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Calculus of Variations

Uniqueness and symmetry of ground states for higher-order equations

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Abstract We establish uniqueness and radial symmetry of ground states for higher-order nonlinear Schrödinger and Hartree equations whose higher-order differentials have small coefficients. As an application, we obtain error estimates for higher-order approximations to the pseudo-relativistic ground state. Our proof adapts the strategy of Lenzmann (Anal PDE 2:1–27, 2009) using local uniqueness near the limit of ground states in a variational problem. However, in order to bypass difficulties from lack of symmetrization tools for higher-order differential operators, we employ the contraction mapping argument in our earlier work (Choi et al. 2017. arXiv:1705.09068) to construct radially symmetric real-valued solutions, as well as improving local uniqueness near the limit.

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1 Introduction

Higher-order elliptic equations, whose higher-order differentials have small coefficients, arise in various physical contexts. For instance, in nonlinear optics, the envelope dynamics of wave trains in a weakly nonlinear medium is given by the equation

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$$i\epsilon^2\partial_t\psi = \omega(\epsilon\partial)\psi - \epsilon^2|\psi|^2\psi,$$

where $\varepsilon > 0$, $\psi = \psi(t,x) : \mathbb{R}\times\mathbb{R}^d \to \mathbb{C}$ and $\omega(\partial)$ denotes the Fourier multiplier operator with a symbol $a = a(\xi) : \mathbb{R}^d \to \mathbb{R}$. Looking for a stationary solution, inserting the ansatz $\psi(t,x) = e^{i\mu t}u(x)$ with $\mu > 0$, we obtain the time-independent equation

$$\omega(\epsilon\partial)u + \epsilon^2\mu u = \epsilon^2|u|^2u.$$

When high-frequency dispersion is negligible and the medium is isotropic,[1] the above equation can be approximated by the second-order equation

$$-\Delta u + \mu u = |u|^2 u$$

(see [15]). However, if high frequency dispersion is weak but not negligible, one should consider a higher-order equation whose differential operator is a Taylor polynomial of $\omega(\epsilon\partial)$. Here, higher-order terms have small coefficients.

In astrophysics, the mean-field limit of stationary boson stars is described by the pseudo-relativistic nonlinear Hartree equation

$$\left(\sqrt{-c^2\Delta + m^2c^4} - mc^2\right)u + \mu u = \left(|x|^{-1} * |u|^2\right) \quad \text{in } \mathbb{R}^3, \tag{1.1}$$

where $u = u(x) : \mathbb{R}^3 \to \mathbb{C}$, $m > 0$ is the particle mass and $c > 0$ stands for the speed of light. In applications, taking the formal Taylor polynomial of the pseudo-relativistic operator

$$\left(\sqrt{-c^2\Delta + m^2c^4} - mc^2\right) = mc^2\left(\sqrt{1 - \frac{\Delta}{m^2c^2}} - 1\right) = \frac{1}{2m}(-\Delta) - \frac{1}{8m^3c^2}(-\Delta)^2 + \cdots, \tag{1.2}$$

the higher-order model

$$\left(\sum_{j=1}^{J}\frac{(-1)^{j-1}\alpha_j}{m^{2j-1}c^{2j-2}}(-\Delta)^j\right)u + \mu u = \left(|x|^{-1} * |u|^2\right)u \quad \text{in } \mathbb{R}^3, \tag{1.3}$$

where $\alpha_j = \frac{(2j-2)!}{j!(j-1)!2^{2j-1}}$, is employed to avoid possible complication from having a non-local operator (see [4,5] and the references therein).

Moreover, given a previously known second-order model

$$-\Delta u + \mu u = f(|u|^2)u,$$

a higher-order equation is sometimes introduced as a refinement taking additional physical effects in account. In this case, it is natural to put small coefficients on higher-order differentials, like

$$-\Delta u + \epsilon\Delta^2 u + \mu u = f(|u|^2)u,$$

for consistency with the second-order model.

The purpose of this paper is to provide a general strategy to prove uniqueness and radial symmetry of ground states for a certain class of higher-order elliptic equations including the above examples.

Before proceeding, it should be pointed out that proving uniqueness and symmetry of ground states for higher-order equations is in general quite challenging. That is because some of useful tools, such as the diamagnetic inequality, the Pólya–Szegö inequality, the

[1] With $\omega(0) = \nabla_{\xi_j}\omega(0) = 0$ and $\partial_{\xi_j}\partial_{\xi_k}\omega(0) = \delta_{jk}$ by a suitable change of variables.

moving plane method and the shooting game argument, might not be available. Recall that for second-order equations, the standard variational approach employs the diamagnetic inequality $\|\nabla(|u|)\|_{L^2} \leq \|\nabla u\|_{L^2}$ in the first step in order to obtain a non-negative ground state from a hypothetical possibly sign-changing ground state, and then symmetrization tools are applied to prove symmetry and uniqueness. When the symbol of a pseudo-differential operator is a Bernstein function, e.g., the pseudo-relativistic operator (1.2), the diamagnetic inequality as well as symmetrization tools can be recovered by a beautiful argument in [6,13] involving the Bernstein's theorem. However, this method does not work for higher-order operators.

In fact, some of analytic tools have been developed for higher-order operators e.g., for polyharmonic operators, and there might be a way to apply them for uniqueness and symmetry. For a comprehensive overview, we refer to the book by Gazzola et al. [7]. Nevertheless, they cannot be directly applied to the above examples. Even worse, the desired diamagnetic inequality does not seem to hold for higher-order differential operators, because even if u is smooth, its second derivative $\nabla_{x_j}^2(|u|)$ could be very singular near the set $\{x : u(x) = 0\}$. In this paper, we go around the lack of the analytic tools rather than making an effort to build them up, by taking the advantage of higher-order differentials having small coefficients.

From now on, for concreteness of exposition, we restrict ourselves to the higher-order nonlinear Schrödinger equation (NLS)

$$P_\epsilon u + u = |u|^{2k} u \quad \text{in } \mathbb{R}^d, \tag{1.4}$$

where $k \in \mathbb{N}$ and $u = u(x) : \mathbb{R}^d \to \mathbb{C}$, and the three-dimensional higher-order nonlinear Hartree equation (NLH)

$$P_\epsilon u + u = \left(|x|^{-1} * |u|^2\right) u \quad \text{in } \mathbb{R}^3, \tag{1.5}$$

where $u = u(x) : \mathbb{R}^3 \to \mathbb{C}$. For an even integer J and $\epsilon \geq 0$ (including zero), the higher-order differential operator P_ϵ is defined by

$$P_\epsilon = P_\epsilon^J =: -\Delta + \sum_{|\alpha|=3}^{J} c_\alpha \epsilon^{|\alpha|-2} (i\nabla)^\alpha,$$

where $\alpha = (\alpha_1, \ldots, \alpha_d) \in (\mathbb{Z}_{\geq 0})^d$ denotes a multi-index and $(i\nabla)^\alpha = i^{|\alpha|} \nabla_{x_1}^{\alpha_1} \ldots \nabla_{x_d}^{\alpha_d}$. We assume that the family of operators $\{P_\epsilon\}_{0 \leq \epsilon \leq 1}$ is uniformly elliptic in the sense that there exists $\gamma > 0$, independent of $\epsilon \geq 0$, such that

$$1 + P_\epsilon \geq \gamma(1 - \Delta). \tag{1.6}$$

For NLS (1.4), we further assume that $1 \leq d \leq 3$ and

$$\begin{cases} k \in \mathbb{N} & \text{if } d = 1, 2, \\ k = 1 & \text{if } d = 3 \end{cases} \tag{1.7}$$

so that the odd-power nonlinearity is H^1-subcritical. We remark that as $\epsilon \to 0$, the higher-order NLS (1.4) formally converges to the standard second-order NLS

$$-\Delta u + u = |u|^{2k} u, \tag{1.8}$$

while the higher-order NLH (1.5) converges to the second-order NLH

$$-\Delta u + u = \left(|x|^{-1} * |u|^2\right) u. \tag{1.9}$$

A solution to the higher-order NLS (1.4) (resp., the higher-order NLH (1.5)) is called a *ground state* if it is a minimizer for the action functional

$$I_\epsilon(u) := \begin{cases} \frac{1}{2}\int_{\mathbb{R}^d}(P_\epsilon+1)u\bar{u}\,dx - \frac{1}{2k+2}\int_{\mathbb{R}^d}|u|^{2k+2}dx & \text{(for NLS (1.4))} \\ \frac{1}{2}\int_{\mathbb{R}^3}(P_\epsilon+1)u\bar{u}\,dx - \frac{1}{4}\int_{\mathbb{R}^3}\left(|x|^{-1}*|u|^2\right)|u|^2dx & \text{(for NLH (1.5)).} \end{cases}$$

restricted to the constraint

$$\langle I'_\epsilon(u), u\rangle_{L^2} = 0 \quad \text{and} \quad u \neq 0,$$

where I'_ϵ is the Frechét derivative of I_ϵ. When $\epsilon = 0$, it is known that the second-order NLS (1.8) (resp., the second-order NLH (1.9)) has a smooth radially symmetric positive ground state, denoted by Q_0, and it is unique up to translation and phase shift[2] (see [1,8,10] for NLS, and [12] for NLH). Moreover, the ground state Q_0 is non-degenerate (see [10] for NLS and [11] for NLH). Indeed, linearizing the equation near the ground state Q_0, we obtain the linearized operator $\mathcal{L} = \begin{pmatrix} \mathcal{L}_0^+ & 0 \\ 0 & \mathcal{L}_0^- \end{pmatrix}$ with the identification $a + bi \leftrightarrow \binom{a}{b}$, where the linear operators $\mathcal{L}_0^\pm : H^2(\mathbb{R}^d;\mathbb{R}) \to L^2(\mathbb{R}^d;\mathbb{R})$ are defined by

$$\mathcal{L}_0^+ h := \begin{cases} -\Delta h + h - (2k+1)Q_0^{2k}h & \text{(for NLS (1.8))} \\ -\Delta h + h - 2\left(|x|^{-1}*(Q_0 h)\right)Q_0 - \left(|x|^{-1}*Q_0^2\right)h & \text{(for NLH (1.9))} \end{cases}$$

and

$$\mathcal{L}_0^- h := \begin{cases} -\Delta h + h - Q_0^{2k}h & \text{(for NLS (1.8))} \\ -\Delta h + h - \left(|x|^{-1}*Q_0^2\right)h & \text{(for NLH (1.9)),} \end{cases}$$

By *non-degeneracy*, we mean that the kernels of $\mathcal{L}_0^\pm$ are explicitly given by

$$\begin{cases} \mathrm{Ker}\mathcal{L}_0^+ = \mathrm{span}\left\{\partial_{x_1}Q_0, \ldots, \partial_{x_d}Q_0\right\}, \\ \mathrm{Ker}\mathcal{L}_0^- = \mathrm{span}\left\{Q_0\right\}. \end{cases}$$

When $\epsilon > 0$, by standard variational arguments, one can show that the higher-order NLS (1.4) (resp., the higher-order NLH (1.5)) possesses a ground state Q_ϵ^J and that it converges to Q_0 as $\epsilon \to 0$ (see Proposition 2.1).

Our main theorem establishes uniqueness and radial symmetry of ground states for the higher-order equations (1.4) and (1.5). We recall we only care the case $1 \le d \le 3$ and

$$\begin{cases} k \in \mathbb{N} & \text{if } d = 1, 2, \\ k = 1 & \text{if } d = 3 \end{cases}$$

for NLS, and $d = 3$ for NLH.

Theorem 1.1 (Uniqueness and symmetry) *Suppose that* (1.6)*, as well as* (1.7) *for NLS and $d = 3$ for NLH), holds. Then, there exists $\epsilon_0 > 0$ such that for each $0 < \epsilon \le \epsilon_0$, there exists a smooth radially symmetric and real-valued ground state Q_ϵ^J for the higher-order NLS* (1.4) *(resp., the higher-order NLH* (1.5)*), and it is unique up to translation and phase shift. Moreover, the ground state Q_ϵ^J is non-degenerate in the sense of Proposition* 3.2 *below.*

[2] We say that Q is a unique solution up to translation and phase shift if for any solution u, there exist $x_0 \in \mathbb{R}^d$ and $\theta \in \mathbb{R}$ such that $u(x) = e^{i\theta}Q(x - x_0)$.

Remark 1.2 An alternative concept of ground states to the higher order equations (1.4) or (1.5) can be given by a minimizer of the physical energy $E_\epsilon(u)$ given by $I_\epsilon(u) - \frac{1}{2}\|u\|_{L^2}^2$ subject to the fixed mass $\|u\|_{L^2}^2 = N$, provided the minimizer exists. In second order case $\epsilon = 0$, a simple scaling argument says that this concept of ground states to (1.4) or (1.5) coincides with our one in Theorem 1.1. It is worth mentioning here that it is not known yet whether this also happens or not for the higher order case $\epsilon > 0$.

For the proof, we follow the roadmap in the important work by Lenzmann [11], where uniqueness of ground states for the pseudo-relativistic NLH (1.1) is established. The robust approach of Lenzmann [11] can be summarized in two steps.

- Step 1 Construct a ground state for the pseudo-relativistic NLH (1.1), and prove its convergence to the ground state Q_0 for the second-order NLH (1.9) as $c \to \infty$ up to translation and phase shift. Here, by construction (involving variational techniques), a pseudo-relativistic ground state must be positive and radially symmetric.
- Step 2 Prove uniqueness of a radially symmetric real-valued solution to the pseudo-relativistic NLH near the ground state Q_0. The proof of this local uniqueness relies on the non-degeneracy of the ground state Q_0, which is indeed one of the main contributions of Lenzmann [11].
- Conclusion If $c \geq 1$ is large enough, then a pseudo-relativistic ground state is close to Q_0, so it is unique up to translation and phase shift.

As for higher-order equations, however, we cannot make use of radial symmetry of a ground state for the proof of uniqueness, but we have to prove it instead, since we do not have symmetrization tools at hand. In order to overcome these obstacles, we employ several new ingredients, including the contraction mapping argument in our earlier work [3] and the improved local uniqueness near the ground state Q_0. Our proof can be summarized as follows.

- Step 1 Construct a ground state Q_ϵ^J for the higher-order equation (1.4) (resp., (1.5)), and prove its convergence to the ground state Q_0 for the second-order equation (1.8) (resp., (1.9)) as $\epsilon \to 0$ up to translation and phase shift. We remark that contrary to Step 1 in [11], due to lack of symmetrization tools, it is not known that Q_ϵ^J is radially symmetric and real-valued.
- Step 2 Construct a radially symmetric real-valued solution u_ϵ for the higher-order equation converging to the ground state Q_0 by the contraction mapping argument. By construction, the solution u_ϵ does not have any variational character as a ground state.
- Step 3 Prove uniqueness (up to translation and phase shift) for the higher-order equation near the ground state Q_0 without assuming that solutions are radially symmetric or real-valued.
- Conclusion If $\epsilon > 0$ is small enough, then two solutions Q_ϵ^J and u_ϵ are close to the ground state Q_0. Thus, identifying them by uniqueness, we conclude that Q_ϵ^J is a unique radially symmetric real-valued ground state.

For the proof of local uniqueness in Step 2, we assume that there is a solution $\tilde{u}_\epsilon$, and then modify it by translation and phase shift to be perpendicular to the kernel of the linearized operator around u_ϵ. Then, we prove that the modified $\tilde{u}_\epsilon$ is indeed u_ϵ itself. This argument, choosing the best modulation parameters, seems quite natural in the context of orbital stability [16]. However, to the best of authors' knowledge, such a local uniqueness and its proof seem new.

Next, we consider the pseudo-relativistic NLH (1.1) and the higher-order NLH (1.3) in the non-relativistic regime ($c \gg 1$). In this case, it is shown in Lenzmann [11] that the pseudo-relativistic NLH (1.1) has a radially symmetric positive ground state Q_c, which is unique up to translation and phase shift. On the other hand, by the main theorem of this paper, the higher-order NLH (1.3) also has a radially symmetric and real-valued ground state Q_c^J, and it is unique up to translation and phase shift. Then, the contraction mapping argument in [3] can be applied to compare two ground states. As a consequence, we obtain the following error estimates for the higher-order approximations to the pseudo-relativistic ground state.

Theorem 1.3 (Higher-order approximations to a pseudo-relativistic ground state) *Let J be an odd number, and $c > 0$ be a sufficiently large number. We denote by Q_c (resp., Q_c^J) the unique radially symmetric, real-valued ground state for the pseudo-relativistic NLH* (1.1) *(resp., the higher-order NLH* (1.3)*). Then,*

$$\|Q_c^J - Q_c\|_{H^1} \lesssim \frac{1}{c^{2J}}.$$

Remark 1.4 The higher-order Schrödinger operator in (1.3) is introduced as a higher-order approximation to the pseudo-relativistic operator $\sqrt{-c^2\Delta + m^2c^4} - mc^2$, provided that high-frequencies are not dominant. In [4,5], the error estimates for the higher-order approximation to the linear evolution has been discussed. Theorem 1.3 first provides a precise error estimate for the higher-order approximation to the pseudo-relativistic ground state, which the simplest nonlinear object.

1.1 Notations

We denote the potential energy functional by

$$\mathcal{N}(u) := \begin{cases} \dfrac{1}{2k+2}\displaystyle\int_{\mathbb{R}^d} |u|^{2k+2} dx & \text{(for NLS)} \\ \dfrac{1}{4}\displaystyle\int_{\mathbb{R}^3} \left(|x|^{-1} * |u|^2\right)|u|^2 dx & \text{(for NLH).} \end{cases}$$

Then, the nonlinearity of the equation is given as its Frechét derivative

$$\mathcal{N}'(u) := \begin{cases} |u|^{2k}u & \text{(for NLS)} \\ \left(|x|^{-1} * |u|^2\right)u & \text{(for NLH).} \end{cases}$$

We denote by $H^1_{P_\epsilon} = H^1_{P_\epsilon}(\mathbb{R}^3; \mathbb{C})$ the Hilbert space equipped with the inner product

$$\langle f, g\rangle_{H^1_{P_\epsilon}} := \int_{\mathbb{R}^3} (P_\epsilon + 1) f(x)\overline{g(x)} dx,$$

1.2 Organization of the paper

In Sect. 2, we prove existence of ground states Q_ϵ^J's for the higher-order NLS (resp., the higher-order NLH) and their convergence to the ground state Q_0 for the second-order equation. In Sect. 3, we provide the non-degeneracy estimates, which are the key analytic tools in this paper. Using them, in Section 4, we construct radially symmetric real-valued solutions u_ϵ's converging to the ground state Q_0 for the second-order equation. In Sect. 5, we establish local uniqueness for higher-order equations near the ground state Q_0, and then identifying Q_ϵ^J and u_ϵ, we prove the main theorem (Theorem 1.1). Finally, in Sect. 6, we prove the error estimates for the higher-order approximation to the pseudo-relativistic ground state (Theorem 1.3).

2 Construction of ground states, and their limit

By the standard variational method, we construct ground states for higher-order equations (Proposition 2.1), and show their convergence to the ground state for the second-order equation (Proposition 2.3). In addition, we prove that in general, convergence to the ground state for the second-order equation in a low regularity norm can be upgraded to that in high regularity norms (Proposition 2.5).

Proposition 2.1 *(Existence of a ground state) Suppose that* (1.6) *(as well as* (1.7) *for NLS) holds. Then, for any $\epsilon > 0$, the higher-order equation* (1.4) *(resp.,* (1.5)*) possesses a ground state $Q_\epsilon^J \in H^1_{P_\epsilon}$.*

Throughout this section, we denote the order of nonlinearity by

$$p = \begin{cases} 2k+1 & \text{for NLS (1.4)} \\ 3 & \text{for NLH (1.5)}. \end{cases}$$

We observe that by algebra,

$$\langle \mathcal{N}'(u), u\rangle_{L^2} = (p+1)\mathcal{N}(u).$$

Hence, if u is admissible for the variational problem (2.3), equivalently

$$0 = \langle I_\epsilon'(u), u\rangle_{L^2} = \|u\|^2_{H^1_{P_\epsilon}} - (p+1)\mathcal{N}(u) \quad \left(\Leftrightarrow \|u\|^2_{H^1_{P_\epsilon}} = (p+1)\mathcal{N}(u)\right), \tag{2.1}$$

then the action functional can be written as

$$I_\epsilon(u) = \frac{1}{2}\|u\|^2_{H^1_{P_\epsilon}} - \mathcal{N}(u) = \frac{p-1}{2(p+1)}\|u\|^2_{H^1_{P_\epsilon}} = \frac{p-1}{2}\mathcal{N}(u). \tag{2.2}$$

For each $\epsilon \geq 0$ (including $\epsilon = 0$), the *ground state energy level* is defined by

$$\mathcal{C}_\epsilon := \inf\left\{ I_\epsilon(u) \mid u \in H^1_{P_\epsilon}\setminus\{0\} \text{ and } \langle I_\epsilon'(u), u\rangle_{L^2} = 0\right\}. \tag{2.3}$$

The following lemma is useful to prove the proposition.

Lemma 2.2 *Suppose that* (1.6) *(as well as* (1.7) *for NLS) holds. Then, $\mathcal{C}_\epsilon$ is strictly positive, and*

$$\limsup_{\epsilon\to 0} \mathcal{C}_\epsilon \leq \mathcal{C}_0. \tag{2.4}$$

Proof For NLS (1.4), by the Sobolev inequality with (1.6), we have

$$\mathcal{N}(u) = \frac{1}{2k+2}\|u\|^{2k+2}_{L^{2k+2}} \leq C'\|u\|^{2k+2}_{H^1_{P_\epsilon}}, \tag{2.5}$$

while for NLH (1.5), by the Hardy–Littlewood–Sobolev inequality and the Sobolev inequality with (1.6),

$$\mathcal{N}(u) \leq \frac{1}{4}\left\|\frac{1}{|x|} * |u|^2\right\|_{L^6} \left\||u|^2\right\|_{L^{6/5}} \leq C\left\||u|^2\right\|^2_{L^{6/5}} = C\|u\|^4_{L^{12/5}} \leq C'\|u\|^4_{H^1_{P_\epsilon}}. \tag{2.6}$$

Then, inserting the above inequality to the constraint (2.1), we get

$$0 \geq \|u\|_{H^1_{P_\epsilon}}^2 \left(1 - (p+1)C'\|u\|_{H^1_{P_\epsilon}}^{p-2}\right) \quad \left(\Leftrightarrow \|u\|_{H^1_{P_\epsilon}}^2 \geq ((p+1)C')^{-\frac{2}{p-2}}\right)$$

Thus, by (2.2), $I(u_\epsilon) = \frac{p-1}{2(p+1)}\|u\|_{H^1_{P_\epsilon}}^2 \geq \frac{p-1}{2(p+1)}((p+1)C')^{-\frac{2}{p-2}}$. Taking the infimum, we prove the lower bound on $\mathcal{C}_\epsilon$.

To show (2.4), we observe that the ground state Q_0 for the limit equation (1.8) (resp., (1.9)) is almost admissible for the variational problem (2.3) for sufficiently small $\epsilon > 0$, because

$$\begin{aligned}\langle I_\epsilon'(Q_0), Q_0\rangle_{L^2} &= \langle (P_\epsilon + 1)Q_0 - \mathcal{N}'(Q_0), Q_0\rangle_{L^2}\\ &= \langle (-\Delta + 1)Q_0 - \mathcal{N}'(Q_0), Q_0\rangle_{L^2} + \langle (P_\epsilon - (-\Delta))Q_0, Q_0\rangle_{L^2}\\ &= 0 + o_\epsilon(1) = o_\epsilon(1),\end{aligned}$$

where in the third identity, we used that $Q_0 \in H^\ell$ for all $\ell \in \mathbb{N}$. Hence, for each $\epsilon > 0$, there exists $t_\epsilon = 1 + o_\epsilon(1)$ such that $t_\epsilon Q_0$ is admissible. Then, it follows from the definition of the level set $\mathcal{C}_\epsilon$ and (2.2) that

$$\mathcal{C}_\epsilon \leq I_\epsilon(t_\epsilon Q_0) = \frac{p-1}{2}\mathcal{N}(t_\epsilon Q_0) = t_\epsilon^{p+1} \cdot \frac{p-1}{2}\mathcal{N}(Q_0) = (1 + o_\epsilon(1)) \cdot I_0(Q_0).$$

Thus, taking $\limsup_{\epsilon\to 0}$, we prove (2.4). □

Proof of Proposition 2.1 Let $\{u_n\}_{n=1}^\infty \subset H^1_{P_\epsilon}$ be a minimizing sequence for $I_\varepsilon(u)$ subject to the constraint $\langle I_\epsilon'(u), u\rangle_{L^2} = 0$ with $u \neq 0$, which is, by (2.2), bounded in $H^1_{P_\epsilon}$. We consider the Levy concentration function of u_n (see [14])

$$M_n(r) := \sup_{x\in\mathbb{R}^d} \int_{B_r(x)} |u_n|^2\, dx,$$

where $B_r(x)$ denotes the Euclidean ball of the radius r centered at x.

Suppose that there exists some $r > 0$ such that $M_n(r) \to 0$ as $n \to \infty$. It is shown in [14] that $\{u_n\}_{n=1}^\infty$ converges to zero in $L^p(\mathbb{R}^d)$ for every $2 < p < 2^*$, where $\frac{1}{2^*} = \max\{\frac{d-2}{2d}, 0\}$. Thus, by (2.2) and (2.5) (resp., (2.6)), it follows that $I_\epsilon(u_n) = \frac{p-1}{2}\mathcal{N}(u_n) \to 0$ as $n \to \infty$, but this contradicts to that $\mathcal{C}_\epsilon > 0$ (see Lemma 2.2).

Now, passing to a subsequence, we assume that

$$M_0 := \lim_{n\to\infty} M(1) > 0.$$

Then, there exists a sequence $\{x_n\}_{n=1}^\infty \subset \mathbb{R}^d$ such that for sufficiently large n,

$$\int_{B_1(x_n)} |u_n|^2\, dx > \frac{M_0}{2}. \tag{2.7}$$

Translating the sequence, we introduce another minimizing sequence $\{v_n\}_{n=1}^\infty$ given by $v_n(x) = u_n(x + x_n)$, which is bounded in $H^1_{P_\epsilon}$. Let v_0 be the weak subsequential limit of $\{v_n\}_{n=1}^\infty$ in $H^1_{P_\epsilon}$ as $n \to \infty$. Note that $v_0 \neq 0$, since $\{v_n\}_{n=1}^\infty$ is locally compact in $L^2(\mathbb{R}^d)$ and it satisfies (2.7).

We claim that v_0 is admissible for the minimization problem (2.3), i.e.,

$$\langle I_\epsilon'(v_0), v_0\rangle_{L^2} = 0.$$

In order to prove the claim by contradiction, we assume that

$$\delta := \langle I'_\epsilon(v_0), v_0\rangle_{L^2} > 0,$$

and then applying the well-known Brezis-Lieb lemma, we decompose $v_n = v_0 + w_n$ such that

$$\begin{aligned}\|v_n\|^2_{H^1_{P_\epsilon}} &= \|v_0\|^2_{H^1_{P_\epsilon}} + \|w_n\|^2_{H^1_{P_\epsilon}} + o_n(1),\\ \mathcal{N}(v_n) &= \mathcal{N}(v_0) + \mathcal{N}(w_n) + o_n(1),\end{aligned} \tag{2.8}$$

and consequently,

$$\langle I'_\epsilon(w_n), w_n\rangle_{L^2} = \langle I'_\epsilon(v_n), v_n\rangle_{L^2} - \langle I'_\epsilon(v_0), v_0\rangle_{L^2} + o_n(1) = -\delta + o_n(1).$$

We observe that $f_n(t) := \langle I'_\epsilon(tw_n), tw_n\rangle_{L^2}$ is a polynomial of the form $a_n t^2 - b_n t^{p+1}$ with $a_n, b_n > 0$, and that $f_n(1) \leq -\frac{\delta}{2}$ for large n. Hence, there exist a small $\eta \in (0,1)$ and a sequence $\{t_n\}_{n=1}^\infty$, with $0 < t_n \leq 1-\eta$, such that $\{t_n w_n\}_{n=1}^\infty$ is admissible, i.e., $\langle I'_\epsilon(t_n w_n), t_n w_n\rangle_{L^2} = 0$. Then, by (2.2) and (2.8), we prove that

$$\begin{aligned}I_\epsilon(t_n w_n) &= \frac{p-1}{2(p+1)}\|t_n w_n\|^2_{H^1_{P_\epsilon}} = t_n^2 \cdot \frac{p-1}{2(p+1)}\|w_n\|^2_{H^1_{P_\epsilon}}\\ &\leq (1-\eta)^2 \cdot \frac{p-1}{2(p+1)}\|v_n\|^2_{H^1_{P_\epsilon}} + o_n(1)\\ &= (1-\eta)^2 I_\epsilon(v_n) + o_n(1) = (1-\eta)^2\mathcal{C}_\epsilon + o_n(1).\end{aligned}$$

However, this contradicts to minimality of $\mathcal{C}_\epsilon$. If $\delta := \langle I'_\epsilon(v_0), v_0\rangle_{L^2} < 0$, repeating the same argument but switching the role of v_0 with w_n, we can again deduce a contradiction. Therefore, the claim is proved.

Finally, by the lower semi-continuity of the norm $\|\cdot\|_{H^1_{P_\epsilon}}$, we show that v_0 achieves the minimal energy,

$$I_\epsilon(v_0) = \frac{p-1}{2(p+1)}\|v_0\|_{H^1_{P_\epsilon}} \leq \frac{p-1}{2(p+1)}\lim_{n\to\infty}\|v_n\|_{H^1_{P_\epsilon}} = \lim_{n\to\infty} I_\epsilon(v_n) = \mathcal{C}_\epsilon.$$

This completes the proof by setting $Q_\epsilon^J := v_0$. □

Proposition 2.3 (Convergence of ground states) *Suppose that* (1.6) *(as well as* (1.7) *for NLS) holds. Let* $\{Q_\epsilon^J\}_{\epsilon>0}$ *be the family of ground states for the higher-order equation* (1.4) *(resp.,* (1.5)*) given by Proposition 2.1. Then,*

$$\lim_{\epsilon\to 0}\|Q_\epsilon^J - \tilde{Q}_0\|_{H^1_{P_\epsilon}} = 0,$$

where $\tilde{Q}_0$ *is a ground state to the second-order equation* (1.8) *(resp.,* (1.9)*).*

Proof By (1.6), (2.2) and Lemma 2.3, we see that $\{Q_\epsilon^J\}_{\epsilon>0}$ is bounded in H^1,

$$\gamma\|Q_\epsilon^J\|^2_{H^1} \leq \|Q_\epsilon^J\|^2_{H^1_{P_\epsilon}} = \frac{2(p+1)}{p-1}I_\epsilon(Q_\epsilon^J) = \frac{2(p+1)}{p-1}\mathcal{C}_\epsilon = \frac{2(p+1)}{p-1}\mathcal{C}_0 + o_\epsilon(1).$$

Hence, Q_ϵ^J weakly subsequentially converges to $\tilde{Q}_0$ in H^1 as $\epsilon \to 0$. As in the proof of Proposition 2.1, one can show that

$$\int_{B_1(0)} |Q_\epsilon^J|^2\, dx \geq \frac{M_0}{2},$$

which implies $\tilde{Q}_0$ is nontrivial.

We claim that $\tilde{Q}_0$ is a smooth solution to (1.8) (resp., (1.9)). To show the claim, we recall that for any $\phi \in C_c^\infty$,

$$\langle (P_\epsilon + 1) Q_\epsilon^J - \mathcal{N}'(Q_\epsilon^J), \phi \rangle_{L^2} = 0.$$

However, by the weak convergence of Q_ϵ^J, up to a subsequence, we have

$$\begin{aligned} \lim_{\varepsilon \to 0} \langle (P_\epsilon + 1) Q_\epsilon^J, \phi \rangle_{L^2} &= \lim_{\varepsilon \to 0} \langle Q_\epsilon^J, (P_\epsilon + 1)\phi \rangle_{L^2} \\ &= \lim_{\varepsilon \to 0} \langle Q_\epsilon^J, (-\Delta + 1)\phi \rangle_{L^2} = \langle \tilde{Q}_0, (-\Delta + 1)\phi \rangle_{L^2} \end{aligned}$$

and

$$\lim_{\epsilon \to 0} \langle \mathcal{N}'(Q_\epsilon), \phi \rangle_{L^2} = \langle \mathcal{N}'(\tilde{Q}_0), \phi \rangle_{L^2}.$$

Thus, sending $\epsilon \to 0$, we show that

$$\langle (-\Delta + 1) \tilde{Q}_0 - \mathcal{N}'(\tilde{Q}_0), \phi \rangle_{L^2} = 0.$$

In other words, $\tilde{Q}_0$ is a weak solution to (1.8) (resp., (1.9)). Then, by the elliptic regularity (see [9], one can show that $\tilde{Q}_0 \in H^\ell$ for every $\ell \in \mathbb{N}$.

Next, using (2.2), we write

$$\begin{aligned} \mathcal{C}_0 \le I_0(\tilde{Q}_0) &= \frac{p-1}{2(p+1)} \|\tilde{Q}_0\|_{H^1}^2 \\ &\le \frac{p-1}{2(p+1)} \left(\|\tilde{Q}_0\|_{H^1_{P_\epsilon}}^2 + \|Q_\epsilon^J - \tilde{Q}_0\|_{H^1_{P_\epsilon}}^2 \right) \\ &= \frac{p-1}{2(p+1)} \|Q_\epsilon^J\|_{H^1_{P_\epsilon}}^2 - \frac{p-1}{p+1} \cdot \mathrm{Re} \langle \tilde{Q}_0, Q_\epsilon^J - \tilde{Q}_0 \rangle_{H^1_{P_\epsilon}} \\ &= I_\epsilon(Q_\epsilon^J) - \frac{p-1}{p+1} \cdot \mathrm{Re} \langle \tilde{Q}_0, Q_\epsilon^J - \tilde{Q}_0 \rangle_{H^1_{P_\epsilon}}. \end{aligned} \tag{2.9}$$

However, since $\tilde{Q}_0$ is smooth and $Q_\epsilon^J \rightharpoonup \tilde{Q}_0$ in H^1 as $\epsilon \to 0$, we have

$$\langle \tilde{Q}_0, Q_\epsilon^J - \tilde{Q}_0 \rangle_{H^1_{P_\epsilon}} = \langle \tilde{Q}_0, Q_\epsilon^J - \tilde{Q}_0 \rangle_{H^1} + \langle (P_\epsilon - (-\Delta)) \tilde{Q}_0, Q_\epsilon^J - \tilde{Q}_0 \rangle_{L^2} \to 0$$

as $\epsilon \to 0$. Thus, by (2.2) again and Lemma 2.2, we get

$$\mathcal{C}_0 \le \mathcal{C}_\epsilon + o_\epsilon(1) \le \mathcal{C}_0 + o_\epsilon(1).$$

Sending $\epsilon \to 0$ in (2.9), we conclude that $\tilde{Q}_0$ achieves the minimum value $\mathcal{C}_0$ of the action functional I_0 and that $\|Q_\epsilon^J - \tilde{Q}_0\|_{H^1_{P_\epsilon}} \to 0$ as $\epsilon \to 0$. □

Remark 2.4 Let Q_0 be the radially symmetric positive ground state for (1.8) (resp., (1.9)). By uniqueness of a ground state to the second-order equation, there exist $\theta \in \mathbb{R}$ and $x_0 \in \mathbb{R}^d$ such that $Q_0(x) = e^{i\theta} \tilde{Q}_0(x - x_0)$. Then, the modified profile $e^{i\theta} Q_\epsilon^J(\cdot - x_0)$, which is also a ground state, converges to Q_0 as $\epsilon \to 0$.

Proposition 2.5 (Upgraded convergence) *Suppose that* (1.6) *(as well as* (1.7) *for NLS) holds. For* $\epsilon > 0$, *let* $u_\epsilon \in H^1_{P_\epsilon}$ *be a solution to the higher-order equation* (1.4) *(resp.,* (1.5)*), which is not necessarily a ground state. Let* Q_0 *be the unique ground state to the second-order equation* (1.8) *(resp.,* (1.9)*).*

(1) If $u_\epsilon \to Q_0$ in H^1 as $\epsilon \to 0$, then $u_\epsilon \to Q_0$ in H^ℓ as $\epsilon \to 0$ for all $\ell \in \mathbb{N}$.
(2) As a consequence, for all $\ell \in \mathbb{N}$, $\|u_\epsilon\|_{H^\ell}$ is bounded uniformly in $0 < \epsilon \le 1$.

Proof We prove the proposition by induction. Let $r_\epsilon = u_\epsilon - Q_0$ be the difference between two solutions. Suppose that $\|r_\epsilon\|_{H^\ell} \to 0$ for some $\ell \in \mathbb{N}$. Then, by the equations, we have

$$\begin{aligned}(P_\epsilon + 1)r_\epsilon &= (-\Delta - P_\epsilon)Q_0 + (P_\epsilon + 1)u_\epsilon - (-\Delta + 1)Q_0\\ &= (-\Delta - P_\epsilon)Q_0 + \mathcal{N}'(u_\epsilon) - \mathcal{N}'(Q_0).\end{aligned}$$

Thus, it follows from (1.6) that

$$\begin{aligned}\gamma\|r_\epsilon\|_{H^{\ell+1}} &\le \|(P_\epsilon + 1)r_\epsilon\|_{H^{\ell-1}}\\ &\le \|(-\Delta - P_\epsilon)Q_0\|_{H^{\ell-1}} + \|\mathcal{N}'(Q_0 + r_\epsilon) - \mathcal{N}'(Q_0)\|_{H^{\ell-1}}.\end{aligned}$$

For the first term on the right hand side, by smoothness of Q_0, $\|(-\Delta - P_\epsilon)Q_0\|_{H^{\ell-1}} \to 0$. For the second term, distributing derivatives and then applying Lemma A.2, one can show that

$$\|\mathcal{N}'(Q_0 + r_\epsilon) - \mathcal{N}'(Q_0)\|_{H^{\ell-1}} \lesssim \left\{\|Q_0\|_{H^\ell} + \|r_\epsilon\|_{H^\ell}\right\}^{p-1}\|r_\epsilon\|_{H^\ell} \to 0,$$

where $p = 2k+1$ for (1.4) (resp., $p = 3$ for (1.5)). Therefore, we conclude that $\|r_\epsilon\|_{H^{\ell+1}} \to 0$. □

3 Non-degeneracy estimates

Let $\{u_\epsilon\}_{\epsilon>0}$ be a family of real-valued solutions to the higher-order equation such that $u_\epsilon \to Q_0$ in H^1 (as well as in H^ℓ for all $\ell \in \mathbb{N}$ by Proposition 2.5) as $\epsilon \to 0$, where Q_0 is the unique radially symmetric positive ground state for the second-order equation (1.8) (resp., (1.9)). For notational convenience, we denote $u_0 := Q_0$. For $0 \le \epsilon \le \epsilon_0$ (including 0), we consider the linear operators $\mathcal{L}_\epsilon^\pm : H^1_{P_\epsilon} \to H^{-1}_{P_\epsilon}$, defined by

$$\begin{cases}\mathcal{L}_\epsilon^+ : = P_\epsilon + 1 - \mathcal{N}^+_{u_\epsilon}\\ \mathcal{L}_\epsilon^- : = P_\epsilon + 1 - \mathcal{N}^-_{u_\epsilon},\end{cases} \tag{3.1}$$

where

$$\mathcal{N}_u^+(g) := \begin{cases}(2k+1)u^{2k}g & \text{(for NLS)}\\ 2(|x|^{-1} * (ug))u + (|x|^{-1} * u^2)g & \text{(for NLH)}\end{cases} \tag{3.2}$$

and

$$\mathcal{N}_u^-(g) := \begin{cases}u^{2k}g & \text{(for NLS)}\\ (|x|^{-1} * u^2)g & \text{(for NLH).}\end{cases} \tag{3.3}$$

These linear operators naturally appear as the real and the imaginary parts of the linearized operator at the solution u_ϵ. Factorizing out the differential operator $(1 + P_\epsilon)$ in a symmetric form, we write

$$\mathcal{L}_\epsilon^\pm = \sqrt{1 + P_\epsilon}(\mathrm{Id} - \mathcal{A}_\epsilon^\pm)\sqrt{1 + P_\epsilon}, \tag{3.4}$$

where

$$\begin{cases}\mathcal{A}_\epsilon^+ : = \dfrac{1}{\sqrt{1 + P_\epsilon}}\mathcal{N}^+_{u_\epsilon}\dfrac{1}{\sqrt{1 + P_\epsilon}}\\ \mathcal{A}_\epsilon^- : = \dfrac{1}{\sqrt{1 + P_\epsilon}}\mathcal{N}^-_{u_\epsilon}\dfrac{1}{\sqrt{1 + P_\epsilon}}.\end{cases}$$

In this section, we prove non-degeneracy of the solution u_ϵ, and obtain uniform lower bounds for the linear operators $(\mathrm{Id}-\mathcal{A}_\epsilon^\pm)$, which are our main analytic tools.

To begin with, we consider the base case $\epsilon=0$. By the non-degeneracy of the ground state Q_0 and the relation (3.4) (see [10] for NLS and [11] for NLH), we have

$$\mathrm{Ker}(\mathrm{Id}-\mathcal{A}_0^+)=\mathrm{span}\Big\{\partial_{x_1}\sqrt{1-\Delta}\,Q_0,\cdots,\partial_{x_d}\sqrt{1-\Delta}\,Q_0\Big\}$$

and

$$\mathrm{Ker}(\mathrm{Id}-\mathcal{A}_0^-)=\mathrm{span}\Big\{\sqrt{1-\Delta}\,Q_0\Big\}.$$

By the equation, the operator $\mathcal{A}_0^\pm$ sends an element of $\mathrm{Ker}(\mathrm{Id}-\mathcal{A}_0^\pm)$ to the same function, and thus $(\mathrm{Id}-\mathcal{A}_0^\pm)$ maps $(\mathrm{Ker}(\mathrm{Id}-\mathcal{A}_0^\pm))^\perp\subset L^2(\mathbb{R}^d;\mathbb{R})$ to itself, where $A^\perp\subset H$ denotes the subspace orthogonal to A in the Hilbert space H. Moreover, the operator $(\mathrm{Id}-\mathcal{A}_0^\pm)$ satisfies the following lower bounds.

Proposition 3.1 *(Non-degeneracy estimates; base case) There exists $\beta_0>0$ such that*

$$\|(Id-\mathcal{A}_0^\pm)g\|_{L^2(\mathbb{R}^d;\mathbb{R})}\geq\beta_0\|g\|_{L^2(\mathbb{R}^d;\mathbb{R})}$$

for all $g\in(Ker(Id-\mathcal{A}_0^\pm))^\perp\subset L^2(\mathbb{R}^d;\mathbb{R})$.

Proof We claim that both $\frac{1}{\sqrt{1-\Delta}}\mathcal{N}_{Q_0}^+\frac{1}{\sqrt{1-\Delta}}$ and $\frac{1}{\sqrt{1-\Delta}}\mathcal{N}_{Q_0}^-\frac{1}{\sqrt{1-\Delta}}$ are compact on L^2. Indeed, for NLS, the integral kernel of $\mathcal{N}_{Q_0}^+\frac{1}{\sqrt{1-\Delta}}$ (or $\mathcal{N}_{Q_0}^-\frac{1}{\sqrt{1-\Delta}}$, respectively) is given by

$$(2k+1)Q_0(x)^{2k}G_{-1}(x-y)\quad\Big(\text{or } Q_0(x)^{2k}G_{-1}(x-y),\ \text{respectively}\Big),$$

where $G_{-1}(x)=((1+|\xi|^2)^{-1})^\vee(x)$ is the Bessel potential. For NLH, it is given by

$$2\int_{\mathbb{R}^3}\frac{Q_0(x)Q_0(z)}{|x-z|}G_{-1}(z-y)dz+(|x|^{-1}*Q_0^2)(x)G_{-1}(x-y)$$
$$\Big(\text{or } (|x|^{-1}*Q_0^2)(x)G_{-1}(x-y),\ \text{respectively}\Big).$$

All of the above kernels are contained in $L^2(\mathbb{R}_x^d\times\mathbb{R}_y^d)$, because Q_0 is smooth and rapidly decaying. Therefore, the associated operators are Hilbert–Schmidt (so, compact on $L^2(\mathbb{R}^d)$). Since composition of a compact operator and a bounded operator is compact, this proves the claim. As a consequence, by the Fredholm alternative, the proposition is proved. □

Next, we show that the non-degeneracy of the ground state Q_0 is stable along the family of solutions which converges to the ground state Q_0.

Proposition 3.2 (Stability of non-degeneracy) *Let $\{u_\epsilon\}_{\epsilon>0}$ be a family of real-valued solutions to the higher-order equation (1.4) (resp., (1.5)) such that $u_\epsilon\to Q_0$ in H^1 as $\epsilon\to0$. Then, there exists $\epsilon_0>0$ such that*

$$Ker\mathcal{L}_\epsilon^+=span\Big\{\partial_{x_1}u_\epsilon,\ldots,\partial_{x_d}u_\epsilon\Big\}\quad and\quad Ker\mathcal{L}_\epsilon^-=span\Big\{u_\epsilon\Big\}$$

for $0<\epsilon\leq\epsilon_0$. Equivalently, we have

$$Ker(Id-\mathcal{A}_\epsilon^+)=span\Big\{\partial_{x_1}\sqrt{1+P_\epsilon}\,u_\epsilon,\ldots,\partial_{x_d}\sqrt{1+P_\epsilon}\,u_\epsilon\Big\}\tag{3.5}$$

and

$$Ker(Id-\mathcal{A}_\epsilon^-)=span\Big\{\sqrt{1+P_\epsilon}\,u_\epsilon\Big\}.\tag{3.6}$$

Proof Following the argument in the proof of [11, Theorem 3], we prove (3.5) only, because (3.6) can be proved by the same way.

By the equation, it is easy to see that each $\partial_{x_j}\sqrt{1+P_\epsilon}u_\epsilon$ is contained in the kernel of $(\mathrm{Id}-\mathcal{A}_\epsilon^+)$. Therefore, it suffices to show that the dimension of $\mathrm{Ker}(\mathrm{Id}-\mathcal{A}_\epsilon^+)$ is $\leq d$. We recall that $\mathrm{Ker}(\mathrm{Id}-\mathcal{A}_\epsilon^+)=\mathrm{Im}(\mathcal{P}_\epsilon)$, where $\mathcal{P}_\epsilon$ is the projection operator given by

$$\mathcal{P}_\epsilon := \frac{1}{2\pi i}\oint_{|z|=c}(\mathrm{Id}-\mathcal{A}_\epsilon^+-z\mathrm{Id})^{-1}dz$$

for some sufficiently small $c>0$. We observe that by Lemma A.2,

$$\begin{aligned}\|\mathcal{A}_\epsilon^+-\mathcal{A}_0^+\|_{L^2\to L^2} &= \left\|\frac{1}{\sqrt{1+P_\epsilon}}\mathcal{N}_{u_\epsilon}^+\frac{1}{\sqrt{1+P_\epsilon}}-\frac{1}{\sqrt{1-\Delta}}\mathcal{N}_{Q_0}^+\frac{1}{\sqrt{1-\Delta}}\right\|_{L^2\to L^2}\\ &\leq \left\|\left(\frac{1}{\sqrt{1+P_\epsilon}}-\frac{1}{\sqrt{1-\Delta}}\right)\mathcal{N}_{u_\epsilon}^+\frac{1}{\sqrt{1+P_\epsilon}}\right\|_{L^2\to L^2}\\ &\quad+\left\|\frac{1}{\sqrt{1-\Delta}}(\mathcal{N}_{u_\epsilon}^+-\mathcal{N}_{Q_0}^+)\frac{1}{\sqrt{1+P_\epsilon}}\right\|_{L^2\to L^2}\\ &\quad+\left\|\frac{1}{\sqrt{1-\Delta}}\mathcal{N}_{Q_0}^+\left(\frac{1}{\sqrt{1+P_\epsilon}}-\frac{1}{\sqrt{1-\Delta}}\right)\right\|_{L^2\to L^2}\to 0,\end{aligned} \tag{3.7}$$

and consequently, $\|\mathcal{P}_\epsilon-\mathcal{P}_0\|_{L^2\to L^2}\to 0$ as $\epsilon\to 0$. Suppose that $\mathrm{Rank}(\mathcal{P}_\epsilon)>\mathrm{Rank}(\mathcal{P}_0)$. Then, there exist L^2-orthonormal vectors $v_1,\ldots,v_{d+1}$ such that $\mathcal{P}_\epsilon v_j=v_j$. Hence, $\mathcal{P}_0v_1,\ldots,\mathcal{P}_0v_{d+1}$ are almost orthogonal, and they are linearly independent, which contradicts to the assumption. Therefore, we conclude that $\mathrm{Rank}(\mathcal{P}_\epsilon)\leq\mathrm{Rank}(\mathcal{P}_0)=d$. □

Using the non-degeneracy, we prove the inequality analogous to Proposition 3.1.

Proposition 3.3 *(Non-degeneracy estimates; general case) Let $\{u_\epsilon\}_{\epsilon>0}$ be a family of real-valued solutions to the higher-order equation (1.4) (resp., (1.5)) such that $u_\epsilon\to Q_0$ in H^1 as $\epsilon\to 0$. Then, there exist $\epsilon_0>0$ and $\beta>0$ such that if $0<\epsilon\leq\epsilon_0$, then*

$$\|(Id-\mathcal{A}_\epsilon^\pm)g\|_{L^2(\mathbb{R}^d;\mathbb{R})}\geq\beta\|g\|_{L^2(\mathbb{R}^d;\mathbb{R})}$$

for all $g\in(Ker(Id-\mathcal{A}_\epsilon^\pm))^\perp\subset L^2(\mathbb{R}^d;\mathbb{R})$, which is equivalent to

$$\|\mathcal{L}_\epsilon^\pm g\|_{H_{P_\epsilon}^{-1}(\mathbb{R}^d;\mathbb{R})}\geq\beta\|g\|_{H_{P_\epsilon}^1(\mathbb{R}^d;\mathbb{R})}$$

for all $g\in(Ker\mathcal{L}_\epsilon^\pm)^\perp\subset H_{P_\epsilon}^1(\mathbb{R}^d;\mathbb{R})$.

Proof We show the proposition only for $\mathcal{A}_\epsilon^+$, since the other inequality can be proved exactly by the same way.

Let $\beta=\frac{\beta_0}{4}>0$, where β_0 is given in Proposition 3.1. For $g\in L^2(\mathbb{R}^d;\mathbb{R})$ and $\epsilon\geq 0$, we denote by $g_\epsilon^\perp$ the orthogonal projection of g to $(\mathrm{Ker}(\mathrm{Id}-\mathcal{A}_\epsilon^+))^\perp\subset L^2(\mathbb{R}^d;\mathbb{R})$, precisely

$$g_\epsilon^\perp := g-\sum_{j=1}^d\langle g,e_{j;\epsilon}\rangle_{L^2}e_{j;\epsilon},$$

where $e_{j;\epsilon}:=\frac{\partial_{x_j}\sqrt{1+P_\epsilon}u_\epsilon}{\|\partial_{x_j}\sqrt{1+P_\epsilon}u_\epsilon\|_{L^2}^2}$. We fix $g\in(\mathrm{Ker}(\mathrm{Id}-\mathcal{A}_\epsilon^+))^\perp$. Then, we decompose

$$(\mathrm{Id}-\mathcal{A}_\epsilon^+)g=(\mathrm{Id}-\mathcal{A}_0^+)g+(\mathcal{A}_0^+-\mathcal{A}_\epsilon^+)g$$

$$= (\mathrm{Id} - \mathcal{A}_0^+)g_0^\perp + (\mathrm{Id} - \mathcal{A}_0^+)(g - g_0^\perp) + (\mathcal{A}_0^+ - \mathcal{A}_\epsilon^+)g.$$

By the triangle inequalities and Proposition 3.1, we get

$$\begin{aligned}
\|(\mathrm{Id} - \mathcal{A}_\epsilon^+)g\|_{L^2} &\geq \|(\mathrm{Id} - \mathcal{A}_0^+)g_0^\perp\|_{L^2} - \|(\mathrm{Id} - \mathcal{A}_0^+)(g - g_0^\perp)\|_{L^2} - \|(\mathcal{A}_0^+ - \mathcal{A}_\epsilon^+)g\|_{L^2} \\
&\geq 4\beta\|g_0^\perp\|_{L^2} - \|g - g_0^\perp\|_{L^2} - \|\mathcal{A}_0^+(g - g_0^\perp)\|_{L^2} - \|(\mathcal{A}_0^+ - \mathcal{A}_\epsilon^+)g\|_{L^2} \\
&\geq 4\beta\|g\|_{L^2} - (4\beta + 1 + \|\mathcal{A}_0^+\|_{L^2\to L^2})\|g_\epsilon^\perp - g_0^\perp\|_{L^2} \quad (\text{by } g_\epsilon^\perp = g) \\
&\quad - \|\mathcal{A}_0^+ - \mathcal{A}_\epsilon^+\|_{L^2\to L^2}\|g\|_{L^2}.
\end{aligned} \tag{3.8}$$

On the other hand, we have

$$\begin{aligned}
\|g_\epsilon^\perp - g_0^\perp\|_{L^2} &\leq \sum_{j=1}^d \left\| \langle g, e_{j;\epsilon}\rangle_{L^2} e_{j;\epsilon} - \langle g, e_{j;0}\rangle_{L^2} e_{j;0} \right\|_{L^2} \\
&\leq \sum_{j=1}^d |\langle g, e_{j;\epsilon} - e_{j;0}\rangle_{L^2}| + |\langle g, e_{j;0}\rangle_{L^2}| \left\| e_{j;\epsilon} - e_{j;0} \right\|_{L^2} \\
&\leq 2\|g\|_{L^2} \sum_{j=1}^d \left\| e_{j;\epsilon} - e_{j;0} \right\|_{L^2} \leq o_\epsilon(1)\|g\|_{L^2},
\end{aligned}$$

because by Proposition 2.5,

$$\begin{aligned}
&\left\| \partial_{x_j}\sqrt{1 + P_\epsilon}u_\epsilon - \partial_{x_j}\sqrt{1 - \Delta}u_0 \right\|_{L^2} \\
&\leq \left\| (\sqrt{1 + P_\epsilon} - \sqrt{1 - \Delta})\partial_{x_j}u_\epsilon \right\|_{L^2} + \left\| \partial_{x_j}\sqrt{1 - \Delta}(u_\epsilon - u_0) \right\|_{L^2} \to 0
\end{aligned}$$

as $\epsilon \to 0$ and it implies $\|e_{j;\epsilon} - e_{j;0}\|_{L^2} \to 0$. Moreover, by (3.7), $\|\mathcal{A}_0^+ - \mathcal{A}_\epsilon^+\|_{L^2\to L^2} \to 0$ as $\epsilon \to 0$. Inserting these to (3.8), we prove the proposition. □

By a little modification, we can also show the following inequality.

Lemma 3.4 *There exists $\epsilon_0 > 0$ such that if $0 < \epsilon \leq \epsilon_0$, then*

$$Id - \frac{1}{\sqrt{1 + P_\epsilon}} \mathcal{N}_{Q_0}^+ \frac{1}{\sqrt{1 + P_\epsilon}}$$

is invertible on $L^2_{rad}(\mathbb{R}^d; \mathbb{R})$. Moreover, its inverse is uniformly bounded,

$$\left\| \left(Id - \frac{1}{\sqrt{1 + P_\epsilon}} \mathcal{N}_{Q_0}^+ \frac{1}{\sqrt{1 + P_\epsilon}} \right)^{-1} \right\|_{L^2_{rad}(\mathbb{R}^d;\mathbb{R}) \to L^2_{rad}(\mathbb{R}^d;\mathbb{R})} \leq \frac{2}{\beta_0},$$

where $\beta_0 > 0$ is given by Proposition 3.1.

Proof By Proposition 3.1, the operator $(\mathrm{Id} - \frac{1}{\sqrt{1-\Delta}}\mathcal{N}_{Q_0}^+\frac{1}{\sqrt{1-\Delta}})$ is invertible, because its kernel in L^2_{rad} is empty ($\partial_{x_j} Q_0$'s are not radially symmetric). On the other hand, repeating the proof of (3.7), one can show that the difference

$$\begin{aligned}
&\frac{1}{\sqrt{1 + P_\epsilon}} \mathcal{N}_{Q_0}^+ \frac{1}{\sqrt{1 + P_\epsilon}} - \frac{1}{\sqrt{1 - \Delta}} \mathcal{N}_{Q_0}^+ \frac{1}{\sqrt{1 - \Delta}} \\
&= \left(\frac{1}{\sqrt{1 + P_\epsilon}} - \frac{1}{\sqrt{1 - \Delta}} \right) \mathcal{N}_{Q_0}^+ \frac{1}{\sqrt{1 + P_\epsilon}} + \frac{1}{\sqrt{1 - \Delta}} \mathcal{N}_{Q_0}^+ \left(\frac{1}{\sqrt{1 + P_\epsilon}} - \frac{1}{\sqrt{1 - \Delta}} \right)
\end{aligned}$$

can be arbitrarily small in the operator norm on $L^2_{rad}(\mathbb{R}^d;\mathbb{R})$, provided that $\epsilon > 0$ is small enough. Therefore, we conclude that if $0 < \epsilon \le \epsilon_0$, then $(\mathrm{Id} - \frac{1}{\sqrt{1+P_\epsilon}}\mathcal{N}^+_{Q_0}\frac{1}{\sqrt{1+P_\epsilon}})$ is invertible, and its inverse is uniformly bounded. □

4 Construction of a solution by contraction

In this section, by the contraction mapping argument in [3] which relies on the non-degeneracy estimates in the previous section, we construct a radially symmetric real-valued solution u_ϵ to the higher-order equation, with small $\epsilon > 0$, that converges to the ground state Q_0 as $\epsilon \to 0$, where Q_0 is the unique radially symmetric real-valued ground state for the second-order equation.

Proposition 4.1 (Construction of a solution by contraction) *Suppose that* (1.6) *(as well as* (1.7) *for NLS) holds. Then, there exists* $\epsilon_0 > 0$ *such that a sequence of radially symmetric real-valued solutions* $\{u_\epsilon\}_{0<\epsilon\le\epsilon_0}$ *to the higher-order NLS* (1.4) *(resp., the higher-order NLH* (1.5)*) exists, with the convergence*

$$\lim_{\epsilon\to 0}\|u_\epsilon - Q_0\|_{H^1_{P_\epsilon}} = 0.$$

Proof Step 1. Reformulation of the equation Let $\epsilon > 0$ be sufficiently small. Suppose that u_ϵ is a radially symmetric real-valued solution to the higher-order equation. Then, the difference

$$r_\epsilon := u_\epsilon - Q_0$$

solves the equation

$$\begin{aligned}(P_\epsilon+1)r_\epsilon &= (P_\epsilon+1)u_\epsilon - (P_\epsilon+1)Q_0\\ &= (-\Delta - P_\epsilon)Q_0 + (P_\epsilon+1)u_\epsilon - (-\Delta+1)Q_0\\ &= (-\Delta - P_\epsilon)Q_0 + \mathcal{N}'(u_\epsilon) - \mathcal{N}'(Q_0)\\ &= (-\Delta - P_\epsilon)Q_0 + \mathcal{N}'(Q_0+r_\epsilon) - \mathcal{N}'(Q_0).\end{aligned}$$

Moving the linear terms with respect to r_ϵ on the right hand side to the left, we write

$$\left(P_\epsilon + 1 - \mathcal{N}^+_{Q_0}\right) r_\epsilon = (-\Delta - P_\epsilon)Q_0 + \mathcal{N}'(Q_0+r_\epsilon) - \mathcal{N}'(Q_0) - \mathcal{N}^+_{Q_0}(r_\epsilon)$$

(see (3.2) for the definition of $\mathcal{N}^+_{Q_0}$). Then, inverting the operator

$$\left(P_\epsilon + 1 - \mathcal{N}^+_{Q_0}\right) = \sqrt{1+P_\epsilon}\left(\mathrm{Id} - \frac{1}{\sqrt{1+P_\epsilon}}\mathcal{N}^+_{Q_0}\frac{1}{\sqrt{1+P_\epsilon}}\right)\sqrt{1+P_\epsilon}$$

by Lemma 3.4, we reformulate the higher-order equation as

$$\begin{aligned}r_\epsilon &= \left(P_\epsilon + 1 - \mathcal{N}^+_{Q_0}\right)^{-1}\left\{(-\Delta - P_\epsilon)Q_0 + \mathcal{N}'(Q_0+r_\epsilon) - \mathcal{N}'(Q_0) - \mathcal{N}^+_{Q_0}(r_\epsilon)\right\}\\ &=: \Phi(r_\epsilon).\end{aligned}$$

Step 2. Construction of a solution We set

$$\delta_\epsilon := \frac{4}{\beta_0}\|(-\Delta - P_\epsilon)u_0\|_{H^{-1}_{P_\epsilon}},$$

where $\beta_0 > 0$ is the constant given in Lemma 3.4. Then, by (1.6), we have

$$\delta_\epsilon \le \frac{4}{\beta_0} \sum_{|\alpha|=3}^{J} |c_\alpha| \epsilon^{|\alpha|-2} \|\nabla^\alpha Q_0\|_{H^{-1}_{P_\epsilon}} \le \frac{4}{\beta_0 \gamma} \sum_{|\alpha|=3}^{J} |c_\alpha| \epsilon^{|\alpha|-2} \|\nabla^\alpha Q_0\|_{H^{-2}} \to 0$$

as $\epsilon \to 0$. Let ϵ_0 be a sufficiently small number to be chosen so that all of the following estimates hold. Suppose that $0 < \epsilon \le \epsilon_0$ (and thus $\delta_\epsilon > 0$ is also small enough). If $\|r\|_{H^1_{P_\epsilon}}, \|\tilde{r}\|_{H^1_{P_\epsilon}} \le \delta_\epsilon$, then by Lemma 3.4 and Lemma A.2,

$$\begin{aligned} \|\Phi(r)\|_{H^1_{P_\epsilon}} &= \Bigg\| \left(\mathrm{Id} - \frac{1}{\sqrt{1+P_\epsilon}} \mathcal{N}^+_{Q_0} \frac{1}{\sqrt{1+P_\epsilon}} \right)^{-1} \frac{1}{\sqrt{1+P_\epsilon}} \\ &\qquad \left\{ (-\Delta - P_\epsilon) Q_0 + \mathcal{N}'(Q_0 + r) - \mathcal{N}'(Q_0) - \mathcal{N}^+_{Q_0}(r) \right\} \Bigg\|_{L^2} \\ &\le \frac{2}{\beta_0} \|(-\Delta - P_\epsilon) Q_0\|_{H^{-1}_{P_\epsilon}} + \frac{2}{\beta_0} \left\| \mathcal{N}'(Q_0 + r) - \mathcal{N}'(Q_0) - \mathcal{N}^+_{Q_0}(r) \right\|_{H^{-1}_{P_\epsilon}} \\ &\le \frac{2}{\beta_0} \|(-\Delta - P_\epsilon) u_0\|_{H^{-1}_{P_\epsilon}} + \frac{1}{2} \|r\|_{H^1_{P_\epsilon}} \le \delta_\epsilon \end{aligned}$$

and similarly,

$$\begin{aligned} \|\Phi(r) - \Phi(\tilde{r})\|_{H^1_{P_\epsilon}} &\le \frac{2}{\beta_0} \Big\| \left(\mathcal{N}'(Q_0 + r) - \mathcal{N}'(Q_0) - \mathcal{N}^+_{Q_0}(r) \right) \\ &\qquad - \left(\mathcal{N}'(Q_0 + \tilde{r}) - \mathcal{N}'(Q_0) - \mathcal{N}^+_{Q_0}(\tilde{r}) \right) \Big\|_{H^{-1}_{P_\epsilon}} \\ &\le \frac{1}{2} \|r - \tilde{r}\|_{H^1_{P_\epsilon}}. \end{aligned}$$

Therefore, we conclude that Φ is contractive, and it has a unique fixed point, denoted by r_ϵ, on the ball of radius δ_ϵ centered at 0 in $H^1_{P_\epsilon}$. As a consequence, $u_\epsilon = Q_0 + r_\epsilon$ solves the higher-order equation, and $\|u_\epsilon - Q_0\|_{H^1_{P_\epsilon}} = \|r_\epsilon\|_{H^1_{P_\epsilon}} \to 0$ as $\epsilon \to 0$. □

5 Local uniqueness

The solution u_ϵ, given by Proposition 4.1, is unique in a small ball of radially symmetric real-valued functions whose radius may depend on $\epsilon > 0$. In this section, we upgrade this uniqueness to that in a small ball of all complex-valued functions whose radius is independent of $\epsilon > 0$.

Proposition 5.1 (Uniqueness) *Suppose that* (1.6) *(as well as* (1.7) *for NLS) holds. Then, there exist* $\delta > 0$ *and* $\epsilon_0 > 0$ *such that if* $0 < \epsilon \le \epsilon_0$, *then the solution* u_ϵ *to the higher-order equation* (1.4) *(resp.,* (1.5)*), constructed in Proposition 4.1, is unique in a* δ*-ball centered at* u_0 *in* $H^1_{P_\epsilon}(\mathbb{R}^d; \mathbb{C})$ *up to translation and phase shift.*

Proof We prove the proposition only for the higher-order NLH, because the proof for the higher-order NLS follows similarly. Let $\epsilon_0 > 0$ be a small number given in Proposition 3.3, and let $\delta > 0$ be sufficiently small numbers to be chosen later. Suppose that $0 < \epsilon \le \epsilon_0$ and $\tilde{u}_\epsilon$ is another solution to the higher-order equation in a δ-ball centered at u_0 in $H^1_{P_\epsilon}(\mathbb{R}^3; \mathbb{C})$.

First, we aim to show that the imaginary part of $\tilde{u}_\epsilon$ is orthogonal to u_ϵ up to phase shift. To this end, we consider

$$\tilde{U}_\epsilon = \frac{\overline{\langle \tilde{u}_\epsilon, u_\epsilon \rangle_{H^1_{P_\epsilon}}}}{|\langle \tilde{u}_\epsilon, u_\epsilon \rangle_{H^1_{P_\epsilon}}|} \tilde{u}_\epsilon,$$

which also solves the higher-order equation. Here, since u_ϵ and $\tilde{u}_\epsilon$ are assumed to be sufficiently close to u_0, the denominator $\langle \tilde{u}_\epsilon, u_\epsilon \rangle_{H^1_{P_\epsilon}} \neq 0$. Note that $\tilde{u}_\epsilon \mapsto \tilde{U}_\epsilon$ is a natural action, because if $\tilde{u}_\epsilon$ is simply a rotated u_ϵ on the complex plane, then this action rotates it back to u_ϵ. Moreover, we have

$$\begin{aligned}
\left\| \tilde{U}_\epsilon - u_\epsilon \right\|^2_{H^1_{P_\epsilon}} &= \|\tilde{U}_\epsilon\|^2_{H^1_{P_\epsilon}} + \|u_\epsilon\|^2_{H^1_{P_\epsilon}} - 2\mathrm{Re}\langle \tilde{U}_\epsilon, u_\epsilon \rangle_{H^1_{P_\epsilon}} \\
&= \|\tilde{u}_\epsilon\|^2_{H^1_{P_\epsilon}} + \|u_\epsilon\|^2_{H^1_{P_\epsilon}} - 2|\langle \tilde{u}_\epsilon, u_\epsilon \rangle_{H^1_{P_\epsilon}}| \\
&\leq \left| \|\tilde{u}_\epsilon\|^2_{H^1_{P_\epsilon}} + \|u_\epsilon\|^2_{H^1_{P_\epsilon}} - 2\mathrm{Re}\langle \tilde{u}_\epsilon, u_\epsilon \rangle_{H^1_{P_\epsilon}} \right| = \|\tilde{u}_\epsilon - u_\epsilon\|^2_{H^1_{P_\epsilon}} \\
&\leq \left\{ \|\tilde{u}_\epsilon - u_0\|_{H^1_{P_\epsilon}} + \|u_\epsilon - u_0\|_{H^1_{P_\epsilon}} \right\}^2 \leq 4\delta^2
\end{aligned}$$

and

$$\langle \mathrm{Im}(\tilde{U}_\epsilon), u_\epsilon \rangle_{H^1_{P_\epsilon}} = \mathrm{Im} \left\{ \frac{\overline{\langle \tilde{u}_\epsilon, u_\epsilon \rangle_{H^1_{P_\epsilon}}}}{|\langle \tilde{u}_\epsilon, u_\epsilon \rangle_{H^1_{P_\epsilon}}|} \langle \tilde{u}_\epsilon, u_\epsilon \rangle_{H^1_{P_\epsilon}} \right\} = \mathrm{Im}\Big\{ |\langle \tilde{u}_\epsilon, u_\epsilon \rangle_{H^1_{P_\epsilon}}| \Big\} = 0.$$

Therefore, replacing $\tilde{u}_\epsilon$ by $\tilde{U}_\epsilon$ and δ by $\frac{\delta}{2}$, we may assume that the imaginary part of $\tilde{u}_\epsilon$ is orthogonal to u_ϵ in $H^1_{P_\epsilon}$.

We denote the difference between two solutions by

$$r_\epsilon := \tilde{u}_\epsilon - u_\epsilon = v_\epsilon + i w_\epsilon \quad (\Leftrightarrow \tilde{u}_\epsilon = (u_\epsilon + v_\epsilon) + i w_\epsilon),$$

where v_ϵ and w_ϵ are real-valued, and $\langle w_\epsilon, u_\epsilon \rangle_{H^1_{P_\epsilon}} = 0$. When u_ϵ and $\tilde{u}_\epsilon$ are solutions to the higher-order NLH, then the difference r_ϵ satisfies

$$\begin{aligned}
(P_\epsilon + 1) r_\epsilon &= \left(|x|^{-1} * |\tilde{u}_\epsilon|^2\right) \tilde{u}_\epsilon - \left(|x|^{-1} * |u_\epsilon|^2\right) u_\epsilon \\
&= \left(|x|^{-1} * \left(u_\epsilon^2 + 2u_\epsilon v_\epsilon + |r_\epsilon|^2\right)\right) ((u_\epsilon + v_\epsilon) + i w_\epsilon) - \left(|x|^{-1} * |u_\epsilon|^2\right) u_\epsilon \\
&= \left(|x|^{-1} * \left(2u_\epsilon v_\epsilon + |r_\epsilon|^2\right)\right) u_\epsilon + \left(|x|^{-1} * \left(u_\epsilon^2 + 2u_\epsilon v_\epsilon + |r_\epsilon|^2\right)\right)(v_\epsilon + i w_\epsilon).
\end{aligned}$$

Moving the linear terms on the right hand side to the left then using (3.4), the imaginary part of the equation (for w_ϵ) can be written as

$$\mathcal{L}_\epsilon^- w_\epsilon = \left(|x|^{-1} * \left(2u_\epsilon v_\epsilon + |r_\epsilon|^2\right)\right) w_\epsilon.$$

Then, by Proposition 3.3 and the nonlinear estimate (Lemma A.2), we prove that

$$\begin{aligned}
\beta \|w_\epsilon\|_{H^1_{P_\epsilon}} &\leq \|\mathcal{L}_\epsilon^- w_\epsilon\|_{H^{-1}_{P_\epsilon}} = \left\| \left(|x|^{-1} * \left(2u_\epsilon v_\epsilon + |r_\epsilon|^2\right)\right) w_\epsilon \right\|_{H^{-1}_{P_\epsilon}} \\
&\leq C \left(\|u_\epsilon\|_{H^1_{P_\epsilon}} + \|r_\epsilon\|_{H^1_{P_\epsilon}} \right) \|r_\epsilon\|_{H^1_{P_\epsilon}} \|w_\epsilon\|_{H^1_{P_\epsilon}} \\
&\leq C \left(\|u_0\|_{H^1_{P_\epsilon}} + 2\delta \right) \delta \|w_\epsilon\|_{H^1_{P_\epsilon}}.
\end{aligned}$$

Therefore, choosing small δ such that $C(\|u_0\|_{H^1_{P_\epsilon}} + 2\delta)\delta < \beta$, we conclude that $w_\epsilon = 0$.

By a suitable phase shift, we may assume that $\tilde{u}_\epsilon$ is real-valued. Furthermore, by translating $\tilde{u}_\epsilon$ so that $\|\tilde{u}_\epsilon(\cdot - a) - u_\epsilon\|_{H^1_{P_\epsilon}} = \|\tilde{u}_\epsilon - u_\epsilon(\cdot + a)\|_{H^1_{P_\epsilon}}$ is minimized, equivalently

$$\frac{\partial}{\partial x_j}\bigg|_{a=0} \|\tilde{u}_\epsilon - u_\epsilon(\cdot + a)\|^2_{H^1_{P_\epsilon}} = 0,$$

we may assume that $\tilde{u}_\epsilon$ is orthogonal to $\partial_{x_j} u_\epsilon$ in $H^1_{P_\epsilon}$ for all $j = 1, 2, 3$. Now, we write the equation for the difference $r_\epsilon = \tilde{u}_\epsilon - u_\epsilon$,

$$\begin{aligned}(P_\epsilon + 1)r_\epsilon &= \mathcal{N}'(\tilde{u}_\epsilon) - \mathcal{N}'(u_\epsilon) = \mathcal{N}'(u_\epsilon + r_\epsilon) - \mathcal{N}'(u_\epsilon)\\ \Rightarrow \mathcal{L}^+_\epsilon r_\epsilon &= \mathcal{N}'(u_\epsilon + r_\epsilon) - \mathcal{N}'(u_\epsilon) - \mathcal{N}^+_{u_\epsilon}(r_\epsilon),\end{aligned}$$

where $\mathcal{N}^+_{u_\epsilon}$ is defined in (3.2). Since $r_\epsilon = \tilde{u}_\epsilon - u_\epsilon$ is orthogonal to $\partial_{x_j} u_\epsilon$ in $H^1_{P_\epsilon}$, by Proposition 3.3 and the nonlinear estimate (Lemma A.2) again, we obtain

$$\begin{aligned}\beta\|r_\epsilon\|_{H^1_{P_\epsilon}} &\le \|\mathcal{L}^+_\epsilon r_\epsilon\|_{H^{-1}_{P_\epsilon}} = \left\| \left(|x|^{-1} * r_\epsilon^2\right) u_\epsilon + \left(|x|^{-1} * \left(2u_\epsilon r_\epsilon + r_\epsilon^2\right)\right) r_\epsilon \right\|_{H^{-1}_{P_\epsilon}}\\ &\le \tilde{C}\left(\|u_\epsilon\|_{H^1_{P_\epsilon}} + \|r_\epsilon\|_{H^1_{P_\epsilon}}\right)\|r_\epsilon\|^2_{H^1_{P_\epsilon}}\\ &\le \tilde{C}\left(\|u_0\|_{H^1_{P_\epsilon}} + 2\delta\right)\delta\|r_\epsilon\|_{H^1_{P_\epsilon}}.\end{aligned}$$

Choosing even smaller $\delta > 0$ such that $\tilde{C}(\|u_0\|_{H^1_{P_\epsilon}} + 2\delta)\delta < \beta$ if necessary, we prove that $r_\epsilon = 0$. Therefore, we conclude that $\tilde{u}_\epsilon = u_\epsilon$ up to translation and phase shift. □

Now, we are ready to prove the main theorem.

Proof of Theorem 1.1 By Proposition 2.3, if $\epsilon > 0$ is small enough, then ground states Q^J_ϵ's are close to the reference ground state Q_0 in $H^1_{P_\epsilon}$ (modifying the sequence by translation and phase shift if necessary). However, by uniqueness in Proposition 5.1, Q^J_ϵ is identified with the radially symmetric real-valued solution u_ϵ, constructed in Proposition 4.1. Moreover, by Proposition 3.2, it is non-degenerate. Therefore, we prove the main theorem. □

6 Proof of Theorem 1.3

We proceed exactly as in the proof of Proposition 4.1. We denote by

$$r^J_c := Q^J_c - Q_c$$

the difference between two solutions. Then, it satisfies

$$\begin{aligned}(P^J_c + 1)r^J_c &= (P^J_c + 1)Q^J_c - (P^J_c + 1)Q_c\\ &= \left(P_c - P^J_c\right)Q_c + (P^J_c + 1)Q^J_c - (P_c + 1)Q_c\\ &= \left(P_c - P^J_c\right)Q_c + \mathcal{N}'(Q^J_c) - \mathcal{N}'(Q_c)\\ &= \left(P_c - P^J_c\right)Q_c + \mathcal{N}'(Q_c + r_c)' - \mathcal{N}'(Q_c),\end{aligned}$$

where $P_c = \sum_{j=1}^{J} \frac{(-1)^{j-1}\alpha_j}{m^{2j-1}c^{2j-2}}$ and $\alpha_j = \frac{(2j-2)!}{j!(j-1)!2^{2j-1}}$. Moving the linear terms on the right hand side to the left as above, we write

$$\mathcal{L}_c^{J,+} r_c^J = \left(P_c - P_c^J\right) Q_c + \mathcal{N}'(Q_c + r_c)' - \mathcal{N}'(Q_c) - \mathcal{N}_{Q_c}^{+}(r_c), \tag{6.1}$$

where

$$\mathcal{L}_c^{J,+} = P_c^J + 1 - \mathcal{N}_{Q_c}^{+} = \sqrt{1 + P_c^J}\left(\mathrm{Id} - \frac{1}{\sqrt{1+P_c^J}}\mathcal{N}_{Q_c}^{+}\frac{1}{\sqrt{1+P_c^J}}\right)\sqrt{1+P_c^J}.$$

Repeating the proof of Lemma 3.4 together with $Q_c \to Q_0$ in H^1 as $c \to \infty$, one can show that

$$\begin{aligned}
&\mathrm{Id} - \frac{1}{\sqrt{1+P_c^J}}\mathcal{N}_{Q_c}^{+}\frac{1}{\sqrt{1+P_c^J}} \\
&= \left(\mathrm{Id} - \frac{1}{\sqrt{1-\Delta}}\mathcal{N}_{Q_0}^{+}\frac{1}{\sqrt{1-\Delta}}\right) + \left(\frac{1}{\sqrt{1-\Delta}}\mathcal{N}_{Q_0}^{+}\frac{1}{\sqrt{1-\Delta}} - \frac{1}{\sqrt{1+P_c^J}}\mathcal{N}_{Q_c}^{+}\frac{1}{\sqrt{1+P_c^J}}\right)
\end{aligned}$$

is invertible on $L^2_{rad}(\mathbb{R}^d;\mathbb{R})$ and its inverse is uniformly bounded for sufficiently large $c \geq 1$. Hence, applying the trivial embedding $H^1_{P_c^J} \hookrightarrow H^1$ (from Lemma B.1) and its dual embedding, we obtain

$$\begin{aligned}
\|r_c^J\|_{H^1} \lesssim \|r_c^J\|_{H^1_{P_c^J}} &= \left\|(\mathcal{L}_c^{J,+})^{-1}\left\{\left(P_c - P_c^J\right)Q_c + \mathcal{N}'(Q_c+r_c)' - \mathcal{N}'(Q_c) - \mathcal{N}_{Q_c}^{+}(r_c)\right\}\right\|_{H^1_{P_c^J}} \\
&\lesssim \left\|\left(P_c - P_c^J\right)Q_c + \mathcal{N}'(Q_c+r_c)' - \mathcal{N}'(Q_c) - \mathcal{N}_{Q_c}^{+}(r_c)\right\|_{H^{-1}_{P_c^J}} \\
&\lesssim \left\|\left(P_c - P_c^J\right)Q_c\right\|_{H^{-1}} + \left\|\mathcal{N}'(Q_c+r_c)' - \mathcal{N}'(Q_c) - \mathcal{N}_{Q_c}^{+}(r_c)\right\|_{H^{-1}},
\end{aligned}$$

where the implicit constants do not depend on $c \geq 1$. Therefore, it follows from the nonlinear estimates (Lemma A.2) that for sufficiently large $c \geq 1$,

$$\|r_c^J\|_{H^1} \lesssim \left\|\left(P_c - P_c^J\right)Q_c\right\|_{H^{-1}} \lesssim \frac{1}{c^{2J}}\|Q_c\|_{H^{2J+1}},$$

because by Taylor's theorem,

$$\left|\left(\sqrt{c^2 s + m^2 c^4} - mc^2\right) - \sum_{j=1}^{J}\frac{(-1)^{j-1}\alpha_j}{m^{2j-1}c^{2j-2}}s^j\right| \lesssim \frac{s^{J+1}}{c^{2J}}.$$

Finally, by the uniform bound on high Sobolev norm of Q_c (see Proposition 2.5 or [2]), we conclude that $\|Q_c^J - Q_c\|_{H^1} = \|r_c^J\|_{H^1} \lesssim_J c^{-2J}$.

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Appendix A: Nonlinear estimates

We show the nonlinear estimates which are used in the contraction mapping argument.

Lemma A.1 (Nonlinear estimates) *Let* $u \in H^1$ *be real-valued. For any* $\eta > 0$*, there exists* $\delta_0 > 0$*, depending on* $\|u\|_{H^1(\mathbb{R}^d;\mathbb{R})}$ *and* η*, such that if* $0 < \delta \leq \delta_0$ *and*

$$\|r\|_{H^1(\mathbb{R}^d;\mathbb{R})}, \|\tilde{r}\|_{H^1(\mathbb{R}^d;\mathbb{R})} \leq \delta,$$

then

$$\left\|\mathcal{N}'(u+r) - \mathcal{N}'(u) - \mathcal{N}_u^+(r)\right\|_{L^2(\mathbb{R}^d;\mathbb{R})} \leq \eta\|r\|_{H^1(\mathbb{R}^d;\mathbb{R})}$$

and

$$\left\|\left(\mathcal{N}'(u+r) - \mathcal{N}'(u) - \mathcal{N}_u^+(r)\right) - \left(\mathcal{N}'(u+\tilde{r}) - \mathcal{N}'(u) - \mathcal{N}_u^+(\tilde{r})\right)\right\|_{L^2(\mathbb{R}^d;\mathbb{R})} \leq \eta\|r-\tilde{r}\|_{H^1(\mathbb{R}^d;\mathbb{R})}.$$

The above lemma follows from the multilinear estimates.

Lemma A.2 (Multilinear estimates) *We have*

$$\left\|\left(\frac{1}{|x|} * (\phi_1\phi_2)\right)\phi_3\right\|_{L^2(\mathbb{R}^3;\mathbb{R})} \lesssim \prod_{j=1}^{3} \|\phi_j\|_{H^1(\mathbb{R}^3;\mathbb{R})}.$$

Moreover, if $d = 1, 2$ *and* $k \in \mathbb{N}$ *or if* $d = 3$ *and* $k = 1$*, then*

$$\left\|\prod_{j=1}^{2k+1} \phi_j\right\|_{L^2(\mathbb{R}^d;\mathbb{R})} \lesssim \prod_{j=1}^{2k+1} \|\phi_j\|_{H^1(\mathbb{R}^d;\mathbb{R})}.$$

Proof By the Hölder, Young's and Sobolev inequalities, we prove that

$$\begin{aligned}
\left\|\left(\frac{1}{|x|} * (\phi_1\phi_2)\right)\phi_3\right\|_{L^2(\mathbb{R}^3;\mathbb{R})} &\leq \left\|\left(\frac{1}{|x|} * (\phi_1\phi_2)\right)\right\|_{L^9(\mathbb{R}^3;\mathbb{R})} \|\phi_3\|_{L^{18/7}(\mathbb{R}^3;\mathbb{R})} \\
&\lesssim \|\phi_1\phi_2\|_{L^{9/7}(\mathbb{R}^3;\mathbb{R})} \|\phi_3\|_{L^{18/7}(\mathbb{R}^3;\mathbb{R})} \\
&\lesssim \prod_{j=1}^{3} \|\phi_j\|_{L^{18/7}(\mathbb{R}^3;\mathbb{R})} \lesssim \prod_{j=1}^{3} \|\phi_j\|_{H^1(\mathbb{R}^3;\mathbb{R})}
\end{aligned}$$

and similarly,

$$\left\|\prod_{j=1}^{2k+1} \phi_j\right\|_{L^2(\mathbb{R}^d;\mathbb{R})} \leq \prod_{j=1}^{2k+1} \|\phi_j\|_{L^{2(2k+1)}(\mathbb{R}^d;\mathbb{R})} \lesssim \prod_{j=1}^{2k+1} \|\phi_j\|_{H^1(\mathbb{R}^d;\mathbb{R})}.$$

Proof of Lemma A.1 Suppose that $\|r\|_{H^1}, \|\tilde{r}\|_{H^1} \leq \|u\|_{H^1}$. For the Hartree nonlinearity, by algebra, we write

$$\mathcal{N}'(u+r) - \mathcal{N}'(u) - \mathcal{N}_u^+(r) = \left(\frac{1}{|x|} * r^2\right)u + 2\left(\frac{1}{|x|} * (ur)\right)r + \left(\frac{1}{|x|} * r^2\right)r$$

and

$$\left(\mathcal{N}'(u+r) - \mathcal{N}'(u) - \mathcal{N}_u^+(r)\right) - \left(\mathcal{N}'(u+\tilde{r}) - \mathcal{N}'(u) - \mathcal{N}_u^+(\tilde{r})\right)$$

$$= \left(\frac{1}{|x|} * ((r+\tilde{r})(r-\tilde{r}))\right) u + 2\left(\frac{1}{|x|} * (u(r-\tilde{r}))\right) r + 2\left(\frac{1}{|x|} * (u\tilde{r})\right)(r-\tilde{r})$$
$$+ \left(\frac{1}{|x|} * ((r+\tilde{r})(r-\tilde{r}))\right) r + \left(\frac{1}{|x|} * \tilde{r}^2\right)(r-\tilde{r})$$

Thus, by the multilinear estimate (Lemma A.2),

$$\left\|\mathcal{N}'(u+r) - \mathcal{N}'(u) - \mathcal{N}_u^+(r)\right\|_{L^2} \leq C\Big(\|u\|_{H^1} + \|r\|_{H^1}\Big)\|r\|_{H^1}^2 \leq 2C\delta\|u\|_{H^1}\|r\|_{H^1}$$

and

$$\begin{aligned}
&\left\|\big(\mathcal{N}'(u+r) - \mathcal{N}'(u) - \mathcal{N}_u^+(r)\big) - \big(\mathcal{N}'(u+\tilde{r}) - \mathcal{N}'(u) - \mathcal{N}_u^+(\tilde{r})\big)\right\|_{L^2} \\
&\leq C\Big(\|u\|_{H^1} + \|r\|_{H^1} + \|\tilde{r}\|_{H^1}\Big)\Big(\|r\|_{H^1} + \|\tilde{r}\|_{H^1}\Big)\|r-\tilde{r}\|_{H^1} \\
&\leq 6C\delta\|u\|_{H^1}\|r-\tilde{r}\|_{H^1}.
\end{aligned}$$

Then, taking $\delta_0 = \eta \min\{\frac{1}{6C\|u\|_{H^1}}, \|u\|_{H^1}\}$, we prove the lemma for the Hartree nonlinearity.

Similarly, for the polynomial nonlinearity, by the multilinear estimate (Lemma A.2),

$$\begin{aligned}
\left\|\mathcal{N}'(u+r) - \mathcal{N}'(u) - \mathcal{N}_u^+(r)\right\|_{L^2} &= \left\|\sum_{j=2}^{2k+1}\binom{2k+1}{j} u^{2k+1-j} r^j\right\|_{L^2} \\
&\leq \sum_{j=2}^{2k+1}\binom{2k+1}{j}\left\|u^{2k+1-j} r^j\right\|_{L^2} \\
&\leq C\sum_{j=2}^{2k+1}\binom{2k+1}{j}\|u\|_{H^1}^{2k+1-j}\|r\|_{H^1}^j \\
&\leq C_k\delta\|u\|_{H^1}^{2k-1}\|r\|_{H^1}
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\big(\mathcal{N}'(u+r) - \mathcal{N}'(u) - \mathcal{N}_u^+(r)\big) - \big(\mathcal{N}'(u+\tilde{r}) - \mathcal{N}'(u) - \mathcal{N}_u^+(\tilde{r})\big)\right\|_{L^2} \\
&= \left\|\sum_{j=3}^{2k+1}\binom{2k+1}{j} u^{2k+1-j}(r-\tilde{r})(r^{j-1} + r^{j-2}\tilde{r} + \cdots + \tilde{r}^{j-1})\right\|_{L^2} \\
&\leq \sum_{j=3}^{2k+1}\binom{2k+1}{j}\left\|u^{2k+1-j}(r-\tilde{r})(r^{j-1} + r^{j-2}\tilde{r} + \cdots + \tilde{r}^{j-1})\right\|_{L^2} \\
&\leq C\sum_{j=3}^{2k+1}\binom{2k+1}{j}\|u\|_{H^1}^{2k+1-j}\|r-\tilde{r}\|_{H^1}\left(\|r\|_{H^1}^{j-1} + \|r\|_{H^1}^{j-2}\|\tilde{r}\|_{H^1} + \cdots + \|\tilde{r}\|_{H^1}^{j-1}\right) \\
&\leq C_k\delta\|u\|_{H^1}^{2k-1}\|r-\tilde{r}\|_{H^1}
\end{aligned}$$

for some constant $C_k > 0$. Then, taking $\delta_0 = \eta \min\{\frac{1}{2C_k\|u\|_{H^1}^{2k-1}}, \|u\|_{H^1}\}$, we complete the proof of the lemma for the polynomial nonlinearity. □

Appendix B: Uniform lower bound for higher-order operators in (1.3)

Lemma B.1 (Uniform lower bound for higher-order operators in (1.3)) *For any* $\xi \in \mathbb{R}^3$, *we have*

$$\sum_{j=1}^{2k-1} \frac{(-1)^{j-1}\alpha_j}{m^{2j-1}c^{2j-2}}|\xi|^{2j} \geq \frac{|\xi|^2}{2m},$$

where $a_j = \frac{(2j-2)!}{j!(j-1)!2^{2j-1}}$.

Proof By change of variables $\frac{\xi}{m} \mapsto \xi$, it suffices to prove the lemma assuming $m = 1$. The inequality is trivial when $k = 1$. Suppose that $k \geq 2$. Splitting the positive and the negative terms and then applying the Cauchy–Schwarz inequality for the negative terms, we obtain

$$\begin{aligned}\sum_{j=1}^{2k-1} \frac{(-1)^{j-1}\alpha_j}{c^{2j-2}}|\xi|^{2j} &= \sum_{j=1}^{k} \frac{\alpha_{2j-1}}{c^{4j-4}}|\xi|^{4j-2} - \sum_{j=1}^{k-1} \frac{\alpha_{2j}}{c^{4j-2}}|\xi|^{4j}\\ &\geq \sum_{j=1}^{k} \frac{\alpha_{2j-1}}{c^{4j-4}}|\xi|^{4j-2}\\ &\quad - \frac{1}{2}\sum_{j=1}^{k-1}\left\{\frac{(\alpha_{2j})^2}{\alpha_{2j-1}\alpha_{2j+1}}\frac{\alpha_{2j-1}}{c^{4j-4}}|\xi|^{4j-2} + \frac{\alpha_{2j+1}}{c^{4j}}|\xi|^{4j+2}\right\}.\end{aligned}$$

Since $\frac{(\alpha_{2j})^2}{\alpha_{2j-1}\alpha_{2j+1}} = \cdots = \frac{(4j-3)(2j+1)}{(4j-1)2j} \leq 1$ for all $j \geq 1$, it is bounded below from

$$\begin{aligned}&\sum_{j=1}^{k} \frac{\alpha_{2j-1}}{c^{4j-4}}|\xi|^{4j-2} - \frac{1}{2}\sum_{j=1}^{k-1}\left\{\frac{\alpha_{2j-1}}{c^{4j-4}}|\xi|^{4j-2} + \frac{\alpha_{2j+1}}{c^{4j}}|\xi|^{4j+2}\right\}\\ &= \sum_{j=1}^{k} \frac{\alpha_{2j-1}}{c^{4j-4}}|\xi|^{4j-2} - \frac{1}{2}\sum_{j=1}^{k-1} \frac{\alpha_{2j-1}}{c^{4j-4}}|\xi|^{4j-2} - \frac{1}{2}\sum_{j=2}^{k} \frac{\alpha_{2j-1}}{c^{4j-4}}|\xi|^{4j-2}\\ &= \frac{1}{2}|\xi|^2 + \frac{\alpha_{2k-1}}{2c^{4k-4}}|\xi|^{4k-2} \geq \frac{1}{2}|\xi|^2.\end{aligned}$$

□

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