frac{1}{|x_n|}} \\ \dfrac{\partial v_n}{\partial \nu} = h_n(|x_n|x)|x|^{\alpha}e^{v_n} & \text{on } L_{\frac{1}{|x_n|}} \end{cases} \tag{23}$$

with the energy conditions

$$\int_{B^+_{\frac{1}{|x_n|}}} V_n(|x_n|x)|x|^{2\alpha}e^{2v_n} \le C, \quad \int_{L_{\frac{1}{|x_n|}}} h_n(|x_n|x)|x|^{\alpha}e^{v_n} \le C.$$

Suppose that along a subsequence $\frac{x_n}{|x_n|} \to x_0 \in \bar{\mathbb{R}}^2_+$ with $|x_0| = 1$. Hence x_0 define a blow-up point for v_n as we have

$$v_n(\frac{x_n}{|x_n|}) = u_n(x_n) + (\alpha+1)\log|x_n| \to +\infty.$$

Moreover functions $V_n(|x_n|x)|x|^{2\alpha}$ and $h_n(|x_n|x)|x|^{\alpha}$ are uniformly bounded from above and below near x_0.

Consequently, if $x_0 \in \mathbb{R}^2_+$, by [2] we have for sufficiently small $\delta > 0$,

$$\limsup_{n\to+\infty} \int_{B^+_\delta(\frac{x_n}{|x_n|})} V_n(|x_n|x)|x|^{2\alpha}e^{2v_n} \ge 4\pi,$$

and if $x_0 \in \partial\mathbb{R}^2_+$, by [5] we have for sufficiently small $\delta > 0$,

$$\limsup_{n\to+\infty} \left(\int_{B^+_\delta(\frac{x_n}{|x_n|})} V_n(|x_n|x)|x|^{2\alpha}e^{2v_n} + \int_{L_\delta(\frac{x_n}{|x_n|})} h_n(|x_n|x)|x|^{\alpha}e^{v_n} \right) \ge 2\pi.$$

A simple change of variables leads to the conclusion. □

On the other hand, if (21) fails to hold, i.e. $\sup_{\bar{B}^+_R}\{u_n(x) + (\alpha+1)\log|x|\} \le C$, we will treat this situation in the following Lemma.

Lemma 3.2 *Assume that $\{u_n\}$ is a sequence of solutions to (1) for $R > 0$ and u_n satisfies (2) and (4). If*

$$\sup_{\bar{B}^+_R}\{u_n(x) + (\alpha+1)\log|x|\} \le C, \tag{24}$$

then we have

$$u_n(0) = \max_{\bar{B}^+_R} u_n + O(1) \text{ as } n \to +\infty.$$

Proof Let $u_n(x_n) = \max_{\bar{B}^+_R} u_n \to +\infty$ and $\varepsilon_n = e^{-\frac{u_n(x_n)}{\alpha+1}} \to 0$. By (24) we get

$$\frac{|x_n|}{\varepsilon_n} = O(1), \text{ as } n \to +\infty.$$

In $\bar{B}^+_{\frac{R}{\varepsilon_n}}$ we define

$$\xi_n(x) = u_n(\varepsilon_n x) + (\alpha+1)\log\varepsilon_n.$$

Then ξ_n satisfies

$$\begin{cases} -\Delta \xi_n = V_n(\varepsilon_n x)|x|^{2\alpha} e^{2\xi_n} & \text{in } B^+_{\frac{R}{\varepsilon_n}}, \\ \dfrac{\partial \xi_n}{\partial \nu} = h_n(\varepsilon_n x)|x|^{\alpha} e^{\xi_n} & \text{on } L_{\frac{R}{\varepsilon_n}}, \\ \max\limits_{\bar{B}^+_{\frac{R}{\varepsilon_n}}} \xi_n = \xi_n(\frac{x_n}{\varepsilon_n}) = 0, \end{cases} \tag{25}$$

with the energy conditions

$$\int_{B^+_{\frac{R}{\varepsilon_n}}} V_n(\varepsilon_n x)|x|^{2\alpha} e^{2\xi_n} \le C, \quad \int_{L_{\frac{R}{\varepsilon_n}}} h_n(\varepsilon_n x)|x|^{\alpha} e^{\xi_n} \le C.$$

Then necessarily alternative (i) in Theorem 1.1 must hold, in other words, ξ_n is uniformly bounded in $L^\infty_{loc}(\bar{\mathbb{R}}^2_+)$. In particular,

$$u_n(0) - u_n(x_n) = \xi_n(0) = O(1), \text{ as } n \to +\infty.$$

□

Remark 3.3 In fact in Lemma 3.2, we have additionally that along a subsequence $\xi_n \to \xi$ uniformly in $C^2_{loc}(\mathbb{R}^2_+) \cap C^1_{loc}(\bar{\mathbb{R}}^2_+ \backslash \{0\}) \cap C^0_{loc}(\bar{\mathbb{R}}^2_+)$. Without loss of generality we always assume that

$$V(0) = h(0) = 1$$

in the sequel. Then by the classification results in [8] we know that ξ takes the form

$$\xi = \log \frac{2(\alpha+1)\lambda^{(\alpha+1)}}{|x^{\alpha+1} - y_0|^2 + \lambda^{2(\alpha+1)}}, \lambda > 0, \text{ some } y_0 \in \bar{\mathbb{R}}^2_+. \tag{26}$$

In addition, $\int_{\mathbb{R}^2_+} |x|^{2\alpha} e^{2\xi} + \int_{\partial\mathbb{R}^2_+} |x|^{\alpha} e^{\xi} = 2\pi(1+\alpha)$. In the forth section, we can further obtain by assistant with the Harnack inequality

$$\lim_{n\to\infty} \left(\int_{B^+_R} V_n |x|^{2\alpha} e^{2u_n} dx + \int_{L_R} h_n |x|^{\alpha} e^{u_n} ds \right) = 2\pi(1+\alpha)$$

provided that the assumptions of Lemma hold.

In general, the assumption "$\sup\limits_{\bar{B}^+_R}\{u_n(x) + (\alpha+1)\log|x|\} \le C$" does not always hold. So we must distinguish between the situation whether (24) holds or not. In particular we have the following Tarantello's decomposition Proposition.

Proposition 3.4 *Assume that $\{u_n\}$ is a sequence of solutions to (1) for $R = 1$ and u_n satisfies (2) and (4). Then there exists $\varepsilon_0 \in \left(0, \frac{1}{2}\right)$ such that the following alternatives hold:*

$$\text{either } (i) \sup_{\bar{B}^+_{2\varepsilon_0}} \{u_n(x) + (\alpha+1)\log|x|\} \le C \tag{27}$$

or (ii) *there exist finite sequences* $\{x_{j,n}\} \in \bar{B}^+_1 \backslash \{0\}$, $j = 1, \ldots, m$, *such that*

1. $$x_{j,n} \to 0,\ u_n(x_{j,n}) + (\alpha+1)\log|x_{j,n}| \to +\infty; \tag{28}$$
2. $$\sup_{D_n} \{u_n(x) + (\alpha+1)\log|x|\} \le C \tag{29}$$
   where $D_n = \{\bar{B}^+_{2\varepsilon_0|x_{1,n}|}\} \cup \{\bar{B}^+_1 \backslash \bar{B}^+_{\frac{1}{2\varepsilon_0}|x_{m,n}|}\}$;
3. *If* $m \ge 2$, *then* $\dfrac{|x_{j,n}|}{|x_{j+1,n}|} \to 0$, *as* $n \to +\infty$, *and*
   $$\sup_{\bar{B}^+_{2\varepsilon_0|x_{j+1,n}|} \backslash \bar{B}^+_{\frac{1}{2\varepsilon_0}|x_{j,n}|}} \{u_n(x) + (\alpha+1)\log|x|\} \le C. \tag{30}$$

Proof If (27) fails to hold for every $\varepsilon_0 \in (0, \frac{1}{2})$, then we find a sequence $x_n \subset \bar{B}^+_1$ and

$$u_n(x_n) + (\alpha+1)\log|x_n| \to +\infty.$$

Then by (22), we have

$$x_n \to 0 \text{ and } \limsup_{n\to+\infty}\left(\int_{B^+_{\delta|x_n|}(x_n)} V_n(x)|x|^{2\alpha}e^{2u_n} + \int_{L_{\delta|x_n|}(x_n)} h_n|x|^{\alpha}e^{u_n}\right) \ge 2\pi,$$

$\forall \delta > 0$. Setting

$$v_n(x) = u_n(|x_n|x) + (\alpha+1)\log|x_n|. \tag{31}$$

Next we consider the new sequences v_n in $\bar{B}^+_{2\varepsilon_0}$. We repeat the alternative above for the sequence v_n in $\bar{B}^+_{2\varepsilon_0}$. If $\sup_{\bar{B}^+_{2\varepsilon_0}}\{v_n(x) + (\alpha+1)\log|x|\} \le C$ holds with a suitable $\varepsilon_0 \in (0, \frac{1}{2})$, then in this case we can set $x_{1,n} = x_n$. If there exists a sequence x'_n, $v_n(x'_n) + (\alpha+1)\log|x'_n| \to +\infty$, then in this case there exists a second sequence $\tilde{x}_n = |x_n|x'_n \subset \bar{B}^+_1$, such that

$$\frac{|\tilde{x}_n|}{|x_n|} \to 0 \text{ and } u_n(\tilde{x}_n) + (\alpha+1)\log|\tilde{x}_n| \to +\infty.$$

Consequently by (22),

$$\limsup_{n\to+\infty}\left(\int_{B^+_{\delta|\tilde{x}_n|}(\tilde{x}_n)} V_n|x|^{2\alpha}e^{2u_n} + \int_{L_{\delta|\tilde{x}_n|}(\tilde{x}_n)} h_n|x|^{\alpha}e^{u_n}\right) \ge 2\pi.$$

In addition, notice that $\bar{B}^+_{\delta|\tilde{x}_n|}(\tilde{x}_n)$ and $\bar{B}^+_{\delta|x_n|}(x_n)$ do not intersect for $\delta \in (0, 1)$ and n large.

Next we consider the new scaling sequences

$$v'_n(x) = u_n(|\tilde{x}_n|x) + (\alpha+1)\log|\tilde{x}_n|.$$

We make the same alternative above for the new sequence v'_n. We see that each time the new iterated sequence v'_n fails to verify (27), we contribute with at least an account of 2π to the blow-up value $m(0)$. So necessarily after a number of steps we find $\varepsilon_0 \in \left(0, \frac{1}{2}\right)$ and a sequence $\{x_{1,n}\} \subset \bar{B}_1^+$:

$$x_{1,n} \to 0, u_n(x_{1,n}) + (\alpha+1)\log|x_{1,n}| \to +\infty,$$
$$\text{and} \sup_{\bar{B}^+_{2\varepsilon_0|x_{1,n}|}} \{u_n(x) + (\alpha+1)\log|x|\} \le C.$$

Now, for $\varepsilon_0 \in \left(0, \frac{1}{2}\right)$, we repeat an analogous alternative for u_n on the set $\bar{B}_1^+ \backslash \bar{B}^+_{\frac{1}{2\varepsilon_0}|x_{1,n}|}$. If

$$\sup_{\bar{B}_1^+ \backslash \bar{B}^+_{\frac{|x_{1,n}|}{2\varepsilon_0}}} \{u_n(x) + (\alpha+1)\log|x|\} \le C,$$

in this case we then obtain the first sequences $x_{1,n}$. If there exists a sequence $\{y_n\} \subset \bar{B}_1^+ \backslash \{0\}$ such that

$$\frac{|x_{1,n}|}{|y_n|} \to 0 \text{ and } u_n(y_n) + (\alpha+1)\log|y_n| \to +\infty. \tag{32}$$

By (22) we have

$$y_n \to 0 \text{ and } \limsup_{n\to+\infty} \left(\int_{B^+_{\delta|y_n|}(y_n)} V_n|x|^{2\alpha} e^{2u_n} + \int_{L_{\delta|y_n|}(y_n)} h_n|x|^{\alpha} e^{u_n} \right) \ge 2\pi \tag{33}$$

for $\forall \delta > 0$. Our next task is to obtain the second sequence $x_{2,n}$ for $\varepsilon_0 \in (0, \frac{1}{2})$. In this direction, we consider

$$\sup_{\bar{B}^+_{2\varepsilon_0|y_n|} \backslash \bar{B}^+_{\frac{|x_{1,n}|}{2\varepsilon_0}}} \{u_n(x) + (\alpha+1)\log|x|\}. \tag{34}$$

If (34) is uniformly bounded for any $\varepsilon_0 \in (0, \frac{1}{2})$ then we would let $x_{2,n} = y_n$ and adjust according ε_0 in order to ensure (30) with $j = 1$. Otherwise we would replace y_n with a new sequence y_n' which have the same properties (32) and (33), but $\frac{|y_n'|}{|y_n|} \to 0$, as $n \to \infty$. Moreover each time when such a new sequences exist, we at least contribute an amount of 2π to the blow-up value $m(0)$. So making the same alternative for such new sequence, this procedure must stop a number of steps. And we arrive to one for which (34) is uniformly bounded for every $\varepsilon_0 \in (0, \frac{1}{2})$. Such sequence will define $x_{2,n}$. So we obtain the desired proproties (29) (30) for $j = 1$ by adjust $\varepsilon_0 \in (0, \frac{1}{2})$. At this point we iterate the argument above by replacing $x_{1,n}$ with the new sequence $x_{2,n}$.

Either (28) (29) (30) hold for $m = 2$, or we obtain a third sequence for which we can verify (29) (30) for $j = 1, 2$. Since the blow-up value $m(0)$ is finite, so only a finite number of sequence $x_{j,n}$ satisfying (29) (30) are allowed. Then after a finite number of steps we arrive to the desired conclusion. □

4 A version of "sup + inf" type inequality

In this section we will show a version of "sup + inf" type inequality for Liouville equation under the Neumann boundary condition. This inequality concerns the case where the sequence u_n is subject to alternative (ii) of Proposition 3.4. It is the key part for the proof of Theorem 1.2.

Proposition 4.1 *Assume that* $\{u_n\}$ *is a sequence of solutions to* (1) *with* $R = 1$ *which satisfying* (2) *and* (3). *Suppose that there exists* $\varepsilon_0 > 0$ *and a sequence* $\{x_n\} \subset \bar{B}_1^+$ *such that*

(i) $x_n \to 0, u_n(x_n) + (\alpha+1)\log|x_n| \to +\infty$;
(ii) $\sup_{\bar{B}^+_{2\varepsilon_0|x_n|}} \{u_n(x) + (\alpha+1)\log|x|\} \le C$;

Set $v_n(x) = u_n(|x_n|x) + (\alpha+1)\log|x_n|$. *Then passing to a subsequence, we have*

$$\text{either (a)}\quad \max_{\bar{B}^+_{\varepsilon_0}} v_n \to -\infty \quad \text{and} \ \inf_{\bar{B}_1^+} u_n \le \max_{\bar{B}^+_{r_0|x_n|}} v_n + (\alpha+1)\log|x_n| + C,$$
$$\text{or (b)}\quad v_n(0) \to +\infty \ \text{and} \ \inf_{\bar{B}_1^+} u_n \le -u_n(0) + C.$$

for suitable constant $r_0 > 0$.

Proof We use a moving plane technique to obtain our conclusion. Similar arguments also be used in [1,13]. As usual we identify $x = (x_1, x_2) \in \mathbb{R}^2$ and $x_1 + ix_2 \in \mathbb{C}$, where $\mathbb{C}$ is complex plane. Recalling (2), without loss of generality, suppose $A \ge B$ and $a \le b$.

In polar coordinates, define

$$\omega_n(t,\theta) = u_n(e^{t+i\theta}) + (\alpha+1)t - \frac{A}{a}e^t \tag{35}$$

for $(t,\theta) \in Q = (-\infty, 0] \times [0,\pi]$. A simple calculation shows that

$$\begin{cases} -\Delta\omega_n = \tilde{V}_n(t,\theta)e^{2\omega_n} + \frac{A}{a}e^t, \\ \frac{\partial\omega_n}{\partial\nu}|_{\theta=0} = \tilde{h}_n(t,0)e^{\omega_n(t,0)} \\ \frac{\partial\omega_n}{\partial\nu}|_{\theta=\pi} = \tilde{h}_n(t,\pi)e^{\omega_n(t,\pi)} \end{cases} \tag{36}$$

where $\tilde{V}_n(t,\theta) = V_n(e^{t+i\theta})e^{\frac{2A}{a}e^t}$ and $\tilde{h}_n(t,\theta) = h_n(e^{t+i\theta})e^{\frac{A}{a}e^t}$.

Since for fixed n

$$\begin{aligned} &\omega_n(2\mu - t,\theta) - \omega_n(t,\theta) \\ &= u_n(e^{2\mu-t+i\theta}) + 2(\alpha+1)(\mu - t) - \frac{A}{a}e^{2\mu-t} - u_n(e^{t+i\theta}) + \frac{A}{a}e^t, \end{aligned}$$

we have

$$\omega_n(2\mu - t,\theta) - \omega_n(t,\theta) \le (\alpha+1)\mu + C(n), \forall t \in [\frac{\mu}{2}, 0], \theta \in [0,\pi].$$

Furthermore,

$$\frac{\partial}{\partial t}\omega_n(t,\theta) \ge (\alpha+1) - C(n)e^{\frac{\mu}{2}}, \forall t < \frac{\mu}{2}, \theta \in [0,\pi].$$

for suitable $C(n) > 0$ depending on n. Thus we can choose λ sufficiently negative (depending on n) such that $\forall \mu \le \lambda$:

$$\omega_n(2\mu - t,\theta) - \omega_n(t,\theta) < 0 \text{ for } t \in [\frac{\mu}{2}, 0], \theta \in [0,\pi]$$
$$\frac{\partial}{\partial t}\omega_n(t,\theta) > 0 \text{ for } t < \frac{\mu}{2}, \theta \in [0,\pi].$$

Therefore we get, for fixed n, there exists $\lambda < 0$(depending on n) such that

$$\forall\mu < \lambda, \omega_n(2\mu - t,\theta) - \omega_n(t,\theta) < 0, \ \text{for } \mu < t < 0 \text{ and } \theta \in [0,\pi]. \tag{37}$$

Consequently we can define

$$\lambda_n = \sup\{\lambda \leq 0 : (37) \text{ holds }\}.$$

We claim that

$$\min_{\theta\in[0,\pi]} \omega_n(0,\theta) \leq \max_{\theta\in[0,\pi]} \omega_n(2\lambda_n,\theta). \tag{38}$$

To prove the claim, we let $\psi_n(t,\theta) = \omega_n(2\lambda_n - t,\theta) - \omega_n(t,\theta)$. Hence by (37) we get $\psi_n \leq 0$. By using assumption (2) we obtain

$$\frac{\partial}{\partial t}(\tilde{V}_n(t,\theta)e^{\xi} + \frac{A}{a}e^t) \geq 0 \text{ and } \frac{\partial}{\partial t}(\tilde{h}_n(t,\theta)e^{\xi}) \geq 0, \tag{39}$$

for $\forall\xi \in \mathbb{R}$. By virtue of (39) we have

$$\begin{cases} \Delta\psi_n \geq 0, \\ \frac{\partial\psi_n}{\partial\nu}|_{\theta=0} \leq 0, \\ \frac{\partial\psi_n}{\partial\nu}|_{\theta=\pi} \leq 0, \end{cases} \tag{40}$$

for $(t,\theta) \in [\lambda_n, 0] \times [0,\pi]$. Suppose by the contradiction that

$$\max_{\theta\in[0,\pi]} \omega_n(2\lambda_n,\theta) < \min_{\theta\in[0,\pi]} \omega_n(0,\theta).$$

By the strong maximum principle, Hopf Lemma and a result in Appendix we have $\psi_n(t,\theta) < 0$ in $(\lambda_n, 0) \times [0,\pi]$ and $\frac{\partial\psi_n(t,\theta)}{\partial t}|_{t=\lambda_n} < 0$ for $\theta \in [0,\pi]$. On the other hand, from the definition of λ_n, there exists a sequence $\lambda_{n,k} \to \lambda_n$, as $k \to +\infty$, such that

$$\max_{[\lambda_{n,k},0]\times[0,\pi]} (\omega_n(2\lambda_{n,k} - t,\theta) - \omega_n(t,\theta)) > 0.$$

Set x_k is the maximum point of $\omega_n(2\lambda_{n,k} - t,\theta) - \omega_n(t,\theta)$ in $[\lambda_{n,k}, 0] \times [0,\pi]$. From continuity, we have $x_k \to x_0$ and x_0 lies on $\{\lambda_n\}\times[0,\pi]$. In addition, we have $\frac{\partial\psi_n(t,\theta)}{\partial t}|_{x_0} = 0$. Thus we get a contradiction. So we arrive to the conclusion (38).

Next we want to estimate λ_n. To this purpose, let us note that v_n satisfies

$$\begin{cases} -\Delta v_n = V_n(|x_n|x)|x|^{2\alpha}e^{2v_n} & \text{in } B^+_{2\varepsilon_0}, \\ \frac{\partial v_n}{\partial\nu} = h_n(|x_n|x)|x|^{\alpha}e^{v_n} & \text{on } L_{2\varepsilon_0}, \end{cases} \tag{41}$$

and

$$\sup_{\bar{B}^+_{2\varepsilon_0}}\{v_n(x) + (\alpha+1)\log|x|\} \leq C, \tag{42}$$

and

$$\int_{B^+_{2\varepsilon_0}} V_n(|x_n|x)|x|^{2\alpha}e^{2v_n} \leq C, \quad \int_{L_{2\varepsilon_0}} h_n(|x_n|x)|x|^{\alpha}e^{v_n} \leq C.$$

Thus in view of (42) and Lemma 3.2 we have

$$\text{either} \quad v_n(0) = \max_{\bar{B}^+_{2\varepsilon_0}} v_n + O(1) \to +\infty, \text{ as } n \to +\infty; \tag{43}$$

$$\text{or} \quad \max_{\bar{B}^+_{2\varepsilon_0}} v_n < +\infty. \tag{44}$$

In order to proceed further, we distinguish two cases.

Case 1. (43) holds, and necessarily $u_n(0) \to +\infty$.

In this case, we set

$$\varepsilon_n = e^{-\frac{u_n(0)}{\alpha+1}} \to 0 \text{ and } \frac{|x_n|}{\varepsilon_n} = e^{\frac{v_n(0)}{\alpha+1}} \to +\infty.$$

We also set $\xi_n(x) = u_n(\varepsilon_n x) + (\alpha+1)\log\varepsilon_n$. Then in $\bar{B}^+_{2\varepsilon_0\frac{|x_n|}{\varepsilon_n}}$, ξ_n satisfies

$$\begin{cases} -\Delta\xi_n = V_n(\varepsilon_n x)|x|^{2\alpha}e^{2\xi_n} & \text{in } B^+_{2\varepsilon_0\frac{|x_n|}{\varepsilon_n}}, \\ \dfrac{\partial\xi_n}{\partial\nu} = h_n(\varepsilon_n x)|x|^{\alpha}e^{\xi_n} & \text{on } L_{2\varepsilon_0\frac{|x_n|}{\varepsilon_n}}, \end{cases}$$

and $\xi_n(0) = 0$. In addition, in view of (43), we have

$$\begin{aligned} \max_{\bar{B}^+_{2\varepsilon_0\frac{|x_n|}{\varepsilon_n}}} \xi_n &= \max_{\bar{B}^+_{2\varepsilon_0|x_n|}} u_n + (\alpha+1)\log\varepsilon_n \\ &= \max_{\bar{B}^+_{2\varepsilon_0}} v_n - (\alpha+1)\log|x_n| + (\alpha+1)\log\varepsilon_n \\ &= v_n(0) - (\alpha+1)\log|x_n| + (\alpha+1)\log\varepsilon_n + O(1) \\ &= \xi_n(0) + O(1) = O(1). \end{aligned}$$

Therefore we argue as in Lemma 3.2 and Remark 3.3 to conclude that

$$\xi_n(x) = u_n(\varepsilon_n x) + (\alpha+1)\log\varepsilon_n \to \xi \tag{45}$$

uniformly in $C^2_{loc}(\mathbb{R}^2_+) \cap C^1_{loc}(\bar{\mathbb{R}}^2_+\backslash\{0\}) \cap C^0_{loc}(\bar{\mathbb{R}}^2_+)$, where ξ takes the form (26) and satisfies $\xi(0) = 0$.

Claim.

$$\lambda_n \le \log\varepsilon_n + O(1) \tag{46}$$

with $\varepsilon_n = e^{-\frac{u_n(0)}{\alpha+1}}$ and

$$\inf_{\bar{B}^+_1} u_n \le -u_n(0) + O(1) \tag{47}$$

as $n \to +\infty$, and then we obtain part (b) of our Proposition.

To establish this Claim, recalling (45), we may use Case (1) to obtain

$$\xi_n(x) = u_n(\varepsilon_n x) + (\alpha+1)\log\varepsilon_n \to \xi = \log\frac{2(\alpha+1)\lambda^{(\alpha+1)}}{|x^{\alpha+1} - y_0|^2 + \lambda^{2(\alpha+1)}}, \tag{48}$$

uniformly in $C^2_{loc}(\mathbb{R}^2_+) \cap C^1_{loc}(\bar{\mathbb{R}}^2_+\backslash\{0\}) \cap C^0_{loc}(\bar{\mathbb{R}}^2_+)$, with $\lambda > 0$, $y_0 \in C$ such that $2(\alpha+1)\lambda^{\alpha+1} = |y_0|^2 + \lambda^{2(\alpha+1)}$.

For $(t,\theta) \in Q$, let

$$\omega(t,\theta) = \xi(e^{t+i\theta}) + (\alpha+1)t = \log\frac{2(\alpha+1)\lambda^{(\alpha+1)}e^{(\alpha+1)t}}{|x^{\alpha+1} - y_0|^2 + \lambda^{2(\alpha+1)}}. \tag{49}$$

Set $y_0 = (|y_0|\cos\theta_0, |y_0|\sin\theta_0)$ and $\tau = (\frac{1}{2(\alpha+1)\lambda^{\alpha+1}})^{\frac{1}{\alpha+1}}$. Since

$$e^{2(\alpha+1)\log\frac{1}{\sqrt{\tau}}} = |y_0|^2 + \lambda^{2(\alpha+1)} = 2(\alpha+1)\lambda^{\alpha+1},$$

we have

$$e^{2(\alpha+1)t}\cdot e^{2(\alpha+1)\log\frac{1}{\sqrt{\tau}}} + 2(\alpha+1)\lambda^{\alpha+1} = 2(\alpha+1)\lambda^{\alpha+1}\cdot e^{2(\alpha+1)t} + e^{2(\alpha+1)\log\frac{1}{\sqrt{\tau}}},$$

and further have

$$e^{2(\alpha+1)(\log\frac{1}{\sqrt{\tau}}+t)}+|y_0|^2+\lambda^{2(\alpha+1)}=e^{2(\alpha+1)\log\frac{1}{\sqrt{\tau}}}+e^{2(\alpha+1)t}(|y_0|^2+\lambda^{2(\alpha+1)}).$$

Then by a direct computation we can obtain

$$\frac{1}{e^{2(\alpha+1)t}(|e^{2(\alpha+1)(\log\frac{1}{\sqrt{\tau}}-t+i\theta)}-y_0|^2+\lambda^{2(\alpha+1)})}=\frac{1}{|e^{2(\alpha+1)(\log\frac{1}{\sqrt{\tau}}+t+i\theta)}-y_0|^2+\lambda^{2(\alpha+1)}}.$$

This implies $\omega(\log\frac{1}{\sqrt{\tau}}-t,\theta)=\omega(\log\frac{1}{\sqrt{\tau}}+t,\theta)$ for $(t,\theta)\in Q$ and $\omega(t,\theta)$ is symmetric with respect to $t=\log\frac{1}{\sqrt{\tau}}$, $\tau=(\frac{1}{2(\alpha+1)\lambda^{\alpha+1}})^{\frac{1}{\alpha+1}}$. On the other hand, if we let $t_1<t_2<\log\frac{1}{\sqrt{\tau}}$, then we have

$$\tau^{-(1+\alpha)}>e^{2(\alpha+1)t_2}>e^{(\alpha+1)t_1}\cdot e^{(\alpha+1)t_2}.$$

Furthermore we have

$$\begin{aligned}&\tau^{-(1+\alpha)}\cdot e^{(\alpha+1)t_1}+e^{(\alpha+1)t_1}\cdot e^{2(\alpha+1)t_2}\\&\quad<\tau^{-(1+\alpha)}\cdot e^{(\alpha+1)t_2}+e^{2(\alpha+1)t_1}\cdot e^{(\alpha+1)t_2},\end{aligned}$$

and

$$\begin{aligned}&e^{(\alpha+1)t_1}\cdot e^{2(\alpha+1)t_2}+e^{2(\alpha+1)t_1}(|y_0|^2+\lambda^{2(\alpha+1)})\\&\quad<e^{(\alpha+1)t_2}\cdot e^{2(\alpha+1)t_1}+e^{2(\alpha+1)t_2}(|y_0|^2+\lambda^{2(\alpha+1)}).\end{aligned}$$

Then by a direct calculation we can get

$$\frac{e^{(\alpha+1)t_1}}{|e^{2(\alpha+1)(t_1+i\theta)}-y_0|^2+\lambda^{2(\alpha+1)}}<\frac{e^{(\alpha+1)t_2}}{|e^{2(\alpha+1)(t_2+i\theta)}-y_0|^2+\lambda^{2(\alpha+1)}}.$$

This implies that $\omega(t,\theta)$ is increasing for $t<\log\frac{1}{\sqrt{\tau}}$ and then attain its maximum at $t=\log\frac{1}{\sqrt{\tau}}$.

By the definition of $\omega(t,\theta)$, we have

$$\omega(t,\theta)\le(\alpha+1)t+C. \tag{50}$$

In addition, by (35),

$$\begin{aligned}\omega_n(t+\log\varepsilon_n,\theta)&=u_n(e^{t+\log\varepsilon_n+i\theta})+(\alpha+1)(t+\log\varepsilon_n)-\varepsilon_n\frac{A}{a}e^t\\&=\xi_n+(\alpha+1)t-\varepsilon_n\frac{A}{a}e^t.\end{aligned}$$

Then in view of (48), (49), for every fixed $s\in\mathbb{R}$ we have

$$\sup_{t\le s,\theta\in[0,\pi]}|\omega_n(t+\log\varepsilon_n,\theta)-\omega(t,\theta)|\to0,\ \text{as } n\to+\infty. \tag{51}$$

From (51) and $\omega(t,\theta)$ attain its maximum at $t=\log\frac{1}{\sqrt{\tau}}$, for large n, we have

$$\sup_{t\le4+\log\frac{1}{\sqrt{\tau}},\theta\in[0,\pi]}|\omega_n(t+\log\varepsilon_n,\theta)-\omega(t,\theta)|<1, \tag{52}$$

and

$$\omega_n(4+\log\frac{1}{\sqrt{\tau}}+\log\varepsilon_n,\theta)<\omega_n(\log\frac{1}{\sqrt{\tau}}+\log\varepsilon_n,\theta),\forall\theta\in[0,\pi]. \tag{53}$$

By (53), we see that for large n, if we set $\lambda = \log\varepsilon_n + \log\frac{1}{\sqrt{\tau}} + 2$ and $t = \log\varepsilon_n + \log\frac{1}{\sqrt{\tau}} + 4$, (37) fails to hold. As a consequence, (46) follows. Hence using (46), (50), (52) for large n, we can estimate

$$\begin{aligned}\omega_n(2\lambda_n,\theta) &\le \omega(2\lambda_n - \log\varepsilon_n,\theta) + 1 \le (\alpha+1)(2\lambda_n - \log\varepsilon_n) + C\\ &\le (\alpha+1)\log\varepsilon_n + O(1) = -u_n(0) + O(1).\end{aligned}$$

Then in view of (35), (38), we have

$$\begin{aligned}\inf_{\bar{B}_1^+} u_n = \inf_{S_1^+} u_n = \min_{\theta\in[0,\pi]}\omega_n(0,\theta) + \frac{A}{a} &\le \max_{\theta\in[0,\pi]}\omega_n(2\lambda_n,\theta) + \frac{A}{a}\\ &\le -u_n(0) + O(1).\end{aligned}$$

Case 2. (44) holds.

In this case, duo to the assumption (i) we have firstly

$$v_n(\frac{x_n}{|x_n|}) \to +\infty, \text{ as } n \to +\infty.$$

Suppose that along a subsequence,

$$\frac{x_n}{|x_n|} \to x_0, \text{ with } |x_0| = 1.$$

Therefore v_n admits a blow-up point x_0. Then we apply Theorem 1.1 to v_n to get that v_n must verify alternative (iii) in Theorem 1.1. Moreover by (44), $0 \notin S$. Consequently,

$$\max_{\bar{B}_{\varepsilon_0}^+} v_n \to -\infty, \text{ as } n \to +\infty. \tag{54}$$

We choose s_0 small enough such that x_0 is an only blow-up point for v_n in $B_{s_0}(x_0)\cap\bar{\mathbb{R}}^2_+$.

If $x_0 \in \mathbb{R}^2_+$, we can choose s_0 small enough such that $B_{s_0}(x_0) \subset \mathbb{R}^2_+$. Let $y_n \in \bar{B}_{s_0}(x_0)$, and $v_n(y_n) = \max_{\bar{B}_{s_0}(x_0)} v_n$. Then $y_n \to x_0$ and $v_n(y_n) \to +\infty$. Set

$$\delta_n = e^{-v_n(y_n)} \to 0, \quad \xi_n(x) = v_n(y_n + \delta_n x) + \log\delta_n.$$

Then we have

$$\begin{cases} -\Delta\xi_n = U_n e^{2\xi_n} & \text{in } B_{\frac{s_0}{2\delta_n}},\\ \max_{\bar{B}_{\frac{s_0}{2\delta_n}}}\xi_n = \xi_n(0) = 0, \end{cases} \tag{55}$$

with the energy condition

$$\int_{B_{\frac{s_0}{2\delta_n}}} U_n e^{2\xi_n} \le C,$$

where $U_n(x) = |y_n + \delta_n x|^{2\alpha} V_n(|x_n|y_n + |x_n|\delta_n x) \to 1$ in $B_L(0)$ for all $L > 0$. Then along a subsequence, by the classification results in [6] we have

$$\xi_n(x) \to \xi(x) = \log\frac{1}{(1+\frac{1}{8}|z|^2)^2} \text{ uniformly in } C^2_{loc}(\mathbb{R}^2). \tag{56}$$

Now we need consider the following two situations.

If $x_0 \in \partial\mathbb{R}^2_+$, then $B_{s_0}(x_0) \cap \mathbb{R}^2_+ = B^+_{s_0}(x_0)$. Let $y_n \in \bar{B}^+_{s_0}(x_0)$, and $v_n(y_n) = \max_{\bar{B}^+_{s_0}(x_0)} v_n$. Denote $y_n = (y_{n,1}, y_{n,2})$. Then $y_n \to x_0$ and $v_n(y_n) \to +\infty$. Set

$$\delta_n = e^{-v_n(y_n)} \to 0, \quad \xi_n(x) = v_n(y_n + \delta_n x) + \log \delta_n.$$

Then we have

$$\begin{cases} -\Delta \xi_n = U_n e^{2\xi_n} & \text{in } B_{\frac{s_0}{2\delta_n}} \cap \{t > -\frac{y_{n,2}}{\delta_n}\}, \\ \dfrac{\partial \xi_n}{\partial \nu} = H_n e^{\xi_n} & \text{on } B_{\frac{s_0}{2\delta_n}} \cap \{t = -\frac{y_{n,2}}{\delta_n}\}, \\ \max_{\bar{B}^+_{\frac{s_0}{2\delta_n}}} \xi_n = \xi_n(0) = 0, \end{cases} \tag{57}$$

with the energy coniditions

$$\int_{B_{\frac{s_0}{2\delta_n}} \cap \{t > -\frac{y_{n,2}}{\delta_n}\},} U_n e^{2\xi_n} \le C, \quad \int_{B_{\frac{s_0}{2\delta_n}} \cap \{t = -\frac{y_{n,2}}{\delta_n}\}} H_n e^{\xi_n} \le C,$$

where $U_n(x) = |y_n + \delta_n x|^{2\alpha} V_n(|x_n| y_n + |x_n| \delta_n x) \to 1$ in B^+_S and $H_n(x) = |y_n + \delta_n x|^{\alpha} h_n(|x_n| y_n + |x_n| \delta_n x) \to 1$ on L_S for all $S > 0$. Now we need consider the following two situations.

(1): $\frac{|y_{n,2}|}{\delta_n} \to +\infty$. Then along a subsequence, by the classification results in [6] we have

$$\xi_n(x) \to \xi(x) = \log \frac{1}{(1 + \frac{1}{8}|z|^2)^2} \text{ uniformly in } C^2_{loc}(\mathbb{R}^2), \tag{58}$$

with $\xi(0) = \max_{\mathbb{R}^2} \xi = 0$.

(2): $\frac{|y_{n,2}|}{\delta_n} \to \Lambda < +\infty$. Also by the classification results in [11] we have

$$\begin{aligned} \xi_n(x) \to \xi(x) = \log \frac{2\lambda}{\lambda^2 + (x_1 - s_0)^2 + (x_2 + \Lambda + \lambda)^2} \\ \text{uniformly in } C^2_{loc}(\mathbb{R}^2_{-\Lambda} \cap C^1_{loc}(\bar{\mathbb{R}}^2_{-\Lambda}), \end{aligned} \tag{59}$$

with $\xi(0) = \max_{\bar{\mathbb{R}}^2_{-\Lambda}} \xi = 0$.

Claim.

$$\lambda_n \le \log |x_n| + O(1), \text{ as } n \to +\infty. \tag{60}$$

To establish this claim, we first notice that (54). By using the convergence properties (56), (58) and (59), we have for suitable small $\sigma > 0$ and n large,

$$v_n(y_n + \delta_n x) \le v_n(y_n) - 2\sigma, \forall x : \{\frac{1}{2} \le |x| \le 3\} \bigcap \bar{\mathbb{R}}^2_{-\Lambda}. \tag{61}$$

Let $\rho_n \in (0, +\infty)$ and $\theta_n \in [0, 2\pi]$ be the polar coordinate for y_n, i.e.

$$\rho_n e^{i\theta_n} = y_n.$$

Since $y_n \to x_0$ and $|x_0| = 1$, we have $\rho_n \to 1$ as $n \to +\infty$. Recalling the definition of ω_n, we have for all $s > 0$

$$\begin{aligned}&\omega_n(\log|x_n| + \log\rho_n + 2\log(1+s), \theta_n)\\&\quad = u_n(|x_n|\rho_n(1+s)^2 e^{i\theta_n}) + (\alpha+1)\log[|x_n|\rho_n(1+s)^2] - \frac{A}{a}|x_n|\rho_n(1+s)^2\\&\quad = v_n((1+s)^2 y_n) + (\alpha+1)\log[\rho_n(1+s)^2] - \frac{A}{a}|x_n|\rho_n(1+s)^2.\end{aligned}$$

Thus we obtain

$$\begin{aligned}&\omega_n(\log|x_n| + \log\rho_n + 2\log(1+\delta_n), \theta_n)\\&\quad = v_n((1+\delta_n)^2 y_n) + (\alpha+1)\log[\rho_n(1+\delta_n)^2] - \frac{A}{a}|x_n|\rho_n(1+\delta_n)^2,\end{aligned}$$

and

$$\omega_n(\log|x_n| + \log\rho_n), \theta_n) = v_n(y_n) + (\alpha+1)\log\rho_n - \frac{A}{a}|x_n|\rho_n.$$

Since $\delta_n \to 0$, then for n large, we can use (61) to obtain

$$\begin{aligned}&\omega_n(\log|x_n| + \log\rho_n + 2\log(1+\delta_n), \theta_n) - \omega_n(\log|x_n| + \log\rho_n, \theta_n)\\&\quad = v_n(y_n + \delta_n(2y_n + \delta_n y_n)) - v_n(y_n) + (\alpha+1)\log(1+\delta_n)^2\\&\quad -\frac{A}{a}|x_n|\rho_n[(1+\delta_n)^2 - 1] < -\sigma.\end{aligned} \tag{62}$$

Consequently, for $\theta = \theta_n$, when $\lambda = \log|x_n| + \log\rho_n + \log(1+\delta_n)$ and $t = \log|x_n| + \log\rho_n + 2\log(1+\delta_n)$, (37) fails to hold. And (60) is established.

From (38), (60) we have

$$\begin{aligned}\inf_{\bar{B}_1^+} u_n = \inf_{S_1^+} u_n &= \min_{\theta\in[0,\pi]} \omega_n(0,\theta) + \frac{A}{a} \le \max_{\theta\in[0,\pi]} \omega_n(2\lambda_n,\theta) + \frac{A}{a}\\&\le \max_{\theta\in[0,\pi]} v_n\left(\frac{e^{2\lambda_n + i\theta}}{|x_n|}\right) + (\alpha+1)(2\lambda_n - \log|x_n|) + \frac{A}{a}\\&\le \max_{\bar{B}^+_{r_0|x_n|}} v_n + (\alpha+1)\log|x_n| + C\end{aligned}$$

for suitable constant $r_0 > 0$. The Proposition is completely established. □

We shall need the following version of Proposition 4.1.

Corollary 4.2 *Under the assumptions of Proposition 4.1, for every $r \in (0,1]$, we have*

either (a) $\max_{\bar{B}^+_{\varepsilon_0}} v_n \to -\infty$ *and* $\inf_{\bar{B}^+_r} u_n \le \max_{\bar{B}^+_{r_0|x_n|}} v_n + (\alpha+1)\log|x_n| - 2(\alpha+1)\log r + C$,

or (b) $v_n(0) \to +\infty$ *and* $\inf_{\bar{B}^+_r} u_n \le -u_n(0) - 2(\alpha+1)\log r + C$,

for suitable $r_0 > 0$ and C.

Proof For $r \in (0,1)$, in $\bar{B}^+_{\frac{1}{r}}$ we define

$$u_{n,r}(x) = u_n(rx) + (\alpha+1)\log r. \tag{63}$$

Then $u_{n,r}$ satisfies

$$\begin{cases} -\Delta u_{n,r} = V_{n,r}(x)|x|^{2\alpha}e^{2u_{n,r}} & \text{in } B_1^+, \\ \dfrac{\partial u_{n,r}}{\partial \nu} = h_{n,r}(x)|x|^{\alpha}e^{u_{n,r}} & \text{on } L_1, \end{cases}$$

where $V_{n,r}(x) = V_n(rx)$ and $h_{n,r}(x) = h_n(rx)$. Notice that

$$u_{n,r}(\frac{1}{r}x_n) + (\alpha+1)\log(\frac{1}{r}x_n) = u_n(x_n) + (\alpha+1)\log|x_n| \to +\infty,$$

and $V_{n,r}(x)$ and $h_{n,r}(x)$ still satisfy (2) in $\bar{B}_1^+$. If we set $x_{n,r} = \frac{x_n}{r}$ and

$$v_{n,r}(x) = u_{n,r}(|x_{n,r}|x) + (\alpha+1)\log|x_{n,r}| = v_n(x),$$

then by applying Proposition 4.1 to $u_{n,r}(x)$ and $v_{n,r}(x)$ we conclude that

$$\begin{aligned} &either\ (a)\ \max_{\bar{B}^+_{\varepsilon_0}} v_n \to -\infty, \text{ and } \inf_{\bar{B}_1^+} u_{n,r} \le \max_{\bar{B}^+_{r_0|x_n|}} v_n + (\alpha+1)\log\frac{|x_n|}{r} + C, \\ &or\ (b)\ v_n(0) \to +\infty \text{ and } \inf_{\bar{B}_1^+} u_{n,r} \le -u_{n,r}(0) + C. \end{aligned}$$

So when case (a) holds, we have

$$\inf_{\bar{B}_r^+} u_n \le \max_{\bar{B}^+_{r_0|x_n|}} v_n + (\alpha+1)\log|x_n| - 2(\alpha+1)\log r + C.$$

When case (b) holds, then

$$\inf_{\bar{B}_r^+} u_n \le -u_n(0) - 2(\alpha+1)\log r + C.$$

□

As already mentioned, Proposition 4.1 play a crucial role in proving Theorem 1.2 as it also implies the following result.

Corollary 4.3 *In addition to the assumptions of Proposition 4.1, we suppose further that*

$$\sup_{\bar{B}^+_{2r_n}\setminus\bar{B}^+_{\frac{\delta_n}{2}}} (u_n(x) + (\alpha+1)\log|x|) \le C, \tag{64}$$

with

$$\gamma|x_n| \le \delta_n < r_n < \frac{1}{2},$$

for $\gamma > 0$ suitable constant. Then along a subsequence,

$$\int_{B^+_{r_n}\setminus B^+_{\delta_n}} V_n|x|^{2\alpha}e^{2u_n} + \int_{L_{r_n}\setminus L_{\delta_n}} h_n|x|^{\alpha}e^{u_n} \to 0,\ as\ n \to +\infty.$$

Proof For given $r \in (\delta_n, r_n)$, define $u_{n,r}$ as in (63). So

$$\begin{cases} -\Delta u_{n,r} = V_n(rx)|x|^{2\alpha}e^{2u_{n,r}} := f_{n,r} & \text{in } B_2^+\setminus B^+_{\frac{1}{2}}, \\ \dfrac{\partial u_{n,r}}{\partial \nu} = h_n(rx)|x|^{\alpha}e^{u_{n,r}} := g_{n,r} & \text{on } L_2\setminus L_{\frac{1}{2}}, \\ \sup_{\bar{B}_2^+\setminus\bar{B}^+_{\frac{1}{2}}} u_{n,r} \le C. \end{cases} \tag{65}$$

And by (64) we have

$$\|f_{n,r}\|_{L^\infty(\bar{B}_2^+\setminus\bar{B}_{\frac{1}{2}}^+)} \le C \text{ and } \|g_{n,r}\|_{L^\infty(L_2\setminus L_{\frac{1}{2}})} \le C.$$

Thus we can use Harnack inequality to conclude that there exists a constant $\beta \in (0, 1)$ such that

$$\sup_{S_r^+} u_n \le \beta \inf_{S_r^+} u_n + (\alpha+1)(\beta-1)\log r + C. \tag{66}$$

According to Corollary 4.2, we must treat two situations.

For case (54), i.e.

$$\max_{\bar{B}_{\varepsilon_0}^+} v_n \to -\infty, \tag{67}$$

and Corollary 4.2 implies that

$$\inf_{S_r^+} u_n = \inf_{\bar{B}_r^+} u_n \le \max_{\bar{B}_{r_0|x_n|}^+} v_n + (\alpha+1)\log|x_n| - 2(\alpha+1)\log r + C. \tag{68}$$

Hence if we insert (68) in (66) we obtain

$$\begin{aligned}\int_{B_{r_n}^+\setminus B_{\delta_n}^+} V_n|x|^{2\alpha}e^{2u_n} &\le Ce^{2\beta\max_{\bar{B}_{\varepsilon_0}^+} v_n}|x_n|^{2(\alpha+1)\beta}\left(\frac{1}{\delta_n^{2(\alpha+1)\beta}}-\frac{1}{r_n^{2(\alpha+1)\beta}}\right)\\ &\le C\gamma^{-2(\alpha+1)\beta}e^{2\beta\max_{\bar{B}_{\varepsilon_0}^+} v_n} \to 0, \text{ as } n\to+\infty.\end{aligned}$$

Similarly,

$$\begin{aligned}\int_{L_{r_n}\setminus L_{\delta_n}} h_n|x|^{\alpha}e^{u_n} &\le Ce^{\beta\max_{\bar{B}_{\varepsilon_0}^+} v_n}|x_n|^{(\alpha+1)\beta}\left(\frac{1}{\delta_n^{(\alpha+1)\beta}}-\frac{1}{r_n^{(\alpha+1)\beta}}\right)\\ &\le C\gamma^{-(\alpha+1)\beta}e^{\beta\max_{\bar{B}_{\varepsilon_0}^+} v_n} \to 0, \text{ as } n\to+\infty.\end{aligned}$$

In case (43), we have

$$v_n(0) \to +\infty, \tag{69}$$

and Corollary 4.2 implies that

$$\inf_{S_r^+} u_n \le -u_n(0) - 2(\alpha+1)\log r + C. \tag{70}$$

Inserting (70) in (66), we obtain

$$\begin{aligned}\int_{B_{r_n}^+\setminus B_{\delta_n}^+} V_n|x|^{2\alpha}e^{2u_n} &\le Ce^{-\beta u_n(0)}\left(\frac{1}{\delta_n^{2(\alpha+1)\beta}}-\frac{1}{r_n^{2(\alpha+1)\beta}}\right)\\ &\le C\gamma^{-2(\alpha+1)\beta}e^{-\beta v_n(0)} \to 0, \text{ as } n\to+\infty.\end{aligned}$$

Similarly,

$$\begin{aligned}\int_{L_{r_n}\setminus L_{\delta_n}} h_n|x|^{\alpha}e^{u_n} &\le Ce^{-\beta u_n(0)}\left(\frac{1}{\delta_n^{(\alpha+1)\beta}}-\frac{1}{r_n^{(\alpha+1)\beta}}\right)\\ &\le C\gamma^{-(\alpha+1)\beta}e^{-\beta v_n(0)} \to 0, \text{ as } n\to+\infty.\end{aligned}$$

Thus we achieve the proof of this Corollary. □

5 Proof of Theorem 1.2

Now we come to prove Theorem 1.2.

Proof of theorem 1.2 We will consider separately alternatives (i) and (ii) for u_n in Proposition 3.4. Firstly we consider the case where alternative (i) holds in Proposition 3.4. We have the following Claim.

Claim 1 If (27) holds, then $m(0) = 2\pi(1+\alpha)$.

Note that by Lemma 3.2, the validity of (27) implies

$$u_n(0) = \max_{\bar{B}^+_{2\varepsilon_0}} u_n + O(1), \text{ as } n \to +\infty.$$

So we set

$$\varepsilon_n = e^{-\frac{u_n(0)}{\alpha+1}} \to 0, \text{ as } n \to +\infty,$$

and as in Lemma 3.2 and Remark 3.3, along a subsequence,

$$\xi_n(x) = u_n(\varepsilon_n x) + (\alpha+1)\log \varepsilon_n \to \xi \text{ uniformly in } C^2_{loc}(\mathbb{R}^2_+) \cap C^1_{loc}(\bar{\mathbb{R}}^2_+ \setminus \{0\}) \cap C^0_{loc}(\bar{\mathbb{R}}^2_+),$$

here ξ described in (26).

Since

$$\int_{\mathbb{R}^2_+} V(0)|x|^{2\alpha} e^{2\xi} + \int_{\partial\mathbb{R}^2_+} h(0)|x|^{\alpha} e^{\xi} = 2\pi(1+\alpha),$$

we can find $R_n \to +\infty$ such that, along a subsequence,

$$\int_{B^+_{R_n\varepsilon_n}} V_n|x|^{2\alpha} e^{2u_n} + \int_{L_{R_n\varepsilon_n}} h_n|x|^{\alpha} e^{u_n} \to 2\pi(1+\alpha), \text{ as } n \to +\infty.$$

For every $r \in (R_n\varepsilon_n, \varepsilon_0)$, we can also apply Harnack inequality as in the proof of Corollary 4.3 to derive

$$\sup_{S^+_r} u_n \leq \beta \inf_{S^+_r} u_n + (\alpha+1)(\beta-1)\log r + C \tag{71}$$

with $\beta \in (0, 1)$.

Moreover by the alternative (b) in Proposition 4.1 and Corollary 4.2 we derive that

$$\inf_{S^+_r} u_n = \inf_{\bar{B}^+_r} u_n \leq -u_n(0) - 2(\alpha+1)\log r + C. \tag{72}$$

Combine (72) with (71), we get the estimate

$$V_n|x|^{2\alpha} e^{2u_n} \leq C\frac{e^{-\beta u_n(0)}}{r^{2(\alpha+1)\beta+2}} \text{ and } h_n|x|^{\alpha} e^{u_n} \leq C\frac{e^{-\beta u_n(0)}}{r^{(\alpha+1)\beta+1}}, \tag{73}$$

on S_r^+. Consequently,

$$\int_{B^+_{\varepsilon_0}\setminus B^+_{R_n\varepsilon_n}} V_n|x|^{2\alpha}e^{2u_n} \le Ce^{-\beta u_n(0)}\left(\frac{1}{(R_n\varepsilon_n)^{2(\alpha+1)\beta}}-\frac{1}{\varepsilon_0^{2(\alpha+1)\beta}}\right)$$
$$\le \frac{C}{R_n^{2(\alpha+1)\beta}} \to 0, \text{ as } n\to+\infty.$$
$$\int_{L_{\varepsilon_0}\setminus L_{R_n\varepsilon_n}} h_n|x|^{\alpha}e^{u_n} \le Ce^{-\beta u_n(0)}\left(\frac{1}{(R_n\varepsilon_n)^{(\alpha+1)\beta}}-\frac{1}{\varepsilon_0^{(\alpha+1)\beta}}\right)$$
$$\le \frac{C}{R_n^{(\alpha+1)\beta}} \to 0, \text{ as } n\to+\infty.$$

So we have

$$\begin{aligned}&\int_{B^+_{\varepsilon_0}} V_n|x|^{2\alpha}e^{2u_n}+\int_{L_{\varepsilon_0}} h_n|x|^{\alpha}e^{u_n}\\&=\int_{B^+_{R_n\varepsilon_n}} V_n|x|^{2\alpha}e^{2u_n}+\int_{L_{R_n\varepsilon_n}} h_n|x|^{\alpha}e^{u_n}+o(1)\\&=2\pi(1+\alpha)+o(1).\end{aligned}$$

Hence by (4), letting $n\to+\infty$, the desired conclusion follows.

We are left to treat the case where alternative (ii) holds in Proposition 3.4. In this case, we can apply Corollary 4.3 and derive

$$\int_{B_1^+\setminus B^+_{\frac{1}{2\varepsilon_0}|x_{m,n}|}} V_n|x|^{2\alpha}e^{2u_n}+\int_{L_1\setminus L_{\frac{1}{2\varepsilon_0}|x_{m,n}|}} h_n|x|^{\alpha}e^{u_n}\to 0, \text{ as } n\to+\infty.$$

And similarly for $m\ge 2$,

$$\int_{B^+_{2\varepsilon_0|x_{j+1,n}|}\setminus B^+_{\frac{1}{2\varepsilon_0}|x_{j,n}|}} V_n|x|^{2\alpha}e^{2u_n}+\int_{L_{2\varepsilon_0|x_{j+1,n}|}\setminus L_{\frac{1}{2\varepsilon_0}|x_{j,n}|}} h_n|x|^{\alpha}e^{u_n}\to 0$$

as $n\to+\infty$ for $j=1,\dots,m-1$. Consequently,

$$\begin{aligned}&\int_{B_1^+} V_n|x|^{2\alpha}e^{2u_n}+\int_{L_1} h_n|x|^{\alpha}e^{u_n}\\&=\int_{B^+_{2\varepsilon_0|x_{1,n}|}} V_n|x|^{2\alpha}e^{2u_n}+\Sigma_{j=1}^m\int_{B^+_{\frac{1}{2\varepsilon_0}|x_{j,n}|}\setminus B^+_{2\varepsilon_0|x_{j,n}|}} V_n|x|^{2\alpha}e^{2u_n}\\&\quad+\int_{L_{2\varepsilon_0|x_{1,n}|}} h_n|x|^{\alpha}e^{u_n}+\Sigma_{j=1}^m\int_{L_{\frac{1}{2\varepsilon_0}|x_{j,n}|}\setminus L_{2\varepsilon_0|x_{j,n}|}} h_n|x|^{\alpha}e^{u_n}+o(1)\end{aligned}\tag{74}$$

as $n\to+\infty$. Set

$$D_0=\bar{B}^+_{\frac{1}{2\varepsilon_0}}\setminus\bar{B}^+_{2\varepsilon_0}.$$

And set

$$v_{j,n}(x)=u_n(|x_{j,n}|x)+(\alpha+1)\log|x_{j,n}|,\ j=1,\dots,m.$$

Then we see that

$$\begin{cases} -\Delta v_{j,n} = V_{j,n} e^{2v_{j,n}} & \text{in } B^+_{\frac{1}{2\varepsilon_0}} \backslash B^+_{2\varepsilon_0}, \\ \dfrac{\partial v_{j,n}}{\partial \nu} = h_{j,n} e^{v_{j,n}} & \text{on } L_{\frac{1}{2\varepsilon_0}} \backslash L_{2\varepsilon_0}, \end{cases} \tag{75}$$

with the energy conditions

$$\int_{B^+_{\frac{1}{2\varepsilon_0}} \backslash \bar{B}^+_{2\varepsilon_0}} V_{j,n} e^{2v_{j,n}} \leq C, \quad \int_{L_{\frac{1}{2\varepsilon_0}} \backslash L_{2\varepsilon_0}} h_{j,n} e^{v_{j,n}} \leq C,$$

where $V_{j,n}(x) = |x|^{2\alpha} V_n(|x_{j,n}|x)$ and $h_{j,n}(x) = |x|^{\alpha} h_n(|x_{j,n}|x)$ satisfy

$$0 < a_1 \leq V_{j,n} \leq C, |\nabla V_{j,n}| \leq A_1; 0 < b_1 \leq h_{j,n} \leq C, |\nabla h_{j,n}| \leq B_1 \text{ in } D_0.$$

Now we set

$$\beta_0 = \lim_{n \to +\infty} \left(\int_{B^+_{2\varepsilon_0}} V_{1,n} e^{2v_{1,n}} + \int_{L_{2\varepsilon_0}} h_{1,n} e^{v_{1,n}} \right),$$

$$\beta_j = \lim_{n \to +\infty} \left(\int_{B^+_{\frac{1}{2\varepsilon_0}} \backslash \bar{B}^+_{2\varepsilon_0}} V_{j,n} e^{2v_{j,n}} + \int_{L_{\frac{1}{2\varepsilon_0}} \backslash L_{2\varepsilon_0}} h_{j,n} e^{v_{j,n}} \right).$$

So that by (74), we have

$$m(0) = \beta_0 + \Sigma_{j=1}^{m} \beta_j.$$

Claim 2 Either $\beta_0 = 0$ or $\beta_0 = 2\pi(1+\alpha)$.

In fact, by Proposition 4.1, we see that either $\max_{\bar{B}^+_{2\varepsilon_0}} v_{1,n} \to -\infty$ or $v_{1,n}(0) \to +\infty$. If $\max_{\bar{B}^+_{2\varepsilon_0}} v_{1,n} \to -\infty$, then $\beta_0 = 0$ in this case. If $v_{1,n}(0) \to +\infty$, we see that 0 is the only blow-up point of $v_{1,n}$ in $\bar{B}^+_{2\varepsilon_0}$ since $\sup_{\bar{B}^+_{2\varepsilon_0}} \{v_{1,n} + (\alpha+1)\log|x|\} \leq C$. We can apply Theorem 1.1 and conclude that $\lim_{n\to+\infty} (\int_{B^+_{2\varepsilon_0}} V_{1,n} e^{2v_{1,n}} + \int_{L_{2\varepsilon_0}} h_{1,n} e^{v_{1,n}}) = \beta_0$. Furthermore, since $\sup_{\bar{B}^+_{2\varepsilon_0}} \{v_{1,n} + (\alpha+1)\log|x|\} \leq C$ we can use Claim 1 above for $v_{1,n}$ and obtain $\beta_0 = 2\pi(1+\alpha)$ in this case.

Claim 3 $\beta_j \in 2\pi\mathbb{N}, \quad j = 1, 2, \ldots, m.$

In fact, (28) implies

$$v_{j,n}\left(\frac{x_{j,n}}{|x_{j,n}|}\right) \to +\infty, \text{ as } n \to +\infty.$$

And by (29)(30) we have

$$\max_{D_0 \backslash \{\bar{B}^+_{\frac{1}{2\varepsilon_0}} \backslash \bar{B}^+_{2\varepsilon_0}\}} v_{j,n} \leq C.$$

Therefore, the blow-up set S_j of $v_{j,n}$ is nonempty and satisfy:

$$S_j \subset \bar{B}^+_{\frac{1}{2\varepsilon_0}} \backslash \bar{B}^+_{2\varepsilon_0} \subset\subset D_0.$$

At this point, we are in position to apply Li–Shafrir and Zhang–Zhou–Zhou (see [10,18]) results around each point S_j and derive $\beta_j \in 2\pi\mathbb{N}$, $\forall j = 1, \ldots, m$.

Thus by the Claims above, Theorem 1.2 is completely established. □

6 Proof of Theorem 1.3

In this section, we will obtain the precise blow-up value at the singular blow-up point when u_n have the following mild boundary condition:

$$\max_{S_1^+} u_n - \min_{S_1^+} u_n \le C. \tag{76}$$

As we shall see, the behavior of u_n around the blow-up point 0 is very seriously affected by the validity of (76). Now we give the proof of Theorem 1.3.

Proof of Theorem 1.3 Let p_n satisfy

$$\begin{cases} -\Delta p_n = 0 & \text{in } B_1^+, \\ \dfrac{\partial p_n}{\partial \nu} = 0 & \text{on } L_1, \\ p_n = u_n - \min\limits_{S_1^+} u_n, & \text{on } S_1^+. \end{cases} \tag{77}$$

From (76), the maximum principle and Hopf Lemma we have

$$\|p_n\|_{L^\infty(\bar{B}_1^+)} \le C.$$

Set $w_n = u_n - \min\limits_{S_1^+} u_n - p_n$, then w_n satisfies

$$\begin{cases} -\Delta w_n = W_n|x|^{2\alpha}e^{2w_n} & \text{in } B_1^+, \\ \dfrac{\partial w_n}{\partial \nu} = G_n|x|^{\alpha}e^{w_n} & \text{on } L_1, \\ w_n = 0, & \text{on } S_1^+, \end{cases} \tag{78}$$

where $W_n(x) = e^{\min\limits_{S_1^+} 2u_n + 2p_n}$ and $G_n(x) = e^{\min\limits_{S_1^+} u_n + p_n}$. In addition, by (4) we have

$$\int_{B_1^+} W_n(x)|x|^{2\alpha}e^{2w_n}\phi + \int_{L_1} G_n(x)|x|^{\alpha}e^{w_n}\phi \to m(0)\phi(0), \ \text{for every } \phi \in C^\infty(\bar{B}_1^+). \tag{79}$$

By Green's representation formula,

$$\begin{aligned} w_n(x) &= \frac{1}{\pi}\int_{B_1^+} \ln\frac{1}{|x-y|} W_n(y)|y|^{2\alpha}e^{2w_n}dy + \frac{1}{\pi}\int_{B_1^+} R(x,y)W_n(y)|y|^{2\alpha}e^{2w_n}dy \\ &\quad + \frac{1}{\pi}\int_{L_1} \ln\frac{1}{|x-y|} G_n(y)|y|^{\alpha}e^{w_n}dy + \frac{1}{\pi}\int_{L_1} R(x,y)G_n(y)|y|^{\alpha}e^{w_n}dy, \end{aligned} \tag{80}$$

where $R(x, y)$ is the regular part of the Green function. Passing to the limit in (80), we have

$$w_n(x) \to \frac{m(0)}{\pi}\ln\frac{1}{|x|} + m(0)R(x,0). \tag{81}$$

Set $g(x) = \beta R(x, 0) \in C^1(\bar{B}_1^+)$ and

$$w_0(x) = \frac{m(0)}{\pi} \ln \frac{1}{|x|} + g(x). \tag{82}$$

By the Pohozaev identity for w_n in $\bar{B}_r^+$, we have

$$\begin{aligned} & r \int_{S_r^+} \left(|\frac{\partial w_n}{\partial \nu}|^2 - \frac{1}{2} |\nabla w_n|^2 \right) d\sigma \\ & = (1+\alpha) \int_{B_r^+} W_n |x|^{2\alpha} e^{2w_n} dx + (1+\alpha) \int_{L_r} G_n |x|^\alpha e^{w_n} d\sigma \\ & \quad - \frac{r}{2} \int_{S_r^+} W_n |x|^{2\alpha} e^{2w_n} d\sigma + \frac{1}{2} \int_{B_r^+} |x|^{2\alpha} e^{2w_n} (x \cdot \nabla W_n) dx \\ & \quad - G_n(x_1, 0) |x_1|^\alpha x_1 e^{w_n(x_1, 0)} \Big|_{x_1=-r}^{x_1=r} + \int_{-r}^{r} \frac{\partial G_n(x_1, 0)}{\partial x_1} |x_1|^\alpha x_1 e^{w_n(x_1, 0)} dx_1 \end{aligned} \tag{83}$$

Let $n \to \infty$ in (83), and using (79) and (81) we find the identity:

$$r \int_{S_r^+} \left(|\frac{\partial w_0}{\partial \nu}|^2 - \frac{1}{2} |\nabla w_0|^2 \right) d\sigma = m(0)(1+\alpha) + o_r(1). \tag{84}$$

Inserting (82) into (84), we obtain

$$r \int_{S_r^+} \left(|\frac{\partial w_0}{\partial \nu}|^2 - \frac{1}{2} |\nabla w_0|^2 \right) d\sigma = \frac{m(0)^2}{2\pi} + o_r(1).$$

Letting $r \to 0$, then we have $m(0) = 2\pi(1+\alpha)$. □

Appendix

Identifying $x = (x_1, x_2) \in \mathbb{R}^2$. Suppose $\Omega = (a, 0) \times (0, b)$ with $a < 0$, $b > 0$ is an open rectangle in $\mathbb{R}^2$. Let $x_0 = (a, b)$. Suppose u satisfy

$$\begin{cases} \Delta u \geq 0, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} \leq 0, & \text{on } x_2 = b, \\ u(x) = 0, & \text{on } \{a\} \times [0, b], \\ u(x) \leq 0 & \text{in } \bar{\Omega} \\ u(x) < 0, & \text{on } \{0\} \times [0, b], \end{cases}$$

Then we have $\frac{\partial u}{\partial x_1}|_{x_0} < 0$.

Proof Firstly we choose a point $y \in (a, 0) \times \{b\}$ which is the center of the circle whose radius is $|x_0 - y| = R$. And let $R < \min\{b, \frac{|a|}{2}\}$. Set $B_R(y) \cap \Omega = B_R^-(y)$. For $0 < \rho < R$, we introduce an auxiliary function v by defining

$$v(x) = e^{-\gamma r^2} - e^{-\gamma R^2}$$

where $r = |x - y| > \rho$ and γ is a positive constant yet to be determined. Direct calculation gives

$$\Delta v(x) = e^{-\gamma r^2} (4\gamma^2 r^2 - 2\gamma).$$

Hence γ may be chosen large enough so that $\Delta v \geq 0$ throughout the annular region $A = B_R^-(y) - B_\rho^-(y)$. By the strong maximum principle and Hopf Lemma we have $u(x) < 0$ in Ω and $u(x) < 0$ in $(a, 0) \times \{b\}$. Since $u - u(x_0) < 0$ on $\partial B_\rho^-(y) \cap \{0 < x_2 < b\}$, there is a constant $\varepsilon > 0$ small enough for which $u - u(x_0) + \varepsilon v \leq 0$ on $\partial B_\rho^-(y) \cap \{0 < x_2 < b\}$. This inequality is also satisfied on $\partial B_R^-(y) \cap \{0 < x_2 < b\}$. Suppose there exists a point y_1 on $\partial B_R^-(y) \backslash \partial B_\rho^-(y) \cap \{x_2 = b\}$ satisfies $u(y_1) - u(x_0) + \varepsilon v(y_1) = \max_{\partial B_R^-(y) \backslash \partial B_\rho^-(y) \cap \{x_2 = b\}} (u - u(x_0) + \varepsilon v)$. Since $\Delta(u - u(x_0) + \varepsilon v) \geq 0$ in A, so we have $\frac{\partial(u - u(x_0) + \varepsilon v)}{\partial x_2}|_{y_1} > 0$ by Hopf Lemma. But since $\frac{\partial v}{\partial x_2}|_{x_2 = b} = 0$ then we have $\frac{\partial(u - u(x_0) + \varepsilon v)}{\partial x_2}|_{x_2 = b} \leq 0$, which is the desired contradiction. The weak maximum principle implies that $u - u(x_0) + \varepsilon v \leq 0$ throughout A. Then we have

$$\frac{\partial u}{\partial x_1}|_{x_0} \leq -\varepsilon \frac{\partial v}{\partial x_1}|_{x_0} = 2\varepsilon\gamma a e^{-\gamma(a^2+b^2)} < 0.$$

□

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