



# Mean curvature flow of arbitrary co-dimensional Reifenberg sets

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## Abstract

We study the existence and uniqueness of smooth mean curvature flow, in arbitrary dimension and co-dimension, emanating from so called  $k$ -dimensional  $(\varepsilon, R)$  Reifenberg flat sets in  $\mathbb{R}^n$ . The Reifenberg condition, roughly speaking, says that the set has a weak metric notion of a  $k$ -dimensional tangent plane at every point and scale, but those tangents are allowed to tilt as the scales vary. We show that if the Reifenberg parameter  $\varepsilon$  is small enough, the (arbitrary co-dimensional) level set flow (in the sense of Ambrosio and Sonner in *J Differ Geom* 43(4):694–737, 1996) is non fattening, smooth and attains the initial value in the Hausdorff sense. Our results generalize the ones from Hershkovits (*Geom Topol* 21(1):441–484, 2017), in which the co-dimension one case was studied. We also prove a general (short time) smooth uniqueness result, generalizing the one for evolution of smooth submanifolds, which may be of independent interest, even in co-dimension one.

**Mathematics Subject Classification** 53C44

## 1 Introduction

For a  $k$ -dimensional manifold  $M^k$ , a family of smooth embeddings  $\phi_t : M^k \rightarrow \mathbb{R}^n$  for  $t \in (a, b)$  is said to evolve by mean curvature if it satisfies the equation  $\frac{d}{dt} \phi_t(x) = \vec{H}(\phi_t(x))$ , where  $\vec{H}$  is the mean curvature vector. If a compact submanifold  $M \subseteq \mathbb{R}^n$  is of type  $C^2$ , it follows from standard parabolic PDE theory that there exists a unique mean curvature flow starting from  $M$  for some finite maximal time  $T$ .

The question of mean curvature flow (and geometric flows in general) with rough initial data, i.e. when the  $C^2$  assumption is weakened, has been researched extensively (see e.g. [4,5,7,13,14,18,20]). For the co-dimension one, arbitrary dimensional mean curvature flow, two results form the forefront in that regard: In the case that  $M$  is merely Lipschitz, short time

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existence was proved by Ecker and Huisken in the celebrated paper [5]. More recently, under the assumption that the initial set is  $(n - 1)$ -dimensional  $(\varepsilon, R)$  Reifenberg flat (see Definition 1.3) with  $\varepsilon$  sufficiently small, short time existence and uniqueness was shown in [7] (see also Theorem 1.5). Those two results are very different in nature; The result in [5] allows *any* Lipschitz submanifold as an input, by that also dictating a local graph structure and a finite local area. The result in [7] allows some *higher Hausdorff dimensional* sets which are not graphical at any scale (such as variants of the Koch-snowflake) to be taken as inputs, but it requires  $\varepsilon$  to be *small*. Note however that the Lipschitz assumption implies the Reifenberg property, the Reifenberg parameter  $\varepsilon$  being the Lipschitz constant.

In the high co-dimensional case, the optimal result in the literature, which is due to Wang, speaks about the same objects as Ecker–Huisken’s result, but has the smallness character of the result in [7]. More precisely, it was shown in [20] that there exists some  $\varepsilon_0$  such that if  $M$  is uniformly locally Lipschitz  $k$ -dimensional submanifold of  $\mathbb{R}^n$ , with Lipschitz constant less than  $\varepsilon_0$  (i.e. there exists some  $R > 0$  such that every point has a ball of radius  $R$  around it on which the submanifold is an  $\varepsilon_0$ -Lipschitz graph) then there exists a mean curvature flow emanating from it (in light of the example in [15], the smallness assumption in high co-dimension is necessary). By the discussion above, the high co-dimensional generalization of the result in [7], which will be stated shortly, will form a full (qualitative) generalization of the result in [20].

**Remark 1.1** Although not appearing in the literature, White’s local regularity theorem [21, Theorem 3.1], along with an approximation scheme can also be used to show smoothing. To be more precise, for  $\varepsilon > 0$  sufficiently small, if  $X$  is  $k$ -rectifiable and satisfies

$$\Theta(p, r) := \frac{1}{(4\pi r^2)^{k/2}} \int_X e^{-|y-p|^2/4r^2} d\mathcal{H}^k(y) \leq 1 + \varepsilon \tag{1.2}$$

for every  $y \in \mathbb{R}^n$  and  $r < R$ , and if  $X$  can be shown to be approximated by smooth  $k$ -submanifolds  $M$  (in the Hausdorff and multiplicity one varifolds senses), then it follows from short time existence (for smooth initial data) and [21, Theorem 3.1] that there exists a smooth MCF emanating from  $X$  for time proportional to  $r^2$ . The proof follows from the scheme of the proof of the main result of this work, utilizing [21, Theorem 3.1] and monotonicity. This result also has the “smallness” character that was described above. As will be remarked upon shortly, this will also be generalized (qualitatively) by the main result of this work.

To state our main result, we first need to define the objects to which it applies.

**Definition 1.3** ([17]) Given  $\varepsilon > 0, R > 0$  and  $k \in \mathbb{N}$ , a compact connected set  $X \subseteq \mathbb{R}^n$  is called  **$k$ -dimensional  $(\varepsilon, R)$  Reifenberg flat** if for every  $x \in X$  and  $r < R$  there exists a  $k$ -dimensional plane  $P_{x,r}$  passing through  $x$  such that

$$d_H(B(x, r) \cap P_{x,r}, B(x, r) \cap X) \leq \varepsilon r. \tag{1.4}$$

Here  $d_H$  is the Hausdorff distance and  $B(x, r)$  is a closed ball.

Any  $C^2$   $k$ -submanifold is easily seen to be  $k$ -dimensional  $(\varepsilon, R)$  Reifenberg flat for some  $\varepsilon, R > 0$ . Every uniformly locally Lipschitz  $k$ -submanifold of  $\mathbb{R}^n$  is trivially  $k$ -dimensional  $(\varepsilon, R)$  Reifenberg flat as well. The Reifenberg condition is however general enough to include some sets with Hausdorff dimension larger than  $k$  (see [7,19]).

Another notion that one needs in order to discuss evolution of non smooth initial data is that of a weak solution to the  $k$ -dimensional mean curvature flow. This weak mean curvature flow is called the level set flow, as its original definition, due to Evans and Spruck [6] and

Chen–Giga–Goto [3] (in the co-dimension one case), was via viscosity solutions for the equation of a level set of a function evolving by mean curvature. In co-dimension one, a geometric, avoidance principle based, equivalent definition was given in [9]. In high co-dimension, while a viscosity solution based definition was given in [1], there is no effective geometric definition of weak mean curvature flow (of arbitrary sets), as even smooth flows cease to satisfy avoidance. The definition of the level set flow in [1] is technical, and will thus be postponed to Sect. 3. For now, it suffices to know that the  $k$ -dimensional level set flow is a semi-group action of  $\mathbb{R}_+$  (time) on compact sets  $X \subseteq \mathbb{R}^n$ ,  $(t, X) \mapsto X_t$  which, starting from an initial  $k$ -dimensional submanifold, coincides with smooth  $k$ -dimensional mean curvature flow, for as long as the latter is defined. Up to the specificities of this association, which will be defined formally (and further investigated) in Sect. 3, we can now state our main theorem.

**Theorem 1.5** *There exist some  $\varepsilon_0, c_0 > 0$  such that if  $X$  is  $k$ -dimensional  $(\varepsilon, R)$  Reifenberg flat in  $\mathbb{R}^n$  for  $0 < \varepsilon < \varepsilon_0$  then the  $k$ -dimensional level set flow (in the sense of [1]) emanating from  $X$ ,  $(X_t)_{t \in (0, c_0 R^2)}$ , is a smooth  $k$ -dimensional mean curvature flow, which further attains the initial value  $X$  in the following sense:*

$$\lim_{t \rightarrow 0} d_H(X, X_t) = 0. \tag{1.6}$$

*In fact, there exist some  $c_1, c_2 > 0$  with  $c_1^2 < \frac{1}{8}$  and  $\frac{1}{4c_1} - c_2 > \sqrt{2k}$  such that the following estimates on the evolution hold:*

- (i)  $|A(t)| \leq \frac{c_1}{\sqrt{t}}$ .
- (ii)  $d_H(X_t, X) \leq c_2 \sqrt{t}$ .
- (iii)  $X_t$  has a tubular neighborhood of size  $\frac{\sqrt{t}}{4c_1}$ .

**Remark 1.7** Recall that level set flow should be thought of as (and in some regards is) “the set of all possible evolutions” (see [10, Sect. 10] and [1, Theorem 5.4]). Thus, in addition to existence of a smooth mean curvature flow emanating from  $k$ -dimensional Reifenberg sufficiently flat sets, we get uniqueness in the strongest possible sense.

**Remark 1.8** In light of the discussion preceding the statement of the theorem, this qualitatively generalizes the result from [20]. As the  $\varepsilon_0$  of Theorem 1.5 is smaller than the one from [20], the generalization is only qualitative, i.e. there are still initial submanifolds for which the result in [20] is applicable while Theorem 1.5 is not.

**Remark 1.9** For every  $\delta > 0$ , if  $X$  satisfies the conditions of Remark 1.1 for  $\varepsilon$  sufficiently small, then  $X$  is also  $(\delta, R)$  Reifenberg flat. This can be seen as follows: By upper-semicontinuity of  $\Theta$  and approximation, it suffices to show this for smooth  $X$ . Let  $X^i$  be smooth sets with  $\Theta_{X^i}(p, r) \leq 1 + \frac{1}{i}$  for every  $r < R$ ,  $p \in \mathbb{R}^n$ , but such that  $d_H(B(x_i, r_i) \cap X^i, P \cap B(x_i, r_i)) \geq \delta r_i$  for every  $k$ -plane  $P$ . Shifting  $x_i$  to the origin and rescaling  $r_i$  to size 1 (and still calling those sets  $X^i$ ), it follows from the result of [21] that  $X^i$  sub-converge in the Hausdorff sense to a  $k$ -plane<sup>1</sup>, which is at least  $\delta$  far in  $B(0, 1)$  from any  $k$ -plane, a contradiction.

Thus, the existence part of Theorem 1.5 is (qualitatively) stronger than the (unpublished) one alluded to in Remark 1.1.

<sup>1</sup> By Proposition 2.8 and Theorem 3.1 of [21] it follows that  $X^i_t$  sub-converge smoothly to a plane for every  $t > 0$ . By integrating over time the curvature bounds of [21, Theorem 3.1] for  $X^i_t$ , we get the  $X^i$  converges to the same plane in the Hausdorff sense.

The proof of Theorem 1.5 is naturally divided into two parts: existence, in which we will construct a mean curvature flow satisfying estimates (i)–(iii) of Theorem 1.5 and uniqueness, in which we will show that the resulted flow is actually the level set flow. The proof of the existence part goes along the same lines as the existence part of the main theorem in [7] and will be described in Sect. 2, where the parts that are identical will only be stated, referring to [7] for the proof. The parts which require additional work to the one done in the co-dimension one case will be treated in full. The proof of the uniqueness part, which will be given in Sect. 3, is completely different from the one in [7]. Back there, one could work almost entirely in the realm of smooth solution, utilizing inward and outward approximations [7, Corollary 2.11] and Ilmanen’s avoidance based definition of the level set flow [9]. The uniform estimates on the evolution of the approximations (see [7, Theorem 1.16] and Theorem 2.26) coupled with a separation estimate [7, Theorem 1.22] were then used to show that the flows emanating from the inward and outward approximations remain very close, providing barriers to the level set flow. In high co-dimension there is no notion of “inside” and “outside”, and, as discussed above, there is no avoidance based definition of the level set flow. In order to show uniqueness (and in fact, even define it) we will therefore have to revert to work entirely in the viscosity solution realm. While in co-dimension one the viscosity solution definition of the level set flow is very well known and standard, the high co-dimensional analogue of it, introduced in [1] is far less known. Some part of our work will consist of exploring it a bit further than what was done in [1].

The short time uniqueness of the flow is an immediate corollary of the existence part of Theorem 1.5 and the following general strong smooth uniqueness criterion for the level set flow, which is of interest by its own right.

**Theorem 1.10** *Let  $X \subset \mathbb{R}^n$  be a connected compact set, let  $c_1, c_2 > 0$  be constants and let  $(X_t)_{t \in (0, T]}$  be a smooth  $k$ -dimensional mean curvature flow, satisfying*

- (i)  $|A(t)| \leq \frac{c_1}{\sqrt{t}}$ .
- (ii)  $d_H(X_t, X) \leq c_2\sqrt{t}$ .
- (iii)  $X_t$  has a tubular neighborhood of size  $\frac{\sqrt{t}}{4c_1}$ .

*Assume further that  $c_1^2 \leq \frac{1}{8}$  and  $\frac{1}{4c_1} - c_2 > \sqrt{2k}$ . Then  $X_t$  is the level set flow of  $X$ .*

Theorem 1.10 is a quantitative generalization of the fact that, starting from a smooth submanifold, the level set flow coincides with smooth mean curvature flow. As in the smooth case (see [1, Sect. 3]), the idea is to use the distance to  $X_t$  to construct non-negative lower barriers to the level set equation. As conditions (i)–(iii) are more precise than a smoothness assumption, this should be made more quantitatively; The construction of the sub-solution, as well as the choice of the parabolic neighborhood along the boundary of which the barrier can be seen to be smaller than the solution, should reflect estimates (i)–(iii). In terms of the proof, this means that as opposed to the smooth case, where one could get along by some continuity based arguments, in our case we will be forced to compute some quantities more explicitly, and to use avoidance of balls [which is true in arbitrary co-dimension (see Lemma 3.17)] to get some initial estimates on the behavior of the level set equation (with the right initial data).

**Remark 1.11** As a flow satisfying (i)–(iii) provides approximations at different scales to  $X$ , having such smooth flow implies some regularity of  $X$ . This regularity is far weaker than the one assumed in Theorem 1.5 (c.f. Theorem 2.1), so we expect Theorem 1.10 to be applicable in other situations as well.

## 2 Existence

The proof of the existence of a flow  $(X_t)_{t \in (0, c_0 R^2)}$  satisfying estimates (i)–(iii) of Theorem 1.5 is divided, as in the co-dimension one case of [7], into three steps: construction of approximations at different scales (Sect. 2.1), obtaining uniform estimates on the flows emanating from them, and passage to a limit (Sect. 2.4). The proof of the uniform estimates in turn, consists of three major ingredients: estimates for graphical mean curvature with small initial data on thick cylinders (Sect. 2.2), an interpolation lemma (Sect. 2.3), and an iteration scheme (Sect. 2.4). As stated in the introduction, in this section we will address in full the parts that require different treatment than the one in [7] and mention the parts that remain the same.

### 2.1 Approximation

A guideline to proving estimates on a class of weak objects is to first approximate them by smooth objects, then derive estimates that depend only on quantities that are expressible for the weak objects as well, and finally pass to a limit. The first step in our case is the following approximation theorem, which is essentially from [12, Appendix B], where the hypothesis used are different, but the construction remains the same.

**Theorem 2.1** ([12]) *For every  $\delta > 0$  there exists  $\varepsilon > 0$  such that if  $X \subseteq \mathbb{R}^n$  is  $k$ -dimensional  $(\varepsilon, R)$  Reifenberg flat, then for every  $0 < r < R/10$  there exists a  $k$ -dimensional submanifold  $X^r$  such that*

1.  $d_H(X, X^r) \leq \delta r$ .
2.  $|A^r| \leq \frac{\delta}{r}$ .
3. For every  $x \in X$ ,  $X^r \cap B(x, r) = G \cup B$  where  $G$  is connected and  $B \subseteq B(x, r) - B(x, (1 - \delta)r)$ .

**Remark 2.2** In [7] we utilized a stronger global approximation result from [8], but its proof depended on mollifying the characteristic function of the domain bounded by  $X$ , which only works in co-dimension one.

**Remark 2.3** Reifenberg’s topological disk theorem [17] follows easily from Theorem 2.1; The approximations at comparable scales are graphical above one another, and composing those graphical representations yields the bi-Hölder parametrization (c.f. proof of Corollary 2.6).

**Remark 2.4** While the “approximate tangents” of a  $k$ -dimensional  $(\varepsilon, R)$  Reifenberg flat set vary with point and scales, comparable scales and nearby points have very close approximating tangents. More precisely, for every  $\delta > 0$  there exists  $\varepsilon > 0$  such that if  $X$  is  $k$ -dimensional  $(\varepsilon, R)$  Reifenberg flat, then for every  $r < R/10$  and  $x_1, x_2 \in X$  with  $d(x_1, x_2) < r$ , both  $\|P_{x_1, r} - P_{x_2, r}\| < \delta$  and  $\|P_{x_1, r} - P_{x_1, 10r}\| < \delta$ . Here  $P_{x_1, r}$  is as in Definition 1.3,  $\| - \|$  is the operator norm and we use the standard identification of a  $k$ -subspace with the projection operator to it. This elementary observation is the key property of Reifenberg flat sets that leads to Theorem 2.1.

**Proof of Theorem 2.1** Fix some  $r < R/10$ . By scaling, we may assume that  $r = 1$ , and that  $R \geq 10$ . Let  $\phi : [0, \infty) \rightarrow \mathbb{R}_+$  be a smooth function such that  $\phi|_{[0, 1]} = 1$  and  $\phi|_{[2, \infty)} = 0$ . Let  $G(n - k, n)$  be the  $(n - k)$ -dimensional Grassmannian and  $E(n - k, n)$  be the total space of the tautological vector bundle over  $G(n - k, n)$ . Let  $p_1, \dots, p_L \in X$

be a maximal collection of points such that the balls  $B(p_i, 1/6)$  are disjoint. In particular  $N(X, 1) := \{y \in \mathbb{R}^n \text{ s.t } d(y, X) < 1\} \subseteq \bigcup B(p_i, 2)$ . For every  $i \in \{1, \dots, L\}$  fix an  $(n-k)$ -dimensional plane  $Q_i = P_{p_i,1}^\perp$  where  $P_{p_i,1}$  is as in the definition of a Reifenberg flat set, and set  $\phi_i(y) = \phi(|y - p_i|/2)$ . Now, for every  $y \in N(X, 1)$  define  $O_y = \frac{\sum_{i=1}^L \phi_i(y) Q_i}{\sum_{i=1}^L \phi_i(y)}$  and note that there exists some universal  $I$ , depending only on  $n$ , such that for every  $y \in N(X, 1)$ , at most  $I$  of the summands in the numerator are non zero. Fixing  $x \in X$  and letting  $Q_x = P_{x,1}^\perp$ , we note that for every  $y \in B(x, 2)$ , all the contributors to  $O_y$  are in  $B(x, 6)$ . Thus, by Remark 2.4,  $\|O_y - Q_x\|_{C^3(B(x,2))} \leq \alpha(\varepsilon)$ , where  $\lim_{\varepsilon \rightarrow 0} \alpha(\varepsilon) = 0$ . As  $O_y$  is symmetric and very close to  $Q_x$ , if we let  $\tilde{Q}_y$  be the orthonormal projection to the span of the  $(n-k)$  eigenvectors of  $O_y$  with eigenvalues close to 1, we also have

$$\|\tilde{Q}_y - Q_x\|_{C^3(B(x,2))} \leq \alpha(\varepsilon). \tag{2.5}$$

Finally, set  $\eta(y) = \frac{\sum_{i=1}^L \phi_i(y) \tilde{Q}_y(y-p_i)}{\sum_{i=1}^L \phi_i(y)}$  and define  $\pi : N(X, 1) \rightarrow E(n-k, n)$  by  $\pi(y) = (\pi_1(y), \pi_2(y)) = (\tilde{Q}_y, \eta(y))$ . We let  $X^1$  be the inverse image of the zero section  $\xi$  in  $E(n-k, n)$ .

Let us verify that  $X^1$  indeed satisfies the desired properties. First observe that if  $\pi_2(y) = 0$  then  $y \in N(X, \delta(\varepsilon))$  where  $\lim_{\varepsilon \rightarrow 0} \delta(\varepsilon) = 0$ . Indeed, let  $x \in X$  be the closest point to  $y$  and observe, as before, that if  $p_i$  provides a non zero contribution to  $O_y$  then  $p_i \in B(x, 6)$  and so by the Reifenberg property at scales 1 and 10,  $|Q_x(p_i)| < \delta(\varepsilon)$  and so  $|Q_x(y)| < \delta(\varepsilon)$  by (2.5). Thus  $d(x + P_{x,1}(y), y) < \delta(\varepsilon)$  and by the Reifenberg property  $d(X, y) < \delta(\varepsilon)$ . Moreover, as  $\pi(N(X, 1))$  is transversal to  $\xi$  [again, by (2.5)],  $X^1$  is a  $k$ -dimensional submanifold. Fixing  $x \in X$  we have that  $|\pi_2(x)| < \delta(\varepsilon)$  and so by (2.5) again, there is  $y \in \mathbb{R}^n$  with  $d(y, x) < \delta(\varepsilon)$  such that  $\pi_2(y) = 0$ . Thus, we have established that  $X^1$  is a submanifold that satisfies condition 1.

Take  $x \in X$  and assume w.l.g. that  $x = 0$ . If  $x' \in P_{0,1} \cap B(0, 1)$  we see that  $\|\pi_2\|_{x'+Q_x} - Id\|_{C^3(B(x',3))} < \delta(\varepsilon)$  and so by the quantitative version of the inverse function theorem, there exists a unique point  $y \in (x' + Q_x) \cap B(0, 2)$  with  $\pi_2(y) = 0$ . Thus  $X^1 \cap \{y \mid |P_{0,1}(y)| < 1\} \cap B(0, 2)$  is a graph of a function  $f$  over  $P_{0,1} \cap B(0, 1)$  with  $\|f(x_1, \dots, x_k)\|_{C^3(B^k(0,1))} = \|Q_x(x_1, \dots, x_k, f(x_1, \dots, x_k))\|_{C^3(B^k(0,1))} < \delta(\varepsilon)$ . This completes the proof.  $\square$

**Corollary 2.6** *For every  $\delta > 0$  there exists  $\varepsilon > 0$  such that if  $X \subseteq \mathbb{R}^n$  is  $k$ -dimensional  $(\varepsilon, R)$  Reifenberg flat set then in addition to 1–3 of Theorem 2.1, for every  $x \in X$  and  $s \in (r, R/10)$ ,  $X^r \cap B(x, s)$  can be expressed as  $X^r \cap B(x, s) = G \cup B$  where  $G$  is connected and  $B \subseteq B(x, s) - B(x, (1 - 5\delta)s)$ .*

**Proof** Just like in [7, Lemmas 4.4 and 4.9] (see also Lemma 2.23), conditions (1)–(3) of Theorem 2.1 imply that for  $\varepsilon > 0$  sufficiently small,  $X^s$  has a tubular neighborhood of radius  $s/4$  and that  $X^{s/4}$  is a graph of a function  $f_s$  over  $X^s$  with  $|f_s(x)| \leq 2\delta s$ . Defining  $f : X^s \rightarrow X^{4-k_s}$  by  $f(y) = f_{4-k+1_s} \circ f_{4-k+2_s} \circ \dots \circ f_s(y)$  we see that for every  $y \in X^s$ ,  $d(f(y), y) < 4\delta s$ . This, together with property (3) for  $X^s$  completes the proof.  $\square$

## 2.2 Estimates for graphical mean curvature flow with small initial data on thick cylinders

In this section, we will generalize the proof of the main estimate for graphical mean curvature flow with small initial data on thick cylinders [7, Theorem 5.1] to the arbitrary co-dimensional setting.

**Theorem 2.7** *There exist some  $c > 0$  such that for every  $\delta > 0$  and  $M > 0$ , there exist positive  $\tau_0 = \tau_0(M, \delta) \ll 1$  and  $\lambda_0 = \lambda_0(M, \delta) \ll 1$  such that for every  $0 < \lambda < \lambda_0$  there exists some  $\varepsilon_0 = \varepsilon_0(\delta, M, \lambda)$  such that for every  $0 < \tau < \tau_0$  and  $\varepsilon < \varepsilon_0$  the following holds: If  $u : B^k(p, r) \times [0, \tau r^2] \rightarrow \mathbb{R}^{n-k}$  is a graph moving by mean curvature such that:*

1. For every  $(x, t) \in B(p, r) \times [0, \tau r^2]$

$$|\nabla u(x, t)| \leq M\varepsilon, \quad |u(x, t)| \leq M^2\beta. \tag{2.8}$$

2. For every  $x \in B(p, r)$  we have

$$|\nabla u(x, \lambda \tau r^2)| \leq \varepsilon, \quad |u(x, \lambda \tau r^2)| \leq \beta. \tag{2.9}$$

Then:

$$|A(p, \tau r^2)| \leq (c + \delta) \frac{\varepsilon}{\sqrt{\tau r}}, \quad |A(p, \tau r^2)| \leq c \frac{\beta}{\tau r^2} + \delta \frac{\varepsilon}{\sqrt{\tau r}}. \tag{2.10}$$

As in the proof of the co-dimension one case, the idea is to regard the graphical mean curvature flow equation as a non-homogeneous heat equation. The controlled growth of the function and its gradient (condition 1 in Theorem 2.7), and the thickness of the cylinder ( $\tau(M) \ll 1$ ) allows one to derive a Schauder type estimate for the non-homogeneous heat equation [7, Theorem 5.12] for which the homogeneous part “does not see the boundary” and depends only on the initial slice (i.e. on  $\varepsilon, \beta$ , but not on  $M$ ). In that regard, the estimate one gets for the homogeneous part are (up to a multiplicative constant) like the ones obtained for physical solutions to the heat equation on the full space. Proving Theorem 2.7 then reduces to showing that the Hölder norm of the non-linearity behaves sub-linearly.

In the co-dimension one case, the major step towards obtaining those estimates was a Hölder gradient estimate for  $u$ , which was proved by tracing the dependences of the constants in the proof of Hölder gradient regularity for parabolic quasilinear equations of general type [16]. This led to showing that some Hölder norm of  $\nabla u$  is at most linear in the  $C^0$  norm of  $\nabla u$  (when the latter is small). Such argument is not valid in the high co-dimensional case (as there is no such general Hölder gradient estimate), but by virtue of a compactness argument, one gets a weaker result, which will nevertheless suffice for our purposes.

**Theorem 2.11** *There exists some  $\tau_0 > 0$  such that for every  $\delta > 0$  there exists  $\varepsilon > 0$  such that for every  $0 < \tau < \tau_0$  the following holds: If  $u : B^k(p, r) \times [0, \tau r^2] \rightarrow \mathbb{R}^{n-k}$  is a solution to the graphical mean curvature flow with  $\|Du\| < \varepsilon$  then, setting  $B^\tau(p, r) = B(p, (1 - 1000\sqrt{\tau_0})r) \times [0, \tau r^2]$ ,*

$$\sup_{z_1, z_2 \in B^\tau(p, r)} d_{z_1, z_2}^\alpha \frac{\|Du(z_1) - Du(z_2)\|}{d(z_1, z_2)^\alpha} < \delta, \quad \sup_{z \in B^\tau(p, r)} d_z \|D^2u(z)\| < \delta. \tag{2.12}$$

Here

$$d((x_1, t_1), (x_2, t_2)) = \sqrt{|x_1 - x_2|^2 + |t_1 - t_2|}, \tag{2.13}$$

$d_{z_1} = d(z_1, \partial(B(p, r) \times [0, \tau r^2]))$  [note that this is not the distance to the boundary of  $B^\tau(p, r)$ ] and  $d_{z_1, z_2} = \min(d_{z_1}, d_{z_2})$ .

**Proof** We argue by contradiction. Assume w.l.g. that  $r = 1$  and  $p = 0$  and note that the first inequality is trivial when  $d(z_1, z_2) \geq d_{z_1, z_2}$  and follows by integration from the second inequality otherwise. Suppose that there exist some  $\delta$  such that for every  $m$  we can find  $\tau^m < \tau_0$  and a solution of the graphical MCF  $u^m : B^k(0, 1) \times [0, \tau^m] \rightarrow \mathbb{R}^{n-k}$  with  $\|Du^m\| < 1/m$  and  $z_m \in B^{\tau^m}(0, 1)$  such that

$$d_{z_m} \|D^2u^m(z_m)\| \geq \delta. \tag{2.14}$$



Setting  $z_m = (x_m, t_m)$ , the closest boundary point to  $z_m$  is  $(x_m, 0)$ . Let  $\lambda_m = \sqrt{\frac{\tau_0}{t_m}} \geq 1$  and define

$$v^m(y, s) = \lambda_m [u^m(x_m + y/\lambda_m, s/\lambda_m^2) - u^m(x_m, 0)]. \tag{2.15}$$

Note that  $v^m(0, 0) = 0$  and set  $\xi_m = (y_m, s_m) = (0, \tau_0)$ . By the definition of  $B^{\tau^m}(0, 1)$ , we conclude that  $v^m$  is defined (at least) on  $B(0, 1000\sqrt{\tau_0}) \times [0, \tau_0]$ , satisfies the graphical mean curvature flow equation, and while  $\|Dv^m\| \leq 1/m$ , for  $\xi_m = (0, \tau_0)$  we have

$$\|D^2v^m(\xi_m)\| \geq \delta/\sqrt{\tau_0}. \tag{2.16}$$

On the other hand, by the estimate from [20], the sequence sub-converges to a solution of the graphical mean curvature flow which is on one hand constant and on the other, has non-vanishing second derivative. □

Now, the graphical mean curvature flow equation has the form

$$\partial_t u - \Delta u = a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} = N(Du, D^2u) \tag{2.17}$$

where

$$a^{ij} = \left[ \left( \delta_{kl} + \left\langle \frac{\partial u}{\partial x_k}, \frac{\partial u}{\partial x_l} \right\rangle \right)^{-1} \right]^{ij} - \delta^{ij}. \tag{2.18}$$

Note that  $a^{ij}$  is a rational function in the gradient of  $u$ , where the numerator  $P(Du)$  has neither free coefficients, nor terms that are linear in  $Du$ . Thus, by Theorem 2.11 (and the estimates from [20]), for every  $\delta$  there exists  $\varepsilon_0 > 0$  such that for every  $\tau < \tau_0$ , for every solution of the graphical MCF on  $B(p, r) \times [0, \tau r^2]$  with  $\|Du\| < \varepsilon$  we have, for every  $x, y \in B^{\tau}(p, r)$

$$d_x |N(x)| \leq \delta\varepsilon, \quad d_{x,y}^{1+\alpha} \frac{|N(x) - N(y)|}{d(x, y)^\alpha} \leq \delta\varepsilon. \tag{2.19}$$

As discussed above, those estimates for the non-linearity, together with [7, Theorem 5.12] imply Theorem 2.7.

### 2.3 Extension and interpolation

In this section we include two very simple results. The first regards the extension of curvature bounds forward in time for arbitrary compact submanifolds, while the second is an interpolation result, at the presence of curvature bounds and Hausdorff bounds.

By [20, Lemma 2.1], the evolution of the second fundamental form in arbitrary co-dimension satisfies the inequality

$$\frac{d}{dt} |A|^2 \leq \Delta |A|^2 + C(k, n) |A|^4. \tag{2.20}$$

Thus, by the maximum principle, and the fact that the curvature must blow up at a singularity, one has the following extension lemma:

**Lemma 2.21** *If  $M$  is a compact  $k$ -dimensional submanifold of  $\mathbb{R}^n$  with  $|A| \leq \alpha$  then there exists a mean curvature flow  $M_t$  starting from  $M$  that exists for (at least)  $0 \leq t \leq \frac{1}{C(k,n)\alpha^2}$ , such that the norm of the second fundamental form satisfies the estimate*

$$|A(t)| \leq \frac{\alpha}{\sqrt{1 - C(k, n)\alpha^2 t}}. \tag{2.22}$$



The following elementary interpolation is central in our argument:

**Lemma 2.23** (Interpolation) *Fix  $\delta > 0, \alpha > 0$ . There exists  $\beta_0 > 0$  such that for every  $\beta < \beta_0$  the following holds: Assume  $p \in X \subseteq B^n(p, r)$  where  $X$  is a  $k$ -submanifold with*

1.  $|A| \leq \frac{\alpha}{r}$ .
2.  $d_H(P \cap B^n(p, r), X \cap B^n(p, r)) \leq \beta r$  for  $P = \text{span}\{e_1, \dots, e_k\}$ .

*Then inside the cylinder  $C_{\delta, \beta} = B^k(p, (1 - \delta)r) \times [-\beta r, \beta r]^{n-k}$ , the connected component of  $p$  is a graph of a function  $u$  over  $P$  and we have the estimate*

$$|\nabla u| \leq \sqrt{3\alpha\beta} \tag{2.24}$$

(and  $|u| \leq \beta r$ ).

**Proof** Assume w.l.g.  $r = 1$  and  $p = 0$  and denote  $Q = P^\perp$ . For  $\beta$  sufficiently small  $C_{\delta/4, \beta} \subseteq B(0, 1)$  and  $\alpha\beta < 1$ . Now, let  $x \in C_{\delta/2, \beta}$  and let  $\gamma(t)$  be a unit speed geodesic with  $\gamma(0) = p$ . We may assume w.l.g., by possibly changing the parametrization according to  $t \mapsto -t$ , that  $\langle \gamma'(0), e_n \rangle = \max_{v \in Q, \|v\|=1} \langle \gamma'(0), v \rangle$  and that  $x_n(\gamma(t)) \geq 0$ . Letting  $f(t) = x_n(\gamma(t))$  we find  $f'(t) = \langle \gamma'(t), e_n \rangle$  and  $f''(t) = \langle \gamma''(t), e_n \rangle = \langle \gamma''(t), e_n - \langle \gamma'(t), e_n \rangle \gamma'(t) \rangle \geq -\alpha \sqrt{1 - f'(t)^2}$ . The equality case of the above ODE for  $f'(t)$  corresponds to a circle of radius  $\frac{1}{\alpha}$ . Letting  $\mu(t) : \mathbb{R} \rightarrow \mathbb{R}^2$  be a clockwise and unit speed parametrized circle of radius  $\frac{1}{\alpha}$  with  $\mu(0) = (0, 0)$  and  $\langle \mu'(0), e_2 \rangle = f'(0)$ , we see that as long as  $x_2(\mu(t))$  is increasing, and as long as  $\gamma(t) \in C_{\delta/2, \beta}$ ,  $x_n(\gamma(t)) \geq x_2(\mu(t))$ . For  $\beta$  sufficiently small (depending on  $\alpha, \delta$ )  $x_2(\mu(t))$  will reach its maximum at time  $0 < T < \delta/4$  so the extra condition  $\gamma(t) \in C_{\delta/2, \beta}$  is redundant. Thus  $x_2(\mu(t)) \leq x_n(\gamma(t)) \leq \beta$ , and an easy calculation for circles in the plane gives the bound

$$\tan \angle(T_x X, P) \leq \frac{\sqrt{2\beta\alpha - \alpha^2\beta^2}}{1 - \alpha\beta} \leq \sqrt{3\alpha\beta} \tag{2.25}$$

for  $\beta$  sufficiently small.

What remains to be shown is that the connected component of  $p$  is indeed a graph. Assume there exist  $x_1, x_2 \in X \cap C_{\delta, \beta}$  with  $x_1 \neq x_2$  but  $P(x_1) = P(x_2)$ . Observe that by (2.25),  $X \cap \overline{C_{\delta, \beta}}$  is a submanifold with boundary. Let  $\gamma : [0, a] \rightarrow X \cap \overline{C_{\delta, \beta}}$  be a minimizing geodesic between  $x_1$  and  $x_2$ . Such a geodesic is always  $C^1$  and is smooth for as long  $\gamma(t)$  is away from the boundary. For such  $t$  however  $\|P(\gamma''(t))\| \leq \sqrt{3\alpha\beta\alpha}$  by (2.25) and so for  $\beta$  sufficiently small, and as  $\gamma'(0)$  is almost parallel to  $P$ ,  $P(\gamma(t))$  is almost a straight line until it hits the boundary (at some  $t < 4$ ). Since  $\gamma(t)$  is  $C^1$ , and intersects the boundary with an exterior normal component, this is a contradiction.

To see that for every  $y \in B^n(0, 1 - \delta)$  there is some  $x \in X$  with  $P(x) = y$ , note that by the Hausdorff condition, we can find  $\bar{x} \in X \cap B(0, (1 - \delta/2))$  with  $d(\bar{x}, y) \leq \beta$  (when  $\beta$  is small). Taking  $\bar{y} = P(\bar{x})$  we see, again, by (2.25) for  $\bar{x}$ , and the fact that the curvature scale  $\frac{1}{\alpha}$  is far bigger than  $\beta$ , that there will exist a point over  $y$  as well.  $\square$

### 2.4 Construction of a flow

For  $\varepsilon$  sufficiently small, if  $X$  is  $k$ -dimensional  $(\varepsilon, R)$  Reifenberg flat, Theorem 2.1 and Corollary 2.6 provide us with smooth approximations to  $X$  at different scales. The extension lemma, Lemma 2.21, the interpolation lemma, Lemma 2.23, and the a-priori estimate, Theorem 2.7 can substitute the corresponding result of [7] in the iteration scheme of [7, Sects. 3.2, 3.3]. Thus, just like there we obtain the following uniform estimates.

**Theorem 2.26** (Uniform estimates) *There exist some  $\varepsilon$  and  $c_0, c_1, c_2, c_3$  such that if  $X$  is  $k$ -dimensional ( $\varepsilon, R$ ) Reifenberg flat, and considering the approximating surfaces  $X^r$  from Theorem 2.1 and Corollary 2.6, each  $X^r$  flows smoothly by  $k$ -dimensional mean curvature for time  $t \in [0, c_0 R^2]$  and for every  $t \in [c_3 r^2, c_0 R^2]$  we have: (1)  $|A^r(t)| \leq \frac{c_1}{\sqrt{t}}$  where  $A^r(t)$  the second fundamental form of  $X_t^r$ . (2)  $d_H(X_t^r, X) \leq c_2 \sqrt{t}$ . (3) For every  $x \in X$  and  $s \in (\frac{\sqrt{t}}{c_1}, R/4)$  we have*

$$B(x, s) \cap X_t^r = G \cup B \tag{2.27}$$

where  $G$  is connected and  $B \cap B(x, \frac{9}{10}s) = \emptyset$ . Moreover, the constants  $c_1, c_2$  satisfy  $c_1^2 \leq \frac{1}{8}$  and  $\frac{1}{4c_1} - c_2 > \sqrt{2k}$ .

By the uniform estimates one can pass to a sub-limit flow  $X_t$  as in Theorem 1.5. Note that the condition (iii) of Theorem 1.5 follows easily from 1 to 3 of Theorem 2.26 (see [7, Lem 4.4]).

### 3 Uniqueness

In this section, we will prove Theorem 1.10 which, together with the existence part of Theorem 1.5, imply the full Theorem 1.5. In Sect. 3.1 we will recall the definition and some properties of the high co-dimensional level set flow from [1]. In Sect. 3.2 we will recall and further explore the behavior of the associated level set operator on distance functions from smooth evolutions by mean curvature. Section 3.3 will be devoted to the proof of Theorem 1.10.

#### 3.1 Preliminaries

Let us start by introducing some notations. For every  $0 \neq p \in \mathbb{R}^n$  define  $P_p$  to be the projection to the orthogonal complement of  $p$ . Given an  $n \times n$  symmetric matrix  $A$  and such  $p$ , let  $X = P_p A P_p$ . If we denote the eigenvalues of  $X$  corresponding to eigenvectors orthogonal to  $p$  by  $\lambda_1(X) \leq \dots \leq \lambda_{n-1}(X)$ , define

$$F(p, A) = \sum_{i=1}^k \lambda_i(X). \tag{3.1}$$

In [1], the level sets of positive viscosity solution to the equation

$$\frac{d}{dt}u = F(\nabla u, \nabla^2 u) \tag{3.2}$$

were used to give a definition for weak mean curvature flow. Before diving into formalities, we will try to convince the reader that this approach is plausible. Let  $(M_t)_{t \in [0, T]}$  be a smooth family of  $k$ -dimensional submanifolds of  $\mathbb{R}^n$ . Let  $u : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}_+$  be a smooth function such that for every  $t \in [0, T]$ ,  $M_t = \{x \in \mathbb{R}^n \text{ s.t. } u(x, t) = 0\}$  and such that  $\nabla u \neq 0$  on a neighborhood of  $M_t$  (away from  $M_t$ ), and  $u \geq \delta_0 > 0$  outside that neighborhood. For  $\varepsilon$  sufficiently small,  $M_t^\varepsilon = \{x \in \mathbb{R}^n \text{ s.t. } u(x, t) = \varepsilon\}$  would be a smooth ‘‘tubular’’ hypersurface around  $M_t$ . We would expect it to have  $(n - k - 1)$  principal curvatures that are very large, corresponding to ellipsoids in the orthogonal complement of  $T_p M_t$  for  $p \in M_t$ . The other  $k$  principal curvatures should be very close to the ones of  $M_t$  w.r.t. the normal of  $M_t^\varepsilon$ , as for every geodesics  $\gamma(s)$  of  $M_t$  and every point on  $x \in M_t^\varepsilon$  which is closest to  $\gamma(0)$ , there should be an almost geodesic curve in  $M_t^\varepsilon$  which is orthogonal to the above mentioned

ellipsoids which “traces  $\gamma$ ”. The second fundamental form of  $M_t^\varepsilon$  at  $x \in M_t^\varepsilon$  is given by  $A(x, t) = \frac{1}{\|\nabla u\|} P_{\nabla u(x,t)} \nabla^2 u(x, t) P_{\nabla u(x,t)}$ , so we expect  $\frac{1}{\|\nabla u\|} F(\nabla u, \nabla^2 u)$  to be very close to  $-\vec{H}(\pi(x), t) \cdot \frac{\nabla u}{\|\nabla u\|}$ , where  $\vec{H}(\pi(x), t)$  is the mean curvature of  $M_t$  at the point closest to  $x$ . On the other hand, the normal velocity of  $M_t^\varepsilon$  at  $x$  is given by  $-u_t/\|\nabla u\|$  and so equation (3.2) tells us that, parameterizing the flow of  $M_t^\varepsilon$  by  $\phi^\varepsilon(x, t)$  and letting  $\nu(x, t)$  be the normal to  $M_t^\varepsilon$  at  $x$ , we have

$$\frac{d}{dt} \phi^\varepsilon(x, t) \cdot \nu = \vec{H}(\pi(x), t) \cdot \nu + O(\varepsilon). \tag{3.3}$$

Thus, the entire ellipsoid around  $\pi(x)$  moves approximately by  $\vec{H}(\pi(x))$ , which would correspond to a motion of  $M_t$  by  $k$ -dimensional mean curvature flow.

We proceed by defining the level set flow and by collecting some of its properties from [1]. Let  $u_0$  be a non-negative, uniformly continuous function. Theorem 2.4 in [1] states that there exists a unique uniformly continuous, positive viscosity solution to Eq. (3.2)  $u : \mathbb{R}^n \times [0, \infty)$  such that  $u(-, 0) = u_0$  (for the definition of viscosity solution, see [1, Definition 2.1]).

**Definition 3.4** [1, Definition 2.6] Let  $X \subseteq \mathbb{R}^n$  be a closed set. Let  $u_0$  be a non-negative, uniformly continuous function such that  $X = \{x \in \mathbb{R}^n \text{ s.t. } u_0(x) = 0\}$ . Letting  $u$  be the solution of the IVP of equation (3.2) with  $u(x, 0) = u_0(x)$ , the  $k$ -dimensional **level set flow** of  $X$  is defined to be  $X_t = \{x \in \mathbb{R}^n \text{ s.t. } u(x, t) = 0\}$ .

A-priori, this definition may depend on the choice of  $u_0$ , but Theorem 2.5 of [1] shows that different choices yield the same result.

The following three properties of the level set flow and the level set equation would be of importance to us.

**Property A** If  $X$  is a compact smooth  $k$ -dimensional submanifolds of  $\mathbb{R}^n$  and the smooth mean curvature flow  $(X_t)_{t \in [0, T]}$  is defined for all  $t \in [0, T]$  then  $X_t$  is also the level set flow of  $X$  (see [1, Corollary 3.9]).

**Property B** If  $u, v : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}_+$  are two non-negative, uniformly continuous solutions to (3.2) then  $\|u - v\|_{L^\infty(\mathbb{R}^n \times [0, T])} \leq \|u - v\|_{L^\infty(\mathbb{R}^n \times \{0\})}$  (see [1, Theorem 2.2]).

**Property C** Let  $\Omega \subseteq \mathbb{R}^n \times [0, T]$  be a bounded domain, and let  $v, u$  be non-negative, uniformly continuous functions on  $\mathbb{R}^n \times [0, T]$  such that  $v$  is a sub-solution to (3.2) in  $\Omega$  and  $u$  is a solution of (3.2) in  $\Omega$ . If  $v \leq u$  on  $\partial_{\text{par}}\Omega$  then  $v \leq u$  on  $\Omega$ . Here  $\partial_{\text{par}}\Omega$  is the parabolic boundary of  $\Omega$  (see remark below).

**Remark 3.5** In [1, Theorem 2.2] it was shown that  $F$  satisfies the assumptions of [11, Theorem 2.1], and of Theorem 4.1 of [3], both of which imply Property C for domains of the form  $\Omega = D \times [0, T]$  where  $D \subseteq \mathbb{R}^n$  is a compact domain. The proof of Theorem 4.1 in [3] works just as well for an arbitrary bounded domain  $\Omega$ , while compactness also gives Proposition 2.3 of [11], from which point the proof of Theorem 2.1 of [11] works for arbitrary such  $\Omega$  as well. This is, of course, not surprising at all, as the weak maximum principle is a statement about the interior.

### 3.2 Evolution of distances

Let  $(M_t)_{t \in [0, T]}$  be a family of compact  $k$ -submanifolds evolving by smooth mean curvature flow. For each  $0 \leq t \leq T$ ,  $M_t$  has a tubular neighborhood at which the distance function from  $M_t$ ,  $r_t$  is smooth (away from  $M_t$ ). Studying the properties of  $F(\nabla r_t, \nabla^2 r_t)$  and of

$\frac{d}{dt}r_t - F(\nabla r_t, \nabla^2 r_t)$  will occupy the first part of this section. In the second part, we will give a simple lower bound, which is based on avoidance of balls, for the solution of the level set equation starting from a distance function from an arbitrary set.

**Lemma 3.6** *Let  $M$  be a compact smooth,  $k$ -dimensional submanifold in  $\mathbb{R}^n$  with normal injectivity radius  $\rho$  and let  $r$  be the distance function from  $M$ . For every  $0 < s \leq \rho$ , let  $M^s$  be the smooth co-dimension 1 level set  $M^s = \{y \in \mathbb{R}^n \text{ s.t. } r(y) = s\}$  and let  $A^s$  be the second fundamental form of  $M^s$  with respect to the interior normal. Let  $x \in M$ ,  $v \in SN_x M$  (the unit normal space to  $M$  at  $x$ ),  $P = T_x M$  and  $Q = (P \oplus v)^\perp$ . The following hold:*

1. *for every  $0 < s < \rho$  the principal directions of  $M^s$  at  $x + sv$  are independent of  $s$  and split according to  $P$  and  $Q$ .*
2.  *$A^s|_Q = \frac{1}{s}Id$  while  $A^s|_P$  is bounded inside the tubular neighborhood, and if  $v_1, \dots, v_k \in P$  are the principal directions of  $A^{-v}(\cdot) = \langle A(\cdot), -v \rangle$  with eigenvalues  $\beta_1, \dots, \beta_k$  then*

$$A^s|_P(v_i, v_i) = \frac{\beta_i}{1 + s\beta_i}, \tag{3.7}$$

*and  $A^s|_P(v, v) < \frac{1}{s}$  for every  $v \in P$  with  $\|v\| = 1$ .*

**Remark 3.8** Most of the above lemma is stated and proved in [1, Theorem 3.2]. Equation (3.7) was proved for some constants  $\beta_i$ , without identifying them as the principal directions of  $-A(\cdot) \cdot v$ . This fact was not needed for the applications therein, and was stated there as a conjecture (which the authors did not care much about).

**Proof of Lemma 3.6**  $(x + P^\perp) \cap M^s$  consist of a sphere of radius  $s$  in  $x + P^\perp$ , whose normal at  $x + sv$  is  $-v$ , so  $A^s|_Q = \frac{1}{s}$  as stated. Fix  $v = \sum a_i v_i \in P$ , and let  $\gamma(t)$  be a unit speed geodesic in  $M$  with  $\gamma(0) = x$  and  $\dot{\gamma}(0) = v$ , and let  $\nu(t)$  be a normal field along  $\gamma(t)$ , which solves the linear system of ODEs  $\nu(0) = v$  and  $N_{\gamma(t)}M(\dot{\nu}(t)) = 0$  in the normal bundle to  $M$ . In particular  $\|\nu(t)\| = \|\nu(0)\| = 1$ , so  $\mu(t) := \gamma(t) + s\nu(t)$  is a curve in  $M^s$  such that

$$\|\dot{\mu}(0)\|^2 = \|v + sA^{-v}(v, v_i)v_i\|^2 = \sum (a_i(1 + s\beta_i))^2 \tag{3.9}$$

and

$$A^s(\dot{\mu}(0), \dot{\mu}(0)) = \langle \dot{\mu}(0), -v \rangle = A^{-v}(v, v) + s\|\dot{\nu}(0)\|^2 = \sum a_i^2 \beta_i + s \sum (a_i \beta_i)^2. \tag{3.10}$$

As  $s < \rho < \frac{1}{|\beta_i|}$ ,  $1 + s\beta_i > 0$  and so

$$\frac{\sum a_i^2 \beta_i + s \sum (a_i \beta_i)^2}{\sum (a_i(1 + s\beta_i))^2} < \frac{1}{s} \tag{3.11}$$

holds. Additionally,

$$A^s(v_i, v_i) = \frac{\beta_i + s\beta_i^2}{(1 + s\beta_i)^2} = \frac{\beta_i}{1 + s\beta_i}. \tag{3.12}$$

□

**Lemma 3.13** *Let  $N_t$  be a tubular neighborhood of a compact smooth  $k$ -dimensional mean curvature flow  $M_t$ ,  $r(x, t)$  be the distance function from  $M_t$  and  $v_1(x, t), \dots, v_k(x, t)$  be the*

principal directions of  $M_t$  at  $\pi(x, t) \in M_t$  w.r.t. the normal  $-\nabla r(x)$  (where  $\pi(x, t)$  is the nearest point projection to  $M_t$ ). Then

$$\frac{d}{dt}r - F(\nabla r, \nabla^2 r) = \left( \sum_{i=1}^k \frac{\langle A(v_i, v_i), -\nabla r \rangle^2}{1 + r \langle A(v_i, v_i), -\nabla r \rangle} \right) r \tag{3.14}$$

on  $N_t - M_t$ .

**Proof** Fix  $t, x \in N_t - M_t$  and let  $p = \pi(x, t)$ . By the definition of  $F$  and the fact that  $\|\nabla r\| = 1$ ,  $F(\nabla r(x, t), \nabla^2 r(x, t))$  is the sum of the  $k$  smallest principal curvatures of  $M_t'$  at  $x$ . By Lemma 3.6, those principal curvatures correspond to vectors in  $T_p M_t$ , so since the trace of a matrix is independent of the basis we get, again by Lemma 3.6,

$$F(\nabla r, \nabla^2 r) = \sum_{i=1}^k \frac{\langle A(v_i, v_i), -\nabla r \rangle}{1 + r \langle A(v_i, v_i), -\nabla r \rangle}. \tag{3.15}$$

On the other hand, since  $M_t$  moves by mean curvature, by the first variation of length we get

$$\frac{d}{dt}r(x, t) = \left\langle \bar{H}(p, t), -\nabla r \right\rangle = \sum_{i=1}^k \langle A(v_i, v_i), -\nabla r \rangle. \tag{3.16}$$

The result follows. □

The following lower bound on the solution of the level set flow starting from a distance function from a set will be used in the proof of Theorem 1.10. As discussed above, it is based on the fact that although arbitrary co-dimensional mean curvature flow does not satisfy avoidance, it does satisfy it w.r.t. (co-dimension one) balls, moving according to the sum of their lowest  $k$  principal curvatures (which for balls are, of course, the same). For Brakke flows, this fact was already observed in Brakke’s original manuscript [2]. In a sense, the following lemma shows it for the Ambrosio–Soner level set flow.

**Lemma 3.17** *Let  $X$  be a closed set and let  $p$  be a point with  $d(X, p) = R$ . Let  $g$  be the distance function from  $X$  and consider the level set flow  $u$  starting from  $g$ . If  $R^2 > 2kt$  then*

$$u(p, t) \geq R - \sqrt{2kt}. \tag{3.18}$$

**Proof** Let  $\tilde{g}(y)$  be a function which equals  $\min\{d(X, y), R - \sqrt{2kt}\}$  on  $\mathbb{R}^n - B(p, \sqrt{2kt})$  and  $R - d(p, y)$  on  $B(p, \sqrt{2kt})$ . Letting  $\tilde{u}$  be the solutions to the level set equation corresponding to  $\tilde{g}$ , for every  $0 \leq c < R - \sqrt{2kt}$ ,  $\{x \mid u(x, t) = c\} = \{x \mid \tilde{u}(x, t) = c\}$  while by continuity and by the known evolution of spheres  $\tilde{u}(p, t) = R - \sqrt{2kt}$ . Thus  $u(p, t) \geq R - \sqrt{2kt}$ . □

### 3.3 Conclusion

**Proof of Theorem 1.10** Set  $X_0 = X$ , consider the functions  $r(x, t) = \text{dist}(x, X_t)$  and  $v(x, t) = r^2(x, t)/\sqrt{t}$  and let  $N = \{(x, t) \in \mathbb{R}^n \times [0, T] \mid r(x, t) < \frac{\sqrt{t}}{4c_1}\}$  and  $N_t = N \cap (\mathbb{R}^n \times \{t\})$ .  $v$  is smooth in  $N$  and by Lemma 3.13

$$\frac{d}{dt}v - F(\nabla v, \nabla^2 v) = 2 \left( \sum_{i=1}^k \frac{\langle A(v_i, v_i), -\nabla r \rangle^2}{1 + r \langle A(v_i, v_i), -\nabla r \rangle} - \frac{1}{4t} \right) v \tag{3.19}$$

where  $v_i$  are the principal directions of  $X_t$  w.r.t. the normal  $-\nabla r(x)$ . Furthermore by the assumptions of the theorem, for every  $(x, t) \in N$  we have

$$|r \langle A(v_i, v_i), -\nabla r \rangle| \leq \frac{c_1}{\sqrt{t}} \cdot \frac{\sqrt{t}}{4c_1} < \frac{1}{2}, \quad \sum_{i=1}^k \langle A(v_i, v_i), -\nabla r \rangle^2 \leq \frac{c_1^2}{t}, \tag{3.20}$$

and since  $c_1^2 \leq 1/8$  we find that  $v$  is a sub-solution to equation (3.2) in  $N$ .

Let  $d(x)$  be the distance function from  $X = X_0$  and let  $u$  be the solution to (3.2) with the initial data  $d$ . For  $(x, t) \in \partial_{\text{par}}N$  we have

$$d(x) \geq r(x, t) - d_H(X, X_t) \geq \left( \frac{1}{4c_1} - c_2 \right) \sqrt{t} > \sqrt{2k}t \tag{3.21}$$

so by Lemma 3.17

$$u(x, t) \geq \left( \frac{1}{4c_1} - c_2 - \sqrt{2k} \right) \sqrt{t} \geq \alpha v(x, t) \tag{3.22}$$

for  $\alpha = 16c_1^2 \left( \frac{1}{4c_1} - c_2 - \sqrt{2k} \right) > 0$ . Thus, by Property C,  $u \geq \alpha v$  on  $N$ . The first part of the above argument also shows that  $u > 0$  outside of  $N$ . In particular, as the level set flow of  $X, \tilde{X}_t$ , is defined to be the zero set of  $u$ , and as  $u > 0$  on  $\mathbb{R}^n - N_t$  and  $u \geq \alpha v > 0$  on  $N_t - X_t$  we see that  $\tilde{X}_t \subseteq X_t$ .

The inclusion  $X_t \subseteq \tilde{X}_t$  is simpler (and in the co-dimension one case, follows immediately from Ilmanen’s definition). Suppose  $u(x_0, t_0) = \delta > 0$  for some  $0 \leq t_0 \leq T$  and  $x_0 \in X_{t_0}$ . For every  $s < t_0$ , the distance function from  $X_s, d^s(x) = r(x, s)$  satisfies  $\|d^s - d\|_{L^\infty(\mathbb{R}^n)} \leq c_2\sqrt{s}$ . Denote by  $u^s$  the solution for the level set equation emanating from  $d^s$ . Then

$$0 < \delta = u(x_0, t_0) \leq |u(x_0, t_0) - u(x_0, t_0 - s)| + |u(x_0, t_0 - s) - u^s(x_0, t_0 - s)| \tag{3.23}$$

where we have used the fact that  $u^s(x_0, t_0 - s) = 0$  by Property A. The first term of the right hand side goes to zero as  $s \rightarrow 0$  since  $u$  is continuous, while the second term also goes to zero since  $\|u^s - u\|_{L^\infty(\mathbb{R}^n \times [0, T])} \leq \|d^s - d\|_{L^\infty(\mathbb{R}^n)}$  by Property B. This is a contradiction, so if  $x_0 \in X_{t_0}$  we must have  $u(x_0, t_0) = 0$ , i.e.  $x_0 \in \tilde{X}_{t_0}$ .  $\square$

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