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Calculus of Variations

Infinitely many solutions for a nonlinear Schrödinger equation with general nonlinearity

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Abstract

We prove the existence of infinitely many solutions for

$$-\Delta u + V(x)u = f(u) \quad \text{in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N),$$

where $V(x)$ satisfies $\lim_{|x|\to\infty} V(x) = V_\infty > 0$ and some conditions. We require conditions on $f(u)$ only around 0 and at ∞.

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1 Introduction

In this paper, we consider the following nonlinear Schrödinger equation:

$$-\Delta u + V(x)u = f(u) \quad \text{in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N). \tag{1.1}$$

Here $N \geq 3$ and we assume that the potential function $V(x)$ satisfies the following:

(V.1) $V \in C^1(\mathbb{R}^N, (0, \infty))$.

(V.2) $\lim_{|x|\to\infty} V(x) = V_\infty > 0$.

(V.3) There exists $\eta \in L^2(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ such that

$$\big(x \cdot \nabla V(x)\big) \leq \eta(x)^2 \quad \text{for all } x \in \mathbb{R}^N.$$

(V.4) There exists $\rho \in (0, 1)$ such that, for any $\alpha > 0$,

$$\lim_{|x|\to\infty} \inf_{y \in B(x,\rho|x|)} \big(x \cdot \nabla V(y)\big)\, e^{\alpha|x|} = \infty.$$

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Here $B(x,r)=\left\{\, y\in\mathbb{R}^N \,\middle|\, |y-x|<r \,\right\}$ for $x\in\mathbb{R}^N$, $r>0$.

We assume that the nonlinearity $f(u)$ satisfies the following:

(f.1) $f\in C^1(\mathbb{R},\mathbb{R})$ and $f(-s)=-f(s)$.
(f.2) $f'(0)=0$.
(f.3) There exists $p\in(1,(N+2)/(N-2))$ such that $\lim_{s\to\infty} f'(s)/s^{p-1}=0$.
(f.4) $\lim_{s\to\infty} f(s)/s=\infty$.

Under the assumptions (V.1)–(V.2), (f.1)–(f.3), the solutions of (1.1) are the critical points of the functional $I\in C^2(H^1(\mathbb{R}^N),\mathbb{R})$ defined by

$$I(u)=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2+V(x)u^2\,dx-\int_{\mathbb{R}^N}F(u)\,dx,$$

where $F(u)=\int_0^u f(s)\,ds$.

Many researchers have studied (1.1) under the assumption (V.2) for $f(u)=|u|^{p-1}u$ or more general $f(u)$ (cf. [1–6,11,13,14,16,17], and their references). When we look for critical points of $I(u)$ by variational approach, we generally need the compactness of Palais–Smale sequences (we denote (PS)-sequences in short) for $I(u)$. If $V(x)\le V_\infty$ ($\not\equiv V_\infty$), then, by the concentration compactness arguments, we obtain the compactness of (PS)-sequences at a mountain pass level (cf. [13,14,17]). However, for (PS)-sequences at higher energy levels, it is not easy to get the compactness. To get the compactness, in the case $f(u)=|u|^{p-1}u$, Cerami–Devillanova–Solimini [3] introduced assumptions such as (V.4) (also see Remark 1.2) and balanced sequences which are sequences of solutions of the following equation on a ball:

$$-\Delta u+V(x)u=f(u)\quad \text{in } B(0,n), \qquad u\in H_0^1\left(B(0,n)\right). \tag{1.2}$$

The balanced sequences are not (PS)-sequences for $I(u)$ but play a similar role to the (PS)-sequences. In fact, the balanced sequences also satisfy concentration-compactness type properties. Moreover, under the assumption (V.4), every balanced sequence is relatively compact (cf. [3, Proposition 2.1]). Consequently, they succeeded to obtain the existence of infinitely many solutions of (1.1).

On the other hand, in the procedure for getting the compactness of (PS)-sequence, in almost all cases, we need the $H^1(\mathbb{R}^N)$-boundedness of (PS)-sequence. For general $f(u)$ as (f.1)–(f.4), it is a problem how to get the bounded (PS)-sequence. As an assumption guaranteeing the boundedness of any (PS)-sequences, the following Ambrosetti-Rabinowitz condition (AR) is well-known.

(AR) There exists $\mu>2$ such that $\mu F(u)\le f(u)u$ for all $u\in\mathbb{R}$.

There are many researches to obtain the bounded (PS)-sequence under weaker conditions than (AR) (cf. [9–12,16,19], and their references). In our knowledge, one of the weakest assumptions is (V.3) and (f.4) which were introduced by Jeanjean–Tanaka [11]. By the monotonicity trick, they obtained a (PS)-sequence which is a sequence of solutions of

$$-\Delta u+V(x)u=\lambda_n f(u)\quad \text{in } \mathbb{R}^N, \qquad u\in H^1(\mathbb{R}^N), \tag{1.3}$$

where $\lambda_n\to 1-0$ ($n\to\infty$). To get the $H^1(\mathbb{R}^N)$-boundedness of this (PS)-sequence, they used (V.3), (f.4), and the Pohozaev identity for the solutions of (1.3). Consequently, for $N\ge 2$, they obtained a positive solution of (1.1) under the conditions (V.1)–(V.3), $V(x)\le V_\infty$ ($\not\equiv V_\infty$), and

(f.1′) $f\in C([0,\infty),\mathbb{R})$.

(f.2′) $\lim_{s\to+0} f(s)/s = 0$.
(f.3′) There exists $p \in (1, (N+2)/(N-2))$ if $N \geq 3$, $p \in (1, \infty)$ if $N = 2$ such that $\lim_{s\to\infty} f(s)/s^p = 0$.
(f.4) $\lim_{s\to\infty} f(s)/s = \infty$.

The main result of this paper is the following theorem which is considered as a development from [3] and [11].

Theorem 1.1 *Assume* $N \geq 3$, (V.1)–(V.4) *and* (f.1)–(f.4). *Then* (1.1) *have infinitely many solutions.*

Remark 1.2 If $V(x)$ satisfies the following (i)–(ii), $V(x)$ satisfies (V.3)–(V.4).

(i) There exist $c_0, c_1, r_1 > 0$ and $\ell_0 \geq \ell_1 > N$ such that

$$c_0|x|^{-\ell_0} \leq \left(x \cdot \nabla V(x)\right) \leq c_1|x|^{-\ell_1} \quad \text{for all } |x| \geq r_1.$$

(ii) There exist $c_2, r_2 > 0$ such that

$$\left|\left(\frac{\xi}{|\xi|} \cdot \nabla V(x)\right)\right| \leq c_2\left(\frac{x}{|x|} \cdot \nabla V(x)\right) \text{ for all } |x| \geq r_2 \text{ and } \xi \text{ with } (\xi \cdot x) = 0.$$

The radial functions V always satisfy (ii). Cerami–Devillanova–Solimini [3] assumed (ii) and

$$\lim_{|x|\to\infty} \left(x \cdot \nabla V(x)\right) e^{\alpha|x|} = \infty \quad \text{for any } \alpha > 0 \tag{1.4}$$

instead of (V.4). (V.4) follows from (ii) and (1.4). In fact, for $y \in B(x, \rho|x|)$ ($\rho \in (0,1)$), we set $\xi = x - \frac{(y\cdot x)}{|y|^2}y$. Then $x = \frac{(y\cdot x)}{|y|^2}y + \xi \in (\text{span}\{y\}) \oplus (\text{span}\{y\})^\perp$. Moreover, since $|y - x| \leq \rho|x|$ and $(1-\rho)|x| \leq |y| \leq (1+\rho)|x|$, we have

$$\frac{1}{1+\rho}|y|^2 \leq (y \cdot x) \leq \frac{1}{1-\rho}|y|^2, \qquad |\xi| \leq \frac{\rho}{1-\rho}|y|.$$

From (ii), for $y \in B(x, \rho|x|)$ and $|y| \geq r_2$,

$$\begin{aligned}\left(x \cdot \nabla V(y)\right) &= \frac{(y\cdot x)}{|y|^2}\left(y \cdot \nabla V(y)\right) + |\xi|\left(\frac{\xi}{|\xi|} \cdot \nabla V(y)\right)\\ &\geq \frac{(y\cdot x)}{|y|^2}\left(y \cdot \nabla V(y)\right) - c_2|\xi|\left(\frac{y}{|y|} \cdot \nabla V(y)\right)\\ &\geq \left(\frac{1}{1+\rho} - c_2\frac{\rho}{1-\rho}\right)\left(y \cdot \nabla V(y)\right).\end{aligned}$$

Thus, choosing $\rho \in (0,1)$ such that $\frac{1}{1+\rho} - c_2\frac{\rho}{1-\rho} > 0$, from (1.4), we obtain (V.4).

Here we emphasize that we can not prove Theorem 1.1 by only combining the methods of [3] and [11]. In fact, if we use a balanced sequence which is a sequence of solutions of (1.2), then we don't know the boundedness of that sequence. On the other hand, we can not obtain infinitely many solutions of (1.3) because of the compactness problem. Therefore the sequences of solutions of (1.2) or (1.3) are not proper to show Theorem 1.1. From those reasons, we need introduce another sequence which satisfies both boundedness and compactness. Just to state only conclusions, this sequence is obtained as a sequence of solutions of

$$-\Delta u + V(x)u = \lambda_n f(u) + (\lambda_n - 1)\, g(u) \quad \text{in } B(0,n), \qquad u \in H_0^1\left(B(0,n)\right), \tag{1.5}$$

where $\lambda_n \to 1+0$ $(n \to \infty)$. Here $g(u)$ is an auxiliary function which is defined in Sect. 2. We obtain a solution of (1.5) as a critical point of a functional which is modified in the quadratic term of $I|_{H_0^1(B(0,n))}(u)$. Thanks to this modification, we can guarantee both boundedness and compactness of the sequence of solutions. This modification is an important idea in this paper.

We also obtain the following two results as by-products of Theorem 1.1.

Theorem 1.3 *Assume* $N \geq 3$, (V.1)–(V.4), (f.1′)–(f.3′), *and* (f.4). *Then* (1.1) *has a positive solution.*

Next, we assume that $\Omega \subset \mathbb{R}^N$ satisfies the following condition.

(Ω) $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $0 \in \Omega$, and $(x \cdot \nu(x)) > 0$ for all $x \in \partial\Omega$, where $\nu(x)$ is the outward unit normal vector at $x \in \partial\Omega$.

We set $\Omega_R = \left\{ x \in \mathbb{R}^N \middle| R^{-1}x \in \Omega \right\}$ for $R \geq 1$.

Theorem 1.4 *Assume* $N \geq 3$, (V.1)–(V.4), (f.1)–(f.4), *and* (Ω). *For any* $k \in \mathbb{N}$, *there exists* $R_0 = R_0(k) > 0$ *such that if* $R > R_0$, *then*

$$-\Delta u + V(x)u = f(u) \quad \text{in } \Omega_R, \qquad u \in H_0^1(\Omega_R) \tag{1.6}$$

has at least k distinct pairs of nontrivial solutions $\pm u_j$ $(j = 1, \ldots, k)$.

This paper consists as follows: In Sect. 2, we modify the functional $I(u)$ and define balanced sequences as sequences of critical points of modified functional. We also present propositions which bring the boundedness and compactness to balanced sequences. Those propositions are proved in Sects. 4–6. In Sect. 3, we prove Theorems 1.1 and 1.3. In Sect. 4, we prove a proposition about the boundedness. Through Sects. 5 and 6 , we prove propositions about the compactness. In Sect. 6, we also prove Theorem 1.4.

2 Preliminaries

In this section, through several subsections, we give balanced sequences which satisfy the boundedness and compactness. In Sect. 2.1, we define notations and a modified functional. In Sect. 2.2, we show properties of the modified functional. In Sect. 2.3, we state propositions about $H^1(\mathbb{R}^N)$-boundedness. In Sect. 2.4, we construct balanced sequences as sequences of critical points of the modified functional. We also state about the compactness for the balanced sequences.

2.1 Notation and modified functional

We use the following notations:

$$\langle u, v \rangle = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v + V(x)uv\,dx, \quad \|u\| = \langle u, u \rangle^{1/2},$$

$$\|u\|_r = \left(\int_{\mathbb{R}^N} |u|^r\,dx \right)^{1/r} \text{ for } r \in [1, \infty), \quad \|u\|_\infty = \operatorname*{ess\,sup}_{x \in \mathbb{R}^N} |u(x)|,$$

$$B(x, R) = \left\{ y \in \mathbb{R}^N \mid |y - x| < R \right\}, \quad B_R = B(0, R).$$

We remark that $B_\infty = \mathbb{R}^N$ and we regard $u \in H_0^1(B_R)$ as $u \in H^1(\mathbb{R}^N)$ by expanding $u = 0$ on $\mathbb{R}^N \setminus B_R$. Then we also regard $\|\cdot\|$ and $\|\cdot\|_r$ as norm on $H_0^1(B_R)$ and $L^r(B_R)$, respectively. We set

$$V_0 = \inf_{x \in \mathbb{R}^N} V(x), \quad V_1 = \sup_{x \in \mathbb{R}^N} V(x).$$

$0 < V_0 \le V_\infty \le V_1 < \infty$ follows from the assumptions (V.1) and (V.2). Then we have

$$\|u\|_2^2 \le \frac{1}{V_0} \int_{\mathbb{R}^N} V(x) u^2 \, dx \le \frac{1}{V_0} \|u\|^2 \quad \text{for all } u \in H^1(\mathbb{R}^N). \tag{2.1}$$

We take an auxiliary function $g(s)$ which satisfies

(g.1) $g \in C^1(\mathbb{R}, \mathbb{R})$, $g(s) \ge 0$ for all $s > 0$, and $g(-s) = -g(s)$ for all $s \in \mathbb{R}$.
(g.2) $g'(0) = 0$.
(g.3) There exists $s_0 > 0$ such that $g(s) = 0$ for all $s \ge s_0$.
(g.4) $f(s) + g(s) > 0$ for all $s > 0$.

Remark 2.1 If $f(s) > 0$ for all $s > 0$, then $g(s) \equiv 0$ satisfies (g.1)–(g.4). Otherwise, we can construct $g(s)$ as follows. We define $\widetilde{g}(s)$ by

$$\widetilde{g}(s) = \int_0^s \left(-f'(t)\right)_+ \, dt + s^p \quad \text{for all } s \ge 0.$$

Then $\widetilde{g} \in C^1([0, \infty), [0, \infty))$ and $\widetilde{g}(0) = \widetilde{g}'(0) = 0$. Since

$$\widetilde{g}(s) \ge \int_0^s (-f'(t)) \, dt + s^p = -f(s) + s^p \quad \text{for all } s \ge 0,$$

we have $f(s) + \widetilde{g}(s) > 0$ for all $s > 0$. From (f.4), there exists $s_0 > 1$ such that $f(s) > 0$ for all $s \ge s_0 - 1$. Thus we take an odd function $g \in C^1(\mathbb{R}, \mathbb{R})$ satisfying

$$g(s) = \begin{cases} \widetilde{g}(s) & \text{for } 0 \le s \le s_0 - 1, \\ 0 & \text{for } s_0 \le s, \end{cases}$$
$$g(s) \ge 0 \quad \text{for } s_0 - 1 \le s \le s_0.$$

Then, $g(s)$ satisfies (g.1)–(g.4).

For $q = 2^* = 2N/(N-2) > 2$ and $L \ge 1$, we define

$$a_L(s) = s - (s - L)_+^q \quad \text{for all } s \ge 0.$$

By using $a_L(s)$, we modify the quadratic term of $I(u)$ as follows. For $L \ge 1$ and $u \in H^1(\mathbb{R}^N)$, we define $J_L \in C^2(H^1(\mathbb{R}^N), \mathbb{R})$ by

$$J_L(u) = a_L\left(\frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^N} G(u) \, dx\right) - \int_{\mathbb{R}^N} F(u) + G(u) \, dx,$$

where $G(u) = \int_0^u g(s) \, ds$. We remark that $J_L(u)$ is written as

$$J_L(u) = \frac{1}{2}\|u\|^2 - \left(\frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^N} G(u) \, dx - L\right)_+^q - \int_{\mathbb{R}^N} F(u) \, dx. \tag{2.2}$$

Therefore $J_L(u)$ satisfies

$$J_L(u) = I(u) \quad \text{if } \frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^N} G(u) \, dx \le L,$$
$$J_L(u) \le I(u) \quad \text{for all } u \in H^1(\mathbb{R}^N).$$

Our required sequence satisfying the $H^1(\mathbb{R}^N)$-boundedness and the compactness will given as a sequence of critical points of $J_L : H_0^1(B_{R_n}) \to \mathbb{R}$ $(R_n \to \infty)$.

At the end of this subsection, we state properties of $a_L(s)$ which are used later.

Lemma 2.2 $a_L(s) \in C^2([0,\infty),\mathbb{R})$ *satisfies the following:*

(i) $a_L'(s) \le 1$ *for all* $s \ge 0$.
(ii) *If* $a_L'(s_1) = 0$, *then* $a_L(s_1) > L$.

Proof (i) follows from $a_L'(s) = 1 - q(s-L)_+^{q-1}$. We show (ii). Let $s_1 > 0$ satisfy $a_L'(s_1) = 1 - q(s_1 - L)_+^{q-1} = 0$. Then, we see that $s_1 > L$ and

$$(s_1 - L)_+^{q-1} = (s_1 - L)^{q-1} = \frac{1}{q}.$$

Thus

$$a_L(s_1) = s_1 - (s_1 - L)^q = s_1 - \frac{1}{q}(s_1 - L) = \left(1 - \frac{1}{q}\right)(s_1 - L) + L > L.$$

Hence, (ii) holds. □

2.2 The properties of modified functional

In this section, we state some properties of $J_L(u)$. First, thanks to the modification, we can easily obtain the boundedness of $\{ u \in H^1(\mathbb{R}^N) \mid J_L(u) \ge 0 \}$.

Lemma 2.3 *Assume* $L \ge 1$. *There exists a constant* $C_0 = C_0(L) > 0$ *such that*

$$\|u\| \le C_0 \quad \text{for all } u \in H^1(\mathbb{R}^N) \text{ with } J_L(u) \ge 0.$$

Proof From (f.1)–(f.3), there exists a constant $c_1 > 0$ such that $|F(s)| \le \frac{V_0}{2}|s|^2 + \frac{c_1}{p+1}|s|^{p+1}$ for all $s \in \mathbb{R}$. Then, from (2.1), for some $c_2 > 0$, we have

$$\left| \int_{\mathbb{R}^N} F(u)\,dx \right| \le \frac{V_0}{2}\|u\|_2^2 + \frac{c_1}{p+1}\|u\|_{p+1}^{p+1} \le \frac{1}{2}\|u\|^2 + c_2\|u\|^{p+1}.$$

Thus, from (2.2), we get

$$\begin{aligned} J_L(u) &\le \|u\|^2 - \left(\frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^N} G(u)\,dx - L\right)_+^q + c_2\|u\|^{p+1} \\ &\le \|u\|^2 - \left(\frac{1}{2}\|u\|^2 - L\right)_+^q + c_2\|u\|^{p+1} = h(\|u\|). \end{aligned} \tag{2.3}$$

Here we set $h(s) = s^2 - \left(\frac{1}{2}s^2 - L\right)_+^q + cs^{p+1}$. Since $\lim_{s\to\infty} h(s) = -\infty$, there exists a constant $C_0 = C_0(L) > 0$ such that

$$h(t) < 0 \quad \text{for all } t \in [C_0, \infty). \tag{2.4}$$

From (2.3) and (2.4), $J_L(u) \ge 0$ implies $\|u\| \le C_0$. □

For $R \geq 1$, we also consider functional $J_L : H_0^1(B_R) \to \mathbb{R}$ that is restricted on $H_0^1(B_R)$. (We use same notation J_L.) We see that $J_L \in C^2(H_0^1(B_R), \mathbb{R})$ and

$$J_L'(u)\varphi = a_L'\left(\frac{1}{2}\|u\|^2 + \int_{B_R} G(u)\,dx\right)\left(\langle u, \varphi\rangle + \int_{B_R} g(u)\varphi\,dx\right) - \int_{B_R} f(u)\varphi + g(u)\varphi\,dx \quad \text{for all } \varphi \in H_0^1(B_R).$$

Thus, if u is a critical point of $J_L : H_0^1(B_R) \to \mathbb{R}$, then u is a solution of

$$a_L'(\beta)\left(-\Delta u + V(x)u\right) = f(u) + \left(1 - a_L'(\beta)\right)g(u) \quad \text{in } B_R, \qquad u \in H_0^1(B_R),$$

where $\beta = \frac{1}{2}\|u\|^2 + \int_{B_R} G(u)\,dx$.

Moreover $J_L(u)$ satisfies the Palais–Smale condition.

Lemma 2.4 *Assume $L \geq 1$ and $R \in [1, \infty)$. Then, for any $c \in (0, L]$, $J_L : H_0^1(B_R) \to \mathbb{R}$ satisfies* (PS)$_c$*-condition, that is, every* (PS)*-sequence of J_L at level c has a convergent subsequence.*

Proof Let $(u_n)_{n=1}^\infty \subset H_0^1(B_R)$ satisfy

$$J_L'(u_n) \to 0 \quad \text{in } \left(H_0^1(B_R)\right)^*, \quad J_L(u_n) \to c \in (0, L].$$

From Lemma 2.3, since $\|u_n\|$ is bounded, there exist a subsequence $(u_n)_{n=1}^\infty$ (we use same notation), $\alpha \geq 0$, and $u_0 \in H_0^1(B_R)$ such that

$$\|u_n\| \to \alpha, \tag{2.5}$$

$$u_n \to u_0 \quad \text{weakly in } H_0^1(B_R) \text{ and strongly in } L^{p+1}(B_R). \tag{2.6}$$

From $J_L'(u_n)u_0 \to 0$, we have

$$\begin{aligned} J_L'(u_n)u_0 &= a_L'\left(\frac{1}{2}\|u_n\|^2 + \int_{B_R} G(u_n)\,dx\right)\left(\langle u_n, u_0\rangle + \int_{B_R} g(u_n)u_0\,dx\right) \\ &\quad - \int_{B_R} f(u_n)u_0 + g(u_n)u_0\,dx \\ &\to a_L'(\beta)\left(\|u_0\|^2 + \int_{B_R} g(u_0)u_0\,dx\right) - \int_{B_R} f(u_0)u_0 + g(u_0)u_0\,dx = 0. \end{aligned} \tag{2.7}$$

Here we set

$$\beta = \frac{1}{2}\alpha^2 + \int_{B_R} G(u_0)\,dx. \tag{2.8}$$

Also, from $J_L'(u_n)u_n \to 0$, we have

$$\begin{aligned} J_L'(u_n)u_n &= a_L'\left(\frac{1}{2}\|u_n\|^2 + \int_{B_R} G(u_n)\,dx\right)\left(\|u_n\|^2 + \int_{B_R} g(u_n)u_n\,dx\right) \\ &\quad - \int_{B_R} f(u_n)u_n + g(u_n)u_n\,dx \\ &\to a_L'(\beta)\left(\alpha^2 + \int_{B_R} g(u_0)u_0\,dx\right) - \int_{B_R} f(u_0)u_0 + g(u_0)u_0\,dx = 0. \end{aligned} \tag{2.9}$$

Thus, subtracting (2.9) from (2.7), we get

$$a_L'(\beta)\left(\|u_0\|^2-\alpha^2\right)=0.$$

If $a_L'(\beta)\neq 0$, then we have $\|u_0\|^2=\alpha^2$. Thus, from (2.5) and (2.6), we see that $u_n\to u_0$ strongly in $H_0^1(B_R)$ and the proof is completed. Therefore we show $a_L'(\beta)\neq 0$ by contradiction, and suppose $a_L'(\beta)=0$. From (2.7), we have

$$\int_{B_R} f(u_0)u_0+g(u_0)u_0\,dx=0.$$

Since $f(s)s+g(s)s>0$ $(s\neq 0)$ from (g.1) and (g.4), we see that $u_0=0$. From $J_L(u_n)\to c\in(0,L]$ and (2.6), we have

$$J_L(u_n)=a_L\left(\frac{1}{2}\|u_n\|^2+\int_{B_R} G(u_n)\,dx\right)-\int_{B_R} F(u_n)+G(u_n)\,dx\to a_L\left(\frac{1}{2}\alpha^2\right)=c\leq L. \tag{2.10}$$

On the other hand, (2.8) implies $a_L'(\beta)=a_L'\left(\frac{1}{2}\alpha^2\right)=0$. Thus, by (ii) of Lemma 2.2, $a_L\left(\frac{1}{2}\alpha^2\right)>L$. This contradicts (2.10). Thus, $a_L'(\beta)\neq 0$, and the proof was finished. □

Moreover $J_L: H_0^1(B_R)\to\mathbb{R}$ has a mountain pass geometry which does not depend on $L\geq 1$ and $R\in[1,\infty]$.

Lemma 2.5 *For $L\geq 1$ and $R\in[1,\infty]$, $J_L: H_0^1(B_R)\to\mathbb{R}$ satisfies the following:*

(i) $J_L(0)=0$.

(ii) *There exist $\delta>0$ and $\rho>0$ which are independent of $L\geq 1$ and $R\in[1,\infty]$ such that*

$$J_L(u)\geq\delta \quad \text{for all } u\in H_0^1(B_R) \text{ with } \|u\|=\rho.$$

(iii) *For any $k\in\mathbb{N}$, there exist subspace $E_k\subset H_0^1(B_1)\subset H_0^1(B_R)$ and $r_k>0$ which are independent of $L\geq 1$ and $R\in[1,\infty]$ such that $E_k\subset E_{k+1}$ and*

$$J_L(u)\leq 0 \quad \text{for all } u\in E_k \text{ with } \|u\|\geq r_k.$$

Proof Since the oddness of f and g implies $f(0)=g(0)=0$, (i) is trivial. From (g.1)–(g.3), for some $c_1>0$, we have $|G(s)|\leq c_1|s|^2$ for all $s\in\mathbb{R}$. Thus, there is a constant $c_2>0$ such that

$$\left(\frac{1}{2}\|u\|^2+\int_{\mathbb{R}^N} G(u)\,dx-L\right)_+^q\leq\left(\frac{1}{2}\|u\|^2+\int_{\mathbb{R}^N} G(u)\,dx\right)^q\leq c_2\|u\|^{2q}. \tag{2.11}$$

From (f.1)–(f.3), for some $c_3>0$, we have $|F(s)|\leq\frac{V_0}{4}|s|^2+c_3|s|^{p+1}$ for all $s\in\mathbb{R}$. Thus, from (2.1), there is a constant $c_4>0$ such that

$$\int_{\mathbb{R}^N} F(u)\,dx\leq\frac{1}{4}\|u\|^2+c_4\|u\|^{p+1}. \tag{2.12}$$

From (2.2) and (2.11)–(2.12), we have

$$J_L(u)\geq\frac{1}{4}\|u\|^2-c_2\|u\|^{2q}-c_4\|u\|^{p+1}.$$

Since $2 < p+1 < 2q$, by Young's inequality, for some $c_5 > 0$,

$$J_L(u) \geq \frac{1}{8}\|u\|^2 - c_5\|u\|^{2q} = \frac{1}{8}\|u\|^2\left(1 - 8c_5\|u\|^{2(q-1)}\right).$$

Thus, setting $\rho = \left(\frac{1}{16c_5}\right)^{\frac{1}{2(q-1)}} > 0$ and $\delta = \frac{1}{16}\left(\frac{1}{16c_5}\right)^{\frac{1}{q-1}} > 0$, we get (ii). Next, we show (iii). We choose $w_1, \dots, w_k \in C_0^\infty(B_1) \setminus \{0\}$ such that $\operatorname{supp} w_i \cap \operatorname{supp} w_j = \emptyset$ for $i \neq j$, and set

$$E_k = \operatorname{span}\{w_1, \dots, w_k\}.$$

From (2.2), for any $w \in E_k$ with $\|w\| = 1$, we have

$$J_L(tw) \leq \frac{1}{2}t^2\|w\|^2 - \int_{B_1} F(tw)\,dx \leq t^2\left(\frac{1}{2} - \int_{B_1}\frac{F(tw)}{(tw)^2}w^2\,dx\right).$$

From (f.1) and (f.4), we have

$$\lim_{|t|\to\infty}\int_{B_1}\frac{F(tw)}{(tw)^2}w^2\,dx = \infty,$$

where the above limit is uniformly with respect to $w \in E_k$ with $\|w\| = 1$. Thus we see that (iii) holds. □

For the critical points of $J_L : H_0^1(B_R) \to \mathbb{R}$, we have the following.

Proposition 2.6 *Assume $L \geq 1$ and $R \in [1, \infty]$. If $u \in H_0^1(B_R)$ satisfies $J_L'(u) = 0$ in $(H_0^1(B_R))^*$, then it holds that*

$$a_L'\left(\frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^N} G(u)\,dx\right) \in (0, 1].$$

Moreover, for $\lambda = a_L'\left(\frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^N} G(u)\,dx\right)^{-1} \geq 1$, u is a solution of

$$-\Delta u + V(x)u = \lambda f(u) + (\lambda - 1)g(u) \quad \text{in } B_R, \qquad u \in H_0^1(B_R). \tag{2.13}$$

Proof We set $\beta = \frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^N} G(u)\,dx \geq 0$. Then (i) of Lemma 2.2 asserts $a_L'(\beta) \leq 1$. Since $u \in H_0^1(B_R)$ satisfies $J_L'(u)u = 0$, we have

$$a_L'(\beta)\int_{B_R}|\nabla u|^2 + V(x)u^2 + g(u)u\,dx = \int_{B_R} f(u)u + g(u)u\,dx. \tag{2.14}$$

Since $f(s)s + g(s)s \geq 0$ and $g(s)s \geq 0$ from (g.1) and (g.4), we get $a_L'(\beta) \geq 0$. We suppose $a_L'(\beta) = 0$ by contradiction to show $a_L'(\beta) > 0$. Then, from (2.14), we have

$$\int_{B_R} f(u)u + g(u)u\,dx = 0.$$

Since $f(s)s + g(s)s > 0$ $(s \neq 0)$, we get $u = 0$. This implies $\beta = \frac{1}{2}\|u\|^2 + \int_{B_R} G(u)\,dx = 0$. Thus, $a_L(\beta) = 0$ holds. On the other hand, by (ii) of Lemma 2.2, $a_L'(\beta) = 0$ implies $a_L(\beta) > L$. This contradicts $a_L(\beta) = 0$. Therefore $a_L'(\beta) \in (0, 1]$. Set $\lambda = a_L'(\beta)^{-1} \geq 1$. For any $\varphi \in H_0^1(B_R)$, it holds that

$$J_L'(u)\varphi = \frac{1}{\lambda}\int_{B_R}\nabla u \cdot \nabla\varphi + V(x)u\varphi + g(u)\varphi\,dx - \int_{B_R} f(u)\varphi + g(u)\varphi\,dx = 0.$$

This means that u is a solution of (2.13). □

2.3 $H^1(\mathbb{R}^N)$-boundedness for modified functional on $\mathbb{R}^N$

In our approach, we have to invalidate the modification finally. Therefore we require apriori estimate for critical points of $J_L : H^1(\mathbb{R}^N) \to \mathbb{R}$ and this estimate must be independent of large $L > 0$. For the original functional $I : H^1(\mathbb{R}^N) \to \mathbb{R}$, Jeanjean-Tanaka [11] had gotten apriori estimate as follows.

Proposition 2.7 (cf. [11, Proposition 4.2]) *Assume* (f.1)–(f.4) *and* (V.1)–(V.3). *If* $(u_n)_{n=1}^{\infty} \subset H^1(\mathbb{R}^N)$ *satisfies* $\overline{\lim}_{n\to\infty} I(u_n) < \infty$ *and* $I'(u_n) = 0$ *in* $(H^1(\mathbb{R}^N))^*$, *then* $(u_n)_{n=1}^{\infty}$ *is bounded in* $H^1(\mathbb{R}^N)$.

Proof This follows from [11, Proposition 4.2] and its proofs. We can apply the almost same proofs of [11, Proposition 4.2] to $(u_n)_{n=1}^{\infty} \subset H^1(\mathbb{R}^N)$ satisfying $\overline{\lim}_{n\to\infty} I(u_n) < \infty$ and $I'(u_n) = 0$ in $(H^1(\mathbb{R}^N))^*$. We remark that [11, Proposition 4.2] required that V satisfies $\sup_{x\in\mathbb{R}^N} V(x) \le V_\infty$. However, this assumption is not essential and we can easily remove it. □

We discuss similar argument of Proposition 2.7 to modified functional $J_L : H^1(\mathbb{R}^N) \to \mathbb{R}$ in next proposition. For J_L, we also obtain the following apriori estimate which is independent of large $L > 0$.

Proposition 2.8 *Assume* (f.1)–(f.4) *and* (V.1)–(V.3). *For any* $b > 0$, *there exist constants* $C_1 = C_1(b) > 0$ *and* $\mathcal{L}_1 = \mathcal{L}_1(b) > b$ *such that, for any* $L \ge \mathcal{L}_1$, *we have*

$$\frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^N} G(u)\,dx \le C_1 \quad \textit{for all } u \in K_{L,b},$$

where $K_{L,b} = \left\{ u \in H^1(\mathbb{R}^N) \middle| 0 \le J_L(u) \le b,\ J_L'(u) = 0 \textit{ in } (H^1(\mathbb{R}^N))^* \right\}$.

Remark 2.9 In the proofs of Propositions 2.7 and 2.8 , we don't use $f, g \in C^1(\mathbb{R}, \mathbb{R})$ but use only $f, g \in C(\mathbb{R}, \mathbb{R})$ (see Sect. 4). Thus, under the assumptions (f.1′)–(f.3′), (f.4), and (V.1)–(V.3), Propositions 2.7 and 2.8 still hold.

We prove Proposition 2.8 in Sect. 4. As a corollary of Proposition 2.8, we obtain the following.

Corollary 2.10 *For any* $b > 0$, *there exists a constant* $\mathcal{L}_2 = \mathcal{L}_2(b) > b$ *such that, for any* $L \ge \mathcal{L}_2$, *if* $u \in K_{L,b}$, *then we have*

$$0 \le I(u) = J_L(u) \le b, \quad I'(u) = 0 \quad \textit{in } \left(H^1(\mathbb{R}^N)\right)^*.$$

Proof We set $\mathcal{L}_2 = \max\{\mathcal{L}_1,\ C_1 + 1\}$ where $\mathcal{L}_1$ and C_1 are constants which were given in Proposition 2.8. For any $L \ge \mathcal{L}_2$, if $u \in K_{L,b}$, then we have

$$\frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^N} G(u)\,dx \le C_1 \le \mathcal{L}_2 - 1 \le L - 1.$$

Therefore, we see that $J_L \equiv I$ in a neighborhood of u and we get the conclusion. □

2.4 Balanced sequence and the compactness

In this section, we consider about balanced sequences and the compactness.

Definition 2.11 Suppose $\mathcal{I} \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$. If $u_n \in H_0^1(B_{R_n})$ $(R_n \to \infty)$ satisfies $\sup_{n\in\mathbb{N}} |\mathcal{I}(u_n)| < \infty$ and $\mathcal{I}'(u_n) = 0$ in $(H_0^1(B_{R_n}))^*$, then we say that $(u_n)_{n=1}^{\infty}$ is a balanced sequence for $\mathcal{I}$.

We construct balanced sequences for $J_L : H^1(\mathbb{R}^N) \to \mathbb{R}$. For any $k \in \mathbb{N}$, let subspace $E_k \subset H_0^1(B_1)$ and $r_k > 0$ be as in (iii) of Lemma 2.5. For $L \geq 1$ and $R \in [1, \infty]$, we define minimax values as follows:

$$D^k = \{u \in E_k \mid \|u\| \leq r_k\},$$

$$\Gamma_R^k = \left\{ \gamma \in C\left(D^k, H_0^1(B_R)\right) \,\middle|\, \begin{array}{l} \gamma(-u) = -\gamma(u) \text{ for all } u \in D^k, \\ \gamma(u) = u \text{ for all } u \in \partial D^k \end{array} \right\},$$

$$b_{L,R}^k = \inf_{\gamma\in\Gamma_R^k} \max_{u\in D^k} J_L(\gamma(u)), \tag{2.15}$$

$$b_R^k = \inf_{\gamma\in\Gamma_R^k} \max_{u\in D^k} I(\gamma(u)). \tag{2.16}$$

Lemma 2.12 *For $k \in \mathbb{N}$, $L \geq 1$, and $R \in [1, \infty)$, we have the following:*

(i) $b_{L,R}^k \leq b_R^k \leq b_1^k$.
(ii) $b_{L,R}^k$ *is a critical value of* $J_L : H_0^1(B_R) \to \mathbb{R}$ *if* $b_{L,R}^k \leq L$.

Proof Since $\Gamma_1^k \subset \Gamma_R^k$, we have $b_R^k \leq b_1^k$. Also, since $J_L(u) \leq I(u)$, we have $b_{L,R}^k \leq b_R^k$. Thus, we get (i). From Lemmas 2.4 and 2.5 , by a standard method, we see that $b_{L,R}^k$ is a critical value of $J_L : H_0^1(B_R) \to \mathbb{R}$. □

Then there exist critical points having the estimates from the below of the Morse indexes.

Lemma 2.13 *For $k \in \mathbb{N}$, $L \geq b_1^k$, and $R \in [1, \infty)$, there exists $w_{L,R}^k \in H_0^1(B_R)$ such that*

$$0 \leq J_L\left(w_{L,R}^k\right) \leq b_{L,R}^k \leq b_1^k,$$

$$J_L'\left(w_{L,R}^k\right) = 0 \quad \textit{in} \ \left(H_0^1(B_R)\right)^*,$$

$$\text{index}_0\, J_L''\left(w_{L,R}^k\right) \geq k,$$

where

$$\text{index}_0\, J_L''(w_{L,R}^k) = \max\left\{ \dim H \,\middle|\, \begin{array}{l} H \subset H_0^1(B_R) \textit{ is a subspace such that} \\ J_L''(w_{L,R}^k)(h,h) \leq 0 \textit{ for } h \in H_0^1(B_R) \end{array} \right\}.$$

Proof This follows from [18, Theorem B]. We remark that [18, Theorem B] is true, if we replace the assumption (I_4) in [18] to the following (I_4)$'$.

(I_4)$'$ For any u with $I'(u) = 0$, $I''(u)$ is represented as $I''(u) = a_u id + K_u$, where $a_u > 0$ and K_u is a compact operator.

In fact, in the proof of [18, Theorem B], we use only $(\mathrm{I}_4)'$. $J_L(u)$ satisfies $(\mathrm{I}_4)'$ because $J_L''(u)$ is written as $\langle J_L''(u)\varphi, \psi\rangle = \langle a_L'(P(u))\varphi, \psi\rangle + \langle K_u\varphi, \psi\rangle$, where $P(u) = \frac{1}{2}\|u\|^2 + \int_{B_R} G(u)\,dx$ and

$$\langle K_u\varphi, \psi\rangle = a_L''\left(P(u)\right) P'(u)\varphi P'(u)\psi$$
$$- \int_{B_R} f'(u)\varphi\psi\,dx - \left(1 - a_L'\left(P(u)\right)\right) \int_{B_R} g'(u)\varphi\psi\,dx.$$

From Proposition 2.6, $J_L'(u) = 0$ implies $a_L'(P(u)) > 0$. We can find that $K_u : H_0^1(B_R) \to H_0^1(B_R)$ is a compact operator. Thus we get Lemma 2.13. □

The following proposition guarantees the compactness of $(w_{L,R}^k)_{R\geq 1}$.

Proposition 2.14 *We assume* (f.1)–(f.4), (V.1), (V.2), and (V.4). *Let* $u_n \in H_0^1(B_{R_n})$ $(R_n \to \infty)$ *satisfy*

$$0 \leq J_L(u_n) \leq L, \quad J_L'(u_n) = 0 \quad in \ \left(H_0^1(B_{R_n})\right)^*. \tag{2.17}$$

Then there exist a subsequence $(u_n)_{n=1}^\infty$ *(we use same notation) and* $u_0 \in H^1(\mathbb{R}^N)$ *such that*

$$\|u_n - u_0\| \to 0 \quad and \quad J_L'(u_0) = 0 \quad in \ \left(H^1(\mathbb{R}^N)\right)^*.$$

For the original functional $I(u)$, the following similar compactness holds.

Proposition 2.15 *We assume* (f.1)–(f.4), (V.1), (V.2)*, and* (V.4). *Let* $u_n \in H_0^1(B_{R_n})$ $(R_n \to \infty)$ *be bounded in* $H^1(\mathbb{R}^N)$ *and satisfy*

$$\varlimsup_{n\to\infty} I(u_n) < \infty \quad and \quad I'(u_n) = 0 \quad in \ \left(H_0^1(B_{R_n})\right)^*.$$

Then there exist a subsequence $(u_n)_{n=1}^\infty$ *(we use same notation) and* $u_0 \in H^1(\mathbb{R}^N)$ *such that*

$$\|u_n - u_0\| \to 0 \quad and \quad I'(u_0) = 0 \quad in \ \left(H^1(\mathbb{R}^N)\right)^*.$$

Proof The proof is almost same as the proof of Proposition 2.14. Thus we omit it. □

Remark 2.16 In the proofs of Propositions 2.14 and 2.15 , we don't use differentiability of f or g (see Sects. 5 and 6). Thus, under the assumptions (f.1′)–(f.3′), (f.4), (V.1)–(V.2), and (V.4), Propositions 2.14 and 2.15 still hold.

In order to prove Proposition 2.14, we use the concentration compactness arguments. The key of getting the compactness is the assumption (V.4). We argue with the concentration compactness in Sect. 5 and prove Proposition 2.14 in Sect. 6.

3 Proof of main theorems

First, we give a proof of Theorem 1.1. We need the following lemma which is similar with [3, Lemma A.1].

Lemma 3.1 *If* $w \in H^1(\mathbb{R}^N)$ *satisfies* $I'(w) = 0$*, then there exists a finite dimensional subspace* $M \subset H^1(\mathbb{R}^N)$ *such that*

$$I''(w)(h, h) \geq \frac{1}{2}\|h\|^2 \quad for\ all\ h \in M^\perp.$$

Proof We suppose, by contradiction, that there exists a sequence $(h_n)_{n=1}^{\infty} \subset H^1(\mathbb{R}^N)$ such that

$$\langle h_m, h_n \rangle = \delta_{mn}, \quad I''(w)(h_n, h_n) < \frac{1}{2}\|h_n\|^2 \quad \text{for all } m, n \in \mathbb{N}. \tag{3.1}$$

Here we have

$$\begin{aligned} I''(w)(h_n, h_n) &= \|h_n\|^2 - \int_{\mathbb{R}^N} f'(w) h_n^2 \, dx \\ &\geq \frac{3}{4}\|h_n\|^2 + \int_{\mathbb{R}^N} \left(\frac{1}{4} V(x) - f'(w) \right) h_n^2 \, dx. \end{aligned} \tag{3.2}$$

From (3.1)–(3.2), we have

$$\int_{\mathbb{R}^N} \left(\frac{1}{4} V(x) - f'(w) \right) h_n^2 \, dx < -\frac{1}{4}\|h_n\|^2 = -\frac{1}{4}. \tag{3.3}$$

Since $|w|$ satisfies $-\Delta |w| + V(x)|w| \leq f(|w|)$ in $\mathbb{R}^N$, by a subsolution estimate (cf. Lemma 6.1), $\|w\|_{L^{p+1}(\mathbb{R}^N \setminus B_R)} \to 0$ $(R \to \infty)$ implies $\|w\|_{L^\infty(\mathbb{R}^N \setminus B_R)} \to 0$ $(R \to \infty)$. Thus, there exists $r > 0$ such that

$$\frac{1}{4} V(x) - f'(w) > 0 \quad \text{if } |x| \geq r.$$

Therefore, there exists $c(x) \in C_0^\infty(B_r)$ such that

$$\frac{1}{4} V(x) - f'(w) \geq c(x) \quad \text{in } \mathbb{R}^N. \tag{3.4}$$

From (3.3)–(3.4), we have

$$\int_{B_r} c(x) h_n^2 \, dx \leq \int_{\mathbb{R}^N} \left(\frac{1}{4} V(x) - f'(w) \right) h_n^2 \, dx < -\frac{1}{4}. \tag{3.5}$$

On the other hand, since $h_n \to 0$ weakly in $H^1(\mathbb{R}^N)$, we have $h_n \to 0$ in $L^2(B_r)$ and

$$\int_{B_r} c(x) h_n^2 \, dx \to 0 \quad \text{as } n \to \infty.$$

This contradicts (3.5). Thus we get the conclusion. □

Now we prove Theorem 1.1.

Proof of Theorem 1.1 For any $k \in \mathbb{N}$, we define a minimax value b_1^k as (2.16). We choose and fix $L^k = \mathcal{L}_2(b_1^k) > 0$ in Corollary 2.10. We consider the modified functional $J_{L^k}(u)$ and define minimax values $b_{L^k,R}^k$ as (2.15). From Lemma 2.13, for $R \in [1, \infty)$, there exists $w_{L^k,R}^k \in H_0^1(B_R)$ such that

$$\begin{aligned} & 0 \leq J_{L^k}\left(w_{L^k,R}^k\right) \leq b_{L^k,R}^k \leq b_1^k, \\ & J'_{L^k}\left(w_{L^k,R}^k\right) = 0 \quad \text{in } \left(H_0^1(B_R)\right)^*, \\ & \text{index}_0\, J''_{L^k}\left(w_{L^k,R}^k\right) \geq k. \end{aligned}$$

Here we set

$$b^k = \varliminf_{R \to \infty} J_{L^k}\left(w_{L^k,R}^k\right).$$

From Proposition 2.14, there exist a subsequence $(w^k_{L^k,R_n})_{n=1}^{\infty}$ $(R_n \to \infty)$ and $w^k_{L^k} \in H^1(\mathbb{R}^N)$ such that

$$\begin{aligned} &\left\| w^k_{L^k,R_n} - w^k_{L^k} \right\| \to 0, \\ &J_{L^k}\left(w^k_{L^k}\right) = b^k, \\ &J'_{L^k}\left(w^k_{L^k}\right) = 0 \quad \text{in } \left(H^1(\mathbb{R}^N)\right)^*. \end{aligned} \tag{3.6}$$

From the choice of $L^k = \mathcal{L}_2(b^k_1)$ in Corollary 2.10, $w^k_{L^k}$ satisfies

$$\begin{aligned} &\frac{1}{2}\left\|\omega^k_{L^k}\right\|^2 + \int_{\mathbb{R}^N} G\left(\omega^k_{L^k}\right) dx < L^k - 1 \\ &b^k = I\left(w^k_{L^k}\right) = J_{L_k}\left(w^k_{L^k}\right) \le b^k_1, \\ &I'\left(w^k_{L^k}\right) = J'_{L_k}\left(w^k_{L^k}\right) = 0 \quad \text{in } \left(H^1(\mathbb{R}^N)\right)^*. \end{aligned} \tag{3.7}$$

Thus $w^k_{L^k}$ is a critical point of $I(u)$. If we get $b^k \to \infty$ as $k \to \infty$, then the proof of Theorem 1.1 is finished. To show $b^k \to \infty$ by contradiction, suppose that there exists $\overline{b} > 0$ such that

$$b^k \le \overline{b} \quad \text{for all } k \in \mathbb{N}.$$

Then, from Proposition 2.7, $(w^k_{L^k})_{k=1}^{\infty}$ is bounded in $H^1(\mathbb{R}^N)$. Furthermore, from Proposition 2.15, there exist a subsequence $(w^k_{L^k})_{k=1}^{\infty}$ (we use same notation) and $\overline{w} \in H^1(\mathbb{R}^N)$ such that

$$\left\| w^k_{L^k} - \overline{w} \right\| \to 0 \text{ as } k \to \infty, \quad I'(\overline{w}) = 0 \text{ in } \left(H^1(\mathbb{R}^N)\right)^*.$$

From Lemma 3.1, there exists a finite dimensional subspace $M \subset H^1(\mathbb{R}^M)$ such that

$$I''(\overline{w})(h,h) \ge \frac{1}{2}\|h\|^2 \quad \text{for all } h \in M^{\perp}. \tag{3.8}$$

We set $k_0 = \dim M$. Since I is C^2, there exists $k_1 > k_0$ such that

$$\left\| I''(\overline{w}) - I''\left(w^{k_1}_{L^{k_1}}\right) \right\| \le \frac{1}{8}.$$

From (3.6), there exists $R_n > 0$ such that

$$\left\| I''\left(w^{k_1}_{L^{k_1}}\right) - I''\left(w^{k_1}_{L^{k_1},R_n}\right) \right\| \le \frac{1}{8}, \quad \text{index}_0\, I''\left(w^{k_1}_{L^{k_1},R_n}\right) \ge k_1.$$

From the definition of index$_0$, there exists a finite dimensional subspace $\widehat{M} \subset H^1_0(B_R)$ with $\dim \widehat{M} = k_1$ such that

$$I''\left(w^{k_1}_{L^{k_1},R_n}\right)(h,h) \le 0 \quad \text{for all } h \in \widehat{M}.$$

Since $\dim M = k_0 < k_1 = \dim \widehat{M}$, there exists $h \in M^{\perp} \cap \widehat{M}$ with $\|h\| = 1$. From (3.8), we have

$$\begin{aligned}\frac{1}{2}\|h\|^2 &\leq I''(\overline{w})(h,h)\\ &= \left[\left(I''(\overline{w}) - I''\left(w^{k_1}_{L^{k_1}}\right)\right) + \left(I''\left(w^{k_1}_{L^{k_1}}\right) - I''\left(w^{k_1}_{L^{k_1},R_n}\right)\right) + I''\left(w^{k_1}_{L^{k_1},R_n}\right)\right](h,h)\\ &\leq \left\|I''(\overline{w}) - I''\left(w^{k_1}_{L^{k_1}}\right)\right\| \|h\|^2 + \left\|I''\left(w^{k_1}_{L^{k_1}}\right) - I''\left(w^{k_1}_{L^{k_1},R_n}\right)\right\| \|h\|^2 \leq \frac{1}{4}\|h\|^2.\end{aligned}$$

This is a contradiction. Thus we get $b^k \to \infty$ and the proof was finished. □

Remark 3.2 In the above proof, from (3.6)–(3.7), there exists $R^k > 0$ such that, for any $R > R^k$, $b^k - 1 \leq J_{L^k}(\omega^k_{L^k,R})$ and

$$\frac{1}{2}\left\|\omega^k_{L^k,R}\right\|^2 + \int_{\mathbb{R}^N} G\left(\omega^k_{L^k,R}\right)dx < L_k - \frac{1}{2}. \tag{3.9}$$

Indeed, if (3.9) does not hold, there exists $(R_n)_{n=1}^{\infty}$ with $R_n \to \infty$ such that

$$\frac{1}{2}\left\|\omega^k_{L^k,R_n}\right\|^2 + \int_{\mathbb{R}^N} G\left(\omega^k_{L^k,R_n}\right)dx \geq L_k - \frac{1}{2}. \tag{3.10}$$

On the other hand, similarly as in the proof of Theorem 1.1, taking a subsequence if necessary, there exists $w^k_{L^k} \in H^1(\mathbb{R}^N)$ such that (3.6) and (3.7) hold. Taking $n \to \infty$ in (3.10), we have

$$\frac{1}{2}\left\|\omega^k_{L^k}\right\|^2 + \int_{\mathbb{R}^N} G\left(\omega^k_{L^k}\right)dx \geq L_k - \frac{1}{2}.$$

This contradicts (3.7). From (3.9), $w^k_{L^k,R}$ satisfies

$$\begin{aligned}&b^k - 1 \leq I\left(w^k_{L^k,R}\right) = J_{L^k}\left(w^k_{L^k,R}\right) \leq b^k_{L^k,R} \leq b^k_1,\\ &I'\left(w^k_{L^k,R}\right) = J'_{L^k}\left(w^k_{L^k,R}\right) = 0 \quad \text{in } (H^1_0(B_R))^*.\end{aligned}$$

Thus $w^k_{L^k,R}$ $(R \geq R^k)$ is a solution of (1.6) with $\Omega_R = B_R$. Since $\lim_{k\to\infty} b^k = \infty$, we obtain Theorem 1.4 for the case $\Omega = B_R$.

As a by-product of Theorem 1.1, we can obtain Theorem 1.3. We state only outline of the proof.

Outline of proof of Theorem 1.3 We can show Theorem 1.3 as a similar way to the proof of Theorem 1.1. To obtain positive solutions, we put $f \equiv 0$ on $(-\infty, 0)$. Under (f.1′)–(f.3′), we take an auxiliary function $g(s) = (-f(s))_+ + |s|^{p-1}s$ near $s = 0$. We also define $J_L(u)$ as in (2.2). We note that Lemmas 2.2, 2.4, 2.5, Propositions 2.6, 2.8, and 2.14 still hold. Indeed, in this setting, (g.4) does not hold. However, we have $f(s)s + g(s)s \geq 0$ for any $s \in \mathbb{R}$ and $s = 0$ is an isolated solution of $f(s)s + g(s)s = 0$. Thus $\int_{B_R} f(u)u + g(u)u\,dx = 0$ and $u \in H^1_0(B_R)$ imply $u = 0$ in B_R. Hence these lemmas and propositions hold. We define the following minimax value

$$\begin{aligned}b_{L,R} &= \inf_{\gamma\in\Gamma_R} \max_{u\in[0,1]} J_L(\gamma(u)),\\ \Gamma_R &= \left\{\gamma \in C\left([0,1], H^1_0(B_R)\right) \middle| \gamma(0) = 0,\ \gamma(1) = e\right\}.\end{aligned}$$

Here we choose $e \in H_0^1(B_1)$ satisfying $J_L(e) < 0$. Let $w_{L,R} \in H_0^1(B_R)$ be a critical point for $b_{L,R}$. By the similar way to the proof of Theorem 1.1, for $R_n \to \infty$, $(w_{L,R_n})_{n=1}^{\infty}$ is a balanced sequence and, after extracting a subsequence (we use same notation), we can show that $(w_{L,R_n})_{n=1}^{\infty}$ converges to a nontrivial critical point $w_L \in H^1(\mathbb{R}^N)$ of $I(u)$. Moreover, by the maximum principle, we see that $w_{L,R_n} > 0$. Thus we got a positive solution of (1.1).

□

4 Apriori estimate for critical points of J_L

In this section, we prove Proposition 2.8. The fundamental idea of the proof comes from [11, Proposition 4.2]. First, we show the following.

Lemma 4.1 *Let $u \in H^1(\mathbb{R}^N)$ satisfy $J_L'(u) = 0$ in $(H^1(\mathbb{R}^N))^*$. Then*

$$NL\left(\frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^N} G(u)\,dx - L\right)_+^{q-1} + \|\nabla u\|_2^2 \le NJ_L(u) + \frac{1}{2}\int_{\mathbb{R}^N} u^2\eta^2\,dx. \tag{4.1}$$

Here $\eta \in L^2(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ is the function in (V.3).

Proof From Proposition 2.6, u is a solution of

$$-\Delta u + V(x)u = \lambda f(u) + (\lambda - 1)g(u) \quad \text{in } \mathbb{R}^N, \qquad u \in H^1(\mathbb{R}^N),$$

where $\lambda = a_L'\left(\frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^N} G(u)\,dx\right)^{-1} \ge 1$. Thus u satisfies the following Pohozaev identity;

$$\begin{aligned}\left(\frac{1}{2} - \frac{1}{N}\right)\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2\,dx + \frac{1}{2N}\int_{\mathbb{R}^N}(x\cdot\nabla V(x))u^2\,dx \\ = \int_{\mathbb{R}^N} \lambda F(u) + (\lambda-1)G(u)\,dx.\end{aligned} \tag{4.2}$$

Set $\beta = \frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^N} G(u)\,dx$. (4.2) is written as

$$a_L'(\beta)\left(\beta - \frac{1}{N}\|\nabla u\|_2^2 + \frac{1}{2N}\int_{\mathbb{R}^N}(x\cdot\nabla V(x))u^2\,dx\right) = \int_{\mathbb{R}^N} F(u) + G(u)\,dx. \tag{4.3}$$

From (4.3), we have

$$\begin{aligned}J_L(u) &= a_L(\beta) - \int_{\mathbb{R}^N} F(u) + G(u)\,dx \\ &= a_L(\beta) - a_L'(\beta)\left(\beta - \frac{1}{N}\|\nabla u\|_2^2 + \frac{1}{2N}\int_{\mathbb{R}^N}\left(x\cdot\nabla V(x)\right)u^2\,dx\right).\end{aligned}$$

From $0 < a'(\beta) \le 1$ and the definition of η, we have

$$\begin{aligned}a_L(\beta) - a_L'(\beta)\left(\beta - \frac{1}{N}\|\nabla u\|_2^2\right) &= J_L(u) + \frac{a_L'(\beta)}{2N}\int_{\mathbb{R}^N}(x\cdot\nabla V(x))u^2\,dx \\ &\le J_L(u) + \frac{1}{2N}\int_{\mathbb{R}^N} u^2\eta^2\,dx.\end{aligned} \tag{4.4}$$

Now we calculate the left hand side of (4.4).

$$a_L(\beta) - a_L'(\beta)\left(\beta - \frac{1}{N}\|\nabla u\|_2^2\right) = \beta - (\beta - L)_+^q - \left(1 - q(\beta - L)_+^{q-1}\right)\left(\beta - \frac{1}{N}\|\nabla u\|_2^2\right)$$
$$= \frac{1}{N}\|\nabla u\|_2^2 + (\beta - L)_+^{q-1}\left(-\beta + L + q\beta - \frac{q}{N}\|\nabla u\|_2^2\right).$$

Recalling $\beta = \frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^N} G(u)\,dx$, we have

$$\begin{aligned} &-\beta + L + q\beta - \frac{q}{N}\|\nabla u\|_2^2 \\ &= \left(\frac{N-2}{2N}q - \frac{1}{2}\right)\|\nabla u\|_2^2 + L + (q-1)\left(\frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2\,dx + \int_{\mathbb{R}^N} G(u)\,dx\right) \\ &\geq L. \end{aligned} \tag{4.5}$$

Combining (4.4) and (4.5), we get

$$J_L(u) + \frac{1}{2N}\int_{\mathbb{R}^N} u^2\eta^2\,dx \geq \frac{1}{N}\|\nabla u\|_2^2 + L(\beta - L)_+^{q-1}.$$

Thus we got (4.1). □

Next we show the following lemma by the argument in [11, Proposition 4.2].

Lemma 4.2 *For any $b > 0$, there exists a constant $C_2 = C_2(b) > 0$ such that*

$$\int_{\mathbb{R}^N} u^2\eta^2\,dx \leq C_2 \quad \text{for all } u \in K_{L,b},$$

where $K_{L,b}$ is defined in Proposition 2.8.

Proof Let $u \in K_{L,b}$. We set $\lambda = a_L'\left(\frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^N} G(u)\,dx\right)^{-1} \geq 1$. From $J_L'(u)(u\eta^2) = 0$, it holds that

$$\begin{aligned} \int_{\mathbb{R}^N} \nabla u \cdot \nabla(u\eta^2) + V(x)u^2\eta^2\,dx &= \lambda\int_{\mathbb{R}^N} (f(u) + g(u))u\eta^2\,dx - \int_{\mathbb{R}^N} g(u)u\eta^2\,dx \\ &\geq \int_{\mathbb{R}^N} (f(u) + g(u))u\eta^2\,dx - \int_{\mathbb{R}^N} g(u)u\eta^2\,dx \\ &= \int_{\mathbb{R}^N} f(u)u\eta^2\,dx. \end{aligned} \tag{4.6}$$

Here we used $(f(s) + g(s))s \geq 0$ and $\lambda \geq 1$. From (f.1) and (f.4), for any $M > 0$, there exists $c_M > 0$ such that

$$f(s)s \geq Ms^2 - c_M \quad \text{for all } s \in \mathbb{R}.$$

Thus we have

$$\int_{\mathbb{R}^N} f(u)u\eta^2\,dx \geq M\int_{\mathbb{R}^N} u^2\eta^2\,dx - c_M\|\eta\|_2^2. \tag{4.7}$$

On the other hand, we have

$$\int_{\mathbb{R}^N} V(x)u^2\eta^2\,dx \leq V_1\int_{\mathbb{R}^N} u^2\eta^2\,dx \tag{4.8}$$

and

$$\begin{aligned}\int_{\mathbb{R}^N} \nabla u \cdot \nabla(u\eta^2)\,dx &\le \int_{\mathbb{R}^N} |\nabla u|^2\eta^2\,dx + \int_{\mathbb{R}^N} 2|\nabla u|\,|\nabla\eta|\,|u\eta|\,dx \\ &\le \int_{\mathbb{R}^N} |\nabla u|^2\left(\eta^2 + |\nabla\eta|^2\right)dx + \int_{\mathbb{R}^N} u^2\eta^2\,dx \\ &\le \left(\|\eta\|_\infty^2 + \|\nabla\eta\|_\infty^2\right)\|\nabla u\|_2^2 + \int_{\mathbb{R}^N} u^2\eta^2\,dx. \qquad (4.9)\end{aligned}$$

From (4.1) and $J_L(u) \le b$, we have

$$\|\nabla u\|_2^2 \le Nb + \frac{1}{2}\int_{\mathbb{R}^N} u^2\eta^2\,dx. \qquad (4.10)$$

Combining (4.6), (4.7), (4.8), (4.9), and (4.10), we get

$$\begin{aligned}M\int_{\mathbb{R}^N} u^2\eta^2\,dx - c_M\|\eta\|_2^2 \le& V_1\int_{\mathbb{R}^N} u^2\eta^2\,dx + \left(\|\eta\|_\infty^2 + \|\nabla\eta\|_\infty^2\right)Nb \\ &+ \frac{\left(\|\eta\|_\infty^2 + \|\nabla\eta\|_\infty^2\right)}{2}\int_{\mathbb{R}^N} u^2\eta^2\,dx + \int_{\mathbb{R}^N} u^2\eta^2\,dx.\end{aligned}$$

Thus we have

$$\left(M - V_1 - 1 - \frac{\|\eta\|_\infty^2 + \|\nabla\eta\|_\infty^2}{2}\right)\int_{\mathbb{R}^N} u^2\eta^2\,dx \le \left(\|\eta\|_\infty^2 + \|\nabla\eta\|_\infty^2\right)Nb + c_M\|\eta\|_2^2.$$

Since $M > 0$ is arbitrary, we set $M = V_1 + 2 + \left(\|\eta\|_\infty^2 + \|\nabla\eta\|_\infty^2\right)/2$. Then

$$\int_{\mathbb{R}^N} u^2\eta^2\,dx \le \left(\|\eta\|_\infty^2 + \|\nabla\eta\|_\infty^2\right)Nb + c_M\|\eta\|_2^2.$$

Thus we get the conclusion. □

Here we observe that, when L is large, λ is restricted.

Lemma 4.3 *For any $b > 0$, there exists a constant $\mathcal{L}_1 = \mathcal{L}_1(b) > b$ such that, for any $L \ge \mathcal{L}_1$,*

$$\lambda = a_L'(\beta)^{-1} \in [1, 2] \quad \textit{for all } u \in K_{L,b},$$

where $\beta = \frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^N} G(u)\,dx$.

Proof We set $\mathcal{L}_1 = 2q\left(b + \frac{C_2}{2N}\right)$. From (4.1) and Lemma 4.2, we have

$$L(\beta - L)_+^{q-1} \le b + \frac{1}{2N}\int_{\mathbb{R}^N} u^2\eta^2\,dx \le b + \frac{C_2}{2N} = \frac{\mathcal{L}_1}{2q} \le \frac{L}{2q}.$$

Thus $a_L'(\beta) = 1 - q(\beta - L)_+^{q-1} \ge 1 - \frac{q}{2q} = \frac{1}{2}$ holds. □

Next we show the boundedness for $\|u\|$.

Lemma 4.4 *For any $b > 0$, there exists a constant $C_3 = C_3(b) > 0$ such that, for any $L \ge \mathcal{L}_1$,*

$$\|u\| \le C_3 \quad \textit{for all } u \in K_{L,b}.$$

Proof From (f.1)–(f.3) and (g.1)–(g.3), for any $\epsilon > 0$, there exists $c_\epsilon > 0$ such that

$$|f(s)|,\ |g(s)| \le \epsilon|s| + c_\epsilon |s|^{\frac{N+2}{N-2}} \quad \text{for all } s \in \mathbb{R}.$$

Let $u \in K_{L,b}$ and $L \ge \mathcal{L}_1$. Then $\lambda = a_L' \left(\frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^N} G(u)\,dx\right)^{-1} \in [1, 2]$ by Lemma 4.3. From $J_L'(u)u = 0$, we have

$$\begin{aligned}
\|u\|^2 &= \int_{\mathbb{R}^N} \lambda f(u)u + (\lambda - 1)g(u)u\,dx \\
&\le 3\left(\epsilon\|u\|_2^2 + c_\epsilon \|u\|_{\frac{2N}{N-2}}^{\frac{2N}{N-2}}\right) \\
&\le 3\left(\epsilon\|u\|_2^2 + c_\epsilon C\|\nabla u\|_2^{\frac{2N}{N-2}}\right).
\end{aligned}$$

From Lemmas 4.1 and 4.2 ,

$$\|\nabla u\|_2^2 \le Nb + \frac{C_2}{2}.$$

Thus we see that $\|u\|$ is bounded. □

Finally, we prove Proposition 2.8.

Proof of Proposition 2.8 From (g.1)–(g.3), there exists $c > 0$ such that $G(s) \le cs^2$ for all $s \in \mathbb{R}$. Thus we have

$$\frac{1}{2}\|u\|^2 + \int_{\mathbb{R}^N} G(u)\,dx \le \frac{1}{2}\|u\|^2 + c\|u\|_2^2.$$

From Lemma 4.4, we obtain Proposition 2.8. □

5 Concentration compactness

In this section, in order to prove Proposition 2.14, we argue about the concentration compactness for balanced sequences. We assume that $(u_n)_{n=1}^\infty$ with $u_n \in H_0^1(B_{R_n})$ is a balanced sequence which satisfies

$$0 \le J_L(u_n) \le L, \quad J_L'(u_n) = 0 \text{ in } \left(H_0^1(B_{R_n})\right)^*. \tag{5.1}$$

Then, from Lemma 2.3 and (5.1), $(u_n)_{n=1}^\infty$ is bounded in $H^1(\mathbb{R}^N)$ and from Proposition 2.6, u_n satisfies

(a) $J_L(u_n) \le L$.
(b) $J_L'(u_n)\varphi = 0$ for all $\varphi \in H_0^1(B_{R_n})$.
(c) $0 < a_L'\left(\frac{1}{2}\|u_n\|^2 + \int_{\mathbb{R}^N} G(u_n)\,dx\right) \le 1$.

The main theorem of this section is the following.

Theorem 5.1 *We assume* (f.1)–(f.4), (V.1), and (V.2). *Let $(u_n)_{n=1}^\infty$ with $u_n \in H_0^1(B_{R_n})$ be a bounded sequence in $H^1(\mathbb{R}^N)$ and satisfy* (a)–(c). *Then, there exist a subsequence $(u_n)_{n=1}^\infty$ (we use same notation), $\ell \in \mathbb{N} \cup \{0\}$, $u_0 \in H^1(\mathbb{R}^N)$, $\omega^1, \ldots, \omega^\ell \in H^1(\mathbb{R}^N) \setminus \{0\}$, $(z_n^k)_{n=1}^\infty \subset \mathbb{R}^N$ with $z_n^k \in B_{R_n}$, $|z_n^k| \to \infty$ $(k = 1, \ldots, \ell)$ and $|z_n^k - z_n^{k'}| \to \infty$ $(k \ne k')$, and $\lambda_0 \ge 1$ such that*

(i) $\left\| u_n - u_0 - \sum_{k=1}^{\ell} \omega^k(\cdot + z_n^k) \right\| \to 0.$

(ii) u_0 *is a solution of*

$$-\Delta u + V(x)u = \lambda_0 f(u) + (\lambda_0 - 1)g(u) \text{ in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N). \tag{5.2}$$

(iii) ω^k $(k = 1, \dots, \ell)$ *are solutions of*

$$-\Delta u + V_\infty u = \lambda_0 f(u) + (\lambda_0 - 1)g(u) \text{ in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N). \tag{5.3}$$

In particular, when $\ell = 0$, *we have* $\|u_n - u_0\| \to 0$ *and* $J_L'(u_0) = 0$.

We state some lemmas which are repeatedly used.

Lemma 5.2 *Let* $(u_n)_{n=1}^\infty$ *and* $(v_n)_{n=1}^\infty$ *be bounded in* $H^1(\mathbb{R}^N)$ *and*

$$v_n \to 0 \quad \text{strongly in } L^{p+1}(\mathbb{R}^N). \tag{5.4}$$

Then we have

$$\int_{\mathbb{R}^N} f(u_n)v_n\,dx \to 0, \quad \int_{\mathbb{R}^N} g(u_n)v_n\,dx \to 0 \quad (n \to \infty). \tag{5.5}$$

Proof Since $(u_n)_{n=1}^\infty$ and $(v_n)_{n=1}^\infty$ are bounded in $H^1(\mathbb{R}^N)$, there exists $M > 0$ such that

$$\|u_n\| \le M, \qquad \|u_n\|_{p+1} \le M, \qquad \|v_n\| \le M.$$

From (f.1)–(f.3) and (g.1)–(g.3), for any $\epsilon > 0$, there exists $c_\epsilon > 0$ such that

$$|f(s)|,\ |g(s)| \le \epsilon|s| + c_\epsilon|s|^p \quad \text{for all } s \in \mathbb{R}.$$

Thus we have

$$\begin{aligned}\left|\int_{\mathbb{R}^N} f(u_n)v_n\,dx\right|, \left|\int_{\mathbb{R}^N} g(u_n)v_n\,dx\right| &\le \epsilon \int_{\mathbb{R}^N} |u_n||v_n|\,dx + c_\epsilon \int_{\mathbb{R}^N} |u_n|^p|v_n|\,dx \\ &\le \epsilon\|u_n\|_2\|v_n\|_2 + c_\epsilon\|u_n\|_{p+1}^p\|v_n\|_{p+1} \\ &\le \epsilon M^2 + c_\epsilon M^p\|v_n\|_{p+1}.\end{aligned}$$

From (5.4), we see that

$$\varlimsup_{n\to\infty}\left|\int_{\mathbb{R}^N} f(u_n)v_n\,dx\right|, \ \varlimsup_{n\to\infty}\left|\int_{\mathbb{R}^N} g(u_n)v_n\,dx\right| \le \epsilon M^2.$$

Since $\epsilon > 0$ is arbitrary, we get (5.5). □

Lemma 5.3 ([13, Lemma 1.1]) *Assume* $p \in (1, (N+2)/(N-2))$ *if* $N \ge 3$, $p \in (1, \infty)$ *if* $N = 1, 2$. *Let* $(u_n)_{n=1}^\infty \subset H^1(\mathbb{R}^N)$ *be bounded in* $H^1(\mathbb{R}^N)$ *and satisfy*

$$\sup_{z\in\mathbb{R}^N} \int_{B(z,1)} |u_n|^{p+1}\,dx \to 0 \quad (n \to \infty).$$

Then we have $\|u_n\|_{p+1} \to 0$ *as* $n \to \infty$.

Proof This follows from [13, Lemma 1.1]. □

We prove Theorem 5.1 through several steps.

Proof of Theorem 5.1 Let $(u_n)_{n=1}^{\infty}$ be a sequence satisfying the assumption of Theorem 5.1. In this proof, we repeatedly choose subsequence of $(u_n)_{n=1}^{\infty}$. Thus, for simplicity, we use same notation for subsequence. Since $(u_n)_{n=1}^{\infty}$ is bounded in $H^1(\mathbb{R}^N)$, there exist a subsequence $(u_n)_{n=1}^{\infty}$, $u_0 \in H^1(\mathbb{R}^N)$, and $\beta \geq 0$ such that

$$\begin{aligned} &u_n \to u_0 \quad \text{weakly in } H^1(\mathbb{R}^N), \\ &\frac{1}{2}\|u_n\|^2 + \int_{\mathbb{R}^N} G(u_n) \to \beta. \end{aligned} \tag{5.6}$$

Here we define functionals $\Phi(u)$ and $\Psi(u)$ as follows:

$$\begin{aligned} \Phi(u) &= a_L'(\beta)\left(\frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} G(u)\,dx\right) - \int_{\mathbb{R}^N} F(u) + G(u)\,dx, \\ \Psi(u) &= a_L'(\beta)\left(\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + V_\infty u^2\,dx - \int_{\mathbb{R}^N} G(u)\,dx\right) - \int_{\mathbb{R}^N} F(u) + G(u)\,dx. \end{aligned}$$

Step 1 *Then we have*

$$0 \leq a_L'(\beta) \leq 1, \tag{5.7}$$

$$\Phi'(u_0)\varphi = 0 \quad \textit{for all } \varphi \in H^1(\mathbb{R}^N). \tag{5.8}$$

Proof of Step 1 Since a_L' is continuous, (5.7) follows from (c). From (b), for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned} J_L'(u_n)\varphi &= a_L'\left(\frac{1}{2}\|u_n\|^2 + \int_{\mathbb{R}^N} G(u_n)\,dx\right)\left(\langle u_n, \varphi\rangle + \int_{\mathbb{R}^N} g(u_n)\varphi\,dx\right) \\ &\quad - \int_{\mathbb{R}^N} f(u_n)\varphi + g(u_n)\varphi\,dx \\ &\to a_L'(\beta)\left(\langle u_0, \varphi\rangle + \int_{\mathbb{R}^N} g(u_0)\varphi\,dx\right) - \int_{\mathbb{R}^N} f(u_0)\varphi + g(u_0)\varphi\,dx \\ &= \Phi'(u_0)\varphi = 0. \end{aligned}$$

Thus we get (5.8). □

Step 2 *We set $v_n^1 = u_n - u_0$. Then, either* (A) *or* (B) *holds.*
(A) $(v_n^1)_{n=1}^{\infty}$ *satisfies*

$$\sup_{z\in\mathbb{R}^N} \int_{B(z,1)} |v_n^1|^{p+1}\,dx \to 0 \quad (n \to \infty).$$

Then, we have

$$0 < a_L'(\beta) \leq 1, \tag{5.9}$$

$$\begin{aligned} &u_n \to u_0 \quad \textit{strongly in } H^1(\mathbb{R}^N), \\ &J_L'(u_0)\varphi = 0 \quad \textit{for all } \varphi \in H^1(\mathbb{R}^N). \end{aligned} \tag{5.10}$$

In particular, for $\lambda_0 = a_L'(\beta)^{-1} \geq 1$, u_0 is a solution of (5.2).
(B) *There exist a subsequence $(v_n^1)_{n=1}^{\infty}$ and a sequence $(z_n^1)_{n=1}^{\infty} \subset \mathbb{R}^N$ with $z_n^1 \in B_{R_n}$ for each n such that*

$$\int_{B(z_n^1,1)} |v_n^1|^{p+1}\,dx \to d_1 > 0 \quad (n \to \infty). \tag{5.11}$$

Then, after extracting a subsequence, there exists $\omega^1 \in H^1(\mathbb{R}^N) \setminus \{0\}$ *such that*

$$|z_n^1| \to \infty, \tag{5.12}$$

$$u_n(\cdot + z_n^1) \to \omega^1 \ \text{ weakly in } H^1(\mathbb{R}^N) \text{ and } \|\omega^1\|_{p+1}^{p+1} \geq d_1, \tag{5.13}$$

$$\Psi'(\omega^1)\varphi = 0 \quad \text{for all } \varphi \in H^1(\mathbb{R}^N), \tag{5.14}$$

$$0 < a_L'(\beta) \leq 1. \tag{5.15}$$

In particular, for $\lambda_0 = a_L'(\beta)^{-1} \geq 1$*,* u_0 *is a solution of* (5.2) *and* ω^1 *is a solution of* (5.3)*.*

Proof of Case (A) *of Step* 2. From Lemma 5.3, we have

$$\|v_n^1\|_{p+1} \to 0 \qquad (n \to \infty). \tag{5.16}$$

We suppose by contradiction that $a_L'(\beta) = 0$. From (5.8), we have

$$\begin{aligned} 0 &= \Phi'(u_0)u_0 \\ &= a_L'(\beta)\left(\|u_0\|^2 + \int_{\mathbb{R}^N} g(u_0)u_0\,dx\right) - \int_{\mathbb{R}^N} f(u_0)u_0 + g(u_0)u_0\,dx \\ &= -\int_{\mathbb{R}^N} f(u_0)u_0 + g(u_0)u_0\,dx. \end{aligned}$$

Since $(f(s) + g(s))s > 0$ $(s \neq 0)$ from (g.1) and (g.4), we see that $u_0 = 0$ in $\mathbb{R}^N$. Since $v_n^1 = u_n - u_0 = u_n$, (5.16) implies

$$\|u_n\|_{p+1} \to 0 \qquad (n \to \infty). \tag{5.17}$$

From (a), we have

$$J_L(u_n) = a_L\left(\frac{1}{2}\|u_n\|^2 + \int_{\mathbb{R}^N} G(u_n)\,dx\right) - \int_{\mathbb{R}^N} F(u_n) + G(u_n)\,dx \leq L.$$

Since $\lim_{n\to\infty} \int_{\mathbb{R}^N} F(u_n) + G(u_n)\,dx = 0$ by (5.17), we get $a_L\left(\beta\right) \leq L$. However $a_L'(\beta) = 0$ implies $a_L\left(\beta\right) > L$ by (ii) of Lemma 2.2. This is a contradiction. Thus (5.9) holds. Next, we show (5.10). By Lemma 5.2, we have

$$\begin{aligned} J_L'(u_n)v_n^1 &= a_L'\left(\frac{1}{2}\|u_n\|^2 + \int_{\mathbb{R}^N} G(u_n)\,dx\right)\langle u_n, v_n^1\rangle + o(1), \\ \Phi'(u_0)v_n^1 &= a_L'(\beta)\,\langle u_0, v_n^1\rangle + o(1), \end{aligned}$$

where $o(1) \to 0$ as $n \to \infty$. Thus

$$\begin{aligned} &J_L'(u_n)v_n^1 - \Phi'(u_0)v_n^1 \\ &\quad = a_L'(\beta)\|v_n^1\|^2 + \left\{a_L'\left(\frac{1}{2}\|u_n\|^2 + \int_{\mathbb{R}^N} G(u_n)\,dx\right) - a_L'(\beta)\right\}\langle u_n, v_n^1\rangle + o(1). \end{aligned}$$

From (b) and (5.8), we see that $J_L'(u_n)v_n^1 - \Phi'(u_0)v_n^1 \to 0$. Consequently, we obtain

$$\lim_{n\to\infty} a_L'(\beta)\|v_n^1\|^2 = 0.$$

Since $a_L'(\beta) > 0$, we get (5.10) which implies $a_L'(\beta) = a_L'\left(\frac{1}{2}\|u_0\| + \int_{\mathbb{R}^N} G(u_0)\,dx\right)$. Thus, we have $J_L'(u_0) = \Phi'(u_0)$. From (5.8), we get $J_L'(u_0) = 0$. From (5.9), u_0 is a solution of (5.2) with $\lambda_0 = a_L(\beta)^{-1} \geq 1$. □

Proof of Case (B) *of Step* 2. First, we show (5.12). From (5.6), $v_n^1 = u_n - u_0 \to 0$ weakly in $H^1(\mathbb{R}^N)$. Thus

$$v_n^1 = u_n - u_0 \to 0 \quad \text{strongly in } L^{p+1}_{\text{loc}}(\mathbb{R}^N). \tag{5.18}$$

If $(z_n^1)_{n=1}^\infty$ is bounded, then (5.11) and (5.18) contradict each other. Thus (5.12) holds. Since $(v_n^1(\cdot + z_n^1))_{n=1}^\infty$ is bounded in $H^1(\mathbb{R}^N)$, after extracting a subsequence, there exists $\omega^1 \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that

$$v_n^1\left(\cdot + z_n^1\right) \to \omega^1 \text{ weakly in } H^1(\mathbb{R}^N) \text{ and } \|\omega^1\|_{p+1}^{p+1} \geq d_1.$$

Thus we have

$$u_n\left(\cdot + z_n^1\right) = v_n^1\left(\cdot + z_n^1\right) + u_0\left(\cdot + z_n^1\right) \to \omega^1 \quad \text{weakly in } H^1(\mathbb{R}^N),$$

and (5.13) holds. Here we show that $(z_n^1)_{n=1}^\infty$ satisfies

$$\text{dist}\left(z_n^1, \partial B_{R_n}\right) \to \infty. \tag{5.19}$$

By a contrary, we assume $\underline{\lim}_{n\to\infty} \text{dist}(z_n^1, \partial B_{R_n}) =: r^1 < \infty$. We may also assume $\lim_{n\to\infty} \frac{z_n^1}{|z_n^1|} =: e^1 \in \mathbb{R}^N$. Set $H_1 = \{x \in R^N \mid (x \cdot e^1) < r^1\}$. Then H_1 is a half space in $\mathbb{R}^N$. For any $\varphi \in C_0^\infty(H_1)$, $\varphi(\cdot - z_n^1) \in C_0^\infty(B_{R_n})$ for large n. From (b), we have

$$\begin{aligned}
&J_L'(u_n)\varphi\left(\cdot - z_n^1\right) \\
&= a_L'\left(\frac{1}{2}\|u_n\|^2 + \int_{B_{R_n}} G(u_n)\,dx\right)\left(\langle u_n, \varphi\left(\cdot - z_n^1\right)\rangle + \int_{B_{R_n}} g(u_n)\varphi\left(\cdot - z_n^1\right)\,dx\right) \\
&\quad - \int_{B_{R_n}} f(u_n)\varphi\left(\cdot - z_n^1\right) + g(u_n)\varphi\left(\cdot - z_n^1\right)\,dx \\
&= a_L'\left(\frac{1}{2}\|u_n\|^2 + \int_{B_{R_n}} G(u_n)\,dx\right) \\
&\left(\int_{H_1} \nabla u_n\left(\cdot + z_n^1\right)\cdot\nabla\varphi + V\left(x + z_n^1\right)u_n\left(\cdot + z_n^1\right)\varphi\,dx + \int_{H_1} g\left(u_n\left(\cdot + z_n^1\right)\right)\varphi\,dx\right) \\
&\quad - \int_{H_1} f\left(u_n\left(\cdot + z_n^1\right)\right)\varphi + g\left(u_n\left(\cdot + z_n^1\right)\right)\varphi\,dx \\
&\to a_L'\left(\beta\right)\left(\int_{H_1} \nabla\omega^1\cdot\nabla\varphi + V_\infty\omega^1\varphi\,dx + \int_{H_1} g(\omega^1)\varphi\,dx\right) \\
&\quad - \int_{H_1} f(\omega^1)\varphi + g(\omega^1)\varphi\,dx \\
&= 0.
\end{aligned} \tag{5.20}$$

If $a_L'\left(\beta\right) = 0$, then $\int_{H_1} f(\omega^1)\omega^1 + g(\omega^1)\omega^1\,dx = 0$ that implies $\omega^1 = 0$. This is a contradiction. Thus $a_L'\left(\beta\right) > 0$ and ω^1 is a non-trivial solution of

$$-\Delta u + V_\infty u = \lambda_0 f(u) + (\lambda_0 - 1)g(u) \quad \text{in } H_1 \quad u \in H_0^1(H_1) \tag{5.21}$$

where $\lambda_0 = a_L'(\beta)^{-1} \geq 1$. However, since (5.21) has only a trivial solution by [7], this is a contradiction. Thus (5.19) holds. From (5.19), for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, $\varphi(\cdot + z_n^1) \in C_0^\infty(B_{R_n})$

for large n. By similar calculations to (5.20), we find that

$$J_L'(u_n)\varphi\left(\cdot - z_n^1\right) \to \Psi(\omega^1)\varphi = 0$$

and $a_L'(\beta) > 0$. Thus (5.14) and (5.15) hold. From (5.8), (5.14), and (5.15), u_0 and ω^1 are solutions of (5.2) and (5.3) respectively. □

In Step 2, if the case (A) occurs, Theorem 5.1 holds as $\ell = 0$. If the case (B) occurs, we proceed next step.

Step 3 *We set*

$$v_n^2 = v_n^1 - \omega^1\left(\cdot - z_n^1\right) = u_n - u_0 - \omega^1\left(\cdot - z_n^1\right).$$

Then, either (A) *or* (B) *holds.*
(A) $(v_n^2)_{n=1}^\infty$ *satisfies*

$$\sup_{z\in\mathbb{R}^N}\int_{B(z,1)} |v_n^2|^{p+1}\,dx \to 0 \quad (n\to\infty). \tag{5.22}$$

Then, we have $\|v_n^2\| \to 0$.
(B) *There exist a subsequence* $(v_n^2)_{n=1}^\infty$ *and a sequence* $(z_n^2)_{n=1}^\infty \subset \mathbb{R}^N$ *with* $z_n^2 \in B_{R_n}$ *for each n such that*

$$\int_{B(z_n^2,1)} |v_n^2|^{p+1}\,dx \to d_2 > 0 \quad (n\to\infty). \tag{5.23}$$

Then, after extracting a subsequence, there exists $\omega^2 \in H^1(\mathbb{R}^N)\setminus\{0\}$ *such that*

$$|z_n^2| \to \infty, \quad |z_n^2 - z_n^1| \to \infty, \tag{5.24}$$

$$u_n(\cdot + z_n^2) \to \omega^2 \ \ \textit{weakly in } H^1(\mathbb{R}^N) \textit{ and } \|\omega^2\|_{p+1}^{p+1} \ge d_2, \tag{5.25}$$

$$\Psi'(\omega^2)\varphi = 0 \quad \textit{for all } \varphi \in H^1(\mathbb{R}^N). \tag{5.26}$$

In particular, ω^2 *is a solution of* (5.3) *with* $\lambda_0 = a_L'(\beta)^{-1} \ge 1$.

Proof of Case (A) *of Step* 3. From (5.22) and Lemma 5.3, we see that $\|v_n^2\|_{p+1} \to 0$ as $n\to\infty$. By Lemma 5.2, we have

$$J_L'(u_n)v_n^2 = a_L'\left(\frac{1}{2}\|u_n\|^2 + \int_{\mathbb{R}^N} G(u_n)\,dx\right)\langle u_n, v_n^2\rangle + o(1),$$

$$\Phi'(u_0)v_n^2 = a_L'(\beta)\langle u_0, v_n^2\rangle + o(1),$$

$$\Psi'\left(\omega^1\left(\cdot - z_n^1\right)\right)v_n^2 = a_L'(\beta)\int_{\mathbb{R}^N} \nabla\omega^1\left(\cdot - z_n^1\right)\cdot\nabla v_n^2 + V_\infty\omega^1\left(\cdot - z_n^1\right)v_n^2\,dx + o(1),$$

where $o(1)\to 0$ as $n\to\infty$. Thus

$$\begin{aligned} &J_L'(u_n)v_n^2 - \Phi'(u_0)v_n^2 - \Psi'\left(\omega^1\left(\cdot - z_n^1\right)\right)v_n^2 \\ &\quad = a_L'(\beta)\|v_n^2\|^2 + \left\{a_L'\left(\frac{1}{2}\|u_n\|^2 + \int_{\mathbb{R}^N} G(u_n)\,dx\right) - a_L'(\beta)\right\}\langle u_n, v_n^2\rangle \\ &\qquad + a_L'(\beta)\int_{\mathbb{R}^N}\left(V(x) - V_\infty\right)\omega^1(\cdot - z_n^1)v_n^2\,dx + o(1). \end{aligned}$$

From (b) and (5.13),

$$
\begin{aligned}
J_L'(u_n)\, v_n^2 &= J_L'(u_n)u_n - J_L'(u_n)u_0 - J_L'(u_n)\omega^1\left(\cdot - z_n^1\right) \\
&= -J_L'(u_n)u_0 - J_L'\left(u_n\left(\cdot + z_n^1\right)\right)\omega^1 \\
&\to 0 - \Psi'(\omega_1)\omega_1 = 0 \qquad (n \to \infty).
\end{aligned}
$$

Since $J_L'(u_n)v_n^2 - \Phi'(u_0)v_n^2 - \Psi'(\omega^1(\cdot - z_n^1))v_n^2 \to 0$, we obtain $\lim_{n\to\infty} a_L'(\beta)\|v_n^2\|^2 = 0$. Thus we get $\|v_n^2\| \to 0$. □

Proof of Case (B) *of Step* 3. First, we show (5.24). From (5.6) and (5.13), we see that

$$
\begin{aligned}
v_n^2 &= u_n - u_0 - \omega^1(\cdot - z_n^1) \to 0 \quad \text{weakly in } H^1(\mathbb{R}^N), \\
v_n^2(\cdot + z_n^1) &= u_n(\cdot + z_n^1) - u_0(\cdot + z_n^1) - \omega^1 \to 0 \quad \text{weakly in } H^1(\mathbb{R}^N).
\end{aligned}
$$

Thus we have

$$
v_n^2 \to 0 \quad \text{strongly in } L^{p+1}_{\text{loc}}(\mathbb{R}^N), \tag{5.27}
$$

$$
v_n^2(\cdot + z_n^1) \to 0 \quad \text{strongly in } L^{p+1}_{\text{loc}}(\mathbb{R}^N). \tag{5.28}
$$

If $(z_n^2)_{n=1}^\infty$ is bounded, then (5.23) and (5.27) contradict each other. If $(z_n^2 - z_n^1)_{n=1}^\infty$ is bounded, then (5.23) and (5.28) contradict each other. Thus (5.24) holds. Since $(v_n^2(\cdot + z_n^2))_{n=1}^\infty$ is bounded in $H^1(\mathbb{R}^N)$, after extracting a subsequence, there exists $\omega^2 \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that

$$
v_n^2\left(\cdot + z_n^2\right) \to \omega^2 \quad \text{weakly in } H^1(\mathbb{R}^N) \text{ and } \|\omega^2\|_{p+1}^{p+1} \ge d_2.
$$

Then (5.25) follows from

$$
u_n\left(\cdot + z_n^2\right) = v_n^2\left(\cdot + z_n^2\right) + u_0\left(\cdot + z_n^2\right) - \omega^1\left(\cdot + z_n^2 - z_n^1\right) \to \omega^2 \quad \text{weakly in } H^1(\mathbb{R}^N).
$$

Also, by similar calculations to (5.19), we get $\lim_{n\to\infty} \operatorname{dist}(z_n^2, \partial B_{R_n}) = \infty$. Thus, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, we obtain

$$
J_L'(u_n)\,\varphi\left(\cdot - z_n^2\right) \to \Psi'\left(\omega^2\right)\varphi = 0.
$$

Thus (5.26) holds. □

In Step 3, if the case (A) occurs, Theorem 5.1 holds as $\ell = 1$. If the case (B) occurs, we repeat similar arguments. That is, the following induction holds.

Step 4 *We suppose that there exist a subsequence of* $(u_n)_{n=1}^\infty$, $m \in \mathbb{N} \cup \{0\}$, $u_0 \in H^1(\mathbb{R}^N)$, $\omega^1, \ldots, \omega^m \in H^1(\mathbb{R}^N) \setminus \{0\}$, $(z_n^k)_{n=1}^\infty \subset \mathbb{R}^N$ *with* $z_n^k \in B_{R_n}$, $|z_n^k| \to \infty$ $(k = 1, \ldots, m)$ *and* $|z_n^k - z_n^{k'}| \to \infty$ $(k \ne k')$ *such that*

$$
\begin{aligned}
& u_n \to u_0 \text{ weakly in } H^1(\mathbb{R}^N), \\
& u_n(\cdot + z_n^k) \to \omega^k \text{ weakly in } H^1(\mathbb{R}^N) \text{ and } \|\omega^k\|_{p+1}^{p+1} \ge d_k > 0 \quad (k = 1, \ldots, m), \\
& \Phi'(u_0)\varphi = 0 \text{ and } \Psi'(\omega^k)\varphi = 0 \text{ for all } \varphi \in H^1(\mathbb{R}^N) \quad (k = 1, \ldots, m).
\end{aligned}
$$

We set

$$
v_n^{m+1} = u_n - u_0 - \sum_{k=1}^m \omega^k\left(\cdot - z_n^k\right).
$$

Then, either (A) *or* (B) *holds.*

(A) $(v_n^{m+1})_{n=1}^\infty$ *satisfies*

$$\sup_{z\in\mathbb{R}^N}\int_{B(z,1)}\left|v_n^{m+1}\right|^{p+1}dx\to 0\qquad(n\to\infty).$$

Then, we have $\|v_n^{m+1}\|\to 0$.
(B) *There exist a subsequence* $(v_n^{m+1})_{n=1}^\infty$ *and a sequence* $(z_n^{m+1})_{n=1}^\infty\subset\mathbb{R}^N$ *such that*

$$\int_{B(z_n^{m+1},1)}\left|v_n^{m+1}\right|^{p+1}dx\to d_{m+1}>0\qquad(n\to\infty).$$

Then, after extracting a subsequence, there exists $\omega^{m+1}\in H^1(\mathbb{R}^N)\setminus\{0\}$ *such that*

$$\begin{aligned}&\left|z_n^{m+1}\right|\to\infty,\quad\left|z_n^{m+1}-z_n^k\right|\to\infty\qquad(k=1,\ldots,m),\\&u_n\left(\cdot+z_n^{m+1}\right)\to\omega^{m+1}\ \textit{weakly in } H^1(\mathbb{R}^N)\textit{ and }\left\|\omega^{m+1}\right\|_{p+1}^{p+1}\ge d_{m+1}>0,\\&\Psi'\left(\omega^{m+1}\right)\varphi=0\quad\textit{for all }\varphi\in H^1(\mathbb{R}^N).\end{aligned}$$

In particular, ω^{m+1} *is a solution of* (5.3) *with* $\lambda_0=a_L'(\beta)^{-1}\ge 1$.

Since the proof of Step 4 is almost same as Step 3, we omit it.

As long as the case (B) occurs, we repeat Step 4. If the case (A) occurs, Theorem 5.1 holds as $\ell=m$. Finally, after repeating Step 4 a finite times, we observe that the case (A) always occurs.

Step 5 *When Step 4 is repeated a finite times, the case* (A) *occurs.*

Proof of Step 5 We suppose, by contradiction, that the case (B) of Step 4 repeated infinite time. Then, there exist a subsequence $(u_n)_{n=1}^\infty$, $u_0\in H^1(\mathbb{R}^N)$, $(\omega^k)_{k=1}^\infty\subset H^1(\mathbb{R}^N)\setminus\{0\}$, $(z_n^k)_{n=1}^\infty\subset\mathbb{R}^N$ with $|z_n^k|\to\infty$ $(k\in\mathbb{N})$ and $|z_n^k-z_n^{k'}|\to\infty$ $(k\neq k')$ such that

$$u_n\to u_0 \text{ weakly in } H^1(\mathbb{R}^N),\tag{5.29}$$

$$\begin{aligned}&u_n\left(\cdot+z_n^k\right)\to\omega^k\text{ weakly in }H^1(\mathbb{R}^N)\text{ and }\left\|\omega^k\right\|_{p+1}^{p+1}\ge d_k>0\quad(k\in\mathbb{N}),\\&\Phi'(u_0)\varphi=0\text{ and }\Psi'\left(\omega^k\right)\varphi=0\text{ for all }\varphi\in H^1(\mathbb{R}^N)\quad(k\in\mathbb{N}).\end{aligned}\tag{5.30}$$

From (5.29) and (5.30), for any $m\in\mathbb{N}$, we see that

$$\begin{aligned}0&\le\overline{\lim_{n\to\infty}}\left\|u_n-u_0-\sum_{k=1}^{m-1}\omega^k\left(\cdot-z_n^k\right)\right\|_{H^1(\mathbb{R}^N)}^2\\&=\overline{\lim_{n\to\infty}}\|u_n\|_{H^1(\mathbb{R}^N)}^2-\|u_0\|_{H^1(\mathbb{R}^N)}^2-\sum_{k=1}^{m-1}\|\omega^k\|_{H^1(\mathbb{R}^N)}^2,\end{aligned}$$

where $\|u\|_{H^1(\mathbb{R}^N)}^2=\|\nabla u\|_2^2+\|u\|_2^2$ which is equivalent to $\|\cdot\|$. Thus we have

$$\sum_{k=1}^{\infty}\|\omega^k\|_{H^1(\mathbb{R}^N)}^2\le\overline{\lim_{n\to\infty}}\|u_n\|_{H^1(\mathbb{R}^N)}^2-\|u_0\|_{H^1(\mathbb{R}^N)}^2<\infty.$$

On the other hand, from (f.2) and (g.2), 0 is an isolated critical point of $\Psi(u)$. Thus there exists $\delta_\beta > 0$ such that $\|\omega^k\| \geq \delta_\beta$ $(k \in \mathbb{N})$. This is a contradiction. Thus (B) of Step 4 is not repeated infinite time. □

Through Step 1 to Step 5, the proof of Theorem 5.1 was completed. □

Remark 5.4 In Theorem 5.1, we also have $J_L(u_n) \to \Phi(u_0) + \sum_{k=1}^{\ell} \Psi(\omega^k)$.

6 The compactness for balanced sequence

In this section, we prove Proposition 2.14. The fundamental idea of the proof comes from [3, Proposition 4.1].

Proof of Proposition 2.14 Let $u_n \in H_0^1(B_{R_n})$ satisfy (2.17). Then, $(u_n)_{n=1}^{\infty}$ satisfies the assumptions of Theorem 5.1. It is sufficient to show that, adding assumption (V.4) to Theorem 5.1, then only $\ell = 0$ occurs. Suppose, by contradiction, that Theorem 5.1 holds for $\ell \geq 1$. Since $u_n = 0$ in $\mathbb{R}^N \setminus B_{R_n}$, choosing a subsequence (we use same notation) and replacing the order $k = 1, \dots, \ell$, we may assume $|z_n^1| \leq |z_n^2| \leq \cdots \leq |z_n^\ell| \leq R_n$. Furthermore, choosing a subsequence, we can also assume that there exist $d_k \in [0, \infty]$ with

$$\lim_{n\to\infty} \frac{\left|z_n^k - z_n^1\right|}{\left|z_n^1\right|} = d_k \quad (k = 1, 2, \dots, \ell).$$

We set r_n and $d > 0$ such that

$$11 r_n = |z_n^1|, \quad d = \min\left\{ 1,\ 11\rho/6,\ d_k \middle| d_k > 0 \right\}.$$

Here ρ is a constant defined by (V.4). Then, for large n, it holds the following.

(i) If $d_k = 0$, then $B(z_n^k, dr_n) \subset B(z_n^1, 2dr_n)$.
(ii) If $d_k > 0$, then $B(z_n^k, dr_n) \subset \mathbb{R}^N \setminus B(z_n^1, 9dr_n)$.

From (i) of Theorem 5.1, we see that

$$\|u_n\|_{L^{p+1}(B(z_n^1, 9dr_n) \setminus B(z_n^1, 2dr_n))} \to 0. \tag{6.1}$$

Since $u_n \in H_0^1(B_{R_n})$ is a solution of

$$-\Delta u_n + V(x) u_n = \lambda_0 f(u_n) + (\lambda_0 - 1) g(u_n) \quad \text{in } B_{R_n},$$

$|u_n| \in H^1(\mathbb{R}^N)$ (expanding 0 on $\mathbb{R}^N \setminus B_{R_n}$) is a subsolution of

$$-\Delta u_n + V(x) u_n = \lambda_0 f(u_n) + (\lambda_0 - 1) g(u_n) \quad \text{in } \mathbb{R}^N. \tag{6.2}$$

Here we use the subsolution estimate below.

Lemma 6.1 *Let Ω be a domain and $\widetilde{V} \in L_{\text{loc}}^{\frac{p+1}{p-1}}(\Omega)$. Suppose that $u \in H^1(\Omega)$ satisfies*

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx - \widetilde{V}(x) u \varphi \, dx \leq 0 \quad \textit{for all } \varphi \in H_0^1(\Omega) \textit{ with } \varphi \geq 0.$$

Then, for any $B(x_0, 2r) \subset \Omega$, there exist constants $C = C(p, N, r) > 0$ and $\sigma = \sigma(p, N) > 0$ such that

$$\|u_+\|_{L^\infty(B(x_0, r))} \leq C \left(1 + \|\widetilde{V}_+\|^\sigma_{L^{\frac{p+1}{p-1}}(B(x_0, 2r))} \right) \|u_+\|_{L^{p+1}(B(x_0, 2r))}.$$

Proof This was shown in [15, Theorem 2.26]. (Also see [8, Theorem 8.15].) □

From (6.1) and Lemma 6.1, we see that

$$\|u_n\|_{L^\infty(B(z_n^1,8dr_n)\setminus B(z_n^1,3dr_n))} \to 0. \tag{6.3}$$

Since $|u_n|$ is a subsolution of (6.2), from (6.3), (f.2), and (g.2), by the comparison theorem, there exist constants $C > 0$ and $\mu > 0$ such that

$$\|u_n\|_{L^\infty(B(z_n^1,7dr_n)\setminus B(z_n^1,4dr_n))} \le Ce^{-\mu r_n}. \tag{6.4}$$

Furthermore, replacing $C > 0$ and $\mu > 0$, we also have

$$\|u_n\|_{L^2(B(z_n^1,7dr_n)\setminus B(z_n^1,4dr_n))} \le Ce^{-\mu r_n}, \tag{6.5}$$

$$\|u_n\|_{L^{p+1}(B(z_n^1,7dr_n)\setminus B(z_n^1,4dr_n))} \le Ce^{-\mu r_n}. \tag{6.6}$$

Lemma 6.2 *There exist constants $C' > 0$ and $\mu' > 0$ such that*

$$\big\|\nabla|u_n|\big\|_{L^2(B(z_n^1,6dr_n)\setminus B(z_n^1,5dr_n))} \le C'e^{-\mu' r_n}.$$

Proof From (f.1)–(f.3) and (g.1)–(g.3), there exists $c_1 > 0$ such that

$$|\lambda_0 f(u) + (\lambda_0 - 1)g(u)| \le |u| + c_1|u|^p \quad \text{for all } u \in \mathbb{R}. \tag{6.7}$$

We take a cut-off function $\psi_n \in C_0^\infty(\mathbb{R}^n, [0,1])$ satisfying

$$\begin{aligned} &\psi_n(x) = 1 \quad \text{for } x \in B(z_n^1, 6dr_n) \setminus B(z_n^1, 5dr_n),\\ &\psi_n(x) = 0 \quad \text{for } x \notin B(z_n^1, 7dr_n) \setminus B(z_n^1, 4dr_n),\\ &|\nabla\psi_n(x)| \le 1 \quad \text{for } x \in \mathbb{R}^N. \end{aligned}$$

Since $|u_n|$ is a subsolution of (6.2), for any $\varphi \in H^1(\mathbb{R}^N)$ with $\varphi \ge 0$, we have

$$\int_{\mathbb{R}^N} \nabla|u_n|\cdot\nabla\varphi + V(x)|u_n|\varphi - \lambda_0 f(u_n)\varphi - (\lambda_0 - 1)g(u_n)\varphi\, dx \le 0.$$

Setting $\varphi = |u_n|\psi_n$ in the above, from (6.7), we have

$$\begin{aligned} &\int_{B(z_n^1,7dr_n)\setminus B(z_n^1,4dr_n)} |\nabla|u_n||^2\psi_n + \big(\nabla|u_n|\cdot\nabla\psi_n\big)\,|u_n| - V(x)|u_n|^2\psi_n\, dx\\ &\le \int_{B(z_n^1,7dr_n)\setminus B(z_n^1,4dr_n)} |u_n|^2\psi_n + c_1|u_n|^{p+1}\psi_n\, dx. \end{aligned}$$

Thus we get

$$\begin{aligned} &\int_{B(z_n^1,6dr_n)\setminus B(z_n^1,5dr_n)} |\nabla|u_n||^2\, dx \le \int_{B(z_n^1,7dr_n)\setminus B(z_n^1,4dr_n)} |\nabla|u_n||^2\psi_n\, dx\\ &\le \|\nabla|u_n|\|_{L^2(B(z_n^1,7dr_n)\setminus B(z_n^1,4dr_n))}\|u_n\|_{L^2(B(z_n^1,7dr_n)\setminus B(z_n^1,4dr_n))}\\ &\quad + \big(\|V\|_\infty + 1\big)\,\|u_n\|^2_{L^2(B(z_n^1,7dr_n)\setminus B(z_n^1,4dr_n))} + c_1\|u_n\|^{p+1}_{L^{p+1}(B(z_n^1,7dr_n)\setminus B(z_n^1,4dr_n))}. \end{aligned}$$

Lemma 6.2 follows from (6.5) and (6.6). □

Lemma 6.3 *There exist constants $C'' > 0$, $\mu'' > 0$, and $s_n \in (5dr_n, 6dr_n)$ such that*

$$\|\nabla u_n\|_{L^2(\partial B(z_n^1,s_n))} = \big\|\nabla|u_n|\big\|_{L^2(\partial B(z_n^1,s_n))} \le C''e^{-\mu'' r_n}. \tag{6.8}$$

Proof From Lemma 6.2, it holds that

$$\begin{aligned}\left(C'\right)^2 e^{-2\mu' r_n} &\geq \left\|\nabla |u_n|\right\|^2_{L^2(B(z_n^1,6dr_n)\setminus B(z_n^1,5dr_n))}\\ &= \int_{5dr_n\leq|x-z_n^1|\leq 6dr_n} |\nabla|u_n||^2\,dx\\ &= \int_{5dr_n}^{6dr_n}\left(\int_{|x-z_n^1|=r} |\nabla|u_n||^2\,d\sigma\right)dr\\ &= \int_{5dr_n}^{6dr_n} \|\nabla|u_n|\|^2_{L^2(\partial B(z_n^1,r))}\,dr.\end{aligned}$$

Since $r\mapsto \|\nabla|u_n|\,\|_{L^2(\partial B(z_n^1,r))}$ is continuous, by the mean value theorem for integration, there exists $s_n\in(5dr_n,6dr_n)$ such that

$$\left(6dr_n-5dr_n\right)\|\nabla|u_n|\|^2_{L^2(\partial B(z_n^1,s_n))}\leq (C')^2e^{-2\mu' r_n}.$$

Thus we see that Lemma 6.3 holds. □

Here, we use the following local Pohozaev identity.

Lemma 6.4 ([3]) *Let $\Omega\subset\mathbb{R}^N$ be a bounded domain with piecewise smooth boundary and $\nu\in\mathbb{R}^N$ be the outward unit normal vector on $\partial\Omega$. We suppose that $V\in C(\mathbb{R}^N,\mathbb{R})$ and $h\in C(\mathbb{R},\mathbb{R})$. If $u\in C^2(\Omega)\cap C^1(\overline{\Omega})$ satisfies*

$$-\Delta u+V(x)u=h(u)\quad in\ \Omega, \tag{6.9}$$

then, for any $\xi\in\mathbb{R}^N$, it holds that

$$\begin{aligned}\frac{1}{2}\int_\Omega\left(\xi\cdot\nabla V(x)\right)u^2\,dx &= \frac{1}{2}\int_{\partial\Omega}(\xi\cdot\nu)|\nabla u|^2\,d\sigma-\int_{\partial\Omega}(\nabla u\cdot\nu)(\xi\cdot\nabla u)\,d\sigma\\ &\quad-\int_{\partial\Omega}(\xi\cdot\nu)\left(V(x)\frac{u^2}{2}-H(u)\right)d\sigma,\end{aligned}\tag{6.10}$$

where $H(u)=\int_0^u h(\tau)\,d\tau$.

Proof Multiplying $(\xi\cdot\nabla u)$ to the both sides of (6.9), integrating over Ω, we get (6.10). (Also see [3, Lemma 4.1].) □

Applying Lemma 6.4 to u_n as $\Omega=B(z_n^1,s_n)\cap B_{R_n}$, $h(u)=\lambda_0 f(u)+(\lambda_0-1)g(u)$, and $\xi=z_n^1$, we calculate as below.

$$\begin{aligned}&\frac{1}{2}\int_{B(z_n^1,s_n)\cap B_{R_n}}\left(z_n^1\cdot\nabla V(x)\right)u_n^2\,dx\\ &\quad=\frac{1}{2}\int_{\Gamma_1\cup\Gamma_2}\left(z_n^1\cdot\nu\right)|\nabla u_n|^2\,d\sigma-\int_{\Gamma_1\cup\Gamma_2}(\nabla u_n\cdot\nu)\left(z_n^1\cdot\nabla u_n\right)d\sigma\\ &\qquad-\int_{\Gamma_1\cup\Gamma_2}\left(z_n^1\cdot\nu\right)\left(V(x)\frac{u_n^2}{2}-\lambda_0F(u_n)-(\lambda_0-1)G(u_n)\right)d\sigma\\ &\quad=:\int_{\Gamma_1}(\mathrm{I})\,d\sigma+\int_{\Gamma_2}(\mathrm{II})\,d\sigma,\end{aligned}\tag{6.11}$$

where $\Gamma_1 = (\partial B(z_n^1, s_n)) \cap B_{R_n}$, $\Gamma_2 = B(z_n^1, s_n) \cap (\partial B_{R_n})$. We note $s_n \le 6dr_n < 11r_n = |z_n^1| \le R_n$. Since $\nu = \nu(x) \in \mathbb{R}^N$ is the outward unit normal vector at $x \in \partial\Omega$, $\nu(x) = x$ on Γ_2. Thus we have

$$\left(z_n^1 \cdot \nu(x)\right) > 0 \quad \text{for all } x \in \Gamma_2. \tag{6.12}$$

Moreover, since $u_n = 0$ and $\nu = -\frac{\nabla u_n}{|\nabla u_n|}$ on Γ_2, we see that

$$\int_{\Gamma_2} (\mathrm{II})\, d\sigma = -\frac{1}{2}\int_{\Gamma_2} \left(z_n^1 \cdot \nu\right) |\nabla u_n|^2\, d\sigma \le 0.$$

From (6.4) and (6.8), for $\mu_0 \in (0, 2\min\{\mu, \mu''\})$, we see that

$$\overline{\lim_{n\to\infty}}\, e^{\mu_0 r_n} \times \left|\int_{\Gamma_1} (\mathrm{I})\, d\sigma\right| < \infty.$$

Thus we have

$$\overline{\lim_{n\to\infty}}\, e^{\mu_0 r_n} \int_{B(z_n^1, s_n)\cap B_{R_n}} \left(z_n^1 \cdot \nabla V(x)\right) u_n^2\, dx < \infty. \tag{6.13}$$

On the other hand, from (i) of Theorem 5.1, we have

$$\underline{\lim_{n\to\infty}} \int_{B(z_n^1, s_n)\cap B_{R_n}} u_n^2\, dx \ge \|\omega^1\|_{L^2(\mathbb{R}^N)}^2 > 0.$$

From (V.4) and $s_n < 6dr_n \le \rho|z_n^1|$, the left hand side of (6.11) satisfies

$$\begin{aligned} &e^{\mu_0 r_n} \frac{1}{2}\int_{B(z_n^1, s_n)\cap B_{R_n}} \left(z_n^1 \cdot \nabla V(x)\right) u_n^2\, dx \\ &\quad \ge \frac{1}{2} e^{\mu_0 |z_n^1|/11} \left(\inf_{x\in B(z_n^1, \rho|z_n^1|)} \left(z_n^1 \cdot \nabla V(x)\right)\right) \int_{B(z_n^1, s_n)\cap B_{R_n}} u_n^2\, dx \\ &\quad \to \infty \qquad (n \to \infty). \end{aligned}$$

This contradicts (6.13). Consequently, we see that $\ell = 0$, and Proposition 2.14 was proved. □

At the last, we give outline of the proof of Theorem 1.4.

Outline of proof of Theorem 1.4 In order to prove Theorem 1.1, we used the approximating problem on B_R. But, even if we approximate by a problem on Ω_R, the proof of Theorem 1.1 is exactly the same if (6.12) hold. Moreover, (6.12) holds by the assumption (Ω). Indeed, by (Ω), there exist $\delta, C > 0$ such that

$$\Omega \subset B_C \quad \text{and} \quad y \cdot \nu_\Omega(x) > 0 \text{ for } x \in \partial\Omega,\ |x - y| < \delta, \tag{6.14}$$

where $\nu_\Omega(x)$ is the outward unit normal vector of Ω at x. We take d with $Cd < \delta$. Since $z_n^1 \in B_{CR_n}$, we obtain

$$s_n \le 6dr_n < \frac{\delta|z_n^1|}{C} \le \delta R_n.$$

Thus, for $x \in \Gamma_2 = B(z_n^1, s_n) \cap \partial\Omega_{R_n}$, we have

$$\frac{1}{R_n}\left(z_n^1 \cdot \nu_{\Omega_{R_n}}(x)\right) = \frac{z_n^1}{R_n} \cdot \nu_\Omega\left(\frac{x}{R_n}\right), \quad \left|\frac{z_n^1}{R_n} - \frac{x}{R_n}\right| \le \frac{s_n}{R_n} \le \delta. \tag{6.15}$$

(6.12) follows from (6.14) and (6.15). Thus, we can prove Theorem 1.4 in the same way as Remark 3.2. □

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