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Calculus of Variations

The influence of a metasolution on the behaviour of the logistic equation with nonlocal diffusion coefficient

Tarcyana S. Figueiredo-Sousa[1] · **Cristian Rodrigo-Morales**[1] · **Antonio Suárez**[1]

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Abstract In this paper we use the bifurcation method and fixed point arguments to study a logistic equation with nonlocal diffusion coefficient. We prove the existence of an unbounded continuum of positive solutions that bifurcates from the trivial solution. The global behaviour of this continuum depends strongly on the value of the nonlocal diffusion coefficient at infinity as well as the relative position between the refuge of the species and the weight of the diffusion coefficient. Moreover, we show the complexity of the structure of the set of positive solutions using fixed point arguments.

Mathematics Subject Classification 35B09 · 35B32 · 35J60

1 Introduction

In this paper we study the following nonlocal elliptic equation

$$\begin{cases} -a\left(\displaystyle\int_\Omega q(x)u^p\right)\Delta u = \lambda u - b(x)u^2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded and regular domain, $p > 0$, $\lambda \in \mathbb{R}$, $b \in C^1(\overline{\Omega})$, is a non-negative and non-trivial function, $a \in C(\mathbb{R})$ is a positive function and $q(x)$ is a bounded,

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✉ Antonio Suárez
suarez@us.es

Tarcyana S. Figueiredo-Sousa
tarfig@alum.us.es

Cristian Rodrigo-Morales
cristianm@us.es

[1] Dpto. de Ecuaciones Diferenciales y Análisis Numérico, Fac. de Matemáticas, Univ. de Sevilla, Calle Tarfia s/n, Seville, Spain

non-negative and non-trivial function in Ω. This equation models the behavior of a species, whose population density is u and its habitat is Ω. We are assuming that Ω is surrounded by inhospitable areas due to homogeneous Dirichlet boundary condition. The reaction term is the classical logistic term: λ denotes the growth rate of the species and b represents the limiting effects of crowding in the population u. We will consider two cases:

(Hb_1) $b(x) \geq b_0 > 0, \quad \forall x \in \overline{\Omega}$, for some positive constant $b_0 > 0$, or
(Hb_2) $b(x) \geq 0$ in Ω and $b(x) \equiv 0, \forall x \in \Omega_0 \subset\subset \Omega$, where Ω_0 is a proper subdomain of Ω.

For simplicity, we assume that Ω_0 has only one connected component. In the first case, this limiting effect acts in all the domain. However, when b verifies (Hb_2), there is a region, Ω_0, where the species grows freely, this set is called a *refuge* for u.

Finally, in (1) the velocity of the diffusion, the spatial movement of the species, is non-local; that is, it depends on the total population in its habitat. Hence, if a is an increasing function, the species has the tendency to leave crowded zones, while if a is decreasing this means that the species is attracted by the growing population, see for instance [2]. We will assume that a is continuous and positive function defined on $\mathbb{R}$, $a(s) > 0$ for all $s \in \mathbb{R}$, that satisfies

$$0 \leq a_L \leq a(s) \leq a_M \leq +\infty, \quad \text{for } s \in \mathbb{R}, \quad a(0) > 0,$$

where

$$a_L = \inf_{s\in[0,\infty)} a(s), \quad a_M = \sup_{s\in[0,\infty)} a(s),$$

and we define

$$a(\infty) := \lim_{s\to\infty} a(s).$$

Also, to simplify some of the proofs of the work, we will assume that $a'(s) = 0$ for s, at most, in a discrete set of $\mathbb{R}$.

In order to state our main results we need to introduce some notations. Given a domain $D \subset \Omega$ we denote by λ_1^D the principal eigenvalue of the Laplacian operator under homogeneous Dirichlet boundary conditions. We denote $\lambda_1 = \lambda_1^{\Omega}$.

In the local case, that is, $a \equiv 1$, the main results are (see Sect. 2 and [8]):

1. If b verifies (Hb_1), then there exists a positive solution if and only if $\lambda > \lambda_1$. In such case, the positive solution is unique.
2. If b verifies (Hb_2), then there exists a positive solution if and only if $\lambda \in (\lambda_1, \lambda_1^{\Omega_0})$. In this case, the positive solution is unique, and if we denote it by u_λ, it verifies

$$\lim_{\lambda\to\lambda_1^{\Omega_0}} u_\lambda = \mathcal{M}(x),$$

where $\mathcal{M}$ is the "metasolution", that is,

$$\mathcal{M}(x) = \begin{cases} +\infty & \text{in } \overline{\Omega}_0, \\ L(x) & \text{in } \Omega\backslash\overline{\Omega}_0, \end{cases}$$

and L is the minimal large solution (see Sect. 2) of

$$\begin{cases} -\Delta L(x) = \lambda_1^{\Omega_0} L(x) - b(x)L(x)^2 & \text{in } \Omega\backslash\overline{\Omega}_0, \\ L(x) = 0 & \text{on } \partial\Omega, \\ L(x) = \infty & \text{on } \partial\Omega_0. \end{cases}$$

In both cases, if the growth rate is small, then the unique solution is the trivial one. Moreover, when the refuge exists, for large value of the growth rate, the species does not exist because it blows up in Ω_0.

Now, we recall the known-results of (1). First, it should be noted that Eq. (1) has been studied only for b verifying (Hb_1) in [3,4] and [10] (see also [12]). In [4] it was proved the existence of positive solution for λ large when

$$0 < a_L \le a(s) \le a_M < \infty.$$

This result was improved in [3] showing the existence of positive solution for $\lambda > a_M\lambda_1$. In both papers, the Schauder fixed point theorem was used. In [10], the bifurcation method was employed to study (1) when

$$a(s) = c_1 + c_2 s^\alpha, \quad c_i > 0,\ \alpha > 0.$$

For this particular choice of the function a, the authors in [10] proved the existence of positive solution for $\lambda > c_1\lambda_1$ provided that $2 > \max\{\alpha p + 1, p - 1\}$.

Our results improve the above results. In addition, we assume nor $a_L > 0$ neither $a_M < \infty$. Moreover, we do not impose any restriction in p. Furthermore, we study the case (Hb_2), that, to our knowledge, is new in the literature. In fact, the structure of the set of positive solutions could be really complex (see for instance Figs. 5 and 6, that sketch examples of bifurcation diagrams).

Our first main result is concerning to the case b verifying (Hb_1). In this case, the result is analogous to the local case:

1. If b verifies (Hb_1), then there exists a positive solution if $\lambda > a(0)\lambda_1$. Moreover, if b is constant and a is increasing, then the positive solution is unique.

However, when b verifies (Hb_2) the results are completely different to the local case. In fact, the results depend strongly on the behaviour of the function a and on the following integral

$$I := \int_\Omega q(x)\mathcal{M}^p(x)dx.$$

Hence, we can summarize our main results in this case: Assume that b verifies (Hb_2):

1. There exists an unbounded continuum $\mathcal{C}$ in $\mathbb{R} \times L^\infty(\Omega)$ of positive solutions of (1) emanating from the trivial solution at $\lambda = a(0)\lambda_1$.
2. If $I = \infty$, then there exists a positive solution if
$$\lambda \in \left(\min\left\{a(0)\lambda_1, a(\infty)\lambda_1^{\Omega_0}\right\}, \max\left\{a(0)\lambda_1, a(\infty)\lambda_1^{\Omega_0}\right\}\right).$$
In this case, the values $a(\infty) = 0$ and $a(\infty) = \infty$ are allowed. In fact, the continuum $\mathcal{C}$ goes to infinity at $\lambda = a(\infty)\lambda_1^{\Omega_0}$.
3. Assume that $I < \infty$ and consider the real equation
$$g(s) := \frac{s}{a^p(s)} = I. \tag{2}$$
   (a) Assume that (2) does not have positive solution, that is, $g(s) < I$ for all $s \ge 0$. Then, (1) possesses at least one positive solution for $\lambda \in (a(0)\lambda_1, \infty)$.
   (b) Assume that there exist $s_1 < s_2 < \cdots < s_m$, $m \ge 1$, simple roots of (2) and consider
$$\Lambda_i = \lambda_1^{\Omega_0}\left(\frac{s_i}{I}\right)^{1/p}, \quad i = 1, \ldots, m.$$

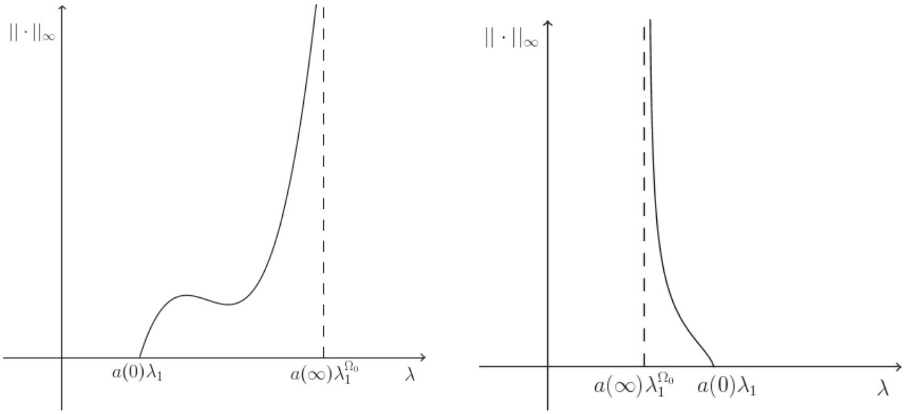

Fig. 1 Bifurcation diagrams when b verifies (Hb_2), $I = \infty$ and $0 < a(\infty) < \infty$. In the left, $a(0)\lambda_1 < a(\infty)\lambda_1^{\Omega_0}$ and in the right $a(\infty)\lambda_1^{\Omega_0} < a(0)\lambda_1$

Then:

i. The unbounded continuum $\mathcal{C}$ of positive solutions of (1) goes to infinity at $\lambda = \Lambda_1$. As consequence, there exists at least one positive solution if

$$\lambda \in (\min\{a(0)\lambda_1, \Lambda_1\}, \max\{a(0)\lambda_1, \Lambda_1\}).$$

ii. If $m = 2k + 1$, $k \geq 0$, (1) possesses at least a positive solution for

$$\lambda \in \bigcup_{j=1}^{k} (\Lambda_{2j}, \Lambda_{2j+1}),$$

and (1) does not have positive solution for λ large.

iii. If $m = 2k$, $k \geq 1$, (1) possesses at least a positive solution for

$$\lambda \in \bigcup_{j=1}^{k-1} (\Lambda_{2j}, \Lambda_{2j+1}) \cup (\Lambda_{2k}, \infty).$$

In Figs. 1, 2, 3, 4, 5 and 6 we show different possibilities of the bifurcation diagrams.

Moreover, we have studied in detail the bifurcation direction from the trivial solution (Sect. 4) and we have detailed the cases when a is increasing and a decreasing, showing the bifurcations diagrams as well as the uniqueness of positive solution.

Let us compare our main results with the local case:

1. In the case that b verifies (Hb_1) the existence results are very similar (in both case there exists positive solution for $\lambda > a(0)\lambda_1$); however, in the nonlocal case there may be multiplicity of positive solutions, in fact, thanks to the bifurcation direction, we can assure the existence of two positive solutions in some cases.
2. Assume that b verifies (Hb_2). In this case, we distinguish two cases: $I = \infty$ and $I < \infty$.

   (a) $I = \infty$. This case occurs when $Q_+ := \{x \in \Omega : q(x) > 0\} \cap \Omega_0 \neq \emptyset$, that is, the nonlocal diffusion coefficient takes into account the refuge of the species. Unlike the local case, it can exist positive solution for λ small (if the diffusion for large values of the population is small, $a(\infty) = 0$) or for λ large (if the diffusion for large values of the population is large, $a(\infty) = \infty$).

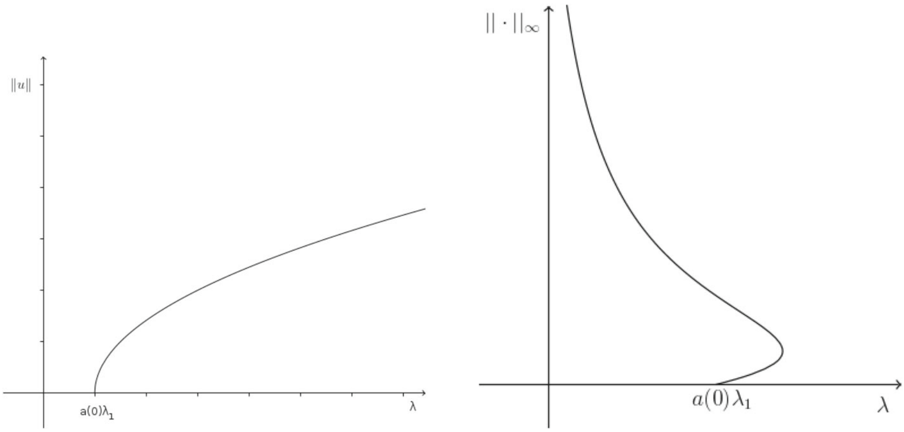

Fig. 2 Bifurcation diagrams in the case when b verifies (Hb_2) and $I = \infty$. At the left $a(\infty) = \infty$, at the right $a(\infty) = 0$. The first diagram also appears when b verifies (Hb_1)

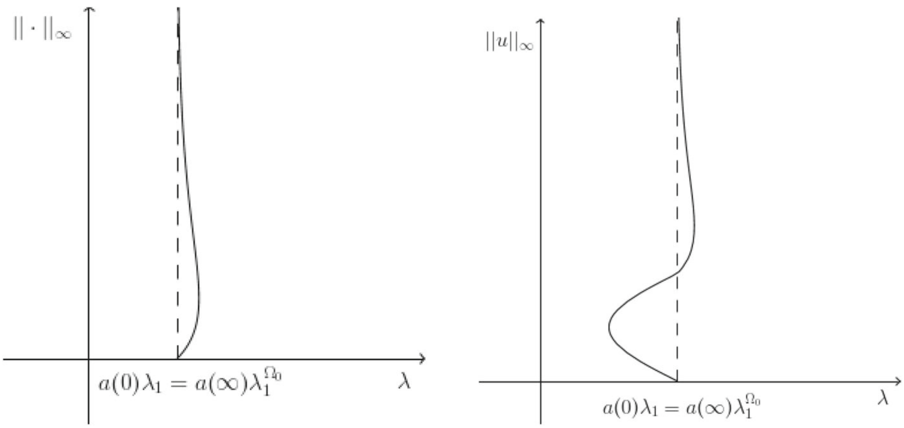

Fig. 3 Possible bifurcation diagrams when b verifies (Hb_2), $I = \infty$ and $a(0)\lambda_1 = a(\infty)\lambda_1^{\Omega_0}$. In the first case, the bifurcation direction is supercritical, while in the second one it is subcritical

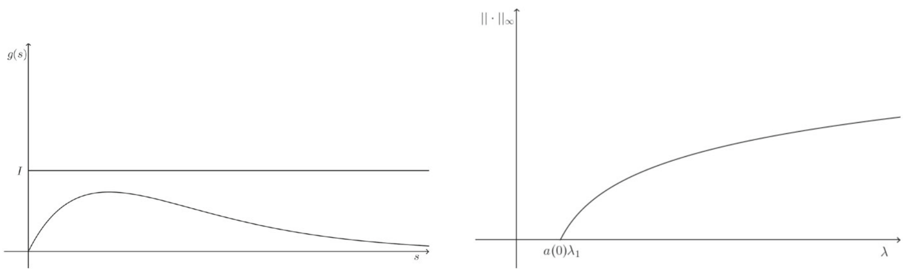

Fig. 4 Case b verifies (Hb_2) and $I < \infty$ and $g(s) < I$. To the left: representation of the function $g(s)$ and to the right, its corresponding bifurcation diagram

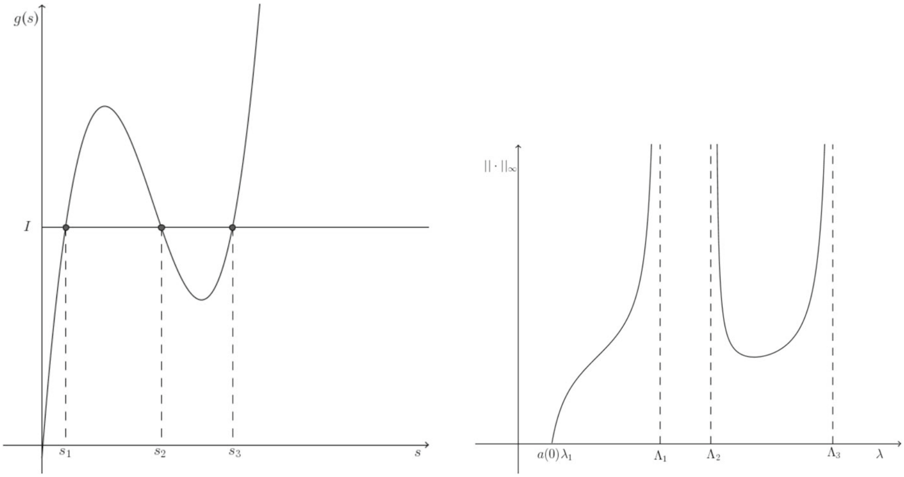

Fig. 5 Case when b verifies (Hb_2), $I < \infty$ and $g(s) = I$ in an odd number of points. Representation of g and its corresponding bifurcation diagram in the case $a(0)\lambda_1 < \Lambda_1$

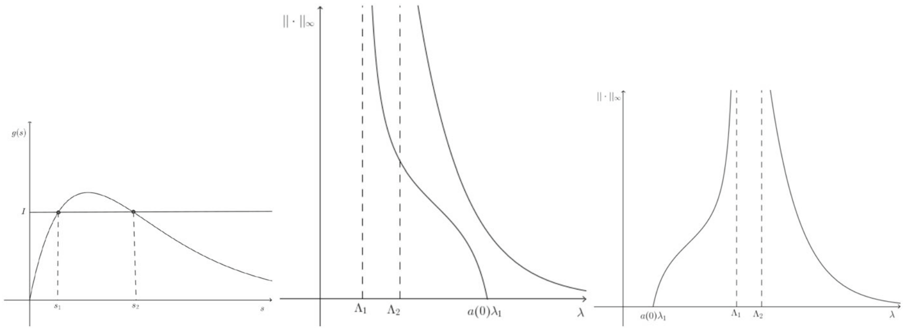

Fig. 6 Case when b verifies (Hb_2), $I < \infty$ and $g(s) = I$ in an even number of points. Representation of g, and corresponding bifurcation diagrams: in the first case $a(0)\lambda_1 > \Lambda_2$ and in the second one, $a(0)\lambda_1 < \Lambda_1$

(b) $I < \infty$. This case occurs when $Q_+ \cap \Omega_0 = \emptyset$, that is, the refuge of the species is not seen by the diffusion. In this case, the structure of the set of positive solutions is complex, but in no case there exists positive solution for λ small. Moreover, in order to exist positive solutions for λ large is necessary that a goes to infinity faster than $s^{1/p}$. On the contrary, positive solutions do not exist for λ large.

The structure of the paper is as follows. First, in Sect. 2, we will introduce some basic notations and terminology necessary along the paper. Section 3 is devoted to recall the local case. In Sect. 4, we apply the bifurcation method to (1). We prove the existence of an unbounded continuum of positive solutions of (1) emanating from the trivial solution at $\lambda = a(0)\lambda_1$. Moreover, we study in detail the direction of this bifurcation. Finally, in Sect. 5 we prove the main results of our paper. For that, we employ an adequate fixed point argument together the bifurcation results.

2 Preliminaries

2.1 An eigenvalue problem

We start the section with some results related to eigenvalue problems that will be needed throughout this paper. Consider the following eigenvalue problem

$$\begin{cases} -d\Delta u + c(x)u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $c \in L^\infty(\Omega)$ and $d > 0$ is a positive constant.

We denote the principal eigenvalue of (3) by $\lambda_1[-d\Delta + c]$, that is, an eigenvalue with a positive eigenfunction associated to it.

Moreover, given a subdomain $D \subseteq \Omega$, we denote $\lambda_1^D[-d\Delta+c]$ as the principal eigenvalue in D. In the case $c \equiv 0$ and $d = 1$, we denote $\lambda_1^D = \lambda_1^D[-\Delta]$. When $D = \Omega$, we delete the superscript to avoid confusion.

The following result shows some properties of the principal eigenvalue, which are direct consequences of its variational characterization, see for instance [9].

Proposition 1 *We have the following properties:*

1. *The map $d \in (0, \infty) \mapsto \lambda_1[-d\Delta + c] \in \mathbb{R}$ is continuous and increasing.*
2. *The map $c \in L^\infty(\Omega) \mapsto \lambda_1[-d\Delta + c] \in \mathbb{R}$ is continuous and increasing.*
3. *If $D \subset \Omega$ is a subdomain, then $\lambda_1[-d\Delta + c] < \lambda_1^D[-d\Delta + c]$.*

2.2 Sub-supersolution method for nonlocal equation

We introduce the sub-supersolution method for a nonlocal equation of the following general form

$$\begin{cases} -a\left(\int_\Omega q(x)u^p\right)\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function. There are different definitions of sub-supersolution of (4) depending on the properties of a and f, see for instance [1]. We present here the sub-supersolution method of [5], Sect. 5, which allows more generality on a and f.

Definition 1 We say that the pair $(\underline{u}, \overline{u})$, with $\underline{u}, \overline{u} \in H^1(\Omega) \cap L^\infty(\Omega)$, is a pair of sub-supersolution of (4) if

a) $\underline{u} \le \overline{u}$ in Ω,
b) $\underline{u} \le 0 \le \overline{u}$ on $\partial\Omega$,
c)

$$a\left(\int_\Omega q(x)u^p\right)\int_\Omega \nabla\underline{u}\cdot\nabla\varphi \le \int_\Omega f(x,\underline{u})\varphi$$
$$a\left(\int_\Omega q(x)u^p\right)\int_\Omega \nabla\overline{u}\cdot\nabla\varphi \ge \int_\Omega f(x,\overline{u})\varphi,$$

for all $\varphi \in H_0^1(\Omega)$, $\varphi \ge 0$ in Ω and for all $u \in [\underline{u}, \overline{u}]$, where $[\underline{u}, \overline{u}] = \{u \in L^\infty(\Omega) : \underline{u} \le u \le \overline{u}\}$.

The main result reads as follows:

Theorem 1 *Assume that there exists a pair of sub-supersolution of* (4)*. Then, there exists a solution* $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *of* (4) *such that*

$$u \in [\underline{u}, \overline{u}].$$

3 Local logistic equation

In this section we consider the following local logistic problem

$$\begin{cases} -\Delta u = \mu u - b(x)u^2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

In the following result we recall the main results concerning to the existence and uniqueness of positive solution of (5) as well as its main properties, see for instance [7,8] and references therein.

Theorem 2 *1. If* $b(x)$ *satisfies* (Hb_1)*, there exists a unique positive solution of* (5) *if and only if* $\mu > \lambda_1$.

2. *If* $b(x)$ *satisfies* (Hb_2)*, there exists a unique positive solution of* (5) *if and only if*

$$\lambda_1 < \mu < \lambda_1^{\Omega_0}. \tag{6}$$

Moreover, if we denote the unique positive solution of (5) *by* $\theta_{[\mu,b]}$*, we have that the map* $\mu \in (\lambda_1, \lambda_1^{\Omega_0}) \to \theta_{[\mu,b]} \in C^2(\overline{\Omega})$ *is continuous, increasing, differentiable and*

$$\lim_{\mu \downarrow \lambda_1} \|\theta_{[\mu,b]}\|_\infty = 0, \quad \lim_{\mu \uparrow \lambda_1^{\Omega_0}} \|\theta_{[\mu,b]}\|_r = \infty, \tag{7}$$

where $1 \leq r \leq \infty$*. Furthermore,*

$$\lim_{\mu \uparrow \lambda_1^{\Omega_0}} \theta_{[\mu,b]}(x) \begin{cases} +\infty & \text{if } x \in \overline{\Omega}_0, \\ < \infty & \text{if } x \in \Omega \setminus \overline{\Omega}_0. \end{cases}$$

In fact, for any open set $D \subset \Omega \setminus \overline{\Omega}_0$ *it holds that*

$$\lim_{\mu \uparrow \lambda_1^{\Omega_0}} \theta_{[\mu,b]} = L \quad \text{in } C^2(\overline{D}),$$

where L *is the minimal solution of*

$$\begin{cases} -\Delta L(x) = \lambda_1^{\Omega_0} L(x) - b(x)L(x)^2 & \text{in } \Omega \setminus \overline{\Omega}_0, \\ L(x) = 0 & \text{on } \partial\Omega, \\ L(x) = \infty & \text{on } \partial\Omega_0. \end{cases} \tag{8}$$

Now, we define a generalized function which will play a fundamental role in our work. Indeed, we define the *metasolution* (see [8]), that is

$$\mathcal{M}(x) := \begin{cases} \infty & \text{in } \overline{\Omega}_0, \\ L(x) & \text{in } \Omega \setminus \overline{\Omega}_0. \end{cases} \tag{9}$$

4 Bifurcation results

In this section we apply the bifurcation theory to prove the existence of an unbounded continuum of positive solutions of (1).

4.1 Global bifurcation

In order to write (1) as a fixed point equation, we introduce the operator $\mathcal{L} : L^{\infty}(\Omega) \to L^{\infty}(\Omega)$, defined by

$$\mathcal{L}(f) := u,$$

where u is the unique solution of the linear elliptic equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Lemma 1 *The operator $\mathcal{L}$ is compact and strictly positive. Moreover, if $f \in L^{\infty}(\Omega)$, then there exists $C > 0$ such that*

$$\|\mathcal{L} f\|_{\infty} \leq C \|f\|_{\infty}.$$

Now, we prove the existence of an unbounded continuum of positive solutions that bifurcates from the trivial solution $u \equiv 0$ at $\lambda = a(0)\lambda_1$.

Theorem 3 *There exists an unbounded continuum $\mathcal{C}$ in $\mathbb{R} \times C(\overline{\Omega})$ of positive solutions of (1) emanating from $(\lambda, u) = (a(0)\lambda_1, 0)$.*

Proof Observe that (1) is equivalent to

$$u = T_{\lambda}(u) := \frac{\lambda}{a(0)} \mathcal{L} u + h(\lambda, u),$$

where

$$h(\lambda, u) := h_1(\lambda, u) + h_2(u)$$

and

$$h_1(\lambda, u) = \lambda \mathcal{L} \left(\left(\frac{1}{a\left(\int_{\Omega} q(x)(u^+)^p\right)} - \frac{1}{a(0)} \right) u^+ \right),$$

$$h_2(u) = -\mathcal{L} \left(\frac{b(x)}{a\left(\int_{\Omega} q(x)(u^+)^p\right)} u^2 \right)$$

and $u^+ = \max\{u, 0\}$. Indeed, observe that u is a non-negative and non-trivial solution of (1) if and only if $u = T_{\lambda}(u)$.

Observe that h is a continuous function and $h(\lambda, u) = o(\|u\|_{\infty})$ for $\|u\|_{\infty}$ near to 0 uniformly on bounded λ intervals. Indeed, by Lemma 1, it yields that

$$\begin{aligned} \frac{\|h(\lambda, u)\|_{\infty}}{\|u\|_{\infty}} &\leq \|h_1(\lambda, u)\|_{\infty} \frac{1}{\|u\|_{\infty}} + \|h_2(u)\|_{\infty} \frac{1}{\|u\|_{\infty}} \\ &\leq C \left(\lambda \left| \frac{1}{a\left(\int_{\Omega} q(x)(u^+)^p\right)} - \frac{1}{a(0)} \right| + \frac{\|b(x)\|_{\infty}}{\left|a\left(\int_{\Omega} q(x)(u^+)^p\right)\right|} \|u\|_{\infty} \right) \to 0, \end{aligned}$$

as $\|u\|_\infty \to 0$. Hence, we can apply Theorem 1.3 in [11] and conclude that there exists a connected component $\mathcal{C}$ of non-negative and non-trivial solutions that emanates from $(a(0)\lambda_1, 0)$, which is unbounded. Finally, by the strong maximum principle, any non-negative and non-trivial solution of (1) is positive in Ω. This concludes the proof. □

Observe that, by elliptic regularity, any solution $u \in L^\infty(\Omega)$, in fact it belongs to $W^{2,p}(\Omega)$ for all $p > 1$, and then, $u \in C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0, 1)$.

4.2 Bifurcation direction

In this section we analyze the bifurcation direction from the trivial solution. Recall that the bifurcation direction is called supercritical (resp. subcritical) if for any sequence of positive solutions (λ_n, u_n) of (1) such that $\lambda_n \to a(0)\lambda_1$ and $\|u_n\|_\infty \to 0$, then $\lambda_n > a(0)\lambda_1$ (resp. $\lambda_n < a(0)\lambda_1$).

First, we show some important properties of the positive solutions of (1).

Proposition 2 *Let (λ, u) be a positive solution of (1).*

1. *If we denote by*

$$d = a\left(\int_\Omega q(x)u^p(x)dx\right),$$

then,

$$\frac{u}{d} = \theta_{\left[\frac{\lambda}{d}, b\right]}.$$

2. *It holds*

$$a\left(\int_\Omega q(x)u^p(x)dx\right)\lambda_1 < \lambda < a\left(\int_\Omega q(x)u^p(x)dx\right)\lambda_1^{\Omega_0}.$$

Proof 1. Dividing the Eq. (1) by d^2, we obtain

$$-\Delta\left(\frac{u}{d}\right) = \frac{\lambda}{d}\left(\frac{u}{d}\right) - b(x)\left(\frac{u}{d}\right)^2,$$

whence the first paragraph follows.

2. By the monotonicity of principal eigenvalue, see Proposition 1, we obtain that

$$\lambda = \lambda_1\left[-a\left(\int_\Omega q(x)u^p\right)\Delta + b(x)u\right] > \lambda_1\left[-a\left(\int_\Omega q(x)u^p\right)\Delta\right] = \\ = a\left(\int_\Omega q(x)u^p\right)\lambda_1.$$

Using again Proposition 1, it becomes apparent that

$$\lambda = \lambda_1\left[-a\left(\int_\Omega q(x)u^p\right)\Delta + b(x)u\right] < \lambda_1^{\Omega_0}\left[-a\left(\int_\Omega q(x)u^p\right)\Delta\right] = \\ = a\left(\int_\Omega q(x)u^p\right)\lambda_1^{\Omega_0}.$$

□

Based on Proposition 2, the following non-existence result of positive solution of (1) follows.

Proposition 3 *Let* (λ, u) *be a positive solution of* (1).

a) If $b(x)$ *verifies* (Hb_1), *then* $\lambda > a_L\lambda_1$.
b) If $b(x)$ *verifies* (Hb_2), *then* $a_L\lambda_1 < \lambda < a_M\lambda_1^{\Omega_0}$.

Remark 1 It is worth emphasizing that in Proposition 3 we assume neither a is bounded nor a strictly positive. Therefore, for example, when a is unbounded the above result reads with $a_M = \infty$.

The following result ascertains the bifurcation direction of the continuum $\mathcal{C}$ that emanates from $(\lambda, u) = (a(0)\lambda_1, 0)$.

Theorem 4 *Assume that* $a \in C^1(\mathbb{R})$. *Denote by* φ_1 *the positive eigenfunction associated to* λ_1, *normalized by* $||\varphi_1||_2 = 1$. *It holds:*

a) If $p > 1$, *the bifurcation direction is supercritical.*
b) Assume that $p = 1$, *then:*

(b_1) *If*

$$a'(0) > -\frac{\displaystyle\int_\Omega b(x)\varphi_1^3}{\lambda_1\displaystyle\int_\Omega q(x)\varphi_1}, \tag{10}$$

then the bifurcation direction is supercritical.

(b_2) *If*

$$a'(0) < -\frac{\displaystyle\int_\Omega b(x)\varphi_1^3}{\lambda_1\displaystyle\int_\Omega q(x)\varphi_1}, \tag{11}$$

then the bifurcation direction is subcritical.

c) Assume that $p < 1$, *then:*

(c_1) *If* $a'(0) > 0$, *then the bifurcation direction is supercritical.*
(c_2) *If* $a'(0) < 0$, *then the bifurcation direction is subcritical.*

Proof First, we consider the paragraphs a) and b). Thus, assume that $p \geq 1$. We will use the Crandall-Rabinowitz Theorem, see [6]. For that, we define the map:

$$\mathcal{F} : \mathbb{R} \times C_0^2(\overline{\Omega}) \to C^0(\overline{\Omega}),$$
$$\mathcal{F}(\lambda, u) = a\left(\int_\Omega q(x)u^p\right)\Delta u + \lambda u - b(x)u^2.$$

It is easily seen that $\mathcal{F} \in C^1(\mathbb{R} \times C^2(\overline{\Omega}); C^0(\overline{\Omega}))$ and it follows directly that

$$\mathcal{F}_u(\lambda, u)(v) = a'\left(\int_\Omega q(x)u^p\right) p\left(\int_\Omega q(x)u^{p-1}v\right)\Delta u + a\left(\int_\Omega q(x)u^p\right)\Delta v + \lambda v - 2b(x)uv, \tag{12}$$

$$\mathcal{F}_{u\lambda}(\lambda, u)v = v. \tag{13}$$

By definition of $\mathcal{F}$, (12) and (13), we get that

$$\mathcal{F}(\lambda, 0) = 0, \qquad \forall \lambda \in \mathbb{R},$$
$$L_0 := \mathcal{F}_u(\lambda, 0)(v) = a(0)\Delta v + \lambda v,$$
$$L_1 := \mathcal{F}_{u\lambda}(\lambda, 0)(v) = v.$$

Then, it follows that

$$Ker(L_0) = \{v \in C_0^2(\overline{\Omega})\backslash\{0\};\ a(0)\Delta v + \lambda v = 0\} \neq \emptyset.$$

Since $a(0)\lambda_1$ is a simple eigenvalue of $(-a(0)\Delta)$, then

$$Ker(\mathcal{F}_u(a(0)\lambda_1, 0)) = \langle \varphi_1 \rangle,$$

and

$$\dim\,(Ker(\mathcal{F}_u(a(0)\lambda_1, 0))) = \text{cod}\,(\text{Rg}(\mathcal{F}_u(a(0)\lambda_1, 0)) = 1.$$

By the Fredholm Alternative Theorem, it follows that

$$\text{Rg}(\mathcal{F}_u(a(0)\lambda_1, 0)) = \left\{u \in L^2(\Omega);\ \int_\Omega \varphi_1 u = 0\right\}.$$

On the other hand, it results that

$$\mathcal{F}_{u\lambda}(a(0)\lambda_1, 0)(\varphi_1) = \varphi_1.$$

Hence,

$$\mathcal{F}_{u\lambda}(a(0)\lambda_1, 0)(\varphi_1) \notin \text{Rg}\,(\mathcal{F}_u(a(0)\lambda_1, 0)),$$

due to $\|\varphi_1\|_2 = 1$.

Then, we can apply the Crandall-Rabinowitz Theorem (see [6]) to conclude that there exist $\epsilon > 0$ and two C^1 maps

$$\mu : (-\epsilon, \epsilon) \to \mathbb{R}, \qquad\qquad v : (-\epsilon, \epsilon) \to Z,$$

where Z is the topological complement of $Ker(L_0)$ in $C_0^2(\overline{\Omega})$, $\mu(0) = 0$, $v(0) = 0$ and for each $s \in (-\epsilon, \epsilon)$

$$\begin{cases} \lambda(s) = a(0)\lambda_1 + \mu(s), \\ u(s) = s(\varphi_1 + v(s)), \end{cases} \tag{14}$$

such that $(\lambda(s), u(s))$ are non-trivial solutions of (1), that is

$$\mathcal{F}(\lambda(s), u(s)) = 0 \quad s \in (-\epsilon, \epsilon).$$

Moreover, there exists $\rho > 0$ such that if $\mathcal{F}(\lambda, u) = 0$ and $(\lambda, u) \in B((a(0)\lambda_1, 0), \rho)$ then either $u \equiv 0$ or $(\lambda, u) = (\lambda(s), u(s))$ for some $s \in (-\epsilon, \epsilon)$; $B((a(0)\lambda_1, 0), \rho)$ denotes the ball centered in $(a(0)\lambda_1, 0)$ and radius $\rho > 0$ in $\mathbb{R} \times C_0^2(\overline{\Omega})$. Observe that $u(s)$ is positive for $s \in (0, \epsilon)$.

On the other hand, the Taylor expansion of the function $a\,(t)$ is given by

$$a\,(t) = a(0) + ta'(0) + \text{o}(t). \tag{15}$$

Replacing (14) and (15) into (1), we obtain that

$$-\left(a(0) + s^p a'(0) \int_\Omega q(x)(\varphi_1 + v(s))^p + o(s^p)\right) \Delta(s\varphi_1 + sv(s)) =$$
$$(a(0)\lambda_1 + \mu(s))(s(\varphi_1 + v(s)) - b(x)(s(\varphi_1 + v(s)))^2.$$

Multiplying by φ_1, integrating in Ω and rearranging terms, we have that

$$\begin{aligned} &s^p(a'(0)\lambda_1\int_\Omega q(x)(\varphi_1+v(s))^p\int_\Omega \varphi_1^2 \\ &\quad + a'(0)\lambda_1\int_\Omega q(x)(\varphi_1+v(s))^p\int_\Omega \varphi_1 v(s)) + o(s^p) \\ &= \mu(s)\int_\Omega(\varphi_1+v(s))\varphi_1 - s\int_\Omega b(x)(\varphi_1+v(s))^2\varphi_1. \end{aligned}$$

Then, a straightforward manipulation leads to

$$\frac{\mu(s)}{s^p} = \frac{a'(0)\lambda_1\displaystyle\int_\Omega q(x)(\varphi_1+v(s))^p + s^{1-p}\int_\Omega b(x)(\varphi_1+v(s))^2\varphi_1}{\displaystyle\int_\Omega(\varphi_1+v(s))\varphi_1} + o(s).$$

Now, when $p>1$ we have that $\mu(s)>0$ for for $s>0$ and small, and then $\lambda(s)>a(0)\lambda_1$ and as consequence the direction is supercritical. This concludes the first paragraph.

When $p=1$, then

$$\lim_{s\to 0}\frac{\mu(s)}{s} = a'(0)\lambda_1\int_\Omega q(x)\varphi_1 + \int_\Omega b(x)\varphi_1^3.$$

Paragraph b) is now an immediate consequence.

Now assume that $p<1$. Suppose that there exists a sequence of positive solutions $\{(\lambda_n,u_n)\}$ of (1) such that

$$\lambda_n\to a(0)\lambda_1 \quad \text{and} \quad \|u_n\|_\infty\to 0.$$

Assume that $a'(0)>0$, it follows that

$$a(s)>a(0) \quad 0<s<\epsilon. \tag{16}$$

Since $\epsilon > \int_\Omega q(x)u_n^p > 0$, for n large, and from (16), we have that

$$a\left(\int_\Omega q(x)u_n^p\right) > a(0).$$

Hence, by Proposition 1 we get

$$\begin{aligned} \lambda_n &= \lambda_1\left[-a\left(\int_\Omega q(x)u_n^p\right)\Delta + b(x)u_n\right] > \lambda_1\left[-a\left(\int_\Omega q(x)u_n^p\right)\Delta\right] \\ &> \lambda_1\left[-a(0)\Delta\right] = a(0)\lambda_1. \end{aligned}$$

Assume now that $a'(0)<0$. We argue by contradiction: assume that there exists a sequence (λ_n,u_n) of positive solutions of (1) such that $\lambda_n\to a(0)\lambda_1$, $\|u_n\|_\infty\to 0$ with $\lambda_n>a(0)\lambda_1$. Let $C>0$, with C large enough, such that

$$a'(0) < -\frac{\displaystyle\int_\Omega b(x)\varphi_1^3}{C\lambda_1\displaystyle\int_\Omega q(x)\varphi_1}. \tag{17}$$

We fix this value of C in the rest of the proof. Consider the problem

$$\begin{cases} -a\left(C\int_\Omega q(x)u\right)\Delta u = u(\lambda - b(x)u) \text{ in } \Omega, \\ u = 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega. \end{cases} \tag{18}$$

Observe that $p = 1$ in (18), and then by (17), we can use paragraph (b_2) and conclude that the bifurcation direction for (18) is subcritical. However, we will prove that there exists a sequence of positive solutions (μ_n, w_n) of (18) with $\mu_n > \lambda_1 a(0)$, $\mu_n \to \lambda_1 a(0)$ and $\|w_n\|_\infty \to 0$.

Take $\mu_n = \lambda_n$. From $a'(0) < 0$, it follows that

$$a(0) > a(s), \quad 0 < s < \epsilon_0. \tag{19}$$

Since $p < 1$, we obtain that there exists $n_0 \in \mathbb{R}$ such that

$$q(x)u_n^p \geq Cq(x)u_n \quad \forall n \geq n_0.$$

Since $a'(0) < 0$, then a is decreasing near 0. Then, as $\|u_n\|_\infty \to 0$, we have that

$$a\left(\int_\Omega q(x)u_n^p\right) < a\left(C\int_\Omega q(x)u_n\right). \tag{20}$$

We are going to apply the sub-supersolution method to (18). We take

$$\overline{u} = u_n, \quad \underline{u} = \varepsilon\varphi_1,$$

where $\varepsilon > 0$ will be chosen later. First, observe that since $\lambda_n \geq \lambda_0 > 0$ for some positive λ_0,

$$f(x,s) = s(\lambda_n - b(x)s) > 0 \quad s \in [\underline{u}, \overline{u}].$$

Then, since a is decreasing in $[\underline{u}, \overline{u}]$ and $f(x,s) > 0$, Definition 1 is equivalent to

$$-a\left(\int_\Omega Cq(x)\underline{u}\right)\Delta\underline{u} - f(x,\underline{u}) \leq 0 \leq -a\left(\int_\Omega Cq(x)\overline{u}\right)\Delta\overline{u} - f(x,\overline{u}).$$

We start showing that $\overline{u} = u_n$ is a supersolution of (18). Indeed, by (20)

$$-a\left(C\int_\Omega q(x)u_n\right)\Delta(u_n) > -a\left(\int_\Omega q(x)u_n^p\right)\Delta(u_n) = u_n(\lambda_n - b(x)u_n).$$

Now, we are going to see that $\underline{u} = \epsilon\varphi_1$ is a subsolution of (18) provided of ϵ is small enough. Indeed, $\epsilon\varphi_1$ is subsolution if

$$b(x)\epsilon\varphi_1 + a\left(\epsilon\int_\Omega Cq(x)\varphi_1\right)\lambda_1 \leq \lambda_n.$$

By (19), for ϵ small, we have that

$$b(x)\epsilon\varphi_1 + a\left(\epsilon\int_\Omega Cq(x)\varphi_1\right)\lambda_1 \leq b_M\epsilon + a(0)\lambda_1 < \lambda_n$$
$$\Leftrightarrow \epsilon \leq \frac{\lambda_n - a(0)\lambda_1}{b_M},$$

where $b_M = \max_{x\in\overline{\Omega}} b(x)$. Therefore, there exists positive solution w_n of (18) for $\lambda_n > a(0)\lambda_1$ such that

$$\epsilon\varphi_1 \leq w_n \leq u_n \quad \text{in } \Omega.$$

This is a contradiction. □

5 Logistic equation with nonlocal diffusion

5.1 b verifies (Hb_1)

First, we focus on the case where b verifies (Hb_1).

Theorem 5 *Assume that b satisfies* (Hb_1). *Then, there exists a positive solution of (1) if* $\lambda > a(0)\lambda_1$. *Moreover, if a is increasing and b is constant, then there exists a unique positive solution of* (1).

Proof First, it should be remembered that $b(x) \geq b_0$ for all $x \in \overline{\Omega}$. We will prove that

$$u < \frac{\lambda}{b_0} \quad \text{in } \Omega. \tag{21}$$

Indeed, we define

$$\Omega_1 = \left\{ x \in \Omega; u(x) > \frac{\lambda}{b_0} \right\}.$$

Thus,

$$-a\left(\int_\Omega q(x)u^p\right)\Delta u = \lambda u - b(x)u^2 \leq 0 \quad \text{in } \Omega_1, \quad u = \frac{\lambda}{b_0} \quad \text{on } \partial\Omega_1,$$

implying that $u \leq \dfrac{\lambda}{b_0}$ in Ω_1. We arrive at a contradiction. Hence $\Omega_1 = \emptyset$ and we conclude $u \leq \lambda/b_0$ in Ω. The strong maximum principle proves (21).

Hence, by Theorem 3 we know the existence of an unbounded continuum of positive solutions $\mathcal{C}$ that bifurcates from the trivial solution at $\lambda = a(0)\lambda_1$. Moreover, (1) does not possess positive solutions for $\lambda \leq a_L\lambda_1$. As consequence of (21), it follows the existence of positive solution for $\lambda > a(0)\lambda_1$.

Assume now that a is increasing and b is constant. Let u and v be positive solutions of (1), with $u \neq v$. We distinguish two cases:

1. Assume that $\int_\Omega q(x)u^p = \int_\Omega q(x)v^p$. Then, u and v are positive solutions of

$$-\Delta v = \frac{\lambda v - bv^2}{a\left(\int_\Omega q(x)u^p\right)} \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega. \tag{22}$$

   Therefore, since (22) has a unique positive solution, it follows that $u = v$ in Ω.

2. Assume now that $\int_\Omega q(x)u^p < \int_\Omega q(x)v^p$. Observe that since b is constant, it follows by (21) that

$$\lambda v - bv^2 > 0 \text{ in } \Omega. \tag{23}$$

   Moreover, since a is increasing and by (23), we obtain that

$$-\Delta v = \frac{\lambda v - bv^2}{a\left(\int_\Omega q(x)v^p\right)} < \frac{\lambda v - bv^2}{a\left(\int_\Omega q(x)u^p\right)}.$$

Therefore, v is subsolution of (22), which implies that $u > v$, a contradiction.

□

5.2 b verifies (Hb_2)

Now, we deal with the case when b verifies (Hb_2). For that, we define

$$I := \int_{\Omega} q(x)(\mathcal{M}(x))^p dx, \tag{24}$$

where $\mathcal{M}(x)$ is defined in (9).

The following result shows (when $I = \infty$) that the structure of the set of positive solutions depends strongly on the behavior of the function a at ∞.

Theorem 6 *Let $I = \infty$ and assume that b verifies (Hb_2).*

1. *If $0 < a(\infty) < \infty$, then there exists at least one positive solution of (1) for*
$$\lambda \in (\min\{a(0)\lambda_1, a(\infty)\lambda_1^{\Omega_0}\}, \max\{a(0)\lambda_1, a(\infty)\lambda_1^{\Omega_0}\}).$$
2. *If $a(\infty) = \infty$, then there exists at least one positive solution of (1) for $\lambda > a(0)\lambda_1$.*
3. *If $a(\infty) = 0$, then there exists at least one positive solution of (1) for $\lambda \in (0, a(0)\lambda_1)$.*
Moreover, in all the cases, there exist sequences $(\lambda_n, u_{\lambda_n}), (\lambda'_n, u_{\lambda'_n}) \in \mathcal{C}$, then
$$\lim_{\lambda_n \to a(0)\lambda_1} \|u_{\lambda_n}\|_\infty = 0, \qquad \lim_{\lambda'_n \to a(\infty)\lambda_1^{\Omega_0}} \|u_{\lambda'_n}\|_\infty = \infty. \tag{25}$$

Proof From Theorem 3 there exists an unbounded continuum $\mathcal{C}$ in $\mathbb{R} \times L^\infty(\Omega)$ of positive solutions of (1) that emanates at $\lambda = a(0)\lambda_1$ from the trivial solution. By Proposition 3, (1) does not admit positive solutions for $\lambda \le a_L\lambda_1$.

1. Assume that $0 < a(\infty) < \infty$, then $0 < a_L < a_M < \infty$. Again, by Proposition 3, if there exists positive solution of (1) then $\lambda < a_M\lambda_1^{\Omega_0}$, and hence, since $\mathcal{C}$ is unbounded, there exists a sequence of positive solutions $(\lambda_n, u_n) \in \mathcal{C}$, with $\lambda_n \to \lambda^* \in (0, \infty)$ and $\|u_n\|_\infty \to \infty$. We define
$$d_n = a\left(\int_\Omega q(x)u_n^p\right).$$
Then, by Proposition 2
$$\frac{u_n}{d_n} = \theta_{\left[\frac{\lambda_n}{d_n}, b\right]}. \tag{26}$$
Therefore,
$$d_n = a\left(d_n^p \int_\Omega q(x)\theta^p_{\left[\frac{\lambda_n}{d_n}, b\right]}\right). \tag{27}$$
On the other hand, since u_n is a positive solution of (1), then again by Proposition 2 we get
$$\lambda_1 < \frac{\lambda_n}{d_n} < \lambda_1^{\Omega_0}.$$

Since $\lambda_n \to \lambda^* \in (0, \infty)$, it follows that d_n is bounded above and below. Then, since $\|u_n\|_\infty \to \infty$ and using (26), we conclude that $\|\theta_{\left[\frac{\lambda_n}{d_n}, b\right]}\|_\infty \to +\infty$, and hence

$$\frac{\lambda_n}{d_n} \to \lambda_1^{\Omega_0}.$$

We conclude that

$$d_n \to \frac{\lambda^*}{\lambda_1^{\Omega_0}} \quad \text{and} \quad \theta_{\left[\frac{\lambda_n}{d_n}, b\right]} \to \mathcal{M}.$$

Taking limit in (27), we obtain

$$\frac{\lambda^*}{\lambda_1^{\Omega_0}} = a\left(\left(\frac{\lambda^*}{\lambda_1^{\Omega_0}}\right)^p I\right) = a(\infty) \Rightarrow \lambda^* = a(\infty)\lambda_1^{\Omega_0}. \tag{28}$$

2. Assume that $a(\infty) = \infty$. In such case, $a_L > 0$ and then if (λ, u) is a positive solution of (1) we get by Proposition 3 that $0 < a_L\lambda_1 < \lambda$. The proof will proceed by contradiction. Suppose that there exists a sequence of positive solutions $(\lambda_n, u_n) \in \mathcal{C}$ of (12) such that $\lambda_n \to \lambda^* < \infty$ and $\|u_n\|_\infty \to \infty$. With a similar argument to the used in the first paragraph, according to (28), we obtain that $\lambda^* = a(\infty)\lambda_1^{\Omega_0} < \infty$. We arrive at a contradiction.
3. Assume that $a(\infty) = 0$, and hence, $a_M < \infty$ and then by Proposition 3 if (λ, u) is a positive solution of (1) we get that $\lambda < a_M\lambda_1^{\Omega_0}$. Assume now by contradiction that there exists a sequence of positive solutions $(\lambda_n, u_n) \in \mathcal{C}$ of (12) such that $\lambda_n \to \lambda^*$ and $\|u_n\|_\infty \to \infty$ with $\lambda^* > 0$. Again, by (28), we can infer that $\lambda^* = a(\infty)\lambda_1^{\Omega_0} = 0$. A contradiction.

□

Now, we handle the case $I < \infty$. In this case, the structure of the set of positive solutions of (1) depends on the real equation

$$g(s) = I, \tag{29}$$

where the function $g : [0, \infty) \to [0, \infty)$ is defined by

$$g(s) := \frac{s}{(a(s))^p}. \tag{30}$$

The first result deals with the case that (29) does not have solutions. First, we need to prove the following result

Lemma 2 *Assume that $I < \infty$ and that there exists a sequence (λ_n, u_n) of positive solutions of (1) such that $\|u_n\|_\infty \to \infty$ and $\lambda_n \to \lambda_* < \infty$. Then, $\lambda_* > 0$ and there exists $s_* > 0$ such that $g(s_*) = I$, in fact,*

$$s_* = \left(\frac{\lambda_*}{\lambda_1^{\Omega_0}}\right)^p I.$$

Proof With the same notation as in Proposition 2, we have that

$$\lambda_1 < \frac{\lambda_n}{d_n} < \lambda_1^{\Omega_0},$$

where

$$d_n = a\left(\int_\Omega q(x)u_n(x)^p dx\right).$$

Then,

$$u_n = d_n\theta_{\left[\frac{\lambda_n}{d_n},b\right]}, \tag{31}$$

and hence,

$$d_n = a\left(d_n^p \int_\Omega q(x)\theta^p_{\left[\frac{\lambda_n}{d_n},b\right]}\right). \tag{32}$$

Since $\lambda_* < \infty$, d_n is bounded above, and then, by (31), if $\|u_n\|_\infty \to \infty$ we have that

$$\frac{\lambda_n}{d_n} \to \lambda_1^{\Omega_0}.$$

Passing to the limit in (32), we obtain:

$$\frac{\lambda_*}{\lambda_1^{\Omega_0}} = a\left(\left(\frac{\lambda_*}{\lambda_1^{\Omega_0}}\right)^p I\right).$$

Then,

$$a(s_*) = \left(\frac{s_*}{I}\right)^{1/p},$$

and consequently, $g(s_*) = I$. This concludes the result. □

Theorem 7 *Let $I < \infty$. If there does not exist solution of* (29)*, then there exists at least one positive solution of* (1) *for $\lambda > a(0)\lambda_1$.*

Proof We know that there exists an unbounded continuum $\mathcal{C}$ of positive solutions emanating from $(\lambda, u) = (a(0)\lambda_1, 0)$. Assume by contradiction that there exists a sequence $(\lambda_n, u_n) \in \mathcal{C}$ of positive solutions such that $\lambda_n \to \lambda_* < \infty$ and $\|u_n\|_\infty \to \infty$. Then, by Lemma 2 there exists a positive solution of (29), a contradiction. □

The next result will be the cornerstone in the rest of the work.

Proposition 4 *Assume that (λ, u) is a positive solution of* (1)*. Then,*

$$g\left(\int_\Omega q(x)u^p(x)dx\right) < I.$$

Proof Observe that

$$\begin{aligned} g\left(\int_\Omega q(x)u^p(x)dx\right) &= \frac{\int_\Omega q(x)u^p(x)dx}{\left(a\left(\int_\Omega q(x)u^p(x)dx\right)\right)^p} \\ &= \int_\Omega q(x)\left(\frac{u(x)}{a\left(\int_\Omega q(x)u^p(x)dx\right)}\right)^p dx \\ &= \int_\Omega q(x)\theta^p_{\left[\frac{\lambda}{d},b\right]} < \int_\Omega q(x)\mathcal{M}^p(x)dx = I, \end{aligned}$$

where

$$d = a\left(\int_\Omega q(x)u^p(x)dx\right).$$

□

In the following result, we show a non-existence result.

Proposition 5 *If there exists* $\overline{s} > 0$ *such that*

$$g(s) > I \quad \text{for } s > \overline{s}, \tag{33}$$

then, there exists $\overline{\lambda} > 0$ *such that* (1) *does not possess positive solution for* $\lambda > \overline{\lambda}$.

Proof Assume by contradiction that there exists a sequence of positive solutions (λ_n, u_n) of (1) for $\lambda_n \to \infty$. We know by Proposition 2 that

$$\lambda_1 < \frac{\lambda_n}{a\left(\int_\Omega q(x)u_n^p(x)dx\right)} < \lambda_1^{\Omega_0}.$$

Hence, $a\left(\int_\Omega q(x)u_n^p(x)dx\right) \to \infty$, which implies that $\int_\Omega q(x)u_n^p(x)dx \to \infty$. Then by (33),

$$g\left(\int_\Omega q(x)u_n^p(x)dx\right) > I,$$

this is a contradiction with Proposition 4. □

The following result shows that the $Proj_{\mathbb{R}}(\mathcal{C})$ is bounded when there exists solutions of (29). Here, given $(\lambda, u) \in \mathcal{C}$ we denote $Proj_{\mathbb{R}}(\lambda, u) = \lambda$.

Proposition 6 *If there exists* $s^* > 0$ *such that* $g(s^*) = I$*, then* $Proj_{\mathbb{R}}(\mathcal{C}) \subset (0, \Lambda^*)$ *for some* $\Lambda^* < \infty$*. In fact, if we denote* $s_1 > 0$ *the least solution of* (29) *and*

$$\Lambda_1 = \lambda_1^{\Omega_0}\left(\frac{s_1}{I}\right)^{1/p},$$

then, there exists a sequence $(\lambda_n, u_n) \in \mathcal{C}$ *such that* $\|u_n\|_\infty \to \infty$ *and* $\lambda_n \to \Lambda_1$.

Proof We define the continuous map

$$\mathcal{H} : \mathbb{R} \times L^\infty(\Omega) \to \mathbb{R}, \quad \mathcal{H}(\lambda, u) = \int_\Omega q(x)u^p(x)dx.$$

Hence, since $\mathcal{C}$ is connected, we obtain that $\mathcal{H}(\mathcal{C})$ is a connected set in $\mathbb{R}$.

Assume by contradiction that there exists a sequence $(\lambda_n, u_{\lambda_n}) \in \mathcal{C}$ such that $\lambda_n \to +\infty$. We know by Proposition 2 that

$$\lambda_n < \lambda_1^{\Omega_0} a\left(\int_\Omega q(x)u_{\lambda_n}(x)^p dx\right),$$

then $a\left(\int_\Omega q(x)u_{\lambda_n}(x)^p dx\right) \to \infty$. We conclude that $\int_\Omega q(x)u_{\lambda_n}^p(x)dx \to +\infty$; that is,

$$\mathcal{H}(\lambda_n, u_{\lambda_n}) \to +\infty.$$

On the other hand, $\mathcal{H}(a(0)\lambda_1, 0) = 0$, and hence, we can conclude that

$$[0, +\infty) \subset \mathcal{H}(\mathcal{C}).$$

Thus, there exists λ^* such that

$$s^* = \int_\Omega q(x) u_{\lambda^*}^p(x) dx.$$

Then, by Proposition 4, we get that

$$g(s^*) = g\left(\int_\Omega q(x) u_{\lambda^*}^p(x) dx\right) < I,$$

a contradiction.

Since $\mathcal{H}(\mathcal{C})$ is a connected set in $\mathbb{R}$ and observe that $\mathcal{H}(a(0)\lambda_1, 0) = 0$, it follows by Proposition 4 that

$$\mathcal{H}(\mathcal{C}) \subset [0, s_1). \tag{34}$$

On the other hand, we know that there exists a sequence of positive solutions $(\lambda_n, u_{\lambda_n}) \in \mathcal{C}$ of (12) such that $\lambda_n \to \lambda_* < \infty$ and $\|u_{\lambda_n}\|_\infty \to \infty$. Moreover, observe that by the proof of Lemma 4 we get that

$$\lim_{n\to\infty} \mathcal{H}(\lambda_n, u_{\lambda_n}) = \lim_{n\to\infty} \int_\Omega q(x) u_{\lambda_n}^p(x) dx = \lim_{n\to\infty} d_n^p \int_\Omega q(x) \theta^p_{\left[\frac{\lambda_n}{d_n}, b\right]} = \left(\frac{\lambda_*}{\lambda_1^{\Omega_0}}\right)^p I,$$

and then by (34)

$$\left(\frac{\lambda_*}{\lambda_1^{\Omega_0}}\right)^p I \leq s_1 = \left(\frac{\Lambda_1}{\lambda_1^{\Omega_0}}\right)^p I.$$

This implies that $\lambda_* = \Lambda_1$. □

Now, we study the case where $I < \infty$ and there exist positive solutions of (29). In the first result, we show the existence of positive solutions between $a(0)\lambda_1$ and Λ_1.

Theorem 8 *Let $I < \infty$ and assume that there exists $s_0 > 0$ solution of* (29). *Then, there exists at least a positive solution of* (1) *for*

$$\lambda \in (\min\{a(0)\lambda_1, \Lambda_1\}, \max\{a(0)\lambda_1, \Lambda_1\}).$$

Moreover, there exist sequences $(\lambda_n, u_{\lambda_n}), (\lambda'_n, u_{\lambda'_n}) \in \mathcal{C}$, then

$$\lim_{\lambda_n \to a(0)\lambda_1} \|u_{\lambda_n}\|_\infty = 0, \quad \lim_{\lambda'_n \to \Lambda_1} \|u_{\lambda'_n}\|_\infty = \infty.$$

Proof We know the existence of an unbounded continuum $\mathcal{C}$ of positive solutions emanating at $\lambda = a(0)\lambda_1$ from the trivial solution. Proposition 6 concludes the result. □

Now, we need to introduce and study the following map.

Proposition 7 *Fix $\lambda > 0$ and define the function $h_\lambda : [0, \infty) \to [0, \infty)$ by*

$$h_\lambda(s) := \int_\Omega q(x) \left(\theta_{\left[\frac{\lambda}{a(s)}, b\right]}\right)^p. \tag{35}$$

The function h_λ is well defined in the set

$$\Lambda_\lambda = \left\{ s \in [0, \infty) : \frac{\lambda}{\lambda_1^{\Omega_0}} < a(s) < \frac{\lambda}{\lambda_1} \right\}.$$

Moreover, h_λ is continuous in Λ_λ and

1. *Let $s^n, s_n \in \Lambda_\lambda$ such that $s^n \to s^*$ and $s_n \to s_* > 0$ such that $a(s^*) = \frac{\lambda}{\lambda_1^{\Omega_0}}$ and $a(s_*) = \frac{\lambda}{\lambda_1}$. Then,*
   (a)
   $$\lim_{s^n \to s^*} h_\lambda(s^n) = I. \tag{36}$$
   (b)
   $$\lim_{s_n \to s_*} h_\lambda(s_n) = 0. \tag{37}$$
2. *It holds that*
   $$h_\lambda(s) \le I, \quad \forall s \in [0, \infty).$$

Proof h_λ is well defined and is continuous in Λ_λ by Theorem 2.

1. (a) Again, by Theorem 2, it follows that
   $$\lim_{s^n \to s^*} h_\lambda(s^n) = \lim_{s^n \to s^*} \int_\Omega q(x) \left(\theta_{\left[\frac{\lambda}{a(s^n)}, b\right]} \right)^p = \int_\Omega q(x) (\mathcal{M}(x))^p = I.$$
   (b) Analogously,
   $$\lim_{s_n \to s_*} h_\lambda(s_n) = \lim_{s_n \to s_*} \int_\Omega q(x) \left(\theta_{\left[\frac{\lambda}{a(s_n)}, b\right]} \right)^p = 0.$$
2. By definition and Theorem 2, we get
   $$h_\lambda(s) = \int_\Omega q(x) \left(\theta_{\left[\frac{\lambda}{a(s)}, b\right]} \right)^p \le \int_\Omega q(x) (\mathcal{M}(x))^p = I.$$

□

In the following result we prove that Eq. (1) is in fact equivalent to find a fixed point to the real equation $h_\lambda(s) = g(s)$. Specifically,

Proposition 8 *If there exists $s^* > 0$ such that $h_\lambda(s^*) = g(s^*)$, then there exists at least one positive solution of (1) for $\lambda \in (a(s^*)\lambda_1, a(s^*)\lambda_1^{\Omega_0})$.*

Conversely, if u is a positive solution of (1), then there exists $s^* > 0$ such that $h_\lambda(s^*) = g(s^*)$, in fact,

$$s^* = \int_\Omega q(x) u^p(x) dx.$$

Proof By (30) and (35), we obtain

$$\frac{s^*}{(a(s^*))^p} = g(s^*) = h_\lambda(s^*) = \int_\Omega q(x) \left(\theta_{\left[\frac{\lambda}{a(s^*)}, b\right]} \right)^p.$$

Hence,

$$s^* = (a(s^*))^p \int_\Omega q(x) \left(\theta_{\left[\frac{\lambda}{a(s^*)},b\right]}\right)^p = \int_\Omega q(x) \left(a(s^*)\theta_{\left[\frac{\lambda}{a(s^*)},b\right]}\right)^p. \tag{38}$$

Observe that

$$u = a(s^*)\theta_{\left[\frac{\lambda}{a(s^*)},b\right]}$$

is solution of (1). Indeed, using (38)

$$\begin{aligned} -a\left(\int_\Omega q(x)u^p\right)\Delta u &= -a\left(\int_\Omega q(x)\left(a(s^*)\theta_{\left[\frac{\lambda}{a(s^*)},b\right]}\right)^p\right)\Delta(a(s^*)\theta_{\left[\frac{\lambda}{a(s^*)},b\right]}) \\ &= (a(s^*))^2\left(\frac{\lambda}{a(s^*)}\theta_{\left[\frac{\lambda}{a(s^*)},b\right]} - b(x)(\theta_{\left[\frac{\lambda}{a(s^*)},b\right]})^2\right) \\ &= \lambda(a(s^*)\theta_{\left[\frac{\lambda}{a(s^*)},b\right]}) - b(x)\left(a(s^*)\theta_{\left[\frac{\lambda}{a(s^*)},b\right]}\right)^2 \\ &= \lambda u - b(x)u^2. \end{aligned}$$

Conversely, assume that u is a positive solution of (1). Then, denoting by $d = a(\int_\Omega q(x)u^p(x)dx)$, we get

$$\frac{u}{d} = \theta_{[\lambda/d,b]},$$

and then,

$$\int_\Omega q(x)u^p(x)dx = d^p \int_\Omega q(x)\theta^p_{[\lambda/d,b]}.$$

Hence,

$$\frac{\displaystyle\int_\Omega q(x)u^p(x)dx}{\left(a\left(\displaystyle\int_\Omega q(x)u^p(x)dx\right)\right)^p} = \int_\Omega q(x)\theta^p_{[\mu,b]},$$

where

$$\mu = \frac{\lambda}{a\left(\displaystyle\int_\Omega q(x)u^p(x)dx\right)}.$$

This concludes the result. □

The following result shows that if $g(s) < I$ between two roots of (29) $\alpha_1 < \alpha_2$, that is $g(\alpha_1) = g(\alpha_2)$, then we can find positive solution in the interval $\lambda \in (\Lambda_1, \Lambda_2)$.

Proposition 9 *If $g(s) < I$ for all $s \in (\alpha_1, \alpha_2)$ with*

$$g(\alpha_1) = g(\alpha_2) = I,$$

then, there exists at least a positive solution of (1) for $\lambda \in (\Lambda_1, \Lambda_2)$, where $\Lambda_i = \left(\frac{\alpha_i}{I}\right)^{\frac{1}{p}} \lambda_1^{\Omega_0}$, $i \in \{1, 2\}$.

Proof Since $g(\alpha_1) = g(\alpha_2) = I$, we know that

$$a(\alpha_1) = \left(\frac{\alpha_1}{I}\right)^{\frac{1}{p}} < \left(\frac{\alpha_2}{I}\right)^{\frac{1}{p}} = a(\alpha_2). \tag{39}$$

Take $\lambda \in (\Lambda_1, \Lambda_2)$, that is

$$\frac{\lambda}{\lambda_1^{\Omega_0}} \in \left(\left(\frac{\alpha_1}{I}\right)^{\frac{1}{p}}, \left(\frac{\alpha_2}{I}\right)^{\frac{1}{p}}\right) = (a(\alpha_1), a(\alpha_2)).$$

By continuity of a, there exists $s_* \in (\alpha_1, \alpha_2)$ such that $a(s_*) = \dfrac{\lambda}{\lambda_1^{\Omega_0}}$.

Observe that $g(s) < I$ for all $s \in (\alpha_1, \alpha_2)$ is equivalent to

$$\left(\frac{s}{I}\right)^{1/p} < a(s) \quad \text{for all } s \in (\alpha_1, \alpha_2).$$

We define the sets

$$\overline{S} := \left\{\overline{s}_j \in [\alpha_1, \alpha_2];\ a(\overline{s}_j) = \frac{\lambda}{\lambda_1^{\Omega_0}}\right\} \quad \text{and} \quad \underline{S} := \left\{\underline{s}_j;\ a(\underline{s}_j) = \frac{\lambda}{\lambda_1}\right\}.$$

Since $a'(s) = 0$ in, at most, a discrete set, we can order $\underline{S}$ and $\overline{S}$ such that

$$\overline{s_1} < \overline{s_2} < \dots \quad \text{and} \quad \underline{s_1} < \underline{s_2} < \dots$$

We know that $s_* \in \overline{S}$, and so, $\overline{S} \neq \emptyset$.

Now, we separate the proof in two different cases:

1. We suppose $\underline{S} \cap [\alpha_1, \alpha_2] = \emptyset$. Then, we take $\overline{s}_{j_0} = \max_{j\in\mathbb{N}} \overline{s}_j \in [\alpha_1, \alpha_2]$. By definition,

$$a(\overline{s}_{j_0}) = \frac{\lambda}{\lambda_1^{\Omega_0}}.$$

Since $\underline{S} \cap [\alpha_1, \alpha_2] = \emptyset$, $a(s) < \frac{\lambda}{\lambda_1}$ for all $s \in [\alpha_1, \alpha_2]$ and hence there exists $\delta > 0$ such that

$$\frac{\lambda}{\lambda_1^{\Omega_0}} < a(s) < \frac{\lambda}{\lambda_1}, \quad s \in (\overline{s}_{j_0}, \alpha_2 + \delta). \tag{40}$$

By Proposition 7, we obtain

$$\lim_{s \downarrow \overline{s}_{j_0}} h_\lambda(s) = I.$$

By Proposition 7, $h_\lambda(\alpha_2 + \delta) \leq I$ and $h_\lambda(\alpha_2) < I$. Then,

$$h_\lambda(\overline{s}_{j_0}) - g(\overline{s}_{j_0}) > 0 \quad \text{and} \quad h_\lambda(\alpha_2) - g(\alpha_2) < 0,$$

and hence there exists $s^* \in (\overline{s}_{j_0}, \alpha_2) \subset (\alpha_1, \alpha_2)$ such that

$$g(s^*) = h_\lambda(s^*).$$

Moreover, by (40) observe that s^* is such that $\lambda \in (\lambda_1 a(s^*), \lambda_1^{\Omega_0} a(s^*))$. Hence, by Proposition 8 there exists at least a positive solution of (1).

2. Assume that $\underline{S} \cap [\alpha_1, \alpha_2] \neq \emptyset$. We take $\overline{s}_1 < \overline{s}_2$.

(a) Suposse $\underline{S} \cap (\overline{s}_1, \overline{s}_2) \neq \emptyset$. Take

$$\underline{s}_{j0} = \min\{\underline{S} \cap (\overline{s}_1, \overline{s}_2)\}.$$

Consider now $[\overline{s}_1, \underline{s}_{j0}]$. It is clear that

$$\frac{\lambda}{\lambda_1^{\Omega_0}} < a(s) < \frac{\lambda}{\lambda_1} \quad \text{for all } s \in [\overline{s}_1, \underline{s}_{j0}].$$

Then, by Proposition 7, we obtain

$$\lim_{s \downarrow \overline{s}_1} h_\lambda(s) = I, \quad \text{and} \quad \lim_{s \to \underline{s}_{j0}} h_\lambda(s) = 0.$$

Moreover, $g(\underline{s}_{j0}) > 0$ and $g(\overline{s}_1) < I$. Hence,

$$h_\lambda(\overline{s}_1) - g(\overline{s}_1) > 0 \quad \text{and} \quad h_\lambda(\underline{s}_{j0}) - g(\underline{s}_{j0}) < 0,$$

and then, there exists $s^* \in (\overline{s}_1, \underline{s}_{j0}) \subset (\overline{s}_1, \overline{s}_2)$ such that $h_\lambda(s^*) = g(s^*)$. Hence, by Proposition 8 there exists at least a positive solution of (1).

(b) Suposse $\underline{S} \cap (\overline{s}_1, \overline{s}_2) = \emptyset$. Then, we take $\overline{s}_2 < \overline{s}_3$. If $\underline{S} \cap (\overline{s}_2, \overline{s}_3) \neq \emptyset$, then we can repeat the previous reasoning. If $\underline{S} \cap (\overline{s}_2, \overline{s}_3) = \emptyset$ we consider $\overline{s}_3 < \overline{s_4}$. Hence, we can continue this argument until $\underline{S} \cap [\overline{s}_{m_0}, \overline{s}_{m_0+1}] \neq \emptyset$, for some m_0.

This completes the proof. □

Proposition 10 *If $g(s) < I$ for all $s \in (\alpha_1, +\infty)$ with*

$$g(\alpha_1) = I.$$

Then there exists positive solution of (1) for $\lambda \in (\Lambda_1, +\infty)$, where

$$\Lambda_1 = \left(\frac{\alpha_1}{I}\right)^{\frac{1}{p}} \lambda_1^{\Omega_0}.$$

Proof Take $\lambda > \Lambda_1$, then $\lambda/\lambda_1^{\Omega_0} > \left(\frac{\alpha_1}{I}\right)^{\frac{1}{p}} = a(\alpha_1)$. Since $g(s) < I$ for $s > \alpha_1$ it follows that $\lim_{s\to\infty} a(s) = \infty$. Then, there exist $s^*, s^{**} > \alpha_1$ such that $\lambda/\lambda_1^{\Omega_0} = a(s^*)$ and $\lambda/\lambda_1 = a(s^{**})$. Now, the argument carried out in the above Proposition can be adapted to cover this case. □

Now, we are ready to prove the main result of this section.

Theorem 9 *Assume that b verifies (Hb_2), $I < \infty$ and that there exist $s_1 < s_2 < \cdots < s_m$, $m \geq 1$, simple roots of (29) and consider*

$$\Lambda_i = \lambda_1^{\Omega_0} \left(\frac{s_i}{I}\right)^{1/p}, \quad i = 1, \ldots, m.$$

Then:

1. *From $\lambda = a(0)\lambda_1$ an unbounded continuum of positive solutions of (1) bifurcates from the trivial solution and it goes to infinity at $\lambda = \Lambda_1$. As consequence, there exists at least a positive solution of (1) if*

$$\lambda \in (\min\{a(0)\lambda_1, \Lambda_1\}, \max\{a(0)\lambda_1, \Lambda_1\}).$$

2. *If* $m = 2k + 1$, $k \geq 0$, (1) *possesses at least a positive solution for*

$$\lambda \in \bigcup_{j=1}^{k} (\Lambda_{2j}, \Lambda_{2j+1}),$$

and (1) *does not have positive solution for* λ *large.*

3. *If* $m = 2k$, $k \geq 1$, (1) *possesses at least a positive solution for*

$$\lambda \in \bigcup_{j=1}^{k-1} (\Lambda_{2j}, \Lambda_{2j+1}) \cup (\Lambda_{2k}, \infty).$$

Moreover, if a *is increasing then for any* Λ_{2j+1}, $j = 0, \ldots, k-1$, *there exists a sequence of positive solutions* $(\lambda_n, u_{\lambda_n})$ *of* (1) *such that* $\lambda_n \to \Lambda_{2j+1}$ *and*

$$\|u_{\lambda_n}\|_\infty \to \infty.$$

Proof 1. The first paragraph follows by Theorem 8.

2. Assume now that $m = 2k+1$. Then, $g(s_{2j}) = g(s_{2j+1})$ and $g(s) < I$ for $s \in (s_{2j}, s_{2j+1})$ for $j = 1, \ldots, k$. Then, we can apply Proposition 9 for $\alpha_1 = s_{2j}$ and $\alpha_2 = s_{2j+1}$. Furthermore, according to Proposition 5, (1) does not possess positive solutions for λ large.
3. When $m = 2k$, the result follows by Propositions 9 and 10 with $\alpha_1 = s_{2j}$ and $\alpha_2 = s_{2j+1}$ and $\alpha_1 = s_{2k}$, respectively.

Finally, assume that a is increasing. We are going to show that a bifurcation to infinity occurs at Λ_{2j+1}. Indeed, take $\lambda_n \uparrow \Lambda_{2j+1}$. Then,

$$\frac{\lambda_n}{\lambda_1^{\Omega_0}} \uparrow a(s_{2j+1}).$$

Then, for each n, take the unique $s_n < s_{2j+1}$ (recall that a is increasing) such that

$$a(s_n) = \frac{\lambda_n}{\lambda_1^{\Omega_0}}.$$

It is apparent that $\frac{\lambda_n}{\lambda_1} > a(s_{2j+1})$ for n large, and then there exists a unique $s^n > s_{2j+1}$ such that

$$a(s^n) = \frac{\lambda_n}{\lambda_1}.$$

Hence, $g(s_n) - h_{\lambda_n}(s_n) < 0$ and $g(s^n) - h_{\lambda_n}(s^n) > 0$. We can conclude the existence of $s_n^* \in (s_n, s^n)$ such that

$$g(s_n^*) = h_{\lambda_n}(s_n),$$

and by Proposition 8 there exists a positive solution. In fact, the positive solution is

$$u_n = a(s_n^*)\theta_{\left[\frac{\lambda_n}{a(s_n^*)}, b\right]}.$$

But, observe that when $\lambda_n \to \Lambda_{2j+1}$ then $s_n \to s_{2j+1}$ and hence $s_n^* \to s_{2j+1}$. Thus,

$$\frac{\lambda_n}{a(s_n^*)} \to \frac{\Lambda_{2j+1}}{a(s_{2j+1})} = \lambda_1^{\Omega_0}.$$

This ends the proof. ◻

Remark 2 The condition that the roots s_j are simple is only necessary to write Theorem 9 more clearly and to have that if $g(s) < I$ for $s \in (s_j, s_{j+1})$, then $g(s) > I$ for $s \in (s_{j+1}, s_{j+2})$. However, if, for instance, $g(s) < I$ for $s \in (s_j, s_{j+1}) \cup (s_{j+1}, s_{j+2})$, then it can be proved the existence of positive solution for $\lambda \in (\Lambda_j, \Lambda_{j+1}) \cup (\Lambda_{j+1}, \Lambda_{j+2})$.

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