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Calculus of Variations

Equivalence of viscosity and weak solutions for the normalized $p(x)$-Laplacian

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Abstract We show that viscosity solutions to the normalized $p(x)$-Laplace equation coincide with distributional weak solutions to the strong $p(x)$-Laplace equation when p is Lipschitz and $\inf p > 1$. This yields $C^{1,\alpha}$ regularity for the viscosity solutions of the normalized $p(x)$-Laplace equation. As an additional application, we prove a Radó-type removability theorem.

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1 Introduction

In this paper, we study viscosity solutions to the *normalized $p(x)$-Laplace equation* which is defined by

$$-\Delta_{p(x)}^N u := -\Delta u - \frac{p(x)-2}{|Du|^2}\Delta_\infty u = 0, \tag{1.1}$$

where

$$\Delta_\infty u := \left\langle D^2 u Du, Du \right\rangle.$$

There has been recent interest in normalized equations, see e.g. [5,9,15]. We are partly motivated by the connection to stochastic tug-of-war games [23,24] as the case of space dependent probabilities leads to (1.1) [3].

The objective of this work is to show that viscosity solutions to (1.1) coincide with solutions in the distributional weak sense, when the equation is rewritten in an appropriate divergence

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formulation. One approach to this kind of equivalence results [10,13] is based on the uniqueness of solutions. However, it seems difficult to use uniqueness in our case because the uniqueness of solutions is an open problem for the Eq. (1.1) as pointed out in [14]. The equation (1.1) is in non-divergence form. In order to find the appropriate weak formulation, we note that for $u \in C^2(\Omega)$ with non-vanishing gradient it holds that

$$-|Du|^{p(x)-2}\,\Delta^N_{p(x)}u = -\mathrm{div}\left(|Du|^{p(x)-2}\,Du\right) + |Du|^{p(x)-2}\log(|Du|)\,Du\cdot Dp.$$

Thus the weak formulation of (1.1) should be the *strong* $p(x)$*-Laplace equation*

$$-\Delta^S_{p(x)}u := -\mathrm{div}(|Du|^{p(x)-2}\,Du) + |Du|^{p(x)-2}\log|Du|\,Du\cdot Dp = 0. \tag{1.2}$$

Our main result, Theorem 5.9, is that viscosity solutions to (1.1) coincide with weak solutions to (1.2) when the function p is Lipschitz with $\inf p > 1$. With these assumptions weak solutions to (1.2) in a domain are locally $C^{1,\alpha}$ continuous [25]. Thus our equivalence result yields local $C^{1,\alpha}$ regularity also for viscosity solutions to (1.1). As an application, we prove a Radó-type removability theorem for the strong $p(x)$-Laplacian. The theorem follows from the equivalence result since in the definition of a viscosity solution we may ignore the test functions whose gradient vanishes. The equivalence result also implies that the equation (1.2) is homogeneous: if u is a solution, so is λu. This is not completely obvious and was established in [1].

That viscosity solutions to (1.1) are weak solutions to (1.2) is proven by applying the method of [11]. The idea is to approximate a viscosity solution through a sequence of inf-convolutions, show that the inf-convolutions are essentially weak supersolutions, and then pass to the limit.

First, in Lemma 5.3 we show that the inf-convolution u_ε of a viscosity supersolution u to (1.1) is still, in essence, a viscosity supersolution up to some error. This fact is a key part of our proof. If there was no x-dependence in (1.1), it would be straightforward to see that the inf-convolution of a viscosity supersolution is still a viscosity supersolution. This is because a test function that touches the inf-convolution from below also touches the original function from below at a nearby point once we add some constant to it. From this it would follow that the inf-convolution is a supersolution to the original equation. However, the Eq. (1.1) has x-dependence caused by $p(x)$. Thus the inf-convolution no longer satisfies the original equation.

In Lemma 5.5 we use the standard mollification on u_ε and p to deduce from Lemma 5.3 that u_ε is "almost" a weak solution to $-\Delta^S_{p(x)}u_\varepsilon \geq 0$. Applying Caccioppoli type estimates and vector inequalities we are then able to deduce that the sequence of inf-convolutions converges to the viscosity supersolution in $W^{1,p(\cdot)}_{loc}(\Omega)$ as $\varepsilon \to 0$. This allows us to pass to the limit and conclude that the function u satisfies $-\Delta^S_{p(x)}u \geq 0$ in the weak sense.

Due to the variable exponent, the operator $\Delta^S_{p(x)}$ can be singular in some subsets and degenerate in others. Therefore we apply different arguments in the cases $p(x) < 2$ and $p(x) \geq 2$, and finally need to be able to combine them.

The equivalence of weak and viscosity solutions to the usual p-Laplace equation was first proven by Juutinen, Lindqvist and Manfredi [13]. Later Julin and Juutinen [11] presented a more direct way to show that viscosity solutions to $-\Delta_p u = f$ are also weak solutions. This proof was adapted in [4] to show that viscosity solutions to $-\Delta^N_p u = f$ coincide with weak solutions to $-\Delta_p u = |Du|^{p-2} f$ when $p \geq 2$. Similar arguments were also used in [20] to study the equivalence of solutions to $-\Delta_p u = f(x, u, Du)$. The variable exponent case was explored in [14] where the equivalence of weak and viscosity solutions was proven for the $p(x)$-Laplace equation using techniques of [13].

As mentioned, the Eq. (1.1) appears in stochastic tug-of-war games. Let us illustrate this in the case where $p > 2$ is a constant by considering the following two-player, zero-sum game from [21]. A step size $\varepsilon > 0$ is fixed and a token is placed at x_0 in a domain Ω. The players toss a biased coin that is heads with the probability $\alpha = \frac{p-2}{p+N}$ and tails with the probability $\beta = 1 - \alpha$. If the outcome is heads, the following tug-of-war step is played: a fair coin is tossed and the winning player is allowed to move the token to any position $x_1 \in B_\varepsilon(x_0)$. If the outcome is tails, the token moves to a random position in $x_1 \in B_\varepsilon(x_0)$. Once the token exits the domain, the game ends and player I pays player II according to the final location of the token. When the players optimize over their strategies, we obtain a value of the game. Then, as the step-size approaches zero, the value function converges uniformly to a viscosity solution of $-\Delta_p^N u = 0$ in Ω. This result can be extended to the general case $1 < p(x) < \infty$, see [3,23].

The Eq. (1.2) was introduced by Adamowicz and Hästö [1] in connection with mappings of finite distortion. Unlike the standard $p(x)$-Laplace equation, the Eq. (1.2) is homogeneous and its solutions satisfy a classical Harnack inequality [2]. The Eq. (1.2) has been further studied for example in [22,25,26].

The paper is organized as follows: in Sect. 2 we recall the variable exponent Lebesgue and Sobolev spaces. Section 3 contains the rigorous definitions of solutions to equations (1.1) and (1.2). In Sect. 4 we show that weak solutions of (1.2) are viscosity solutions to (1.1) and the converse statement is proven in Sect. 5. Finally, in Sect. 6 we formulate and prove a Radó-type removability theorem for weak solutions of (1.2).

2 Variable exponent lebesgue and sobolev spaces

We briefly recall basic facts about these spaces. For general reference see e.g. [7]. Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set and let $p : \Omega \to (1, \infty)$ be a measurable function. We denote

$$p_{\max} := \operatorname*{ess\,sup}_{x \in \Omega} p(x) \quad \text{and} \quad p_{\min} := \operatorname*{ess\,inf}_{x \in \Omega} p(x).$$

The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ is defined as the set of measurable functions $u : \Omega \to \mathbb{R}$ for which the $p(\cdot)$-modular

$$\varrho_{p(\cdot)}(u) := \int_\Omega |u|^{p(x)}\, dx$$

is finite. It is a Banach space equipped with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} := \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

Given that $p_{\max} < \infty$ or $\varrho_{p(\cdot)}(u) > 0$, the norm and the modular satisfy the inequality (see [7, p75])

$$\min \left\{ \varrho_{p(\cdot)}(u)^{\frac{1}{p_{\min}}}, \varrho_{p(\cdot)}(u)^{\frac{1}{p_{\max}}} \right\} \leq \|u\|_{L^{p(\cdot)}(\Omega)} \leq \max \left\{ \varrho_{p(\cdot)}(u)^{\frac{1}{p_{\min}}}, \varrho_{p(\cdot)}(u)^{\frac{1}{p_{\max}}} \right\}. \tag{2.1}$$

A version of Hölder's inequality holds [7, p81]: if $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ for a.e. $x \in \Omega$, then

$$\int_\Omega |u|\,|v|\,dx \le 2\,\|u\|_{L^{p(\cdot)}(\Omega)}\,\|v\|_{L^{p'(\cdot)}(\Omega)}.$$

As a consequence of the Hölder's inequality we have that

$$\|u\|_{L^{q(\cdot)}(\Omega)} \le 2\,(1+|\Omega|)\,\|u\|_{L^{p(\cdot)}(\Omega)}$$

for all $u \in L^{p(\cdot)}(\Omega)$ if $q(x) \le p(x)$ for a.e. $x \in \Omega$.

If $1 < p_{\min} \le p_{\max} < \infty$, then $L^{p(\cdot)}(\Omega)$ is reflexive and the dual of $L^{p(\cdot)}(\Omega)$ is $L^{p'(\cdot)}(\Omega)$.

The *variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ is the set of functions in $u \in L^{p(\cdot)}(\Omega)$ for which the weak gradient Du belongs in $L^{p(\cdot)}(\Omega)$. It is a Banach space equipped with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} := \|u\|_{L^{p(\cdot)}(\Omega)} + \|Du\|_{L^{p(\cdot)}(\Omega)}.$$

The space $W_0^{1,p}(\Omega)$ is the closure of compactly supported Sobolev functions in the space $W^{1,p(\cdot)}(\Omega)$. A function belongs to the the local Lebesgue space $L_{loc}^{p(\cdot)}(\Omega)$ if it belongs to $L^{p(\cdot)}(\Omega')$ for all $\Omega' \Subset \Omega$. The space $W_{loc}^{1,p(\cdot)}(\Omega)$ is defined analogically.

3 The strong and normalized $p(x)$-Laplace equations

In this section, we define weak solutions to the strong $p(x)$-Laplace equation and viscosity solutions to the normalized $p(x)$-Laplace equation.

From now on we assume that p is Lipschitz continuous and $p_{\min} > 1$.

Definition 3.1 A function $u \in W_{loc}^{1,p(\cdot)}(\Omega)$ is a *weak supersolution* to $-\Delta_{p(x)}^S u \ge 0$ in Ω if

$$\int_\Omega |Du|^{p(x)-2}\,Du\cdot D\varphi + |Du|^{p(x)-2}\log(|Du|)\,Du\cdot Dp\,\varphi\,dx \ge 0$$

for all non-negative $\varphi \in W^{1,p(\cdot)}(\Omega)$ with compact support. We say that u is a *weak subsolution* to $-\Delta_{p(x)}^S u \le 0$ if $-u$ is a supersolution and that u is a *weak solution* to $-\Delta_{p(x)}^S u = 0$ if u is both supersolution and subsolution.

Lemma 3.2 *It is enough to consider $C_0^\infty(\Omega)$ test functions in the previous definition.*

Proof Assume that $\varphi \in W^{1,p(\cdot)}(\Omega)$ has a compact support in an open set $\Omega' \Subset \Omega$. Since p is log-Hölder continuous and bounded as a Lipschitz function, there is a sequence of functions $\varphi_j \in C_0^\infty(\Omega')$ such that $\varphi_j \to \varphi$ in $W^{1,p(\cdot)}(\Omega')$ (see [7, p347]). We set $\psi_j := \varphi - \varphi_j$. Then it is enough to show that

$$\int_{\Omega'} |Du|^{p(x)-2}\,Du\cdot D\psi_j\,dx + \int_{\Omega'} |Du|^{p(x)-2}\log(|Du|)\,Du\cdot Dp\,\psi_j\,dx \to 0$$

as $j \to \infty$. The first integral convergences to zero by Hölder's inequality so we focus on the second integral. We may assume that $N > 1$. We set $q(x) := \frac{p(x)}{p(x)-1+\frac{1}{N}}$. Using the inequality $a^s \log a \le N a^{s+\frac{1}{N}} + \frac{1}{s}$ for $a, s > 0$ we get

$$\int_{\Omega'} |Du|^{p(x)-1}\,|\log|Du||\,|Dp|\,\left|\psi_j\right|\,dx$$

$$\leq \|Dp\|_{L^\infty(\Omega')}\left(\int_{\Omega'}\frac{|\psi_j|}{p(x)-1}\,dx + N\int_{\Omega'}|Du|^{p(x)-1+\frac{1}{N}}\left|\psi_j\right|\,dx\right)$$

$$\leq C(p,\Omega)\left(\left\|\psi_j\right\|_{L^{p(\cdot)}(\Omega')} + \left\||Du|^{p(x)-1+\frac{1}{N}}\right\|_{L^{q(\cdot)}(\Omega')}\left\|\psi_j\right\|_{L^{q'(\cdot)}(\Omega')}\right).$$

We take $r \in (1,N)$ such that $q'^{+} \leq r^* := \frac{Nr}{N-r}$. Then we have $q'(x) = \frac{Np(x)}{N-1} \leq \min(p^*(x), r^*)$, where $p^*(x) := \frac{Np(x)}{N-p(x)}$. Therefore

$$\left\|\psi_j\right\|_{L^{q'(\cdot)}(\Omega')} \leq 2\left(1+|\Omega|\right)\left\|\psi_j\right\|_{L^{\min(p^*(\cdot),r^*)}(\Omega')}.$$

Since $\psi_j \in W_0^{1,\min(p(\cdot),r)}(\Omega')$, we have by a variable exponent version of the Sobolev inequality (see e.g. [7, p265])

$$\left\|\psi_j\right\|_{L^{\min(p^*(\cdot),r^*)}(\Omega')} \leq C\left\|D\psi_j\right\|_{L^{\min(p(\cdot),r)}(\Omega')} \leq 2C(1+|\Omega|)\left\|D\psi_j\right\|_{L^{p(\cdot)}(\Omega')}.$$

These estimates imply the claim since $\left\|\psi_j\right\|_{W^{1,p}(\Omega')} \to 0$ as $j \to \infty$. □

In order to define viscosity solutions to $-\Delta^N_{p(x)}u = 0$, we set

$$F(x,\eta,X) := -\left(\operatorname{tr}X + \frac{p(x)-2}{|\eta|^2}\langle X\eta,\eta\rangle\right)$$

for all $(x,\eta,X) \in \Omega \times \left(\mathbb{R}^N \setminus \{0\}\right) \times S^N$ where S^N is the set of symmetric $N \times N$ matrices. We also recall the concept of semi-jets. The *subjet of a function* $u : \Omega \to \mathbb{R}$ *at* x is defined by setting $(\eta, X) \in J^{2,-}u(x)$ if

$$u(y) \geq u(x) + \eta\cdot(y-x) + \frac{1}{2}\langle X(y-x),(y-x)\rangle + o(|y-x|^2) \text{ as } y \to x. \tag{3.1}$$

The *closure of a subjet* is defined by setting $(\eta, X) \in \overline{J}^{2,-}u(x)$ if there is a sequence $(\eta_i, X_i) \in J^{2,-}u(x_i)$ such that $(x_i,\eta_i,X_i) \to (x,\eta,X)$. The *superjet* $J^{2,+}u(x)$ and its closure $\overline{J}^{2,+}u(x)$ are defined in the same manner except that the inequality (3.1) is reversed.

Definition 3.3 A lower semicontinuous function $u : \Omega \to \mathbb{R}$ is a *viscosity supersolution* to $-\Delta^N_{p(x)}u \geq 0$ in Ω if, whenever $(\eta, X) \in J^{2,-}u(x)$ with $x \in \Omega$ and $\eta \neq 0$, then

$$F(x,\eta,X) \geq 0.$$

A function u is a *viscosity subsolution* to $-\Delta^N_{p(x)}u \leq 0$ if $-u$ is a viscosity supersolution, and a *viscosity solution* to $-\Delta^N_{p(x)}u = 0$ if it is both viscosity super- and subsolution.

Remark Observe that in the previous definition we require nothing in the case $(0, X) \in J^{2,-}u(x)$.

Viscosity solutions may be equivalently defined using the jet-closures or test functions. The next proposition follows easily from the proof of Proposition 2.6 in [18].

Proposition 3.4 *Let* $u : \Omega \to \mathbb{R}$ *be lower semicontinuous. Then the following conditions are equivalent.*

- (i) *The function* u *is a viscosity supersolution to* $-\Delta^N_{p(x)}u \geq 0$ *in* Ω.
- (ii) *Whenever* $(\eta, X) \in \overline{J}^{2,-}u(x)$ *with* $x \in \Omega$, $\eta \neq 0$, *we have* $F(x,\eta,X) \geq 0$.
- (iii) *Whenever* $\varphi \in C^2(\Omega)$ *is such that* $\varphi(x) = u(x)$, $D\varphi(x) \neq 0$ *and* $\varphi(y) < u(y)$ *for all* $y \neq x$, *it holds* $F(x, D\varphi(x), D^2\varphi(x)) \geq 0$.

When φ is as in the third condition above, we say that φ *touches* u *from below at* x.

4 Weak solutions are viscosity solutions

We show that if u is a weak solution to $-\Delta_{p(x)}^S u = 0$, then it is a viscosity solution to $-\Delta_{p(x)}^N u = 0$.

Juutinen, Lukkari and Parviainen [14] showed that weak solutions to the standard $p(x)$-Laplace equation are also viscosity solutions. This was accomplished with the help of the comparison principle. For if u is a weak supersolution to $-\Delta_{p(x)} u \geq 0$ that is not a viscosity supersolution, then there is a test function $\varphi \in C^2$ touching u from below at x so that $-\Delta_{p(x)}\varphi < 0$ in some ball $B(x)$. Lifting φ slightly produces a new function $\tilde{\varphi}$ still satisfying $-\Delta_{p(x)}\tilde{\varphi} < 0$ in $B(x)$ and $\tilde{\varphi} \leq u$ in $\partial B(x)$. Comparison principle now implies that $\tilde{\varphi} \leq u$ in $B(x)$ which is a contradiction since $\tilde{\varphi}(x) > \varphi(x) = u(x)$.

Our difficulty is that, to the best of our knowledge, the comparison principle is an open problem for the strong $p(x)$-Laplacian. Our strategy is therefore to consider a ball so small that the gradient of the test function does not vanish. Then the comparison principle holds and we arrive at a contradiction.

Theorem 4.1 *If $u \in W_{loc}^{1,p(\cdot)}(\Omega)$ is a weak solution to $-\Delta_{p(x)}^S u = 0$, then it is a viscosity solution to $-\Delta_{p(x)}^N u = 0$ in Ω.*

Proof Zhang and Zhou [25] showed that weak solutions of $-\Delta_{p(x)}^S u = 0$ are in $C^1(\Omega)$. Therefore it suffices to show that if $u \in C^1(\Omega)$ is a weak supersolution to $-\Delta_{p(x)}^S u \geq 0$, then it is also a viscosity supersolution to $-\Delta_{p(x)}^N u \geq 0$. Assume on the contrary that there is $\varphi \in C^2(\Omega)$ touching u from below at $x_0 \in \Omega$, $D\varphi(x_0) \neq 0$ and

$$0 > -h > F(x_0, D\varphi(x_0), D^2\varphi(x_0)).$$

Then by continuity there is $r > 0$ such that in $B_r(x_0)$ it holds

$$-h\,|D\varphi|^{p(x)-2} \geq -\,|D\varphi|^{p(x)-2}\left(\Delta\varphi + \frac{p(x)-2}{|D\varphi|^2}\Delta_\infty\varphi\right). \tag{4.1}$$

Since $Du(x_0) = D\varphi(x_0) \neq 0$, we may also assume that there is $m > 0$ such that

$$\inf_{x\in B_r(x_0)} |D\varphi|^{p(x)-2} \geq m \tag{4.2}$$

and

$$\operatorname*{ess\,sup}_{x\in B_r(x_0)} |Dp| \left| |D\varphi|^{p(x)-2}\log(|D\varphi|)\,D\varphi - |Du|^{p(x)-2}\log(|Du|)\,Du \right| \leq \frac{hm}{2}. \tag{4.3}$$

Let $l := \min_{x\in\partial B_r(x_0)}(u-\varphi) > 0$ and set $\psi(x) := \max(\varphi(x) + l - u(x), 0)$. Then $\psi \in W_0^{1,2}(B_r(x_0))$ so there are $\psi_j \in C_0^\infty(B_r(x_0))$ such that $\psi_j \to \psi$ in $W^{1,2}(B_r(x_0))$. Let p_j be the standard mollification of p. Multiplying (4.1) by ψ and integrating over $B_r(x_0)$ yields

$$\begin{aligned} &-h\int_{B_r(x_0)} |D\varphi|^{p(x)-2}\,\psi\,dx \\ &\geq \int_{B_r(x_0)} -\,|D\varphi|^{p(x)-2}\left(\Delta\varphi + \frac{p(x)-2}{|D\varphi|^2}\Delta_\infty\varphi\right)\psi\,dx \\ &= \lim_{j\to\infty}\int_{B_r(x_0)} -\,|D\varphi|^{p_j(x)-2}\left(\Delta\varphi + \frac{p_j(x)-2}{|D\varphi|^2}\Delta_\infty\varphi\right)\psi_j\,dx, \end{aligned} \tag{4.4}$$

where the last equality holds because $\psi_j \to \psi$ in $W^{1,2}(B_r(x_0))$ and $p_j \to p$ uniformly in $B_r(x_0)$. Calculating the divergence of $|D\varphi|^{p_j(x)-2} D\varphi$ and integrating by parts we get

$$\begin{aligned}
&\int_{B_r(x_0)} -|D\varphi|^{p_j(x)-2}\left(\Delta\varphi + \frac{p_j(x)-2}{|D\varphi|^2}\Delta_\infty\varphi\right)\psi_j\,dx \\
&= \int_{B_r(x_0)} -\mathrm{div}\left(|D\varphi|^{p_j(x)-2} D\varphi\right)\psi_j + |D\varphi|^{p_j(x)-2}\log(|D\varphi|)\,D\varphi\cdot Dp_j\,\psi_j\,dx \\
&= \int_{B_r(x_0)} |D\varphi|^{p_j(x)-2} D\varphi\cdot\left(D\psi_j + \log(|D\varphi|)\,Dp_j\,\psi_j\right)dx.
\end{aligned} \tag{4.5}$$

By the convergence of ψ_j and p_j, it follows from (4.4) and (4.5) that

$$-h\int_{B_r(x_0)} |D\varphi|^{p(x)-2}\psi\,dx \geq \int_{B_r(x_0)} |D\varphi|^{p(x)-2} D\varphi\cdot(D\psi + \log(|D\varphi|)\,Dp\,\psi)\,dx. \tag{4.6}$$

Since u is a weak supersolution to $\Delta^S_{p(x)}u = 0$ and $\psi \in W^{1,p(\cdot)}(\Omega)$ has a compact support in Ω, we have

$$\int_{B_r(x_0)} |Du|^{p(x)-2} Du\cdot(D\psi + \log|Du|\,Dp\,\psi)\,dx \geq 0. \tag{4.7}$$

Denoting $A := \{x \in B_r(x_0) : \psi(x) > 0\}$ and combining (4.6) and (4.7) we arrive at

$$\begin{aligned}
&\int_A \left(|D\varphi|^{p(x)-2} D\varphi - |Du|^{p(x)-2} Du\right)\cdot(D\varphi - Du)\,dx \\
&\leq \int_A \left||Du|^{p(x)-2}\log(|Du|)\,Du - |D\varphi|^{p(x)-2}\log(|D\varphi|)\,D\varphi\right||Dp|\,\psi\,dx \\
&\quad - h\int_A |D\varphi|^{p(x)-2}\psi\,dx \\
&\leq -\frac{hm}{2}\int_A \psi\,dx,
\end{aligned} \tag{4.8}$$

where the last inequality follows from (4.2) and (4.3). Since

$$\left(|a|^{p(x)-2}a - |b|^{p(x)-2}b\right)\cdot(a-b) \geq 0$$

for any two vectors $a, b \in \mathbb{R}^N$ when $p(x) > 1$, it follows from (4.8) that $|A| = 0$. But this is impossible since $\varphi(x_0) = u(x_0)$ and $l > 0$. □

5 Viscosity solutions are weak solutions

We show that if u is a viscosity supersolution to $-\Delta^N_{p(x)}u \geq 0$, then it is a weak supersolution to $-\Delta^S_{p(x)}u \geq 0$. The same statement for subsolutions then follows by analogy.

We recall the usual partial ordering for symmetric $N \times N$ matrices by setting $X \leq Y$ if $\langle X\xi, \xi\rangle \leq \langle Y\xi, \xi\rangle$ for all $\xi \in \mathbb{R}^N$. For a matrix X we also set $\|X\| := \max\{|\lambda| : \lambda \text{ is an eigenvalue of } X\}$ and for vectors $\xi, \eta \in \mathbb{R}^N$ we use the notation $\xi \otimes \eta := \xi\eta'$, i.e. $\xi \otimes \eta$ is an $N \times N$ matrix whose (i, j) entry is $\xi_i\eta_j$.

Definition 5.1 [Inf-convolution] Let $q \geq 2$ and $\varepsilon > 0$. The inf-convolution of a bounded function $u \in C(\Omega)$ is defined by

$$u_\varepsilon(x) := \inf_{y\in\Omega} \left\{ u(y) + \frac{1}{q\varepsilon^{q-1}} |x-y|^q \right\}. \tag{5.1}$$

The inf-convolution is well known to provide good approximations of viscosity supersolutions and often one only needs to consider it for $q = 2$ (see e.g. [6]). However, as the authors in [11] observed, considering large enough q essentially cancels the singularity in the usual p-Laplace operator when $1 < p < 2$. In similar fashion it also cancels the singularity of the operator $\Delta^S_{p(x)}$. This is due to the property (v) in the next lemma. We also list some other properties of the inf-convolution.

Lemma 5.2 *Let $u \in C(\Omega)$ be a bounded function. Then the inf-convolution u_ε as defined in* (5.1) *has the following properties.*

(i) *We have $u_\varepsilon \leq u$ in Ω and $u_\varepsilon \to u$ locally uniformly in Ω as $\varepsilon \to 0$.*
(ii) *There exists $r(\varepsilon) > 0$ such that*

$$u_\varepsilon(x) = \inf_{y\in B_{r(\varepsilon)}(x)\cap\Omega} \left\{ u(y) + \frac{1}{q\varepsilon^{q-1}} |x-y|^q \right\}$$

and $r(\varepsilon) \to 0$ as $\varepsilon \to 0$. In fact we can choose $r(\varepsilon) = \left(q\varepsilon^{q-1}\operatorname{osc}_\Omega u\right)^{\frac{1}{q}}$.
(iii) *The function u_ε is semi-concave in $\Omega_{r(\varepsilon)}$, that is, the function $x \mapsto u_\varepsilon(x) - \frac{q-1}{2\varepsilon^{q-1}} r(\varepsilon)^{q-2} |x|^2$ is concave.*
(iv) *If $x \in \Omega_{r(\varepsilon)} := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > r(\varepsilon)\}$, then there exists a point $x_\varepsilon \in B_{r(\varepsilon)}(x)$ such that $u_\varepsilon(x) = u(x_\varepsilon) + \frac{1}{q\varepsilon^{q-1}} |x - x_\varepsilon|^q$.*
(v) *If $(\eta, X) \in J^{2,-}u_\varepsilon(x)$ with $x \in \Omega_{r(\varepsilon)}$, then $\eta = \frac{(x-x_\varepsilon)}{\varepsilon^{q-1}} |x_\varepsilon - x|^{q-2}$ and $X \leq \frac{q-1}{\varepsilon} |\eta|^{\frac{q-2}{q-1}} I$, where x_ε is as in* (iv).

These properties are well known, see appendix of [11] and also [17] where more general "flat inf-convolution" is considered. Regardless, we give a proof of (v) based on [16, p53] due to its critical role in the proof of Lemma 5.5.

Proof of property (v) in Lemma 5.2 Let $(\eta, X) \in J^{2,-}u_\varepsilon(x)$. Then there is a function $\varphi \in C^2(\mathbb{R}^N)$ such that it touches u_ε from below at x and $D\varphi(x) = \eta$, $D^2\varphi(x) = X$. Therefore for all $y, z \in \Omega$ we have

$$u(y) + \frac{|y-z|^q}{q\varepsilon^{q-1}} - \varphi(z) \geq u_\varepsilon(z) - \varphi(z) \geq 0.$$

Choosing $y = x_\varepsilon$, we obtain

$$\varphi(z) - \frac{|x_\varepsilon - z|^q}{q\varepsilon^{q-1}} \leq u(x_\varepsilon) \text{ for all } z \in \Omega.$$

Since $\varphi(x) = u_\varepsilon(x) = u(x_\varepsilon) + \frac{|x_\varepsilon - x|^q}{q\varepsilon^{q-1}}$, the above inequality means that the function

$$z \mapsto \varphi(z) - \frac{|x_\varepsilon - z|^q}{q\varepsilon^{q-1}} =: \varphi(z) - \psi(z)$$

has a maximum at x. Thus $\eta = D\psi(x) = \frac{(x-x_\varepsilon)}{\varepsilon^{q-1}} |x_\varepsilon - x|^{q-2}$ and

$$X \leq D^2\psi(x) = \frac{1}{\varepsilon^{q-1}} |x_\varepsilon - x|^{q-4} \left((q-2)(x_\varepsilon - x) \otimes (x_\varepsilon - x) + |x_\varepsilon - x|^2 I\right)$$

$$
\begin{aligned}
&\leq \frac{1}{\varepsilon^{q-1}} |x_\varepsilon - x|^{q-4} \left((q-2) \|(x_\varepsilon - x) \otimes (x_\varepsilon - x)\| I + |x_\varepsilon - x|^2 I \right) \\
&= \frac{q-1}{\varepsilon^{q-1}} |x_\varepsilon - x|^{q-2} I \\
&= \frac{q-1}{\varepsilon^{q-1}} \left(\varepsilon |\eta|^{\frac{1}{q-1}} \right)^{q-2} I \\
&= \frac{q-1}{\varepsilon} |\eta|^{\frac{q-2}{q-1}} I.
\end{aligned}
$$

□

We will show that the inf-convolution provides approximations of viscosity supersolutions to $-\Delta^N_{p(x)} u \geq 0$. If there was no x-dependence in the equation, it would be straightforward to show that the inf-convolution of a supersolution is still a supersolution. However, the equation $-\Delta^N_{p(x)} u \geq 0$ has x-dependence caused by $p(x)$. Regardless, in [10, Thm 3] it is shown that with some assumptions on G, the inf-convolution u_ε of a viscosity supersolution to $G(x, u, Du, D^2u) \geq 0$ is still a viscosity supersolution to $G(x, u_\varepsilon, Du_\varepsilon, D^2u_\varepsilon) \geq E(\varepsilon)$, where $E(\varepsilon) \to 0$ as $\varepsilon \to 0$.

We prove a modified version of this theorem for the solutions of $-\Delta^N_{p(x)} u \geq 0$. The important modification is the term $|\eta|^{\min(p(x)-2,0)}$ in (5.2) as it cancels a singular gradient term that appears due to the error term in the proof of Lemma 5.5, see (5.14). Another difference is that we consider inf-convolution with the exponent $q \geq 2$.

Lemma 5.3 *Assume that u is a uniformly continuous viscosity supersolution to $-\Delta^N_{p(x)} u \geq 0$ in Ω. Then, whenever $(\eta, X) \in J^{2,-} u_\varepsilon(x)$, $\eta \neq 0$ and $x \in \Omega_{r(\varepsilon)}$, it holds*

$$
|\eta|^{\min(p(x)-2,0)} F(x, \eta, X) \geq E(\varepsilon), \tag{5.2}
$$

where $E(\varepsilon) \to 0$ as $\varepsilon \to 0$. The error function E depends only on p, q and the modulus of continuity of u.

Proof Fix $x \in \Omega_{r(\varepsilon)}$ and $(\eta, X) \in J^{2,-} u_\varepsilon(x)$, $\eta \neq 0$. Then by Lemma 5.2 there is $x_\varepsilon \in B_{r(\varepsilon)}(x)$ such that

$$
u_\varepsilon(x) = u(x_\varepsilon) + \frac{|x_\varepsilon - x|^q}{q\varepsilon^{q-1}} \tag{5.3}
$$

and $\eta = \frac{(x - x_\varepsilon)}{\varepsilon^{q-1}} |x_\varepsilon - x|^{q-2}$. There exists a function $\varphi \in C^2(\mathbb{R}^N)$ such that it touches u_ε from below at x and $D\varphi(x) = \eta$, $D^2\varphi(x) = X$. By the definition of inf-convolution

$$
u(y) - \varphi(z) + \frac{|y - z|^q}{q\varepsilon^{q-1}} \geq u_\varepsilon(z) - \varphi(z) \geq 0 \quad \text{for all } y, z \in \Omega_{r(\varepsilon)}. \tag{5.4}
$$

Since by (5.3) we have $u(x_\varepsilon) = \varphi(x) - \frac{|x_\varepsilon - x|^q}{q\varepsilon^{q-1}}$, it follows from (5.4) that the expression $u(y) - \varphi(z) + \frac{|y-z|^q}{q\varepsilon^{q-1}}$ reaches its minimum at $(y, z) = (x_\varepsilon, x)$. Thus

$$
\max_{(y,z) \in \Omega_{r(\varepsilon)} \times \Omega_{r(\varepsilon)}} -u(y) + \varphi(z) - \frac{|y - z|^q}{q\varepsilon^{q-1}} = -u(x_\varepsilon) + \varphi(x) - \frac{|x_\varepsilon - x|^q}{q\varepsilon^{q-1}}.
$$

We denote $\Phi(y, z) := \frac{1}{q\varepsilon^{q-1}} |y - z|^q$ and invoke the Theorem of sums (see [6]). There exist $Y, Z \in S^N$ such that

$$
(\eta, -Y) \in \overline{J}^{2,-} u(x_\varepsilon), \ (\eta, -Z) \in \overline{J}^{2,+} \varphi(x)
$$

and

$$\begin{pmatrix} Y & 0 \\ 0 & -Z \end{pmatrix} \leq D^2\Phi(x_\varepsilon, x) + \varepsilon^{q-1}\left(D^2\Phi(x_\varepsilon, x)\right)^2 \tag{5.5}$$

where

$$D^2\Phi(x_\varepsilon, x) = \begin{pmatrix} M & -M \\ -M & M \end{pmatrix}$$

with $M = \frac{1}{\varepsilon^{q-1}} |x_\varepsilon - x|^{q-4} \left((q-2)(x_\varepsilon - x) \otimes (x_\varepsilon - x) + |x_\varepsilon - x|^2 I\right)$ and

$$\left(D^2\Phi(x_\varepsilon, x)\right)^2 = 2\begin{pmatrix} M^2 & -M^2 \\ -M^2 & M^2 \end{pmatrix}.$$

The above implies $Y \leq Z \leq -D^2\varphi(x) = -X$. Multiplying (5.5) by the $\mathbb{R}^{2N}$ vector $(\frac{\eta}{|\eta|}\sqrt{p(x_\varepsilon) - 1}, \frac{\eta}{|\eta|}\sqrt{p(x) - 1})$ from both sides yields

$$\frac{(p(x_\varepsilon) - 1)}{|\eta|^2}\langle Y\eta, \eta\rangle - \frac{(p(x) - 1)}{|\eta|^2}\langle Z\eta, \eta\rangle \leq \Lambda^2\left\langle\left(M + 2\varepsilon^{q-1}M^2\right)\frac{\eta}{|\eta|}, \frac{\eta}{|\eta|}\right\rangle, \tag{5.6}$$

where $\Lambda = \sqrt{p(x) - 1} - \sqrt{p(x_\varepsilon) - 1}$. We have

$$\begin{aligned} 0 \leq& F(x_\varepsilon, \eta, -Y) \\ =& F(x, \eta, Z) - F(x_\varepsilon, \eta, Y) - F(x, \eta, Z) \\ =& (p(x_\varepsilon) - 1)\left\langle Y\frac{\eta}{|\eta|}, \frac{\eta}{|\eta|}\right\rangle - (p(x) - 1)\left\langle Z\frac{\eta}{|\eta|}, \frac{\eta}{|\eta|}\right\rangle \\ &+ \operatorname{tr}(Y) - \left\langle Y\frac{\eta}{|\eta|}, \frac{\eta}{|\eta|}\right\rangle - \operatorname{tr}(Z) + \left\langle Z\frac{\eta}{|\eta|}, \frac{\eta}{|\eta|}\right\rangle + F(x, \eta, -Z) \\ \leq& \Lambda^2\left\langle\left(M + 2\varepsilon^{q-1}M^2\right)\frac{\eta}{|\eta|}, \frac{\eta}{|\eta|}\right\rangle + F(x, \eta, X), \end{aligned} \tag{5.7}$$

where we used (5.6) and the fact that $Y \leq Z$ implies

$$\operatorname{tr}(Y - Z) - \left\langle (Y - Z)\frac{\eta}{|\eta|}, \frac{\eta}{|\eta|}\right\rangle \leq 0.$$

We have the estimate

$$\begin{aligned} \|M\| \leq& \frac{1}{\varepsilon^{q-1}} |x_\varepsilon - x|^{q-4}\left((q-2)\|(x_\varepsilon - x) \otimes (x_\varepsilon - x)\| + |x_\varepsilon - x|^2 \|I\|\right) \\ =& \frac{q-1}{\varepsilon^{q-1}} |x_\varepsilon - x|^{q-2}. \end{aligned}$$

Since p is Lipschitz continuous and $p_{\min} > 1$, we have also

$$\Lambda^2 = \frac{|p(x) - p(x_\varepsilon)|^2}{\left|\sqrt{p(x) - 1} + \sqrt{p(x_\varepsilon) - 1}\right|^2} \leq C(p)\,|x - x_\varepsilon|^2.$$

Combining these with (5.7) we get (we may assume that $r(\varepsilon) < 1$)

$$\begin{aligned} -F(x, \eta, X) \leq& \Lambda^2\left(\|M\| + 2\varepsilon^{q-1}\|M\|^2\right) \\ \leq& \Lambda^2\left(\frac{q-1}{\varepsilon^{q-1}}|x_\varepsilon - x|^{q-2} + 2\varepsilon^{q-1}\left(\frac{q-1}{\varepsilon^{q-1}}\right)^2 |x_\varepsilon - x|^{2(q-2)}\right) \end{aligned}$$

$$
\begin{aligned}
\leq & \frac{3\,(q-1)^2}{\varepsilon^{q-1}} \Lambda^2 \left|x_\varepsilon - x\right|^{q-2} \\
\leq & C(p,q) \frac{1}{\varepsilon^{q-1}} \left|x_\varepsilon - x\right|^{q}. \qquad (5.8)
\end{aligned}
$$

Moreover, by uniform continuity of u there is a modulus of continuity ω such that $\omega(t) \to 0$ as $t \to 0$ and $|u(y) - u(z)| \leq \omega(|y-z|)$ for all $y, z \in \Omega$. Hence by (5.3)

$$
\left|x_\varepsilon - x\right| \leq \left(q\varepsilon^{q-1}\left(u(x) - u(x_\varepsilon)\right)\right)^{\frac{1}{q}} \leq q^{\frac{1}{q}} \varepsilon^{\frac{q-1}{q}} \omega(r(\varepsilon))^{\frac{1}{q}}. \qquad (5.9)
$$

We now consider the situations $p(x) \leq 2$ and $p(x) > 2$ separately.
If $p(x) \leq 2$, we multiply (5.8) by $|\eta|^{p(x)-2}$ and estimate using (5.9). We get

$$
\begin{aligned}
-\left|\eta\right|^{p(x)-2} F(x,\eta,X) \leq & C(p,q) \frac{1}{\varepsilon^{q-1}} \left|x_\varepsilon - x\right|^{q} \left|\eta\right|^{p(x)-2} \\
= & C(p,q) \frac{1}{\varepsilon^{q-1}} \left|x_\varepsilon - x\right|^{q} \left| \frac{1}{\varepsilon^{q-1}} \left(x - x_\varepsilon\right) \left|x_\varepsilon - x\right|^{q-2} \right|^{p(x)-2} \\
= & C(p,q) \left(\frac{1}{\varepsilon}\right)^{(q-1)(p(x)-1)} \left|x_\varepsilon - x\right|^{q+(q-1)(p(x)-2)} \\
\leq & C(p,q) \left(\frac{1}{\varepsilon}\right)^{(q-1)(p(x)-1)} \left(q^{\frac{1}{q}} \varepsilon^{\frac{q-1}{q}} \omega(r(\varepsilon))^{\frac{1}{q}}\right)^{q+(q-1)(p(x)-2)} \\
= & C(p,q) \left(\frac{1}{\varepsilon}\right)^{\left(\frac{q-1}{q}\right)(p(x)-2)} \omega(r(\varepsilon))^{\frac{q+(q-1)(p(x)-2)}{q}} \\
\leq & C(p,q) \omega(r(\varepsilon))^{\frac{q+(q-1)(p_{\min}-2)}{q}},
\end{aligned}
$$

where the last inequality is true when $\varepsilon < 1$ is so small that $\omega(r(\varepsilon)) < 1$. This proves (5.2) when $p(x) \leq 2$.
If $p(x) > 2$, we estimate (5.8) directly using (5.9). We get

$$
-F(x,\eta,X) \leq C(p,q) \frac{1}{\varepsilon^{q-1}} \left(q^{\frac{1}{q}} \varepsilon^{\frac{q-1}{q}} \omega(r(\varepsilon))^{\frac{1}{q}}\right)^{q} = C(p,q)\omega(r(\varepsilon))),
$$

which proves (5.2) when $p(x) > 2$. □

Next we will use the previous lemma to show that inf-convolution of a viscosity supersolution to $-\Delta^N_{p(x)} u \geq 0$ in Ω is a weak supersolution to $-\Delta^S_{p(x)} u \geq 0$ in $\Omega_{r(\varepsilon)}$ up to some error term. Before proceeding we make some remarks about the point-wise differentiability of inf-convolution.

Remark 5.4 It follows from semi-concavity that the inf-convolution u_ε is locally Lipschitz in $\Omega_{r(\varepsilon)}$ (see [8, p267]). Therefore it belongs in $W^{1,\infty}_{loc}(\Omega_{r(\varepsilon)})$, is differentiable almost everywhere in $\Omega_{r(\varepsilon)}$, and its derivative agrees with its Sobolev derivative almost everywhere in $\Omega_{r(\varepsilon)}$ (see [8, p155 and p265]).

By Lemma 5.2 the function $\phi(x) := u_\varepsilon(x) - C(q,\varepsilon,u)\left|x\right|^2$ is concave in $\Omega_{r(\varepsilon)}$. Thus Alexandrov's theorem implies that u_ε is twice differentiable almost everywhere in $\Omega_{r(\varepsilon)}$. Furthermore, the proof of Alexandrov's theorem in [8, p273] establishes that if ϕ_j is the standard mollification of ϕ, then $D^2\phi_j \to D^2\phi$ almost everywhere in $\Omega_{r(\varepsilon)}$.

Lemma 5.5 *Assume that u is a uniformly continuous viscosity supersolution to $-\Delta_{p(x)}^N u \geq 0$ in Ω. Let $q > 2$ be so large that $p_{\min} - 2 + \frac{q-2}{q-1} \geq 0$ and let u_ε be the inf-convolution of u as defined in (5.1). Then*

$$\int_{\Omega_{r(\varepsilon)}} |Du_\varepsilon|^{p(x)-2} Du_\varepsilon \cdot (D\varphi + \log|Du_\varepsilon|\, Dp\, \varphi)\, dx \geq E(\varepsilon) \int_{\Omega_{r(\varepsilon)}} |Du_\varepsilon|^{s(x)} \varphi\, dx$$

for all non-negative $\varphi \in W^{1,p(\cdot)}(\Omega_{r(\varepsilon)})$ with compact support, where $E(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $s(x) = \max(p(x) - 2, 0)$.

Proof It is enough to consider $\varphi \in C_0^\infty(\Omega_{r(\varepsilon)})$. This can be proven as Lemma 3.2, but since $u_\varepsilon \in W_{loc}^{1,\infty}(\Omega_{r(\varepsilon)})$, the proof is even simpler.

(Step 1) We show that u_ε satisfies the auxiliary inequality (5.11) for all $0 < \delta < 1$. As mentioned in Remark 5.4, the function $\phi(x) := u_\varepsilon(x) - C(q, \varepsilon, u)\,|x|^2$ is concave in $\Omega_{r(\varepsilon)}$ and therefore we can approximate it by smooth concave functions ϕ_j so that $\left(\phi_j, D\phi_j, D^2\phi_j\right) \to \left(\phi, D\phi, D^2\phi\right)$ almost everywhere in $\Omega_{r(\varepsilon)}$. We define

$$u_{\varepsilon,j}(x) := \phi_j(x) + C(q, \varepsilon, u)\,|x|^2$$

and denote by p_j the standard mollification of p. Since $u_{\varepsilon,j}$ and p_j are smooth, we calculate

$$\begin{aligned}
&\int_{\Omega_{r(\varepsilon)}} -\left(\delta + \left|Du_{\varepsilon,j}\right|^2\right)^{\frac{p_j(x)-2}{2}} \left(\Delta u_{\varepsilon,j} + \frac{p_j(x)-2}{\delta + \left|Du_{\varepsilon,j}\right|^2} \Delta_\infty u_{\varepsilon,j}\right) \varphi\, dx \\
&= \int_{\Omega_{r(\varepsilon)}} -\operatorname{div}\left(\left(\delta + \left|Du_{\varepsilon,j}\right|^2\right)^{\frac{p_j(x)-2}{2}} Du_{\varepsilon,j}\right) \varphi \\
&\quad + \frac{1}{2}\left(\delta + \left|Du_{\varepsilon,j}\right|^2\right)^{\frac{p_j(x)-2}{2}} \log\left(\delta + \left|Du_{\varepsilon,j}\right|^2\right) Du_{\varepsilon,j} \cdot Dp_j\, \varphi\, dx \\
&= \int_{\Omega_{r(\varepsilon)}} \left(\delta + \left|Du_{\varepsilon,j}\right|^2\right)^{\frac{p_j(x)-2}{2}} Du_{\varepsilon,j} \cdot \left(D\varphi + \frac{1}{2}\log\left(\delta + \left|Du_{\varepsilon,j}\right|^2\right) Dp_j\, \varphi\right) dx.
\end{aligned} \tag{5.10}$$

We let $j \to \infty$ in (5.10) and intend to use Fatou's lemma at the LHS and the Dominated convergence theorem at the RHS. This results in the auxiliary inequality

$$\begin{aligned}
&\int_{\Omega_{r(\varepsilon)}} -\left(\delta + |Du_\varepsilon|^2\right)^{\frac{p(x)-2}{2}} \left(\Delta u_\varepsilon + \frac{p(x)-2}{\delta + |Du_\varepsilon|^2} \Delta_\infty u_\varepsilon\right) \varphi\, dx \\
&\quad \leq \int_{\Omega_{r(\varepsilon)}} \left(\delta + |Du_\varepsilon|^2\right)^{\frac{p(x)-2}{2}} Du_\varepsilon \cdot \left(D\varphi + \frac{1}{2}\log\left(\delta + |Du_\varepsilon|^2\right) Dp\, \varphi\right) dx,
\end{aligned} \tag{5.11}$$

where $D^2 u_\varepsilon$ is the Hessian of u_ε in the Alexandrov's sense. We still need to check that the assumptions of the Dominated convergence theorem and Fatou's lemma hold. By Lipschitz continuity of u_ε and p there is $M \geq 1$ such that

$$\sup_j \left\|Du_{\varepsilon,j}\right\|_{L^\infty(\operatorname{supp}\varphi)}, \sup_j \left\|Dp_j\right\|_{L^\infty(\operatorname{supp}\varphi)} \leq M.$$

This justifies our use of the Dominated convergence theorem. In order to justify our use of Fatou's lemma, we notice first that by concavity of ϕ_j we have $D^2 u_{\varepsilon,j} \leq C(q, \varepsilon, u) I$. Thus the integrand at the LHS of (5.10) is clearly bounded from below by a constant independent of j if $Du_{\varepsilon,j} = 0$. If $Du_{\varepsilon,j} \neq 0$, we have

$$\begin{aligned}
&\left(\delta + \left|Du_{\varepsilon,j}\right|^2\right)^{\frac{p_j(x)-2}{2}} \left(\Delta u_{\varepsilon,j} + \frac{p_j(x)-2}{\delta + \left|Du_{\varepsilon,j}\right|^2} \Delta_\infty u_{\varepsilon,j}\right) \\
&= \frac{\left(\delta + \left|Du_{\varepsilon,j}\right|^2\right)^{\frac{p_j(x)-2}{2}}}{\delta + \left|Du_{\varepsilon,j}\right|^2} \left(\left|Du_{\varepsilon,j}\right|^2 \left(\Delta u_{\varepsilon,j} + \frac{p_j(x)-2}{\left|Du_{\varepsilon,j}\right|^2} \Delta_\infty u_{\varepsilon,j}\right) + \delta \Delta u_{\varepsilon,j}\right) \\
&\leq \frac{\delta^{\frac{p_j(x)-2}{2}} + \left(\delta + M^2\right)^{\frac{p_j(x)-2}{2}}}{\delta + \left|Du_{\varepsilon,j}\right|^2} C(q,\varepsilon,u) \left(\left|Du_{\varepsilon,j}\right|^2 \left(N + p_j(x) - 2\right) + \delta N\right) \\
&\leq C(q,\varepsilon,u) \left(\delta^{\frac{p_{\min}-2}{2}} + \left(\delta + M^2\right)^{\frac{p_{\max}-2}{2}}\right) (2N + p_{\max} - 2),
\end{aligned}$$

where the first inequality follows like estimate (5.7) since $p_j \geq p_{\min} > 1$.

(Step 2) We let $\delta \to 0$ in the auxiliary inequality (5.11). The RHS becomes

$$\int_{\Omega_{r(\varepsilon)} \setminus \{Du_\varepsilon = 0\}} |Du_\varepsilon|^{p(x)-2} Du_\varepsilon \cdot (D\varphi + \log|Du_\varepsilon| Dp\, \varphi)\, dx$$

by the Lebesgue's dominated convergence theorem. We intend to apply Fatou's lemma on the LHS. We have $\left(Du_\varepsilon(x), D^2u_\varepsilon(x)\right) \in J^{2,-}u_\varepsilon(x)$ for almost every $x \in \Omega_{r(\varepsilon)}$. Therefore by Lemma 5.3 it holds that

$$|Du_\varepsilon|^{\min(p(x)-2,0)} F(x, Du_\varepsilon, D^2u_\varepsilon) \geq E(\varepsilon) \text{ in } \left\{x \in \Omega_{r(\varepsilon)} : Du_\varepsilon \neq 0\right\} \tag{5.12}$$

and by the property (v) in Lemma 5.2 we have

$$D^2u_\varepsilon \leq \frac{q-1}{\varepsilon} |Du_\varepsilon|^{\frac{q-2}{q-1}} I. \tag{5.13}$$

Observe that since $q > 2$, the condition (5.13) implies that the Hessian D^2u_ε is negative semi-definite in the set where the gradient Du_ε vanishes. Using this fact, Fatou's lemma and (5.12) we get

$$\begin{aligned}
&\liminf_{\delta\to 0} \int_{\Omega_{r(\varepsilon)}} -\left(|Du_\varepsilon|^2 + \delta\right)^{\frac{p(x)-2}{2}} \left(\Delta u_\varepsilon + \frac{p(x)-2}{|Du_\varepsilon|^2 + \delta} \Delta_\infty u_\varepsilon\right) \varphi\, dx \\
&\geq \liminf_{\delta\to 0} \int_{\{Du_\varepsilon \neq 0\}} -\left(|Du_\varepsilon|^2 + \delta\right)^{\frac{p(x)-2}{2}} \left(\Delta u_\varepsilon + \frac{p(x)-2}{|Du_\varepsilon|^2 + \delta} \Delta_\infty u_\varepsilon\right) \varphi\, dx \\
&\quad + \liminf_{\delta\to 0} \int_{\{Du_\varepsilon = 0\}} -\delta^{\frac{p(x)-2}{2}} \Delta u_\varepsilon \varphi\, dx \\
&\geq \int_{\{Du_\varepsilon \neq 0\}} -|Du_\varepsilon|^{p(x)-2} \left(\Delta u_\varepsilon + \frac{p(x)-2}{|Du_\varepsilon|^2} \Delta_\infty u_\varepsilon\right) \varphi\, dx \\
&\geq E(\varepsilon) \int_{\{Du_\varepsilon \neq 0\}} |Du_\varepsilon|^{\max(p(x)-2,0)} \varphi\, dx,
\end{aligned} \tag{5.14}$$

and thus we arrive at the desired inequality. Our use of Fatou's lemma is justified since if $Du_\varepsilon \neq 0$ and $p(x) \leq 2$, we have by (5.13)

$$
\begin{aligned}
&\left(|Du_\varepsilon|^2+\delta\right)^{\frac{p(x)-2}{2}}\left(\Delta u_\varepsilon+\frac{p(x)-2}{|Du_\varepsilon|^2+\delta}\Delta_\infty u_\varepsilon\right)\\
&=\frac{\left(|Du_\varepsilon|^2+\delta\right)^{\frac{p(x)-2}{2}}}{|Du_\varepsilon|^2+\delta}\left(|Du_\varepsilon|^2\left(\Delta u_\varepsilon+\frac{p(x)-2}{|Du_\varepsilon|^2}\Delta_\infty u_\varepsilon\right)+\delta\Delta u_\varepsilon\right)\\
&\le\frac{\left(|Du_\varepsilon|^2+\delta\right)^{\frac{p(x)-2}{2}}}{|Du_\varepsilon|^2+\delta}\frac{q-1}{\varepsilon}\left(|Du_\varepsilon|^{\frac{q-2}{q-1}+2}(N+p(x)-2)+|Du_\varepsilon|^{\frac{q-2}{q-1}}\delta N\right)\\
&\le|Du_\varepsilon|^{p(x)-2+\frac{q-2}{q-1}}\left(\frac{q-1}{\varepsilon}\right)(2N+p(x)-2)\\
&\le\left(\|Du_\varepsilon\|_{L^\infty(\operatorname{supp}\varphi)}+1\right)^{p_{\max}-2+\frac{q-2}{q-1}}\left(\frac{q-1}{\varepsilon}\right)(2N+p_{\max}-2),
\end{aligned}
$$

where the last inequality follows from $p_{\min}-2+\frac{q-2}{q-1}\ge 0$. If $Du_\varepsilon\neq 0$ and $p(x)>2$, we have

$$
\begin{aligned}
&\left(|Du_\varepsilon|^2+\delta\right)^{\frac{p(x)-2}{2}}\left(\Delta u_\varepsilon+\frac{p(x)-2}{|Du_\varepsilon|^2+\delta}\Delta_\infty u_\varepsilon\right)\\
&\le\left(\|Du_\varepsilon\|^2_{L^\infty(\operatorname{supp}\varphi)}+1\right)^{\frac{p_{\max}-2}{2}+\frac{q-2}{q-1}}\left(\frac{q-1}{\varepsilon}\right)(N+p_{\max}-2).
\end{aligned}
$$

In the next two lemmas we use Caccioppoli type estimates and algebraic inequalities to show that the sequence of inf-convolutions converges to the viscosity supersolution in $W^{1,p(\cdot)}_{loc}(\Omega)$.

Lemma 5.6 *Under the assumptions of Lemma 5.5, the function u belongs in $W^{1,p(\cdot)}_{loc}(\Omega)$ and for any $\Omega'\Subset\Omega$ we have $Du_\varepsilon\to Du$ weakly in $L^{p(\cdot)}(\Omega')$ for some subsequence.*

Proof Take a cut-off function $\xi\in C_0^\infty(\Omega')$ such that $0\le\xi\le 1$ in Ω and $\xi\equiv 1$ in Ω'. Then assume that ε is so small that $\operatorname{supp}\xi=:K\subset\Omega_{r(\varepsilon)}$. We define a test function $\varphi:=(L-u_\varepsilon)\xi^{p_{\max}}$ where $L:=\sup_{\varepsilon,x\in\Omega'}|u_\varepsilon(x)|$ is finite since $u_\varepsilon\to u$ locally uniformly. We have

$$
D\varphi=-Du_\varepsilon\,\xi^{p_{\max}}+(L-u_\varepsilon)p^+\xi^{p_{\max}-1}D\xi
$$

and therefore by Lemma 5.5

$$
\begin{aligned}
\int_{\Omega_{r(\varepsilon)}}|Du_\varepsilon|^{p(x)}\,\xi^{p_{\max}}\,dx\le&\int_{\Omega_{r(\varepsilon)}}|Du_\varepsilon|^{p(x)-1}\,\xi^{p_{\max}-1}\,(L-u_\varepsilon)\,p_{\max}\,|D\xi|\,dx\\
&+\int_{\Omega_{r(\varepsilon)}}|Du_\varepsilon|^{p(x)-1}\,|\log|Du_\varepsilon||\,|Dp|\,(L-u_\varepsilon)\,\xi^{p_{\max}}\,dx\\
&+|E(\varepsilon)|\int_{\Omega_{r(\varepsilon)}}|Du_\varepsilon|^{\max(p(x)-2,0)}\,(L-u_\varepsilon)\,\xi^{p_{\max}}\,dx\\
=:&I_1+I_2+I_3.
\end{aligned}
$$

We estimate these integrals using Young's inequality. The first integral is estimated by the facts $\frac{p(x)(p_{\max}-1)}{p(x)-1}\ge p_{\max}$ and $\xi\le 1$ as follows

$$
I_1\le\int_{\Omega_{r(\varepsilon)}}\delta\,|Du_\varepsilon|^{p(x)}\,\xi^{\frac{p(x)(p_{\max}-1)}{p(x)-1}}+\left(\frac{2}{\delta}Lp_{\max}\,|D\xi|\right)^{p(x)}\,dx
$$

$$\leq \delta \int_{\Omega_{r(\varepsilon)}} |Du_\varepsilon|^{p(x)}\, \xi^{p_{\max}}\, dx + C(\delta, p, L, D\xi).$$

To estimate I_2, we also use the inequality $a^s\,|\log a| \leq a^{s+\frac{1}{2}} + \frac{1}{s}$ for $a > 0$ and $s > 0$,

$$\begin{aligned} I_2 &\leq \int_{\Omega_{r(\varepsilon)}} \left(|Du_\varepsilon|^{p(x)-\frac{1}{2}} + \frac{1}{p(x)-1}\right) \xi^{p_{\max}}\, |Dp|\, 2L\, dx \\ &\leq \int_{\Omega_{r(\varepsilon)}} \delta\, |Du_\varepsilon|^{p(x)}\, \xi^{\frac{p_{\max} p(x)}{p(x)-\frac{1}{2}}} + \left(\frac{2}{\delta}\,|Dp|\,L\right)^{2p(x)} + \frac{2L\,|Dp|\,\xi^{p_{\max}}}{p_{\min}-1}\, dx \\ &\leq \delta \int_{\Omega_{r(\varepsilon)}} |Du_\varepsilon|^{p(x)}\, \xi^{p_{\max}}\, dx + C(\delta, p, Dp, L). \end{aligned}$$

The last integral is estimated by the two alternatives in $\max(p(x)-2, 0)$ as follows (we may assume that $|E(\varepsilon)| \leq 1$)

$$\begin{aligned} I_3 &\leq \int_{\Omega_{r(\varepsilon)} \cap \{p(x)>2\}} |Du_\varepsilon|^{p(x)-2}\, \xi^{p_{\max}} 2L\, dx + \int_{\Omega_{r(\varepsilon)} \cap \{p(x)\leq 2\}} 2L \xi^{p_{\max}}\, dx \\ &\leq \int_{\Omega_{r(\varepsilon)} \cap \{p(x)>2\}} \delta\, |Du_\varepsilon|^{p(x)}\, \xi^{\frac{p_{\max} p(x)}{p(x)-2}} + \left(\frac{2}{\delta} L\right)^{\frac{p(x)}{2}} dx + C(p, L) \\ &\leq \delta \int_{\Omega_{r(\varepsilon)}} |Du_\varepsilon|^{p(x)}\, \xi^{p_{\max}}\, dx + C(\delta, p, L). \end{aligned}$$

Taking small δ we conclude that Du_ε is bounded in $L^{p(\cdot)}(\Omega')$ with respect to ε. Since $L^{p(\cdot)}(\Omega')$ is a reflexive Banach space [7, p76 and p89], it follows that there is a function $Du \in L^{p(\cdot)}(\Omega')$ such that $Du_\varepsilon \to Du$ weakly in $L^{p(\cdot)}(\Omega')$ for some subsequence. Consequently $u \in W^{1,p(\cdot)}(\Omega')$ with Du as its weak derivative. □

Lemma 5.7 *Under the assumptions of Lemma 5.5, for any $\Omega' \Subset \Omega$ we have $Du_\varepsilon \to Du$ in $L^{p(\cdot)}(\Omega')$ for some subsequence.*

Proof Take a cut-off function $\xi \in C_0^\infty(\Omega)$ such that $\xi \equiv 1$ in Ω' and define a test function $\varphi := (u - u_\varepsilon)\xi$. Then assume that ε is so small that $\operatorname{supp} \xi =: K \subset \Omega_{r(\varepsilon)}$. Since $\varphi \in W^{1,p(\cdot)}(\Omega_{r(\varepsilon)})$ with compact support it follows from Lemma 5.5 that

$$\begin{aligned} &\int_{\Omega_{r(\varepsilon)}} \left(|Du|^{p(x)-2}\, Du - |Du_\varepsilon|^{p(x)-2}\, Du_\varepsilon\right) \cdot (Du - Du_\varepsilon)\, \xi\, dx \\ &\quad\leq \int_{\Omega_{r(\varepsilon)}} |Du_\varepsilon|^{p(x)-2}\, Du_\varepsilon \cdot D\xi\, (u - u_\varepsilon)\, dx \\ &\qquad + \int_{\Omega_{r(\varepsilon)}} |Du_\varepsilon|^{p(x)-2} \log(|Du_\varepsilon|)\, Du_\varepsilon \cdot Dp\, (u - u_\varepsilon)\xi\, dx \\ &\qquad + |E(\varepsilon)| \int_{\Omega_{r(\varepsilon)}} |Du_\varepsilon|^{\max(p(x)-2,0)}\, (u - u_\varepsilon)\xi\, dx \\ &\qquad + \int_{\Omega_{r(\varepsilon)}} |Du|^{p(x)-2}\, Du \cdot (Du - Du_\varepsilon)\, \xi\, dx \\ &\quad\leq \|u - u_\varepsilon\|_{L^\infty(K)} \int_K \left(C(p_{\min}) + |Du_\varepsilon|^{p(x)}\right) (D\xi + |Dp| + |E(\varepsilon)|)\, dx \\ &\qquad + \int_K |Du|^{p(x)-2}\, Du \cdot (Du - Du_\varepsilon)\, \xi\, dx. \end{aligned} \tag{5.15}$$

According to Lemma 5.6 we have $u_\varepsilon \to u$ locally uniformly and $Du_\varepsilon \to Du$ weakly in $L^{p(\cdot)}(K)$ for a subsequence. Thus by passing to a subsequence we may assume that the right hand side of (5.15) converges to zero. The claim now follows from the inequalities (see e.g. [19, Chapter 12])

$$\begin{aligned}&\left(|a|^{p(x)-2}a-|b|^{p(x)-2}b\right)\cdot(a-b)\\&\quad\geq\begin{cases}(p(x)-1)\,|a-b|^2\left(1+|a|^2+|b|^2\right)^{\frac{p(x)-2}{2}} & p(x)<2\\ 2^{2-p(x)}\,|a-b|^{p(x)} & p(x)\geq 2\end{cases}\end{aligned}$$

for $a,b\in\mathbb{R}^N$. Indeed, we immediately get that $\int_{\Omega'\cap\{p(x)\geq 2\}}|Du-Du_\varepsilon|^{p(x)}\,dx\to 0$. To deal with the set $\{p(x)<2\}$, we first apply the above algebraic inequality and then estimate using Hölder's inequality, the modular inequality (2.1) and the definition of the $\|\cdot\|_{L^{p(\cdot)}}$-norm. We get

$$\begin{aligned}&\int_{\Omega'\cap\{p(x)<2\}}|Du-Du_\varepsilon|^{p(x)}\,dx\\&\leq\int_{\Omega'\cap\{p(x)<2\}}\left(\left(|Du|^{p(x)-2}Du-|Du_\varepsilon|^{p(x)-2}Du_\varepsilon\right)\cdot(Du-Du_\varepsilon)\right)^{\frac{p(x)}{2}}\\&\quad\cdot\left(\frac{1}{p(x)-1}\right)^{\frac{p(x)}{2}}\left(1+|Du|^2+|Du_\varepsilon|^2\right)^{\frac{p(x)(2-p(x))}{4}}\,dx\\&\leq\left\|\left(\left(|Du|^{p(x)-2}Du-|Du_\varepsilon|^{p(x)-2}Du_\varepsilon\right)\cdot(Du-Du_\varepsilon)\right)^{\frac{p(x)}{2}}\right\|_{L^{\frac{2}{p(\cdot)}}(\Omega'\cap\{p(x)<2\})}\\&\quad\cdot\frac{2}{p_{\min}-1}\left\|\left(1+|Du|^2+|Du_\varepsilon|^2\right)^{\frac{p(x)(2-p(x))}{4}}\right\|_{L^{\frac{2}{2-p(\cdot)}}(\Omega'\cap\{p(x)<2\})}\\&\leq\left(\int_{\Omega_{r(\varepsilon)}}\left(|Du|^{p(x)-2}Du-|Du_\varepsilon|^{p(x)-2}Du_\varepsilon\right)\cdot(Du-Du_\varepsilon)\,\xi\,dx\right)^s\\&\quad\cdot\frac{2}{p_{\min}-1}\left(1+\int_{\Omega'\cap\{p(x)<2\}}\left(1+|Du|^2+|Du_\varepsilon|^2\right)^{\frac{p(x)}{2}}\,dx\right),\end{aligned}$$

where $s\in\left\{\frac{p_{\max}}{2},\frac{p_{\min}}{2}\right\}$. The last integral is bounded since the sequence Du_ε is bounded in $L^{p(\cdot)}(\Omega')$ by its weak convergence. The RHS therefore converges to zero by (5.15). □

Next, we use the previous convergence result to pass to the limit in the inequality of Lemma 5.5 and conclude that viscosity supersolutions to $-\Delta^N_{p(x)}u\geq 0$ are weak supersolutions to $-\Delta^S_{p(x)}u\geq 0$.

Theorem 5.8 *If $u\in C(\Omega)$ is a viscosity supersolution to $-\Delta^N_{p(x)}u\geq 0$ in Ω, then u is a weak supersolution to $-\Delta^S_{p(x)}u\geq 0$ in Ω.*

Proof It is clear from the definition of weak supersolutions to $-\Delta^S_{p(x)}u\geq 0$ that we can without loss of generality assume that u is uniformly continuous in Ω by restricting to a smaller domain. Fix a non-negative test function $\varphi\in C_0^\infty(\Omega)$ and take an open $\Omega'\Subset\Omega$ such that $\operatorname{supp}\varphi\subset\Omega'$. Let q and u_ε be as in Lemma 5.5 and assume that ε is so small that $\Omega'\subset\Omega_{r(\varepsilon)}$. Then the claim follows from Lemma 5.5 if we show that

$$\lim_{\varepsilon\to 0}\int_{\Omega'}|Du_\varepsilon|^{p(x)-2}Du_\varepsilon\cdot D\varphi\,dx=\int_{\Omega'}|Du|^{p(x)-2}Du\cdot D\varphi\,dx \tag{5.16}$$

and

$$\lim_{\varepsilon\to 0}\int_{\Omega'} |Du_\varepsilon|^{p(x)-2}\log\left(|Du_\varepsilon|\right) Du_\varepsilon\cdot Dp\,\varphi\,dx$$
$$=\int_{\Omega'} |Du|^{p(x)-2}\log\left(|Du|\right) Du\cdot Dp\,\varphi\,dx \tag{5.17}$$

as well as

$$\lim_{\varepsilon\to 0} E(\varepsilon)\int_{\Omega'} |Du_\varepsilon|^{\max(p(x)-2,0)}\,\varphi\,dx=0. \tag{5.18}$$

By Lemma 5.7 we have that $u_\varepsilon\to u$ in $W^{1,p(\cdot)}(\Omega')$.
Claim Equation (5.16) follows from the inequalities (see e.g. [19, Chapter 12])

$$\left||a|^{p(x)-2}a-|b|^{p(x)-2}b\right|\le\begin{cases}2^{2-p(x)}\,|a-b|^{p(x)-1} & p(x)<2\\ 2^{-1}\left(|a|^{p(x)-2}+|b|^{p(x)-2}\right)|a-b| & p(x)\ge 2\end{cases} \tag{5.19}$$

for $a,b\in\mathbb{R}^N$. Indeed, when ε is so small that $\int_{\Omega'}|Du_\varepsilon-Du|^{p(x)}\,dx<1$ we have by Hölder's inequality and the modular inequality

$$\begin{aligned}&\int_{\Omega'}\left||Du_\varepsilon|^{p(x)-2}Du_\varepsilon-|Du|^{p(x)-2}Du\right|\,dx\\
&\le 2\int_{\Omega'\cap\{p(x)<2\}}|Du_\varepsilon-Du|^{p(x)-1}\,dx\\
&\quad+2^{-1}\int_{\Omega'\cap\{p(x)\ge 2\}}\left(|Du_\varepsilon|^{p(x)-2}+|Du|^{p(x)-2}\right)|Du_\varepsilon-Du|\,dx\\
&\le C(p,\Omega)\left(\int_{\Omega'}|Du_\varepsilon-Du|^{p(x)}\,dx\right)^{\frac{1}{p_{\max}}}\\
&\quad+C(p,\Omega)\left(1+\int_{\Omega'}|Du_\varepsilon|^{p(x)}+|Du|^{p(x)}\,dx\right)\|Du_\varepsilon-Du\|_{L^{p(\cdot)}(\Omega')}.\end{aligned}$$

Claim Equation (5.18) holds since $\int_{\Omega'}|Du_\varepsilon|^{p(x)}\,dx$ is bounded and $E(\varepsilon)\to 0$.
Claim Equation (5.17) follows if we show that

$$\lim_{\varepsilon\to 0}\int_{\Omega'}\left||Du_\varepsilon|^{p(x)-2}\log\left(|Du_\varepsilon|\right)Du_\varepsilon-|Du|^{p(x)-2}\log\left(|Du|\right)Du\right|\,dx=0. \tag{5.20}$$

To this end, fix $0<\epsilon<1$. The mapping $(a,x)\mapsto|a|^{p(x)-2}\log\left(|a|\right)a$ is uniformly continuous in bounded sets of $\mathbb{R}^N\times\Omega'$. Hence there exists $\delta=\delta(\epsilon)<\epsilon$ such that whenever $x\in\Omega'$ and $a,b\in\overline{B}(0,3)$ satisfy $|a-b|<\delta$, it holds

$$\left||a|^{p(x)-2}\log\left(|a|\right)a-|b|^{p(x)-2}\log\left(|b|\right)b\right|\le\epsilon. \tag{5.21}$$

If $|a|,|b|\ge 1$ and $|a-b|<\delta$, then we use (5.19) to get the estimate

$$\begin{aligned}&\left|\,|a|^{p(x)-2}\log\left(|a|\right)a-|b|^{p(x)-2}\log\left(|b|\right)b\right|\\
&\le|b|^{p(x)-1}\left|\log|a|-\log|b|\right|+\left|\log|a|\right|\left||a|^{p(x)-2}a-|b|^{p(x)-2}b\right|\\
&\le|b|^{p(x)}|a-b|+|a|\cdot\begin{cases}2^{2-p(x)}\,|a-b|^{p(x)-1}, & p(x)<2\\ 2^{-1}\left(|a|^{p(x)-2}+|b|^{p(x)-2}\right)|a-b|, & p(x)\ge 2\end{cases}\end{aligned}$$

$$
\begin{aligned}
&\leq (1+2^{-1})\left(|a|^{p(x)}+|b|^{p(x)}\right)|a-b|+2\,|a|\,|a-b|^{p(x)-1}\\
&\leq C\left(|a|^{p(x)}+|b|^{p(x)}\right)\epsilon^{\min(p_{\min}-1,1)}. \qquad (5.22)
\end{aligned}
$$

We denote

$$
F_\varepsilon=\left\{x\in\Omega' : |Du_\varepsilon(x)-Du(x)|\geq\delta\right\}.
$$

The strong convergence of Du_ε to Du in $L^{p(\cdot)}(\Omega')$ implies that $Du_\varepsilon \to Du$ in measure in Ω' (see [7, Lemma 3.2.10]). Thus there is $\varepsilon_0=\varepsilon_0(\delta)$ such that for all $\varepsilon<\varepsilon_0$ it holds $|F_\varepsilon|\leq\delta$. Using the inequality $a^s\,|\log a|\leq a^{s+\frac{1}{2}}+\frac{1}{s}$ for $a, s>0$, we get for all $\varepsilon<\varepsilon_0$

$$
\begin{aligned}
&\int_{F_\varepsilon}\left||Du_\varepsilon|^{p(x)-2}\log(|Du_\varepsilon|)\,Du_\varepsilon-|Du|^{p(x)-2}\log(|Du|)\,Du\right|dx\\
&\leq\int_{F_\varepsilon}\frac{2}{p(x)-1}+|Du_\varepsilon|^{p(x)-\frac{1}{2}}+|Du|^{p(x)-\frac{1}{2}}\,dx\\
&\leq C(p_{\min})\,|F_\varepsilon|+\|1\|_{L^{2p(\cdot)}(F_\varepsilon)}\left(\|Du_\varepsilon\|_{L^{\frac{p(\cdot)}{p(\cdot)-\frac{1}{2}}}(F_\varepsilon)}+\|Du\|_{L^{\frac{p(\cdot)}{p(\cdot)-\frac{1}{2}}}(F_\varepsilon)}\right)\\
&\leq C(p_{\min})\,|F_\varepsilon|+|F_\varepsilon|^{\frac{1}{2p_{\max}}}\left(2+\int_{F_\varepsilon}|Du_\varepsilon|^{p(x)}+|Du|^{p(x)}\,dx\right)\\
&\leq C(p_{\min})\left(1+\int_{\Omega'}|Du_\varepsilon|^{p(x)}+|Du|^{p(x)}\,dx\right)\epsilon^{\frac{1}{2p_{\max}}}. \qquad (5.23)
\end{aligned}
$$

If $x\in\Omega'\setminus F_\varepsilon$, then either $|Du_\varepsilon|, |Du|\leq 3$ or $|Du_\varepsilon|, |Du|\geq 1$. Hence by (5.21) and (5.22) we have

$$
\begin{aligned}
&\int_{\Omega'\setminus F_\varepsilon}\left||Du_\varepsilon|^{p(x)-2}\log(|Du_\varepsilon|)\,Du_\varepsilon-|Du|^{p(x)-2}\log(|Du|)\,Du\right|dx\\
&\qquad\leq C\left(\int_{\Omega'}|Du_\varepsilon|^{p(x)}+|Du|^{p(x)}+1\,dx\right)\epsilon^{\min(p_{\min}-1,1)}. \qquad (5.24)
\end{aligned}
$$

Combining (5.24) and (5.23) proves (5.20) since ϵ was arbitrary. □

Merging Theorems 4.1 and 5.8 yields the following equivalence result.

Theorem 5.9 *A function u is a viscosity solution to $-\Delta^N_{p(x)}u=0$ in Ω if and only if it is a weak solution to $-\Delta^S_{p(x)}u=0$ in Ω.*

Since the weak solutions to the strong $p(x)$-Laplace equation are locally $C^{1,\alpha}$ continuous [25], our equivalence result yields local $C^{1,\alpha}$ regularity also for viscosity solutions of the normalized $p(x)$-Laplace equation.

Corollary 5.10 *If u is a viscosity solution to $-\Delta^N_{p(x)}u=0$ in a bounded domain Ω, then $u\in C^{1,\alpha}(\Omega)$ with $\alpha\in(0,1)$.*

6 An application: a Radó-type removability theorem

The classical theorem of Radó says that if a continuous complex-valued function f defined on a domain $\Omega\subset\mathbb{C}$ is holomorphic in $\Omega\setminus\{f=0\}$, then it is holomorphic in the whole Ω. Similar results have been proven for solutions of partial differential equations. We prove a

Radó-type removability theorem for the strong $p(x)$-Laplace equation. It is worth pointing out that it could be difficult to show this kind of result without appealing to viscosity solutions whereas it is straightforward to do so with the help of the equivalence result. The theorem follows by observing that weak solutions to $\Delta^S_{p(x)} u = 0$ coincide with viscosity solutions of an equation that satisfies the assumptions of a Radó-type removability theorem in [12].

Recall that we ignore the test functions whose gradient vanishes at the point of touching in the Definition 3.3 of viscosity solutions to $-\Delta^N_{p(x)} u = 0$. Sometimes this kind of solutions are called *feeble viscosity solutions* (e.g. [12,17]). We will observe that these feeble viscosity solutions to $-\Delta^N_{p(x)} u = 0$ are exactly the usual viscosity solutions to

$$-\operatorname{tr}(A(x, Du)D^2u) = 0, \tag{6.1}$$

where $A(x, Du) := |Du|^2 I + (p(x) - 2)\, Du \otimes Du$. To be precise, we define the viscosity solutions to (6.1).

Definition 6.1 A lower semicontinuous function u is a *viscosity supersolution* to (6.1) in Ω if, whenever $(\eta, X) \in J^{2,-}u(x)$ with $x \in \Omega$, then

$$-\operatorname{tr}(A(x, \eta)X) \geq 0.$$

A function u is a *viscosity subsolution* to (6.1) if $-u$ is a supersolution, and a *viscosity solution* if it is both viscosity super- and subsolution.

Lemma 6.2 *A function u is a viscosity solution to* $-\Delta^N_{p(x)} u = 0$ *if and only if it is a viscosity solution to* (6.1).

Proof It is enough to consider supersolutions. Take $(\eta, X) \in J^{2,-}u(x)$ with $x \in \Omega$. If $\eta = 0$, then the conditions for both definitions are satisfied, so we may assume that $\eta \neq 0$. Then we have

$$F(x, \eta, X) \geq 0$$

if and only if

$$-\big(|\eta|^2 \operatorname{tr}(X) + (p(x) - 2)\, \langle X\eta, \eta\rangle \big) \geq 0,$$

where

$$\begin{aligned} |\eta|^2 \operatorname{tr}(X) + (p(x) - 2)\, \langle X\eta, \eta\rangle &= |\eta|^2 \operatorname{tr}(X) + (p(x) - 2) \operatorname{tr}(\eta \otimes \eta X) \\ &= \operatorname{tr}\big(\big(|\eta|^2 I + (p(x) - 2)\, \eta \otimes \eta\big) X\big). \end{aligned}$$

Hence the definitions are equivalent. □

Theorem 6.3 (A Radó-type removability theorem) *Let* $u \in C^1(\Omega)$ *be a weak solution to* $-\Delta^S_{p(x)} u = 0$ *in* $\Omega \setminus \{u = 0\}$. *Then u is a weak solution to* $-\Delta^S_{p(x)} u = 0$ *in the whole* Ω.

Proof By Lemma 6.2 and our equivalence result weak solutions to $-\Delta^S_{p(x)} u = 0$ coincide with viscosity solutions to (6.1). Therefore it suffices to show that if u is a viscosity solution to (6.1) in $\Omega \setminus \{u = 0\}$, it is a viscosity solution to (6.1) in the whole Ω. This on the other hand follows from [12, Theorem 2.2]. The matrix A satisfies the assumptions of the theorem as it is symmetric, has continuous entries and $A(x, 0, 0) = 0$ for all $x \in \Omega$. It is also positive semi-definite since for all $\xi \in \mathbb{R}^N$ we have

$$\begin{aligned} \xi' \left(|\eta|^2 I + (p(x) - 2)\, \eta \otimes \eta\right) \xi &\geq \xi' \left(|\eta|^2 I - \eta \otimes \eta\right) \xi \\ &\geq |\xi|^2 \left(|\eta|^2 - \|\eta \otimes \eta\|\right) = 0. \end{aligned}$$

□

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