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Calculus of Variations

Uniqueness of self-shrinkers to the degree-one curvature flow with a tangent cone at infinity

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Abstract Given a smooth, symmetric and homogeneous of degree one function $f(\lambda_1, \ldots, \lambda_n)$ satisfying $\partial_i f > 0 \quad \forall i = 1, \ldots, n$, and a properly embedded smooth cone $\mathcal{C}$ in $\mathbb{R}^{n+1}$, we show that under suitable conditions on f, there is at most one f self-shrinker (i.e. a hypersurface Σ in $\mathbb{R}^{n+1}$ satisfying $f(\kappa_1, \ldots, \kappa_n) + \frac{1}{2} X \cdot N = 0$, where $\kappa_1, \ldots, \kappa_n$ are principal curvatures of Σ) that is asymptotic to the given cone $\mathcal{C}$ at infinity.

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1 Introduction

Let $\mathcal{C}$ be an orientable and properly embedded smooth cone (excluding the vertex O) in $\mathbb{R}^{n+1}$. Suppose that $\mathbf{\Sigma}$ is an orientable and properly embedded smooth hypersurface in $\mathbb{R}^{n+1}$ which satisfies

$$H + \frac{1}{2} X \cdot N = 0 \quad \forall X \in \mathbf{\Sigma}$$

$$\varrho \mathbf{\Sigma} \xrightarrow{C^\infty_{\mathrm{loc}}} \mathcal{C} \quad \text{as } \varrho \searrow 0$$

where N is the unit-normal and $H = -\nabla_{\mathbf{\Sigma}} \cdot N$ is the mean curvature of $\mathbf{\Sigma}$. Then $\mathbf{\Sigma}$ is called a self-shrinker to the mean curvature flow (i.e. $\partial_t X^\perp = HN$) which is smoothly asymptotic to the cone $\mathcal{C}$ at infinity. It follows that the rescaled family of hypersufaces $\left\{\mathbf{\Sigma}_t = \sqrt{-t}\,\mathbf{\Sigma}\right\}$ forms a mean curvature flow starting from $\mathbf{\Sigma}$ (when $t = -1$) and converging locally smoothly to $\mathcal{C}$ as $t \nearrow 0$. Wang in [12] proves the uniqueness of such self-shrinkers by showing that: suppose $\tilde{\mathbf{\Sigma}}$ is also a self-shrinker which is asymptotic to the same cone, then outside a compact set, $\tilde{\mathbf{\Sigma}}_t = \sqrt{-t}\,\tilde{\mathbf{\Sigma}}$ can be regarded as a normal graph of h_t defined on $\mathbf{\Sigma}_t \backslash \bar{B}_R$ for some $R > 0$; moreover, given $\varepsilon > 0$ and choose R large accordingly, there holds

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$$\left|\partial_t \mathrm{h} - \triangle_{\mathbf{\Sigma}_t} \mathrm{h}\right| \leq \varepsilon \left(|\nabla_{\mathbf{\Sigma}_t}| + |\mathrm{h}|\right)$$
$$\mathrm{h}\Big|_{t=0} = 0$$

Using the idea in [6], Wang derives a Carleman's inequality for the heat operator on the flow $\{\mathbf{\Sigma}_t\}$, apply it to the localization of h, and use the unique continuation principle (see [4], for instance) to conclude that $\mathrm{h} = 0$.

On the other hand, Andrews [1] consider the motion of hypersurfaces in $\mathbb{R}^{n+1}$ moved by some degree one curvature (see also [2,3]). More precisely, given a smooth, symmetric and homogeneous of degree-one function $f = f(\lambda_1, \ldots, \lambda_n)$ which satisfies $\partial_i f > 0 \quad \forall\, i$, consider the following evolution of hypersurfaces:

$$\partial_t X^{\perp} = f(\kappa_1, \ldots, \kappa_n)\, N$$

where $\kappa_1, \ldots, \kappa_n$ are the principal curvatures of the evolving hypersurface. For instance, if we take the curvature function to be $f(\lambda_1, \ldots, \lambda_n) = \lambda_1 + \cdots + \lambda_n$, then this corresponds to the mean curvature flow. And we call an orientable C^2 hypersurface Σ in $\mathbb{R}^{n+1}$to be a "f self-shrinkers" to the above "f curvature flow" provided that

$$f(\kappa_1, \ldots, \kappa_n) + \frac{1}{2} X \cdot N = 0$$

holds on Σ. Likewise, the rescaled family of "f self-shrinkers" is a self-similar solution to the f curvature flow; that is, the one-parameter family of hypersurfaces $\left\{\Sigma_t = \sqrt{-t}\,\Sigma\right\}_{t<0}$ is a f curvature flow. In the case when Σ is smoothly asymptotic to the cone $\mathcal{C}$ at infinity, the rescaled flow $\{\Sigma_t\}_{t<0}$ will converge locally smoothly to $\mathcal{C}$ as $t \nearrow 0$.

This paper is an extension of the uniqueness result of [12] and existence result of [10] to the class of f self-shrinkers with a tangent cone $\mathcal{C}$ at infinity. In fact, Wang's idea (of proving the uniqueness for the mean curvature flow) work perfectly for the f curvature flow as well, except that some additional treatment for the nonlinearity of f is required (which is not a concern in Wang's case because the curvature function there is linear). The crucial step is to derive Carleman's inequality for the associated parabolic operator to the f curvature flow under some conditions on the nonlinearity of f , the uniform positivity of $\partial_i f$ and also some curvature bounds of $\mathcal{C}$. For this part, we are motivated by the work of Nguyen [11] and Wu and Zhang [13] for deriving Carleman's inequality for parabolic operator with variable coefficients.

In order to state our main theorem, we have to first introduce some notaions and definitions regarding the f self-shrinkers, the tangent cone of a hypersurface at infinity, and also some basic assumptions on the curvature function f. We put all of these in Sect. 2.

In Sect. 3, we essentially follow [12] to show that if Σ and $\tilde{\Sigma}$ are f self-shrinker which are asymptotic to the given cone $\mathcal{C}$ at infinity, then outside a compact set, $\tilde{\Sigma}_t = \sqrt{-t}\,\tilde{\Sigma}$ can be regarded as a normal graph of h_t defined on $\Sigma_t \backslash \bar{B}_R$ for some $R > 0$, which satisfies some parabolic equation and vanishes at time 0. We also give some estimates on the coefficients of the parabolic operators.

In Sect. 4, we follow the idea of [6] for treating the backward uniqueness of the heat equation (which is also used in [12] to deal with the uniqueness of self-shrinkers of the mean curvature flow) to show that the deviation h_t vanishes outside some compact set. We first use the mean value inequality for parabolic equations and a local type of Carleman's inequalities to show the exponential decay of the deviation h_t as $t \nearrow 0$ as in [11]. Then we are devoted to derive a different type of Carleman's inequalities (based on the estimates of the coefficients of the parabolic operator which we derive in Sect. 3) and use it to show that h_t vanishes outside a compact set. In the end, we use the unique continuation principle to characterize the overlap region of Σ and $\tilde{\Sigma}$.

2 Assumptions and main results

Definition 2.1 (*A regular cone*) Let $\mathcal{C}$ be an orientable and properly embedded smooth cone (excluding the vertex O) in $\mathbb{R}^{n+1}$; that is, $\mathcal{C}$ is an orientable and properly embedded hypersurface in $\mathbb{R}^{n+1}$ satisfying $\varrho\,\mathcal{C} = \mathcal{C} \quad \forall\,\varrho \in \mathbb{R}_+$ and $O \notin \mathcal{C}$.

We then define what it means for a hypersurface to be asymptotic to the cone $\mathcal{C}$ at infinity:

Definition 2.2 (*Tangent cone at infinity*) A C^k hypersurface Σ in $\mathbb{R}^{n+1}$ (with $k \in \mathbb{N}$) is said to be C^k asymptotic to $\mathcal{C}$ at infinity provided that $\varrho\Sigma \overset{C^k_{\text{loc}}}{\longrightarrow} \mathcal{C}$ as $\varrho \searrow 0$ (see [8] for the C^k topology of hypersurfaces in $\mathbb{R}^{n+1}$). In this case, $\mathcal{C}$ is called the tangent cone of Σ at infinity.

For a given C^2 orientable hypersurface Σ in $\mathbb{R}^{n+1}$, its shape operator (or Weingarten map) $A^{\#}$ sends tangent vectors to tangent vectors and is defined by

$$A^{\#}(V) = -D_V N$$

for any tangent vector field V on Σ, where N is the unit-normal of Σ. The second fundamental form A is defined to be a 2 tensor on Σ such that

$$A(V, W) = A^{\#}(V) \cdot W$$

for any tangent vector fields V and W on Σ. The components of $A^{\#}$ and A with respect to a given local frame $\{e_1, \ldots, e_n\}$ of the tangent bundle of Σ are defined by

$$A^{\#}(e_i) = A_i^j e_j, \quad A\left(e_i, e_j\right) = A_{ij}$$

and we are used to denote $A^{\#}$ and A by their components like $A^{\#} \sim A_i^j$ and $A \sim A_{ij}$. Note that $A^{\#}$ is a self-adjoint operator with repect to the dot product restricted to the tangent space (or equivalently, A is a symmetric 2 tensor), so $A^{\#}$ is diagonalizable. The eigenvectors of $A^{\#}$ are called pricipal vectors and its eigenvalues are called principal curvatures, which are denoted by $\kappa_1, \ldots, \kappa_n$. The mean curvature is defined to be $H = \operatorname{tr}\left(A^{\#}\right) = \kappa_1 + \cdots + \kappa_n$, which is a linear, symmetric and homogeneous of degree-one function of the shape operator (or the principal curvatures). In this paper, we consider a more general type of degree-one curvature.

Definition 2.3 (*The degree-one curvature function*) Let $F = F(S)$ be a conjugation-invariant, homogeneous of degree-one function whose domain $\mathbf{\Omega}$ (in the space of $n \times n$ matrices) containing a neighborhood of the set consisting of all the values of shape operator $A^{\#}_{\mathcal{C}}$ of $\mathcal{C}$; besides, F can be written as a C^3 function composed with the the elementary symmetric functions $\mathcal{E}_1, \ldots, \mathcal{E}_n$ (for instance, $\mathcal{E}_1 = \operatorname{tr}$ and $\mathcal{E}_n = \det$) and $\frac{\partial F}{\partial S_i^j} > 0$ (i.e. $\frac{\partial F}{\partial S_i^j}$ is a positive matrix).

Note that by the conjugation-invariant and homogeneous property of F, we may assume that $\mathbf{\Omega}$ is closed under conjugation and homothety; that is, if $S \in \mathbf{\Omega}$, then so are RSR^{-1} and ϱS for any invetible $n \times n$ matrix R and positive number ϱ.

Also, by the condition that F can be written as a C^3 function composed with the the elementary symmetric functions, it induces a symmetric, homogeneous of degree-one function f such that

$$F(S) = f(\lambda_1, \ldots, \lambda_n)$$

whenever $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the matrix S; the function f is defined and C^3 on an open set $\mho$ (in $\mathbb{R}^n$) containing a neighborhood of the set consisting of all the values of the pricinpal curvature vector $\left(\kappa_1^{\mathcal{C}}, \ldots, \kappa_n^{\mathcal{C}}\right)$ of $\mathcal{C}$. Likewise, we may assume that the domain $\mho$ is closed under permutation and homothety.

In fact, at a diagonal matrix $S = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, there holds (see [1]):

$$\frac{\partial F}{\partial S_i^j}(S) = \partial_i f(\lambda_1, \ldots, \lambda_n)\,\delta_{ij} \tag{2.1}$$

$$\frac{\partial^2 F}{\partial S_i^j \partial S_i^l}(S) = \partial_{ii}^2 f(\lambda_1, \ldots, \lambda_n)\,\delta_{ij}\delta_{il} \tag{2.2}$$

$$\frac{\partial^2 F}{\partial S_i^j \partial S_k^l}(S) = \partial_{ik}^2 f(\lambda_1, \ldots, \lambda_n)\,\delta_{ij}\delta_{kl} + \frac{\partial_i f - \partial_k f}{\lambda_i - \lambda_k}\delta_{il}\delta_{kj} \quad \text{if } i \neq k \tag{2.3}$$

Since F is well-defined on conjugacy classes, (2.1), (2.2), (2.3) can be applied to any diagonalizable matrix in $\mathbf{\Omega}$. For instance, by (2.1), we have

$$\frac{\partial F}{\partial S_i^j}\left(A_{\mathcal{C}}^{\#}\right) \sim \partial_i f\left(\kappa_1^{\mathcal{C}}, \ldots, \kappa_n^{\mathcal{C}}\right)\delta_{ij}$$

where $A_{\mathcal{C}}^{\#} \sim \kappa_{\mathcal{C}}^i \delta_{ij}$ is the shape operator (and principal curvatures) of $\mathcal{C}$. Hence, by the condition that $\frac{\partial F}{\partial S_i^j} > 0$ on $\mathbf{\Omega}$, we may assume that $\partial_i f > 0 \quad \forall i = 1, \ldots, n$ on $\mho$.

Now let U be an open neighborhood of the set consisting of the all the shape operator $A_{\mathcal{C}}^{\#}$ of $\mathcal{C}$ at $X_{\mathcal{C}} \in \mathcal{C} \cap \left(B_3 \setminus \bar{B}_{\frac{1}{3}}\right)$ in $\mathbf{\Omega}$. Note that we may assume that U is closed under conjugation and that $\frac{\partial F}{\partial S_i^j}$ is uniformly positive on U; that is, there exist a constant $\lambda \in (0, 1]$ so that

$$\lambda \delta_j^i \leq \frac{\partial F}{\partial S_i^j} \leq \frac{1}{\lambda}\delta_j^i \tag{2.4}$$

Also, we have

$$\begin{aligned} \varkappa &\equiv \sup_{X_{\mathcal{C}} \in \mathcal{C} \cap \left(B_3 \setminus \bar{B}_{\frac{1}{3}}\right)} \left| \nabla_{\mathcal{C}} \left(\frac{\partial F}{\partial S_i^j}\left(A_{\mathcal{C}}^{\#}\right)\right) \right| \\ &= \sup_{X_{\mathcal{C}} \in \mathcal{C} \cap \left(B_3 \setminus \bar{B}_{\frac{1}{3}}\right)} \left| \sum_{k,l} \frac{\partial^2 F}{\partial S_i^j \partial S_k^l}\left(A_{\mathcal{C}}^{\#}\right)\left(\nabla_{\mathcal{C}} A_{\mathcal{C}}^{\#}\right)_k^l \right| \leq C\left(n, \mathcal{C}, \| F \|_{C^2(U)}\right) \end{aligned} \tag{2.5}$$

where $A_{\mathcal{C}}^{\#}$ and $\nabla_{\mathcal{C}} A_{\mathcal{C}}^{\#}$ are the shape operator of $\mathcal{C}$ and its covariant derivative at $X_{\mathcal{C}}$, respectively; $B_\varrho = B_\varrho^{n+1}$ is the ball of radius ϱ in $\mathbb{R}^{n+1}$. A more precise estimate of $\varkappa$ is given (see 4.97) in the case when $\mathcal{C}$ is rotationally symmetric.

Now let's define the F self-shrinker (or f self-shrinker):

Definition 2.4 (*F self-shrinker*) An oriented C^2 hypersurface Σ (excluding its boundary) in $\mathbb{R}^{n+1}$ is called a F self-shrinker (or f self-shrinker) provided that F is defined on the shape operator $A^{\#}$ of Σ (i.e. $A^{\#} \in \mathbf{\Omega}$) and satisfies

$$F\left(A^{\#}\right) + \frac{1}{2} X \cdot N = 0$$

where X is the position vector, N is the unit-normal, and $A^{\#}$ is the shape operator of Σ; or equivalently, f is defined on the principal curvatures of Σ (i.e. $(\kappa_1, \ldots, \kappa_n) \in \mho$) and satisfies

$$f(\kappa_1, \ldots, \kappa_n) + \frac{1}{2} X \cdot N = 0$$

where $\kappa_1, \ldots, \kappa_n$ are the principal curvatures of Σ.

Note that the rescaled family of F self-shrinkers forms a self-similar solution to the F curvature flow. More precisely, the one-parameter family $\left\{\Sigma_t = \sqrt{-t}\,\Sigma\right\}_{-1 \leqslant t < 0}$ is a motion of a hypersurface moved by F curvature vector. That is,

$$\partial_t X^{\perp} = F\left(A^{\#}\right) N$$

where $\partial_t X^{\perp}$ is the normal projection of $\partial_t X$. Besides, for each time slice $\Sigma_t = \sqrt{-t}\,\Sigma$, there holds

$$F\left(A^{\#}\right) + \frac{X \cdot N}{2(-t)} = 0$$

We will prove the following uniqueness result F self-shrinkers with a tangent cone in Sect. 4:

Theorem 2.1 (Uniqueness of self-shrikers with a conical end) *Assume that $\varkappa \leq 6^{-4} \lambda^3$ [in (2.4), (2.5)]. Then for any properly embedded F self-shrinkers Σ and $\tilde{\Sigma}$ which are C^5 asymptotic to the cone $\mathcal{C}$ at infinity, there exists $R = R\left(\Sigma, \tilde{\Sigma}, \mathcal{C}, U, \| F \|_{C^3(U)}, \lambda, \varkappa\right) \geq 1$ so that $\Sigma \backslash B_R = \tilde{\Sigma} \backslash B_R$. Moreover, let*

$$\Sigma^0 = \left\{X \in \Sigma \cap \tilde{\Sigma} \,\middle|\, \Sigma \text{ coincides with } \tilde{\Sigma} \text{ in a neighborhood of } X\right\}$$

then Σ^0 is a nonempty hypersurface, which satisfies $\partial \Sigma^0 \subseteq \left(\partial \Sigma \cup \partial \tilde{\Sigma}\right)$.

Remark 2.1 In the case of [12], $F = \mathcal{E}_1$ (or equivalentsly, $f(\lambda_1, \ldots, \lambda_n) = \lambda_1 + \cdots + \lambda_n$) is a linear function, so [by (2.5), (2.2), (2.3)] $\varkappa \equiv 0$ and the hypothesis of Theorem 2.1 is trivially satisfied. On the other hand, consider

$$F = \mathcal{E}_1 \pm \epsilon \frac{\mathcal{E}_n}{\mathcal{E}_{n-1}}$$

or equivalently,

$$f(\lambda_1, \ldots, \lambda_n) = (\lambda_1 + \cdots + \lambda_n) \pm \epsilon \frac{\prod_{i=1}^n \lambda_i}{\sum_{i=1}^n \left(\prod_{j \neq i} \lambda_j\right)}$$

and take $\mathcal{C}$ to be a rotationally symmetric cone. Then by Theorem 2.1 and (4.97) in the last section, the uniqueness holds when $0 < \epsilon \ll 1$.

In the last section, we assume $\mathcal{C}$ to be rotationally symmetric, say

$$\mathcal{C} = \left\{ (\sigma s\, \nu, s) \,\middle|\, \nu \in \mathbf{S}^{n-1}, s \in \mathbb{R}_+ \right\}$$

for some constant $\sigma > 0$, where $\mathbf{S}^{n-1}$ is the unit-sphere in $\mathbb{R}^n$. In this case, its principal curvatures at each point are

$$\kappa_1^{\mathcal{C}} = \cdots = \kappa_{n-1}^{\mathcal{C}} = \frac{1}{\sigma s \sqrt{1+\sigma^2}}, \quad \kappa_n^{\mathcal{C}} = 0$$

Therefore, the condition that the curvature function f is defined on a neighborhood of the set consisting of all the values of the pricinpal curvature vector $\left(\kappa_1^{\mathcal{C}}, \ldots, \kappa_n^{\mathcal{C}}\right)$ of $\mathcal{C}$ in Definition 3 is equivalent to requiring its domain $\mho$ to contain a neighborhood of $\left(\overrightarrow{1}, 0\right) = (1, \ldots, 1, 0) \in \mathbb{R}^n$, since $\mho$ is closed under permutation and homothety.

3 Deviation between two *F* self-shrinkers with the same asymptotic behaviour at infinity

Let Σ be a properly embedded F self-shrinker (in Definition 2.4) which is C^5 asymptotic to the cone $\mathcal{C}$ at infinity.

By Definition 2.2, $\varrho\Sigma$ can be arbitrary C^5 close to $\mathcal{C}$ on any fixed bounded set of $\mathbb{R}^{n+1}$ which is away from the origin (e.g. on $B_2 \backslash \bar{B}_{\frac{1}{2}}$) as long as ϱ is sufficiently small, so any "rescaled C^5 quantities" of $\Sigma\backslash\bar{B}_R$ can estimated by that of $\mathcal{C}$ for $R \gg 1$. Below we will show these in detail.

First of all, there exists $R \gg 1$ (depending on $\Sigma, \mathcal{C}$) such that outside a compact set, Σ is a normal graph over $\mathcal{C}\backslash\bar{B}_R$, say $X = \boldsymbol{\Psi}(X_{\mathcal{C}}) = X_{\mathcal{C}} + \boldsymbol{\psi} N_{\mathcal{C}}$, where $X_{\mathcal{C}}$ is the position vector of $\mathcal{C}$ and $N_{\mathcal{C}}$ is the unit-normal of $\mathcal{C}$ at $X_{\mathcal{C}}$. Consequently, we can define the "normal projecton" Π (to be the inverse map of $\boldsymbol{\Psi}$) which sends $X \in \Sigma$ to $X_{\mathcal{C}} \in \mathcal{C}$. Moreover, by the rescaling argument, we may assume that $\mathcal{H}^n\left(\Sigma \cap \left(B_{2\mathrm{r}} \backslash \bar{B}_{\mathrm{r}}\right)\right) \leq C(n, \mathcal{C})\, \mathrm{r}^n$ for all $\mathrm{r} \geq R$ (i.e. Σ has polynomial volume growth).

On the other hand, fix $\hat{X}_{\mathcal{C}} \in \mathcal{C}\backslash\bar{B}_R$, $|\hat{X}_{\mathcal{C}}|^{-1}\mathcal{C} = \mathcal{C}$ is locally (near $|\hat{X}_{\mathcal{C}}|^{-1}\hat{X}_{\mathcal{C}}$) a graph over the tangent hyperplanethe $T_{|\hat{X}_{\mathcal{C}}|^{-1}\hat{X}_{\mathcal{C}}}\mathcal{C}$, so by Definition 2.2, $|\hat{X}_{\mathcal{C}}|^{-1}\Sigma$ must also be a local graph over $T_{|\hat{X}_{\mathcal{C}}|^{-1}\hat{X}_{\mathcal{C}}}\mathcal{C}$ and is C^5 close to the corresponding graph of $|\hat{X}_{\mathcal{C}}|^{-1}\mathcal{C} = \mathcal{C}$. Furthermore, we may choose a uniform constant $\rho \in (0, \frac{1}{8}]$ (depending on the dimension n, the volume and the C^3 bound of the curvature of $\mathcal{C} \cap \left(B_3 \backslash \bar{B}_{\frac{1}{3}}\right)$) so that near $|\hat{X}_{\mathcal{C}}|^{-1}\hat{X}_{\mathcal{C}}$, the graphes of $|\hat{X}_{\mathcal{C}}|^{-1}\mathcal{C} = \mathcal{C}$ and $|\hat{X}_{\mathcal{C}}|^{-1}\Sigma$ are defined on $B^n_{\rho|\hat{X}_{\mathcal{C}}|} = \left\{x \in \mathbb{R}^n \,\middle|\, |x| < \rho|\hat{X}_{\mathcal{C}}|\right\} \subset T_{|\hat{X}_{\mathcal{C}}|^{-1}\hat{X}_{\mathcal{C}}}\mathcal{C}$ and the C^1 norm of the local graph of $\mathcal{C}$ is small. By undoing the rescaling, it tranlates into the following: there exists $R = R(\Sigma, \mathcal{C}) \geq 1$ so that near each $\hat{X}_{\mathcal{C}} \in \mathcal{C}\backslash\bar{B}_R$, $\mathcal{C}$ and Σ can be respectively parameterized by

$$X_{\mathcal{C}} = X_{\mathcal{C}}(x) \equiv \hat{X}_{\mathcal{C}} + (x, \mathrm{w}(x))$$
$$X = X(x) \equiv \hat{X}_{\mathcal{C}} + (x, \mathrm{u}(x))$$

for $x = (x_1, \ldots, x_n) \in B^n_{\rho|\hat{X}_\mathcal{C}|}$, such that $\mathrm{w}(0) = 0$, $\partial_x \mathrm{w}(0) = 0$ and

$$|\hat{X}_\mathcal{C}|^{-1} \parallel \mathrm{w} \parallel_{L^\infty(B^n_{\rho|\hat{X}_\mathcal{C}|})} + \parallel \partial_x \mathrm{w} \parallel_{L^\infty(B^n_{\rho|\hat{X}_\mathcal{C}|})} \leq \frac{1}{16} \tag{3.1}$$

$$|\hat{X}_\mathcal{C}| \parallel \partial_x^2 \mathrm{w} \parallel_{L^\infty(B^n_{\rho|\hat{X}_\mathcal{C}|})} + \cdots + |\hat{X}_\mathcal{C}|^4 \parallel \partial_x^5 \mathrm{w} \parallel_{L^\infty(B^n_{\rho|\hat{X}_\mathcal{C}|})} \leq C(n, \mathcal{C}) \tag{3.2}$$

$$\begin{aligned} &|\hat{X}_\mathcal{C}|^{-1} \parallel \mathrm{u} - \mathrm{w} \parallel_{L^\infty(B^n_{\rho|\hat{X}_\mathcal{C}|})} + \parallel \partial_x \mathrm{u} - \partial_x \mathrm{w} \parallel_{L^\infty(B^n_{\rho|\hat{X}_\mathcal{C}|})} \\ &\quad + |\hat{X}_\mathcal{C}| \parallel \partial_x^2 \mathrm{u} - \partial_x^2 \mathrm{w} \parallel_{L^\infty(B^n_{\rho|\hat{X}_\mathcal{C}|})} + \cdots + |\hat{X}_\mathcal{C}|^4 \parallel \partial_x^5 \mathrm{u} - \partial_x^5 \mathrm{w} \parallel_{L^\infty(B^n_{\rho|\hat{X}_\mathcal{C}|})} \leq \frac{1}{16} \end{aligned} \tag{3.3}$$

where we assume the unit-normal of $\mathcal{C}$ at $\hat{X}_\mathcal{C}$ to be $(0, 1)$ for ease of notation (and hence $\Pi(X(0)) = \hat{X}_\mathcal{C}$). Note that (3.1) is the rescale of the smallness of the C^1 norm of the local graph of $\mathcal{C}$, while (3.3) is the rescale of the small C^5 difference between the local graphes of $|\hat{X}_\mathcal{C}|^{-1}\mathcal{C}$ and $|\hat{X}_\mathcal{C}|^{-1}\Sigma$.

By Definition 2.2 and the rescaling argument, the same thing holds for each rescaled hypersurface $\Sigma_t = \sqrt{-t}\,\Sigma$, $t \in [-1, 0)$ as well. That is, outside a compact set, Σ_t is a normal graph over $\mathcal{C}\setminus\bar{B}_R$ (with $R \gg 1$ depending on $\Sigma, \mathcal{C}$); besideis, near each $\hat{X}_\mathcal{C} \in \mathcal{C}\setminus\bar{B}_R$, Σ_t is a graph over $T_{|\hat{X}_\mathcal{C}|^{-1}\hat{X}_\mathcal{C}}\mathcal{C}$ and can be parametrized by

$$X_t(x) = X(x, t) \equiv \hat{X}_\mathcal{C} + (x, \mathrm{u}_t(x)) = \hat{X}_\mathcal{C} + (x, \mathrm{u}(x, t))$$

which satisfies

$$\begin{aligned} &|\hat{X}_\mathcal{C}|^{-1} \parallel \mathrm{u}(\cdot, t) - w \parallel_{L^\infty(B^n_{\rho|\hat{X}_\mathcal{C}|})} + \parallel \partial_x \mathrm{u}(\cdot, t) - \partial_x \mathrm{w} \parallel_{L^\infty(B^n_{\rho|\hat{X}_\mathcal{C}|})} \\ &+ |\hat{X}_\mathcal{C}| \parallel \partial_x^2 \mathrm{u}(\cdot, t) - \partial_x^2 \mathrm{w} \parallel_{L^\infty(B^n_{\rho|\hat{X}_\mathcal{C}|})} \\ &+ \cdots + |\hat{X}_\mathcal{C}|^4 \parallel \partial_x^5 \mathrm{u}(\cdot, t) - \partial_x^5 \mathrm{w} \parallel_{L^\infty(B^n_{\rho|\hat{X}_\mathcal{C}|})} \leq \frac{1}{16} \end{aligned} \tag{3.4}$$

We call $t \mapsto X(x, t) = \hat{X}_\mathcal{C} + (x, \mathrm{u}(x, t))$ is the "vertical parametrization" of the flow $\{\Sigma_t\}_{-1\leq t<0}$. Note that by (3.1), (3.4) and $0 < \rho \leq \frac{1}{8}$, we have

$$\frac{3}{4}|\hat{X}_\mathcal{C}| \leq |X(x, t)| = |\hat{X}_\mathcal{C} + (x, \mathrm{u}(x, t))| \leq \frac{5}{4}|\hat{X}_\mathcal{C}|$$

for $x \in B^n_{\rho|\hat{X}_\mathcal{C}|}$, $t \in [-1, 0)$; that is, $|X|$ is comparable with $|\hat{X}_\mathcal{C}|$. Also, we still have the following polynomial volume growth for Σ_t:

$$\mathcal{H}^n\left(\Sigma_t \cap \left(B_{2\mathrm{r}}\setminus\bar{B}_\mathrm{r}\right)\right) \leq C(n, \mathcal{C})\,\mathrm{r}^n \tag{3.5}$$

for all $\mathrm{r} \geq R$.

On the other hand, Σ is a F self-shrinker, which we can use to improve (3.4). To see this, observe that under the condtions of being a F self-shrinker and having a tangent cone $\mathcal{C}$ at infinity, the rescaled flow $\left\{\Sigma_t = \sqrt{-t}\,\Sigma\right\}_{-1\leq t<0}$ moves by F curvature vector and converges (in the locally C^5 sense) to the cone $\mathcal{C}$ as $t \nearrow 0$. In other words, we can define a F curvature flow $\{\Sigma_t\}_{-1\leq t\leq 0}$ with $\Sigma_t = \sqrt{-t}\,\Sigma$ for $t \in [-1, 0)$ and $\Sigma_0 = \mathcal{C}$ which is continuous upto $t = 0$ (in the locally C^5 sense). Besides, near each $\hat{X}_\mathcal{C} \in \mathcal{C}\setminus\bar{B}_R$ (with $R \gg 1$ depending on $\Sigma, \mathcal{C}$), we have the vertical parametrization of the flow (as above) for $t \in [-1, 0]$ and the evolution of u_t satisfies (by Definition 2.4)

$$\partial_t \mathrm{u} = \sqrt{1 + |\partial_x \mathrm{u}|^2}\, F\left(A_i^j(x,t)\right) \quad \text{for } (x_1, \ldots, x_n) \in B^n_{\rho|\hat{X}_{\mathcal{C}}|},\ -1 \le t < 0 \tag{3.6}$$

$$\mathrm{u}(\cdot, t) \xrightarrow{C^5} w \quad \text{on } B^n_{\rho|\hat{X}_{\mathcal{C}}|} \quad \text{as } t \nearrow 0 \tag{3.7}$$

where the shape operator $A_t^{\#}(x) \sim A_i^j(x,t)$ of Σ_t (with respect to the local coordinate frame $\{\partial_1 X_t, \ldots, \partial_n X_t\}$) is equal to

$$A_i^j(x,t) = \partial_i \left(\frac{\partial_j \mathrm{u}(x,t)}{\sqrt{1 + |\partial_x \mathrm{u}|^2}} \right) \tag{3.8}$$

It follows (by using (3.6), (3.4), (3.1), (3.2) and (3.8)) that

$$\begin{aligned}|\partial_t \mathrm{u}| &= |\hat{X}_{\mathcal{C}}|^{-1} \sqrt{1 + |\partial_x \mathrm{u}|^2} \left| F\left(|\hat{X}_{\mathcal{C}}| A_i^j(x,t) \right) \right| \\ &\le |\hat{X}_{\mathcal{C}}|^{-1} \left(1 + \| \partial_x \mathrm{u}_t \|_{L^\infty(B^n_{\rho|\hat{X}_{\mathcal{C}}|})} \right) \| F \|_{L^\infty(U)}\end{aligned}$$

in which we use the homogeneity of F. Similarly, by differentiating (3.6) and using the homogeneity of the derivatives of F, we get

$$\begin{aligned}&|\hat{X}_{\mathcal{C}}| \, \| \partial_t \mathrm{u}(\cdot,t) \|_{L^\infty(B^n_{\rho|\hat{X}_{\mathcal{C}}|})} + |\hat{X}_{\mathcal{C}}|^2 \, \| \partial_t \partial_x \mathrm{u}(\cdot,t) \|_{L^\infty(B^n_{\rho|\hat{X}_{\mathcal{C}}|})} \\ &+ |\hat{X}_{\mathcal{C}}|^3 \, \| \partial_t \partial_x^2 \mathrm{u}(\cdot,t) \|_{L^\infty(B^n_{\rho|\hat{X}_{\mathcal{C}}|})} \\ &+ |\hat{X}_{\mathcal{C}}|^4 \, \| \partial_t \partial_x^3 \mathrm{u}(\cdot,t) \|_{L^\infty(B^n_{\rho|\hat{X}_{\mathcal{C}}|})} \le C\left(n, \mathcal{C}, \| F \|_{C^3(U)}\right)\end{aligned} \tag{3.9}$$

which implies (by (3.9) and (3.6))

$$|\mathrm{u}(\cdot,t) - w| \le \int_t^0 |\partial_t \mathrm{u}(\cdot,\tau)| \le C\left(n, \mathcal{C}, \| F \|_{C^3(U)}\right) |\hat{X}_{\mathcal{C}}|^{-1} (-t)$$

Likewise, integrate the estimates for derivatives in (3.9) to get $\forall\, t \in [-1, 0]$

$$\begin{aligned}&|\hat{X}_{\mathcal{C}}| \, \| \mathrm{u}(\cdot,t) - \mathrm{w} \|_{L^\infty(B^n_{\rho|\hat{X}_{\mathcal{C}}|})} + |\hat{X}_{\mathcal{C}}|^2 \, \| \partial_x \mathrm{u}(\cdot,t) - \partial_x \mathrm{w} \|_{L^\infty\left(B^n_{\rho|\hat{X}_{\mathcal{C}}|}\right)} \\ &+ |\hat{X}_{\mathcal{C}}|^3 \, \| \partial_x^2 \mathrm{u}(\cdot,t) - \partial_x^2 \mathrm{w} \|_{L^\infty\left(B^n_{\rho|\hat{X}_{\mathcal{C}}|}\right)} \\ &+ |\hat{X}_{\mathcal{C}}|^4 \, \| \partial_x^3 \mathrm{u}(\cdot,t) - \partial_x^3 \mathrm{w} \|_{L^\infty\left(B^n_{\rho|\hat{X}_{\mathcal{C}}|}\right)} \le C\left(n, \mathcal{C}, \| F \|_{C^3(U)}\right)(-t)\end{aligned} \tag{3.10}$$

which is the improvement of (3.4) by using the F self-shrinker equation (3.6).

In view of the pull-back metric $g_{ij}(x,t) = \delta_{ij} + \partial_i u(x,t)\, \partial_j u(x,t)$ and the associated Christoffel symbols

$$\Gamma_{ij}^k(x,t) = \frac{\partial_k \mathrm{u}(x,t)\ \partial_{ij}^2 \mathrm{u}(x,t)}{\sqrt{1 + |\partial_x \mathrm{u}(x,t)|^2}} \tag{3.11}$$

together with (3.8), (3.10), the comparablity of $|X|$ and $|\hat{X}_{\mathcal{C}}|$, (2.4), (2.5) and the continuity and homogeneity of F (and its derivatives), there exits $R \ge 1$ (depending on $\Sigma, \mathcal{C}, U, \| F \|_{C^3(U)}, \lambda, \varkappa$) such that for $X_t \in \Sigma_t \backslash \bar{B}_R$, the following hold:

$$|X_t|\, A_t^{\#} \in U \tag{3.12}$$

$$\frac{\lambda}{2}\delta_j^i \le \frac{\partial F}{\partial S_i^j}\left(A_t^{\#}\right) = \frac{\partial F}{\partial S_i^j}\left(|X_t|\, A_t^{\#}\right) \le \frac{2}{\lambda}\delta_j^i \tag{3.13}$$

$$\begin{aligned} &|X_t|\Big|\sum_{k,l}\frac{\partial^2 F}{\partial S_i^j\,\partial S_k^l}\left(A_t^{\#}\right)\left(\nabla_{\Sigma_t} A_t^{\#}\right)_k^l\Big| \\ &= \Big|\sum_{k,l}\frac{\partial^2 F}{\partial S_i^j\,\partial S_k^l}\left(|X_t| A_t^{\#}\right)\cdot\left(|X_t|^2\nabla_{\Sigma_t} A_t^{\#}\right)_k^l\Big| \le 2\varkappa \end{aligned} \tag{3.14}$$

$$|X_t|\,|A_t^{\#}| + |X_t|^2|\nabla_{\Sigma_t} A_t^{\#}| + |X_t|^3\;|\nabla_{\Sigma_t}^2 A_t^{\#}| \le C\,(n,\mathcal{C}) \tag{3.15}$$

where $A_t^{\#}$ is the shape operator of Σ_t at X_t and $\nabla_{\Sigma_t} A_t^{\#}$ is the covariant derivative of $A_t^{\#}$ (with rescpect to Σ_t). Note that F is homogeneous of degree 1, $\frac{\partial F}{\partial S_i^j}$ is of degree 0 and $\frac{\partial^2 F}{\partial S_i^j\partial S_k^l}$ is of degree -1.

Now let $\tilde{\Sigma}$ to be a F self-shrinker which is also C^5 asymptotic to $\mathcal{C}$ at infinity. By the same limting behaviour, $\tilde{\Sigma}$ is C^5close to Σ (in the rescale sense) for $|X| \gg 1$, and hence it can be regarded as a normal graph of a function h defined on Σ. Later we will derive an elliptic equation which is satisfied by h. To this end, we need the following two lemmas (Lemma 3.1 & Lemma 3.2). The first one gives the decay rate of the fuction h and the difference of the shape operators between Σ and $\tilde{\Sigma}$ as $|X| \nearrow \infty$; in the second one, we estimate the coefficients of the differential equation to be satisfied by h.

Lemma 3.1 *There exits* $R = R\left(\Sigma, \tilde{\Sigma}, n, \mathcal{C}, \| F \|_{C^3(U)}\right) \ge 1$ *so that outside a compact set,* $\tilde{\Sigma}$ *is a normal graph over* $\Sigma\backslash\bar{B}_R$ *and can be parametrized as*

$$\tilde{X} = X + hN \quad for \quad X \in \Sigma\backslash\bar{B}_R$$

where N *is the inward unit-normal of* Σ *and* h *is the deviation of* $\tilde{\Sigma}$ *from* Σ*. Besides, there hold*

$$\begin{aligned} &\|\, |X| h \,\|_{L^\infty(\Sigma\backslash\bar{B}_R)} + \|\, |X|^2\nabla_\Sigma h \,\|_{L^\infty(\Sigma\backslash\bar{B}_R)} + \|\, |X|^3\nabla_\Sigma^2 h \,\|_{L^\infty(\Sigma\backslash\bar{B}_R)} \\ &\quad\le C\left(n, \mathcal{C}, \| F \|_{C^3(U)}\right) \end{aligned} \tag{3.16}$$

$$\begin{aligned} &\|\, |X|^3\left(\tilde{A}^{\#} - A^{\#}\right) \|_{L^\infty(\Sigma\backslash\bar{B}_R)} + \|\, |X|^4\left(\nabla_\Sigma\tilde{A}^{\#} - \nabla_\Sigma A^{\#}\right) \|_{L^\infty(\Sigma\backslash\bar{B}_R)} \\ &\quad\le C\left(n, \mathcal{C}, \| F \|_{C^3(U)}\right) \end{aligned} \tag{3.17}$$

$$\begin{aligned} &\|\, |X|^3\nabla_\Sigma^2\tilde{A}^{\#} \,\|_{L^\infty(\Sigma\backslash\bar{B}_R)} \\ &\quad\le C\left(n, \mathcal{C}, \| F \|_{C^3(U)}\right) \end{aligned} \tag{3.18}$$

where $\tilde{A}^{\#}$ *is the shape operator of* $\tilde{\Sigma}$ *at* $\tilde{X} = X + hN$ *and* $\nabla_\Sigma\tilde{A}^{\#}$ *is the covariant derivative of* $\tilde{A}^{\#}$ *(which can be regarded as a 2-tensor on* Σ *via the normal graphic parametrization) with rescpect to* Σ*.*

Proof Choose $R \gg 1$ (depending on $\Sigma, \tilde{\Sigma}, n, \mathcal{C}, \| F \|_{C^3(U)}$) so that $\Sigma\backslash\bar{B}_R$ and $\tilde{\Sigma}\backslash\bar{B}_R$ have the local graph coordinates over tangent hyperplanes of $\mathcal{C}$ with appropriate estimates for the graphes as before. That is, for each $\hat{X} \in \Sigma\backslash\bar{B}_R$, we can respectively parametrize Σ and $\tilde{\Sigma}$ locally (near $\Pi\left(\hat{X}\right) = \hat{X}_{\mathcal{C}} \in \mathcal{C}$) by

$$X = X(x) \equiv \Pi\left(\hat{X}\right) + (x, \mathrm{u}(x))$$
$$\tilde{X} = \tilde{X}(x) \equiv \Pi\left(\hat{X}\right) + (x, \tilde{\mathrm{u}}(x))$$

for $x = (x_1, \ldots, x_n) \in B^n_{\rho|\Pi\left(\hat{X}\right)|}$, which satisfy [by (3.1), (3.2), (3.3) and the comparability of $|\hat{X}|$ and $|\hat{X}_{\mathcal{C}}|$]

$$|\hat{X}|^{-1} \parallel \mathrm{u} \parallel_{L^\infty(B^n_{\rho|\Pi(\hat{X})|})} + \parallel \partial_x \mathrm{u} \parallel_{L^\infty(B^n_{\rho|\Pi(\hat{X})|})} + |\hat{X}| \parallel \partial_x^2 \mathrm{u} \parallel_{L^\infty(B^n_{\rho|\Pi(\hat{X})|})} + \cdots + |\hat{X}|^4 \parallel \partial_x^5 \mathrm{u} \parallel_{L^\infty(B^n_{\rho|\Pi(\hat{X})|})} \leq C\,(n, \mathcal{C}) \tag{3.19}$$

$$|\hat{X}|^{-1} \parallel \tilde{\mathrm{u}} \parallel_{L^\infty(B^n_{\rho|\Pi(\hat{X})|})} + \parallel \partial_x \tilde{\mathrm{u}} \parallel_{L^\infty(B^n_{\rho|\Pi(\hat{X})|})} + |\hat{X}| \parallel \partial_x^2 \tilde{\mathrm{u}} \parallel_{L^\infty(B^n_{\rho|\Pi(\hat{X})|})} + \cdots + |\hat{X}|^4 \parallel \partial_x^5 \tilde{\mathrm{u}} \parallel_{L^\infty(B^n_{\rho|\Pi(\hat{X})|})} \leq C\,(n, \mathcal{C}) \tag{3.20}$$

Also, by applying the triangle inequality to (3.10), we get

$$|\hat{X}| \parallel \tilde{\mathrm{u}} - \mathrm{u} \parallel_{L^\infty(B^n_{\rho|\Pi(\hat{X})|})} + |\hat{X}|^2 \parallel \partial_x \tilde{\mathrm{u}} - \partial_x \mathrm{u} \parallel_{L^\infty(B^n_{\rho|\Pi(\hat{X})|})} + |\hat{X}|^3 \parallel \partial_x^2 \tilde{\mathrm{u}} - \partial_x^2 \mathrm{u} \parallel + |\hat{X}|^4 \parallel \partial_x^3 \tilde{\mathrm{u}} - \partial_x^3 \mathrm{u} \parallel_{L^\infty(B^n_{\rho|\Pi(\hat{X})|})} \leq C\left(n, \mathcal{C}, \parallel F \parallel_{C^3(U)}\right) \tag{3.21}$$

By (3.21), we may assume that $\tilde{\Sigma}$ is a normal graph of h defined on $\Sigma \setminus \bar{B}_R$; that is, for each $x \in B^n_{\frac{\rho}{2}|\Pi\left(\hat{X}\right)|}$, there is a unique $y \in B^n_{\rho|\Pi\left(\hat{X}\right)|}$ such that

$$\Pi\left(\hat{X}\right) + (x, \mathrm{u}\,(x)) + h(x)\frac{(-\partial_x \mathrm{u}, 1)}{\sqrt{1 + |\partial_x \mathrm{u}|^2}} = \Pi\left(\hat{X}\right) + (y, \tilde{\mathrm{u}}\,(y)) \tag{3.22}$$

or equivalently,

$$\left(x - h(x)\frac{\partial_x \mathrm{u}}{\sqrt{1 + |\partial_x \mathrm{u}|^2}},\ \mathrm{u}\,(x) + \frac{h(x)}{\sqrt{1 + |\partial_x \mathrm{u}|^2}}\right) = (y, \tilde{\mathrm{u}}\,(y))$$

where $\frac{(-\partial_x \mathrm{u}, 1)}{\sqrt{1+|\partial_x \mathrm{u}|^2}}$ is the unit normal N of Σ at $\Pi\left(\hat{X}\right) + (x, \mathrm{u}\,(x))$. In other words, h is defined implicitly by the following equation

$$\tilde{\mathrm{u}}\,(\psi(x)) - \left(\mathrm{u} + \frac{h(x)}{\sqrt{1 + |\partial_x \mathrm{u}|^2}}\right) = 0 \tag{3.23}$$

where

$$\psi(x) = x - h(x)\frac{\partial_x \mathrm{u}}{\sqrt{1 + |\partial_x \mathrm{u}|^2}} \tag{3.24}$$

defines a map from $B^n_{\frac{\rho}{2}|\Pi\left(\hat{X}\right)|}$ into $B^n_{\rho|\Pi\left(\hat{X}\right)|}$. Since $|h\,(x)|$ stands for the distance from the point $\Pi\left(\hat{X}\right) + (\psi(x), \tilde{u}\,(\psi(x)))$ on $\tilde{\Sigma}$ (i.e. the RHS of (3.22)) to Σ, we immediately have

$$|h(x)| \leq |\tilde{\mathrm{u}}\,(\psi(x)) - \mathrm{u}(\psi(x))| \leq C\left(n, \mathcal{C}, \parallel F \parallel_{C^3(U)}\right) |\hat{X}|^{-1}$$

To proceed further, first notice that for the unit normal vectors of Σ and $\tilde{\Sigma}$

$$N\,(x) = \frac{(-\partial_x \mathrm{u}, 1)}{\sqrt{1 + |\partial_x \mathrm{u}|^2}}, \quad \tilde{N}\,(x) = \frac{(-\partial_x \tilde{\mathrm{u}}, 1)}{\sqrt{1 + |\partial_x \tilde{\mathrm{u}}|^2}} \tag{3.25}$$

respectively, we may assume, by (3.21), (3.19), that

$$\| \tilde{N} - N \|_{L^\infty(B^n_{\rho|\Pi(\hat{x})|})} + \| N \circ \psi - N \|_{L^\infty(B^n_{\frac{\rho}{2}|\Pi(\hat{x})|})} \leq \frac{1}{3}$$

which implies that for each $x \in B^n_{\frac{\rho}{2}|\Pi\left(\hat{x}\right)|}$,

$$\begin{aligned} \tilde{N}\left(\psi(x)\right) \cdot N(x) &\geq N(x) \cdot N(x) - |\tilde{N}\left(\psi(x)\right) - N(x)|\,|N(x)| \\ &\geq 1 - \left(|\tilde{N}\left(\psi(x)\right) - N\left(\psi(x)\right)| + |N\left(\psi(x)\right) - N(x)|\right) \geq \frac{2}{3} \end{aligned} \tag{3.26}$$

Let

$$\Theta\left(x, s\right) = \tilde{\mathrm{u}}\left(x - s\frac{\partial_x \mathrm{u}}{\sqrt{1 + |\partial_x \mathrm{u}|^2}}\right) - \left(\mathrm{u} + \frac{s}{\sqrt{1 + |\partial_x \mathrm{u}|^2}}\right)$$

then by (3.23), (3.24) and (3.26), we have $\Theta\left(x, h(x)\right) = 0$ and

$$\partial_s \Theta\left(x, h(x)\right) = -\sqrt{1 + |\partial_y \tilde{\mathrm{u}}\left(\psi(x)\right)|^2}\; \tilde{N}\left(\psi(x)\right) \cdot N(x) \leq -\frac{2}{3}$$

Therefore, by the implicit function theorem, we have $h \in C^2\left(B^n_{\frac{\rho}{2}|\Pi\left(\hat{x}\right)|}\right)$. Besides, by doing the implicit differentiation of (3.23) (or $\Theta\left(x, h(x)\right) = 0$), we get

$$\begin{aligned} \frac{1 + \partial_j \tilde{\mathrm{u}} \circ \psi \cdot \partial_j \mathrm{u}}{\sqrt{1 + |\partial_x \mathrm{u}|^2}} \partial_i h &= \left(\partial_i \tilde{\mathrm{u}} \circ \psi - \partial_i \mathrm{u}\right) \\ &\quad - \left(\partial_j \tilde{\mathrm{u}} \circ \psi \cdot \partial_i \frac{\partial_j \mathrm{u}}{\sqrt{1 + |\partial_x \mathrm{u}|^2}} + \partial_j \mathrm{u} \frac{\partial^2_{ij} \mathrm{u}}{\left(1 + |\partial_x \mathrm{u}|^2\right)^{\frac{3}{2}}}\right) h \end{aligned} \tag{3.27}$$

in which we sum over repeated indicies. Note that we can use (3.27), together with (3.19) and (3.21), to estimate $\partial_x h$. For instance, for the first term on the RHS of the equation, we have

$$\begin{aligned} |\partial_i \tilde{\mathrm{u}} \circ \psi - \partial_i \mathrm{u}| &\leq |\partial_i \tilde{\mathrm{u}} \circ \psi - \partial_i \mathrm{u} \circ \psi| + |\partial_i \mathrm{u} \circ \psi - \partial_i \mathrm{u}| \\ &\leq C\left(n, \mathcal{C}, \| F \|_{C^3(U)}\right) |\hat{X}|^{-2} \\ &\quad + \sum_j \int_0^1 \left|\partial^2_{ij} \mathrm{u}\left(x - \theta h \frac{\partial_x \mathrm{u}}{\sqrt{1 + |\partial_x \mathrm{u}|^2}}\right)\right| d\theta\; \frac{|\partial_j \mathrm{u}|}{\sqrt{1 + |\partial_x \mathrm{u}|^2}} |h| \\ &\leq C\left(n, \mathcal{C}, \| F \|_{C^3(U)}\right) |\hat{X}|^{-2} \end{aligned}$$

Thus we get $\| \partial_x h \|_{L^\infty(B^n_{\frac{\rho}{2}|\Pi(\hat{x})|})} \leq C\left(n, \mathcal{C}, \| F \|_{C^3(U)}\right) |\hat{X}|^{-2}$. Similarly, doing the implicit differentiation of (3.27) and using (3.19) and (3.21) yields $\| \partial^2_x h \|_{L^\infty(B^n_{\frac{\rho}{2}|\Pi(\hat{x})|})} \leq C\left(n, \mathcal{C}, \| F \|_{C^3(U)}\right) |\hat{X}|^{-3}$. The bounds on the covariant derivatives of h follow from the the following estimates on the pull-back metric $g_{ij} = \partial_i X \cdot \partial_j X$ and the Christoffel symbols Γ^k_{ij} in (3.11) associated with the local coordinates $x = (x_1, \ldots, x_n)$:

$$\delta_{ij} \leq g_{ij} = 1 + \partial_i \mathrm{u}\, \partial_j \mathrm{u} \leq \frac{5}{4} \delta_{ij} \tag{3.28}$$

$$|\Gamma_{ij}^k| = \frac{|\partial_k \mathrm{u}|}{\sqrt{1 + |\partial_x \mathrm{u}|^2}} |\partial_{ij}^2 \mathrm{u}| \leq C\,(n, \mathcal{C}, F)\; |\hat{X}|^{-1} \tag{3.29}$$

where we have used (3.19). This completes the derivation of (3.16).

As for (3.17), let's first observe that the normal graph reparametrization of $\tilde{\Sigma}$ amounts to the following change of variables:

$$\tilde{X} = \Pi\left(\hat{X}\right) + (y, \tilde{\mathrm{u}}(y)) \quad \text{with} \quad y = \psi(x) = x - h(x) \frac{\partial_x \mathrm{u}}{\sqrt{1 + |\partial_x \mathrm{u}|^2}} \tag{3.30}$$

Note that from (3.30), (3.19) and (3.16), we have

$$\frac{\partial y_k}{\partial x_i} = \delta_i^k - h \cdot \partial_{x_i} \left(\frac{\partial_{x_j} \mathrm{u}}{\sqrt{1 + |\partial_x \mathrm{u}|^2}} \right) - \partial_{x_i} h \frac{\partial_k \mathrm{u}}{\sqrt{1 + |\partial_x \mathrm{u}|^2}} = \delta_i^k + O\left(|\hat{X}|^{-2}\right) \tag{3.31}$$

By taking R sufficiently large, we may assume that $\psi : B^n_{\frac{\rho}{2}|\Pi\left(\hat{x}\right)|} \to \mathrm{Im}\psi \subset B^n_{\rho|\Pi\left(\hat{x}\right)|}$ is a C^2 diffeomorphism and the inverse of $\frac{\partial y_k}{\partial x_i}$ satisfies

$$\frac{\partial x_i}{\partial y_k} = \delta_k^i + O\left(|\hat{X}|^{-2}\right)$$

It follows that the components of shape operators $\tilde{A}^{\#}$ of $\tilde{\Sigma}$ and $A^{\#}$ of Σ with respect to the local coodinates $x = (x_1, \ldots, x_n)$ are respectively equal to

$$\tilde{A}_i^j = \frac{\partial y_k}{\partial x_i} \frac{\partial x_j}{\partial y_l} \partial_{y_k} \left(\frac{\partial_{y_l} \tilde{\mathrm{u}}}{\sqrt{1 + |\partial_y \tilde{\mathrm{u}}|^2}} \right) \Bigg|_{y=\varphi(x)}, \quad A_i^j = \partial_{x_i} \left(\frac{\partial_{x_j} \mathrm{u}}{\sqrt{1 + |\partial_x \mathrm{u}|^2}} \right) \tag{3.32}$$

in which we sum over repeated indicies. Using the triangle inequality, combined with (3.19), (3.21), (3.30), (3.16) and (3.31), we then get from (3.32) that

$$\left| \tilde{A}_i^j - A_i^j \right| \leq C\left(n, \mathcal{C}, \| F \|_{C^3(U)}\right) |\hat{X}|^{-3}$$

Due to (3.28), the above implies that

$$\left| \tilde{A}^{\#} - A^{\#} \right| \leq C\left(n, \mathcal{C}, \| F \|_{C^3(U)}\right) |\hat{X}|^{-3}$$

Also, in view of $\nabla_\Sigma \tilde{A}^{\#} \sim \nabla_r \tilde{A}_i^j$, $\nabla_\Sigma A^{\#} \sim \nabla_r A_i^j$ and

$$\nabla_r \tilde{A}_i^j = \partial_r \tilde{A}_i^j - \Gamma_{ri}^s \tilde{A}_s^j + \Gamma_{rs}^j \tilde{A}_i^s, \quad \nabla_r A_i^j = \partial_r A_i^j - \Gamma_{ri}^s A_s^j + \Gamma_{rs}^j A_i^s \tag{3.33}$$

in which we sum over repeated indicies, we can similarly derive

$$|\nabla_\Sigma \tilde{A}^{\#} - \nabla_\Sigma A^{\#}| \leq C\left(n, \mathcal{C}, \| F \|_{C^3(U)}\right) |\hat{X}|^{-4}$$

This completes (3.17).

Equation (3.18) follows from taking one more derivative of (3.33) and use (3.32), (3.29), (3.19), (3.21) and (3.28). □

Next, we'd like to define a 2-tensor **a** on Σ (outside a compact set), which will be served as the coefficients of the differential equation to be satisfied by the deviation h. Note that by (3.12), Lemma 8 [in particular (3.17)], we may assume that

$$(1-\theta)\,|X|\,A^{\#}+\theta|X|\,\tilde{A}^{\#}\in U\quad \forall\, X\in\Sigma\backslash\bar{B}_R,\ \theta\in[0,1] \tag{3.34}$$

where $\tilde{A}^{\#}$ is the shape operator of $\tilde{\Sigma}$ at $\tilde{X}=X+hN$.

Definition 3.1 In the setting of Lemma 3.1, let's take a local coordinate $x=(x_1,\ldots,x_n)$ of Σ (outside a compact set) so that Σ and $\tilde{\Sigma}$ can be respectively parametrized as

$$X=X\,(x)\,,\quad \tilde{X}\,(x)=X\,(x)+h\,(x)\,N\,(x)$$

where $h\,(x)$is the deviation and $N\,(x)$ is the unit-normal of Σ at $X\,(x)$. Then we define

$$\mathsf{a}^{ij}\,(x)=\sum_k \mathsf{a}^i_k\,(x)\;g^{kj}\,(x)\quad\text{with}$$

$$\mathsf{a}^i_j\,(x)=\int_0^1\frac{\partial F}{\partial S^j_i}\left((1-\theta)\,|X|\,A^{\#}\,(x)+\theta|X|\,\tilde{A}^{\#}\,(x)\right)d\theta$$

and its symmetrization

$$\mathbf{a}^{ij}\,(x)=\frac{1}{2}\left(\mathsf{a}^{ij}\,(x)+\mathsf{a}^{ji}\,(x)\right)$$

where $g^{ij}\,(x)$ is the inverse of the pull-back metric $g_{ij}=\partial_i X\cdot\partial_j X$, $A^{\#}\,(x)\sim A^j_i\,(x)=-\partial_i N\cdot\partial_j X$ is the shape operator of Σ at $X\,(x)$, $\tilde{A}^{\#}_t\,(x)\sim\tilde{A}^j_i\,(x,t)=-\partial_i\tilde{N}\cdot\partial_j\tilde{X}$ is the shape operator of $\tilde{\Sigma}$ at $\tilde{X}\,(x)$ with $\tilde{N}\,(x)$ being the unit-normal of $\tilde{\Sigma}$ at $\tilde{X}\,(x)$.

Note that

$$\begin{aligned}\mathsf{a}^i_j\,(x)&=\int_0^1\frac{\partial F}{\partial S^j_i}\left((1-\theta)\,|X|\,A^{\#}\,(x)+\theta|X|\,\tilde{A}^{\#}\,(x)\right)d\theta\\&=\int_0^1\frac{\partial F}{\partial S^j_i}\left((1-\theta)\,A^{\#}\,(x)+\theta\tilde{A}^{\#}\,(x)\right)d\theta\end{aligned}$$

since $\frac{\partial F}{\partial S^j_i}$ is homogeneous of degree 0; besides, the operator **a** is independent of the choice of local coordinates and hence defines a 2-tensor on Σ.

We have the following estimates for the tensor **a**, which is based on (3.13), (3.14), (3.15), (3.17), (3.18) and the homogeneity of F and its derivatives.

Lemma 3.2 *There exits* $R=R\left(\Sigma,\tilde{\Sigma},\mathcal{C},U,\|\,F\,\|_{C^3(U)},\lambda,\varkappa\right)\geq 1$ *such that*

$$\frac{\lambda}{3}\leq\mathbf{a}\leq\frac{3}{\lambda} \tag{3.35}$$

$$|X|\left|\nabla_\Sigma\mathbf{a}\right|\leq 3\varkappa \tag{3.36}$$

$$|X|^2\left|\nabla^2_\Sigma\mathbf{a}\right|\leq C\left(n,\mathcal{C},\|\,F\,\|_{C^3(U)}\right) \tag{3.37}$$

for all $X\in\Sigma\backslash\bar{B}_R$.

Proof By (3.13), (3.14), (3.34), (3.17), the homogeneity and continuity of F (and its derivatives), there exists $R = R\left(\Sigma, \tilde{\Sigma}, \mathcal{C}, U, \| F \|_{C^3(U)}, \lambda, \varkappa\right) \geq 1$ such that

$$\frac{\lambda}{3}\delta_j^i \leq \mathrm{a}_j^i = \int_0^1 \frac{\partial F}{\partial S_i^j}\left((1-\theta)\,|X|\,A^{\#} + \theta|X|\,\tilde{A}^{\#}\right) d\theta \leq \frac{3}{\lambda}\delta_j^i$$

$$|X|\left|\nabla_r \mathrm{a}_i^j\right| = |X|\left|\int_0^1 \sum_{k,l} \frac{\partial^2 F}{\partial S_i^j \partial S_k^l}\left((1-\theta)\,A^{\#} + \theta\tilde{A}^{\#}\right)\cdot\left((1-\theta)\,\nabla_r A_k^l + \theta\nabla_r \tilde{A}_k^l\right) d\theta\right|$$
$$= \left|\int_0^1 \sum_{k,l} \frac{\partial^2 F}{\partial S_i^j \partial S_k^l}\left((1-\theta)\,|X|\,A^{\#} + \theta|X|\,\tilde{A}^{\#}\right) \cdot\left((1-\theta)\,|X|^2\nabla_r A_k^l + \theta|X|^2\nabla_r \tilde{A}_k^l\right) d\theta\right| \leq 3\varkappa$$

Likewise, with the help of (3.15), (3.18), we can get

$$|X|^2\left|\nabla_\Sigma^2 \mathrm{a}\right| \leq C\left(n, \mathcal{C}, \| F \|_{C^3(U)}\right)$$

The conclusion follows immediately. □

Now we are in a position to derive an elliptic equation satisfied by h.

Proposition 3.1 *There exits* $R = R\left(\Sigma, \tilde{\Sigma}, \mathcal{C}, U, \| F \|_{C^3(U)}, \lambda, \varkappa\right) \geq 1$ *such that the deviation* h *satisfies*

$$\nabla_\Sigma \cdot (\mathbf{a}\, dh) - \frac{1}{2}\left(X \cdot \nabla_\Sigma h - h\right) = O\left(|X|^{-1}\right)|\nabla_\Sigma h| + O\left(|X|^{-2}\right)|h| \tag{3.38}$$

for $X \in \Sigma\backslash\bar{B}_R$, *where*

$$\nabla_\Sigma \cdot (\mathbf{a}\, dh) = \sum_{i,j} \nabla_i\left(\mathbf{a}^{ij}\nabla_j h\right)$$

in local coordinates and the notation $O\left(|X|^{-1}\right)$ *means that*

$$\left|O\left(|X|^{-1}\right)\right| \leq C\left(n, \mathcal{C}, \| F \|_{C^3(U)}\right)|X|^{-1}$$

Proof Fix $\hat{X} \in \Sigma\backslash\bar{B}_R$ and take a local coodinate $x = (x_1, \ldots, x_n)$ of Σ which is normal and pricipal (w.r.t. Σ) at $\hat{X} = X(0)$. That is

$$g_{ij}\Big|_{x=0} = \delta_{ij},\ \Gamma_{ij}^k\Big|_{x=0} = 0,\ A_i^j\Big|_{x=0} = \kappa_i\delta_{ij}$$

where g_{ij} is the pull-back metric, Γ_{ij}^k is the Christoffel symbols and A_i^j is the shape operator of Σ at $X(x)$. Denote the principal direction of Σ at $\hat{X}$ by

$$\partial_i X\Big|_{x=0} = e_i$$

Throughout the proof, we adopt the Einstein summation convension (i.e. summing over repeated indicies). Recall that we regard $\tilde{\Sigma}$ (outside a compact set) as a normal graph over

$\Sigma \backslash \bar{B}_R$ and parametrize it by $\tilde{X} = X(x) + h(x) N(x)$. We then want to compute some geomtric quantities of $\tilde{\Sigma}$ in terms of this local coordinate at $\tilde{X}(0) = \hat{X} + hN\Big|_{\hat{X}}$. First, we compute

$$\partial_i \tilde{X}\Big|_{x=0} = \left(\delta_i^k - A_i^k h\right) \partial_k X + \partial_i h \, N\Big|_{x=0} = (1 - \kappa_i h) \, e_i + \nabla_i h \, N$$
$$\partial_{ij}^2 \tilde{X}\Big|_{x=0} = -\left(A_i^k \nabla_j h + A_j^k \nabla_i h + \nabla_i A_j^k \cdot h\right) e_k + \left(A_{ij} + \nabla_{ij}^2 h - A_{ij}^2 h\right) N \quad (3.39)$$

which (together with Lemma 3.1) gives the metric of $\tilde{\Sigma}$, its inverse and determinant as follows:

$$\tilde{g}_{ij}\Big|_{x=0} = (1 - \kappa_i h)^2 \, \delta_{ij} + \nabla_i h \, \nabla_j h = (1 - \kappa_i h)^2 \left(\delta_{ij} + \frac{\nabla_i h \, \nabla_j h}{(1 - \kappa_i h)^2}\right)$$
$$\tilde{g}^{ij}\Big|_{x=0} = (1 - \kappa_i h)^{-2} \left(\delta_{ij} + \frac{\nabla_i h \, \nabla_j h}{(1 - \kappa_i h)^2}\right)^{-1}$$
$$= (1 + 2\kappa_i h) \, \delta^{ij} + O\left(|\hat{X}|^{-2}\right) |\nabla_\Sigma h| + O\left(|\hat{X}|^{-3}\right) |h|$$
$$\det \tilde{g}\Big|_{x=0} = (1 - \kappa_1 h)^2 \ldots (1 - \kappa_n h)^2 \det \left(\delta_{ij} + \frac{\nabla_i h \, \nabla_j h}{(1 - \kappa_i h)^2}\right)$$
$$= 1 - 2Hh + O\left(|\hat{X}|^{-2}\right) |\nabla_\Sigma h| + O\left(|\hat{X}|^{-3}\right) |h| \quad (3.40)$$

and also the unit-normal of $\tilde{\Sigma}$:

$$\tilde{N}\Big|_{x=0} = (\det \tilde{g})^{-\frac{1}{2}} \, \partial_1 \tilde{X} \wedge \cdots \wedge \partial_n \tilde{X}$$
$$= (\det \tilde{g})^{-\frac{1}{2}} \left(-\sum_{i=1}^n \left(\nabla_i h \prod_{j \neq i} (1 - \kappa_j h)\right) e_i + (1 - \kappa_1 h) \ldots (1 - \kappa_n h) \, N\right)$$
$$= -\sum_{i=1}^n \left(1 + \kappa_i h + O\left(|\hat{X}|^{-2}\right) |\nabla_\Sigma h| + O\left(|\hat{X}|^{-3}\right) |h|\right) \nabla_i h \cdot e_i$$
$$+ \left(1 + O\left(|\hat{X}|^{-2}\right) |\nabla_\Sigma h| + O\left(|\hat{X}|^{-3}\right) |h|\right) N \quad (3.41)$$

By (3.39), (3.40), (3.41) and Lemma 3.1, we compute the shape operator of $\tilde{\Sigma}$at $\tilde{X}(0)$:

$$\tilde{A}_i^j\Big|_{x=0} = \tilde{A}_{ik} \, \tilde{g}^{kj} = \left(\partial_{ik}^2 \tilde{X} \cdot \tilde{N}\right) \tilde{g}^{kj}$$
$$= \left(A_{ik} + \nabla_{ik}^2 h + O\left(|\hat{X}|^{-2}\right) |\nabla_\Sigma h| + O\left(|\hat{X}|^{-2}\right) |h|\right)$$
$$\left((1 + 2\kappa_j h) \, \delta^{kj} + O\left(|\hat{X}|^{-2}\right) |\nabla_\Sigma h|\right)$$
$$+ \left(A_{ik} + \nabla_{ik}^2 h + O\left(|\hat{X}|^{-2}\right) |\nabla_\Sigma h| + O\left(|\hat{X}|^{-2}\right) |h|\right) O\left(|\hat{X}|^{-3}\right) |h|$$
$$= A_i^j + \delta^{kj} \nabla_{ik}^2 h + O\left(|\hat{X}|^{-2}\right) (|\nabla_\Sigma h| + |h|) \quad (3.42)$$

and

$$\tilde{X} \cdot \tilde{N}\Big|_{x=0} = X \cdot N - X \cdot \nabla_\Sigma h + h + O\left(|\hat{X}|^{-1}\right) |\nabla_\Sigma h| + O\left(|\hat{X}|^{-2}\right) |h| \quad (3.43)$$

Thus, in view of the F self-shrinker equation satisfied by Σ and $\tilde{\Sigma}$, we get

$$\begin{aligned} 0 &= F\left(\tilde{A}^{\#}\right) - F\left(A^{\#}\right) + \frac{1}{2}\left(\tilde{X}\cdot\tilde{N} - X\cdot N\right)\Big|_{x=0} \\ &= \int_0^1 \frac{\partial F}{\partial S_i^j}\left((1-\theta)\,A^{\#} + \theta\tilde{A}^{\#}\right) d\theta\cdot\left(\tilde{A}_i^j - A_i^j\right) - \frac{1}{2}\left(X\cdot\nabla_\Sigma h - h\right) \\ &\quad + O\left(|\hat{X}|^{-1}\right)|\nabla_\Sigma h| + O\left(|\hat{X}|^{-2}\right)|h| \\ &= \mathrm{a}_j^i\delta^{jk}\nabla_{ik}^2 h - \frac{1}{2}\left(X\cdot\nabla_\Sigma h - h\right) + O\left(|\hat{X}|^{-1}\right)|\nabla_\Sigma h| + O\left(|\hat{X}|^{-2}\right)|h| \\ &= \mathrm{a}^{ik}\nabla_{ik}^2 h - \frac{1}{2}\left(X\cdot\nabla_\Sigma h - h\right) + O\left(|\hat{X}|^{-1}\right)|\nabla_\Sigma h| + O\left(|\hat{X}|^{-2}\right)|h| \\ &= \langle \mathrm{a}, \nabla_\Sigma^2 h\rangle - \frac{1}{2}\left(X\cdot\nabla_\Sigma h - h\right) + O\left(|\hat{X}|^{-1}\right)|\nabla_\Sigma h| + O\left(|\hat{X}|^{-2}\right)|h| \end{aligned} \tag{3.44}$$

Note that by the symmetry of the Hessian and Lemma 3.2, we have

$$\begin{aligned} \langle \mathrm{a}, \nabla_\Sigma^2 h\rangle &= \mathrm{a}^{ij}\nabla_{ij}^2 h = \frac{1}{2}\left(\mathrm{a}^{ij} + \mathrm{a}^{ji}\right)\nabla_{ij}^2 h = \langle \mathbf{a}, \nabla_\Sigma^2 h\rangle \\ &= \nabla_i\left(\mathbf{a}^{ij}\nabla_j h\right) - \left(\nabla_i \mathbf{a}^{ij}\right)\nabla_j h = \nabla_\Sigma\cdot(\mathbf{a}\, dh) + O\left(|\hat{X}|^{-1}\right)|\nabla_\Sigma h| \end{aligned} \tag{3.45}$$

(3.38) follows from combining (3.44) and (3.45). □

Our goal is to show that h vanishes on $\Sigma\backslash\bar{B}_R$ for some $R \gg 1$, which will be done in the next section through Carleman's inequality. For that purpose, we first observe that for each $t\in[-1,0)$, $\tilde{\Sigma}_t = \sqrt{-t}\,\tilde{\Sigma}$ is (outside a compact set) also a normal graph over $\Sigma_t\backslash\bar{B}_R$ and it can be parametrized as $\tilde{X}_t = X_t + h_t N_t$. For the rest of this section, we will show that each $h_t = h(\cdot,t)$ satisfies a similar equation as $h(\cdot,-1)$ does in Proposition 3.1. Due to the property that $\{\Sigma_t\}_{-1\le t<0}$ form a F curvature flow, it turns out that the evolution of h_t satisfies a parabolic equation. We then give some estimates for the coefficients of the parabolic equations (as in Lemma 3.2) , which is crucial for deriving the Carleman's inequality in the next section.

Now fix $t\in[-1,0)$ and define a 2-tensor $\mathbf{a}_t$ on $\Sigma_t = \sqrt{-t}\,\Sigma$ as in Definition 3.1. First, take a local coordinate $x = (x_1,\ldots,x_n)$ of Σ_t (outside a compact set) so that Σ_t and $\tilde{\Sigma}_t$ can be respectively parametrized as

$$X_t = X_t(x), \quad \tilde{X}_t(x) = X_t(x) + h_t(x)\,N_t(x)$$

We define

$$\mathrm{a}_t^{ij}(x) = \sum_k \mathrm{a}_k^i(x,t)\,g_t^{kj}(x) \quad \text{with } \mathrm{a}_j^i(x,t) = \int_0^1 \frac{\partial F}{\partial S_i^j}\left((1-\theta)\,A_t^{\#}(x) + \theta\tilde{A}_t^{\#}(x)\right)d\theta$$

and its symmetrization

$$\mathbf{a}_t^{ij}(x) = \frac{1}{2}\left(\mathrm{a}_t^{ij}(x) + \mathrm{a}_t^{ji}(x)\right)$$

where $g_t^{ij}(x)$ is the inverse of the pull-back metric $g_{ij}(x,t) = \partial_i X_t(x)\cdot\partial_j X_t(x)$, $A_t^{\#}(x)\sim A_i^j(x,t) = -\partial_i N_t(x)\cdot\partial_j X_t(x)$ is the shape operator of Σ_t at $X_t(x)$ with $N_t(x)$ being the unit-normal of Σ_t at $X_t(x)$, $\tilde{A}_t^{\#}\sim\tilde{A}_i^j(x,t) = -\partial_i\tilde{N}_t(x)\cdot\partial_j\tilde{X}_t(x)$ is the shape operator of $\tilde{\Sigma}_t$ at $\tilde{X}_t(x)$ with $\tilde{N}_t(x)$ being the unit-normal of $\tilde{\Sigma}_t$ at $\tilde{X}_t(x)$.

Then we have the following lemma, which is an analogous of Proposition 3.1 for $\Sigma_t = \sqrt{-t}\,\Sigma,\ t \in [-1,0)$:

Lemma 3.3 *There exits* $R = R\left(\Sigma, \tilde{\Sigma}, \mathcal{C}, U, \| F \|_{C^3(U)}, \lambda, \varkappa\right) \geq 1$ *such that for each* $t \in [-1,0)$*, the deviation* h_t *satisfies*

$$\nabla_{\Sigma_t} \cdot (\mathbf{a}_t\, dh_t) - \frac{1}{2(-t)}\left(X_t \cdot \nabla_{\Sigma_t} h_t - h_t\right) = O\left(|X_t|^{-1}\right)|\nabla_{\Sigma_t} h_t| + O\left(|X_t|^{-2}\right)|h_t| \tag{3.46}$$

for $X_t \in \Sigma_t \setminus \bar{B}_R$*, where* $\nabla_{\Sigma_t} \cdot (\mathbf{a}_t\, dh_t) = \sum_{i,j} \nabla_i \left(\mathbf{a}_t^{ij} \nabla_j h_t\right)$ *and*

$$\left| O\left(|X_t|^{-1}\right)\right| \leq C\left(n, \mathcal{C}, \| F \|_{C^3(U)}\right) |X_t|^{-1}$$

Also, we have

$$\begin{aligned} &\| \, |X_t|\, h_t \|_{L^\infty(\Sigma_t \setminus \bar{B}_R)} + \| \, |X_t|^2 \nabla_{\Sigma_t} h_t \|_{L^\infty(\Sigma_t \setminus \bar{B}_R)} + \| \, |X_t|^3 \nabla^2_{\Sigma_t} h_t \|_{L^\infty(\Sigma_t \setminus \bar{B}_R)} \\ &\quad \leq C\left(n, \mathcal{C}, \| F \|_{C^3(U)}\right)(-t) \end{aligned} \tag{3.47}$$

Proof Fix $t \in [-1,0)$ and $\hat{X}_t \in \Sigma_t \setminus B_R$, then we have $\hat{X} = \frac{\hat{X}_t}{\sqrt{-t}} \in \Sigma \setminus \bar{B}_R$ and

$$\begin{aligned} &\left(\nabla_{\Sigma_t} \cdot (\mathbf{a}_t\, dh_t) - \frac{1}{2(-t)}\left(X_t \cdot \nabla_{\Sigma_t} h_t - h_t\right)\right)\Big|_{\hat{X}_t} \\ &= \frac{1}{\sqrt{-t}}\left(\nabla_\Sigma \cdot (\mathbf{a}\, dh) - \frac{1}{2}(X \cdot \nabla_\Sigma h - h)\right)\Big|_{\hat{X}_t} \\ &= \frac{1}{\sqrt{-t}}\left(O\left(|\hat{X}|^{-1}\right)|\nabla_\Sigma h| + O\left(|\hat{X}|^{-2}\right)|h|\right)\Big|_{\hat{X}_t} \\ &= \left(O\left(|\hat{X}_t|^{-1}\right)|\nabla_{\Sigma_t} h_t| + O\left(|\hat{X}_t|^{-2}\right)|h_t|\right)\Big|_{\hat{X}_t} \end{aligned}$$

Similarly, to derive (3.47), it suffices to rescale (3.16) to get

$$\begin{aligned} &|\hat{X}_t|\,|h_t| + |\hat{X}_t|^2 |\nabla_{\Sigma_t} h_t| + |\hat{X}_t|^3 |\nabla^2_{\Sigma_t} h_t|\Big|_{\hat{X}_t} \\ &= (-t)\left(|\hat{X}|\,|h| + |\hat{X}|^2 |\nabla_\Sigma h| + |\hat{X}|^3 |\nabla^2_\Sigma h|\right)\Big|_{\hat{X}_t} \\ &\leq C\left(n, \mathcal{C}, \| F \|_{C^3(U)}\right)(-t) \end{aligned}$$

□

Next, we define the "normal parametrization" of the flow:

Definition 3.2 $X_t = X(\cdot, t)$ is called a "normal parametrization" for the motion of a hypersurface $\{\Sigma_t\}$ provided that

$$\partial_t X = F\left(A^\#\right) N$$

That is, each particle on the hypersurface moves in normal direction during the flow. (See also Definition 2.4)

In the derivation of the parabolic equation to be satisfied by $h_t = h(\cdot, t)$, we will start with a "radial parametrization" of the flow $\{\Sigma_t\}_{-1 \leq t < 0}$ (i.e. each particles on the hypersurface moves in the radial direction along the flow, see the proof of Propostion 3.2 for more deatails), then we make a trasition to the "normal parametrization" by using a time-dependent tangential diffeomorphism. Note that in general, the "radial parametrization" exists only for a short period of time (unlike the "vertical parametrization"), so later in the proof, we will do a "local" (in spacetime) argument, which is quite sufficient for deriviving the equation.

Proposition 3.2 *There exits* $R = R\left(\Sigma, \tilde{\Sigma}, \mathcal{C}, U, \| F \|_{C^3(U)}, \lambda, \varkappa\right) \geq 1$ *so that in the normal parametrization of the* F *curvature flow* $\{\Sigma_t\}_{-1\leq t<0}$ *, the deviation* h_t *satisfies*

$$\begin{aligned} \mathbf{P}h &\equiv \partial_t h - \nabla_{\Sigma_t} \cdot (\mathbf{a}(\cdot, t)\, dh) \qquad (3.48)\\ &= O\left(|X_t|^{-1}\right) |\nabla_{\Sigma_t} h| + O\left(|X_t|^{-2}\right) |h| \\ h(\cdot, 0) &= 0 \quad as\ t \nearrow 0 \qquad (3.49) \end{aligned}$$

for $X_t \in \Sigma_t \backslash \bar{B}_R$, $-1 \leq t < 0$, *where* $\mathbf{a}(\cdot, t) = \mathbf{a}_t$.

Proof Fix $\hat{t} \in [-1, 0)$, $\hat{X} \in \Sigma_{\hat{t}} \backslash \bar{B}_R$, and take a local coordinate $x = (x_1, \ldots, x_n)$ of $\Sigma_{\hat{t}}$ around $\hat{X}$. Define the "radial parametrization" of the flow starting at time $\hat{t}$ near the point $\hat{X}$ by

$$X(x, t) = \frac{\sqrt{-t}}{\sqrt{-\hat{t}}} X_{\hat{t}}(x)$$

For this parametrization, we can decompose the velocity vector into the normal part and the tangential part as follows:

$$\begin{aligned} \partial_t X(x, t) &= \frac{-1}{2\sqrt{-\hat{t}}\sqrt{-t}} X_{\hat{t}}(x) \\ &= \frac{-1}{2\sqrt{-\hat{t}}\sqrt{-t}} \left(\left(X_{\hat{t}}(x) \cdot N_{\hat{t}}(x)\right) N_{\hat{t}}(x) + \sum_{i,j} g_{\hat{t}}^{ij}(x) \left(X_{\hat{t}}(x) \cdot \partial_j X_{\hat{t}}(x)\right) \partial_i X_{\hat{t}}(x) \right) \\ &= F\left(A_i^j(x, t)\right) N(x, t) - \sum_{i,j} \frac{1}{2(-t)} g^{ij}(x, t) \left(X(x, t) \cdot \partial_j X(x, t)\right) \partial_i X(x, t) \end{aligned} \qquad (3.50)$$

in which we use the F self-shrinker equation of $\Sigma_{\hat{t}} = \sqrt{-\hat{t}}\,\Sigma$ (in Definition 2.4) and the homogeneity of F. Note that the normal part agrees with Definition 4 for the F curvature flow. Now consider the following ODE system:

$$\begin{aligned} \partial_t x_i &= \sum_{i,j} \frac{1}{2(-t)} g^{ij}(x, t) \left(X(x, t) \cdot \partial_j X(x, t)\right) \\ x_i\Big|_{t=\hat{t}} &= \xi_i, \quad i = 1, \ldots, n \end{aligned} \qquad (3.51)$$

Let the solution (which exists at least for a while) to be $x = \varphi_t(\xi)$. In other words, φ_t is the local diffeomorphism on Σ_t generated by the tangent vector field $\frac{1}{2(-t)} X(x, t)^{\top}$. By (3.50) and (3.51), the reparametrization $X(\varphi_t(\xi), t)$ of the flow becomes a normal parametrization.

On the other hand, in the radial parametrization, $h(x,t)=\frac{\sqrt{-t}}{\sqrt{-\hat{t}}}h_{\hat{t}}(x)$. Thus, by (3.51) and Lemma 3.3, we get

$$\begin{aligned}\frac{\partial}{\partial t}\left\{h\left(\varphi_t\left(\xi\right),t\right)\right\} &= \partial_t h\left(x,t\right)+\sum_{i,j}\frac{1}{2\left(-t\right)}g^{ij}\left(x,t\right)\left(X\left(x,t\right)\cdot\partial_j X\left(x,t\right)\right)\partial_i h\left(x,t\right)\Big|_{x=\varphi_t(\xi)}\\ &= \frac{1}{2\left(-t\right)}\left\{-h\left(x,t\right)+X\left(x,t\right)\cdot\nabla_{\Sigma_t}h\right\}\Big|_{x=\varphi_t(\xi)}\\ &= \nabla_{\Sigma_t}\cdot\left(\mathbf{a}\left(\cdot,t\right)dh_t\right)+O\left(|X_t|^{-1}\right)|\nabla_{\Sigma_t}h_t|+O\left(|X_t|^{-2}\right)|h_t|\Big|_{x=\varphi_t(\xi)}\end{aligned}$$

which proves (3.48).

Equation (3.49) follows from (3.47). □

Lastly, we conclude this section by some estimates on the 2-tensor $\mathbf{a}\left(\cdot,t\right)$ on each time-slice Σ_t.

Proposition 3.3 *There exits* $R=R\left(\Sigma,\tilde{\Sigma},\mathcal{C},U,\|F\|_{C^3(U)},\lambda,\varkappa\right)\geq 1$ *so that for* $t\in[-1,0)$*,* $X_t\in\Sigma_t\backslash\bar{B}_R$*, there hold*

$$\frac{\lambda}{3}\leq\mathbf{a}\left(\cdot,t\right)\leq\frac{3}{\lambda} \tag{3.52}$$

$$|X_t|\left|\nabla_{\Sigma_t}\mathbf{a}\left(\cdot,t\right)\right|\leq 3\varkappa \tag{3.53}$$

$$|X_t|^2\left|\nabla^2_{\Sigma_t}\mathbf{a}\left(\cdot,t\right)\right|\leq C\left(n,\mathcal{C},\|F\|_{C^3(U)}\right) \tag{3.54}$$

$$|X_t|^2\left|\partial_t\mathbf{a}\left(\cdot,t\right)\right|\leq C\left(n,\mathcal{C},\|F\|_{C^3(U)}\right) \tag{3.55}$$

where the time derivative in the last term is taken with respect to the normal parametrization of the flow $\{\Sigma_t\}_{-1\leq t<0}$.

Proof We adopt the Einstein summation convension throughout the proof.

By using the rescaling argument and the homogeneity of the derivatives of F, (3.52), (3.53), (3.54) follow from (3.35), (3.36), (3.37), respectively. As for (3.55), note that in normal parametrization, we have

$$\partial_t\mathrm{a}^{ij}\left(t\right)=\partial_t\left(\mathrm{a}^i_k(t)\,g^{kj}_t\right)=\left(\partial_t\mathrm{a}^i_k(t)\right)g^{kj}_t+2\mathrm{a}^i_k(t)\,F\left(A^\#_t\right)A^{kj}_t \tag{3.56}$$

in which we use the following evolution equation for the metric along the F curvature flow $\{\Sigma_t\}_{-1\leq t<0}$ (see [1]):

$$\partial_t g_{ij}\left(t\right)=-2F\left(A^\#_t\right)A_{ij}\left(t\right),\quad \partial_t g^{ij}_t=2F\left(A^\#_t\right)A^{ij}_t \tag{3.57}$$

By the rescaling argument, (3.12), and the homogeneity of F and its derivatives, we can estimate each term in (3.56) by

$$|X_t|^2\left|F\left(A^\#_t\right)A^{ij}_t\right|=\left|F\left(|X_t|\,A^\#_t\right)\cdot|X_t|\,A^{ij}_t\right|\leq C\left(n,\mathcal{C},\|F\|_{C^3(U)}\right)$$

and

$$|X_t|^2|\partial_t \mathrm{a}_j^i| = |X_t|^2 \Big| \int_0^1 \frac{\partial^2 F}{\partial S_i^j \partial S_k^l} \left((1-\theta)\, A_t^\# + \theta \tilde{A}_t{}^\# \right) \cdot \left((1-\theta)\, \partial_t A_k^l + \theta \partial_t \tilde{A}_k^l \right) d\theta \Big|$$
$$= \Big| \int_0^1 \frac{\partial^2 F}{\partial S_i^j \partial S_k^l} \left((1-\theta)\, |X_t|\, A_t^\# + \theta |X_t|\, \tilde{A}_t{}^\# \right)$$
$$\cdot \left((1-\theta)\, |X_t|^3 \partial_t A_k^l + \theta |X_t|^3 \partial_t \tilde{A}_k^l \right) d\theta \Big|$$
$$\leq C\left(n, \mathcal{C}, \| F \|_{C^3(U)}\right) \Big| \int_0^1 \left((1-\theta)\, |X_t|^3 \partial_t A_k^l + \theta |X_t|^3 \partial_t \tilde{A}_k^l \right) d\theta \Big|$$

Thus, to establish (3.55), it suffices to show that

$$|X_t|^3 |\partial_t A_t^\#| \leq C\left(n, \mathcal{C}, \| F \|_{C^3(U)}\right) \tag{3.58}$$
$$|X_t|^3 |\partial_t \tilde{A}_t^\# - \partial_t A_t^\#| \leq C\left(n, \mathcal{C}, \| F \|_{C^3(U)}\right) \tag{3.59}$$

for all $X_t \in \Sigma_t \backslash \bar{B}_R$, $t \in [-1, 0)$.

Firstly, let's recall the evolution equation for the shape operator $A_t^\#$ in the normal parametrization along the flow (see [1]):

$$\partial_t A_i^j(t) = \frac{\partial F}{\partial S_k^l}\left(A_t^\#\right) \cdot g_t^{lm} \nabla_{km}^2 A_i^j + \frac{\partial F}{\partial S_k^l}\left(A_t^\#\right) \cdot \left(A_t^2\right)_k^l A_i^j(t)$$
$$+ \frac{\partial^2 F}{\partial S_k^l \partial S_p^q}\left(A_t^\#\right) \cdot g_t^{jm} \nabla_i A_k^l(t) \nabla_m A_p^q(t) \tag{3.60}$$

which yields (3.58) by the rescaling argument, (3.15) and the homogeneity of F and its derivatives.

Secondly, we would like to compute $\partial_t \left(\tilde{A}_t^\# - A_t^\#\right)$ in the normal parametrization (of $\{\Sigma_t\}_{-1 \leq t < 0}$) by using the same trick as in the proof of Proposition 3.2. Fix $\hat{t} \in [-1, 0)$, $\hat{X} \in \Sigma_{\hat{t}} \backslash \bar{B}_R$, and take a local coordinate $x = (x_1, \ldots, x_n)$ of $\Sigma_{\hat{t}}$ which is normal at $\hat{X} = X(0)$. Consider the radial parametrization of the flow starting at time $\hat{t}$ near the point $\hat{X}$ by $X(x, t) = \frac{\sqrt{-t}}{\sqrt{-\hat{t}}} X_{\hat{t}}(x)$. Then we have

$$\tilde{A}_i^j(x, t) - A_i^j(x, t) = \frac{\sqrt{-\hat{t}}}{\sqrt{-t}} \left(\tilde{A}_i^j\left(x, \hat{t}\right) - A_i^j\left(x, \hat{t}\right) \right)$$

Let $x = \varphi_t(\xi)$ with $\varphi_{\hat{t}} = \mathrm{id}$ to be the local diffeomorphism on Σ_t generated by the tangent vector field $\frac{1}{2(-t)} X(\cdot, t)^\top$ as before. Then the reparametrization $X(\varphi_t(\xi), t)$ of the flow becomes a normal parametrization and we have

$$\partial_t \left(\tilde{A}_i^j(\varphi_t(\xi), t) - A_i^j(\varphi_t(\xi), t) \right) \Big|_{\xi=0, t=\hat{t}} = \left(\partial_t \tilde{A}_i^j - \partial_t A_i^j \right)(\varphi_t(\xi), t)$$
$$+ \frac{1}{2(-t)} g^{kl}(\varphi_t(\xi), t)\, \left(X_t(\varphi_t(\xi), t) \cdot \partial_l X_t(\varphi_t(\xi), t)\right)$$
$$\left(\partial_k \tilde{A}_i^j(\varphi_t(\xi), t) - \partial_k A_i^j(\varphi_t(\xi), t) \right) \Big|_{\xi=0, t=\hat{t}}$$
$$= \frac{1}{2(-\hat{t})} \left\{ \left(\tilde{A}_i^j(\hat{t}) - A_i^j(\hat{t}) \right) + g_{\hat{t}}^{kl} \left(X_{\hat{t}} \cdot \partial_l X_{\hat{t}} \right) \left(\nabla_k \tilde{A}_i^j(\hat{t}) - \nabla_k A_i^j(\hat{t}) \right) \right\} \Big|_{\hat{X}} \tag{3.61}$$

Note that for each $t \in [-1, 0)$, by the rescaling argument and (3.17), we have

$$
\begin{aligned}
&\| \, |X_t|^3 \left(\tilde{A}_t^{\#} - A_t^{\#}\right) \|_{L^\infty(\Sigma_t \setminus \bar{B}_R)} + \| \, |X_t|^4 \left(\nabla_{\Sigma_t} \tilde{A}_t^{\#} - \nabla_{\Sigma_t} A_t^{\#}\right) \|_{L^\infty(\Sigma_t \setminus \bar{B}_R)} \\
&\le \left\{ \| \, |X|^3 \left(\tilde{A}^{\#} - A^{\#}\right) \|_{L^\infty(\Sigma \setminus \bar{B}_R)} + \| \, |X|^4 \left(\nabla_{\Sigma} \tilde{A}^{\#} - \nabla_{\Sigma} A^{\#}\right) \|_{L^\infty(\Sigma \setminus \bar{B}_R)} \right\} (-t) \\
&\le C \left(n, \mathcal{C}, \| \, F \, \|_{C^3(U)}\right) (-t) \qquad (3.62)
\end{aligned}
$$

Combining (3.61) and (3.62) to get (3.59). □

4 Carleman's inequalities and uniqueness of *F* self-shrinkers with a tangent cone

This section is a continuation of the previous section. Here we still assume that Σ and $\tilde{\Sigma}$ are properly embedded F self-shrinkers (in Definition 2.4) which are C^5 asymptotic to the cone $\mathcal{C}$ at infinity, and they induce F curvature flows $\{\Sigma_t\}_{-1 \le t \le 0}$ and $\left\{\tilde{\Sigma}_t\right\}_{-1 \le t \le 0}$ with $\Sigma_t = \sqrt{-t}\,\Sigma$, $\tilde{\Sigma}_t = \sqrt{-t}\,\tilde{\Sigma}$ for $t \in [-1, 0)$ and $\Sigma_0 = \mathcal{C} = \tilde{\Sigma}_0$. We also consider the deviation $h_t = h(\cdot, t)$ of $\tilde{\Sigma}_t$ from Σ_t for $t \in [-1, 0]$ (we set $h_0 = 0$), which is defined on $\Sigma_t \setminus \bar{B}_R$ with $R \gg 1$ (depending on $\Sigma, \tilde{\Sigma}, \mathcal{C}, U, \| \, F \, \|_{C^3(U)}, \lambda, \varkappa$). For the function h, recall that we have Proposition 3.2 and Proposition 3.3. Note that the Einstein summation convension is adopted throughout this section (i.e. summing over repeated indicies).

At the beginning, we will like to improve the decay rate of h_t as $t \nearrow 0$ in (3.47) to exponential decay. To achieve that, we need Proposition 4.1, which is due to [4] and [11] for different cases. The proof (of Proposition 4.1) will be included here for readers' convenience, and it is based on two crucial lemmas. The first one is a mean value inequality for parabolic equations from [9].

Lemma 4.1 (Mean value inequality) *Let $P = \partial_t - \partial_i \left(a^{ij}(x, t)\, \partial_j\right)$ be a differential operator such that $a_t^{ij} = a^{ij}(\cdot, t) \in C^1\left(B_1^n\right)$ for $t \in [-1, 0]$, $a^{ij} = a^{ji}$, and*

$$
\boldsymbol{\lambda} \delta^{ij} \le a^{ij} \le \frac{1}{\boldsymbol{\lambda}} \delta^{ij}
$$
$$
|a^{ij}(x, t) - a^{ij}\left(\tilde{x}, \tilde{t}\right)| \le L \left(|x - \tilde{x}| + |t - \tilde{t}|^{\frac{1}{2}}\right)
$$

for some $\boldsymbol{\lambda} \in (0, 1]$, $L > 0$, where $B_1^n = \left\{x \in \mathbb{R}^n \middle| \, |x| < 1\right\}$.

Suppose that $\mathrm{u} \in C^{2,1}\left(B_1^n \times [-T, 0]\right)$ satisfies

$$
|P\mathrm{u}| \le L \left(\frac{1}{\sqrt{T}} |\partial_x \mathrm{u}| + \frac{1}{T} |\mathrm{u}|\right)
$$

for some $T \in (0, 1]$, then there holds

$$
|\mathrm{u}(x, t)| + \sqrt{-t}\, |\partial_x \mathrm{u}(x, t)| \le C(n, \boldsymbol{\lambda}, L) \fint_{Q(x, t;\, \sqrt{-t})} |\mathrm{u}|
$$

for $(x, t) \in Q\left(0, 0; \frac{T}{2}\right)$, where $Q(x, t;\, r) = B_r^n(x) \times (-r^2, 0)$ is the parabolic cylinder centered at (x, t) and $\fint_{\mathcal{D}}$ means the average of a function on the domain $\mathcal{D}$.

Remark 4.1 To prove the above lemma, we may consider the following change of variables:

$$(x,t)=\left(\sqrt{T}\,\tilde{x},T\,\tilde{t}\right)$$

In the new variables, the equation in Lemma 4.1 becomes

$$\left|\partial_{\tilde{t}}\mathrm{u}-\partial_{\tilde{x}_i}\left(a^{ij}\left(\sqrt{T}\,\tilde{x},\ T\,\tilde{t}\right)\partial_{\tilde{x}_j}\mathrm{u}\right)\right|\le L\left(|\partial_{\tilde{x}}\mathrm{u}|+|\mathrm{u}|\right)$$

for $\tilde{x}\in B^n_{1/\sqrt{T}}$, $\tilde{t}\in[-1,0]$. Then apply the standard theorem from [9] to the new equation.

The second lemma is a local type of Carleman's inequalities from [5].

Lemma 4.2 (Local Carleman's inequality) *Let $P=\partial_t-\partial_i\left(a^{ij}(x,t)\,\partial_j\right)$ be a differential operator such that $a^{ij}_t=a^{ij}(\cdot,t)\in C^1\left(B^n_1\right)$ for $t\in[-1,0]$, $a^{ij}=a^{ji}$, $a^{ij}(0,0)=\delta^{ij}$ and*

$$\boldsymbol{\lambda}\delta^{ij}\le a^{ij}\le\frac{1}{\boldsymbol{\lambda}}\delta^{ij}$$
$$|a^{ij}(x,t)-a^{ij}\left(\tilde{x},\tilde{t}\right)|\le L\left(|x-\tilde{x}|+|t-\tilde{t}|^{\frac{1}{2}}\right)$$

for some $\boldsymbol{\lambda}\in(0,1]$, $L>0$, where $B^n_1=\left\{x\in\mathbb{R}^n\,\middle|\,|x|<1\right\}$.

Then for any fixed constant $M\ge 4$, there exists a non-increasing function $\boldsymbol{\varphi}:\left(-\frac{4}{M},0\right)\to\mathbb{R}_+$ satisfying $\frac{-t}{\sigma}\le\boldsymbol{\varphi}(t)\le -t$ for some constant $\sigma=\sigma(n,\boldsymbol{\lambda},L)\ge 1$, so that for any constant $\delta\in\left(0,\frac{1}{M}\right)$ and function $\mathrm{v}\in C^{2,1}_c\left(B^n_1\times(-\frac{2}{M},0]\right)$, there holds

$$\begin{aligned}
&M^2\int \mathrm{v}^2\boldsymbol{\varphi}_\delta^{-M}\boldsymbol{\Phi}_\delta\,dx\,dt+M\int|\partial_x\mathrm{v}|^2\boldsymbol{\varphi}_\delta^{1-M}\boldsymbol{\Phi}_\delta\,dx\,dt\\
&\le\sigma\int|P\mathrm{v}|^2\boldsymbol{\varphi}_\delta^{1-M}\boldsymbol{\Phi}_\delta\,dx\,dt\\
&\quad+(\sigma M)^M\sup_{t<0}\int\left(|\partial_x\mathrm{v}|^2+\mathrm{v}^2\right)dx+\sigma M\int\mathrm{v}^2\boldsymbol{\varphi}_\delta^{-M}\boldsymbol{\Phi}_\delta\,dx\Big|_{t=0}
\end{aligned}$$

where $\boldsymbol{\varphi}_\delta(t)=\boldsymbol{\varphi}(t-\delta)$ and $\boldsymbol{\Phi}_\delta(x,t)=\boldsymbol{\Phi}(x,t-\delta)=\frac{1}{(4\pi(-t+\delta))^{\frac{n}{2}}}\exp\left(-\frac{|x|^2}{4(-t+\delta)}\right)$.

Remark 4.2 Note that the last term on the RHS of the above inequality vanishes provided that $\mathrm{v}\Big|_{t=0}=0$.

Now we state the proposition (of showing the exponential deay) and then follow [4,11] to give it a proof:

Proposition 4.1 (Exponential decay/Unique continuation principle) *Let $P=\partial_t-\partial_i\left(a^{ij}(x,t)\,\partial_j\right)$ be a differential operator such that $a^{ij}_t=a^{ij}(\cdot,t)\in C^1\left(B^n_1\right)$ for $t\in[-1,0]$, $a^{ij}=a^{ji}$, and*

$$\boldsymbol{\lambda}\delta^{ij}\le a^{ij}\le\frac{1}{\boldsymbol{\lambda}}\delta^{ij}$$
$$|a^{ij}(x,t)-a^{ij}\left(\tilde{x},\tilde{t}\right)|\le L\left(|x-\tilde{x}|+|t-\tilde{t}|^{\frac{1}{2}}\right)$$

for some $\boldsymbol{\lambda}\in(0,1]$, $L>0$, where $B^n_1=\left\{x\in\mathbb{R}^n\,\middle|\,|x|<1\right\}$.

Suppose that $\mathrm{u} \in C^{2,1}\left(B_1^n \times [-T, 0]\right)$ *satisfies*

$$|P\mathrm{u}| \leq L\left(\frac{1}{\sqrt{T}}|\partial_x \mathrm{u}| + \frac{1}{T}|\mathrm{u}|\right) \tag{4.1}$$

for some $T \in (0, 1]$*, and that either u vanishes at* $(0, 0)$ *to infinite order (see [4]), i.e.*

$$\forall k \in \mathbb{N} \quad \exists C_k > 0 \quad s.t.\ |\mathrm{u}(x, t)| \leq C_k \left(|x| + \sqrt{-t}\right)^k \tag{4.2}$$

or u *vanishes identically at* $t = 0$ *(see* [11]*), i.e.*

$$\mathrm{u}\Big|_{t=0} = 0 \tag{4.3}$$

Then there exist $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(n, \boldsymbol{\lambda}, L) > 0$*,* $\boldsymbol{\alpha} = \boldsymbol{\alpha}(n, \boldsymbol{\lambda}, L) \in (0, 1)$ *so that*

$$\begin{aligned} &|\mathrm{u}(x, t)| + |\partial_x \mathrm{u}(x, t)| \\ &\leq \boldsymbol{\Lambda}\, e^{\frac{1}{\Lambda t}} \left(\| \partial_x \mathrm{u} \|_{L^\infty(B_1 \times [-T, 0])} + \| \mathrm{u} \|_{L^\infty(B_1 \times [-T, 0])}\right) \end{aligned} \tag{4.4}$$

for $x \in B_{1/4}^n$*,* $t \in [-\boldsymbol{\alpha} T, 0)$*.*

Remark 4.3 Later we will apply Proposition 4.1 under the condition (4.3) to show the exponential decay of the deviation h as $t \nearrow 0$. On the other hand, the proposition implies that under the condtion (4.2), the function u in (4.1) must vanish identically at $t = 0$; in particular, it implies that u vanishes identically in the case when u is time-independent. Such phenomenon is called the "unique continuation principle" and will be used at the end of this section.

Proof For simplicity, we may assume that $a^{ij}(0, 0) = \delta^{ij}$. Otherwise, we can do change of variables like $\tilde{x} = a^{ij}(0, 0)^{-\frac{1}{2}} x$ to achieve that.

In the proof, we will focus on dealing with the case of (4.2), since the same argument work for the case of (4.3) with only a slight difference, which we will point out on the way of proof.

Fix a constant $M \in [\frac{4L^2(n+\sigma)}{T}, \infty)$ (to be chosen), where $\sigma = \sigma(n, \boldsymbol{\lambda}, L) \geq 1$ is the constant that appears in Lemma 4.1. Then for any $\epsilon \in \left(0, \min\left\{\frac{1}{M}, 1\right\}\right)$, choose smooth cut-off functions $\zeta = \zeta(x)$, $\eta_\epsilon = \eta_\epsilon(t)$ and $\eta = \eta(t)$ such that

$$\begin{gathered} \chi_{B_{1/2}^n} \leq \zeta \leq \chi_{B_1^n}, \quad \| \zeta \|_{C^2} \leq 4 \\ \chi_{\left[\frac{-1}{M}, -\epsilon\right]} \leq \eta_\epsilon \leq \chi_{\left[\frac{-2}{M}, -\frac{\epsilon}{2}\right]}, \quad \chi_{\left[\frac{-1}{M}, 0\right]} \leq \eta \leq \chi_{\left[\frac{-2}{M}, 0\right]}, \quad \eta_\epsilon \nearrow \eta \quad \text{as } \epsilon \searrow 0 \\ |\partial_t \eta_\epsilon| \leq 2M \chi_{\left[\frac{-2}{M}, \frac{-1}{M}\right]} + \frac{3}{\epsilon}\chi_{[-\epsilon, 0]} \end{gathered}$$

where $\chi_{B_1^n}$ is the characteristic function of B_1^n. Let $\mathrm{v}_\epsilon(x, t) = \zeta(x)\, \eta_\epsilon(t)\, \mathrm{u}(x, t)$ be a localization of u, which satisfies $\mathrm{v}_\epsilon\Big|_{t=0} = 0$ and convergers pointwisely to $\mathrm{v}(x, t) = \zeta(x)\, \eta(t)\, \mathrm{u}(x, t)$ as $\epsilon \searrow 0$. By the product rule, we have

$$\begin{aligned} P\mathrm{v}_\epsilon &= P\left(\zeta(x)\, \eta_\epsilon(t)\, \mathrm{u}(x, t)\right) \\ &= \zeta(x)\, \eta_\epsilon(t)\, P\mathrm{u} + \left(P\left(\zeta(x)\, \eta_\epsilon(t)\right)\right) \mathrm{u} - 2a^{ij} \partial_i \left(\zeta(x)\, \eta_\epsilon(t)\right) \partial_j \mathrm{u} \\ &= \zeta(x)\, \eta_\epsilon(t)\, P\mathrm{u} + \left((\partial_t \eta_\epsilon)\, \zeta(x) - \eta_\epsilon(t)\, \partial_i\left(a^{ij} \partial_j \zeta\right)\right) \mathrm{u} - 2a^{ij} \eta_\epsilon(t)\, \partial_i \zeta\, \partial_j \mathrm{u} \end{aligned}$$

By (4.1), it follows that

$$
\begin{aligned}
|P\mathrm{v}_\epsilon| \le\ & \zeta\eta_\epsilon L\left(\frac{1}{\sqrt{T}}|\partial_x \mathrm{u}| + \frac{1}{T}|\mathrm{u}|\right) \\
& + C(\boldsymbol{\lambda}, L)\left(|\partial_x \mathrm{u}| + |\mathrm{u}|\right)\chi_{B_1\setminus B_{\frac{1}{2}}}(x) + 2LM|\mathrm{u}|\chi_{\left[\frac{-2}{M},\frac{-1}{M}\right]}(t) + \frac{2L}{\epsilon}|\mathrm{u}|\chi_{[-\epsilon,0]}(t) \\
\le\ & L\left(\frac{1}{\sqrt{T}}|\partial_x \mathrm{v}_\epsilon| + \frac{1}{T}|\mathrm{v}_\epsilon|\right) + C(\boldsymbol{\lambda}, L)M\left(|\partial_x \mathrm{u}| + |\mathrm{u}|\right)\chi_E(x,t) \\
& + \frac{2L}{\epsilon}|\mathrm{u}|\chi_{[-\epsilon,0]}(t) \qquad (4.5)
\end{aligned}
$$

where

$$E = \left\{(x,t) \in B_1^n \times [-1,0) \,\Big|\, \frac{1}{2} \le |x| \le 1 \text{ or } \frac{-2}{M} \le t \le \frac{-1}{M}\right\}$$

Note that in the case of (4.3), it suffices to consider v (without using the ϵ cut-off) in order to make the function vanishing at $t = 0$. By (4.1).

Then for each $\delta \in \left(0, \frac{1}{M}\right)$, by Lemma 4.1 (applied to v_ϵ) and (4.5), there holds

$$
\begin{aligned}
& M^2\int \mathrm{v}_\epsilon^2 \boldsymbol{\varphi}_\delta^{-M}\boldsymbol{\Phi}_\delta\,dx\,dt + M\int |\partial_x \mathrm{v}_\epsilon|^2\boldsymbol{\varphi}_\delta^{1-M}\boldsymbol{\Phi}_\delta\,dx\,dt \\
& \le 2\sigma L^2\int\left(\frac{\mathrm{v}_\epsilon^2}{T^2} + \frac{|\partial_x \mathrm{v}_\epsilon|^2}{T}\right)\boldsymbol{\varphi}_\delta^{1-M}\boldsymbol{\Phi}_\delta\,dx\,dt \\
& \quad + 2C(\boldsymbol{\lambda}, L)\sigma M^2\int_E\left(|\partial_x \mathrm{u}|^2 + \mathrm{u}^2\right)\boldsymbol{\varphi}_\delta^{1-M}\boldsymbol{\Phi}_\delta\,dx\,dt \\
& \quad + \frac{4\sigma L^2}{\epsilon^2}\int_{-\epsilon}^0\int_{B_1}\mathrm{u}^2\boldsymbol{\varphi}_\delta^{1-M}\boldsymbol{\Phi}_\delta\,dx\,dt + (\sigma M)^M\sup_t\int\left(|\partial_x \mathrm{v}_\epsilon|^2 + \mathrm{v}_\epsilon^2\right)dx
\end{aligned}
$$

By our choice of M, the first term on the RHS of the above inequality can be absorbed by its LHS. Thus, we get

$$
\begin{aligned}
& M^2\int \mathrm{v}_\epsilon^2\boldsymbol{\varphi}_\delta^{-M}\boldsymbol{\Phi}_\delta\,dx\,dt \le C(\boldsymbol{\lambda}, L)\sigma M^2\int_E\left(|\partial_x \mathrm{u}|^2 + \mathrm{u}^2\right)\boldsymbol{\varphi}_\delta^{1-M}\boldsymbol{\Phi}_\delta\,dx\,dt \\
& + 4(\sigma M)^M\sup_{-T\le t\le 0}\int_{B_1}\left(|\partial_x \mathrm{u}|^2 + \mathrm{u}^2\right)dx + \frac{5\sigma L^2}{\epsilon^2}\int_{-\epsilon}^0\int_{B_1}\mathrm{u}^2\boldsymbol{\varphi}_\delta^{1-M}\boldsymbol{\Phi}_\delta\,dx\,dt \qquad (4.6)
\end{aligned}
$$

Now choose an integer $k \ge M + \frac{n}{2}$, then by (4.2) the last term on the RHS of (4.6) can be estimated by

$$
\begin{aligned}
& \frac{4\sigma L^2}{\epsilon^2}\int_{-\epsilon}^0\int_{B_1}\mathrm{u}^2\boldsymbol{\varphi}_\delta^{1-M}\boldsymbol{\Phi}_\delta\,dx\,dt \\
& \le \frac{4\sigma L^2}{\epsilon^2}\int_{-\epsilon}^0\int_{B_1}\frac{C_k\left(|x| + \sqrt{-t}\right)^{2(M+\frac{n}{2})}}{\left(\frac{-t+\delta}{\sigma}\right)^{M-1}}\,\frac{\exp\left(\frac{-|x|^2}{4(-t+\delta)}\right)}{(4\pi(-t+\delta))^{\frac{n}{2}}}\,dx\,dt \\
& \le C(n, C_k, \sigma, M, L) \\
& \quad \frac{1}{\epsilon^2}\int_{-\epsilon}^0\left\{\int_{B_1}\left(\frac{|x|^2}{-t+\delta} + 1\right)^{M+\frac{n}{2}}\exp\left(\frac{-|x|^2}{4(-t+\delta)}\right)dx\right\}(-t+\delta)\,dt
\end{aligned}
$$

$$\leq C\left(n, C_k, \sigma, M, L\right) \frac{1}{\epsilon^2} \int_{-\epsilon}^{0} \left\{ \int_0^{\infty} \left(|\xi|^2 + 1\right)^{M+\frac{n}{2}} \exp\left(\frac{-|\xi|^2}{4}\right) d\xi \right\} (-t+\delta)^{\frac{n}{2}+1}\, dt$$

$$\leq C\left(n, C_k, \sigma, M, L\right) \frac{(\epsilon+\delta)^{\frac{n}{2}+2} - \delta^{\frac{n}{2}+2}}{\epsilon^2} \tag{4.7}$$

In view of (4.7), apply the monotone convergence theorem to (4.6) by first letting $\delta \searrow 0$ and then $\epsilon \searrow 0$ to arrive at

$$\begin{aligned}
&\int_{B_{\frac{1}{2}} \times \left(\frac{-1}{M}, 0\right)} \mathrm{u}^2 \boldsymbol{\varphi}^{-M} \boldsymbol{\Phi}\, dx\, dt \\
&\leq C(\Lambda, L)\, \sigma \int_E \left(|\partial_x \mathrm{u}|^2 + \mathrm{u}^2\right) \boldsymbol{\varphi}^{1-M} \boldsymbol{\Phi}\, dx\, dt + (4\sigma M)^M \sup_{-T \leq t \leq 0} \int_{B_1} \left(|\partial_x \mathrm{u}|^2 + \mathrm{u}^2\right) dx \\
&\leq C(n, \Lambda, L) \\
&\quad \left(\sigma \int_E \boldsymbol{\varphi}^{1-M} \boldsymbol{\Phi}\, dx\, dt + (\sigma M)^M\right) \left(\| \partial_x \mathrm{u} \|_{L^{\infty}(B_1 \times [-T,0])} + \| \mathrm{u} \|_{L^{\infty}(B_1 \times [-T,0])}\right)
\end{aligned} \tag{4.8}$$

Note that in the case of (4.3), we can get (4.8) directly from taking the limit as $\delta \searrow 0$ without using (4.7).

Next, we would like to estimate the first term on the RHS of (4.8). For $(x,t) \in E$, either $\frac{-2}{M} \leq t \leq \frac{-1}{M}$, in which case we have

$$\boldsymbol{\varphi}^{1-M} \boldsymbol{\Phi}(x,t) \leq \left(\frac{-t}{\sigma}\right)^{1-M} \frac{1}{(4\pi(-t))^{\frac{n}{2}}} \leq \frac{(\sigma M)^{M-1+\frac{n}{2}}}{(4\pi\sigma)^{\frac{n}{2}}} \tag{4.9}$$

or $\frac{1}{2} \leq |x| \leq 1$ and $\frac{-2}{M} \leq t < 0$, in which case we have

$$\begin{aligned}
\boldsymbol{\varphi}^{1-M} \boldsymbol{\Phi}(x,t) &\leq \left(\frac{\sigma M}{(-t)\, M}\right)^{M-1} \frac{M^{\frac{n}{2}}}{(4\pi(-t)\, M)^{\frac{n}{2}}} \exp\left(\frac{-M}{16(-tM)}\right) \\
&= \frac{(\sigma M)^{M-1} \left(\frac{M}{4\pi}\right)^{\frac{n}{2}}}{(-tM)^{M-1+\frac{n}{2}} \exp\left(\frac{M/16}{-tM}\right)} \leq (\sigma M)^{M-1} \left(\frac{M}{4\pi}\right)^{\frac{n}{2}} \left(\frac{M-1+\frac{n}{2}}{e\, M/16}\right)^{M-1+\frac{n}{2}} \\
&\leq \left(\frac{16\sigma}{\mathrm{e}} \left(M - 1 + \frac{n}{2}\right)\right)^{M-1+\frac{n}{2}}
\end{aligned} \tag{4.10}$$

Note that in (4.10) we use the fact that the function $\vartheta(\xi) = \xi^{M-1+\frac{n}{2}} \exp\left(\frac{M/16}{\xi}\right)$ achieves its minimum on $\mathbb{R}_+$ at $\xi = \frac{M/16}{M-1+\frac{n}{2}}$.

On the other hand, for any $(y,s) \in B_{\frac{1}{4}} \times [\frac{-1}{8M}, 0)$, the parabolic cylinder $Q\left(y, s; \sqrt{-s}\right) = B^n_{\sqrt{-s}}(y) \times (2s, s)$ is contained in $B^n_{1/2} \times \left(\frac{-1}{M}, 0\right)$ and hence the LHS of (4.8) is bounded below by

$$\int_{B^n_{1/2} \times \left(\frac{-1}{M}, 0\right)} \mathrm{u}^2 \boldsymbol{\varphi}^{-M} \boldsymbol{\Phi}\, dx\, dt \geq \frac{\exp \frac{-1/4}{-8s}}{(4\pi)^{\frac{n}{2}} (-2s)^{M+\frac{n}{2}}} \int_{Q(y,s;\sqrt{-s})} \mathrm{u}^2\, dx\, dt \tag{4.11}$$

Combining (4.8), (4.9), (4.10), (4.11), we conclude that for $(y,s)\in Q\left(0,0;\ \frac{-1}{8M}\right)$,

$$\begin{aligned}\int_{Q(y,s;\,\sqrt{-s})} & \mathrm{u}^2\,dx\,dt\\ &\le C(n,\boldsymbol{\lambda},L,\sigma)\left(\frac{64\sigma}{\mathrm{e}}(-sM)\right)^{M-1+\frac{n}{2}}\left(\|\,\partial_x\mathrm{u}\,\|_{L^\infty(B_1\times[-T,0])}+\|\,\mathrm{u}\,\|_{L^\infty(B_1\times[-T,0])}\right)\end{aligned}\tag{4.12}$$

Now let $\beta=\frac{1}{2}\left(\frac{64\sigma}{\mathrm{e}}\right)^{-1}$. For each $(y,s)\in B^n_{1/4}\times[\frac{-\beta}{4L^2(n+\sigma)}T,0)$, we choose $M=\frac{\beta}{-s}$ so that $M\ge\frac{4L^2(n+\sigma)}{T}$ (and note that $\frac{-1}{8M}\le s<0$). By (4.12), we get

$$\begin{aligned}⨍_{Q(y,s;\,\sqrt{-s})} & \mathrm{u}^2\,dx\,dt\\ &\le C(n,\Lambda,L,\sigma)(-s)^{-\frac{n}{2}-1}\left(\frac{1}{2}\right)^{-\frac{\beta}{s}-1+\frac{n}{2}}\left(\|\,\partial_x\mathrm{u}\,\|_{L^\infty(B_1\times[-T,0])}+\|\,\mathrm{u}\,\|_{L^\infty(B_1\times[-T,0])}\right)\\ &\le C(n,\boldsymbol{\lambda},L,\sigma)\left(2^{\frac{\beta}{2}}\right)^{\frac{1}{s}}\left(\|\,\partial_x\mathrm{u}\,\|_{L^\infty(B_1\times[-T,0])}+\|\,\mathrm{u}\,\|_{L^\infty(B_1\times[-T,0])}\right)\end{aligned}\tag{4.13}$$

Let $\boldsymbol{\alpha}=\frac{\beta}{4L^2(n+\sigma)}$, $\boldsymbol{\Lambda}=\max\left\{C(n,\boldsymbol{\lambda},L,\sigma),\left(\frac{\beta}{2}\ln 2\right)^{-1}\right\}$, then (4.4) follows from (4.13) and Lemma 4.1. ☐

Combining Propositions 3.2, 3.3 and 4.1, we can show the exponential decay of h_t as $t\nearrow 0$ as in [12] (see also [11]).

Proposition 4.2 (Exponential decay of the deviation) *There exist* $R=R\left(\Sigma,\tilde{\Sigma},\mathcal{C},U,\|\,F\,\|_{C^3(U)},\lambda,\varkappa\right)\ge 1$, $\Lambda=\Lambda\left(n,\mathcal{C},\|\,F\,\|_{C^3(U)},\lambda\right)>0$, $\alpha=\alpha\left(n,\mathcal{C},\|\,F\,\|_{C^3(U)},\lambda\right)\in(0,1)$ *such that for* $X\in\Sigma_t\backslash\bar{B}_R$, $t\in[-\alpha,0)$, *there holds*

$$|\nabla_{\Sigma_t}h|+|h|\le\Lambda\,\exp\left(\frac{|X|^2}{\Lambda t}\right)$$

Proof Fix $\hat{X}\in\Sigma\backslash\bar{B}_R$ with $R=R\left(\Sigma,\tilde{\Sigma},\mathcal{C},U,\|\,F\,\|_{C^3(U)},\lambda,\varkappa\right)\ge 1$, first we would like to show that near $\hat{X}$, there is a "normal parametrization" for the flow $\{\Sigma_t\}$ for $t\in[-1,0]$.

Recall that in the beginning of Sect. 3, we show that there exists a constant $\boldsymbol{\rho}=\boldsymbol{\rho}(n,\mathcal{C})\in(0,1)$ so that near $\hat{X}$, each Σ_t is the graph of the function $\mathrm{u}_t=\mathrm{u}(\cdot,t)$ defined on $B^n_{\boldsymbol{\rho}|\hat{X}|}\subset T_{\hat{X}_{\mathcal{C}}}\mathcal{C}$ for $t\in[-1,0]$, where $\hat{X}_{\mathcal{C}}=\Pi\left(\hat{X}\right)$ is the the normal projection of $\hat{X}$ onto $\mathcal{C}$. Note that $|\hat{X}_{\mathcal{C}}|$ is comparable with $|\hat{X}|$. In other words, locally near $\hat{X}$, we have the following "vertical parametrization" of the flow $\{\Sigma_t\}_{-1\le t\le 0}$:

$$X=X(x,t)\equiv\hat{X}_{\mathcal{C}}+(x,\mathrm{u}(x,t))$$

Here we assume that the unit-normal of $\mathcal{C}$ at $\hat{X}_{\mathcal{C}}$ to be $(0,1)$ for ease of notation. For this vertical parametrization, we may decompose the velocity vector into normal and tangential components as follows:

$$\partial_t X=F\left(A^{\#}(x,t)\right)N(x,t)+\sum_{i=1}^{n}\frac{\partial_i\mathrm{u}\,\partial_t\mathrm{u}}{1+|\partial_x\mathrm{u}|^2}\,\partial_i X$$

where $A^{\#}(x,t)$, $N(x,t)$ are the shape operator and the unit-normal of Σ_t at $X(x,t)$, respectively. Note that the normal component is given by Definition 2.4 for the F curvature flow.

Next, we would like to do suitable change of variables to go from this "vertical parametrization" to the "noramal parametrization" of the flow (see Definition 3.2). For that purpose, we use the same trick as in Proposition 3.2. Let $x = \phi_t(\xi)$ with $\phi_{-1} = \mathrm{id}$ to be the local diffeomorphism on Σ_t generated by the following tangent vector field:

$$\mathcal{V}(x,t) = -\sum_{i=1}^{n} \frac{\partial_i \mathrm{u}\, \partial_t \mathrm{u}}{1+|\partial_x \mathrm{u}|^2}\, \partial_i X \equiv -\sum_{i=1}^{n} \mathcal{V}^i(x,t)\, \partial_i X$$

That is, $\phi_t(\xi) = \phi(\xi,t)$ satisfies

$$\partial_t \phi_t = \left(\mathcal{V}^1(\phi_t,t),\ldots,\mathcal{V}^n(\phi_t,t)\right), \quad \phi_{-1}(\xi) = \xi \tag{4.14}$$

in which, by (3.4) and (3.9), we have

$$|\mathcal{V}^i| \le C\left(n,\mathcal{C},\|F\|_{C^3(U)}\right)|\hat{X}|^{-1} \quad \forall i = 1,\ldots,n \tag{4.15}$$

Thus, by taking R sufficiently large, ϕ_t is well-defined for $\xi \in B^n_{\frac{\varrho}{2}|\hat{X}|}$, $t \in [-1,0]$. It follows that the reparametrization $X = X(\phi_t(\xi),t)$ of the flow becomes a "normal parametrization" near $\hat{X}$ for $t \in [-1,0]$; that is,

$$\frac{\partial}{\partial t}\left(X(\phi_t(\xi),t)\right) = F\left(A^{\#}(\phi_t(\xi),t)\right) N(\phi_t(\xi),t)$$

Let $g_{ij}(\xi,t) = \partial_{\xi_i}(X(\phi_t(\xi),t)) \cdot \partial_{\xi_j}(X(\phi_t(\xi),t))$ be the pull-back metric associated with this "normal parametrization", then by the evolution equation for the metric in [1], the homogeneity of F and the condition that $\phi_{-1} = \mathrm{id}$, we have

$$\begin{aligned}\partial_t g_{ij}(\xi,t) &= -2F\left(A^{\#}(\phi_t(\xi),t)\right) A_{ij}(\phi_t(\xi),t) &(4.16)\\ &= -2\Big|X(\phi_t(\xi),t)\Big|^{-1} F\left(\Big|X(\phi_t(\xi),t)\Big| A^{\#}(\phi_t(\xi),t)\right) A_{ij}(\phi_t(\xi),t)\\ g_{ij}(\xi,-1) &= \delta_{ij} + \partial_i \mathrm{u}(\xi,-1)\, \partial_j \mathrm{u}(\xi,-1) &(4.17)\end{aligned}$$

where the second fundamental form $A_t(x) \sim A_{ij}(x,t)$ is equal to

$$A_{ij}(x,t) = \frac{\partial^2_{ij}\mathrm{u}(x,t)}{\sqrt{1+|\partial_x \mathrm{u}(x,t)|^2}} \tag{4.18}$$

By (4.18), (3.1), (3.2), (3.3), (3.12) and the comparability of $|X(x,t)|$ and $|\hat{X}|$, the ℓ^2 norm of the matrix $\partial_t g_{ij}(\xi,t)$ satisfies

$$|\partial_t g_{ij}(\xi,t)| \le C\left(n,\mathcal{C},\|F\|_{C^1(U)}\right)|\hat{X}|^{-2} \tag{4.19}$$

So by (4.17), (3.1), (3.3) and (4.19), the pull-back metric $g_{ij}(\xi,t)$ is equivalent to the dot product δ_{ij}.

Let $\Gamma^k_{ij}(\xi,t)$ be the Christoffel symbols associated with the metric $g_{ij}(\xi,t)$, then we have

$$\partial_t \Gamma^k_{ij} = \frac{1}{2} g^{kl}\left(\nabla_i \dot{g}_{lj} + \nabla_j \dot{g}_{il} - \nabla_l \dot{g}_{ij}\right) \tag{4.20}$$

$$\Gamma^k_{ij}(\xi,-1) = \frac{\partial_k \mathrm{u}(\xi,-1)\, \partial^2_{ij}\mathrm{u}(\xi,-1)}{\sqrt{1+|\partial_x \mathrm{u}(\xi,-1)|^2}} \tag{4.21}$$

where $\dot{g}_{ij} = \partial_t g_{ij} = -2F\left(A^{\#}\right) A_{ij}$. Similarly, and also by (3.15), the homogeneity of the derivative of F, the equivalence of g_{ij} and δ_{ij}, we have

$$|\partial_t \Gamma_{ij}^k| \leq C\left(n, \mathcal{C}, \| F \|_{C^1(U)}\right) |\hat{X}|^{-3}$$
$$|\Gamma_{ij}^k(\xi, -1)| \leq C\left(n, \mathcal{C}, \| F \|_{C^1(U)}\right) |\hat{X}|^{-1}$$

which implies

$$|\Gamma_{ij}^k(\xi, t)| \leq C\left(n, \mathcal{C}, \| F \|_{C^1(U)}\right) |\hat{X}|^{-1} \tag{4.22}$$

Now consider the deviation h in the local coordinates (ξ, t), then the equation in Proposition 3.2 becomes

$$\left|\partial_t h - \left\{\partial_{\xi_i}\left(\mathbf{a}^{ij}(\xi, t)\, \partial_{\xi_j} h\right) + \Gamma_{ik}^i(\xi, t)\, \mathbf{a}^{kj}(\xi, t)\, \partial_{\xi_j} h\right\}\right|$$
$$\leq C\left(n, \mathcal{C}, \| F \|_{C^3(U)}\right)\left(|\hat{X}|^{-1}|\partial_\xi h| + |\hat{X}|^{-2}|h|\right) \tag{4.23}$$
$$h(\xi, 0) = 0 \tag{4.24}$$

where $\mathbf{a}^{ij}(\xi, t) = \mathbf{a}^{ji}(\xi, t)$ satisfies (by Proposition 3.3 and (4.22))

$$\frac{\lambda}{C\left(n, \mathcal{C}, \| F \|_{C^3(U)}\right)}\delta^{ij} \leq \frac{\lambda}{3} g^{ij}(\xi, t) \leq \mathbf{a}^{ij}(\xi, t) \leq \frac{3}{\lambda} g^{ij}(\xi, t)$$
$$\leq \frac{C\left(n, \mathcal{C}, \| F \|_{C^3(U)}\right)}{\lambda}\delta^{ij} \tag{4.25}$$
$$|\hat{X}|\left|\partial_\xi \mathbf{a}^{ij}(\xi, t)\right| + |\hat{X}|^2\left|\partial_t \mathbf{a}^{ij}(\xi, t)\right| \leq C\left(n, \mathcal{C}, \| F \|_{C^3(U)}, \lambda\right) \tag{4.26}$$

Thus, by (4.22), (4.25), (4.17) and (4.19), the equation (4.23) is equivalent to

$$\left|\partial_t h - \partial_{\xi_i}\left(\mathbf{a}^{ij}(\xi, t)\, \partial_{\xi_j} h\right)\right|$$
$$\leq C\left(n, \mathcal{C}, \| F \|_{C^3(U)}, \lambda\right)\left(|\hat{X}|^{-1}|\partial_\xi h| + |\hat{X}|^{-2}|h|\right) \tag{4.27}$$

for $(\xi, t) \in B^n_{\frac{\rho}{2}|\hat{X}|} \times [-1, 0]$.

Let's consider the following change of variables:

$$(\xi, t) = \Xi\left(\bar{\xi}, \bar{t}\right) \equiv \left(\left(\frac{\rho}{2}|\hat{X}|\right)\bar{\xi},\ \left(\frac{\rho}{2}|\hat{X}|\right)^2 \bar{t}\right)$$

and let $\bar{h} = h \circ \psi, \bar{\mathbf{a}}^{ij} = \mathbf{a}^{ij} \circ \psi$. Then (4.27) and (4.24) in the new variables become

$$\left|\partial_{\bar{t}} \bar{h} - \partial_{\bar{\xi}_i}\left(\bar{\mathbf{a}}^{ij}\left(\bar{\xi}, \bar{t}\right) \partial_{\bar{\xi}_j} \bar{h}\right)\right| \leq C\left(n, \mathcal{C}, \| F \|_{C^3(U)}, \lambda, \boldsymbol{\rho}\right)\left(|\partial_{\bar{\xi}} \bar{h}| + |\bar{h}|\right) \tag{4.28}$$
$$\bar{h}\Big|_{\bar{t}=0} = 0 \tag{4.29}$$

and (4.25), (4.26) are translated into

$$\frac{\lambda}{C\left(n, \mathcal{C}, \| F \|_{C^3(U)}\right)}\delta^{ij} \leq \bar{\mathbf{a}}^{ij}\left(\bar{\xi}, \bar{t}\right) \leq \frac{C\left(n, \mathcal{C}, \| F \|_{C^3(U)}\right)}{\lambda}\delta^{ij} \tag{4.30}$$
$$\left|\partial_{\bar{\xi}} \bar{\mathbf{a}}^{ij}\left(\bar{\xi}, \bar{t}\right)\right| + \left|\partial_{\bar{t}} \bar{\mathbf{a}}^{ij}\left(\bar{\xi}, \bar{t}\right)\right| \leq C\left(n, \mathcal{C}, \| F \|_{C^3(U)}, \lambda, \boldsymbol{\rho}\right) \tag{4.31}$$

for $\bar{\xi} \in B_1^n,\ \bar{t} \in \left[-\left(\frac{\rho}{2}|\hat{X}|\right)^{-2}, 0\right]$.

Applying Proposition 20 to $\bar{h}\left(\bar{\xi},\bar{t}\right)$, we may conclude that there exist $\tilde{\Lambda}=\tilde{\Lambda}\left(n,\mathcal{C},\parallel F\parallel_{C^3(U)},\lambda\right)>0$, $\alpha=\alpha\left(n,\mathcal{C},\parallel F\parallel_{C^3(U)},\lambda\right)\in(0,1)$ for which the following holds:

$$\begin{aligned}&|\partial_{\bar{\xi}}\bar{h}|+|\bar{h}|\\&\leq\tilde{\Lambda}\,\exp\left(\frac{1}{\tilde{\Lambda}\bar{t}}\right)\left(\parallel\partial_{\bar{\xi}}\bar{h}\parallel_{L^\infty\left(B_1^n\times\left[-\left(\frac{\rho}{2}|\hat{X}|\right)^{-2},0\right]\right)}+\parallel\bar{h}\parallel_{L^\infty\left(B_1^n\times\left[-\left(\frac{\rho}{2}|\hat{X}|\right)^{-2},0\right]\right)}\right)\end{aligned}\tag{4.32}$$

for $\left(\bar{\xi},\bar{t}\right)\in B^n_{1/4}\times[-\alpha\left(\frac{\rho}{2}|\hat{X}|\right)^{-2},0)$. By undoing change of variables, (4.32) becomes

$$\begin{aligned}&\frac{\rho}{2}\,|\hat{X}|\,|\partial_\xi h|\,+\,|h|\\&\leq\tilde{\Lambda}\,\exp\left(\frac{|\hat{X}|^2}{\tilde{\Lambda}t}\right)\left(\frac{\rho}{2}|\hat{X}|\parallel\partial_\xi h\parallel_{L^\infty\left(B^n_{\frac{\rho}{2}|\hat{X}|}\times[-1,0]\right)}+\parallel h\parallel_{L^\infty\left(B^n_{\frac{\rho}{2}|\hat{X}|}\times[-1,0]\right)}\right)\end{aligned}\tag{4.33}$$

for $(\xi,t)\in B^n_{\frac{\rho}{8}|\hat{X}|}\times[-\alpha,0)$. Note that the pull-back metric $g_{ij}\,(\xi,t)$ is equivalent to the dot product δ_{ij} and that $|X\,(x,t)\,|$ is comparable with $|\hat{X}|$. The conclusion follows immediately. □

Next, we'd like to go from the exponential decay to identically vanishing of the deviation h outside a compact set. To this end, we have to derive a different type of Carleman's inequality on the flow $\{\Sigma_t\}_{-1\leq t\leq 0}$, which is done through two lemmas. The first lemma is a modification of the integral equality in [4].

Lemma 4.3 *Let* $(\mathrm{M},\boldsymbol{g}_t)$ *be a flow of Riemannian manifolds and* P *be a differential operator on the flow defined by*

$$P\mathrm{v}=\partial_t\mathrm{v}-\nabla_{\boldsymbol{g}_t}\cdot(a_t\,d\mathrm{v})\equiv\partial_t\mathrm{v}-\nabla_i\left(a^{ij}\,(\cdot,t)\,\nabla_j\mathrm{v}\right)$$

where $a_t=a(\cdot,t)$ *is a symmetric 2-tensor on* M. *Then given functions* $G,\Psi\in C^{2,1}\,(\mathrm{M}\times[-T,0])$ *with* $G>0$*, define a function* Φ *as*

$$\begin{aligned}\Phi&=\frac{\partial_t G+\nabla_i\left(a^{ij}\nabla_j G\right)+\frac{1}{2}tr\,(\partial_t\boldsymbol{g})\;G}{G}\\&=\partial_t\ln G+\nabla_i\left(a^{ij}\nabla_j\ln G\right)+a^{ij}\nabla_i\ln G\,\nabla_j\ln G+\frac{1}{2}tr\,(\partial_t\boldsymbol{g})\end{aligned}\tag{4.34}$$

and a 2-tensor Υ *as*

$$\begin{aligned}\Upsilon^{ij}&=a^{ik}a^{jl}\nabla^2_{kl}\ln G-\frac{1}{2}\partial_t a^{ij}\\&\quad+\frac{1}{2}\left(a^{ik}\nabla_k a^{jl}+a^{jk}\nabla_k a^{il}-a^{lk}\nabla_k a^{ij}\right)\,\nabla_l\ln G\end{aligned}\tag{4.35}$$

It follows that for any $\mathrm{u} \in C_c^{2,1}(\mathrm{M} \times [-T, 0])$*, there holds*

$$\int_{\mathrm{M}} \left\{ \left(2\Upsilon^{ij} - (\Phi - \Psi)\, a^{ij} \right) \nabla_i \mathrm{u}\, \nabla_j \mathrm{u} + \frac{1}{2} \left(\partial_t \Psi - \nabla_i \left(a^{ij} \nabla_j \Psi \right) + (\Phi - \Psi)\, \Psi \right) \mathrm{u}^2 \right\} G\, d\mu_t$$
$$= \int_{\mathrm{M}} 2\, P\mathrm{u} \left(\partial_t \mathrm{u} + a^{ij} \nabla_i \ln G\, \nabla_j \mathrm{u} + \frac{1}{2} \Psi \mathrm{u} \right) G\, d\mu_t$$
$$- \int_{\mathrm{M}} 2 \left(\partial_t \mathrm{u} + a^{ij} \nabla_i \ln G\, \nabla_j \mathrm{u} + \frac{1}{2} \Psi \mathrm{u} \right)^2 G\, d\mu_t$$
$$- \partial_t \left\{ \int_{\mathrm{M}} \left(a^{ij} \nabla_i \mathrm{u}\, \nabla_j \mathrm{u} - \frac{1}{2} \Psi \mathrm{u}^2 \right) G\, d\mu_t \right\} \tag{4.36}$$

where μ_t *is the volume form of* $\left(\mathrm{M}, \boldsymbol{g}_t\right)$.

Proof Let's begin with

$$\partial_t \left\{ \int_M a^{ij} \nabla_i \mathrm{u}\, \nabla_j \mathrm{u}\, G\, d\mu_t \right\}$$
$$= \int_{\mathrm{M}} \left\{ 2a^{ij} \nabla_j \mathrm{u}\, \nabla_i \partial_t \mathrm{u}\, G + a^{ij} \nabla_i \mathrm{u}\, \nabla_j \mathrm{u} \left(\partial_t G + \frac{1}{2} \mathrm{tr}\,(\partial_t g)\; G \right) + \partial_t a^{ij} \nabla_i \mathrm{u}\, \nabla_j \mathrm{u}\, G \right\} d\mu_t \tag{4.37}$$

in which we use the commutativity

$$\partial_t\, d\mathrm{u} = d\, \partial_t \mathrm{u}, \quad d\mathrm{u} \sim \nabla_i \mathrm{u}$$

and the evolution equation of the volume form:

$$\partial_t\, d\mu_t = \frac{1}{2} \mathrm{tr}\,(\partial_t \boldsymbol{g})\; d\mu_t \tag{4.38}$$

Applying integation by parts on $\left(\mathrm{M}, \boldsymbol{g}_t\right)$, (4.37) becomes

$$\int_{\mathrm{M}} -2 \left(\nabla_i \left(a^{ij} \nabla_j \mathrm{u} \right) + a^{ij} \nabla_i \ln G\, \nabla_j \mathrm{u} \right) \partial_t \mathrm{u}\, G\, d\mu_t$$
$$+ \int_{\mathrm{M}} a^{ij} \nabla_i \mathrm{u}\, \nabla_j \mathrm{u} \left(\partial_t G + \nabla_k \left(a^{kl} \nabla_l G \right) + \frac{1}{2} \mathrm{tr}\,(\partial_t g)\; G \right) d\mu_t$$
$$- \int_{\mathrm{M}} a^{ij} \nabla_i \mathrm{u}\, \nabla_j \mathrm{u}\, \nabla_k \left(a^{kl} \nabla_l G \right) d\mu_t + \int_{\mathrm{M}} \partial_t a^{ij} \nabla_i \mathrm{u}\, \nabla_j \mathrm{u}\, G\, d\mu_t \tag{4.39}$$

By (4.34), integrating by parts twice and the symmetry of a_t, (4.39) becomes

$$-2 \int_{\mathrm{M}} \left(\nabla_i \left(a^{ij} \nabla_j \mathrm{u} \right) + a^{ij} \nabla_i \ln G\, \nabla_j \mathrm{u} \right) \partial_t \mathrm{u}\, G\, d\mu_t + \int_{\mathrm{M}} a^{ij} \nabla_i \mathrm{u}\, \nabla_j \mathrm{u}\, \Phi G\, d\mu_t$$
$$+ \int_{\mathrm{M}} \left\{ \nabla_k a^{ij} \nabla_i \mathrm{u}\, \nabla_j \mathrm{u}\, a^{kl} \nabla_l \ln G \right.$$
$$\left. - 2\nabla_j \left(a^{ij} \nabla_i \mathrm{u} \right) \nabla_k \mathrm{u}\, a^{kl} \nabla_l \ln G - 2a^{ij} \nabla_i \mathrm{u}\, \nabla_k \mathrm{u}\, \nabla_j a^{kl} \nabla_l \ln G \right\} G\, d\mu_t$$
$$-2 \int_{\mathrm{M}} a^{ij} \nabla_i \mathrm{u}\, \nabla_k \mathrm{u}\, a^{kl} \nabla_{jl}^2 G\, d\mu_t + \int_{\mathrm{M}} \partial_t a^{ij} \nabla_i \mathrm{u}\, \nabla_j \mathrm{u}\, G\, d\mu_t \tag{4.40}$$

Then we reorganize (4.40) (in order to make up the term $P\mathrm{u}$) to get

$$
\begin{aligned}
&2\int_{\mathrm{M}}\left\{\left(\partial_t \mathrm{u}-\nabla_i\left(a^{ij}\nabla_j \mathrm{u}\right)\right)\left(\partial_t \mathrm{u}+a^{kl}\nabla_k \ln G\,\nabla_l \mathrm{u}\right)-(\partial_t \mathrm{u})^2-2a^{ij}\nabla_i \ln G\,\nabla_j \mathrm{u}\,\partial_t \mathrm{u}\right\}G\,d\mu_t\\
&+\int_{\mathrm{M}}\Phi a^{ij}\nabla_i \mathrm{u}\,\nabla_j \mathrm{u}\,G\,d\mu_t-2\int_{\mathrm{M}}a^{ij}a^{kl}\left(\nabla_{jl}^2 \ln G+\nabla_j \ln G\,\nabla_l \ln G\right)\nabla_i \mathrm{u}\,\nabla_k \mathrm{u}\,G\,d\mu_t\\
&+\int_{\mathrm{M}}\left\{a^{kl}\nabla_k a^{ij}\,\nabla_l \ln G\nabla_i \mathrm{u}\,\nabla_j \mathrm{u}-2a^{ij}\nabla_j a^{kl}\,\nabla_l \ln G\,\nabla_i \mathrm{u}\,\nabla_k \mathrm{u}+\partial_t a^{ij}\nabla_i \mathrm{u}\,\nabla_j \mathrm{u}\right\}G\,d\mu_t
\end{aligned}
\tag{4.41}
$$

By (4.35), (4.41) becomes

$$
\begin{aligned}
&2\int_{\mathrm{M}}\left\{\left(\partial_t \mathrm{u}-\nabla_i\left(a^{ij}\nabla_j \mathrm{u}\right)\right)\left(\partial_t \mathrm{u}+a^{kl}\nabla_k \ln G\,\nabla_l \mathrm{u}\right)-\left(\partial_t \mathrm{u}+a^{ij}\nabla_i \ln G\,\nabla_j \mathrm{u}\right)^2\right\}G\,d\mu_t\\
&+\int_{\mathrm{M}}\Phi a^{ij}\nabla_i \mathrm{u}\,\nabla_j \mathrm{u}\,G\,d\mu_t-2\int_{\mathrm{M}}\Upsilon^{ij}\nabla_i \mathrm{u}\,\nabla_j \mathrm{u}\,G\,d\mu_t\\
&=2\int_{\mathrm{M}}P\mathrm{u}\left(\partial_t \mathrm{u}+a^{ij}\nabla_i \ln G\,\nabla_j \mathrm{u}+\frac{1}{2}\Psi \mathrm{u}\right)G\,d\mu_t\\
&\quad-\int_{\mathrm{M}}\left(\partial_t \mathrm{u}-\nabla_i\left(a^{ij}\nabla_j \mathrm{u}\right)\right)\Psi \mathrm{u}\,G\,d\mu_t\\
&\quad-2\int_{\mathrm{M}}\left(\partial_t \mathrm{u}+a^{ij}\nabla_i \ln G\,\nabla_j \mathrm{u}+\frac{1}{2}\Psi \mathrm{u}\right)^2 G\,d\mu_t\\
&\quad+2\int_{\mathrm{M}}\left(\partial_t \mathrm{u}+a^{ij}\nabla_i \ln G\,\nabla_j \mathrm{u}+\frac{1}{2}\Psi \mathrm{u}\right)\Psi \mathrm{u}\,G\,d\mu_t\\
&\quad-\frac{1}{2}\int_{\mathrm{M}}\Psi^2 \mathrm{u}^2 G\,d\mu_t-\int_{\mathrm{M}}\left(2\Upsilon^{ij}-\Phi a^{ij}\right)\nabla_i \mathrm{u}\,\nabla_j \mathrm{u}\,G\,d\mu_t
\end{aligned}
\tag{4.42}
$$

For the second term of (4.42), by the product rule and integration by parts, we get

$$
\begin{aligned}
&-\int_{\mathrm{M}}\left(\partial_t \mathrm{u}-\nabla_i\left(a^{ij}\nabla_j \mathrm{u}\right)\right)\mathrm{u}\,\Psi G\,d\mu_t\\
&=-\frac{1}{2}\int_{\mathrm{M}}\left(\partial_t \mathrm{u}^2-\nabla_i\left(a^{ij}\nabla_j \mathrm{u}^2\right)+2a^{ij}\nabla_i \mathrm{u}\,\nabla_j \mathrm{u}\right)\Psi G\,d\mu_t\\
&=\frac{1}{2}\int_{\mathrm{M}}\left(\partial_t \Psi\,G+\Psi\left(\partial_t G+\frac{1}{2}\mathrm{tr}\,(\partial_t g)\;G\right)\right)\mathrm{u}^2\,d\mu_t\\
&\quad-\partial_t\left(\int_{\mathrm{M}}\frac{1}{2}\Psi^2 \mathrm{u}^2 G\,d\mu_t\right)-\int_{\mathrm{M}}a^{ij}\nabla_i \mathrm{u}\,\nabla_j \mathrm{u}\,\Psi G\,d\mu_t\\
&\quad+\frac{1}{2}\int_{\mathrm{M}}\left\{\nabla_j\left(a^{ij}\nabla_i \Psi\right)G+2a^{ij}\nabla_i G\,\nabla_j \Psi+\Psi\nabla_j\left(a^{ij}\nabla_i G\right)\right\}\mathrm{u}^2\,d\mu_t\\
&=\frac{1}{2}\int_{\mathrm{M}}\left(\partial_t \Psi+\nabla_j\left(a^{ij}\nabla_i \Psi\right)+\Phi\Psi+a^{ij}\nabla_i \ln G\,\nabla_j \Psi\right)\mathrm{u}^2 G\,d\mu_t\\
&\quad-\int_{\mathrm{M}}\Psi a^{ij}\nabla_i \mathrm{u}\,\nabla_j \mathrm{u}\,G\,d\mu_t\\
&\quad-\partial_t\left(\int_{\mathrm{M}}\frac{1}{2}\Psi^2 \mathrm{u}^2 G\,d\mu_t\right)
\end{aligned}
\tag{4.43}
$$

Likewise, for the fourth term of (4.42), we have

$$
\begin{aligned}
&2\int_{\mathrm{M}}\left(\partial_t \mathrm{u}+a^{ij}\nabla_i \ln G\,\nabla_j \mathrm{u}+\frac{1}{2}\Psi \mathrm{u}\right)\Psi \mathrm{u}\, G\, d\mu_t\\
&=\int_{\mathrm{M}}\partial_t \mathrm{u}^2\,\Psi G\, d\mu_t+\int_{\mathrm{M}} a^{ij}\nabla_i G\,\nabla_j \mathrm{u}^2\,\Psi\, d\mu_t+\int_{\mathrm{M}}\Psi^2\mathrm{u}^2 G\, d\mu_t\\
&=-\int_{\mathrm{M}}\left(\partial_t\Psi\, G+\Psi\left(\partial_t G+\frac{1}{2}\mathrm{tr}\partial_t g\, G\right)\right)\mathrm{u}^2\, d\mu_t+\partial_t\left(\int_{\mathrm{M}}\Psi \mathrm{u}^2 G\, d\mu_t\right)\\
&\quad+\int_{\mathrm{M}}\Psi^2\mathrm{u}^2 G\, d\mu_t\\
&\quad-\int_{\mathrm{M}}\left(\nabla_j\left(a^{ij}\nabla_i G\right)\Psi+a^{ij}\nabla_i G\,\nabla_j\Psi\right)\mathrm{u}^2\, d\mu_t\\
&=-\int_{\mathrm{M}}\left(\partial_t\Psi+\Phi\Psi+a^{ij}\nabla_i\ln G\,\nabla_j\Psi-\Psi^2\right)\mathrm{u}^2 G\, d\mu_t+\partial_t\left(\int_{\mathrm{M}}\Psi\mathrm{u}^2 G\, d\mu_t\right)
\end{aligned}
\tag{4.44}
$$

Combining (4.42), (4.43), (4.44) to get (4.36). □

We hereafter consider the Riemannian manifold in Lemma 4.3 to be each time-slice Σ_t with the induced metric g_t evolving (in "normal parametrization") like $\partial_t g = -2F\left(A^{\#}\right)A$ (see [1]) and the differential operator (in Lemma 4.3) to be the one in Proposition 3.2.

For the second lemma, we choose suitable weight function G and auxiliary function Ψ in Lemma 4.3 in order to bound the LHS of (4.36) from below. The choice of G is due to [6] and [12]. As for Ψ, it is not shown in [12] but is used here to deal with the last term in (4.35), which comes from the nonlinear nature of F (see Definition 3.1). Note that in the linear case when $F(S) = \mathrm{tr}(S)$ (see [12]), the coefficients of the differenital operator in Proposition 3.2 becomes $\mathbf{a}^{ij} = g^{ij}$; besides, (4.35) is reduced to

$$\Upsilon^{ij} = g^{ik}g^{jl}\nabla^2_{kl}\ln G - HA^{ij}$$

The idea of using an auxiliary function for the nonlinear case is motivated by [11].

Lemma 4.4 *Assume that* $\varkappa \le 6^{-4}\lambda^3$ *in (2.1) and (2.2). Then there exists* $R = R\left(\Sigma, \tilde{\Sigma}, \mathcal{C}, U, \| F \|_{C^3(U)}, \lambda, \varkappa\right) \ge 1$ *so that for any constants* $M \ge 1,\ \tau \in (0, 1]$, *let*

$$G = G_{M,\tau} := \exp\left(M(t+\tau)|X|^{\frac{3}{2}} + |X|^2\right) \tag{4.45}$$

$$
\begin{aligned}
\Psi = \Psi_{M,\tau} :=& \left(\frac{3}{2}M(t+\tau)|X|^{-\frac{1}{2}}+2\right)^2 \mathbf{a}^{ij}(X\cdot\partial_i X)\left(X\cdot\partial_j X\right)+M|X|^{\frac{3}{2}}\\
&+\frac{1}{2}\left(\frac{3}{2}M(t+\tau)|X|^{-\frac{1}{2}}+2\right)\left(tr(\mathbf{a})-\frac{\lambda}{3}\right)\\
&+\left(tr(\mathbf{a})-\frac{\lambda}{3}\right)+\frac{3}{4}M(t+\tau)|X|^{-\frac{5}{2}}\left(tr(\mathbf{a})|X|^2-\mathbf{a}^{ij}(X\cdot\partial_i X)\left(X\cdot\partial_j X\right)\right)
\end{aligned}
\tag{4.46}
$$

(note that $G > 0$ *and* $\Psi \ge 0$*), there hold*

$$2\Upsilon^{ij} - (\Phi - \Psi)\,\mathbf{a}^{ij} \ge \tfrac{\lambda^2}{9} g^{ij} \tag{4.47}$$

$$\tfrac{1}{2}\left(\partial_t\Psi - \nabla_i\left(\mathbf{a}^{ij}\nabla_j\Psi\right) + (\Phi-\Psi)\Psi\right) \ge \tfrac{\lambda^2}{9}|X|^2 \tag{4.48}$$

for $X \in \Sigma_t \backslash \bar{B}_R,\ t \in [-\tau, 0)$*, where* $\mathrm{tr}\,(\mathbf{a}) = g_{ij}\mathbf{a}^{ij}$*,* Φ *and* Υ^{ij} *are defined in (4.34) and (4.35), respectively, with the covariant derivative is taken w.r.t.* Σ_t*,* $\partial_t g = -2F\left(A^{\#}\right) A$*, and* $a^{ij} = \mathbf{a}^{ij}$.

Remark 4.4 In view of Proposition 3.3, the hypothesis that $\varkappa \le 6^{-4}\lambda^3$ amounts to requiring the smallness of $|X|\,|\nabla_{\Sigma_t}\mathbf{a}|$ (compared with the ellipticity of $\mathbf{a}$). Similar hypothesis also appears in [11] and [13] when using Carleman's inequalities to prove the backward uniqueness of parabolic equations.

Proof Let's start with computing the covariant derivatives of $\ln G$:

$$\nabla_i \ln G = \left(\frac{3}{2}M\,(t+\tau)\,|X|^{-\frac{1}{2}} + 2\right)(X \cdot \partial_i X) \tag{4.49}$$

$$\begin{aligned}\nabla^2_{ij} \ln G = &\left(\frac{3}{2}M\,(t+\tau)\,|X|^{-\frac{1}{2}} + 2\right)\left(g_{ij} + X \cdot N\, A_{ij}\right)\\ &- \frac{3}{4}M\,(t+\tau)\,|X|^{-\frac{5}{2}}\left(|X|^2 g_{ij} - (X \cdot \partial_i X)\left(X \cdot \partial_j X\right)\right)\\ &+ 2t\left(\frac{3}{2}M\,(t+\tau)\,|X|^{-\frac{1}{2}} + 2\right) F\left(A^{\#}\right) A_{ij}\end{aligned} \tag{4.50}$$

and its evolution

$$\begin{aligned}\partial_t \ln G &= M|X|^{\frac{3}{2}} + \left(\frac{3}{2}M\,(t+\tau)\,|X|^{-\frac{1}{2}} + 2\right)(X \cdot \partial_t X)\\ &= M|X|^{\frac{3}{2}} + 2t\left(\frac{3}{2}M\,(t+\tau)\,|X|^{-\frac{1}{2}} + 2\right) F\left(A^{\#}\right)^2\end{aligned} \tag{4.51}$$

in which we use the F curvature flow equation in normal parametrization (see Definition 3.2)

$$\partial_t X = F\left(A^{\#}\right) N$$

and the F self-shrinker equation for $\Sigma_t = \sqrt{-t}\,\Sigma$ (in Definition 2.4):

$$X \cdot N = 2t F\left(A^{\#}\right)$$

Thus, by (4.34), (4.49), (4.50) and (4.51), we have

$$\begin{aligned}\Phi = &\left(\frac{3}{2}M\,(t+\tau)\,|X|^{-\frac{1}{2}} + 2\right)^2 \mathbf{a}^{ij}\,(X \cdot \partial_i X)\left(X \cdot \partial_j X\right) + M|X|^{\frac{3}{2}}\\ &+ \frac{1}{2}\left(\frac{3}{2}M\,(t+\tau)\,|X|^{-\frac{1}{2}} + 2\right)\mathrm{tr}\,(\mathbf{a})\\ &+ \mathrm{tr}\,(\mathbf{a}) + \frac{3}{4}M\,(t+\tau)\,|X|^{-\frac{5}{2}}\left(\mathrm{tr}\,(\mathbf{a})\,|X|^2 - \mathbf{a}^{ij}\,(X \cdot \partial_i X)\left(X \cdot \partial_j X\right)\right)\\ &+ \left(\frac{3}{2}M\,(t+\tau)\,|X|^{-\frac{1}{2}} + 2\right)\\ &\left\{\left(\nabla_i \mathbf{a}^{ij}\right)\left(X \cdot \partial_j X\right) + 2t F\left(A^{\#}\right)\left(F\left(A^{\#}\right) + \mathbf{a}^{ij} A_{ij}\right)\right\} - F\left(A^{\#}\right) H\end{aligned} \tag{4.52}$$

which, together with (4.46), implies that

$$\begin{aligned}\Phi-\Psi=&\frac{\lambda}{2}\left(\frac{3}{2}M\left(t+\tau\right)|X|^{-\frac{1}{2}}+2\right)+\frac{\lambda}{3}\\&+\left(\frac{3}{2}M\left(t+\tau\right)|X|^{-\frac{1}{2}}+2\right)\\&\left\{\left(\nabla_k\mathbf{a}^{kl}\right)\left(X\cdot\partial_lX\right)+2tF\left(A^{\#}\right)\left(F\left(A^{\#}\right)+\mathbf{a}^{kl}A_{kl}\right)\right\}-F\left(A^{\#}\right)H\end{aligned}\tag{4.53}$$

By (4.35), (4.49), (4.50) and (4.53),

$$\begin{aligned}&2\Upsilon^{ij}-(\Phi-\Psi)\,\mathbf{a}^{ij}\\&=\left(\frac{3}{2}M\left(t+\tau\right)|X|^{-\frac{1}{2}}+2\right)\left(\mathbf{a}^{ik}\mathbf{a}^{jl}g_{kl}-\frac{\lambda}{6}\mathbf{a}^{ij}\right)\\&\quad+\left(2\mathbf{a}^{ik}\mathbf{a}^{jl}g_{kl}-\frac{\lambda}{3}\mathbf{a}^{ij}\right)+\frac{3}{2}M\left(t+\tau\right)|X|^{-\frac{5}{2}}\mathbf{a}^{ik}\mathbf{a}^{jl}\left(|X|^2g_{kl}-\left(X\cdot\partial_kX\right)\left(X\cdot\partial_lX\right)\right)\\&\quad+\left(\frac{3}{2}M\left(t+\tau\right)|X|^{-\frac{1}{2}}+2\right)\left\{\mathbf{a}^{ik}\nabla_k\mathbf{a}^{jl}+\mathbf{a}^{jk}\nabla_k\mathbf{a}^{il}-\mathbf{a}^{lk}\nabla_k\mathbf{a}^{ij}-\mathbf{a}^{ij}\nabla_k\mathbf{a}^{kl}\right\}\left(X\cdot\partial_lX\right)\\&\quad+\left(\frac{3}{2}M\left(t+\tau\right)|X|^{-\frac{1}{2}}+2\right)\left(2\mathbf{a}^{ik}\mathbf{a}^{jl}A_{kl}-\mathbf{a}^{ij}\mathbf{a}^{kl}A_{kl}-F\left(A^{\#}\right)\mathbf{a}^{ij}\right)2t\,F\left(A^{\#}\right)\\&\quad-\partial_t\mathbf{a}^{ij}+F\left(A^{\#}\right)H\,\mathbf{a}^{ij}\end{aligned}\tag{4.54}$$

which can be estimated from below, using (3.52), (3.53), (3.55), (3.12), (3.15) and the homogeneity of F, by

$$\begin{aligned}2\Upsilon^{ij}-(\Phi-\Psi)\,\mathbf{a}^{ij}\geq&\left(\frac{3}{2}M\left(t+\tau\right)|X|^{-\frac{1}{2}}+2\right)\left(\left(\frac{\lambda^2}{18}-36\frac{\varkappa}{\lambda}\right)g^{ij}+O\left(|X|^{-2}\right)\right)\\&+\frac{\lambda^2}{9}g^{ij}+O\left(|X|^{-2}\right)\end{aligned}\tag{4.55}$$

where the notation $O\left(|X|^{-2}\right)$ means that

$$\left|O\left(|X|^{-2}\right)\right|\leq C\left(n,\mathcal{C},\|\,F\,\|_{C^3(U)}\right)|X|^{-2}$$

Then (4.47) follows from (4.45) and the hypothesis ($\varkappa\leq 6^{-4}\lambda^3$) provided that $R\gg 1$ (independing of M and τ).

On the other hand, by (3.52), (3.53), (3.12), (3.15), the homogeneity of F, the hypothesis that $\varkappa\leq 6^{-4}\lambda^3$ (note that $\lambda\in(0,1]$) and $R\gg 1$ (independing of M and τ), we can estimate (4.53) from below by

$$\begin{aligned}\Phi-\Psi&\geq\left(\frac{3}{2}M\left(t+\tau\right)|X|^{-\frac{1}{2}}+2\right)\left(\frac{\lambda}{6}-3\varkappa+O\left(|X|^{-2}\right)\right)+\frac{\lambda}{3}+O\left(|X|^{-2}\right)\\&\geq\left(\frac{3}{2}M\left(t+\tau\right)|X|^{-\frac{1}{2}}+2\right)\frac{\lambda}{9}+\frac{\lambda}{6}\end{aligned}\tag{4.56}$$

Similarly, from the F self-shrinker equation for Σ_t, we can estimate the tangential component of the position vector by

$$\begin{aligned} |X^{\top}|^2 &= |X|^2 - (X \cdot N)^2 = |X|^2 - \left(2t\, F\left(A^{\#}\right)\right)^2 \\ &= |X|^2 - \left(2t\, F\left(|X|\, A^{\#}\right)\right)^2 |X|^{-2} = |X|^2 + O\left(|X|^{-2}\right) \end{aligned} \tag{4.57}$$

Consequently, (4.46) can be estimated (from below), using (3.52) and (4.57), by

$$\begin{aligned} \Psi &\geq \left(\frac{3}{2} M\,(t+\tau)\,|X|^{-\frac{1}{2}} + 2\right)^2 \mathbf{a}^{ij}\,(X \cdot \partial_i X)\left(X \cdot \partial_j X\right) + M\,|X|^{\frac{3}{2}} \\ &\geq \left(\frac{3}{2} M\,(t+\tau)\,|X|^{-\frac{1}{2}} + 2\right)^2 \left(\frac{\lambda}{3}|X|^2 + O\left(|X|^{-2}\right)\right) + M\,|X|^{\frac{3}{2}} \end{aligned} \tag{4.58}$$

Multiplying (4.56) and (4.58) to get

$$\begin{aligned} (\Phi - \Psi)\,\Psi \geq & \left(\frac{3}{2} M\,(t+\tau)\,|X|^{-\frac{1}{2}} + 2\right)^3 \frac{1}{36}\lambda^2 |X|^2 \\ & + \left(\frac{3}{2} M\,(t+\tau)\,|X|^{-\frac{1}{2}} + 2\right)^2 \frac{1}{27}\lambda^2 |X|^2 \\ & + \left(\frac{3}{2} M\,(t+\tau)\,|X|^{-\frac{1}{2}} + 2\right) \frac{\lambda}{9} M\,|X|^{\frac{3}{2}} + \frac{\lambda}{6} M\,|X|^{\frac{3}{2}} \end{aligned} \tag{4.59}$$

To achieve (4.48), let's first rearrange (4.46) to get

$$\begin{aligned} \Psi = & \left(\frac{3}{2} M\,(t+\tau)\,|X|^{-\frac{1}{2}} + 2\right)^2 \mathbf{a}^{kl}\,(X \cdot \partial_k X)\,(X \cdot \partial_l X) + M\,|X|^{\frac{3}{2}} \\ & + \left(\frac{3}{2} M\,(t+\tau)\,|X|^{-\frac{1}{2}} + 2\right)\left(\operatorname{tr}(\mathbf{a}) - \frac{\mathbf{a}^{kl}\,(X \cdot \partial_k X)\,(X \cdot \partial_l X)}{2|X|^2} - \frac{\lambda}{6}\right) \\ & + \frac{\mathbf{a}^{kl}\,(X \cdot \partial_k X)\,(X \cdot \partial_l X)}{|X|^2} - \frac{\lambda}{3} \end{aligned} \tag{4.60}$$

Then we would like to take time-derivative of (4.60) and estimate it by using Proposition 3.3, (3.12), (3.15), the homogeneity of F and its derivatives, the F self-shrinker equaiton for Σ_t (i.e. $X \cdot N = 2t F\left(A^{\#}\right)$) and the F curvature flow equation (i.e. $\partial_t X = F\left(A^{\#}\right) N$)), and also assuming that $R \gg 1$ (depending on λ). Note that we can simplify the compuation by taking "normal coodinates" of Σ_t. For instance, let's compute and estimate the time-derivative of the first term in (4.60):

$$\begin{aligned} & \partial_t \left\{\left(\frac{3}{2} M\,(t+\tau)\,|X|^{-\frac{1}{2}} + 2\right)^2 \mathbf{a}^{kl}\,(X \cdot \partial_k X)\,(X \cdot \partial_l X)\right\} \\ & = 2\left(\frac{3}{2} M\,(t+\tau)\,|X|^{-\frac{1}{2}} + 2\right) \\ & \quad \left\{\frac{3}{2} M\,|X|^{-\frac{1}{2}} + \frac{3}{2} M\,(t+\tau)\left(-\frac{1}{2}|X|^{-\frac{3}{2}}\right)\frac{X \cdot F\left(A^{\#}\right) N}{|X|}\right\} \mathbf{a}^{kl}\,(X \cdot \partial_k X)\,(X \cdot \partial_l X) \\ & \quad + \left(\frac{3}{2} M\,(t+\tau)\,|X|^{-\frac{1}{2}} + 2\right)^2 \\ & \quad \left\{\left(\partial_t \mathbf{a}^{kl}\right)(X \cdot \partial_k X)\,(X \cdot \partial_l X) + 2\mathbf{a}^{kl}\,(X \cdot \partial_k X)\left(X \cdot \partial_l \left(F\left(A^{\#}\right) N\right)\right)\right\} \end{aligned} \tag{4.61}$$

By taking normal coordinates, we may assume that (at the point of consideration) $g_{ij} = \delta_{ij}$ (so the norm is Proposition 3.3 becomes ℓ^2 norm), $\{\partial_1 X, \dots, \partial_n X, N\}$ is an orthonormal basis for $\mathbb{R}^{n+1}$, and the last term in (4.61) can be computed and estimated by

$$\begin{aligned}\partial_l \left(F\left(A^{\#}\right) N\right) &= \frac{\partial F}{\partial S_i^j}\left(A^{\#}\right)\left(\partial_l A_i^j\right) N + F\left(A^{\#}\right)\left(-A_l^k\,\partial_k X\right)\\ &= \frac{\partial F}{\partial S_i^j}\left(|X|\,A^{\#}\right)\left(\nabla_l A_i^j\right) N + |X|^{-1} F\left(|X|\,A^{\#}\right)\left(-A_l^k\,\partial_k X\right) = O\left(|X|^{-2}\right)\end{aligned}$$

so (4.61) can be estimated by

$$\begin{aligned}&\left(\frac{3}{2}M(t+\tau)|X|^{-\frac{1}{2}}+2\right)\left(3M|X|^{-\frac{1}{2}}+M\cdot O\left(|X|^{-\frac{9}{2}}\right)\right)\mathbf{a}^{kl}\left(X\cdot\partial_k X\right)\left(X\cdot\partial_l X\right)\\ &\quad+\left(\frac{3}{2}M(t+\tau)|X|^{-\frac{1}{2}}+2\right)^2 O(1)\end{aligned}$$

By doing the same thing to other terms in (4.60), we arrive at

$$\begin{aligned}\partial_t\Psi &= \left(\frac{3}{2}M(t+\tau)|X|^{-\frac{1}{2}}+2\right)\left(3M|X|^{-\frac{1}{2}}+M\cdot O\left(|X|^{-\frac{9}{2}}\right)\right)\mathbf{a}^{kl}\left(X\cdot\partial_k X\right)\left(X\cdot\partial_l X\right)\\ &\quad+\left(\frac{3}{2}M(t+\tau)|X|^{-\frac{1}{2}}+2\right)^2 O(1)\\ &\quad+M\cdot O\left(|X|^{-\frac{1}{2}}\right)+\left(\frac{3}{2}M(t+\tau)|X|^{-\frac{1}{2}}+2\right)O\left(|X|^{-2}\right)+O\left(|X|^{-2}\right)\\ &\geq \left(\frac{3}{2}M(t+\tau)|X|^{-\frac{1}{2}}+2\right)\left(\frac{2}{3}\lambda M|X|^{\frac{3}{2}}\right)\\ &\quad+\left(\frac{3}{2}M(t+\tau)|X|^{-\frac{1}{2}}+2\right)^2 O(1)+M\cdot O\left(|X|^{-\frac{1}{2}}\right)\end{aligned}\tag{4.62}$$

Similarly, we can compute $\nabla_i\left(\mathbf{a}^{ij}\nabla_j\Psi\right)$ and estimate it by

$$\begin{aligned}\nabla_i\left(\mathbf{a}^{ij}\nabla_j\Psi\right) &= \mathbf{a}^{ij}\nabla_{ij}^2\Psi+\left(\nabla_i\mathbf{a}^{ij}\right)\left(\nabla_j\Psi\right)\\ &= \left(\frac{3}{2}M(t+\tau)|X|^{-\frac{1}{2}}+2\right)^2 O(1)\\ &\quad+\left(\frac{3}{2}M(t+\tau)|X|^{-\frac{1}{2}}+2\right)O\left(|X|^{-2}\right)+M\cdot O\left(|X|^{-\frac{1}{2}}\right)\end{aligned}\tag{4.63}$$

Then (4.48) follows from (4.59), (4.62) and (4.63). □

Using the above two lemmas, we can derive the following Carleman's inequality on the flow $\{\Sigma_t\}_{-1\leq t\leq 0}$ (with $\Sigma_0 = \mathcal{C}$).

Proposition 4.3 (Carleman's inequality) *Assume that* $\varkappa \leq 6^{-4}\lambda^3$ *in (2.4) and (2.5). Then there exists* $R \geq 1$ *(depending on* $\Sigma, \tilde{\Sigma}, \mathcal{C}, U, \|\ F\ \|_{C^3(U)}, \lambda, \varkappa$*) so that for any constants* $M \geq 1$, $\tau \in (0, 1]$, *and one-parameter family of* C^2 *functions* $\mathrm{u}_t = \mathrm{u}(\cdot, t)$ *which is compactly* *supported in* $\Sigma_t \backslash \bar{B}_R$ *for each* $t \in [-\tau, 0]$ *and is differentiable in time, there holds*

$$\frac{\lambda^2}{9}\int_{-\tau}^{0}\int_{\Sigma_t}\left(|\nabla_{\Sigma_t}\mathrm{u}|^2+\mathrm{u}^2\right)G\,d\mathcal{H}^n dt$$
$$\leq \int_{-\tau}^{0}\int_{\Sigma_t}|\mathbf{P}\mathrm{u}|^2 G\,d\mathcal{H}^n dt+\frac{3}{\lambda}\int_{\Sigma_{-\tau}}|\nabla_{\Sigma_{-\tau}}\mathrm{u}_{-\tau}|^2 G\,(\cdot,-\tau)\;d\mathcal{H}^n$$
$$+\frac{1}{2}\int_{\mathcal{C}}\Psi\,(\cdot,0)\;\mathrm{u}^2\,(\cdot,0)\;G\,(\cdot,0)\;d\mathcal{H}^n \tag{4.64}$$

where $\mathcal{H}^n$ *is the n dimensional Hausdorff measure;* $\mathbf{P}$, $G = G_{M,\tau}$ *and* $\Psi = \Psi_{M,\tau}$ *are defined in (3.48), (4.45), (4.46), respectively.*

Proof Apply Lemma 4.3 to the hypersurface Σ_t (with $\partial_t g = -2F\left(A^{\#}\right)A$), the differential operator $\mathbf{P}$ and the function u_t to get

$$\int_{\Sigma_{\mathrm{t}}}\left\{\left(2\Upsilon^{ij}-(\Phi-\Psi)\,\mathbf{a}^{ij}\right)\nabla_i\mathrm{u}\,\nabla_j\mathrm{u}+\frac{1}{2}\left(\partial_t\Psi-\nabla_i\left(a^{ij}\nabla_j\Psi\right)+(\Phi-\Psi)\,\Psi\right)\mathrm{u}^2\right\}G\,d\mathcal{H}^n$$
$$=\int_{\Sigma_{\mathrm{t}}}2\,\mathbf{P}\mathrm{u}\,\left(\partial_t\mathrm{u}+\mathbf{a}^{ij}\nabla_i\ln G\,\nabla_j\mathrm{u}+\frac{1}{2}\Psi\mathrm{u}\right)G\,d\mathcal{H}^n$$
$$-\int_{\Sigma_{\mathrm{t}}}2\left(\partial_t\mathrm{u}+\mathbf{a}^{ij}\nabla_i\ln G\,\nabla_j\mathrm{u}+\frac{1}{2}\Psi\mathrm{u}\right)^2 G\,d\mathcal{H}^n$$
$$-\partial_t\left\{\int_{\Sigma_{\mathrm{t}}}\left(\mathbf{a}^{ij}\nabla_i\mathrm{u}\,\nabla_j\mathrm{u}-\frac{1}{2}\Psi\mathrm{u}^2\right)G\,d\mathcal{H}^n\right\} \tag{4.65}$$

By Cauchy-Schwarz inequality, the RHS of (4.65) is bounded from above by

$$\int_{\Sigma_t}|\mathbf{P}\mathrm{u}|^2 G\,d\mathcal{H}^n dt-\partial_t\left\{\int_{\Sigma_t}\left(\mathbf{a}^{ij}\nabla_i\mathrm{u}\,\nabla_j\mathrm{u}-\frac{1}{2}\Psi\mathrm{u}^2\right)G\,d\mathcal{H}^n\right\} \tag{4.66}$$

By Lemma 4.4 and $R \geq 1$, the LHS of (4.65) is bounded from below by

$$\frac{\lambda^2}{9}\int_{\Sigma_t}\left(|\nabla_{\Sigma_t}\mathrm{u}|^2+\mathrm{u}^2\right)G\,d\mathcal{H}^n \tag{4.67}$$

Combining (4.65), (4.66), (4.67), we get

$$\frac{\lambda^2}{9}\int_{\Sigma_t}\left(|\nabla_{\Sigma_t}\mathrm{u}|^2+\mathrm{u}^2\right)G\,d\mathcal{H}^n$$
$$\leq\int_{\Sigma_t}|\mathbf{P}\mathrm{u}|^2 G\,d\mathcal{H}^n dt-\partial_t\left\{\int_{\Sigma_t}\left(\mathbf{a}^{ij}\nabla_i\mathrm{u}\,\nabla_j\mathrm{u}-\frac{1}{2}\Psi\mathrm{u}^2\right)G\,d\mathcal{H}^n\right\} \tag{4.68}$$

Integrate (4.68) in time from $-\tau$ to 0 and then use (3.52) and $\Psi \geq 0$ to conclude (4.64). □

Now we are ready to show that h vanishes outside a compact set. We basically follow the proof in [6] (which is also used in [12]).

Theorem 4.1 *Suppose that* $\varkappa \leq 6^{-4}\lambda^3$ *in (2.4) and (2.5), then there exists* $\boldsymbol{R} = \boldsymbol{R}\left(\Sigma, \tilde{\Sigma}, \mathcal{C}, U, \|\, F\,\|_{C^3(U)}, \lambda, \varkappa\right) \geq 1$ *so that the deviation* $h\,(\cdot,-1)$ *of* $\tilde{\Sigma}$ *from* Σ *vanishes on* $\Sigma\backslash\bar{B}_{\boldsymbol{R}}$. *In other words,* $\tilde{\Sigma} = \Sigma$ *outside the ball* $B_{\boldsymbol{R}}$.

Proof Choose $R \gg 1$ (depending on $\Sigma, \tilde{\Sigma}, \mathcal{C}, U, \| F \|_{C^3(U)}, \lambda$) so that Proposition 3.2, Proposition 3.3, Proposition 4.2, Proposition 4.3 and (3.15) hold; in particular, we may assume that for all $X \in \Sigma_t \backslash \bar{B}_R$, $t \in [-\tau, 0]$

$$|\mathbf{P}h| \leq \frac{\lambda}{6}\left(|\nabla_{\Sigma_t} h| + |h|\right) \tag{4.69}$$

$$|\nabla_{\Sigma_t} h| + |h| \leq \Lambda \exp\left(\frac{|X|^2}{\Lambda t}\right) \tag{4.70}$$

where $\Lambda = \Lambda\left(n, \mathcal{C}, \| F \|_{C^3(U)}, \lambda\right) > 0$, $\tau \equiv \min\left\{\alpha\left(n, \mathcal{C}, \| F \|_{C^3(U)}, \lambda\right), \frac{1}{\Lambda}\right\}$ (see Proposition 4.2).

For any given $M \geq 1$ and $\mathcal{R} \geq 4R + 1$, choose a smooth cut-off function $\zeta = \zeta(X)$ so that

$$\begin{aligned} \chi_{B_{\mathcal{R}-1} \backslash \bar{B}_{R+1}} \leq \zeta \leq \chi_{B_{\mathcal{R}} \backslash \bar{B}_R} \\ |D\zeta| + |D^2\zeta| \leq 3 \end{aligned} \tag{4.71}$$

Note that $D\zeta$ is supported in $E = \left\{X \in \mathbb{R}^{n+1} \middle| R \leq |X| \leq R + 1 \text{ or } \mathcal{R} - 1 \leq |X| \leq \mathcal{R}\right\}$.

Let $\mathrm{u}(\cdot, t) = \zeta\, h(\cdot, t)$, then $\mathrm{u}(\cdot, t)$ is compactly supported in $\Sigma_t \backslash \bar{B}_R$ for each $t \in [-\tau, 0]$, and we have, by (4.69), (4.70), (4.71)

$$\begin{aligned} \left|\mathbf{P}\mathrm{u}\right| &= \left|\zeta\, \mathbf{P}h - h\, \mathbf{P}\zeta - 2\mathbf{a}^{ij} \nabla_i \zeta\, \nabla_j h\right| \\ &\leq \frac{\lambda}{6}\left(|\nabla_{\Sigma_t} \mathrm{u}| + |\mathrm{u}|\right) + C\left(n, \mathcal{C}, \| F \|_{C^3(U)}, \lambda\right)\left(|\nabla_{\Sigma_t} h| + |h|\right) \chi_E \\ &\leq \frac{\lambda}{6}\left(|\nabla_{\Sigma_t} \mathrm{u}| + |\mathrm{u}|\right) + C\left(n, \mathcal{C}, \| F \|_{C^3(U)}, \lambda\right) \exp\left(\frac{|X|^2}{\Lambda t}\right) \chi_E \end{aligned} \tag{4.72}$$

$$\mathrm{u}(\cdot, 0) = 0 \tag{4.73}$$

By (4.72), (4.73), Proposition 4.3 and (4.70), we get

$$\begin{aligned} &\frac{\lambda^2}{9} \int_{-\tau}^0 \int_{\Sigma_t} \left(|\nabla_{\Sigma_t} \mathrm{u}|^2 + \mathrm{u}^2\right) G\, d\mathcal{H}^n\, dt \\ &\leq \frac{\lambda^2}{18} \int_{-\tau}^0 \int_{\Sigma_t} \left(|\nabla_{\Sigma_t} \mathrm{u}|^2 + \mathrm{u}^2\right) G\, d\mathcal{H}^n\, dt \\ &\quad + C\left(n, \mathcal{C}, \| F \|_{C^3(U)}, \lambda\right) \int_{-\tau}^0 \int_{\Sigma_t \cap E} \exp\left(2\frac{|X|^2}{\Lambda t}\right) G\, d\mathcal{H}^n\, dt \\ &\quad + C\left(n, \mathcal{C}, \| F \|_{C^3(U)}, \lambda\right) \int_{\Sigma_{-\tau}} \exp\left(-2\frac{|X|^2}{\Lambda \tau}\right) G(\cdot, -\tau)\, d\mathcal{H}^n \end{aligned} \tag{4.74}$$

where G is defined in (4.45). Note that by the choice $\tau \leq \frac{1}{\Lambda}$, we can estimate the last two terms on the RHS of (4.74) by

$$\int_{-\tau}^0 \int_{\Sigma_t \cap E} \exp\left(2\frac{|X|^2}{\Lambda t}\right) G\, d\mathcal{H}^n\, dt \leq \int_{-\tau}^0 \int_{\Sigma_t \cap E} \exp\left(M\tau |X|^{\frac{3}{2}} - |X|^2\right) d\mathcal{H}^n\, dt \tag{4.75}$$

and

$$\int_{\Sigma_{-\tau}} \exp\left(-2\frac{|X|^2}{\Lambda \tau}\right) G(\cdot, -\tau)\, d\mathcal{H}^n \leq \int_{\Sigma_{-\tau}} \exp\left(-|X|^2\right) d\mathcal{H}^n \tag{4.76}$$

Consequently, by (4.75), (4.76) and noting that the first term on the RHS of (4.74) can be abosorbed by its LHS, we get from (4.74) that

$$
\begin{aligned}
&\frac{\lambda^2}{18}\int_{-\tau}^{0}\int_{\Sigma_t}\left(|\nabla_{\Sigma_t}\mathrm{u}|^2+\mathrm{u}^2\right)G\,d\mathcal{H}^n dt\\
&\le C\left(n,\mathcal{C},\|\ F\ \|_{C^3(U)},\lambda\right)\int_{-\tau}^{0}\int_{\Sigma_t\cap E}\exp\left(M\tau|X|^{\frac{3}{2}}-|X|^2\right)d\mathcal{H}^n dt\\
&\quad+C\left(n,\mathcal{C},\|\ F\ \|_{C^3(U)},\lambda\right)\int_{\Sigma_{-\tau}}\exp\left(-|X|^2\right)d\mathcal{H}^n\\
&\le C\left(n,\mathcal{C},\|\ F\ \|_{C^3(U)},\lambda\right)\int_{-\tau}^{0}\int_{\Sigma_t\cap(B_{\mathcal{R}-1}\setminus\bar{B}_{\mathcal{R}})}\exp\left(M\tau\mathcal{R}^{\frac{3}{2}}-(\mathcal{R}-1)^2\right)d\mathcal{H}^n dt\\
&\quad+C\left(n,\mathcal{C},\|\ F\ \|_{C^3(U)},\lambda\right)\int_{-\tau}^{0}\int_{\Sigma_t\cap(B_R\setminus\bar{B}_{R+1})}\exp\left(M\tau\,(R+1)^{\frac{3}{2}}-R^2\right)d\mathcal{H}^n dt\\
&\quad+C\left(n,\mathcal{C},\|\ F\ \|_{C^3(U)},\lambda\right)\int_{\Sigma_{-\tau}}\exp\left(-|X|^2\right)d\mathcal{H}^n
\end{aligned}\tag{4.77}
$$

The first term on the RHS of (4.77) goes away as $\mathcal{R}\nearrow\infty$; the last term is bounded from above by $C\left(n,\mathcal{C},\|\ F\ \|_{C^3(U)},\lambda\right)$ because of (3.5). For the LHS of (4.77), we have

$$
\begin{aligned}
\frac{\lambda^2}{18}\int_{-\tau}^{0}\int_{\Sigma_t}\left(|\nabla_{\Sigma_t}\mathrm{u}|^2+\mathrm{u}^2\right)G\,d\mathcal{H}^n dt &\ge \frac{\lambda^2}{18}\int_{-\frac{\tau}{2}}^{0}\int_{\Sigma_t\cap(B_{\mathcal{R}-1}\setminus\bar{B}_{4R})}\mathrm{u}^2G\,d\mathcal{H}^n dt\\
&\ge \frac{\lambda^2}{18}\exp\left(4M\tau R^{\frac{3}{2}}\right)\int_{-\frac{\tau}{2}}^{0}\int_{\Sigma_t\cap(B_{\mathcal{R}-1}\setminus\bar{B}_{4R})}h^2\,d\mathcal{H}^n dt
\end{aligned}
$$

Therefore, let $\mathcal{R}\nearrow\infty$ in (4.77), we arrive at

$$
\begin{aligned}
&\int_{-\frac{\tau}{2}}^{0}\int_{\Sigma_t\setminus\bar{B}_{4R}}h^2\,d\mathcal{H}^n dt\\
&\quad\le\exp\left(-4M\tau R^{\frac{3}{2}}\right)C\left(n,\mathcal{C},\|\ F\ \|_{C^3(U)},\lambda\right)\left\{\exp\left(2\sqrt{2}M\tau R^{\frac{3}{2}}\right)+1\right\}
\end{aligned}\tag{4.78}
$$

Let $M\nearrow\infty$ in (4.78), we get $h_t=h(\cdot,t)$ vanishes on $\Sigma_t\setminus\bar{B}_{4R}$ for $t\in\left[-\frac{\tau}{2},0\right]$, and hence $\tilde{\Sigma}_{-\frac{\tau}{2}}=\sqrt{\frac{\tau}{2}}\,\tilde{\Sigma}$ coincides with $\Sigma_{-\frac{\tau}{2}}=\sqrt{\frac{\tau}{2}}\,\Sigma$ outside B_{4R}, which in turn shows that $\tilde{\Sigma}$ coincides with Σ outside the ball of radius $\boldsymbol{R}=\frac{4R}{\sqrt{\tau/2}}$. □

By the previous theorem and the "unique continuation principle" in Proposition 4.1 (see also Remark 4.3), we have the following conclusion on the overlap region of Σ and $\tilde{\Sigma}$.

Theorem 4.2 *Under the same hypothesis of Theorem 4.1, let*

$$
\Sigma^0=\left\{X\in\Sigma\cap\tilde{\Sigma}\,\middle|\,\Sigma\text{ coincides with }\tilde{\Sigma}\text{ in a neighborhood of }X\right\}
$$

then Σ^0 is a nonempty hypersurface and $\partial\Sigma^0\subseteq\left(\partial\Sigma\cup\partial\tilde{\Sigma}\right)$.

Proof Note that Σ^0 is a nonempty hypersurface follows from Theorem 4.1 and the definition of Σ^0.

Suppose that $\partial\Sigma^0\not\subseteq\left(\partial\Sigma\cup\partial\tilde{\Sigma}\right)$, then pick $\hat{X}\in\partial\Sigma^0\setminus\left(\partial\Sigma\cup\partial\tilde{\Sigma}\right)$ and choose a sequence $\left\{\hat{X}_m\in\Sigma^0\right\}$ converging to $\hat{X}$. Note that $N\left(\hat{X}\right)=\tilde{N}\left(\hat{X}\right)$ since $N\left(\hat{X}_m\right)=$

$\tilde{N}\left(\hat{X}_m\right)$ for all $m \in \mathbb{N}$, where N , $\tilde{N}$ are the unit-normal of Σ and $\tilde{\Sigma}$, respectively. Thus, near $\hat{X}$, Σ and $\tilde{\Sigma}$ can be regarded as graphes of $\mathfrak{u}$ and $\tilde{\mathfrak{u}}$, respectively, over $B^n_{\varrho} \subset T_{\hat{X}}\Sigma = T_{\hat{X}}\tilde{\Sigma}$ for some $\varrho \in (0, 1)$. That is, Σ and $\tilde{\Sigma}$ can be respectively parametrized by

$$X = X(x) \equiv \hat{X} + (x, \mathfrak{u}(x)), \quad \tilde{X} = \tilde{X}(x) \equiv \hat{X} + \left(x, \tilde{\mathfrak{u}}(x)\right) \quad \text{for } x \in B^n_{\varrho}$$

in which we assume that $N\left(\hat{X}\right) = \tilde{N}\left(\hat{X}\right) = (0, 1)$ for ease of notation. Note also that $A_i^j(0) = \tilde{A_i}^j(0)$ since $A_i^j(x_m) = \tilde{A_i}^j(x_m)$ for all $m \in \mathbb{N}$, where x_m is the coordinates of $\hat{X}_m$ (i.e. $X(x_m) = \hat{X}_m$) and

$$\begin{aligned} A^{\#}(x) \sim A_i^j(x) &= \partial_i\left(\frac{\partial_j \mathfrak{u}(x)}{\sqrt{1+|\partial_x \mathfrak{u}|^2}}\right), \\ \tilde{A}^{\#}(x) \sim \tilde{A}_i^j(x) &= \partial_i\left(\frac{\partial_j \tilde{\mathfrak{u}}(x)}{\sqrt{1+|\partial_x \tilde{\mathfrak{u}}|^2}}\right) \end{aligned} \tag{4.79}$$

are the shape operators of Σ and$\tilde{\Sigma}$, respectively. As a result, we may assume (by choosing ϱ small if necessary) that $\tilde{A}_i^j(x)$ is so closed to $A_i^j(x)$ that the set

$$\mathfrak{U} = \left\{(1-\theta)A_i^j(x) + \theta\tilde{A}_i^j(x) \,\middle|\, x \in B^n_{\varrho}, \theta \in [0,1]\right\}$$

is a bounded subset of $\mathbf{\Omega}$ and there holds

$$\bar{\lambda} \le \frac{\partial F}{\partial S_i^j}\left((1-\theta)A^{\#}(x) + \theta\tilde{A}^{\#}(x)\right) \le \frac{1}{\bar{\lambda}}$$

for some $\bar{\lambda} \in (0, 1]$.

From the F shrinker equation in Definition 2.4, we get

$$\begin{aligned} &\sqrt{1+|\partial_x \mathfrak{u}|^2}\,F\left(A_i^j(x)\right) + \frac{1}{2}(\mathfrak{u} - x\cdot\partial_x \mathfrak{u}) = 0, \\ &\sqrt{1+|\partial_x \tilde{\mathfrak{u}}|^2}\,F\left(\tilde{A}_i^j(x)\right) + \frac{1}{2}\left(\tilde{\mathfrak{u}} - x\cdot\partial_x \tilde{\mathfrak{u}}\right) = 0 \end{aligned} \tag{4.80}$$

Substracting (4.80) and using (4.79) and the mean value theorem, we then get an equation for $\mathrm{v} = \tilde{\mathfrak{u}} - \mathfrak{u}$:

$$\mathfrak{a}^{ij}\partial^2_{ij}\mathrm{v} + \mathfrak{b}^j\partial_j\mathrm{v} + \frac{1}{2}\mathrm{v} = 0 \tag{4.81}$$

with

$$\begin{aligned} \mathfrak{a}^{ij}(x) = \int_0^1 &\left\{\frac{\partial F}{\partial S_i^j}\left((1-\theta)A^{\#}(x) + \theta\tilde{A}^{\#}(x)\right)\right. \\ &\left. - \frac{\partial F}{\partial S_i^k}\left((1-\theta)A^{\#}(x) + \theta\tilde{A}^{\#}(x)\right)\frac{\partial_k \mathfrak{u}_\theta\,\partial_j \mathfrak{u}_\theta}{1+|\partial_x \mathfrak{u}_\theta|^2}\right\}d\theta \end{aligned} \tag{4.82}$$

$$\begin{aligned} \mathfrak{b}^j(x) = &-\int_0^1 \frac{\partial F}{\partial S_i^j}\left((1-\theta)A^{\#}(x) + \theta\tilde{A}^{\#}(x)\right)\frac{\partial_k \mathfrak{u}_\theta\,\partial^2_{ik}\mathfrak{u}_\theta}{1+|\partial_x \mathfrak{u}_\theta|^2}\,d\theta \\ &-\int_0^1 \frac{\partial F}{\partial S_i^k}\left((1-\theta)A^{\#}(x) + \theta\tilde{A}^{\#}(x)\right)\frac{\partial_j \mathfrak{u}_\theta\,\partial^2_{ik}\mathfrak{u}_\theta + \partial_k \mathfrak{u}_\theta\,\partial^2_{ij}\mathfrak{u}_\theta}{1+|\partial_x \mathfrak{u}_\theta|^2}\,d\theta \end{aligned}$$

$$+3\int_0^1 \frac{\partial F}{\partial S_i^k}\left((1-\theta)\,A^{\#}(x)+\theta\tilde{A}^{\#}(x)\right)\frac{\partial_j\mathfrak{u}_\theta\,\partial_k\mathfrak{u}_\theta\,\partial_l\mathfrak{u}_\theta\,\partial_{il}^2\mathfrak{u}_\theta}{\left(1+|\partial_x\mathfrak{u}_\theta|^2\right)^{\frac{3}{2}}}\,d\theta$$

$$+\int_0^1 F\left((1-\theta)\,A^{\#}(x)+\theta\tilde{A}^{\#}(x)\right)\frac{\partial_j\mathfrak{u}_\theta}{\sqrt{1+|\partial_x\mathfrak{u}_\theta|^2}}\,d\theta-\frac{1}{2}x_j \tag{4.83}$$

where $\mathfrak{u}_\theta=(1-\theta)\,\mathfrak{u}+\theta\tilde{\mathfrak{u}}$. Note that (4.81) is equivalent to the following divergence form equation:

$$-\partial_i\left(\frac{\mathfrak{a}^{ij}+\mathfrak{a}^{ji}}{2}\partial_j\mathrm{v}\right)=\left(-\partial_i\left(\frac{\mathfrak{a}^{ij}+\mathfrak{a}^{ji}}{2}\right)+\mathfrak{b}^j\right)\partial_j\mathrm{v}+\frac{1}{2}\mathrm{v} \tag{4.84}$$

And by (4.82), (4.83) and (4.79), we have the following estimates for the coefficients of (4.84):

$$\frac{\bar{\lambda}}{1+\parallel\partial_x\mathfrak{u}_\theta\parallel^2_{L^\infty(B^n_\varrho)}}\le\frac{\mathfrak{a}^{ij}+\mathfrak{a}^{ji}}{2}\le C\left(\parallel F\parallel_{C^1(\mathfrak{U})},\ \parallel\mathfrak{u}\parallel_{C^2(B^n_\varrho)}\right) \tag{4.85}$$

$$|\partial_x\mathfrak{a}^{ij}|+|\mathfrak{b}^j|\le C\left(\parallel F\parallel_{C^2(\mathfrak{U})},\ \parallel\mathfrak{u}\parallel_{C^3(B^n_\varrho)}\right) \tag{4.86}$$

On the other hand, since $\hat{X}_m\in\Sigma^0$ and $\hat{X}_m\to\hat{X}$ as $m\nearrow\infty$, v is vanishing at each neighborhood of x_m and $x_m\to 0$ as $m\nearrow\infty$. Thus, by Proposition 4.1 and Remark 4.3, v vanishes on $B^n\left(x_m,\frac{1}{4}(\varrho-|x_m|)\right)$ for all $m\in\mathbb{N}$, which implies that v vanishes on $B^n\left(0,\frac{1}{4}\varrho\right)$. In other words, Σ coincides with $\tilde{\Sigma}$ in a neighborhood of $\hat{X}$, which contradicts with $\hat{X}\in\partial\Sigma^0$. □

Lastly, we would like to estimate $\varkappa$ (defined in (2.5)) in the rotationally symmetric case. For that purpose, we have to compute the covariant derivatives of the second fundamental form of $\mathcal{C}$.

Lemma 4.5 *At each point* $X_{\mathcal{C}}=(\sigma s\,\nu,s)\in\mathcal{C}$ (with $\nu\in\mathbf{S}^{n-1}$, $s>0$), *pick an orthonormal basis* $\left\{e_1^{\mathcal{C}},\ldots,e_n^{\mathcal{C}}\right\}$ *for* $T_{X_{\mathcal{C}}}\mathcal{C}$ *so that* $e_n^{\mathcal{C}}=\frac{(\sigma\nu,1)}{\sqrt{1+\sigma^2}}$, *then we have*

$$A_{\mathcal{C}}\left(e_i^{\mathcal{C}},e_j^{\mathcal{C}}\right)=\kappa_i^{\mathcal{C}}\delta_{ij},\quad \text{with } \kappa_1^{\mathcal{C}}=\cdots=\kappa_{n-1}^{\mathcal{C}}=\frac{1}{\sigma|X_{\mathcal{C}}|},\ \kappa_n^{\mathcal{C}}=0 \tag{4.87}$$

$$\nabla_{\mathcal{C}}A_{\mathcal{C}}\left(e_i^{\mathcal{C}},e_j^{\mathcal{C}},e_n^{\mathcal{C}}\right)=\frac{-1}{\sigma|X_{\mathcal{C}}|^2}\delta_{ij}=-\frac{\kappa_i^{\mathcal{C}}}{|X_{\mathcal{C}}|}\delta_{ij},\quad\forall i,j\neq n \tag{4.88}$$

$$\nabla_{\mathcal{C}}A_{\mathcal{C}}\left(e_i^{\mathcal{C}},e_j^{\mathcal{C}},e_k^{\mathcal{C}}\right)=\nabla_{\mathcal{C}}A_{\mathcal{C}}\left(e_i^{\mathcal{C}},e_n^{\mathcal{C}},e_n^{\mathcal{C}}\right)=\nabla_{\mathcal{C}}A_{\mathcal{C}}\left(e_n^{\mathcal{C}},e_n^{\mathcal{C}},e_n^{\mathcal{C}}\right)=0\quad\forall i,j,k\neq n \tag{4.89}$$

where $A_{\mathcal{C}}$ *is the second fundamental form of* $\mathcal{C}$ *and* $\nabla_{\mathcal{C}}A_{\mathcal{C}}$ *is its covariant derivative. Note that* $A_{\mathcal{C}}$ *and* $\nabla_{\mathcal{C}}A_{\mathcal{C}}$ *are totally symmetric tensors (by Codazzi equation).*

Proof Let's parameterize $\mathcal{C}$ by

$$X_{\mathcal{C}}=(\sigma s\,\nu,s)\quad\text{for } \nu\in\mathbf{S}^{n-1},\ s\in\mathbb{R}_+$$

and take an othornomal local frame $\left\{e_1^{\mathcal{C}},\ldots,e_n^{\mathcal{C}}\right\}$ of $\mathcal{C}$ so that

$$e_n^{\mathcal{C}}=\frac{\partial_s X_{\mathcal{C}}}{|\partial_s X_{\mathcal{C}}|}=\frac{(\sigma\nu,1)}{\sqrt{1+\sigma^2}} \tag{4.90}$$

By the general formula (see [7]) for the principal curvatures of hypersuface of revolution, we get

$$\kappa_1^{\mathcal{C}} = \cdots = \kappa_{n-1}^{\mathcal{C}} = \frac{1}{\sigma s\sqrt{1+\sigma^2}} = \frac{1}{\sigma |X_{\mathcal{C}}|}, \quad \kappa_n^{\mathcal{C}} = 0 \tag{4.91}$$

Since $\left\{e_1^{\mathcal{C}}, \ldots, e_n^{\mathcal{C}}\right\}$ forms a principal basis at each point, so by (4.91) we have

$$\begin{aligned} A_{ii}^{\mathcal{C}} &= \kappa_i^{\mathcal{C}} = \frac{1}{\sigma s\sqrt{1+\sigma^2}} = \frac{1}{\sigma |X_{\mathcal{C}}|} \quad \text{whenever } i \neq n \\ A_{ij}^{\mathcal{C}} &= 0 = A_{nn}^{\mathcal{C}} \quad \text{whenever } i \neq j \end{aligned} \tag{4.92}$$

where $A_{ij}^{\mathcal{C}} \equiv A_{\mathcal{C}}\left(e_i^{\mathcal{C}}, e_j^{\mathcal{C}}\right)$. Also, by the orthonormality of $\left\{e_1^{\mathcal{C}}, \ldots, e_n^{\mathcal{C}}\right\}$ and the product rule, the Christoffel symbols ${}^{\mathcal{C}}\Gamma_{ij}^k \equiv \left(D_{e_i^{\mathcal{C}}}^{\mathcal{C}} e_j^{\mathcal{C}}\right) \cdot e_k^{\mathcal{C}}$ satisfy

$${}^{\mathcal{C}}\Gamma_{ki}^j = \left(D_{e_k^{\mathcal{C}}}^{\mathcal{C}} e_i^{\mathcal{C}}\right) \cdot e_j^{\mathcal{C}} = -\left(D_{e_k^{\mathcal{C}}}^{\mathcal{C}} e_j^{\mathcal{C}}\right) \cdot e_i^{\mathcal{C}} = -{}^{\mathcal{C}}\Gamma_{kj}^i \tag{4.93}$$

Thus, from (4.92) and (4.93), we deduce that whenever $i, j \neq n$ or $i = j = n$, there holds

$$\nabla_k^{\mathcal{C}} A_{ij}^{\mathcal{C}} = D_{e_k^{\mathcal{C}}}\left(A_{ij}^{\mathcal{C}}\right) - {}^{\mathcal{C}}\Gamma_{ki}^j A_{jj}^{\mathcal{C}} - {}^{\mathcal{C}}\Gamma_{kj}^i A_{ii}^{\mathcal{C}} = D_{e_k^{\mathcal{C}}}\left(A_{ij}^{\mathcal{C}}\right) \tag{4.94}$$

By (4.94), (4.92) and (4.90), we get

$$\begin{aligned} \nabla_n^{\mathcal{C}} A_{ij}^{\mathcal{C}} &= D_{e_n^{\mathcal{C}}}\left(\kappa_i^{\mathcal{C}}\delta_{ij}\right) = \frac{1}{\sqrt{1+\sigma^2}}\partial_s\left(\frac{1}{\sigma s\sqrt{1+\sigma^2}}\right)\delta_{ij} \\ &= \frac{-1}{\sigma\left(1+\sigma^2\right)s^2}\delta_{ij} = \frac{-1}{\sigma |X_{\mathcal{C}}|^2}\delta_{ij} \quad \text{if } i, j \neq n \end{aligned}$$

which verifies (4.88).

By (4.94), (4.92) and noting that $|X_{\mathcal{C}}|$ is invariant along $e_k^{\mathcal{C}}$ for $k \neq n$, we get

$$\nabla_k^{\mathcal{C}} A_{ij}^{\mathcal{C}} = D_{e_k^{\mathcal{C}}}\left(\kappa_i^{\mathcal{C}}\delta_{ij}\right) = D_{e_k^{\mathcal{C}}}\left(\frac{1}{\sigma |X_{\mathcal{C}}|}\right)\delta_{ij} = 0 \quad \text{if } i, j, k \neq n \tag{4.95}$$

From (4.94) and (4.92), we have

$$\nabla_i^{\mathcal{C}} A_{nn}^{\mathcal{C}} = D_{e_i^{\mathcal{C}}}\left(A_{nn}^{\mathcal{C}}\right) = 0 \quad \forall i \tag{4.96}$$

Then (4.89) follows from (4.95) and (4.96). □

Combining (2.1), (2.2), (2.3) with Lemma 4.5, we conclude the following:

Proposition 4.4 *The constant $\varkappa$ defined in (2.5) can be estimated by*

$$\varkappa \leq C(n)\left(\left|\partial^2 f\left(\vec{1}, 0\right)\right| + \left|\partial_1 f\left(\vec{1}, 0\right) - \partial_n f\left(\vec{1}, 0\right)\right|\right) \tag{4.97}$$

Proof At each point $X_{\mathcal{C}} \in \mathcal{C}$, take an orthonormal basis $\left\{e_1^{\mathcal{C}}, \ldots, e_n^{\mathcal{C}}\right\}$ for $T_{X_{\mathcal{C}}}\mathcal{C}$ so that $e_n^{\mathcal{C}} = \frac{(\sigma\nu, 1)}{\sqrt{1+\sigma^2}}$. Then by (2.2), (2.3), Lemma 4.5 and the homogeneity of the derivatives of f, we get

$$\left|\frac{\partial^2 F}{\partial S_i^j \partial S_k^l}\left(A_{\mathcal{C}}^{\#}\right)\right| \leq \left(\left|\partial^2 f\left(\kappa_1^{\mathcal{C}},\ldots,\kappa_n^{\mathcal{C}}\right)\right| + \left|\frac{\partial_1 f\left(\kappa_1^{\mathcal{C}},\ldots,\kappa_n^{\mathcal{C}}\right) - \partial_n f\left(\kappa_1^{\mathcal{C}},\ldots,\kappa_n^{\mathcal{C}}\right)}{\kappa_1^{\mathcal{C}} - \kappa_n^{\mathcal{C}}}\right|\right)$$
$$= \frac{1}{\kappa_1^{\mathcal{C}}}\left(\left|\partial^2 f\left(\overrightarrow{1},0\right)\right| + \left|\partial_1 f\left(\overrightarrow{1},0\right) - \partial_k f\left(\overrightarrow{1},0\right)\right|\right)$$

which implies that

$$|X_{\mathcal{C}}|\left|\sum_{k,l}\frac{\partial^2 F}{\partial S_i^j \partial S_k^l}\left(A_{\mathcal{C}}^{\#}\right)\left(\nabla_{\mathcal{C}} A_{\mathcal{C}}^{\#}\right)_k^l\right|$$
$$\leq |X_{\mathcal{C}}|\frac{C(n)}{\kappa_1^{\mathcal{C}}}\left(\left|\partial^2 f\left(\overrightarrow{1},0\right)\right| + \left|\partial_1 f\left(\overrightarrow{1},0\right) - \partial_k f\left(\overrightarrow{1},0\right)\right|\right)\frac{\kappa_1^{\mathcal{C}}}{|X_{\mathcal{C}}|}$$
$$= C(n)\left(\left|\partial^2 f\left(\overrightarrow{1},0\right)\right| + \left|\partial_1 f\left(\overrightarrow{1},0\right) - \partial_k f\left(\overrightarrow{1},0\right)\right|\right)$$

Therefore,

$$\varkappa = \sup_{X_{\mathcal{C}}\in\mathcal{C}\cap\left(B_3\setminus\bar{B}_{\frac{1}{3}}\right)}\left|\sum_{k,l}\frac{\partial^2 F}{\partial S_i^j \partial S_k^l}\left(A_{\mathcal{C}}^{\#}\right)\left(\nabla_{\mathcal{C}} A_{\mathcal{C}}^{\#}\right)_k^l\right|$$
$$\leq C(n)\left(\left|\partial^2 f\left(\overrightarrow{1},0\right)\right| + \left|\partial_1 f\left(\overrightarrow{1},0\right) - \partial_k f\left(\overrightarrow{1},0\right)\right|\right)$$

□

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