Calculus of Variations

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Planar least gradient problem: existence, regularity and anisotropic case

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Abstract We show existence of solutions to the least gradient problem on the plane for boundary data in $BV(\partial\Omega)$. We also provide an example of a function $f \in L^1(\partial\Omega)\backslash (C(\partial\Omega) \cup BV(\partial\Omega))$, for which the solution exists. We also show non-uniqueness of solutions even for smooth boundary data in the anisotropic case for a nonsmooth anisotropy. We additionally prove a regularity result valid also in higher dimensions.

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1 Introduction

The least gradient problem is a problem of minimalization

$$\min\left\{\int_\Omega |Du|, \quad u \in BV(\Omega), \quad u|_{\partial\Omega} = f\right\}$$

arising in the study of convergence of p-Laplace functions with $p \to 1$, see [11], or as a dimensional reduction in free material design problem, see [7], or (in its anisotropic version) in the conductivity imaging problem, see [10].

Here, we may impose certain conditions on Ω and the function f and use different approaches to the boundary condition. In [18] f is assumed to be continuous and the boundary condition is in the sense of traces. The authors also impose a set of geometrical conditions on Ω, which are satisfied by strictly convex sets; in fact, in dimension two they are equivalent to strict convexity. The authors of [10] consider an anisotropic version of the problem and introduce another geometric condition on Ω called the barrier condition; see Definition 5.8.

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Another approach is presented in [13], where boundary datum belongs to $L^1(\partial\Omega)$, but the boundary condition is understood in a weaker sense.

Throughout this paper $\Omega \subset \mathbb{R}^N$ shall be an open bounded set with Lipschitz boundary. When necessary, we will additionally assume convexity or strict convexity of Ω and bounds on the dimension N. We consider the following minimalization problem called the least gradient problem:

$$\min\left\{\int_\Omega |Du|, \quad u \in BV(\Omega), \quad Tu = f\right\}, \tag{1}$$

where T denotes the trace operator $T : BV(\Omega) \to L^1(\partial\Omega)$. Unless specified otherwise, the boundary datum f belongs to $L^1(\partial\Omega)$. Even existence of solutions in this sense is not obvious, as the functional

$$F(u) = \begin{cases} \int_\Omega |Du| & \text{if } u \in BV(\Omega) \text{ and } Tu = f; \\ +\infty & \text{otherwise} \end{cases}$$

is not lower semicontinuous with respect to L^1 convergence. In fact, in [17] the authors have given an example of a function f without a solution to corresponding two-dimensional least gradient problem. It was a characteristic function of a certain fat Cantor set; note that it does not lie in $BV(\partial\Omega)$.

There are two possible ways to deal with Problem (1). The first is the relaxation of the functional F. Such reformulation and its relationship with the original statement is considered in [13] and [12]. Another way is to consider when Problem (1) has a solution in the classical sense and what is its regularity. This paper uses the latter approach and its main result is giving a sufficient condition for existence of solutions of the least gradient problem on the plane:

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, strictly convex set with C^1 boundary. Then for every $f \in BV(\partial\Omega)$ there exists a solution of the least gradient problem.*

Obviously, this condition is not necessary; the construction given in [18] for continuous boundary data does not require the boundary data to have finite total variation. We also provide an example of a function $f \in L^1(\partial\Omega)\backslash(C(\partial\Omega) \cup BV(\partial\Omega))$, for which the solution exists, see Example 4.8.

Another result included in this article provides a certain regularity property. Theorem 1.2 asserts existence of a decomposition of a function of least gradient into a continuous and a locally constant function. It is not a property shared by all BV functions, see [2, Example 4.1].

Theorem 1.2 *Let $\Omega \subset \mathbb{R}^N$, where $N \leq 7$, be an open bounded convex set. Suppose $u \in BV(\Omega)$ is a function of least gradient. Then there exist functions $u_c, u_j \in BV(\Omega)$ such that $u = u_c + u_j$ and $(Du)_c = Du_c$ and $(Du)_j = Du_j$, i.e. one can represent u as a sum of a continuous function and a function with only jump-type derivative. They are of least gradient in Ω. Moreover this decomposition is unique up to an additive constant.*

Let us clarify the notation in the above theorem. Firstly, u_c is a continuous function and u_j is a piecewise constant function; in particular, Du_c does not consist only of Cantor part of the derivative, but also the absolutely continuous part. Secondly, Du_j is the derivative of u_j, while $(Du)_j$ is the jump part of the derivative of u; the same applies to Du_c and $(Du)_c$.

The final chapter takes on the subject of anisotropy. As it was proved in [10], for a uniformly convex anisotropic norm ϕ on $\mathbb{R}^N$ smooth with respect to the Euclidean norm

there is a unique solution to the anisotropic least gradient problem. I consider l_p norms on the plane for $p \in [1, \infty]$ to show that for $p = 1, \infty$, when the anisotropy is neither smooth nor uniformly convex, the solutions need not be unique even for smooth boundary data (see Examples 5.16 and 5.17), whereas for $1 < p < \infty$, when the anisotropy is smooth and uniformly convex, Theorem 1.3 asserts that the only connected minimal surface with respect to the l_p norm is a line segment, similarly to the isotropic solution.

Theorem 1.3 *Let $\Omega \subset \mathbb{R}^2$ be an open bounded convex set. Let the anisotropy be defined by the function $\phi(x, Du) = \|Du\|_p$, where $1 < p < \infty$. Let E be a ϕ-minimal set with respect to Ω, i.e. χ_E is a function of ϕ-least gradient in Ω. Then every connected component of ∂E is a line segment.*

Finally, we observe in Corollary 5.18 that if Ω is an open bounded set with Lipschitz boundary, then Theorem 1.3 implies existence of a unique solution to the anisotropic least gradient problem for continuous boundary data.

2 Preliminaries

2.1 Least gradient functions

Now we shall briefly recall basic facts about least gradient functions. What we need most in this paper is the Miranda stability theorem, see [14], and the relationship between functions of least gradient and minimal surfaces. For more information see [4], [6].

Definition 2.1 Let $\Omega \subset \mathbb{R}^N$ be open. We say that $u \in BV(\Omega)$ is a function of least gradient, if for every compactly supported $v \in BV(\Omega)$ we have

$$\int_\Omega |Du| \leq \int_\Omega |D(u + v)|.$$

If Ω is bounded with Lipschitz boundary, we may instead assume that $v \in BV_0(\Omega)$, i.e. v is a BV function with zero trace on $\partial\Omega$. This equivalence follows from approximation of v by functions of the form $v_n = v\chi_{\Omega_n}$ for properly chosen Ω_n; a full proof can be found in [19, Theorem 2.2].

Definition 2.2 Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. We say that $u \in BV(\Omega)$ is a solution of the least gradient problem in the sense of traces for given $f \in L^1(\partial\Omega)$, if u is a function of least gradient and $Tu = f$.

To underline the difference between the two notions, we recall a stability theorem by Miranda:

Theorem 2.3 ([14, Theorem 3]) *Let $\Omega \subset \mathbb{R}^N$ be open. Suppose $\{u_n\} \subset BV(\Omega)$ is a sequence of least gradient functions in Ω convergent in $L^1_{loc}(\Omega)$ to $u \in BV(\Omega)$. Then u is a function of least gradient in Ω.* □

An identical result for solutions of least gradient problem is impossible, as the trace operator is not continuous in L^1 topology; see Example 4.7. We need an additional assumption regarding traces. A correct formulation would be:

Remark 2.4 Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. Suppose that $f_n \to f \in L^1(\partial\Omega)$. Let u_n be a solution of the least gradient problem for f_n, i.e. u_n is a function of least gradient such that $Tu_n = f_n$. If $u_n \to u$ in $L^1(\Omega)$ and $Tu = f$, then u is a solution of the least gradient problem for f. □

To deal with regularity of solutions of the least gradient problem, it is convenient to consider superlevel sets of u, i.e. sets of the form $\partial\{u > t\}$ for $t \in \mathbb{R}$. The reason for this are the two subsequent results:

Lemma 2.5 *Suppose that $u_1 \in L^1(\Omega)$ and u_2 is Lebesgue measurable. Then $u_1 = u_2$ a.e. if and only if for every $t \in \mathbb{R}$ the superlevel sets of u_1 and u_2 are equal, i.e. $\{u_1 > t\} = \{u_2 > t\}$ up to a set of measure zero.* □

Theorem 2.6 ([4, Theorem 1])
Suppose that $\Omega \subset \mathbb{R}^N$ is open. Let u be a function of least gradient in Ω. Then for every $t \in \mathbb{R}$ the set $\{u > t\}$ is minimal in Ω, i.e. $\chi_{\{u>t\}}$ is of least gradient. □

The two above results also hold, with essentially unchanged proofs, if we take $\{u \geq t\}$ in place of $\{u > t\}$. We will use Theorem 2.6 for both types of superlevel sets; in Sect. 3 it is more convenient to work with $\{u > t\}$, while in Sect. 4 due to application of variants of Sternberg–Williams–Ziemer construction, see Sect. 2.2, it is more convenient to work with $\{u \geq t\}$.

A comprehensive description of the regularity theory for minimal sets can be found in [6]. As sets of bounded perimeter are defined up to a set of measure zero, the regularity results depend on the representative; here, we employ the standard convention:

$$x \in E \text{ if and only if } \limsup_{r \to 0} \frac{|E \cap B(x,r)|}{|B(x,r)|} > 0,$$

i.e. a set E of bounded perimeter consists of points of positive density. Under this convention, [6, Theorem 10.11] implies that in dimensions $2 \leq N \leq 7$ the boundary of a minimal set E is locally an analytical hypersurface. Now, if $u \in BV(\Omega)$ is a function of least gradient, then Theorem 2.6 implies that each set $E_t = \{u \geq t\}$ is minimal; we change the representative of each set E_t to $\widetilde{E}_t$ to follow the convention above. This way we define a function $\widetilde{u}(x) = \sup\{t : x \in \widetilde{E}_t\}$ satisfying $\widetilde{E}_t = \{\widetilde{u} \geq t\}$; but Lemma 2.5 implies that u and $\widetilde{u}$ differ on a set of measure zero. Hence, after a change of representative, we get that the boundary of each superlevel set of u is a union of analytical minimal surfaces; from now on we assume that we deal with such a representative.

2.2 Sternberg–Williams–Ziemer construction

In [18] the authors have shown existence and uniqueness of solutions to the least gradient problem for continuous boundary data and strictly convex Ω (or, to be more precise, the authors assume that $\partial\Omega$ has non-negative mean curvature and is not locally area-minimizing). The proof of existence is constructive and we shall briefly recall it. The main idea is reversing Theorem 2.6 and constructing almost all level sets of the solution. According to the Lemma 2.5 this uniquely determines the solution.

We fix the boundary data $g \in C(\partial\Omega)$. By Tietze theorem it has an extension $G \in C(\mathbb{R}^N \setminus \Omega)$. We may also demand that G has support contained in a ball $B(0, R)$ containing Ω and that $G \in BV(\mathbb{R}^N \setminus \overline{\Omega})$. Let $L_t = (\mathbb{R}^N \setminus \Omega) \cap \{G \geq t\}$. Since $G \in BV(\mathbb{R}^N \setminus \overline{\Omega})$, then

for a.e. $t \in \mathbb{R}$ we have $P(L_t, \mathbb{R}^N \backslash \overline{\Omega}) < \infty$. For such t, let E_t be a set solving the following problems:

$$\min \left\{ P(E, \mathbb{R}^N) : E \backslash \overline{\Omega} = L_t \backslash \overline{\Omega} \right\}, \tag{2}$$

$$\max \left\{ |E \cap \Omega| : E \text{ is a minimizer of (2)} \right\}.$$

Let us note that both of these problems have solutions; let $m \geq 0$ be the infimum in the first problem. Let E_n be a sequence of sets such that $P(E_n, \mathbb{R}^N) \to m$. By compactness of unit ball in $BV(B(0, R))$ and lower semicontinuity of the total variation we obtain on some subsequence $\chi_{E_{n_k}} \to \chi_E$ in $L^1(B(0, R))$, where $E \backslash \overline{\Omega} = L_t \backslash \overline{\Omega}$ and

$$m \leq P(E, \mathbb{R}^N) \leq P(E_{n_k}, \mathbb{R}^N) \to m.$$

Take $M \leq |\Omega|$ be the supremum in the second problem. Take a sequence of sets E_n such that $|E_n \cap \Omega| \to M$. Then on some subsequence $\chi_{E_{n_k}} \to \chi_E$ in $L^1(B(0, R))$, and thus

$$M \geq |E \cap \Omega| \geq |E_{n_k} \cap \Omega| - |E_{n_k} \triangle E| = |E_{n_k} \cap \Omega| - \|\chi_{E_{n_k}} - \chi_E\|_1 \to M - 0.$$

Then we can show existence of a set T of full measure such that for every $t \in T$ we have $\partial E_t \cap \partial \Omega \subset g^{-1}(t)$ and for every $t, s \in T$, $s < t$ the inclusion $\overline{E_t} \subset \text{int } E_s$ holds. It enables us to treat E_t as superlevel sets of a certain function; we define it on $\overline{\Omega}$ by the following formula:

$$u(x) = \sup\{t \in T : x \in \overline{E_t \cap \Omega}\}.$$

It turns out that $u \in C(\overline{\Omega}) \cap BV(\Omega)$ and u is a solution to the least gradient problem for g. Moreover $|\{u \geq t\} \triangle (\overline{E_t \cap \Omega})| = 0$ for a.e. $t \in \mathbb{R}$. Uniqueness proof is based on the following comparison principle:

Theorem 2.7 ([18, Theorem 4.1]) *Suppose that $\Omega \subset \mathbb{R}^N$ is an open bounded set with Lipschitz boundary such that $\partial \Omega$ has non-negative mean curvature and is not locally area-minimizing. Let $g_1, g_2 \in C(\partial \Omega)$ satisfy $g_1 \geq g_2$ on $\partial \Omega$. Then the corresponding minimizers to the least gradient problem satisfy $u_1 \geq u_2$ in Ω.* □

In the existence proof in Sect. 4 we are going to use a particularly simple case of the construction. Suppose that $\Omega \subset \mathbb{R}^2$ and $g \in C^1(\partial \Omega)$. Firstly, notice that we only have to construct the set E_t for almost all t. Secondly, we recall that in dimension two the only connected minimal surface is a line segment; thus, to find the set E_t, let us fix t and look at the preimage $g^{-1}(t)$. We connect its points using line segments with sum of their lengths as small as possible. It can cause problems, for example if we take t to be a global maximum of the function; thus, let us take t to be a regular value (by Sard theorem almost all values are regular), so the preimage $g^{-1}(t)$ contains finitely many points. As the derivative at every point $p \in g^{-1}(t)$ is nonzero, there is at least one line segment $L \subset \partial E_t$ ending in p. By minimality of ∂E_t there can be at most one (we prove this rigorously even for discontinuous g in Proposition 3.5), so there is exactly one line segment $L \subset \partial E_t$ ending in every $p \in g^{-1}(t)$.

A typical example for the construction, attributed to John Brothers, is to let $\Omega = B(0, 1) \subset \mathbb{R}^2$ and take the boundary data to be (in polar coordinates, for fixed $r = 1$) the function $g : [0, 2\pi) \to \mathbb{R}$ given by the formula $g(\theta) = \cos(2\theta)$; see [13, Example 2.7] or [20, Example 3.6].

The proof of existence on numerous occasions relies heavily on the continuity of the boundary data g: firstly, via Tietze extension theorem; secondly, to prove that $\partial E_t \cap \partial \Omega \subset g^{-1}(t)$, where for discontinuous g we would have to add to the right hand side the set of discontinuity points; finally, in order for u to be well-defined we need the fact that $\overline{E_t} \subset \text{int}$

E_s for $s < t$, which does not necessarily hold for discontinuous g. Thus we may not directly apply the above construction to discontinuous boundary data.

2.3 BV on a one-dimensional compact manifold

Using a partition of unity, one may attempt to define BV spaces on arbitrary compact manifolds; such approach is presented in [3]. It is not necessary for us; it suffices to consider one-dimensional case. Let us consider $\Omega \subset \mathbb{R}^2$ open and bounded with C^1 boundary. We consider functions on $\partial\Omega$ integrable with respect to the one-dimensional Hausdorff measure (which are approximately continuous $\mathscr{H}^1$-a.e.). We recall (see [5, Chapter 5.10]) that the one-dimensional BV space on the interval $(a, b) \subset \mathbb{R}$ may be described in the following way:

$$f \in BV((a, b)) \Leftrightarrow \sum |f(x_i) - f(x_{i-1})| \leq M < \infty$$

for every $a < x_0 < \cdots < x_n < b$, where x_i are points of approximate continuity of f. The smallest such constant M turns out to be the usual total variation of f.

  We may extend this definition to the case where we have a one-dimensional manifold diffeomorphic to an open interval if it is properly parametrized, i.e. all tangent vectors have length one. Repeating the proof from [5] we get that this definition coincides with the divergence definition. Then we extend it to the case of a one-dimensional compact connected manifold in the following way:

Definition 2.8 Let $\Omega \subset \mathbb{R}^2$ be an open bounded set with C^1 boundary. We say that $f \in BV(\partial\Omega)$, if after removing from $\partial\Omega$ a point p of approximate continuity of f we have $f \in BV(\partial\Omega\setminus\{p\})$. The norm is defined to be

$$\|f\|_{BV(\partial\Omega)} = \|f\|_{BV(\partial\Omega\setminus\{p\})}.$$

This definition does not depend on the choice of p, as in dimension one the total variation on disjoint intervals is additive, thus for different points p_1, p_2 we get that

$$\|f\|_{BV(\partial\Omega\setminus\{p_1\})} = \|f\|_{BV((p_1, p_2))} + \|f\|_{BV((p_2, p_1))} = \|f\|_{BV(\partial\Omega\setminus\{p_2\})},$$

where (p_1, p_2) is an oriented arc from p_1 to p_2. Thus all local properties of $BV(\partial\Omega)$ hold; we shall recall the most important one for our considerations:

Proposition 2.9 *Let $\Omega \subset \mathbb{R}^2$ be an open bounded set with C^1 boundary and let $E \subset \partial\Omega$ be a set of finite perimeter, i.e. $\chi_E \in BV(\partial\Omega)$. Then $\partial E = \partial^* E = \{p_1, \ldots, p_{2n}\}$ and $P(E, \partial\Omega) = 2n$. Here $\partial^* E$ denotes the reduced boundary of E, i.e. the set where a measure-theoretical normal vector exists; see [5, Chapter 5.7].* □

  However, some global properties need not hold. For example, the decomposition theorem $f = f_{ac} + f_j + f_s$ does not hold; consider $\Omega = B(0, 1)$, $f = \arg(z)$. The main reason is that $\pi_1(\partial\Omega) \neq 0$.

3 Regularity of least gradient functions

In this section we are going to prove several regularity results about functions of least gradient, valid up to dimension 7. We start with a weak form of the maximum principle and later prove a result on decomposition of a least gradient function into a continuous and jump-type part;

this decomposition holds not only at the level derivatives, but also at the level of functions. We will extensively use the regularity theory for area-minimizing sets, the monotonicity formula for minimal surfaces and a version of the maximum principle for minimal graphs. Moreover, as Theorem 2.6 works also for superlevel sets of the form $\{u \geq t\}$, all results in this Section also hold with $\{u \geq t\}$ in place of $\{u > t\}$.

Proposition 3.1 ([6, Theorem 10.11]) *Suppose that* $N \leq 7$ *and* E *is a minimal set in* $B(x, r) \subset \mathbb{R}^N$. *Then* $\partial E \cap B(x, r)$ *is an analytic hypersurface.* $\qquad\square$

Proposition 3.2 ([18, Theorem 2.2]) *Suppose* $E_1 \subset E_2$ *and let* ∂E_1, ∂E_2 *be area-minimizing in an open set* $\Omega \subset \mathbb{R}^N$. *Further, suppose* $x \in \partial E_1 \cap \partial E_2 \cap \Omega$. *Then* ∂E_1 *and* ∂E_2 *agree in some neighbourhood of* x. $\qquad\square$

A well-known consequence of the monotonicity formula for minimal surfaces, see [16, Theorem 17.6], is the following:

Proposition 3.3 ([16, Remark 37.9]) *Let* $\Omega \subset \mathbb{R}^N$, *where* $N \leq 7$, *be open and suppose that* E *is area-minimizing in* Ω. *Then for any compact* $K \subset \Omega$ *there are only finitely many connected components of* ∂E *intersecting* K. $\qquad\square$

The result above is originally stated more generally; this version takes into account the regularity theory for area-minimizing sets as stated in Proposition 3.1.

Firstly, we provide a reformulation of Theorem 2.6 taking into account the regularity theory for area-minimizing sets in low dimensions. It is an analogue of the maximum principle for linear equations; geometrically speaking, the linear weak maximum principle states that every level set touches the boundary. As by Theorem 2.6 for every $t \in \mathbb{R}$ the set $\{u > t\}$ is area-minimizing in Ω, in view of the above results we obtain

Proposition 3.4 (weak maximum principle) *Let* $\Omega \subset \mathbb{R}^N$, *where* $N \leq 7$, *be an open set and suppose* $u \in BV(\Omega)$ *is a function of least gradient. Then for every* $t \in \mathbb{R}$ *we have* $\partial\{u > t\} = \bigcup_{i \in I} S_{t,i}$, *where* $S_{t,i}$ *are smooth minimal surfaces without boundary in* Ω *and this union is locally finite.* $\qquad\square$

For bounded convex sets on the plane it can be strengthened so that the connected components of $\partial\{u > t\}$ cannot intersect even on the boundary of Ω. We recall that bounded convex sets in $\mathbb{R}^N$ automatically have Lipschitz boundary, so the trace operator is well defined.

Proposition 3.5 (weak maximum principle on the plane) *Let* $\Omega \subset \mathbb{R}^2$ *be an open bounded convex set and suppose* $u \in BV(\Omega)$ *is a function of least gradient. Then for every* $t \in \mathbb{R}$ *we have* $\partial\{u > t\} = \bigcup_{i \in I} L_{t,i}$, *where* $L_{t,i}$ *are line segments with ends on* $\partial\Omega$ *and this union is locally finite. Furthermore,* $\overline{L_{t,i}}$ *are pairwise disjoint in* $\overline{\Omega}$.

Proof There is only one thing left to prove: that two connected components of $\partial E_t = \partial\{u > t\}$ do not intersect on $\partial\Omega$. Suppose we have at least two line segments in ∂E_t: $\overline{xy}$ and $\overline{xz}$. Due to Proposition 3.3, there are at most finitely many line segments of the form $\overline{xx'} \subset \partial E_t$ intersecting the triangle $\triangle xyz$ enclosed by the line segments $\overline{xy}$, $\overline{xz}$, $\overline{yz}$ (as Ω is convex, we have $\triangle xyz \subset \Omega$); without loss of generality we may assume that $\overline{xy}$ and $\overline{xz}$ are adjacent.

Consider the function χ_{E_t}. In the triangle $\triangle xyz$ we have $\chi_{E_t} = 1$ and $\chi_{E_t} = 0$ on the opposite side of $\overline{xy}$, $\overline{xz}$ (or the opposite situation, which we handle similarly). Now, we see that the function $\widetilde{\chi_{E_t}} = \chi_{E_t} - \chi_{\triangle xyz}$ has strictly smaller total variation due to the triangle inequality. While the difference between $\widetilde{\chi_{E_t}}$ and χ_{E_t} is not compactly supported in Ω, these functions have the same trace, so in view of the Definition 2.1 we deduce that χ_{E_t} is not a function of least gradient. Thus in every point of $\partial\Omega$ ends at most one line segment from ∂E_t. $\qquad\square$

The next Corollary is a direct consequence of Proposition 3.4.

Corollary 3.6 *Let $\Omega \subset \mathbb{R}^N$, where $N \leq 7$, be an open set and take $u \in BV(\Omega)$ to be a least gradient function. Let $E_t = \{u > t\}$. For each $x \in \Omega$ and $t \in \mathbb{R}$ there exists a ball $B(x, r)$ such that there is at most one connected component of ∂E_t intersecting this ball.* $\square$

Remark 3.7 Even in two dimensions the above result may fail for $x \in \partial\Omega$. Even though Proposition 3.5 excludes the possibility of intersection of two connected components of ∂E_t on $\partial\Omega$, there still may be points in the neighbourhood of which there are infinitely many connected components of ∂E_t; see Examples 4.7 and 4.8.

Now, we recall the definitions of the jump set J_u and the approximate discontinuity set S_u (see [2, Chapter 3]):

Definition 3.8 Let $u \in BV(\Omega)$. We say that $x \in J_u$, the jump set of u, if there exists a unit vector v (called the normal vector) and real numbers $a \neq b$ such that

$$\lim_{r \to 0} \fint_{B(x,r) \cap \{\langle y-x,v \rangle > 0\}} |u(y) - a| dy = 0,$$

$$\lim_{r \to 0} \fint_{B(x,r) \cap \{\langle y-x,v \rangle < 0\}} |u(y) - b| dy = 0.$$

The triple (a, b, v) is uniquely determined up to permutation of a, b and the sign of v.

We say that $x \in S_u$, the approximate discontinuity set of u, if the upper and lower approximate limits of u at x do not coincide, i.e. $u^\wedge(x) \neq u^\vee(x)$.

Proposition 3.9 *Let $\Omega \subset \mathbb{R}^N$, where $N \leq 7$, be an open set and let $u \in BV(\Omega)$ be a function of least gradient in Ω. Then*

$$J_u = S_u = \bigcup_{s,t \in \mathbb{Q}; s \neq t} (\partial^* \{u > s\} \cap \partial^* \{u > t\}).$$

As $J_u = S_u$, due to our choice of representative [9, Theorem 4.1] *implies that u has only jump-type discontinuities, i.e. is continuous on $\Omega \setminus J_u$.*

Proof The inclusion $J_u \subset S_u$ is obvious from the definitions. Now, we will prove that S_u is a subset of the right hand side. Let $x \in S_u$. By definition of S_u for every $t \in (u^\vee(x), u^\wedge(x))$ the density of $\{u > t\}$ at x is neither 0 nor 1, so $x \in \partial_M \{u > t\}$, where ∂_M denotes a measure-theoretical boundary of a set. As for $N \leq 7$ with our choice of representative the boundary of $\{u > t\}$ coincides with its measure-theoretic boundary and its reduced boundary, we obtain the second inclusion: if $x \in S_u$, then $x \in \partial^* \{u > t\}$ for every t in some open interval.

Finally, we will prove that the right hand side is a subset of J_u. Fix $x \in \partial^* \{u > s_0\} \cap \partial^* \{u > t_0\}$. Using Corollary 3.6 we choose a ball $B = B(x, r_0)$ such that there is one connected component S_{t_0} of $\partial \{u > t_0\}$ and S_{s_0} of $\partial \{u > s_0\}$ intersecting this ball. Due to Proposition 3.4 they are minimal surfaces and by Proposition 3.2 we have $S_{t_0} = S_{s_0} = S$. Now we define two auxiliary functions from S to $\mathbb{R}$:

$$u^+(y) = \sup\{t : y \in \partial^* \{u > s_0\} \cap \partial^* \{u > t\}\};$$
$$u^-(y) = \inf\{t : y \in \partial^* \{u > s_0\} \cap \partial^* \{u > t\}\}.$$

Using an argument as above, we notice that due to Proposition 3.2 both u^+ and u^- are constant along S and for each $t \in (u^-, u^+)$ the connected component S_t of $\partial \{u > t\}$ passing

through x coincides with S. In particular the normal vectors at x coincide, so the normal at x does not depend on $t \in (u^-, u^+)$.

We introduce again the notation $E_t = \{u > t\}$. Notice that the set $B \setminus S$ is a union of two disjoint connected open sets B^+ and B^-, where $B^+ = (B \setminus S) \cap E_{t_0}$ and $B^- = (B \setminus S) \setminus E_{t_0}$; as B is convex, this either can be seen directly or (as it is homeomorphic to $\mathbb{R}^N$) as a simple case of the Alexander duality theorem, see [8, Theorem 27.10]. We will show that u^+ and u^- coincide with the trace values of u at x along S from B^+ and B^- respectively. Let $T_S^{B^\pm} u$ denote the trace of u on S from $B^\pm$; in this notation, our goal is to show that $T_S^{B^\pm} u(x) = u^\pm$, so $x \in J_u$.

We will prove the above equality for u^+ (the other case is analogous). Firstly, we use Corollary 3.6 to notice that for every $t \in (u^-, u^+)$ there exists a smaller ball $B(x, r) \subset B$ such that S is the only one connected component of ∂E_t intersecting this ball. Again, $B(x, r) \setminus S$ is a union of two disjoint open sets $B(x, r) \cap B^\pm$ and we have $B(x, r) \cap B^+ \subset E_t$ and $B(x, r) \cap B^- \cap E_t = \emptyset$. As trace is a positive operator, the trace of u on S from $B(x, r) \cap B^+ = (B(x, r) \setminus S) \cap E_t$ is greater or equal to t; thus the trace at x is greater or equal to u^+. We have to prove the opposite bound, i.e. $T_S^{B^+} u(x) \leq u^+$.

Suppose that there is no radius $r > 0$ such that u is constant in $B(x, r) \cap B^+$ (otherwise the result is obvious). Let us notice that

$$u^+ = \liminf_{r \to 0} \left\{ s \in \mathbb{R}, \ s > u^+ : \ \partial E_s \cap B(x, r) \cap B^+ \neq \emptyset \right\}. \tag{3}$$

Let us denote the limit above by s^+; it is clearly greater or equal to u^+. The other inequality follows from definition of u^+: if $s^+ > u^+$, then $x \in \partial\{u > \frac{u^+ + s^+}{2}\}$, contradiction with maximality of u^+.

Now take $s > u^+$ arbitrarily close to u^+. By Corollary 3.6 there is a smaller ball $B(x, \rho)$ such that there is at most one connected component of ∂E_s intersecting this ball. If no connected component of ∂E_s intersects $B(x, \rho)$, then $B(x, \rho) \subset \Omega \setminus E_s$, so $u^+ \leq T_S^{B(x,\rho) \cap B^+} u \leq s$ and we let $s \to u^+$ to obtain that $T_S^{B^+} u(x) \leq u^+$.

If there is a connected component of ∂E_s intersecting $B(x, \rho)$, let us denote it by S'. We have two possibilities: either the subset of $B(x, \rho)$ bounded by S and S' is a subset of $B^+ \cap E_s$ or it is a subset of $B^+ \setminus E_s$. We see that the first case is impossible by taking sufficiently small r in Eq. (3). In the second case we obtain on an even smaller ball $B(x, \rho')$, on which $u^+ < u \leq s$, so again $u^+ \leq T_S^{B(x,\rho') \cap B^+} u \leq s$ and by letting $s \to u^+$ we obtain that $T_S^{B^+} u(x) \leq u^+$. Thus the traces from $B(x, r)^\pm$ are well defined and distinct, so $x \in J_u$. $\square$

This result is optimal, as it fails for least gradient functions in higher dimensions:

Example 3.10 Let $E \subset \mathbb{R}^8$ be an open set, whose boundary is the Simons cone. Then χ_E is a function of least gradient, $0 \notin J_u$ and $0 \in S_u$. Also χ_E is not continuous at 0. $\square$

Corollary 3.11 *Let $\Omega \subset \mathbb{R}^N$, where $N \leq 7$, be an open set and let $u \in BV(\Omega)$ be a least gradient function in Ω. Then $J_u = \bigcup_{k \in \mathscr{J}} S_k$, where S_k are pairwise disjoint minimal surfaces and $\mathscr{J}$ is at most countable.*

Proof We follow the characterisation of J_u from Proposition 3.9. For every t the set $\partial^*\{u > t\}$ is a minimal surface. Proposition 3.2 ensures that if $\partial^*\{u > s\} \cap \partial^*\{u > t\} \neq \emptyset$, then their intersecting connected components $S_{s,i}$, $S_{t,j}$ coincide. Thus connected components of J_u coincide with connected components of $\partial^*\{u > t\}$ for some t, so by weak maximum principle

they are minimal surfaces non-intersecting in Ω without boundary in Ω. From Proposition 3.9 it is clear that this union is at most countable (but not necessarily locally finite). □

Corollary 3.12 *Let $\Omega \subset \mathbb{R}^N$, where $N \leq 7$, be an open set and let $u \in BV(\Omega)$ be a least gradient function in Ω. Let $E_t = \{u > t\}$ and denote by S_k the connected components of ∂E_t. Then the trace of u from both sides is constant along each S_k.*

Proof Take $x \in \partial E_t$. Let S be the (unique) connected component of ∂E_t passing through x. Using Proposition 3.9 we notice that either $S \cap J_u = \emptyset$ or $S \subset J_u$. In the former case u is continuous at every point of S, so by our choice of representative the trace of u is constant along S. Thus we only have to consider the latter case, i.e. $S \subset J_u$. Now, we define u^+ and u^- as in Proposition 3.9; they are constant along S and they coincide with trace of u from both sides of S. □

Now we have all the tools to prove Theorem 1.2. The notation in the theorem and its proof is described in the introductory chapter after the Theorem is formulated.

Proof of Theorem 1.2 1. From Corollary 3.11 $J_u = \bigcup_{k \in \mathscr{J}} S_k$, where S_k are pairwise disjoint minimal surfaces without boundary in Ω and $\mathscr{J}$ is at most countable. Fix any $x_0 \in \Omega \backslash J_u$. As Ω is convex, each of the surfaces S_k divides Ω into two open connected sets. While these two sets do not depend on the choice of x_0, we denote them $\Omega_k'(x_0)$ and $\Omega_k''(x_0)$; this numbering is chosen so that $x_0 \in \Omega_k'(x_0)$. To such an edge we ascribe a weight $a_k(x_0)$ in the following way: from Corollary 3.12 the trace of u from $\Omega_k'(x_0)$ (and $\Omega_k''(x_0)$) along S_k is constant. Now, to each surface S_k we assign the constant (signed) jump $a_k(x_0) = T_{S_{ij}}^{\Omega_k''(x_0)} u - T_{S_{ij}}^{\Omega_k'(x_0)} u$.

2. Let $x \in \Omega \backslash J_u$. Let $\mathscr{P}$ denote the set of such k that $x \in \Omega_k''(x_0)$, i.e. such that x_0 and x lie on the opposite sides of S_k. We define $u_j(\cdot, p, x_0)$ by the formula

$$u_j(x, x_0) = \sum_{k \in \mathscr{P}} a_k(x_0) = \sum_k a_k(x_0) \chi_{\Omega_k''(x_0)}(x). \tag{4}$$

Heuristically, this is the function that counts all the jumps of u along a path P going from x_0 to x; see also Remark 3.15.

3. The function $u_j(x) = u_j(x, x_0)$ does not depend on the choice of x_0 up to an additive constant: choose instead some $x_1 \in \Omega \backslash J_u$. For each k, we either have $x_1 \in \Omega_k'(x_1) = \Omega_k'(x_0)$ or $x_1 \in \Omega_k'(x_1) = \Omega_k''(x_0)$. Take any $x \in \Omega \backslash J_u$ and let $\mathscr{P}'$ denote the set of such k that $x \in \Omega_k''(x_1)$.

   We notice that if $k \in \mathscr{P} \cap \mathscr{P}'$, then $\Omega_k'(x_1) = \Omega_k'(x_0)$, so S_k does not separate x_0 and x_1. By the definition of the jump function a_k we also have $a_k(x_1) = a_k(x_0)$. Moreover, $k \in \mathscr{P} \triangle \mathscr{P}'$ if and only if $\Omega_k'(x_1) = \Omega_k''(x_0)$, i.e. S_k separates x_0 and x_1; in this case the traces in the definitions of $a_k(x_0)$ and $a_k(x_1)$ appear in opposite order and $a_k(x_1) = -a_k(x_0)$. We calculate

$$u_j(x, x_0) = \sum_{k \in \mathscr{P}} a_k(x_0) = \sum_{k \in \mathscr{P} \cap \mathscr{P}'} a_k(x_0) + \sum_{k \in \mathscr{P} \backslash \mathscr{P}'} a_k(x_0) = \sum_{k \in \mathscr{P} \cap \mathscr{P}'} a_k(x_1)$$

$$+ \sum_{k \in \mathscr{P} \backslash \mathscr{P}'} a_k(x_0) + \sum_{k \in \mathscr{P}' \backslash \mathscr{P}} a_k(x_1) - \sum_{k \in \mathscr{P}' \backslash \mathscr{P}} a_k(x_1) = \sum_{k \in \mathscr{P}'} a_k(x_1) + \sum_{k \in \mathscr{P} \backslash \mathscr{P}'} a_k(x_0)$$

$$- \sum_{k \in \mathscr{P}' \backslash \mathscr{P}} a_k(x_1) = u_j(x, x_1) + \sum_{k \in \mathscr{P} \backslash \mathscr{P}'} a_k(x_0) + \sum_{k \in \mathscr{P}' \backslash \mathscr{P}} a_k(x_0)$$

$$= u_j(x, x_1) + \sum_{k \in \mathscr{P} \triangle \mathscr{P}'} a_k(x_0) = u_j(x, x_1) + u_j(x_1, x_0),$$

as $\mathscr{P} \triangle \mathscr{P}'$ is precisely the set of such k that S_k separates x_0 and x_1. Thus, if we chose some x_1 in place of x_0, the function u_j would change by a summand $u_j(x_1, x_0)$. We see that $u_j \in L^1(\Omega)$ and that it takes at most countably many values.

4. We notice that $Du_j = (Du)_j$: as $u_j = \sum_k a_k(x_0)\chi_{\Omega''_k(x_0)}$, the distributional derivative of u_j is concentrated on $J_u = \bigcup_k S_k = \bigcup_k \partial\Omega''_k(x_0)$. We notice that $J_{u_j} = J_u$ and the jumps along connected components of J_u have the same magnitude. We define $u_c = u - u_j$ and see that $(Du_c)_j = 0$.

5. The u_c, u_j defined above are functions of least gradient.
   Suppose that u_j is not a a function of least gradient, i.e. there exists $v \in BV(\Omega)$ with compact support such that $\int_\Omega |D(u_j + v)| < \int_\Omega |Du_j|$. Then we would get

$$\int_\Omega |Du| \leq \int_\Omega |D(u + v)| = \int_\Omega |D(u_c + u_j + v)|$$

$$\leq \int_\Omega |Du_c| + \int_\Omega |D(u_j + v)| < \int_\Omega |Du_c| + \int_\Omega |Du_j| = \int_\Omega |Du|,$$

where the first inequality follows from u being a function of least gradient, and the last equality from measures Du_c and Du_j being mutually singular. The proof for u_c is analogous.

6. The function u_c is continuous. As u_c is of least gradient, then if it isn't continuous at $x \in \Omega$, then by Proposition 3.9 a certain set of the form $\partial\{u_c > t\}$ passes through x. Let S be a connected component of $\partial\{u_c > t\}$ containing x. Again, as u_c is of least gradient, by Corollary 3.12 u_c has a constant nonzero jump along S, which is impossible as $(Du_c)_j = 0$.

7. What is left is to prove uniqueness of such a decomposition. Let $u = u_c^1 + u_j^1 = u_c^2 + u_j^2$. Changing the order of summands we obtain

$$u_c^1 - u_c^2 = u_j^2 - u_j^1,$$

but the distributional derivative of the left hand side is a continuous measure, and the distributional derivative of the right hand side is supported on the set of zero measure with respect to $\mathscr{H}^{N-1}$, so both of them are zero measures. But the condition $Dv = 0$ implies $v = \mathrm{const}$, so the functions u_c^1, u_c^2 differ by an additive constant. □

Example 3.13 In this decomposition u_c is not necessarily continuous up to the boundary. Let us use the complex numbers notation for the plane. We take $\Omega = B(1, 1)$. Let the boundary values be given by the formula $f(z) = \arg(z)$. Then $u = u_c = \arg(z) \in BV(\Omega) \cap C^\infty(\Omega)$, but u isn't continuous at $(0, 0) \in \partial\Omega$. □

Remark 3.14 If J_u is closed relative to Ω, for example if $\mathscr{H}^{N-1}(J_u) < \infty$ (it suffices as each connected component of J_u is a smooth hypersurface, closed relative to Ω), then $\Omega \setminus J_u$ is open, each of its connected components U_i is open and the function u_j is piecewise constant: from Eq. (4) we see that it is constant on U_i.

Remark 3.15 In two dimensions, we may give a clear geometrical meaning to the function u_j. Take a path P, which parametrizes a (finite) polygonal chain with vertices in $\Omega \setminus J_u$ such that $P(0) = x_0$, $P(1) = x$ (for example we may take P to parametrize the line segment $\overline{x_0 x}$). The path P intersects an at most countable family of the line segments S_k. If P passes through S_k from $\Omega'_k(x_0)$ to $\Omega''_k(x_0)$, then we add to u_j the summand $a_k(x_0)$; if in the opposite direction, then we add the summand $-a_k(x_0)$. As we see, some of the summands may cancel

out. Then we may prove that this definition does not depend on the choice of P and on the choice of x_0 up to an additive constant.

Using this line of reasoning in higher dimensions we run into geometric difficulties, such as the fact that even if P parametrizes a line segment, it may intersect S_k multiple times and these intersections might be not transversal, in which case we do not know how to assign the sign of $a_k(x_0)$. The proof of Theorem 1.2 as presented above defines the same function u_j and avoids these geometric difficulties.

4 Existence of solutions on the plane

We will prove existence of solutions on the plane for boundary data in $BV(\partial\Omega)$. We are going to use approximations of the solution in strict topology: firstly, in Proposition 4.1 we will ensure that existence of convergent sequences of approximations in $L^1(\Omega)$ topology is not a problem; then, we will upgrade it to strict convergence in Theorem 4.6 and use the Miranda stability theorem (Theorem 2.3) to end the proof. Later, we will see an example of a discontinuous function f of infinite total variation such that the solution to the corresponding least gradient problem exists.

Proposition 4.1 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. Suppose that $u_n \in BV(\Omega)$ are functions of least gradient with traces f_n, which converge to f in $L^1(\partial\Omega)$. Then u_n has a convergent subsequence $u_{n_k} \to u$ in $L^1(\Omega)$.*

Note that this does not imply that $Tu = f$, as the Proposition only provides convergence in $L^1(\Omega)$.

Proof As the trace operator is a surjection, by the Open Mapping Theorem it is open. Let us fix $\widetilde{f} \in BV(\Omega)$ such that $T\widetilde{f} = f$ and a sequence of positive numbers $\varepsilon_n \to 0$. Then by continuity and openness of T the image of a ball $B(\widetilde{f}, \varepsilon_n)$ contains a smaller ball $B(T\widetilde{f}, \delta_n)$ for another sequence of positive numbers $\delta_n \to 0$. As $f_n \to f$ in $L^1(\partial\Omega)$, there exists a subsequence f_{n_k} such that $f_{n_k} \in B(f, \delta_n) = B(T\widetilde{f}, \delta_n)$, so the set $T^{-1}(f_{n_k})$ is non-empty; there exists a preimage of f_{n_k} by T in $B(\widetilde{f}, \varepsilon_n)$. Let us call it $\widetilde{f}_n$. Obviously $\widetilde{f}_n \to \widetilde{f}$ in $BV(\Omega)$.

Thus, after possibly passing to a subsequence, there exist functions $\widetilde{f}_n, \widetilde{f}$ such that $\widetilde{f}_n \to \widetilde{f}$ in $BV(\Omega)$ and $T\widetilde{f}_n = f_n, T\widetilde{f} = f$. Now we may proceed as in [9, Proposition 3.3]. Let us estimate from above the norm of $\|u_n - \widetilde{f}_n\|_{BV}$:

$$\|u_n - \widetilde{f}_n\|_{BV} = \|u_n - \widetilde{f}_n\|_1 + \int_\Omega |D(u_n - \widetilde{f}_n)| \le (C+1) \int_\Omega |D(u_n - \widetilde{f}_n)|$$

$$\le (C+1)\left(\int_\Omega |Du_n| + \int_\Omega |D\widetilde{f}_n|\right) \le 2(C+1) \int_\Omega |D\widetilde{f}_n| \le M < \infty$$

where the inequalities follow from Poincaré inequality (as $u_n - f_n$ has trace zero), triangle inequality and the fact that u_n is solution of the least gradient problem for f_n. The common bound follows from convergence of $\widetilde{f}_n$.

Thus, by compactness of the unit ball of $BV(\Omega)$ in $L^1(\Omega)$ we get a convergent subsequence $u_{n_k} - \widetilde{f}_{n_k} \to v$ in $L^1(\Omega)$. But $\widetilde{f}_n \to \widetilde{f}$ in $BV(\Omega)$, so as well in $L^1(\Omega)$; thus $u_{n_k} \to v + \widetilde{f} = u$ in $L^1(\Omega)$. □

We are going to need three lemmas. The first two are straightforward and their proofs can be found as a step in the proof of co-area formula, see [5, Section 5.5]. The third one is a convenient version of Fatou lemma.

Lemma 4.2 *Let $f_n \to f$ in $L^1(\Omega)$. Then there exists a subsequence f_{n_k} such that $\chi_{\{f_{n_k} \geq t\}} \to \chi_{\{f \geq t\}}$ in $L^1(\Omega)$ for a.e. $t \in \mathbb{R}$.* □

Lemma 4.3 *Let $f, f_n \in L^\infty(\Omega)$ and suppose that f, f_n form a bounded family in $L^\infty(\Omega)$. If $\chi_{\{f_n \geq t\}} \to \chi_{\{f \geq t\}}$ in $L^1(\Omega)$ for a.e. $t \in \mathbb{R}$, then $f_n \to f$ in $L^1(\Omega)$.* □

Lemma 4.4 *Suppose that $g, g_n : \Omega \to \mathbb{R}$ are nonnegative. If additionally for a.e. $x \in \Omega$ we have $g(x) \leq \liminf_{n \to \infty} g_n(x)$ and $\lim_{n \to \infty} \int_\Omega g_n \, dx = \int_\Omega g \, dx < \infty$, then $g_n \to g$ in $L^1(\Omega)$.*

Proof Let $f_+ = \max(f, 0)$ and $f_- = \max(-f, 0)$. Let us note that

$$\int_\Omega |g - g_n| dx = \int_\Omega (g - g_n)_+ dx + \int_\Omega (g - g_n)_- dx$$

and

$$0 \leftarrow \int_\Omega (g - g_n) dx = \int_\Omega (g - g_n)_+ dx - \int_\Omega (g - g_n)_- dx,$$

so it suffices to prove that $\int_\Omega (g - g_n)_+ dx \to 0$ to show that $g_n \to g$ in $L^1(\Omega)$. Now let us see what happens to (well defined) upper limit of the sequence $\int_\Omega (g - g_n)_+ dx$:

$$0 \leq \limsup_{n \to \infty} \int_\Omega (g - g_n)_+ dx \leq \int_\Omega \limsup_{n \to \infty} (g - g_n)_+ dx = \int_\Omega \limsup_{n \to \infty} \max(g - g_n, 0) dx$$

$$= \int_\Omega \max(g + \limsup_{n \to \infty} (-g_n), 0) dx = \int_\Omega \max(g - \liminf_{n \to \infty} g_n, 0) dx = \int_\Omega 0 \, dx = 0.$$

where inequality follows from the (inverse) Fatou lemma: by definition $0 \leq (g - g_n)_+ \leq g$, and g is integrable, so we can apply the Fatou lemma. To prove equalities we use the fact that $\limsup_{n \to \infty} (-g_n) = -\liminf_{n \to \infty} g_n$ and the assumption that $g \leq \liminf_{n \to \infty} g_n$ a.e. Thus $\int_\Omega (g - g_n)_+ dx \to 0$, so $g_n \to g$ in $L^1(\Omega)$. □

We recall the definition of strict convergence in $BV(\Omega)$.

Definition 4.5 Let $\Omega \subset \mathbb{R}^N$ be an open set. We say that a sequence $u_n \in BV(\Omega)$ converges strictly to a function $u \in BV(\Omega)$, if $u_n \to u$ in $L^1(\Omega)$ and total variations of u_n converge to the total variation of u, i.e.

$$\int_\Omega |Du_n| \to \int_\Omega |Du|.$$

We are going to use two properties of strict convergence: firstly, for any function $u \in BV(\Omega)$ we can find a sequence $u_n \in C^\infty(\Omega) \cap BV(\Omega)$ converging to u in strict topology. Secondly, the trace operator is continuous with respect to strict convergence.

Theorem 4.6 *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, strictly convex set with C^1 boundary and suppose $f \in BV(\partial\Omega)$. Let $f_n \to f$ strictly in $BV(\partial\Omega)$, where f_n are smooth. Denote the unique solution of the least gradient problem for f_n by u_n. Then on some subsequence u_{n_k} we have strict convergence in $BV(\Omega)$ to a function $u \in BV(\Omega)$. In particular $Tu = f$.*

Proof 1. As we have $f_n \to f$ strictly in $BV(\partial\Omega)$, we by definition also convergence in $L^1(\partial\Omega)$. Thus, by Lemma 4.2, after possibly passing to a subsequence we have convergence $\chi_{\{f_n \geq t\}} \to \chi_{\{f \geq t\}}$ for a.e. t.

2. Let $E_t = \{f \geq t\}$ and $E_t^n = \{f_n \geq t\}$. By co-area formula

$$\int_{\partial\Omega} |Df_n| = \int_{\mathbb{R}} P(E_t^n, \partial\Omega)\,dt \to \int_{\mathbb{R}} P(E_t, \partial\Omega)\,dt = \int_{\partial\Omega} |Df|.$$

Lower semicontinuity of the total variation gives us $P(E_t, \partial\Omega) \leq \liminf_{n\to\infty} P(E_t^n, \partial\Omega)$ $< \infty$ for a.e. t. We observe that the conditions in Lemma 4.4 are fulfilled and we obtain convergence $P(E_t^n, \partial\Omega) \to P(E_t, \partial\Omega)$ in $L^1(\mathbb{R})$, so after possibly passing to a subsequence we have pointwise convergence for a.e. t. Consequently $\chi_{\{f_n \geq t\}} \to \chi_{\{f \geq t\}}$ strictly in $BV(\partial\Omega)$.

3. As $\partial\Omega \in C^1$ and $f_n \in C^1(\partial\Omega)$, then by Sard theorem the set $\mathscr{T}$ of such t, which are regular values for all f_n, is of full measure. From now on it is necessary that we are in dimension $N = 2$. Recalling the Sternberg–Williams–Ziemer construction we get that for every $t \in \mathscr{T}$ every point of $\partial E_t^n \cap \partial\Omega$ is an end of at least one line segment; by Proposition 3.5 it is an end of exactly one line segment.

4. Let $t \in \mathscr{T}$. As $\partial\Omega$ is one-dimensional, then $P(E_t^n, \partial\Omega) \in \mathbb{N}$ and $D\chi_{\{f_n \geq t\}}$ is a sum $\sum_{i=1}^{M} \pm\delta_{x_i}$. Furthermore, by Proposition 2.9 and our convention of choosing the representative the set E_t is a union of closed arcs between consecutive points x_i. Similarly E_t^n is a union of closed arcs between consecutive points x_i^n (however, this is obvious as f_n are smooth functions).

5. As $\chi_{\{f_n \geq t\}} \to \chi_{\{f \geq t\}}$ strictly, then for sufficiently large n we have $P(E_t^n, \partial\Omega) = P(E_t, \partial\Omega)$. What is more, their derivatives converge in weak* topology; but we have an exact representation of those derivatives. This gives us convergence $x_i^n \to x_i$ for every i.

6. Now, take the sequence u_n of unique solutions to the least gradient problem for f_n given by the Sternberg–Williams–Ziemer construction. By Proposition 4.1 it has a subsequence (still denoted by u_n) convergent in $L^1(\Omega)$ to $u \in BV(\Omega)$. This gives us convergence $\chi_{\{u_n \geq t\}} \to \chi_{\{u \geq t\}}$ in $L^1(\Omega)$ for a.e. $t \in \mathbb{R}$. Let us fix $t \in \mathscr{T}$ such that the above convergence holds. Let $F_t = \{u \geq t\}$ and $F_t^n = \{u_n \geq t\}$. As by Theorem 2.6 $\chi_{F_t^n}$ are functions of least gradient in Ω, by Theorem 2.3 χ_{F_t} is a also function of least gradient in Ω. By Proposition 3.5 each of the sets ∂F_t^n (and ∂F_t) is a finite union of line segments, pairwise disjoint in $\overline{\Omega}$, connecting certain pairs of points among x_i^n (x_i). By definition of $\mathscr{T}$ every point of $\partial F_t^n \cap \partial\Omega$ (and $\partial F_t \cap \partial\Omega$) is an end of exactly one line segment.

7. Let $\mathscr{A}$ be the set of pairs (i, j), where $i < j$, such that the line segment $\overline{x_i x_j}$ is a subset of ∂F_t. We notice that as $\chi_{F_t^n} \to \chi_{F_t}$ in $L^1(\Omega)$, then for sufficiently big n also the line segment $\overline{x_i^n x_j^n}$ is a subset of ∂F_t^n and only such line segments are connected components of ∂F_t^n. Thus

$$P(F_t^n, \Omega) = \sum_{(i,j)\in\mathscr{A}} \|x_i^n - x_j^n\| \to \sum_{(i,j)\in\mathscr{A}} \|x_i - x_j\| = P(F_t, \Omega).$$

8. Let us see that $P(F_t^n, \Omega) \leq P(\Omega, \mathbb{R}^N)$. Indeed, ∂F_t^n is a finite union of line segments, pairwise disjoint in $\overline{\Omega}$, connecting certain pairs of points among x_i^n. If we choose a different connection between them, for example by drawing a full convex polygon with vertices in x_i^n, by minimality of ∂F_t^n the polygon has a larger perimeter. If we use arcs on $\partial\Omega$ instead, the perimeter would be even larger, as line segments are minimal surfaces in $\mathbb{R}^2$.

9. Since the functions $\chi_{\{u_n \geq t\}}$ converge in $L^1(\Omega)$ for a.e. t to $\chi_{\{u \geq t\}}$, then by Lemma 4.3 we have convergence $u_n \to u$ in $L^1(\Omega)$. Furthermore in Step 7 we proved convergence $P(F_t^n, \Omega) \to P(F_t, \Omega)$ for a.e. t, so by dominated convergence theorem (by Step 8

this sequence is bounded) we have convergence $P(F_t^n, \Omega) \to P(F_t, \Omega)$ in $L^1(\mathbb{R})$. By co-area formula $\int_\Omega |Du_n| \to \int_\Omega |Du|$, which gives that $u_n \to u$ strictly in $BV(\Omega)$. $\qquad \square$

From the above Theorem we immediately obtain Theorem 1.1.

Proof of Theorem 1.1 For each $f \in BV(\partial\Omega)$ we can find a sequence f_n of class $C^\infty(\partial\Omega)$ strictly convergent to f. Let u_n be solutions of the least gradient problem for f_n. Then after possibly passing to subsequence we have that $u_n \to u$ strictly in $BV(\Omega)$; but the trace is continuous in the strict topology, so $Tu = f$. Thus by Miranda stability theorem (Theorem 2.3) u is a function of least gradient, so it is a solution of the least gradient problem for f. $\square$

Example 4.7 Take $\Omega = B(0, 1) \subset \mathbb{R}^2$. As we know from [17], when f is a characteristic function of a certain fat Cantor set, then the least gradient problem has no solution. Thus, we would expect that if we approximated the boundary function and constructed solutions of the least gradient problem for the approximation, then the trace of the limit would be incorrect. To settle this, let f_n be a function of the n-th stage of the Cantor set construction and u_n be the corresponding solution to the least gradient problem; it exists by Theorem 1.1. We will see that $u_n \to 0$ in $L^1(\Omega)$.

Let $\widetilde{f_0}(\theta) = \chi_{[0,1]}$. We construct $\widetilde{f_1}$ by removing from the middle of $[0, 1]$ an interval of length 2^{-2}, i.e. $\widetilde{f_1} = \chi_{[0,3/8]} + \chi_{[5/8,1]}$. In the second stage we remove from the middle of both intervals an interval of length 2^{-4} and obtain $\widetilde{f_2} = \chi_{[0,5/32]} + \chi_{[7/32,3/8]} + \chi_{[5/8,25/32]} + \chi_{[27/32,1]}$. During the n-th stage of construction we remove an interval of length $b_n = 2^{-2n}$ from the middle of all existing 2^{n-1} intervals.

Let us see what is the length of all such intervals. Let a_n be the length of an interval at the n-th stage of construction. Then $a_n = \frac{a_{n-1}}{2} - \frac{1}{2^{2n+1}}$. As $a_0 = 1$, we obtain a direct formula $a_n = \frac{2^n+1}{2^{2n+1}}$.

We map the interval $[0, 1]$ to an arc of length 1 on the circle symmetric with respect to the y axis, i.e. $g(x) = (\cos(\frac{\pi+1}{2} - x), \sin(\frac{\pi+1}{2} - x))$. This way our Cantor set is on the circle and we define $f_n = \widetilde{f_n} \circ g^{-1} : \text{im } g \to \mathbb{R}$. We extend each of the functions f, f_i to the whole $\partial B(0, 1)$ by 0. This situation for $n = 1$ is presented on Fig. 1.

Consider the function f_1. In order to check whether $\partial\{u \geq t\}$ consists of two bases of the trapezoid or two sides, we compare the sums of lengths. We directly compute the sum of lengths of the bases to be $(\cos(\frac{\pi}{2} - \frac{1}{2}) - \cos(\frac{\pi}{2} + \frac{1}{2})) + (\cos(\frac{\pi}{2} - \frac{1}{8}) - \cos(\frac{\pi}{2} + \frac{1}{8})) \approx 1.21$, while the length of the sides equals $2(\cos(\frac{\pi}{2} - \frac{1}{2}) - \cos(\frac{\pi}{2} - \frac{1}{8}))^2 + (\sin(\frac{\pi}{2} - \frac{1}{2}) - \sin(\frac{\pi}{2} - \frac{1}{8}))^2)^{1/2} \approx 0.75$. Hence the sides of the trapezoid minimize $P(E_t, B(0, 1))$ for $t \in (0, 1)$ and u_1, solution of the least gradient problem for f_1, takes value 0 on the trapezoid and takes value 1 only on flaps cut off by the sides of the trapezoid (shown in grey on Fig. 1).

Now, we will repeat the above reasoning at every stage of construction of the Cantor set. At every stage of construction there appear 2^{n-1} identical trapezoids in the flaps on which the value of u_{n-1} was constant and equal to 1; we have to compare the sum of lengths of the sides of these trapezoid versus their bases. We use the law of cosines to notice that for a circle of radius one te length of the chord corresponding to angle α equals $\sqrt{2 - 2\cos\alpha}$; thus we have to check the following inequality:

$$\sqrt{1 - \cos(a_n)} + \sqrt{1 - \cos(b_n)} > 2\sqrt{1 - \cos(a_{n+1})}. \tag{5}$$

On the left hand side is the (rescaled by $\frac{1}{\sqrt{2}}$) sum of lengths of the two bases and on the right hand side is the (rescaled) sum of lengths of the sides. Substitute $x = 2^{-n}$, recall that $b_n = 2^{-2n}$ and use the direct formula for a_n. It changes to the inequality

Fig. 1 Solution of the least gradient problem for the first stage of construction
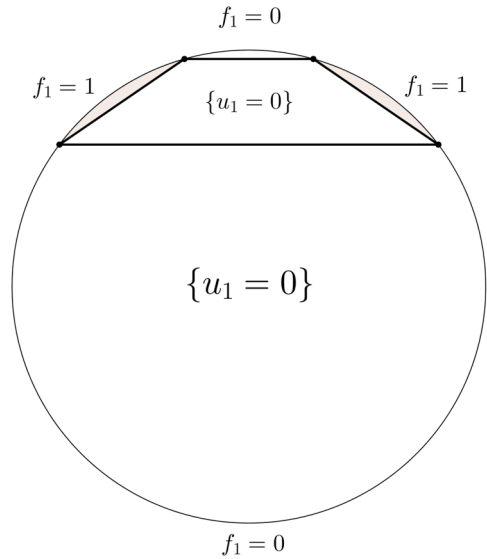
$$\sqrt{1 - \cos\left(\frac{x(x+1)}{2}\right)} + \sqrt{1 - \cos(x^2)} > 2\sqrt{1 - \cos\left(\frac{x(x+2)}{8}\right)}.$$

Actually, we may prove a stronger inequality: let us omit the second summand and square both sides of the resulting inequality. Then it changes to

$$g(x) = 1 - \cos\left(\frac{x(x+1)}{2}\right) - 4 + 4\cos\left(\frac{x(x+2)}{8}\right) > 0.$$

But g satisfies $g(0) = 0$ and its derivative is positive on $(0, 1)$, so $g > 0$ on $(0, 1)$, thus the inequality holds for all $x \in (0, 1)$; hence inequality (5) holds for all n. Thus on every stage of construction the sides of the trapezoids are shorter than the bases.

Now it is easy to determine the solution u_n. We proceed as in the proof of Theorem 4.6. Let us take a sequence of smooth functions f_1^k approximating f_1 in the strict topology on $\partial\Omega$ and another sequence f_2^k approximating f_2 in the same topology; we additionally require that $f_1^k \geq f_2^k$ for each k. We denote the solutions to the approximating problems by u_1^k and u_2^k. Thus, by Theorem 2.7, i.e. the comparison principle for continuous boundary data, we have that $u_1^k \geq u_2^k$; we pass to the limit with $k \to \infty$ to obtain that $u_1 \geq u_2$. Thus, while determining u_2, we only have to take into account two possible configurations: in the two flaps in which $u_1 = 1$ we have to compare the sum of lengths of the sides of the resulting trapezoid versus its bases. As the sides of the trapezoid are shorter, u_2 is nonzero only in four smaller flaps enclosed by an arc on which $f_2 = 1$ and a line segment.

We repeat the above argument at every level of the construction. It follows that the solution u_n equals zero on each trapezoid in this construction and will be nonzero only on 2^n flaps enclosed by arcs on which f_n equals 1 and line segments. In particular, the sequence u_n is nonincreasing and for every point x inside the circle at a sufficiently large stage of construction we would have $u_n(x) = 0$. Thus $u_n \to 0$ a.e.; but it is bounded from above by 1, so the convergence holds also in $L^1(\Omega)$.

Example 4.8 Let us make a slight change to the previous example: consider another fat Cantor set on the circle. More precisely, take a sequence b_n converging very rapidly to zero, so the inequality (5) holds in the opposite direction; it is possible due to the triangle inequality. This results in a Cantor set C of almost full measure.

Let us look again at Fig. 1. As inequality (5) holds in the opposite direction, at the first level of the construction it is more efficient to fix $u_1 = 1$ in both the grey flaps and the trapezoid than to follow the construction in the previous example. Arguing as in the previous example, we see that every stage of construction it is more efficient to remove 2^{n-1} curvilinear flaps bounded by a line segment and an arc on $\partial\Omega$ corresponding to a connected component of $\{f_n = 0\}$ from the set $\{u_{n-1} = 1\}$ than to repeat the construction from the previous example, i.e. add trapezoids to the set $\{u_{n-1} = 0\}$. Thus the set $F_n = \{u_n = 1\}$ is a union of trapezoids arising in all the previous steps and flaps bounded by a line segment and an arc on $\partial\Omega$ corresponding to a connected component of $\{f_n = 1\}$.

The sequence $u_n = \chi_{F_n}$ converges to $u = \chi_F$ a.e. and in $L^1(\Omega)$; by Theorem 2.3 u is a function of least gradient. We have to check that $Tu = f$. Firstly, let us notice that the trace of χ_A, a characteristic function of a set of bounded perimeter $A \subset \mathbb{R}^2$ is again a characteristic function of some set: by [5, Theorem 5.3.2] for $\mathcal{H}^1$-almost all $x \in \partial\Omega$ we write

$$\lim_{r \to 0} \fint_{B(x,r) \cap \Omega} |\chi_A(y) - T\chi_A(x)| dy = \lim_{r \to 0} (|1 - T\chi_A(x)| \frac{|B(x,r) \cap A|}{|B(x,r) \cap \Omega|}$$
$$+ |T\chi_A(x)| \frac{|B(x,r) \cap (\Omega \setminus A)|}{|B(x,r) \cap \Omega|}) = 0,$$

but as $|B(x,r) \cap A| + |B(x,r) \cap (\Omega \setminus A)| = |B(x,r) \cap \Omega|$ we see that it is only possible for $T\chi_A(x) = 0$ or $T\chi_A(x) = 1$.

Now, let us see that by the construction above for every $x \in C$ except for the two extreme points there is a circular sector centered at x such that it lies entirely inside each F_n, so also entirely inside F: as b_n is small, then in every step of construction (except for the first) the angle between $\partial\Omega$ and every line segment cutting off a flap in which $u_n = 0$ is bounded from above by some small α_0. Hence at every point of C (except for the two extreme points) for sufficiently small $r > 0$ there is a circular sector, bounded by two rays such that the angle between them and $\partial\Omega$ equals α_0, in which $u = 1$.

However, it implies that the mean integral $\fint_{B(x,r) \cap \Omega} |\chi_F(y)| dy$ is bounded from below, so $Tu = T\chi_F$ cannot equal 0 on the Cantor set. By the previous paragraph Tu equals one $\mathcal{H}^1$-a.e. on the Cantor set. We also see that for each $x \in \partial\Omega \setminus C$ there is an open neighbourhood of x in which $u_n = 0$ for sufficiently large n, so Tu equals zero on $\partial\Omega \setminus C$. We obtained that there exists a solution to the least gradient problem for a certain discontinuous $f \notin BV(\partial\Omega)$.

5 Anisotropic case

This section is devoted to the anisotropic least gradient problem. We discuss l_p norms on the plane for $p \in [1, \infty]$. We prove a non-uniqueness result for $p = 1, \infty$ and discuss how the solutions look like for $p \in (1, \infty)$. We will use the notation introduced in [12].

Definition 5.1 Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. A continuous function $\phi : \overline{\Omega} \times \mathbb{R}^N \to [0, \infty)$ is called a metric integrand, if it satisfies the following conditions:

1. ϕ is convex with respect to the second variable for a.e. $x \in \overline{\Omega}$;

2. ϕ is homogeneous with respect to the second variable, i.e.

$$\forall x \in \overline{\Omega}, \quad \forall \xi \in \mathbb{R}^N, \quad \forall t \in \mathbb{R} \quad \phi(x, t\xi) = |t|\phi(x, \xi);$$

3. ϕ is bounded in $\overline{\Omega}$, i.e.

$$\exists \Gamma > 0 \quad \forall x \in \overline{\Omega}, \quad \forall \xi \in \mathbb{R}^N \quad 0 \leq \phi(x, \xi) \leq \Gamma|\xi|.$$

In order to introduce the anisotropic total variation and recover some properties of the classical total variation, we will additionally assume that

4. ϕ is elliptic in $\overline{\Omega}$, i.e.

$$\exists \lambda > 0 \quad \forall x \in \overline{\Omega}, \quad \forall \xi \in \mathbb{R}^N \quad \lambda|\xi| \leq \phi(x, \xi).$$

Remark 5.2 These conditions are sufficient for most of the cases considered in scientific work: they are satisfied for the classical least gradient problem, i.e. $\phi(x, \xi) = |\xi|$, as well as for the l_p norms, $p \in [1, \infty]$ and for weighted least gradient problem considered in [10]: a function $\phi(x, \xi) = g(x)|\xi|$, where $g \geq c > 0$.

Definition 5.3 The polar function of ϕ is $\phi^0 : \overline{\Omega} \times \mathbb{R}^N \to [0, \infty)$ defined as

$$\phi^0(x, \xi^*) = \sup \left\{ \langle \xi^*, \xi \rangle : \xi \in \mathbb{R}^N, \ \phi(x, \xi) \leq 1 \right\}.$$

While the definition of the anisotropic total variation as introduced in [1] is slightly more general, we will only consider metric integrands ϕ which are elliptic and continuous in $\overline{\Omega}$. As proved in [1, Chapter 3], in that case we may introduce the following equivalent definition:

Definition 5.4 Let ϕ be a metric integrand continuous and elliptic in $\overline{\Omega}$. For a given function $u \in L^1(\Omega)$ we define its ϕ-total variation in Ω by the formula (another notation used in the literature is $\int_\Omega \phi(x, Du)$):

$$\int_\Omega |Du|_\phi = \sup \left\{ \int_\Omega u \operatorname{div} \mathbf{z} \, dx : \phi^0(x, \mathbf{z}(x)) \leq 1 \text{ a.e.}, \quad \mathbf{z} \in C_c^1(\Omega) \right\}.$$

If $\int_\Omega |Du|_\phi < \infty$, we say that $u \in BV_\phi(\Omega)$. Note that by properties (3) and (4) we have that $\lambda \int_\Omega |Du| \leq \int_\Omega |Du|_\phi \leq \Gamma \int_\Omega |Du|$, so $BV_\phi(\Omega) = BV(\Omega)$ as sets, but they are equipped with different norms. We also define the ϕ-perimeter of a set E to be

$$P_\phi(E, \Omega) = \int_\Omega |D\chi_E|_\phi.$$

Similarly to the classical case, we say that E is a set of bounded ϕ-perimeter in Ω, if $P_\phi(E, \Omega) < \infty$.

Remark 5.5 When ϕ is continuous and elliptic in $\overline{\Omega}$, then similarly to the classical case ([1, Chapter 4]) we recover lower semicontinuity of the ϕ-total variation and the co-area formula. We also recover the approximation by C^∞ functions in the strict topology, even in a stronger form proved by Giusti in [6, Corollaries 1.17, 2.10]: let $v \in BV_\phi(\Omega)$, $Tv = f$. Then there exists a sequence of C^∞ functions v_n such that $v_n \to v$ strictly in $BV_\phi(\Omega)$ such that $Tv = f$.

The approximation by smooth functions in the strict topology entails that we may approximate sets of bounded ϕ-perimeter both in the Lebesgue measure and in ϕ-perimeter by open sets with smooth boundary (also with respect to some given boundary conditions); in the isotropic case, see [2, Theorem 3.42]. The tools used there include the isoperimetric inequality, lower semicontinuity of the total variation and the co-area formula, all of which are valid also in the anisotropic case. □

For an explicit use we shall need the following integral representation ([1,10]):

Proposition 5.6 *Let* $\varphi : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ *be a metric integrand. Then we have an integral representation:*

$$\int_\Omega |Du|_\phi = \int_\Omega \phi(x, v^u(x)) \, |Du|,$$

where v^u *is the Radon–Nikodym derivative* $v^u = \frac{dDu}{d|Du|}$. *In particular, if* $E \subset \Omega$ *and* ∂E *is sufficiently smooth (at least* C^1), *then we have a representation*

$$P_\phi(E, \Omega) = \int_\Omega \phi(x, v_E) \, d\mathcal{H}^{n-1},$$

where v_E *is the external normal to* E. $\qquad\square$

We are interested in the anisotropic version of the least gradient problem. We state it similarly to the isotropic version:

Definition 5.7 Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. We say that $u \in BV_\phi(\Omega)$ is a function of ϕ-least gradient, if for every compactly supported $v \in BV_\phi(\Omega)$ we have

$$\int_\Omega |Du|_\phi \leq \int_\Omega |D(u + v)|_\phi.$$

Again, if ϕ is a metric integrand with continuous extension to $\mathbb{R}^N$, we may instead assume that v is a BV_ϕ function with zero trace on $\partial\Omega$; see [12, Proposition 3.16]. Furthermore, we say that u is a solution to the anisotropic least gradient problem with boundary data f if u is a function of ϕ-least gradient and $Tu = f$.

In the anisotropic case, existence and uniqueness of minimizers in the least gradient problem for continuous boundary data depend not only on the geometry of Ω, but also on the regularity of ϕ. The uniqueness proof is based on a maximum principle and requires uniform convexity and a condition slightly weaker than $W^{3,\infty}$ regularity of an elliptic metric integrand away from $\{\xi = 0\}$; for a precise assumption, see [10, Theorem 1.2]. The existence proof requires ellipticity of the metric integrand ϕ and a *barrier condition*:

Definition 5.8 ([10, Definition 3]) Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. Suppose that ϕ is an elliptic metric integrand. We say that Ω satisfies the barrier condition if for every $x_0 \in \partial\Omega$ and sufficiently small $\varepsilon > 0$, if V minimizes $P_\phi(\cdot, p; \mathbb{R}^N)$ in

$$\{W \subset \Omega : W \backslash B(x_0, \varepsilon) = \Omega \backslash B(x_0, \varepsilon)\}$$

then

$$\partial V \cap \partial\Omega \cap B(x_0, \varepsilon) = \emptyset.$$

In the isotropic case $\phi(x, \xi) = \|\xi\|_2$ this is equivalent, at least for sets with C^2 boundary, to the condition slightly weaker than strict convexity introduced in [18].

Before we proceed, we need one additional result that relates functions of ϕ-least gradient and ϕ-minimal sets, i.e. sets, the characteristic functions of which are of ϕ-least gradient. Its proof in one direction follows the lines of the proof of Theorem 2.6 and in the other direction it is a simple application of the co-area formula.

Proposition 5.9 ([12, Theorem 3.19]) *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. Assume that the metric integrand ϕ has a continuous extension to $\mathbb{R}^N$. Then $u \in BV_\phi(\Omega)$ is a function of ϕ-least gradient in Ω if and only if χ_{E_t} is a function of ϕ-least gradient for almost all $t \in \mathbb{R}$.* □

Definition 5.10 For $p \in [1, \infty)$ we define the l_p norm of a vector on the plane by the formula $\|(x, y)\|_p = (|x|^p + |y|^p)^{1/p}$. For $p = \infty$ it is defined as $\|(x, y)\|_\infty = \sup(|x|, |y|)$.

We will discuss the ϕ-least gradient function with respect to an anisotropy generated by the l_p norms, i.e. $\phi(x, \xi) = \|\xi\|_p$. Firstly, let us see that for $p = 1$ or $p = \infty$ does not satisfy the regularity assumptions of [10, Theorem 1.2], so if the solution to the anisotropic least gradient problem exists, it need not be unique; indeed, we will provide an example of nonuniqueness in Example 5.16. Unfortunately, as Corollary 5.13 clearly shows, the barrier condition is not satisfied and we have to prove existence of minimizers using another means.

We aim to prove that for nonsmooth anisotropy, i.e. for $p = 1$ or $p = \infty$, the solutions in general are not unique; in order to achieve this goal, we will study how do minimal surfaces with respect to the l_p norm look like. The next result is stated for sets with C^1 boundary for two reasons. Firstly, together with approximation by smooth functions in strict topology it is enough to consider sets with C^1 boundary to construct an example with nonuniqueness of minimizers. Secondly, minimal sets with respect to l_1 anisotropy may have singularities on the boundary and an analogue of the next Proposition would be false; see the end of Example 5.16.

Proposition 5.11 *Let $\Omega \subset \mathbb{R}^2$ be an open bounded convex set. Let the anisotropy be defined by the function $\phi(x, Du) = \|Du\|_1$. Let E be a 1-minimal set with respect to Ω, i.e. χ_E is a function of 1-least gradient in Ω. Suppose that $\partial E \in C^1$. Then every connected component of ∂E is a graph of a monotone function (possibly including vertical line segments in the same direction).*

Proof 1. Firstly, want to show that if two points $p_1 = (x_0, y_1)$ and $p_2 = (x_0, y_2)$ with the same first coordinate belong to ∂E, then they are connected by a vertical line segment $\overline{p_1 p_2} \subset \partial E$.

Suppose otherwise. Denote by $\partial E(p_1, p_2)$ the part of ∂E between p_1 and p_2; without loss of generality we may assume there it contains no other points with first coordinate x_0. As ∂E is C^1, at the point $(x, y) \in \partial E$ the Radon–Nikodym derivative ν^{χ_E} is perpendicular to the level set, so it is a vector $(-\sin\theta, \cos\theta)$. Thus $\phi(x, \nu^{\chi_E}) = |\sin\theta| + |\cos\theta|$. We also recall that $|D\chi_E| = \mathscr{H}^1|_{\partial E}$. Using the representation introduced by Proposition 5.6 we calculate

$$
\begin{aligned}
P_\phi(E, \Omega) &= \int_\Omega \phi(x, \nu^{\chi_E})|D\chi_E| = \int_{\partial E} (|\sin\theta| + |\cos\theta|)d\mathscr{H}^1 \\
&= \int_{\partial E \setminus \partial E(p_1, p_2)} (|\sin\theta| + |\cos\theta|)d\mathscr{H}^1 + \int_{\partial E(p_1, p_2)} (|\sin\theta| + |\cos\theta|)d\mathscr{H}^1 \\
&> \int_{\partial E \setminus \partial E(p_1, p_2)} (|\sin\theta| + |\cos\theta|)d\mathscr{H}^1 + \int_{\partial E(p_1, p_2)} 1 d\mathscr{H}^1 \\
&> \int_{\partial E \setminus \partial E(p_1, p_2)} (|\sin\theta| + |\cos\theta|)d\mathscr{H}^1 + \int_{\overline{p_1 p_2}} 1 d\mathscr{H}^1 = P_\phi(F, \Omega),
\end{aligned}
$$

where F is a set with C^1 boundary (possibly except for p_1 and p_2) such that $E \triangle F$ is the set enclosed by $\partial E(p_1, p_2)$ and $\overline{p_1 p_2}$. Thus E was not a 1-minimal set, contradiction.

2. Let (x_0, y_0) and (x_1, y_1) be two points on the same connected component of ∂E. Let us suppose additionally that ∂E does not contain any vertical line segments, i.e. we may represent a level set from the point (x_0, y_0) to (x_1, y_1) as a graph of a C^1 function g. Like before, let us note that at the point $(s, g(s))$ the Radon–Nikodym derivative $\nu^{\chi_{E_t}}$ is a vector $(-\sin\theta, \cos\theta)$, where $g'(s) = \tan\theta$. We have to minimize the integral (we may assume that $x_0 < x_1$):

$$
\begin{aligned}
P_\phi(E, \Omega) &= \int_\Omega \phi(x, \nu^{\chi_E})|D\chi_E| \\
&= \int_{\partial E} (|\sin\theta| + |\cos\theta|) d\mathcal{H}^1 \\
&= \int_{x_0}^{x_1} (|\sin\theta| + |\cos\theta|)\sqrt{1 + (\tan\theta)^2} dx = \int_{x_0}^{x_1} (|\sin\theta| + |\cos\theta|)\frac{1}{|\cos\theta|} dx \\
&= \int_{x_0}^{x_1} (1 + |\tan\theta|) dx = |x_1 - x_0| + \int_{x_0}^{x_1} |g'| dx \geq |x_1 - x_0| + |y_1 - y_0|,
\end{aligned}
$$

As we assumed g to be C^1, the inequality becomes equality if and only if g is monotone. In particular there are multiple functions minimizing this integral.

3. Now we allow ∂E to contain vertical line segments. The difference is purely technical, as we have to divide our integral into two parts. Let us suppose that the (orientated) length of i-th vertical line segment equals λ_i, then we have

$$
\int_{\text{graph part of } \partial E} (|\sin\theta| + |\cos\theta|) d\mathcal{H}^1 + \int_{\text{vertical part of } \partial E} (1 + 0) d\mathcal{H}^1
$$

$$
= \int_{x_0}^{x_1} (1 + |g'|) dx + \sum_{i=1}^\infty |\lambda_i| = \int_{x_0}^{x_1} |g'| dx + |x_1 - x_0| + \sum_{i=1}^\infty |\lambda_i|
$$

$$
\geq |y_1 - y_0 - \sum_{i=1}^\infty \lambda_i| + |x_1 - x_0| + \sum_{i=1}^\infty |\lambda_i| \geq |x_1 - x_0| + |y_1 - y_0|,
$$

where the inequality becomes equality if and only if g is monotone and all the vertical line segments are orientated in the same direction as g'. Again, there are multiple functions minimizing this integral. $\qquad\square$

Corollary 5.12 *Let E be a set such that ∂E is a C^1 Jordan curve, which is a graph of a monotone function (possibly including vertical line segments). Then E is a 1-minimal set: by Remark 5.5 the infimum of $P_1(E, \Omega)$ with respect to some boundary conditions is the same in the class of smooth sets as in the class of sets of bounded 1-perimeter; but due to Proposition 5.11 the infimum is achieved in the class of smooth functions. Thus we may consider only smooth competitors and the calculation in Proposition 5.11 shows that E is a 1-minimal set.*

Corollary 5.13 *Notice that if E is a set such that $\partial E = \overline{xy} \cup \overline{xz}$, where one of the line segments is horizontal and the other is vertical, then E is also a 1-minimal set: let E_n be a set with C^1 boundary such that $E_n \subset E$, $\partial E_n = \partial E$ in $\overline{\Omega} \backslash B(x, 2^{-n})$ and ∂E_n is a circle arc in $B(x, 2^{-n})$. Then $\chi_{E_n} \to \chi_E$ in $L^1(\Omega)$, so by lower semicontinuity of the 1-total variation E is a 1-minimal set. In particular, if we take the intersection point x to be on $\partial\Omega$, we see that the barrier condition fails for $p = 1$ and we cannot expect existence of minimizers to the anisotropic least gradient problem for arbitrary continuous boundary data.*
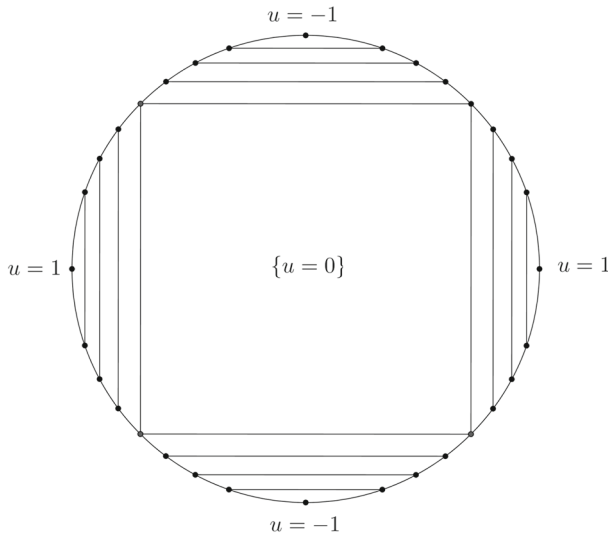
Fig. 2 Solution of this anisotropic least gradient problem for l_1 norm is unique

The next Proposition estabilishes a criterion for existence and uniqueness of minimizers with respect to l_1 norm. Then, in Example 5.16, we provide an example when minimizers exist and are not unique.

Proposition 5.14 *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, strictly convex set. Take $\phi(x, Du) = \|Du\|_1$. Let $f \in C(\partial\Omega)$. Denote by u the solution to the isotropic least gradient problem for f. Then, if the boundaries of superlevel sets of u are parallel to the axes of the coordinate system, then u is a unique solution of the anisotropic least gradient problem with respect to the l_1 norm.*

Proof Let $v \in BV(\Omega)$, $Tv = f$. Denote by u the solution to the isotropic problem. Then

$$\int_\Omega |Dv|_1 \geq \int_\Omega |Dv|_2 \geq \int_\Omega |Du|_2 = \int_\Omega |Du|_1.$$

As $\| \cdot p\|_p$ is nonincreasing in p, we have the first inequality. The second follows from the definition of the Euclidean solution; by its uniqueness the second inequality is strict, if only $u \neq v$. As the boundaries of superlevel sets of u are parallel to the axes of the coordinate system, we have the final equality. It follows that u is a unique solution to the anisotropic least gradient problem.   □

Example 5.15 Let $\Omega = B(0, 1) \subset \mathbb{R}^2$. Take $\phi(x, Du) = \|Du\|_1$. Let $f(\theta) = \cos(2\theta)$. We construct the isotropic solution u_0 using Sternberg–Williams–Ziemer construction. We notice, as shown on Fig. 2, that the boundaries of superlevel sets of u_0 are parallel to the axes of the coordinate system. By Proposition 5.14 the solution to the anisotropic least gradient problem is unique.   □

As the barrier condition fails for $p = 1$, we cannot expect existence of solutions for arbitrary continuous boundary data. However, we may show an example of nonuniqueness of solutions even for the unit ball in the class of smooth boundary data.

Example 5.16 Let $\Omega = B(0,1) \subset \mathbb{R}^2$. Take $\phi(x, Du) = \|Du\|_1$ and let $f \in C^\infty(\partial\Omega)$ be given by the formula $f = \sin(2\theta)$, i.e. the boundary datum is a rotation of the boundary datum from Example 5.15 by $\frac{\pi}{4}$. Then the solution to the anisotropic least gradient problem exists and is not unique.

According to Proposition 5.9, it is sufficient to construct superlevel sets of v such that $P_1(E_t, \Omega)$ is minimal for almost all $t \in \mathbb{R}$; then $\int_\Omega |Dv|_1$ would be minimal as well. We observe that every $t \in (0, 1)$ is regular and its preimage consists of four points of the form $p_1 = (a, b)$, $p_2 = (b, a)$, $p_3 = (-a, -b)$, $p_4 = (-b, -a)$, where $a > b > 0$.

We want to construct a 1-minimal set E_t with boundary value $\chi_{\{f > t\}}$. Arguing as in Corollary 5.12 we see that we only need to consider competitors with smooth boundary. By Proposition 5.11 every connected component of ∂E_t is a graph of a monotone function (possibly including vertical line segments); hence there are two types of competitors: sets, whose boundary is a union of two monotone Jordan curves $p_1 p_2$ and $p_3 p_4$ or the union of two monotone Jordan curves $p_1 p_4$ and $p_2 p_3$.

Now we recall the calculation in Proposition 5.11. Thus the l_1 length of every monotone Jordan curve $p_1 p_2$ equals $2|a - b|$ (and the same for $p_3 p_4$), while the l_1 length of every monotone Jordan curve $p_1 p_4$ equals $2|a + b|$ (and the same for $p_2 p_3$). Thus, if we choose $p_1 p_2$ and $p_3 p_4$ as the boundary of a set E_t, we obtain that E_t is 1-minimal. We may perform an analogous calculation for $t \in (-1, 0)$.

We construct a function $v \in BV_1(\Omega)$ in the following way: let $E_t = \{v > t\}$ be as above. We additionally require that $\overline{E_t} \subset \text{int } E_s$ and define $v(x) = \sup\{t : x \in E_t\}$. Then v is well defined and by Proposition 5.9 it is a function of 1-least gradient.

Obviously this construction leads to multiple solutions. Firstly, we may take ∂E_t to be a union of two line segments; then v coincides with the isotropic solution. Secondly, in the first quadrant ($\{x, y > 0\}$) we may define ∂E_t to be an arc of the circle with centre $(1, 1)$ and passing through p_1 and p_2; we extend this definition similarly to other quadrants. This minimizer is presented on Fig. 3.

Finally, we may take ∂E_t to be the union of a vertical and horizontal line segment: in the first quadrant we define $q = (a, a)$ and take $\partial E_t = \overline{p_1 q} \cup \overline{q p_2}$ and proceed similarly in other quadrants. While ∂E_t is not C^1, we argue as in Corollary 5.13 to see that E_t is a 1-minimal set. Then the zero level set of v has a singularity: $\partial E_0 = \{(x, y) \in \overline{\Omega} : x = 0 \vee y = 0\}$. It is a union of two monotone Jordan curves intersecting at $(0, 0)$. □

Example 5.17 Now let $p = \infty$. If we make a similar calculation, we obtain that the perimeter of a level set connecting points (x_0, y_0) with (x_1, y_1) equals

$$\int_{\partial E_t} \max(|\sin\theta|, |\cos\theta|) d\mathcal{H}^{n-1} = \int_{x_0}^{x_1} \max(|\sin\theta|, |\cos\theta|)\sqrt{1 + (\tan\theta)^2} dx$$

$$= \int_{x_0}^{x_1} \max(|\sin\theta|, |\cos\theta|)\frac{1}{|\cos\theta|} dx = \int_{x_0}^{x_1} \max(1, |\tan\theta|) dx$$

$$= \int_{x_0}^{x_1} \max(1, |g'|) dx \geq \max(|x_1 - x_0|, |y_1 - y_0|),$$

where the inequality becomes equality if and only if $|g'| - 1$ has constant sign; in other words, the angle between the level set and the x coordinate axis is always not greater (or not smaller) than $\frac{\pi}{4}$. Thus, if we take the function $f(\theta) = \cos(2\theta)$, the solution is not unique; we apply this result for $t \in (-1, 0)$ and then apply it again for $t \in (0, 1)$ considering the level set as a function of y. A solution different than the Euclidean one is presented on Fig. 4.

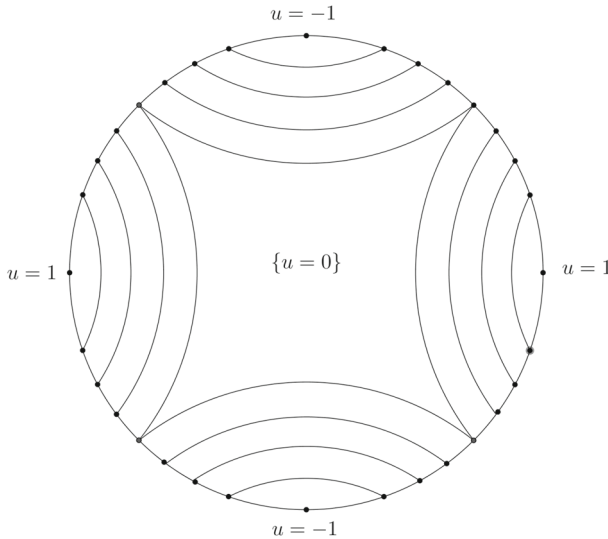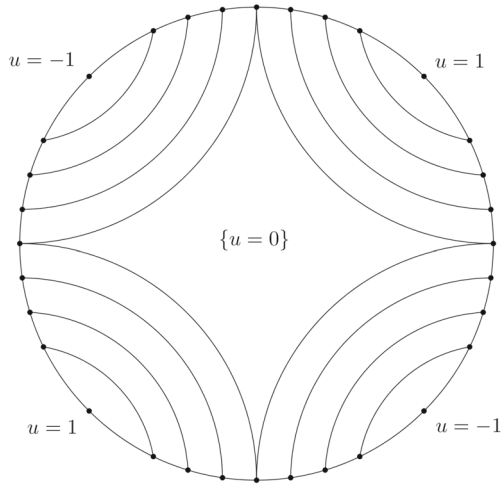**Fig. 3** Solution of this anisotropic least gradient problem with respect to l_1 norm is not unique

Fig. 4 Solutions of anisotropic least gradient problem may be not unique for l_∞ norm

Nevertheless, it may still happen that the solution is unique: it is the case if we take such f that the Euclidean solution has all level sets at an angle $\frac{\pi}{4}$ to the coordinate axes. For example we can take $f(\theta) = \sin(2\theta)$. □

Now we fix $p \in (1, \infty)$. Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, strictly convex set. Take $f \in C(\partial \Omega)$. The l_p norm is uniformly convex and smooth away from $\{\xi = 0\}$, so by [10, Theorem 1.2] there is at most one solution to the anisotropic least gradient problem with boundary data f. Regarding the barrier condition, if $\partial \Omega$ is additionally C^2, we may use a version of the barrier condition in local coordinates, see [10, Remark 3.2] to check that it is satisfied and that in view of [10, Theorem 1.1] a solution to the anisotropic least gradient problem with boundary data f exists.

Our main goal is Theorem 1.3. Apart from its value as a structure theorem for ϕ-least gradient functions, it shows that the barrier condition as stated in Definition 5.8 is satisfied also without the additional assumption that $\partial\Omega$ is C^2: as the connected minimal surfaces with respect to l_p norms for $p \in (1, \infty)$ are the same as in the isotropic case, the barrier condition for a l_p norm for given p is satisfied for the same sets Ω as it is for $p = 2$, i.e. if Ω is an open, bounded, strictly convex set.

Proof of Theorem 1.3 Let (x_0, y_0) and (x_1, y_1) be two points on the same connected component of ∂E. We have to minimize an integral analogous to the previous one (notation stays the same):

$$
\begin{aligned}
L(x, g, g') &= \int_{\partial E_t} (|\sin\theta|^p + |\cos\theta|^p)^{\frac{1}{p}} d\mathcal{H}^{n-1} \\
&= \int_{x_0}^{x_1} (|\sin\theta|^p + |\cos\theta|^p)^{\frac{1}{p}} \sqrt{1 + (\tan\theta)^2} dx \\
&= \int_{x_0}^{x_1} (|\sin\theta|^p + |\cos\theta|^p)^{\frac{1}{p}} \frac{1}{|\cos\theta|} dx = \int_{x_0}^{x_1} (1 + |\tan\theta|^p)^{\frac{1}{p}} dx \\
&= \int_{x_0}^{x_1} (1 + |g'|^p)^{\frac{1}{p}} dx.
\end{aligned}
$$

As we are in dimension two, due to regularity results from [15, Theorem I.3.1] every connected component of the boundary of a ϕ-minimal set is a C^2 manifold, thus we may use the Euler–Lagrange equation for the functional L, which takes form

$$
0 = \frac{\partial L}{\partial g} = \frac{d}{dx}\left(\frac{\partial L}{\partial g'}\right) = \frac{d}{dx}\left(\text{sgn}(g')(|g'|)^{p-1}(1 + |g'|^p)^{\frac{1}{p}-1}\right)
$$

$$
\text{sgn}(g')(|g'|)^{p-1}(1 + |g'|^p)^{\frac{1}{p}-1} = \text{const}.
$$

Taking absolute value and raising both sides to power $\frac{p}{p-1}$ we obtain

$$
\frac{|g'|^p}{1 + |g'|^p} = \text{const} = C,
$$

thus $g' = \text{const}$. Thus the anisotropic minimal surface connecting points (x_0, y_0) and (x_1, y_1) is a line segment. □

We conclude by stating a consequence of Theorem 1.3 for the existence theory of anisotropic least gradient functions. In view of [10, Theorems 1.1–1.2] we obtain

Corollary 5.18 *If $\Omega \subset \mathbb{R}^2$ is an open, bounded, strictly convex set and $f \in C(\partial\Omega)$, then the corresponding anisotropic least gradient problem has a unique solution.* □

However, Theorem 1.3 does not imply that the solution to the anisotropic problem coincides with the isotropic solution. A comment from the referee report has inspired the following example:

Example 5.19 Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, strictly convex set such that points $p_1 = (1, 0)$, $p_2 = (11, 0)$, $p_3 = (0, 11)$ and $p_4 = (0, 1)$ lie on $\partial\Omega$. Take $f \in C(\partial\Omega)$ such that $f > 0$ on the arcs $p_1 p_2 \subset \partial\Omega$ and $p_3 p_4 \subset \partial\Omega$ and $f < 0$ on the two other arcs. We are interested in the zero superlevel set for $p \in (1, \infty)$. We easily see that $\partial\{u \geq 0\}$ is the union of two line segments; either $\partial\{u \geq 0\} = \overline{p_1 p_2} \cup \overline{p_3 p_4}$ or $\partial\{u \geq 0\} = \overline{p_2 p_3} \cup \overline{p_4 p_1}$.

We compare the sums of lengths of these line segments with respect to l_p norms. The sum of lengths of $\overline{p_1 p_2}$ and $\overline{p_3 p_4}$ is independent of p and equals 20. The sum of lengths of $\overline{p_2 p_3}$ and $\overline{p_4 p_1}$ equals $(1^p + 1^p)^{1/p} + (11^p + 11^p)^{1/p} = 12 \cdot 2^{1/p}$. Let $p_0 = \frac{\log(2)}{\log(\frac{5}{3})}$. Then for $p > p_0$ it is smaller than 20, so the trapezoid $p_1 p_2 p_3 p_4 \subset \{u \geq 0\}$; if $p < p_0$, then it is greater than 20, so $p_1 p_2 p_3 p_4 \subset \{u < 0\}$. Finally, for $p = p_0$ the trapezoid $p_1 p_2 p_3 p_4$ is a zero level set of positive measure; this is the only p with this property. In particular, solutions to the anisotropic least gradient problem for the same boundary data can vary with p.

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