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Calculus of Variations

Multiplicity results for elliptic problems with super-critical concave and convex nonlinearties

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Abstract We shall prove a multiplicity result for semilinear elliptic problems with a super-critical nonlinearity of the form,

$$\begin{cases} -\Delta u = |u|^{p-2}u + \mu |u|^{q-2}u, & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^2-boundary and $1 < q < 2 < p$. As a consequence of our results we shall show that, for each $p > 2$, there exists $\mu^* > 0$ such that for each $\mu \in (0, \mu^*)$ problem (1) has a sequence of solutions with a negative energy. This result is already known for the subcritical values of p. In this paper, we shall extend it to the supercritical values of p as well. Our methodology is based on a new variational principle established by one of the authors that allows one to deal with problems beyond the usual locally compactness structure.

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1 Introduction

In this paper we consider the semilinear elliptic problem

$$\begin{cases} -\Delta u = u|u|^{p-2} + \mu u|u|^{q-2}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^2-boundary, $\mu \in \mathbb{R}^+$ and $1 < q < 2 < p$. This problem has received a lot of attention since being first investigated by Ambrosetti et al. [2]. Using the method of sub-super solutions, it is proved in [2] that there exists $\Lambda > 0$ such that (2) has a positive solution $\underline{u}_\mu$ for $0 < \mu \leq \Lambda$. The importance of their results lies in the fact that p can be arbitrarily large. If in addition $p < 2^* := 2n/(n-2)$, then solutions of (2) correspond to critical points of the functional

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p}\int_\Omega |u|^p dx - \frac{\mu}{q}\int_\Omega |u|^q dx, \tag{3}$$

defined on $H_0^1(\Omega)$, and hence variational methods may be applied. In this case a second positive solution $\overline{u}_\mu$ exists for $0 < \mu \leq \Lambda$ as shown in [2], Theorem 2.3. Moreover, there exists $\Lambda > 0$ such that for every $0 < \mu < \Lambda$ problem (2) has infinitely many solutions $\{\underline{u}_{\mu,j}\}_{j\in\mathbb{N}}$ satisfying $I(\underline{u}_{\mu,j}) < 0$, and there exist infinitely many solutions $\{\overline{u}_{\mu,j}\}_{j\in\mathbb{N}}$ satisfying $I(\overline{u}_{\mu,j}) > 0$. In fact, they showed that there exists an additional pair of solutions (which can change sign) for all $0 < \mu < \mu^*$ with μ^* possibly smaller than Λ (see also Ambrosetti et al. [1] and references therein). Their method relied on the standard methods in the critical point theory.
Over the years, the study for the number of positive solutions were furthered by many authors including [5,14,15,19]. It was indeed established that if $1 < q < 2 < p < 2^*$ then there exists $\mu^* > 0$ such that for $0 < \mu < \mu^*$, there are exactly two positive solutions of (2), exactly one positive solution for $\mu = \mu^*$ and no positive solution exists for $\mu > \mu^*$, when Ω is the unit ball in $\mathbb{R}^n$.
In [2,7] the existence of solutions with negative energy has also been proved in the critical case $p = 2^*$ provided $\mu > 0$ is small enough.

Also, Bartsch and Willem [3] showed that for the subcritical case $\mu^* = \infty$ and $I(\underline{u}_{\mu,j}) \to 0$ as $j \to \infty$. In addition they proved that a sequence of solutions $\{\overline{u}_{\mu,j}\}$ with a positive energy also exists for $\mu \leq 0$. Furthermore, Wang [20] proved that the solutions $\underline{u}_{\mu,j}$ not only tend to 0 energetically but also uniformly on Ω. Wang even dealt with more general classes of nonlinear functions $f_\mu(u)$ instead of just $u|u|^{p-2} + \mu u|u|^{q-2}$. The variational structure and the oddness of the nonlinearity, however, are essential to obtain infinitely many solutions $\{\underline{u}_{\mu,j}\}$ and $\{\overline{u}_{\mu,j}\}$ for the subcritical case.

Our main objective in this paper is to prove multiplicity results without imposing any growth condition on the nonlinearity $u|u|^{p-2}$. We shall now state our result in this paper regarding positive solutions of (2).

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2-boundary and assume that $1 < q < 2 < p$. Then there exists $\mu^* > 0$ such that for each $\mu \in (0, \mu^*)$ problem (2) has at least one positive solution $\overline{u} \in W^{2,n}(\Omega)$ with a negative energy.*

This result, however, is already known in [2]. Here we shall provide a different approach based on variational principles on convex closed sets. The next result concerns with the multiplicity of solutions for the super-critical case.

Theorem 1.2 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2-boundary and assume that $1 < q < 2 < p$. Then there exists $\mu^* > 0$ such that for each $\mu \in (0, \mu^*)$ problem (2) has infinitely many distinct nontrivial solutions in $W^{2,n}(\Omega)$ with a negative energy.*

As there is no upper bound for p in Theorem 1.2, thus, this theorem will be an extension of a similar result by Ambrosetti et al. [2] to the supercritical case.

Remark 1.3 Note that the term $u|u|^{p-2}$ can be substituted by any super-linear odd function f that behaves like $f(u) = u|u|^{p-2}$ around $u = 0$ and around $u = +\infty$. The oddness of f is not required in Theorem 1.1, however, f has to be positive on $(0, \infty)$. We would also like to remark that the parameter μ^* is the same in both Theorems 1.1 and 1.2. Finally, it is worth noting that there exists $\Lambda \in (0, \infty)$ such that problem (2) does not have any positive solution for $\lambda > \Lambda$ (see Theorem 2.1 in [2]).

We shall be proving Theorems 1.1 and 1.2 by making use of a new abstract variational principle established recently in [12,13] (see also [10,11] for some other new variational principles and [4] for an application in super-critical Neumann problems). To be more specific, let V be a reflexive Banach space, V^* its topological dual and let K be a non-empty convex and weakly closed subset of V. Assume that $\Psi : V \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex, lower semi-continuous function which is Gâteaux differentiable on K. The Gâteaux derivative of Ψ at each point $u \in K$ will be denoted by $D\Psi(u)$. The restriction of Ψ to K is denoted by Ψ_K and defined by

$$\Psi_K(u) = \begin{cases} \Psi(u), & u \in K, \\ +\infty, & u \notin K. \end{cases} \tag{4}$$

For a given functional $\Phi \in C^1(V, \mathbb{R})$ denote by $D\Phi \in V^*$ its derivative and consider the functional $I_K : V \to (-\infty, +\infty]$ defined by

$$I_K(u) := \Psi_K(u) - \Phi(u).$$

According to Szulkin [18], we have the following definition for critical points of I_K.

Definition 1.4 A point $u_0 \in V$ is said to be a critical point of I_K if $I_K(u_0) \in \mathbb{R}$ and if it satisfies the following inequality

$$\langle D\Phi(u_0), u_0 - v\rangle + \Psi_K(v) - \Psi_K(u_0) \geq 0, \quad \forall\, v \in V, \tag{5}$$

where $\langle ., .\rangle$ is the duality pairing between V and its dual V^*.

We shall now recall the following variational principle established recently in [12].

Theorem 1.5 *Let V be a reflexive Banach space and K be a non-empty convex and weakly closed subset of V. Let $\Psi : V \to \mathbb{R} \cup \{+\infty\}$ be a convex, lower semi-continuous function which is Gâteaux differentiable on K and let $\Phi \in C^1(V, \mathbb{R})$. If the following two assertions hold:*

(i) *The functional $I_K : V \to \mathbb{R} \cup \{+\infty\}$ defined by $I_K(u) := \Psi_K(u) - \Phi(u)$ has a critical point $u_0 \in V$ as in Definition 1.4, and;*
(ii) *there exists $v_0 \in K$ such that $D\Psi(v_0) = D\Phi(u_0)$ in the following sense*

$$\Psi(v) - \Psi(v_0) \geq \langle D\Phi(u_0), v - v_0\rangle, \quad \forall\, v \in V, \tag{6}$$

Then $u_0 \in K$ is a solution of the equation

$$D\Psi(u) = D\Phi(u), \tag{7}$$

in the following sense

$$\Psi(v) - \Psi(u_0) \geq \langle D\Phi(u_0), v - u_0 \rangle, \quad \forall\, v \in V.$$

For the convenience of the reader, by choosing the functions Ψ, Φ and the convex set K in lines with problem (2), we shall provide a proof to a particular case of Theorem 1.5 applicable to this problem.

In the next section we shall recall some preliminaries from convex analysis, critical point theory and Elliptic regularity theory. Section 3 is devoted to the proof of Theorems 1.1 and 1.2.

2 Preliminaries

In this section we recall some important definitions and results from convex analysis [6] and partial differential equations [8].

Let V be a real Banach space and V^* its topological dual and let $\langle ., . \rangle$ be the pairing between V and V^*. The weak topology on V induced by $\langle ., . \rangle$ is denoted by $\sigma(V, V^*)$. A function $\Psi : V \to \mathbb{R}$ is said to be weakly lower semi-continuous if

$$\Psi(u) \leq \liminf_{m \to \infty} \Psi(u_m),$$

for each $u \in V$ and any sequence u_m approaching u in the weak topology $\sigma(V, V^*)$. Let $\Psi : V \to \mathbb{R} \cup \{\infty\}$ be a proper (i.e. $Dom(\Psi) = \{v \in V;\ \Psi(v) < \infty\} \neq \emptyset$) convex function. The subdifferential $\partial\Psi$ of Ψ is defined to be the following set-valued operator: if $u \in Dom(\Psi)$, set

$$\partial\Psi(u) = \{u^* \in V^*;\ \langle u^*, v - u \rangle + \Psi(u) \leq \Psi(v) \quad \text{for all } v \in V\}$$

and if $u \notin Dom(\Psi)$, set $\partial\Psi(u) = \varnothing$. If Ψ is Gâteaux differentiable at u, denote by $D\Psi(u)$ the Gâteaux derivative of Ψ at u. In this case $\partial\Psi(u) = \{D\Psi(u)\}$.

Let I be a function on V satisfying the following hypothesis:
(H): $I = \Psi - \Phi$,*where* $\Phi \in C^1(V, \mathbb{R})$ and $\Psi : V \to (-\infty, +\infty]$ *is proper, convex and lower semi-continuous.*

Definition 2.1 A point $u \in V$ is said to be a critical point of I if $u \in Dom(\Psi)$ and if it satisfies the inequality

$$\langle D\Phi(u), u - v \rangle + \Psi(v) - \Psi(u) \geq 0, \quad \forall\, v \in V, \tag{8}$$

where $D\Phi(u)$ stands for the derivative of Φ at u.

Note that a function satisfying (8) is indeed a solution of the inclusion $D\Phi(u) \in \partial\Psi(u)$.

Proposition 2.2 *If I satisfies (H), then each local minimum of I is necessarily a critical point of I.*

Proof Let u be a local minimum of I. Using the convexity of Ψ, it follows that for all small $t > 0$,

$$0 \leq I\big((1-t)u + tv\big) - I(u) = \Phi\big(u + t(v-u)\big) - \Phi(u) + \Psi\big((1-t)u + tv\big) - \Psi(u)$$
$$\leq \Phi\big(u + t(v-u)\big) - \Phi(u) + t\left(\Psi(v) - \Psi(u)\right).$$

Dividing by t and letting $t \to 0^+$ we obtain (8). □

The critical point theory for functions of the type (H) was established by Szulkin [18]. According to [18], say that I satisfies the compactness condition of Palais-Smale type provided,
(PS): *If* $\{u_m\}$ *is a sequence such that* $I(u_m) \to c \in \mathbb{R}$ *and*

$$\langle D\Phi(u_m), u_m - v\rangle + \Psi(v) - \Psi(u_m) \geq -\epsilon_m \|v - u_m\|_V, \quad \forall\, v \in V, \tag{9}$$

where $\epsilon_m \to 0$, then $\{u_m\}$ *possesses a convergent subsequence.*

In the following we recall an important result about critical points of even functions of the type (H). We shall begin with some preliminaries. Let Σ be the collection of all symmetric subsets of $V\backslash\{0\}$ which are closed in V. A nonempty set $A \in \Sigma$ is said to have *genus* k (denoted $\gamma(A) = k$) if k is the smallest integer with the property that there exists an odd continuous mapping $h : A \to \mathbb{R}^k\backslash\{0\}$. If such an integer does not exist, $\gamma(A) = \infty$. For the empty set $\emptyset$ we define $\gamma(\emptyset) = 0$.

Proposition 2.3 *Let* $A \in \Sigma$. *If* A *is a homeomorphic to* S^{k-1} *by an odd homeomorphism, then* $\gamma(A) = k$.

Proof and a more detailed discussion of the notion of genus can be found in [16,17]. Let Θ be the collection of all nonempty closed and bounded subsets of V. In Θ we introduce the Hausdorff metric distance ([9], §15, VII), given by

$$dist(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\}.$$

The space $(\Theta, dist)$ is complete ([9], §29, IV). Denote by Γ the sub-collection of Θ consisting of all nonempty compact symmetric subsets of V and let

$$\Gamma_j = cl\{A \in \Gamma : 0 \notin A, \gamma(A) \geq j\} \tag{10}$$

(cl is the closure in Γ). It is easy to verify that Γ is closed in Θ, so $(\Gamma, dist)$ and $(\Gamma_j, dist)$ are complete metric spaces. The following Theorem is proved in [18].

Theorem 2.4 *Suppose that* $I : V \to (-\infty, +\infty]$ *satisfies (H) and (PS),* $I(0) = 0$ *and* Φ, Ψ *are even. Define*

$$c_j = \inf_{A\in\Gamma_j} \sup_{u\in A} I(u).$$

If $-\infty < c_j < 0$ *for* $j = 1, \ldots, k$, *then* I *has at least* k *distinct pairs of nontrivial critical points by means of Definition* 2.1.

We shall now recall some notations and results from the theory of Sobolev spaces and Elliptic regularity required in the sequel. Here is the general Sobolev embedding theorem in $W^{k,p}(\Omega)$ (see Lemma 7.26 in [8]).

Theorem 2.5 *Let* Ω *be a bounded* $C^{0,1}$ *domain in* $\mathbb{R}^n$. *Then,*

(i) *If* $kp < n$, *the space* $W^{k,p}(\Omega)$ *is continuously imbedded in* $L^{t^*}(\Omega)$, $t^* = \frac{np}{(n-kp)}$, *and compactly imbedded in* $L^q(\Omega)$ *for any* $q < t^*$.

(ii) *If* $0 \le m < k - \frac{n}{p} < m+1$, *the space* $W^{k,p}(\Omega)$ *is continuously imbedded in* $C^{m,\alpha}(\overline{\Omega})$, $\alpha = k - \frac{n}{p} - m$, *and compactly imbedded in* $C^{m,\beta}(\overline{\Omega})$ *for any* $\beta < \alpha$.

The following inequality is proved in ([8], Lemma 9.17).

Lemma 2.6 *Let* Ω *be a bounded* $C^{1,1}$ *domain in* $\mathbb{R}^n$ *and let the operator* $L = a^{ij}(x)D_{ij}u + b^i(x)D_iu + c(x)u$ *be strictly Elliptic in* Ω *with coefficients* $a^{ij} \in C(\Omega)$, $b^i, c \in L^\infty(\Omega)$, *with* $i, j = 1, \ldots, n$ *and* $c \le 0$. *Then there exists a positive constant C (independent of u) such that*

$$\|u\|_{W^{2,p}(\Omega)} \le C\|Lu\|_{L^p(\Omega)},$$

for all $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, $1 < p < \infty$.

Here is a direct consequence of Lemma 2.6.

Corollary 2.7 *Let* Ω *be a bounded* $C^{1,1}$ *domain in* $\mathbb{R}^n$. *Assume that* $p \ge 2$. *Then there exist constants* Λ_1 *and* Λ_2 *such that*

$$\Lambda_1\|u\|_{W^{2,p}(\Omega)} \le \|\Delta u\|_{L^p(\Omega)} \le \Lambda_2\|u\|_{W^{2,p}(\Omega)},$$

for all $u \in W^{2,p}(\Omega) \cap H_0^1(\Omega)$.

Proof Since $p \ge 2$, it is easily seen that $W^{2,p}(\Omega) \cap H_0^1(\Omega) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Thus, the existence of Λ_1 follows from Lemma 2.6. The existence of Λ_2 follows from the definition of the Sobolev space $W^{2,p}(\Omega)$. □

3 Proofs and further comments

We shall need some preliminary results before proving Theorems 1.1 and 1.2 in this section. We shall consider the Banach space $V = H_0^1(\Omega) \cap L^p(\Omega)$ equipped with the following norm

$$\|u\| := \|u\|_{H_0^1(\Omega)} + \|u\|_{L^p(\Omega)}.$$

Note that the duality pairing between V and its dual V^* is defined by

$$\langle u, u^* \rangle = \int_\Omega u(x)u^*(x)\,dx, \quad \forall\, u \in V, \forall u^* \in V^*.$$

Let $I : V \to \mathbb{R}$ be the Euler–Lagrange functional corresponding to (2),

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p}\int_\Omega |u|^p dx - \frac{\mu}{q}\int_\Omega |u|^q dx.$$

To make use of Theorem 1.5, we shall first define the function $\Phi : V \to \mathbb{R}$ by

$$\Phi(u) = \frac{1}{p}\int_\Omega |u|^p dx + \frac{\mu}{q}\int_\Omega |u|^q dx,$$

Note that $\Phi \in C^1(V; \mathbb{R})$. Define $\Psi : V \to \mathbb{R}$ by

$$\Psi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx.$$

The restriction of Ψ to a convex and weakly closed subset K (to be introduced later) of V is denoted by Ψ_K and defined by

$$\Psi_K(u) = \begin{cases} \Psi(u), & u \in K, \\ +\infty, & u \notin K, \end{cases} \tag{11}$$

Finally, let us introduce the functional $I_K : V \to (-\infty, +\infty]$ defined by

$$I_K(u) := \Psi_K(u) - \Phi(u), \tag{12}$$

which is of the form (H). Note that I_K is indeed the Euler–Lagrange functional corresponding to (2) restricted to K. Here is a simplified version of Theorem 1.5 applicable to problem (2).

Theorem 3.1 *Let $V = H_0^1(\Omega) \cap L^p(\Omega)$, and let K be a convex and weakly closed subset of V. If the following two assertions hold:*

(i) *The functional $I_K : V \to \mathbb{R} \cup \{+\infty\}$ defined in (12) has a critical point $\bar{u} \in V$ as in Definition 2.1, and;*
(ii) *there exists $\bar{v} \in K$ such that $-\Delta \bar{v} = D\Phi(\bar{u}) = \bar{u}|\bar{u}|^{p-2} + \mu \bar{u}|\bar{u}|^{q-2}$ in the weak sense, i.e.,*

$$\int_\Omega \nabla \bar{v}.\nabla \eta \, dx = \int_\Omega D\Phi(\bar{u})\eta \, dx, \quad \forall\, \eta \in V.$$

Then $\bar{u} \in K$ is a weak solution of the equation

$$-\Delta u = u|u|^{p-2} + \mu u|u|^{q-2}. \tag{13}$$

Proof Since $\bar{u}$ is a critical point of $I(u) = \Psi_K(u) - \Phi(u)$, it follows from Definition 2.1 that

$$\Psi_K(v) - \Psi_K(\bar{u}) \geq \langle D\Phi(\bar{u}), v - \bar{u} \rangle, \quad \forall\, v \in V, \tag{14}$$

where $\langle D\Phi(\bar{u}), v - \bar{u} \rangle = \int_\Omega D\Phi(\bar{u})(v - \bar{u})\, dx$. Thus, the inequality (14) translates to

$$\frac{1}{2}\int_\Omega |\nabla v|^2 dx - \frac{1}{2}\int_\Omega |\nabla \bar{u}|^2 dx \geq \int_\Omega D\Phi(\bar{u})(v - \bar{u})\, dx, \quad \forall\, v \in K. \tag{15}$$

It also follows from (ii) in the theorem that there exists $\bar{v} \in K$ such that $-\Delta \bar{v} = D\Phi(\bar{u})$ in the weak sense, i.e.,

$$\int_\Omega \nabla \bar{v}.\nabla \eta \, dx = \int_\Omega D\Phi(\bar{u})\eta \, dx, \quad \forall\, \eta \in V. \tag{16}$$

Thus, by substituting $\eta = \bar{u} - \bar{v}$ in the latter equality one gets

$$\int_\Omega \nabla \bar{v}.\nabla(\bar{u} - \bar{v})\, dx = \int_\Omega D\Phi(\bar{u})(\bar{u} - \bar{v})\, dx. \tag{17}$$

Now by substituting $v = \bar{v}$ in (15) and taking into account the equality (17) we obtain that

$$\frac{1}{2}\int_\Omega |\nabla \bar{v}|^2 dx - \frac{1}{2}\int_\Omega |\nabla \bar{u}|^2 dx \geq \int_\Omega D\Phi(\bar{u})(\bar{v} - \bar{u})\, dx = \int_\Omega \nabla \bar{v}.\nabla(\bar{v} - \bar{u}) dx. \tag{18}$$

On the other hand, it follows from the convexity of Ψ that

$$\frac{1}{2}\int_\Omega |\nabla \bar{u}|^2 dx - \frac{1}{2}\int_\Omega |\nabla \bar{v}|^2 dx \geq \int_\Omega \nabla \bar{v}.\nabla(\bar{u} - \bar{v})\, dx. \tag{19}$$

Taking into account inequalities (18) and (19) we obtain that

$$\frac{1}{2}\int_{\Omega}|\nabla\bar{v}|^2dx-\frac{1}{2}\int_{\Omega}|\nabla\bar{u}|^2dx=\int_{\Omega}\nabla\bar{v}.\nabla(\bar{v}-\bar{u})\,dx. \tag{20}$$

This indeed implies that

$$\int_{\Omega}|\nabla\bar{v}-\nabla\bar{u}|^2dx=0,$$

from which we obtain $\bar{v}=\bar{u}$ for a.e. $x\in\Omega$. Therefore, the result follows from the equality (16). □

We shall use Theorem 3.1 to prove our main results in Theorems 1.1 and 1.2. The convex subset K of V required in Theorem 1.2 is defined as follows

$$K(r):=\Big\{u\in V:\|u\|_{W^{2,n}(\Omega)}\leq r\Big\},$$

for some $r>0$ to be determined later. Also, the convex set K required in the proof of Theorem 1.1 consists of all non-negative functions in $K(r)$ for some $r>0$.

Lemma 3.2 *Let $r>0$ be fixed. The set*

$$K(r):=\Big\{u\in V:\|u\|_{W^{2,n}(\Omega)}\leq r\Big\},$$

is weakly closed in V.

Proof Let $\{u_m\}$ be a sequence in $K(r)$ such that $u_m\rightharpoonup u$ weakly in V. It follows that, up to a subsequence, $u_m(x)\to u(x)$ for a.e. $x\in\Omega$. On the other hand, since $\{u_m\}\subset K(r)$ we have that $\|u_m\|_{W^{2,n}(\Omega)}\leq r$ for all $m\in\mathbb{N}$. Thus, we can conclude that $\{u_m\}$ is bounded in $W^{2,n}(\Omega)$. Going if necessary to a subsequence, there exists $\overline{u}\in W^{2,n}(\Omega)$ such that $u_m\rightharpoonup\overline{u}$ weakly in $W^{2,n}(\Omega)$ and $u_m(x)\to\overline{u}(x)$ for a.e. $x\in\Omega$. It then implies that $u(x)=\overline{u}(x)$ for a.e. $x\in\Omega$ and therefore $u_m\rightharpoonup u$ weakly in $W^{2,n}(\Omega)$. It now follows from the weak lower semi-continuity of the norm in $W^{2,n}(\Omega)$ that

$$\|u\|_{W^{2,n}(\Omega)}\leq\liminf_{m\to\infty}\|u_m\|_{W^{2,n}(\Omega)}\leq r.$$

Thus $u\in K(r)$. □

To apply Theorem 3.1, we shall need to verify both conditions (i) and (ii) in this Theorem. To verify condition (i) in Theorem 1.1 we simply find a minimizer of I_K for some weakly compact and convex subset K of V, and in Theorem 1.2 we shall make use of the abstract Theorem 2.4 to find a sequence of solutions. However, condition (ii) in Theorem 3.1 seems to be rather identical for both Theorems 1.1 and 1.2. Let us first proceed with condition (ii) in Theorem 3.1. In fact, our plan is to show that if $\overline{u}\in K(r)$ then for appropriate choices of r, there exists $v\in K(r)$ such that $-\Delta v=D\Phi(\overline{u})$. We shall do this in a few lemmas.

Lemma 3.3 *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain and assume that $1<q<2<p$. Let d_1 and d_2 be the best constants in the imbeddings $W^{2,n}(\Omega)\hookrightarrow L^{n(p-1)}(\Omega)$ and $W^{2,n}(\Omega)\hookrightarrow L^{n(q-1)}(\Omega)$, respectively. Then*

$$\|D\Phi(u)\|_{L^n(\Omega)}\leq C_1r^{p-1}+\mu C_2r^{q-1},\quad\forall\, u\in K(r),$$

where $C_1=d_1^{p-1}$ and $C_2=d_2^{q-1}$.

Proof By the definition of $D\Phi(u)$ we have

$$\|D\Phi(u)\|_{L^n(\Omega)} = \left\|u|u|^{p-2} + \mu u|u|^{q-2}\right\|_{L^n(\Omega)} \leq \left\|u|u|^{p-2}\right\|_{L^n(\Omega)} + \mu\left\|u|u|^{q-2}\right\|_{L^n(\Omega)}$$
$$\leq \|u\|^{p-1}_{L^{n(p-1)}(\Omega)} + \mu\|u\|^{q-1}_{L^{n(q-1)}(\Omega)}.$$

By Theorem 2.5 the space $W^{2,n}(\Omega)$ is compactly imbedded in $L^{n(p-1)}$ and $L^{n(q-1)}$. Thus,

$$\|D\Phi(u)\|_{L^n(\Omega)} \leq C_1\|u\|^{p-1}_{W^{2,n}(\Omega)} + \mu C_2\|u\|^{q-1}_{W^{2,n}(\Omega)}.$$

It follows from $u \in K(r)$ that

$$\|D\Phi(u)\|_{L^n(\Omega)} \leq C_1 r^{p-1} + \mu C_2 r^{q-1},$$

as desired. □

By a straightforward computation one can easily deduce the following result.

Lemma 3.4 *Let $1 < q < 2 < p$. Assume that C_1 and C_2 are given in Lemma 3.3. Then there exists $\mu^* > 0$ with the following properties.*

1. *For each $\mu \in (0, \mu^*)$, there exist positive numbers $r_1, r_2 \in \mathbb{R}$ with $r_1 < r_2$ such that $r \in [r_1, r_2]$ if and only if $C_1 r^{p-1} + \mu C_2 r^{q-1} \leq r$.*
2. *For $\mu = \mu^*$, there exists one and only one $r > 0$ such that $C_1 r^{p-1} + \mu C_2 r^{q-1} = r$.*
3. *For $\mu > \mu^*$, there is no $r > 0$ such that $C_1 r^{p-1} + \mu C_2 r^{q-1} = r$.*

Remark 3.5 Since the Sobolev space $W^{2,n}(\Omega)$ is compactly embedded into $L^p(\Omega)$, we obtain that

$$V \cap W^{2,n}(\Omega) = H^1_0(\Omega) \cap W^{2,n}(\Omega).$$

It also follows from Corollary 2.7 that $u \to \|\Delta u\|_{L^n(\Omega)}$ is an equivalent norm on $H^1_0(\Omega) \cap W^{2,n}(\Omega)$. For the rest of the paper, we shall then consider this norm, i.e., for each $u \in H^1_0(\Omega) \cap W^{2,n}(\Omega)$,

$$\|u\|_{W^{2,n}(\Omega)} = \|\Delta u\|_{L^n(\Omega)}.$$

We are now in the position to state the following result addressing condition (ii) in Theorem 3.1.

Lemma 3.6 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2-boundary and assume that $1 < q < 2 < p$. We also assume that $\mu^* > 0$ is given in Lemma 3.4 and $\mu \in (0, \mu^*)$. Let r_1, r_2 be given in part (1) of Lemma 3.4. Then for each $r \in [r_1, r_2]$ and each $\overline{u} \in K(r)$ there exists $v \in K(r)$ such that the following equation holds in the weak sense,*

$$-\Delta v = \overline{u}|\overline{u}|^{p-2} + \mu\overline{u}|\overline{u}|^{q-2}. \tag{21}$$

In particular, $v \in W^{2,n}(\Omega) \cap H^1_0(\Omega)$ and Eq. (21) holds pointwise for a.e. $x \in \Omega$.

Proof By standard methods we see that there exists $v \in H^1_0(\Omega)$ which satisfies

$$-\Delta v = D\Phi(\overline{u}) = \overline{u}|\overline{u}|^{p-2} + \mu\overline{u}|\overline{u}|^{q-2}. \tag{22}$$

in the weak sense. Since the right hand side is an element in $L^n(\Omega)$, it follows from the standard regularity results that $v \in W^{2,n}(\Omega) \cap H^1_0(\Omega)$ and (22) holds pointwise for a.e. $x \in \Omega$. Therefore,

$$\|\Delta v\|_{L^n(\Omega)} = \|D\Phi(\overline{u})\|_{L^n(\Omega)}.$$

Thus, by Remark 3.5 we have that

$$\|v\|_{W^{2,n}(\Omega)} = \|\Delta v\|_{L^n(\Omega)} = \|D\Phi(\overline{u})\|_{L^n(\Omega)}.$$

This together with Lemma 3.3 yield that

$$\|v\|_{W^{2,n}(\Omega)} \leq C_1 r^{p-1} + \mu C_2 r^{q-1},$$

By Lemma 3.4, for each $r \in [r_1, r_2]$ we have that $C_1 r^{p-1} + \mu C_2 r^{q-1} \leq r$. Therefore,

$$\|v\|_{W^{2,n}(\Omega)} \leq C_1 r^{p-1} + \mu C_2 r^{q-1} \leq r,$$

as desired. □

Proof of Theorem 1.1 Let μ^* be as in Lemma 3.6 and $\mu \in (0, \mu^*)$. Also, let r_1 and r_2 be as in Lemma 3.6 and define

$$K := \left\{u \in K(r_2); \;\; u(x) \geq 0 \text{ a.e. } x \in \Omega\right\}.$$

Step 1. We show that there exists $\overline{u} \in K$ such that $I_K(\overline{u}) = \inf_{u \in V} I_K(u)$. Then by Proposition 2.2, we conclude that $\overline{u}$ is a critical point of I_K.
Set $\beta := \inf_{u \in V} I_K(u)$. So by definition of Ψ_K for every $u \notin K$, we have $I_K(u) = +\infty$ and therefore $\beta = \inf_{u \in K} I_K(u)$. On the other hand, by Theorem 2.5, the Sobolev space $W^{2,n}(\Omega)$ is compactly embedded in $L^t(\Omega)$ for all $t > 1$. It then follows that for every $u \in K$

$$\begin{aligned}\Phi(u) &= \frac{1}{p}\int_\Omega |u|^p dx + \frac{\mu}{q}\int_\Omega |u|^q dx \\ &\leq c_1\|u\|^p_{W^{2,n}(\Omega)} + c_2\|u\|^q_{W^{2,n}(\Omega)} \leq c_1 r^p + c_2 r^q,\end{aligned}$$

for some positive constants c_1 and c_2. Since $\Psi(u)$ is nonnegative, we have

$$I_K(u) := \Psi_K(u) - \Phi(u) \geq -\left(c_1 r^p + c_2 r^q\right), \quad \forall\, u \in K.$$

So $\beta > -\infty$. Now, suppose that $\{u_m\}$ is a sequence in V such that $I_K(u_m) \to \beta$. So the sequence $\{I_K(u_m)\}$ is bounded and we can conclude by the definition of I_K that the sequence $\{u_m\}$ is bounded in $W^{2,n}(\Omega)$. Using standard results in Sobolev spaces, after passing to a subsequence if necessary, there exists $\overline{u} \in K$ such that $u_m \rightharpoonup \overline{u}$ weakly in $W^{2,n}(\Omega)$ and strongly in V. Therefore, $I_K(u_m) \to I_K(\overline{u})$. So, $I_K(\overline{u}) = \beta = \inf_{u \in V} I_K(u)$, and the proof of *Step 1* is complete.

Step 2. In this step we show that there exists $v \in K$ such that $-\Delta v = \overline{u}|\overline{u}|^{p-2} + \mu\overline{u}|\overline{u}|^{q-2}$. By Lemma 3.6 together with the fact that $\overline{u} \in K(r_2)$ we obtain that there exists $v \in K(r_2)$ such that $-\Delta v = \overline{u}|\overline{u}|^{p-2} + \mu\overline{u}|\overline{u}|^{q-2}$. To show that $v \in K$, we shall need to verify that v is non-negative almost every where. But, this is a simple consequence of the maximum principle and the fact that $-\Delta v = \overline{u}|\overline{u}|^{p-2} + \mu\overline{u}|\overline{u}|^{q-2} \geq 0$.

It now follows from Theorem 3.1 together with *Step 1* and *Step 2* that $\overline{u}$ is a solution of the problem (2). To complete the proof we shall show that $\overline{u}$ is non-trivial by proving that $I_K(\overline{u}) = \inf_{u \in V} I_K(u) < 0$.
Take $0 \neq e \in K$. For $t \in [0, 1]$, we have that $te \in K$ and therefore

$$\begin{aligned}I_K(te) &= \frac{1}{2}\int_\Omega |\nabla te|^2 dx - \frac{1}{p}\int_\Omega |te|^p dx - \frac{\mu}{q}\int_\Omega |te|^q dx \\ &= t^q\left(\frac{t^{2-q}}{2}\int_\Omega |\nabla e|^2 dx - \frac{t^{p-q}}{p}\int_\Omega |e|^p dx - \frac{\mu}{q}\int_\Omega |e|^q dx\right).\end{aligned}$$

Since $1 < q < 2 < p$, $I_K(te)$ is negative for t sufficiently small. Thus, we can conclude that $I_K(\overline{u}) = \inf_{u\in V} I_K(u) < 0$. Thus, $\overline{u}$ is a non-trivial and non-negative solution of (2). Finally, it follows from the strong maximum principle that $\overline{u} > 0$ on Ω. □

Proof of Theorem 1.2 Let μ^* be as in Lemma 3.6 and $\mu \in (0, \mu^*)$. Also, let r_1 and r_2 be as in Lemma 3.6 and define $K = K(r_2)$. We first show that the functional I_K has infinitely many distinct critical points. To do this, we shall employ Theorem 2.4. It is obvious that the function Φ is even and continuously differentiable. Also Ψ_K is a proper, convex and lower semi-continuous even function. So (H) is satisfied. We now verify (PS). If $I_K(u_m) \to c$ for some $c \in \mathbb{R}$, by definition of I_K we can conclude that $\{u_m\}$ is bounded in $W^{2,n}(\Omega)$. Going if necessary to a subsequence, there exists some $\overline{u} \in W^{2,n}(\Omega) \cap H^1_0(\Omega)$ such that $u_m \rightharpoonup \overline{u}$ weakly in $W^{2,n}(\Omega)$. Due to the compact imbeddings of $W^{2,n}(\Omega) \hookrightarrow H^1_0(\Omega)$ and $W^{2,n}(\Omega) \hookrightarrow L^p(\Omega)$ we obtain that $u_m \to \overline{u}$ in V strongly. Thus, (PS) is satisfied.

For each $k \in \mathbb{N}$, considering the definition of Γ_k in (10), we define

$$c_k = \inf_{A\in\Gamma_k} \sup_{u\in A} I(u).$$

We shall now prove that $-\infty < c_k < 0$ for all $k \in \mathbb{N}$. To do this, let us denote by λ_j the jth eigenvalue of $-\Delta$ on $H^1_0(\Omega)$ (counted according to its multiplicity) and by e_j a corresponding eigenfunction satisfying $\int_\Omega \nabla e_i.\nabla e_j\, dx = \delta_{ij}$ where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ii} = 1$. As in the proof of Theorem 1.1, we have that I_K is bounded below. Thus $c_k > -\infty$ for each $k \in \mathbb{N}$. Let

$$A := \left\{ u = \alpha_1 e_1 + \cdots + \alpha_k e_k : \|u\|^2_{H^1_0(\Omega)} = \alpha_1^2 + \cdots + \alpha_k^2 = \rho^2 \right\},$$

for small $\rho > 0$ to be determined. Then $A \in \Gamma_k$ because $\gamma(A) = k$ by Proposition 2.3. Since A is finite dimensional, all norms are equivalent on A. Thus, we can choose ρ small enough so that $A \subseteq K$. Also, we can choose positive constants c_1, c_2 such that $\|u\|_{L^p(\Omega)} > c_1\|u\|_{H^1_0(\Omega)}$ and $\|u\|_{L^q(\Omega)} > c_2\|u\|_{H^1_0(\Omega)}$ for all $u \in A$. Therefore,

$$\begin{aligned} I_K(u) &= \frac{1}{2}\|u\|^2_{H^1_0(\Omega)} - \frac{1}{p}\|u\|^p_{L^p(\Omega)} - \frac{\mu}{q}\|u\|^q_{L^q(\Omega)} \\ &\leq \frac{1}{2}\rho^2 - \frac{1}{p}c_1^p\rho^p - \frac{\mu}{q}c_2^q\rho^q = \rho^q\left(\frac{1}{2}\rho^{2-q} - \frac{1}{p}c_1^p\rho^{p-q} - \frac{\mu}{q}c_2^q\right). \end{aligned}$$

Now we can choose ρ small enough such that $I_K(u) \leq \rho^q\left(\frac{1}{2}\rho^{2-q} - \frac{1}{p}c_1^p\rho^{p-q} - \frac{\mu}{q}c_2^q\right) < 0$ for every $u \in A$. It then follows that $c_k < 0$. Thus, by Theorem 2.4, I_K has has a sequence of distinct critical points $\{u_k\}_{k\in\mathbb{N}}$ by means of Definition 2.1. Also, by Lemma 3.6, for each critical point u_k of I_K there exists $v_k \in K$ such that $-\Delta v_k = D\Phi(u_k)$. It now follows from Theorem 3.1 that $\{u_k\}$ is a sequence of distinct solutions of (2) such that $I_K(u_k) < 0$ for each $k \in \mathbb{N}$. This completes the proof. □

It is evident that Theorem 1.2 can be easily extended to $p-$laplacian problems similar to the problem (2). Indeed, consider

$$\begin{cases} -\Delta_p u = |u|^{s-2}u + \mu|u|^{q-2}u, & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases} \tag{23}$$

By using a similar argument as in the proof of Theorem 1.2 one can prove that, if $1 < q < p < s$ then there exists $\mu^* > 0$ such that for each $\mu \in (0, \mu^*)$ problem (23) has infinitely many distinct nontrivial solutions with a negative energy. In our forthcoming project, we are investigating the existence of two positive solutions in Theorem 1.1 rather than just one. We are also extending both Theorems 1.1 and 1.2 to the fractional laplacian case via the method proposed in this manuscript.

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