Calculus of Variations

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Starshaped compact hypersurfaces with prescribed Weingarten curvature in warped product manifolds

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Abstract Given a compact Riemannian manifold M, we consider a warped product manifold $\bar{M} = I \times_h M$, where I is an open interval in $\mathbb{R}$. For a positive function ψ defined on $\bar{M}$, we generalize the arguments in Guan et al. (Commun. Pure Appl. Math. 68(8):1287–1325, 2015) and Ren and Wang (On the curvature estimates for Hessian equations, 2016. arXiv:1602.06535), to obtain the curvature estimates for Hessian equations $\sigma_k(\kappa) = \psi(V, \nu(V))$. We also obtain some existence results for the starshaped compact hypersurface Σ satisfying the above equation with various assumptions.

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1 Introduction

Assume that Σ^n is a hypersurface in a Riemannian manifold $\bar{M}^{n+1}$. The Weingarten curvature equation is given by

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$$\sigma_k(\kappa(X)) = \psi(X), \quad \forall X \in \Sigma,$$

where X is the position vector field of the hypersurface Σ in $\bar{M}^{n+1}$ and σ_k is the kth elementary symmetric function.

Finding closed hypersurfaces with prescribed Weingarten curvature in Riemannian manifolds attracts many authors' interest. Such results were obtained for the case of prescribing mean curvature by Bakelman and Kantor [10,11] and by Treibergs and Wei [46] in the Euclidean space, for the case of prescribing Gaussian curvature by Oliker [41], and for general Weingarten curvatures by Aleksandrov [1–7], Firey [19], Caffarelli et al. [17]. For Riemannian manifolds, some results have been obtained by Li and Oliker [37] for the unit sphere, Barbosa et al. [13] for space forms, Jin and Li [34] for the hyperbolic space, Andrade et al. [8] for warped product manifolds, Li and Sheng [36] for the Riemannain manifold equipped with a global normal Gaussian coordinate system.

For the hypersurface Σ in the Euclidean space $\mathbb{R}^{n+1}$, the Weingarten curvature equation in general form is defined by

$$\sigma_k(\kappa(X)) = \psi(X, \nu(X)), \quad \forall X \in \Sigma,$$

where $\nu(X)$ is the normal vector field along the hypersurface Σ. In many cases, the curvature estimates are the key part for the above prescribed curvature problems. Let us give a brief review. When $k = 1$, curvature estimate comes from the theory of quasilinear PDEs. If $k = n$, curvature estimate in this case for general $\psi(X, \nu)$ was due to Caffarelli et al. [15]. Ivochkina [32,33] considered the Dirichlet problem of the above equation on domains in $\mathbb{R}^n$, and obtained C^2 estimates there under some extra conditions on the dependence of f on ν. C^2 estimate was also proved for equation of prescribing curvature measures problem in [25,27]. If the function ψ is convex with respect to the normal ν, it is well known that the global C^2 estimate has been obtained by Guan [21]. Recently, Guan et al. [30] obtained global C^2 estimates for a closed convex hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ and then solved the long standing problem (1.2). In the same paper [30], they also proved the estimate for starshaped 2-convex hypersurfaces by introducing some new auxiliary curvature functions. Li et al. [35] substitute the convex by $(k + 1)$- convex for any k Hessian equations. In [42], Ren and the third author completely solved the case $k = n - 1$, that is the global curvature estimates of $n - 1$ convex solutions for $n - 1$ Hessian equations. In [45], Spruck–Xiao extended 2-curvature equations in [30] to space forms and gave a simple proof if the hypersurface in the Euclidean space. We also note the recently important work on the curvature estimates and C^2 estimates developed by Guan [22] and Guan et al. [31].

These type of equations and estimates with generalized right hand sides appear some new geometric applications recently, which will be mentioned in detail in the following. Phong et al. [38,39], generalized the Fu–Yau's equations, which is a complex 2-Hessian equation depending on gradient term on the right hand side. In [39,40], they obtained their C^2 estimates using the idea of [30]. Guan and Lu [28] considered the curvature estimate for isometric embedding system in general Riemannian manifolds, which is also a 2-Hessian equation depending on the normal vector field. The estimates in [30] are applied in [14,47] too.

Let (M^n, g') be a compact Riemannian manifold and I be an open interval in $\mathbb{R}$. The warped product manifold $\bar{M} = I \times_h M$ is endowed with the metric

$$\bar{g}^2 = dt^2 + h^2(t)g', \tag{1.1}$$

where $h : I \longrightarrow \mathbb{R}^+$ is a positive differential function. Given a differentiable function $z : M \longrightarrow I$, its graph is defined as the hypersurface

$$\Sigma = \{X(u) = (z(u), u)|u \in M\}.$$

For the Weingarten curvature equation in general form

$$\sigma_k(\kappa(V)) = f(\kappa(V)) = \psi(V, \nu(V)), \quad \forall V \in \Sigma, \tag{1.2}$$

where $V = h \dfrac{\partial}{\partial t}$ is the position vector field of hypersurface Σ in $\bar{M}$, σ_k is the kth elementary symmetric function, $\nu(V)$ is the inward unit normal vector field along the hypersurface Σ and $\kappa(V) = (\kappa_1, \dots, \kappa_n)$ are principal curvatures of hypersurface Σ at V. Given t_-, t_+ with $t_- < t_+$, we define the annulus domain $\bar{M}_-^+ = \{(t, u) \in \bar{M}|t_- \leq t \leq t_+\}$.

In this article, we will generalize the results in [30,42] to the hypersurfaces in warped product manifolds. The main results of this paper are the followings:

Theorem 1.1 *Let M^n be a compact Riemannian manifold and $\bar{M}$ be the warped product manifold with the metric (1.1). Assume that h is a positive differential function and $h' > 0$. Suppose that ψ satisfies*

(a) $\psi(t, u, \nu(u)) > C_n^k(\kappa(t))^k$ for $t \leq t_-$,
(b) $\psi(t, u, \nu(u)) < C_n^k(\kappa(t))^k$ for $t \geq t_+$,
(c) $\partial_t (h^k \psi(V, \nu)) \leq 0$ for $t_- < t < t_+$,

where $\kappa(t) = h'(t)/h(t)$ and C_n^k is the combinatorial numbers. Then there exists a unique differentiable function $z : M^n \to I$ solve the Eq. (1.2) for $k = 2$ and $k = n - 1$ whose graph Σ is contained in the interior of the region $\bar{M}_-^+$.

For the convex hypersurface in any warped product manifolds, we obtain the global curvature estimates.

Theorem 1.2 *Suppose $\Sigma \longrightarrow \bar{M}^{n+1}$ is a convex compact hypersurface satisfying curvature Eq. (1.2) for some positive function $\psi(V, \nu) \in C^2(\Gamma)$, where Γ is an open neighborhood of unit normal bundle of M in $\bar{M}^{n+1} \times \mathbb{S}^n$. Then there is a constant C depending only on n, k, $|z|_{C^1}$, $\inf \psi$ and $\|\psi\|_{C^2}$, such that*

$$\max_{u \in M} \kappa_i(u) \leq C. \tag{1.3}$$

Since the second fundamental form does not satisfy Codazzi properties for hypersurfaces in warped product manifolds in general, the constant rank theorem is still unknown. Thus, the above estimates only can imply the existence results in the sphere.

Theorem 1.3 *Let $\bar{M}$ be the sphere with sectional curvature $\lambda > 0$ which means the metric $\bar{g}$ of $\bar{M}$ is defined by (1.1), where function h is defined by*

$$h(t) = \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}}. \tag{1.4}$$

Suppose that ψ satisfies

(a) $\psi(t, u, \nu(u)) > \kappa(t)$ for $t \leq t_-$,
(b) $\psi(t, u, \nu(u)) < \kappa(t)$ for $t \geq t_+$,
(c) $(\psi^{-1/k})_{ij} + \lambda \psi^{-1/k} g_{ij} \geq 0$, for any ν,

where $\kappa(t) = h'(t)/h(t) = \sqrt{\lambda} \cot(\sqrt{\lambda t})$ and $t_+ < \pi/2$. *Then there exists a differentiable function* $z : \mathbb{S}^n \rightarrow I$ *solve the Eq.* (1.2) *for any k whose graph* Σ *is a strictly convex hypersurface and is contained in the interior of the region* $\bar{M}_-^+$.

The paper is organized as follows. In Sect. 2, we fix notations and recall some basic formulas for geometric and analytic preliminaries, including the detailed description of the problem. In Sect. 3, the gradient estimates of (1.2) are presented. In Sect. 4, the curvature estimates are proved for the starshaped 2-convex hypersurfaces. In Sects. 5 and 6, C^2 estimates are obtained for convex and $(n-1)$-convex hypersurface in the warped product manifold $\bar{M}$. In the last section, we derive the constant rank theorem and existence results .

2 Preliminaries

2.1 Warped product manifold $\bar{M}$

Let M^n be a compact Riemannian manifold with the metric g' and I be an open interval in $\mathbb{R}$. Assuming $h : I \longrightarrow \mathbb{R}^+$ is a positive differential function and $h' > 0$, the manifold $\bar{M} = I \times_h M$ is called the warped product if it is endowed with the metric

$$\bar{g}^2 = dt^2 + h^2(t)g'. \tag{2.1}$$

In the section, we use Latin lower case letters $i, j, \ldots$ to refer to indices running from 1 to n and $a, b, \ldots$ to indices from 0 to $n - 1$. The Einstein summation convention is used throughout the paper.

The metric in $\bar{M}$ is denoted by $\langle \cdot, \cdot \rangle$. The corresponding Riemannian connection in $\bar{M}$ will be denoted by $\bar{\nabla}$. The usual connection in M will be denoted ∇'. The curvature tensors in M and $\bar{M}$ will be denoted by R and $\bar{R}$, respectively.

Let $e_1, \ldots, e_{n-1}$ be an orthonormal frame field in M and let $\theta_1, \ldots, \theta_n$ be the associated dual frame. The connection forms θ_{ij} and curvature forms Θ_{ij} in M satisfy the structural equations

$$d\theta_i = \sum_j \theta_{ij} \wedge \theta_j, \quad \theta_{ij} = -\theta_{ji}, \tag{2.2}$$

$$d\theta_{ij} - \sum_k \theta_{ik} \wedge \theta_{kj} = \Theta_{ij} = -\frac{1}{2} \sum_{k,l} R_{ijkl} \theta_k \wedge \theta_l. \tag{2.3}$$

An orthonormal frame in $\bar{M}$ may be defined by $\bar{e}_i = (1/h)e_i$, $1 \leq i \leq n-1$, and $\bar{e}_0 = \partial/\partial t$. The associated dual frame is then $\bar{\theta}_i = h\theta_i$ for $1 \leq i \leq n - 1$ and $\bar{\theta}_0 = dt$. A simple computation permits to obtain

Lemma 2.1 *On the leaf M_t, the curvature satisfies*

$$\bar{R}_{ijk0} = 0 \tag{2.4}$$

and the principle curvature is given by

$$\kappa(t) = h'(t)/h(t) \tag{2.5}$$

where the inward unit normal $-\bar{e}_0 = -\partial/\partial t$ *is chosen for each leaf M_t.*

2.2 Hypersurfaces in the warped product manifold $\bar{M}$

Given a differentiable function $z : M \longrightarrow I$, its graph is defined by the hypersurface

$$\Sigma = \{X(u) = (z(u), u)|u \in M\} \tag{2.6}$$

whose tangent space is spanned at each point by the vectors

$$X_i = h\,\bar{e}_i + z_i\,\bar{e}_0, \tag{2.7}$$

where z_i are the components of the differential $\mathrm{d}z = z_i\theta^i$. The unit vector field

$$\nu = \frac{1}{\sqrt{h^2 + |\nabla'z|^2}} \left(\sum_{i=1}^{n} z^i\bar{e}_i - h\bar{e}_0 \right) \tag{2.8}$$

is an unit inner normal vector field to Σ. Here, $|\nabla'z|^2 = z^i z_i$ is the squared norm of $\nabla'z = z^i e_i$. The components of the induced metric in Σ is given by

$$g_{ij} = \langle X_i, X_j \rangle = h^2\delta_{ij} + z_i z_j \tag{2.9}$$

The second fundamental form of Σ with components (a_{ij}) is determined by

$$a_{ij} = \langle \bar{\nabla}_{X_j} X_i, \nu \rangle = \frac{1}{\sqrt{h^2 + |\nabla'z|^2}} \left(-hz_{ij} + 2h'z_i z_j + h^2 h'\delta_{ij} \right)$$

where z_{ij} are the components of the Hessian $\nabla'^2 z = \nabla'\mathrm{d}z$ of z in M.

Now we choose the coordinate systems such that $\{E_0 = \nu, E_1, \ldots, E_n\}$ is an orthonormal frame field in some open set of Σ and $\{\omega_0, \omega_1, \ldots, \omega_n\}$ is its associated dual frame. The connection forms $\{\omega_{ij}\}$ and curvature forms $\{\Omega_{ij}\}$ in Σ satisfy the structural equations

$$\mathrm{d}\omega_i - \sum_j \omega_{ij} \wedge \omega_j = 0, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$\mathrm{d}\omega_{ij} - \sum_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij} = -\frac{1}{2} \sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l.$$

The coefficients a_{ij} of the second fundamental form are given by Weingarten equation

$$\omega_{i0} = \sum_j a_{ij}\,\omega_j.$$

The covariant derivative of the second fundamental form a_{ij} in Σ is defined by

$$\sum_k a_{ijk}\,\omega_k = \mathrm{d}a_{ij} + \sum_l a_{il}\,\omega_{lj} + \sum_l a_{lj}\,\omega_{li},$$

$$\sum_l a_{ijkl}\,\omega_l = \mathrm{d}a_{ijk} + \sum_l a_{ljk}\,\omega_{li} + \sum_l a_{ilk}\,\omega_{lj} + \sum_l a_{ijl}\,\omega_{lk}.$$

The Codazzi equation is a commutation formula for the first order derivative of a_{ij} given by

$$a_{ijk} - a_{ikj} = -\bar{R}_{0ijk} \tag{2.10}$$

and the Ricci identity is a commutation formula for the second order derivative of a_{ij} given by

Lemma 2.2 *Let $\bar{X}$ be a point of Σ and $\{E_0 = \nu, E_1, \ldots, E_n\}$ be an adapted frame field such that each E_i is a principal direction and $\omega_i^k = 0$ at $\bar{X}$. Let (a_{ij}) be the second quadratic form of Σ. Then, at the point $\bar{X}$, we have*

$$
\begin{aligned}
a_{llii} =& a_{iill} - a_{lm}\,(a_{mi}a_{il} - a_{ml}a_{ii}) - a_{mi}\,(a_{mi}a_{ll} - a_{ml}a_{li}) \\
&+ \bar{R}_{0iil;l} - 2a_{ml}\bar{R}_{miil} + a_{il}\bar{R}_{0i0l} + a_{ll}\bar{R}_{0ii0} \\
&+ \bar{R}_{0lil;i} - 2a_{mi}\bar{R}_{mlil} + a_{ii}\bar{R}_{0l0l} + a_{li}\bar{R}_{0li0}.
\end{aligned}
\tag{2.11}
$$

In particular, we have

$$
a_{ii11} - a_{11ii} = a_{11}a_{ii}^2 - a_{11}^2 a_{ii} + 2(a_{ii} - a_{11})\bar{R}_{i1i1} + a_{11}\,\bar{R}_{i0i0} - a_{ii}\,\bar{R}_{1010} + \bar{R}_{i1i0;1} - \bar{R}_{1i10;i}.
\tag{2.12}
$$

2.3 Two functions η and τ

Define the functions $\tau : \Sigma \to \mathbb{R}$ and $\eta : \Sigma \to \mathbb{R}$ by

$$
\tau = -h\langle \nu, \bar{e}_0 \rangle = -\langle V, \nu \rangle \quad \text{and} \quad \eta = -\int h\,\mathrm{d}t,
\tag{2.13}
$$

where $V = h\bar{e}_0 = h\dfrac{\partial}{\partial t}$ is the position vector field and ν is the inner unit normal. Then we have

Lemma 2.3 [8] *The gradient vector fields of the functions η and τ are*

$$
\nabla_{E_i}\eta = -h\langle \bar{e}_0, E_i \rangle E_i,
\tag{2.14}
$$

$$
\nabla_{E_i}\tau = -\sum_j \nabla_{E_j}\eta\, a_{ij},
\tag{2.15}
$$

and the second order derivative of τ and η are given by

$$
\nabla_{E_i, E_j}^2 \eta = \tau a_{ij} - h' g_{ij},
\tag{2.16}
$$

$$
\begin{aligned}
\nabla_{E_i, E_j}^2 \tau &= -\sum_k \tau a_{ik}a_{kj} + h' a_{ij} - \sum_k a_{ikj}\nabla_{E_k}\eta \\
&= -\tau \sum_k a_{ik}a_{kj} + h' a_{ij} - \sum_k (a_{ijk} + \bar{R}_{0ijk})\nabla_{E_k}\eta.
\end{aligned}
\tag{2.17}
$$

2.4 Basic formulae

Assume that $\Sigma \longrightarrow \bar{M}$ is the graph defined as the hypersurface Σ whose points are the form $X(u) = (z(u), u)$ with $u \in M$. This graph is diffeomorphic with M and may be globally oriented by an unit normal vector field ν for which it holds that $\langle \nu, \partial_t \rangle < 0$. Let $\kappa = (\kappa_1, \ldots, \kappa_n)$ be the vector whose components κ_i are the principal curvatures of Σ, that is, the eigenvalues of the second fundamental form $B = (\langle \bar{\nabla}_i E_j, \nu \rangle)$ in Σ.

The elementary symmetric function of order k $(1 \le k \le n)$ of $\kappa = (\kappa_1, \ldots, \kappa_n)$ is defined as following

$$
\sigma_k = \sum_{i_1 < \cdots < i_n} \kappa_{i_1} \ldots \kappa_{i_n}.
\tag{2.18}
$$

Let Γ_k be the connected component of $\{\kappa \in \mathbb{R}^n | \sigma_m > 0, m = 1, \ldots, k\}$ containing the positive cone $\{\kappa \in \mathbb{R}^n | \kappa_1, \ldots, \kappa_n > 0\}$.

Definition 2.4 A positive function $z \in C^2(M^n)$ is said to be admissible for the operator σ_k if for the corresponding hypersurface $\Sigma = \{(z(u), u) | u \in M^n\}$, at every point of Σ with the normal as in (2.8), the principal curvatures $\kappa = (\kappa_1, \ldots, \kappa_n)$ are in Γ_k.

Lemma 2.5 *([9,12,16,20,44]) Let F be a C^2 symmetric function defined in some open set of $Sym(n)$, where $Sym(n)$ is the set of all $n \times n$ symmetric matrices. For any symmetric matrix (b_{ij}), there holds*

$$F^{ij,kl} b_{ij} b_{kl} = \sum_{i,j} \frac{\partial^2 f}{\partial \kappa_i \partial \kappa_j} b_{ii} b_{jj} + \sum_{i \neq j} \frac{f_i - f_j}{\kappa_i - \kappa_j} b_{ij}^2,$$

where the second term on the right-hand side must be interpreted as a limit whenever $\kappa_i = \kappa_j$.

Lemma 2.6 *[25,30] Assume that $k > l$, $W = (w_{ij})$ is a Codazzi tensor which is in Γ_k. Denote $\alpha = \dfrac{1}{k-l}$. Then, for $h = 1, \ldots, n$, we have the following inequality,*

$$-\frac{\sigma_k^{pp,qq}}{\sigma_k}(W) w_{pph} w_{qqh} + \frac{\sigma_l^{pp,qq}}{\sigma_l}(W) w_{pph} w_{qqh}$$
$$\geq \left(\frac{(\sigma_k(W))_h}{\sigma_k(W)} - \frac{(\sigma_l(W))_h}{\sigma_l(W)} \right) \left((\alpha - 1) \frac{(\sigma_k(W))_h}{\sigma_k(W)} - (\alpha + 1) \frac{(\sigma_l(W))_h}{\sigma_l(W)} \right). \quad (2.19)$$

Furthermore, for any $\delta > 0$, we have

$$-\sigma_k^{pp,qq}(W) w_{pph} w_{qqh} + \left(1 - \alpha + \frac{\alpha}{\delta} \right) \frac{(\sigma_k(W))_h^2}{\sigma_k(W)}$$
$$\geq \sigma_k(W)(\alpha + 1 - \delta\alpha) \left[\frac{(\sigma_l(W))_h}{\sigma_l(W)} \right]^2 - \frac{\sigma_k}{\sigma_l}(W) \sigma_l^{pp,qq}(W) w_{pph} w_{qqh}. \quad (2.20)$$

3 Gradient estimates

In this section, we follow the ideas of [17,27] to derive C^1 estimates for the height function z. In other words, we are looking for a lower bound of the support function τ. Firstly, we need the following technical assumption:

$$\frac{\partial}{\partial t} (h(t)^k \psi(V, \nu)) \leq 0, \quad \text{where } V = h(t) \frac{\partial}{\partial t}. \quad (3.1)$$

Lemma 3.1 *Let Σ be a graph in $\bar{M} = I \times_h M$ satisfying (1.2), (3.1) and let z be the height function of Σ. If h has positive lower and upper bounds, then there is a constant C depending on the minimum and maximum values of z such that*

$$|\nabla z| \leq C. \quad (3.2)$$

Proof Set $\chi(z) = -\ln(\tau) + \gamma(-\eta(t))$, where γ is a single variable function to be determined later. Assume that χ achieve its maximum value at point u_0. We claim that V is parallel to its normal ν at u_0 if we choose a suitable γ. We will prove it by contradiction. If not, we can choose a local orthonormal basis $\{E_i\}_{i=1}^n$ such that $\langle V, E_1 \rangle \neq 0$, and $\langle V, E_i \rangle = 0$, $i \geq 2$.

Obviously, $V = \langle V, E_1 \rangle E_1 + \langle V, \nu \rangle \nu$. At point u_0, by the maximum principle we have

$$0 = \nabla_{E_i} \chi(z) = -\frac{\nabla_{E_i} \tau}{\tau} - \gamma' \nabla_{E_i} \eta, \tag{3.3}$$

$$0 \geq \nabla^2_{E_i, E_i} \chi(z)$$
$$= -\frac{\nabla^2_{E_i, E_i} \tau}{\tau} + \frac{|\nabla_{E_i} \tau|^2}{\tau^2} - \gamma' \nabla^2_{E_i, E_i} \eta + \gamma'' |\nabla_{E_i} \eta|^2. \tag{3.4}$$

From (2.15), (2.17) and (3.3), we have

$$0 \geq -\frac{\nabla^2_{E_i, E_i} \tau}{\tau} + \frac{|\nabla_{E_i} \tau|^2}{\tau^2} - \gamma' \nabla^2_{E_i, E_i} \eta + \gamma'' |\nabla_{E_i} \eta|^2$$
$$= -\frac{1}{\tau} \left(-\tau a_{il} a_{li} + h' a_{ii} - (a_{iil} + \bar{R}_{0iil}) \eta_l \right) + \left(\gamma'' + (\gamma')^2 \right) \eta_i^2 - \gamma' \left(\tau a_{ii} - h' g_{ii} \right). \tag{3.5}$$

By (2.15) and (3.3), we get

$$a_{11} = \tau \gamma', \qquad a_{i1} = 0, \qquad i \geq 2. \tag{3.6}$$

Therefore, it is possible to rotate the coordinate system such that $\{E_i\}_{i=1}^n$ are the principal curvature directions of the second fundamental form (a_{ij}), i.e. $a_{ij} = a_{ii} \delta_{ij}$, which means that (σ_k^{ij}) is also diagonal. By multiplying σ_k^{ii} in the inequality (3.5) both sides and taking sum on i from 1 to n, one gets from (3.5) and (3.6)

$$0 \geq \sigma_k^{ii} a_{ii}^2 - \frac{1}{\tau} h' \sigma_k^{ii} a_{ii} + \frac{1}{\tau} \sigma_k^{ii} (a_{iil} + \bar{R}_{0iil}) \eta_l + \left(\gamma'' + (\gamma')^2 \right) \sigma_k^{ii} \eta_i^2$$
$$- \gamma' \left(\tau \sigma^{ii} a_{ii} - h' \sum_{i=1}^n \sigma_k^{ii} \right)$$
$$= \sigma_k^{ii} a_{ii}^2 + \frac{1}{\tau} \sigma_k^{ii} a_{ii1} \eta_1 + \frac{1}{\tau} \sigma_k^{ii} \bar{R}_{0ii1} \eta_1 + \left(\gamma'' + (\gamma')^2 \right) \sigma_k^{11} \eta_1^2$$
$$+ \gamma' h' \sum_{i=1}^n \sigma_k^{ii} - \gamma' \tau k \psi - \frac{1}{\tau} h' k \psi \tag{3.7}$$

where $F^{ii} a_{ii} = k\psi$ is used. Differentiating Eq. (1.2) with respect to E_1 we obtain

$$\sigma_k^{ii} a_{ii1} = d_V \psi(\nabla_{E_1} V) - a_{11} d_\nu \psi(E_1). \tag{3.8}$$

Putting (3.6) and (3.8) into (3.5) yields

$$0 \geq \sigma_k^{ii} a_{ii}^2 + \frac{1}{\tau} \left(d_V \psi(\nabla_{E_1} V) - a_{11} d_\nu \psi(E_1) \right) \eta_1 + \frac{1}{\tau} \sigma_k^{ii} \bar{R}_{0ii1} \eta_1$$
$$+ \left(\gamma'' + (\gamma')^2 \right) \sigma_k^{11} \eta_1^2 + \gamma' h' \sum_{i=1}^n \sigma_k^{ii} - \gamma' \tau k \psi - \frac{1}{\tau} h' k \psi$$
$$= \sigma_k^{ii} a_{ii}^2 - \frac{1}{\tau} \left(k h' \psi + \langle V, E_1 \rangle d_V \psi(\nabla_{E_1} V) \right) + \gamma' d_\nu \psi(E_1) \langle V, E_1 \rangle + \frac{1}{\tau} \sigma_k^{ii} \bar{R}_{0i1i} \langle V, E_1 \rangle$$
$$+ \left(\gamma'' + (\gamma')^2 \right) \sigma_k^{11} \langle V, E_1 \rangle^2 - k \gamma' \tau \psi + \gamma' h' \sum_{i=1}^n \sigma_k^{ii}. \tag{3.9}$$

Since $V = \langle V, E_1 \rangle E_1 + \langle V, \nu \rangle \nu$, we have

$$d_V \psi(V, \nu) = \langle V, E_1 \rangle d_V \psi(\nabla_{E_1} V) + \langle V, \nu \rangle d_V \psi(\nabla_\nu V). \tag{3.10}$$

Putting (3.10) into (3.9) gets

$$
\begin{aligned}
0 \geq{}& \sigma_k^{ii} a_{ii}^2 - \frac{1}{\tau} \left(k h' \psi + d_V \psi(V, \nu) \right) + \gamma' d_\nu \psi(E_1) \langle V, E_1 \rangle + \frac{1}{\tau} \sigma_k^{ii} \bar{R}_{0i1i} \langle V, E_1 \rangle \\
& + \left(\gamma'' + (\gamma')^2 \right) \sigma_k^{11} \langle V, E_1 \rangle^2 - k \gamma' \tau \psi + \gamma' h' \sum_{i=1}^n \sigma_k^{ii} + d_V \psi(\nabla_\nu V) \\
\geq{}& \sigma_k^{ii} a_{ii}^2 + \left(\gamma'' + (\gamma')^2 \right) \sigma_k^{11} \langle V, E_1 \rangle^2 + \gamma' h' \sum_{i=1}^n \sigma_k^{ii} + \frac{1}{\tau} \sigma_k^{ii} \bar{R}_{0i1i} \langle V, E_1 \rangle \\
& + \gamma' d_\nu \psi(E_1) \langle V, E_1 \rangle - k \gamma' \tau \psi + d_V \psi(\nabla_\nu V),
\end{aligned}
\tag{3.11}
$$

where we use the assumption (3.1). Choosing the function $\gamma(t) = \dfrac{\alpha}{t}$ for a positive constant α, we have

$$\gamma'(t) = -\frac{\alpha}{t^2}, \qquad \gamma''(t) = \frac{2\alpha}{t^3}. \tag{3.12}$$

By (3.6) and the choice of function γ, we have $a_{11} \leq 0$. Thus, the Newton–Maclaurin inequality implies

$$\sigma_k^{11} \geq \sigma_{k-1} \geq \frac{k}{(n-k+1)(k-1)} (C_n^k)^{\frac{1}{k}} \psi^{\frac{k-1}{k}}. \tag{3.13}$$

Therefore by the previous three inequalities, we have

$$
\begin{aligned}
0 \geq{}& \sigma_k^{11} a_{11}^2 + \left(\frac{\alpha^2}{t^4} + \frac{2\alpha}{t^3} \right) \sigma_k^{11} \langle V, E_1 \rangle^2 - \frac{\alpha}{t^2} h' \sum_{i=1}^n \sigma_k^{ii} + \frac{1}{\tau} \sigma_k^{ii} \bar{R}_{0i1i} \langle V, E_1 \rangle \\
& - \frac{\alpha}{t^2} d_\nu \psi(E_1) \langle V, E_1 \rangle + \frac{\alpha}{t^2} k \tau \psi + d_V \psi(\nabla_\nu V).
\end{aligned}
\tag{3.14}
$$

Since $V = \langle V, E_1 \rangle E_1 + \langle V, \nu \rangle \nu$, one can find that $V \perp \text{Span}\{E_2, \ldots, E_n\}$. On the other hand, $E_1, \nu \perp \text{Span}\{E_2, \ldots, E_n\}$. It is possible to choose coordinate systems such that $\bar{e}_1 \perp \text{Span}\{E_2, \ldots, E_n\}$, which implies that the pair $\{V, \bar{e}_1\}$ and $\{\nu, E_1\}$ lie in the same plane and

$$\text{Span}\{E_2, \ldots, E_n\} = \text{Span}\{\bar{e}_2, \ldots, \bar{e}_n\}.$$

Therefore, we can choose $E_2 = \bar{e}_2, \ldots, E_n = \bar{e}_n$. The vector ν and E_1 can decompose into

$$
\begin{aligned}
\nu &= \langle \nu, \bar{e}_0 \rangle \bar{e}_0 + \langle \nu, \bar{e}_1 \rangle \bar{e}_1 = -\frac{\tau}{h} \bar{e}_0 + \langle \nu, \bar{e}_1 \rangle \bar{e}_1, \\
E_1 &= \langle E_1, \bar{e}_0 \rangle \bar{e}_0 + \langle E_1, \bar{e}_1 \rangle \bar{e}_1.
\end{aligned}
$$

For (2.4) and $V = \langle V, E_1 \rangle E_1 + \langle V, \nu \rangle \nu$, we obtain

$$
\begin{aligned}
\bar{R}_{0i1i} &= \bar{R}(\nu, E_i, E_1, E_i) \\
&= -\frac{\tau}{h} \langle E_1, \bar{e}_0 \rangle \bar{R}(\bar{e}_0, \bar{e}_i, \bar{e}_0, \bar{e}_i) + \langle \nu, \bar{e}_1 \rangle \langle E_1, \bar{e}_1 \rangle \bar{R}(\bar{e}_1, \bar{e}_i, \bar{e}_1, \bar{e}_i) \\
&= -\frac{\tau}{h} \langle E_1, \bar{e}_0 \rangle \bar{R}(\bar{e}_0, \bar{e}_i, \bar{e}_0, \bar{e}_i) - \tau \frac{\langle \nu, \bar{e}_1 \rangle^2}{\langle E_1, V \rangle} \bar{R}(\bar{e}_1, \bar{e}_i, \bar{e}_1, \bar{e}_i) \\
&= \tau \left(-\frac{1}{h} \langle E_1, \bar{e}_0 \rangle \bar{R}(\bar{e}_0, \bar{e}_i, \bar{e}_0, \bar{e}_i) - \frac{\langle \nu, \bar{e}_1 \rangle^2}{\langle E_1, V \rangle} \bar{R}(\bar{e}_1, \bar{e}_i, \bar{e}_1, \bar{e}_i) \right).
\end{aligned}
\tag{3.15}
$$

The third equality comes from $0 = \langle V, \bar{e}_1 \rangle$. From (3.6), (3.13) and (3.15), (3.14) becomes

$$0 \geq \alpha^2 \sigma_k^{11} (\tau^2 (\gamma')^2 + \frac{\alpha^2}{t^4} \langle V, E_1 \rangle^2) - C_1 \alpha \sigma_{k-1} - C_2 \alpha |d_\nu \psi(e_1)| - |d_V \psi(\nabla_\nu V)|$$

$$\geq C \alpha^2 |V|^2 \sigma_k^{11} - C_1 \alpha \sigma_k^{11} - C_2 \alpha |d_\nu \psi(e_1)| - |d_V \psi(\nabla_\nu V)|$$

where C, C_1, C_2 depends on k, n, the C^0 bound of h and the curvature $\bar{R}$. Thus, we have a contradiction when α is large enough. Hence, V is parallel to the normal ν which implies the lower bound of τ. □

4 C^2 estimates for σ_2

In this section, we study the solution of the following normalized equation

$$F(b) = \binom{n}{2}^{(-1/2)} \sigma_2(\kappa(a))^{1/2} = f(\kappa(a_{ij})) = \overline{\psi}(V, \nu). \tag{4.1}$$

Now we can prove the C^2 estimate for 2-convex hypersurfaces.

Theorem 4.1 *With the assumption of Theorem 1.1, there is a constant C depending only on n, k, t_-, t_+, the C^1 bound of z and $|\bar{\psi}|_{C^2}$, such that*

$$\max_{u \in M} |\kappa_i(u)| \leq C. \tag{4.2}$$

Proof Define the function

$$W(u, \xi) = e^{-\beta \eta} \frac{B(\xi, \xi)}{\tau - a} \tag{4.3}$$

where $\tau \geq 2a$ and β is a large constant to be chosen, ξ is a tangent vector of Σ and B is the second fundamental form. Assume that W is achieved at $X_0 = (z(u_0), u_0)$ along ξ, and we may choose a local orthonormal frame $E_1, \ldots, E_n$ around X_0 such that $\xi = E_1$ and $a_{ij}(X_0) = \kappa_i \delta_{ij}$, where $\kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_n$ are the principal curvatures of Σ at u_0. Thus at u_0, $\ln W = \ln a_{11} - \log(\tau - a) - \beta \eta$ has a local maximum. Therefore,

$$0 = \frac{a_{11i}}{a_{11}} - \frac{\nabla_i \tau}{\tau - a} - \beta \eta_i, \tag{4.4}$$

and

$$0 \geq \frac{a_{11ii}}{a_{11}} - \left(\frac{a_{11i}}{a_{11}}\right)^2 - \frac{\nabla_{ii} \tau}{\tau - a} + \left(\frac{\nabla_i \tau}{\tau - a}\right)^2 - \beta \eta_{ii}. \tag{4.5}$$

Multiplying F^{ii} both sides in (4.5) and using (2.14)–(2.17), we have

$$0 \geq \frac{1}{\kappa_1} F^{ii} a_{11ii} - \frac{1}{\kappa_1^2} F^{ii} (a_{11i})^2 - \frac{1}{\tau - a} F^{ii} \tau_{ii} + F^{ii} \left(\frac{\tau_i}{\tau - a}\right)^2 - \beta F^{ii} \eta_{ii}$$

$$= \frac{1}{\kappa_1} F^{ii} a_{11ii} - \frac{1}{\kappa_1^2} F^{ii} (a_{11i})^2 + \frac{\tau}{\tau - a} F^{ii} \kappa_i^2 - \frac{h'}{\tau - a} \bar{\psi}$$

$$+ \frac{1}{\tau - a} \sum_l F^{ii} (a_{iil} + \bar{R}_{0iil}) \eta_l$$

$$+ \sum_i F^{ii} \left(\frac{\kappa_i \eta_i}{\tau - a}\right)^2 - \beta \tau \bar{\psi} + h' \beta \sum_{i=1}^n F^{ii}. \tag{4.6}$$

The Ricci identity (2.11) yields

$$
\begin{aligned}
F^{ii}a_{ii11} - F^{ii}a_{11ii} &= a_{11}F^{ii}a_{ii}^2 - a_{11}^2 F^{ii}a_{ii} + 2F^{ii}(a_{ii} - a_{11})\bar{R}_{i1i1} \\
&\quad + a_{11}\,F^{ii}\bar{R}_{i0i0} - F^{ii}a_{ii}\bar{R}_{1010} \\
&\quad + F^{ii}\bar{R}_{i1i0;1} - F^{ii}\bar{R}_{1i10;i} \\
&\geq -C_1\kappa_1^2 - C_2\kappa_1\sum_i F^{ii},
\end{aligned}
\tag{4.7}
$$

for sufficiently large κ_1. Inserting (4.7) into (4.6), one gives

$$
\begin{aligned}
0 \geq{}& \frac{1}{\kappa_1}F^{ii}a_{ii11} - \frac{1}{\kappa_1^2}F^{ii}(a_{11i})^2 + \frac{\tau}{\tau - a}F^{ii}\kappa_i^2 + \frac{1}{\tau - a}\sum_l F^{ii}(a_{iil} + \bar{R}_{0iil})\eta_l \\
& + \sum_i F^{ii}\left(\frac{\kappa_i\eta_i}{\tau - a}\right)^2 - C_1\kappa_1 + (h'\beta - C_2)\sum_i F^{ii} - C_3(\beta).
\end{aligned}
\tag{4.8}
$$

Taking covariant derivative with respect to the equation (4.1) yields

$$
F^{ii}a_{iij} = \bar{\psi}_V(\nabla_{E_j}V) - a_{jl}\bar{\psi}_\nu(E_l).
\tag{4.9}
$$

Taking covariant derivative with respect to the Eq. (4.9) again yields

$$
\begin{aligned}
F^{ii}a_{ii11} + F^{ij,kl}a_{ij1}a_{kl1} &= \bar{\psi}_{VV}(\nabla_{E_1}V, \nabla_{E_1}V) + 2a_{1l}\bar{\psi}_{V\nu}(\nabla_{E_1}V, E_l) \\
&\quad - a_{1l1}\bar{\psi}_\nu(E_l) + a_{1k}a_{1l}\bar{\psi}_{\nu\nu}(E_l, E_l) \\
&\geq -C(1 + \kappa_1^2) - a_{1l1}\bar{\psi}_\nu(E_l) \\
&= -C(1 + \kappa_1^2) - (a_{11l} - \bar{R}_{01l1})\bar{\psi}_\nu(E_l) \\
&\geq -C(1 + \kappa_1^2 + \beta\kappa_1) - a_{11l}\bar{\psi}_V(E_l).
\end{aligned}
\tag{4.10}
$$

where we have used the Codazzi equation in the last equality, (4.4) and the bound of the curvature of the ambient manifold in the last inequality.

We also have

$$
\frac{1}{\kappa_1}\sum_l a_{11l}\bar{\psi}_V(E_l) - \sum_l \frac{\eta_l}{\tau - a}F^{ii}a_{iil} = \sum_l \beta\eta_l\bar{\psi}_V(E_l) - \sum_l \frac{\eta_l}{\tau - a}\bar{\psi}_V(\nabla_{E_j}V).
\tag{4.11}
$$

Combining the inequality (4.10) and (4.11), (4.8) gives

$$
\begin{aligned}
0 \geq{}& \frac{1}{\kappa_1}\left(-F^{ij,kl}a_{ij1}a_{kl1}\right) - \frac{1}{\kappa_1^2}F^{ii}(a_{11i})^2 + \frac{\tau}{\tau - a}F^{ii}\kappa_i^2 \\
& + \sum_{i=1}^n F^{ii}\left(\frac{\kappa_i\eta_i}{\tau - a}\right)^2 - C_1\kappa_1 + (h'\beta - C_2)\sum_i F^{ii} - C_3(\beta)
\end{aligned}
\tag{4.12}
$$

In the following, we consider two cases.

Case 1 We suppose that $\kappa_n \leq -\theta\kappa_1$ for some positive constant θ to be chosen later. In this case, using the concavity of F, we discard the term $-\frac{1}{\kappa_1}F^{ij,kl}a_{ij1}a_{kl1}$.

By Young's inequality and (4.4), we have

$$\frac{1}{\kappa_1^2} F^{ii} |a_{11i}|^2 \le (1 + \epsilon^{-1}) \beta^2 F^{ii} |\eta_i|^2 + \frac{(1+\epsilon)}{(\tau-a)^2} F^{ii} |\tau_i|^2$$

$$\le C_4 (1 + \epsilon^{-1}) \beta^2 \sum_i F^{ii} + \frac{(1+\epsilon)}{(\tau-a)^2} F^{ii} |\tau_i|^2 \tag{4.13}$$

for any $\epsilon > 0$, where we have used $|\nabla \eta| \le C$. From (4.12) and (4.13), we obtain

$$0 \ge - C_1 \kappa_1 - C_3(\beta) + \left(\frac{\tau}{\tau-a} - C_5 \epsilon \right) F^{ii} \kappa_i^2 + \left(h'\beta - C_2 - C_4(1+\epsilon^{-1})\beta^2 \right) \sum_i F^{ii}$$

$$\ge - \bar{C}(\kappa_1 + \beta) + C_6 \sum_{i=1}^n F^{ii} \kappa_i^2 - C_7 \beta^2 \sum_i F^{ii}. \tag{4.14}$$

Since $F^{11} \le F^{22} \le \cdots \le F^{nn}$ and $\kappa_n \le -\theta \kappa_1$, we get

$$\sum_{i=1}^n F^{ii} \kappa_i^2 \ge F^{nn} \kappa_n^2 \ge \frac{1}{n} \theta^2 \sum_i F^{ii} \kappa_1^2.$$

Hence,

$$0 \ge -\bar{C}(\kappa_1 + \beta) + \left(C_6 \frac{1}{n} \theta^2 \kappa_1^2 - C_7 \beta^2 \right) \sum_i F^{ii}. \tag{4.15}$$

Since $\sum_i F^{ii} \ge 1$ for sufficiently large κ_1, the inequality (4.15) clearly implies the bound of κ_1 from above.

Case 2 In this case, we assume that $\kappa_n \ge -\theta \kappa_1$. Hence, $\kappa_i \ge \kappa_n \ge -\theta \kappa_1$. We then group the indices $\{1, ..., n\}$ into two sets $I = \{j : F^{jj} \le 4F^{11}\}$ and $J = \{j : F^{jj} > 4F^{11}\}$. Using (4.4), we can infer

$$\frac{1}{\kappa_1^2} \sum_{i \in I} F^{ii} |a_{11i}|^2 \le C_1 (1 + \epsilon^{-1}) \beta^2 F^{11} + \frac{(1+\epsilon)}{(\tau-a)^2} F^{ii} |\tau_i|^2 \tag{4.16}$$

for any $\epsilon > 0$. Therefore it follows from (4.12) that

$$0 \ge - C_1 \kappa_1 - C_3(\beta) - \frac{1}{\kappa_1} F^{ij,kl} a_{ij1} a_{kl1} + \left(\frac{\tau}{\tau-a} - C_5\epsilon \right) F^{ii} \kappa_i^2$$

$$+ \left(h'\beta - C_2 \right) \sum_i F^{ii} - \frac{1}{\kappa_1^2} \sum_{i \in J} F^{ii} (\nabla_i a_{11})^2 - C_4(1+\epsilon^{-1})\beta^2 F^{11}. \tag{4.17}$$

Using Lemma (2.5) and the Codazzi's equation, one gets

$$-\frac{1}{\kappa_1} F^{ij,kl} a_{ij1} a_{kl1} \ge -\frac{2}{\kappa_1} \sum_{j \in J} \frac{f_1 - f_j}{\kappa_1 - \kappa_j} (a_{1j1})^2 = -\frac{2}{\kappa_1} \sum_{j \in J} \frac{f_1 - f_j}{\kappa_1 - \kappa_j} (a_{11j} - \bar{R}_{01j1})^2. \tag{4.18}$$

Following the argument in [34], we may verify that choosing $\theta = \frac{1}{2}$ it holds that for all $j \in J$,

$$-\frac{2}{\kappa_1} \frac{f_1 - f_j}{\kappa_1 - \kappa_j} \ge \frac{f_j}{\kappa_1^2} = \frac{F^{jj}}{\kappa_1^2}. \tag{4.19}$$

Combining (4.17), (4.18) and (4.19), we obtain

$$
\begin{aligned}
0 \geq & -C_1\kappa_1 - C_3(\beta) - 2\frac{1}{\kappa_1^2}\sum_{j\in J} F^{jj} a_{11j}\bar{R}_{01j1} + \left(\frac{\tau}{\tau-a} - C_5\epsilon\right) F^{ii}\kappa_i^2 \\
& + \left(h'\beta - C_2\right)\sum_i F^{ii} - C_4(1+\epsilon^{-1})\beta^2 F^{11} \\
\geq & -\bar{C}(\kappa_1 + \beta) + C_6\sum_{i=1}^n F^{ii}\kappa_i^2 + \left(h'\beta - C\right)\sum_i F^{ii} - C_7\beta^2 F^{11} \\
\geq & (C_8(h'\beta - C_2) - C_1)\kappa_1 + (C_6\kappa_1^2 - C_7\beta^2)F^{11} - \bar{C}_3(\beta),
\end{aligned} \tag{4.20}
$$

by choosing ϵ small and sufficiently large κ_1. Here we also used (4.4) and

$$
\sum_{i=1}^n F^{ii} \geq C\kappa_1.
$$

For $\beta > 0$ sufficiently large, we may obtain an upper bound for κ_1 by (4.20). □

Remark 4.2 The similar idea also has been used in [18,23,43].

5 A global C^2 estimate for convex hypersurfaces in the warped product space

In this section, following the arguments in [30], we can obtain C^2 estimates of convex solutions for the curvature Eq. (1.2) in Σ, namely, proving Theorem 1.2.

Define the following auxiliary function,

$$
\Psi = \frac{1}{2}\ln P(\kappa) - N\log\tau - \beta\eta, \tag{5.1}
$$

where $P(\kappa) = \kappa_1^2 + \cdots + \kappa_n^2 = \sum_{i,j=1}^n a_{ij}^2$, and N, β are two constants to be determined later.

We assume that Ψ achieves its maximum value at $X_0 \in \Sigma$. By a proper rotation, we may assume that (a_{ij}) is a diagonal matrix at the point, and $a_{11} \geq a_{22}\ldots \geq a_{nn}$.

At x_0, covariant differentiate Ψ twice,

$$
0 = \Psi_i = \frac{\sum_{l,j} a_{lj}a_{lji}}{P} - N\frac{\tau_i}{\tau} - \beta\eta_i = \frac{\sum_l \kappa_l a_{lli}}{P} + N\frac{a_{ii}\eta_i}{\tau} - \beta\eta_i = 0, \tag{5.2}
$$

and

$$
\begin{aligned}
0 \geq & \Psi_{ii} \\
\geq & \frac{1}{P}\left(\sum_l \kappa_l a_{llii} + \sum_l a_{lli}^2 + \sum_{p\neq q} a_{pqi}^2\right) - \frac{2}{P^2}\left(\sum_l \kappa_l a_{lli}\right)^2 - N\frac{\tau_{ii}}{\tau} + N\frac{\tau_i^2}{\tau^2} - \beta\eta_{ii} \\
= & \frac{1}{P}\left[\sum_l \kappa_l\left(a_{iill} - a_{lm}\left(a_{mi}a_{il} - a_{ml}a_{ii}\right) - a_{mi}\left(a_{mi}a_{ll} - a_{ml}a_{li}\right)\right)\right. \\
& \left. + \bar{R}_{0iil;l} - 2a_{ml}\bar{R}_{miil} + a_{il}\bar{R}_{0i0l} + a_{ll}\bar{R}_{0ii0}\right.
\end{aligned}
$$

$$+ \bar{R}_{0lil;i} - 2a_{mi}\bar{R}_{mlil} + a_{ii}\bar{R}_{0l0l} + a_{li}\bar{R}_{0li0}) + \sum_l a_{lli}^2 + \sum_{p \neq q} a_{pqi}^2 \bigg]$$

$$- \frac{2}{P^2}\bigg(\sum_l \kappa_l a_{lli}\bigg)^2 + \frac{N}{\tau}\sum_l a_{iil}\eta_l - \frac{Nh'}{\tau}\kappa_i + N\kappa_i^2 + \frac{N}{\tau^2}\kappa_i^2\eta_i^2$$

$$+ \frac{N}{\tau}\sum_l \bar{R}_{0iil}\eta_l + \beta(h'\delta_{ii} - \tau\kappa_i).$$

Multiplying σ_k^{ii} both sides gives

$$0 \geq \frac{1}{P}\bigg[\sum_l \kappa_l\Big(\sigma_k^{ii}a_{iill} - \sigma_k^{ii}a_{lm}(a_{mi}a_{il} - a_{ml}a_{ii}) - \sigma_k^{ii}a_{mi}(a_{mi}a_{ll} - a_{ml}a_{li})$$

$$+ \sigma_k^{ii}\bar{R}_{0iil;l} - 2\sigma_k^{ii}a_{ml}\bar{R}_{miil} + \sigma_k^{ii}a_{il}\bar{R}_{0i0l} + \sigma_k^{ii}a_{ll}\bar{R}_{0ii0} + \sigma_k^{ii}\bar{R}_{0lil;i}$$

$$- 2\sigma_k^{ii}a_{mi}\bar{R}_{mlil} + \sigma_k^{ii}a_{ii}\bar{R}_{0l0l} + \sigma_k^{ii}a_{li}\bar{R}_{0li0}\Big)$$

$$+ \sum_l \sigma_k^{ii}a_{lli}^2 + \sum_{p \neq q}\sigma_k^{ii}a_{pqi}^2\bigg]$$

$$- \frac{2}{P^2}\sigma_k^{ii}\bigg(\sum_l \kappa_l a_{lli}\bigg)^2 + \frac{N}{\tau}\sum_l \sigma_k^{ii}a_{iil}\eta_l - \frac{Nh'}{\tau}kf + N\sigma_k^{ii}\kappa_i^2 + \frac{N}{\tau^2}\sigma_k^{ii}\kappa_i^2\eta_i^2$$

$$+ \frac{N}{\tau}\sum_l \sigma_k^{ii}\bar{R}_{0iil}\eta_l + \beta\bigg(h'\sum_i \sigma_k^{ii} - \tau kf\bigg)$$

$$\geq \frac{1}{P}\bigg[\sum_l \kappa_l\sigma_k^{ii}a_{iill} + kf\sum_l \kappa_l^3 - C(1 + \kappa_1^2)\sum_i \sigma_k^{ii} + \sum_l \sigma_k^{ii}a_{lli}^2 + \sum_{p \neq q}\sigma_k^{ii}a_{pqi}^2\bigg]$$

$$- \frac{2}{P^2}\sigma_k^{ii}\bigg(\sum_l \kappa_l a_{lli}\bigg)^2 + \frac{N}{\tau}\sum_l \sigma_k^{ii}a_{iil}\eta_l + (N-1)\sigma_k^{ii}\kappa_i^2$$

$$+ (C_1\beta - C_2N)\sum_i \sigma_k^{ii} - C(\beta, N).$$

(5.3)

Now covariant differentiate the Eq. (1.2) twice,

$$\sigma_k^{ii}a_{iij} = d_V\psi(\nabla_j V) + d_v\psi(\nabla_j v) = h'd_V\psi(E_j) - a_{jl}d_v\psi(E_l), \quad (5.4)$$

and

$$\sigma_k^{ii}a_{iijj} + \sigma_k^{pq,rs}a_{pqj}a_{rsj}$$
$$= d_V\psi(\nabla_{jj}V) + d_V^2\psi(\nabla_j V, \nabla_j V) + 2d_V d_v\psi(\nabla_j V, \nabla_j v)$$
$$+ d_v^2\psi(\nabla_j v, \nabla_j v) + d_v\psi(\nabla_{jj}v).$$
$$= -\frac{h''}{h}\eta_j d_V\psi(E_j) + h'a_{jj}d_V\psi(v) + (h')^2 d_V^2\psi(E_j, E_j)$$
$$- 2h'a_{jj}d_V d_v\psi(E_j, E_j) + a_{jj}^2 d_v^2\psi(E_j, E_j)$$

$$- \sum_l a_{ljj} d_v \psi(E_l) - a_{jj}^2 d_v \psi(v)$$

$$\geq -C - C\kappa_j^2 - \sum_l a_{ljj} d_v \psi(E_l), \tag{5.5}$$

where the Schwarz inequality is used in the last inequality.

Since

$$- \sigma_k^{pq,rs} a_{pql} a_{rsl} = -\sigma_k^{pp,qq} a_{ppl} a_{qql} + \sigma_k^{pp,qq} a_{pql}^2, \tag{5.6}$$

it follows from (5.2) and (5.4), and Codazzi equation (2.10) implies

$$\frac{1}{P} \sum_{l,j} \kappa_j a_{ljj} d_v \psi(E_l) = \frac{N}{\tau} \sum_j \sigma_k^{ii} a_{iij} \eta_j - \frac{Nh'}{\tau} \sum_j d_V \psi(E_j) \eta_j + \beta \sum_j \eta_j d_v \psi(E_j)$$

$$- \frac{1}{P} \sum_{l,j} \kappa_j \bar{R}_{0jlj} d_v \psi(E_l). \tag{5.7}$$

Denote

$$A_i = \frac{\kappa_i}{P} \left(K(\sigma_k)_i^2 - \sum_{p,q} \sigma_k^{pp,qq} a_{ppi} a_{qqi} \right), \quad B_i = 2 \sum_j \frac{\kappa_j}{P} \sigma_k^{jj,ii} a_{jji}^2,$$

$$C_i = 2 \sum_{j \neq i} \frac{\sigma_k^{jj}}{P} a_{jji}^2, \quad D_i = \frac{1}{P} \sum_j \sigma_k^{ii} a_{jji}^2, \quad E_i = \frac{2\sigma_k^{ii}}{P^2} \left(\sum_j \kappa_j a_{jji} \right)^2.$$

By (5.4) and (5.7), we can infer

$$0 \geq \frac{1}{P} \left[\sum_l \kappa_l \left(-C - C\kappa_l^2 - K(\sigma_k)_l^2 + K(\sigma_k)_i^2 - \sigma_k^{pp,qq} a_{ppl} a_{qql} + 2 \sum_{j \neq l} \sigma_k^{ll,jj} a_{ljl}^2 \right) \right.$$

$$\left. + kf \sum_l \kappa_l^3 + \sum_l \sigma_k^{ii} a_{lli}^2 + 2 \sum_{j \neq i} \sigma_k^{ii} a_{iji}^2 \right]$$

$$- \frac{2}{P^2} \sigma_k^{ii} \left(\sum_l \kappa_l a_{lli} \right)^2 + (N-1) \sigma_k^{ii} \kappa_i^2$$

$$+ (C_1 \beta - C_2 N - C_3) \sum_i \sigma_k^{ii} - C(\beta, N) - \frac{C_4}{\kappa_1}. \tag{5.8}$$

From the Codazzi equation $a_{iji} = a_{iij} - \bar{R}_{0iji}$ and the Cauchy–Schwarz inequality, we have

$$2 \sum_{j \neq l} \sigma_k^{ll,jj} \kappa_l a_{ljl}^2 = 2 \sum_{j \neq l} \sigma_k^{ll,jj} \kappa_l \left(a_{llj} - \bar{R}_{0ljl} \right)^2$$

$$\geq (2 - \delta) \sum_{j \neq l} \kappa_l \sigma_k^{ll,jj} a_{llj}^2 - C_\delta \sum_j \sigma_k^{jj}$$

$$= (2 - \delta) \sum_{j \neq l} \kappa_l \sigma_k^{ll,jj} a_{llj}^2 - C_\delta \sum_{j \neq l} \kappa_l \sigma_k^{ll,jj},$$

and

$$2 \sum_{j \neq i} \sigma_k^{ii} a_{iji}^2 = 2 \sum_{j \neq i} \sigma_k^{ii,jj} \left(a_{iij} - \bar{R}_{0iji} \right)^2 \geq (2 - \delta) \sum_{j \neq i} \sigma_k^{ii} a_{iij}^2 - C_\delta \sum_i \sigma_k^{ii},$$

where δ is a small constant to be determined later and C_δ is a constant depending on δ. Therefore, we obtain

$$
\begin{aligned}
0 \geq \frac{1}{P} & \left[\sum_l \kappa_l \left(-C - C\kappa_l^2 - K(\sigma_k)_l^2 \right) + kf \sum_l \kappa_l^3 \right] \\
& + (N-1)\sigma_k^{ii}\kappa_i^2 + \left(C_1\beta - C_2N - C_3 - C_\delta \frac{1}{P} \right) \sum_i \sigma_k^{ii} - C(\beta, N) - \frac{C_4}{\kappa_1} \\
& + \left(1 - \frac{\delta}{2} \right) \sum_i (A_i + B_i + C_i + D_i - E_i) + \frac{\delta}{2} \sum_i (A_i + D_i) \\
& - \frac{\delta}{2} \frac{2}{P^2} \sigma_k^{ii} \left(\sum_l \kappa_l a_{lli} \right)^2 .
\end{aligned}
\tag{5.9}
$$

According to the proof of Lemma 4.2, Lemma 4.3 and Corollary 4.4 in [30], we have the following alternatives. There exist positive numbers $\delta_2, \delta_3 \ldots, \delta_n$ depending only on k, n, such that either

$$\kappa_i > \delta_i \kappa_1, \forall \ 2 \leq i \leq n,$$

or

$$A_i + B_i + C_i + D_i - E_i \geq 0, \forall \ 1 \leq i \leq n.$$

Thus, in the following, the proof will be divided into two cases.

Case (A): There exists some $2 \leq i \leq k-1$, such that $\kappa_i \geq \delta_i \kappa_1$ and $\kappa_{i+1} \leq \delta_{i+1}\kappa_1$. Choosing K sufficiently large, we have A_i is positive by Lemma 2.6 . By the above alternatives, we can infer

$$\sum_i (A_i + B_i + C_i + D_i - E_i) \geq 0.$$

From (5.2) and Cauchy–Schwarz inequality, we have

$$\sigma_k^{ii} \left(\frac{1}{P^2} \sum_l \kappa_l a_{lli} \right)^2 = \sigma_k^{ii} \left(\frac{N}{\tau}\kappa_i - \beta \right)^2 \eta_i^2 \leq C_5 \left(N^2\sigma_k^{ii}\kappa_i^2 + \beta^2 \sum_i \sigma_k^{ii} \right). \tag{5.10}$$

Inserting (5.10) into (5.9), we get

$$
\begin{aligned}
0 \geq \frac{1}{P} & \left[\sum_l \kappa_l \left(-C - C\kappa_l^2 - K(\sigma_k)_l^2 \right) + kf \sum_l \kappa_l^3 \right] \\
& + N \left(\frac{1}{2} - C_5\delta N \right) \sigma_k^{ii}\kappa_i^2 + \left(\frac{N}{2} - 1 \right) \sigma_k^{ii}\kappa_i^2 \\
& + \left(C_1\beta - C_2N - C_3 - C_\delta \frac{1}{P} - C_5\delta\beta^2 \right) \sum_i \sigma_k^{ii} - C(\beta, N) - \frac{C_4}{\kappa_1} \\
\geq & -\frac{1}{P} \left(C(K) + C(K)\kappa_1^3 \right) + C_6 \left(\frac{N}{2} - 1 \right) \kappa_1
\end{aligned}
$$

$$+ N \left(\frac{1}{2} - C_5 \delta N \right) \sigma_k^{ii} \kappa_i^2 + \left(C_1 \beta - C_2 N - C_3 - C_\delta \frac{1}{P} - C_5 \delta \beta^2 \right)$$

$$\times \sum_i \sigma_k^{ii} - C(\beta, N) - \frac{C_4}{\kappa_1}$$

$$\geq \left(\frac{1}{2} C_6 N - C(K) \right) \kappa_1 + N \left(\frac{1}{2} - C_5 \delta N \right) \sigma_k^{ii} \kappa_i^2$$

$$+ \left(C_1 \beta - C_2 N - C_3 - C_\delta \frac{1}{P} - C_5 \delta \beta^2 \right) \sum_i \sigma_k^{ii} - C(\beta, N) - \frac{C_4}{\kappa_1}, \tag{5.11}$$

where we have used $\sigma_k^{ii} \kappa_i^2 \geq c_0 \kappa_1$. Now let us choose these constants carefully. Firstly, choose N such that

$$C(K) + 1 \leq \frac{1}{2} C_5 N, \text{ and } N \geq 4.$$

Secondly, choose β such that

$$C_1 \beta - C_2 N - C_3 - 3 \geq 0.$$

Thirdly, choose the constant δ satisfying

$$\max\{N^2, \beta^2\} \leq (2C_5 \delta)^{-1}.$$

At last, take sufficiently large κ_1 satisfying

$$\frac{C_\delta}{P} \leq 1.$$

Otherwise we are done. Finally, κ_1 has upper bound by (5.11).

Case(B): If the Case(A) does not hold. That means $\kappa_k \geq \delta_k \kappa_1$. Since $\kappa_l \geq 0$, we have,

$$\sigma_k \geq \kappa_1 \kappa_2 \dots \kappa_k \geq \delta_k^{k-1} \kappa_1^k.$$

This implies the bound of κ_1.

6 A global curvature estimate for $(n-1)$ convex hypersurfaces

For the functions τ and η, we employ the following auxiliary function which is introduced firstly in [30],

$$\Psi = \log \log P - N \ln(\tau) - \beta \eta,$$

where $P = \sum_l e^{\kappa_l}$ and $\{\kappa_l\}_{l=1}^n$ are the eigenvalues of the second fundamental form.

We may assume that the maximum of Ψ is achieved at some point $X_0 \in \Sigma$. After rotating the coordinates, we may assume the matrix (a_{ij}) is diagonal at that point, and we can further assume that $a_{11} \geq a_{22} \dots \geq a_{nn}$. Denote $\kappa_i = a_{ii}$.

Covariant differentiating the function Ψ twice at X_0, we have

$$0 = \Psi_i = \frac{P_i}{P \log P} - N \frac{\tau_i}{\tau} - \beta \eta_i = \frac{1}{P \log P} \sum_l e^{\kappa_l} a_{lli} + N \frac{a_{ij} \eta_j}{\tau} - \beta \eta_i, \tag{6.1}$$

and

$$0 \geq \Psi_{ii}$$

$$= \frac{P_{ii}}{P \log P} - \frac{P_i^2}{P^2 \log P} - \frac{P_i^2}{(P \log P)^2} - N \frac{\tau_{ii}}{\tau} + N \frac{\tau_i^2}{\tau^2} - \beta \eta_{ii}$$

$$= \frac{1}{P \log P} \left[\sum_l e^{\kappa_l} a_{llii} + \sum_l e^{\kappa_l} a_{lli}^2 + \sum_{\alpha \neq \gamma} \frac{e^{\kappa_\alpha} - e^{\kappa_\gamma}}{\kappa_\alpha - \kappa_\gamma} a_{\alpha\gamma i}^2 - \left(\frac{1}{P} + \frac{1}{P \log P} \right) P_i^2 \right]$$

$$+ \frac{N}{\tau} \sum_l a_{iil} \eta_l - \frac{Nh'}{\tau} \kappa_i + N \kappa_i^2 + \frac{N}{\tau^2} \kappa_i^2 \eta_i^2 + \frac{N}{\tau} \sum_l \bar{R}_{0iil} \eta_l - \beta(\tau \kappa_i - h' \delta_{ii})$$

$$= \frac{1}{P \log P} \left[\sum_l e^{\kappa_l} \left(a_{iill} - a_{lm} (a_{mi} a_{il} - a_{ml} a_{ii}) - a_{mi} (a_{mi} a_{ll} - a_{ml} a_{li}) + \bar{R}_{0iil;l} \right. \right.$$

$$\left. -2 a_{ml} \bar{R}_{miil} + a_{il} \bar{R}_{0i0l} + a_{ll} \bar{R}_{0ii0} + \bar{R}_{0lil;i} - 2 a_{mi} \bar{R}_{mlil} + a_{ii} \bar{R}_{0l0l} + a_{li} \bar{R}_{0li0} \right)$$

$$+ \sum_l e^{\kappa_l} a_{lli}^2 + \sum_{\alpha \neq \gamma} \frac{e^{\kappa_\alpha} - e^{\kappa_\gamma}}{\kappa_\alpha - \kappa_\gamma} a_{\alpha\gamma i}^2 - \left(\frac{1}{P} + \frac{1}{P \log P} \right) P_i^2 \right]$$

$$+ \frac{N}{\tau} \sum_l a_{iil} \eta_l - \frac{Nh'}{\tau} \kappa_i + N \kappa_i^2 + \frac{N}{\tau^2} \kappa_i^2 \eta_i^2 + \frac{N}{\tau} \sum_l \bar{R}_{0iil} \eta_l - \beta(\tau \kappa_i - h' \delta_{ii}).$$

Contract with σ_{n-1}^{ii},

$$0 \geq \sigma_{n-1}^{ii} \Psi_{ii}$$

$$= \frac{1}{P \log P} \left[\sum_l e^{\kappa_l} \sigma_{n-1}^{ii} a_{iill} + (n-1) \psi \sum_l e^{\kappa_l} \kappa_l^2 - \sigma_{n-1}^{ii} \kappa_i^2 \sum_l e^{\kappa_l} \kappa_l \right.$$

$$- C(1 + \kappa_1) P \sum_i \sigma_{n-1}^{ii} + \sum_l \sigma_{n-1}^{ii} e^{\kappa_l} a_{lli}^2$$

$$\left. + \sum_{\alpha \neq \gamma} \frac{e^{\kappa_\alpha} - e^{\kappa_\gamma}}{\kappa_\alpha - \kappa_\gamma} \sigma_{n-1}^{ii} a_{\alpha\gamma i}^2 - \left(\frac{1}{P} + \frac{1}{P \log P} \right) \sigma_{n-1}^{ii} P_i^2 \right] \tag{6.2}$$

$$+ \frac{N}{\tau} \sum_l \sigma_{n-1}^{ii} a_{iil} \eta_l - \frac{1}{\tau} (n-1) Nh' \psi + N \sigma_{n-1}^{ii} \kappa_i^2$$

$$+ \frac{N}{\tau^2} \sigma_{n-1}^{ii} \kappa_i^2 \eta_i^2 + \frac{N}{\tau} \sum_l \sigma_{n-1}^{ii} \bar{R}_{0iil} \eta_l - (n-1) \beta \tau \psi + h' \beta \sum_i \sigma_{n-1}^{ii}.$$

Inserting (5.4), (5.5) into (6.2), we obtain

$$0 \geq \sigma_{n-1}^{ii} \Psi_{ii}$$

$$\geq \frac{1}{P \log P} \left[\sum_l e^{\kappa_l} \left(-C - C \kappa_1^2 - K (\sigma_{n-1})_l^2 + K (\sigma_{n-1})_l^2 - \sigma_{n-1}^{pq,rs} a_{pql} a_{rsl} \right) \right.$$

$$- \sum_{l,j} e^{\kappa_l} a_{jll} d_v \psi(E_j) + (n-1) \psi \sum_l e^{\kappa_l} \kappa_l^2 - \sigma_{n-1}^{ii} \kappa_i^2 \sum_l e^{\kappa_l} \kappa_l$$

$$\left. - C(1 + \kappa_1) P \sum_i \sigma_{n-1}^{ii} + \sum_l \sigma_{n-1}^{ii} e^{\kappa_l} a_{lli}^2 \right.$$

$$+ \sum_{\alpha \neq \gamma} \frac{e^{\kappa_\alpha} - e^{\kappa_\gamma}}{\kappa_\alpha - \kappa_\gamma} \sigma_{n-1}^{ii} a_{\alpha \gamma i}^2 - \left(\frac{1}{P} + \frac{1}{P \log P} \right) \sigma_{n-1}^{ii} P_i^2 \Bigg]$$

$$+ \frac{N}{\tau} \sum_l \sigma_{n-1}^{ii} a_{iil} \eta_l + N \sigma_{n-1}^{ii} \kappa_i^2 + \frac{N}{\tau^2} \sigma_{n-1}^{ii} \kappa_i^2 \eta_i^2 + (h' \beta - C) \sum_i \sigma_{n-1}^{ii} - C(\beta, N).$$

$$(6.3)$$

By (6.1) and (5.4), and the Codazzi equation (2.10), we have

$$\frac{1}{P \log P} \sum_{l,j} e^{\kappa_l} a_{jll} d_\nu \psi(E_j) = \frac{N}{\tau} \sum_l \sigma_{n-1}^{ii} a_{iil} \eta_l - \frac{Nh'}{\tau} \sum_l d_V \psi(E_l) \eta_l$$

$$+ \beta \sum_j \eta_j d_\nu \psi(E_j) - \frac{1}{P \log P} \sum_{l,j} e^{\kappa_l} \bar{R}_{0ljl} d_\nu \psi(E_j).$$

$$(6.4)$$

By using (5.6) and (6.3), we get

$$0 \geq \frac{1}{P \log P} \Bigg[\sum_l e^{\kappa_l} \left(-C - C \kappa_1^2 - K (\sigma_{n-1})_l^2 \right) + \sum_l e^{\kappa_i} \left(K (\sigma_{n-1})_i^2 - \sigma_{n-1}^{pp,qq} a_{ppi} a_{qqi} \right)$$

$$+ 2 \sum_{l \neq i} \sigma_{n-1}^{ii,ll} e^{\kappa_l} a_{lil}^2 + (n-1) \psi \sum_l e^{\kappa_l} \kappa_l^2 - \sigma_{n-1}^{ii} \kappa_i^2 \sum_l e^{\kappa_l} \kappa_l$$

$$- C(1 + \kappa_1) P \sum_i \sigma_{n-1}^{ii} + \sum_l \sigma_{n-1}^{ii} e^{\kappa_l} a_{lli}^2$$

$$+ 2 \sum_{l \neq i} \sigma_{n-1}^{ii} \frac{e^{\kappa_l} - e^{\kappa_i}}{\kappa_l - \kappa_i} a_{lil}^2 - \left(\frac{1}{P} + \frac{1}{P \log P} \right) \sigma_{n-1}^{ii} P_i^2 \Bigg]$$

$$+ N \sigma_{n-1}^{ii} \kappa_i^2 + \frac{N}{\tau^2} \sigma_{n-1}^{ii} \kappa_i^2 \eta_i^2 + (\beta h' - C) \sum_i \sigma_{n-1}^{ii} - C(\beta, N) - \frac{C}{\kappa_1}.$$

$$(6.5)$$

From the Codazzi equation (2.10) and Cauchy–Schwarz inequality, we have

$$2(a_{lil})^2 = 2(a_{lli} - \bar{R}_{0lil})^2 \geq (2 - \delta) a_{lli}^2 - C_\delta,$$

where δ is a small constant to be determined later. Denoting

$$A_i = e^{\kappa_i} \left(K (\sigma_{n-1})_i^2 - \sum_{p \neq q} \sigma_{n-1}^{pp,qq} a_{ppi} a_{qqi} \right), \quad B_i = 2 \sum_{l \neq i} \sigma_{n-1}^{ii,ll} e^{\kappa_l} a_{lli}^2,$$

$$C_i = \sigma_{n-1}^{ii} \sum_l e^{\kappa_l} a_{lli}^2; \quad D_i = 2 \sum_{l \neq i} \sigma_{n-1}^{ll} \frac{e^{\kappa_l} - e^{\kappa_i}}{\kappa_l - \kappa_i} a_{lli}^2, \quad E_i = \frac{1 + \log P}{P \log P} \sigma_{n-1}^{ii} P_i^2,$$

we have, by (6.5),

$$
0 \geq \left(1 - \frac{1}{2}\delta\right) \frac{1}{P \log P} \left[A_i + B_i + C_i + D_i - E_i\right] + \frac{\delta}{2} \frac{1}{P \log P} \sum_i (A_i + C_i)
$$
$$
- \frac{\delta}{2} \frac{1 + \log P}{(P \log P)^2} \sigma_{n-1}^{ii} P_i^2 - \frac{C_\delta}{P \ln P} \sum_{l \neq i} \sigma_{n-1}^{ii,ll} e^{\kappa_l} - \frac{C_\delta}{P \ln P} \sum_{l \neq i} \sigma_{n-1}^{ii} \frac{e^{\kappa_l} - e^{\kappa_i}}{\kappa_l - \kappa_i}
$$
$$
+ (N-1) \sigma_{n-1}^{ii} \kappa_i^2 + \frac{N}{\tau^2} \sigma_{n-1}^{ii} \kappa_i^2 \eta_i^2 + (\beta h' - C) \sum_i \sigma_{n-1}^{ii} - C\kappa_1 - C(\beta, N, K) - \frac{C}{\kappa_1}.
$$
$$
(6.6)
$$

By Schwarz inequality, we always have

$$
P_i^2 = \left(\sum_l e^{\kappa_l} a_{lli}\right)^2 \leq P \sum_l e^{\kappa_l} a_{lli}^2,
$$

which implies

$$
\frac{\delta}{2} \frac{1}{P \log P} C_i \geq \frac{\delta}{2} \frac{\log P}{(P \log P)^2} \sigma_{n-1}^{ii} P_i^2. \tag{6.7}
$$

We also have

$$
\sum_{l \neq i} \sigma_{n-1}^{ii,ll} e^{\kappa_l} \leq P \sum_{l \neq i} \sigma_{n-1}^{ii,ll} = 2P \sum_i \sigma_{n-2}^{ii} = 6P\sigma_{n-3}. \tag{6.8}
$$

We divided into several cases to compare with σ_{n-2}.

Case (A) If $\sigma_{n-2} \geq \sigma_{n-3}$, by (6.8), we have, for $n \geq 3$,

$$
\frac{C_\delta}{P \log P} \sum_{l \neq i} \sigma_{n-1}^{ii,ll} e^{\kappa_l} \leq 3n^2 \left(\frac{C_\delta}{\log P} \sum_i \sigma_{n-1}^{ii} + 1\right). \tag{6.9}
$$

Case (B) If $\sigma_{n-2} \leq \sigma_{n-3}$, in Γ_{n-1} cone, since $|\kappa_n| \leq \kappa_1/(n-1)$ by the argument in [42], we have

$$
\kappa_1 \ldots \kappa_{n-2} \leq C_0 \kappa_1 \ldots \kappa_{n-3},
$$

which implies $\kappa_{n-2} \leq C_0$. We further divide into two sub-cases to discuss for index $l = 1, \ldots, n$.

Subcase (B1) If $2|\kappa_l| \leq \kappa_1$, we have

$$
\frac{e^{\kappa_l}}{P} \leq e^{\kappa_l - \kappa_1} \leq e^{-\frac{\kappa_1}{2}} \leq \left[\frac{1}{(n-3)!} \left(\frac{\kappa_1}{2}\right)^{n-3}\right]^{-1}.
$$

The last inequality comes from Taylor expansion. Thus, we have

$$
\frac{C_\delta}{P \log P} \sigma_{n-1}^{ii,ll} e^{\kappa_l} \leq C_1 \frac{C_\delta}{\kappa_1} \leq 1,
$$

for sufficiently large κ_1.

Subcase (B2) For sufficiently large κ_1, if $2|\kappa_l| \geq \kappa_1$, by $\kappa_{n-2} \leq C_0$, we have $1 \leq l \leq n-3$. In this case, we have

$$
\sigma_{n-1}^{ii,ll} \leq C_1 \kappa_1 \ldots \kappa_{l-1} \kappa_{l+1} \ldots \kappa_{n-2} \leq \kappa_1 \ldots \kappa_{l-1} \kappa_l \kappa_{l+1} \ldots \kappa_{n-2} \leq \sigma_{n-2}.
$$

The middle inequality comes from $2\kappa_l \geq \kappa_1 \geq 2C_1$ for sufficiently large κ_1. Thus, we have

$$\frac{C_\delta}{P \log P}\sigma_{n-1}^{ii,ll}e^{\kappa_l} \leq \frac{C_\delta}{\log P}\sigma_{n-2}.$$

Combining cases (B1) and (B2), we also have (6.9).

By mean value theorem, we have some ξ between in κ_i and κ_l satisfying

$$\sum_{l \neq i}\sigma_{n-1}^{ii}\frac{e^{\kappa_l} - e^{\kappa_i}}{\kappa_l - \kappa_i} = \sum_{l \neq i}\sigma_{n-1}^{ii}e^{\xi} \leq (n-1)P\sum_i\sigma_{n-1}^{ii}. \tag{6.10}$$

Hence, using the discussion in [42], we have

$$A_i + B_i + C_i + D_i - E_i \geq 0.$$

Thus, by (6.6), (7.6), (6.9), (6.10), we have

$$0 \geq -\frac{\delta}{2}\frac{1}{(P \log P)^2}\sigma_{n-1}^{ii}P_i^2 + \frac{\delta}{2}\frac{1}{P \log P}\sum_i A_i$$

$$+ (N-1)\sigma_{n-1}^{ii}\kappa_i^2 + \frac{N}{\tau^2}\sigma_{n-1}^{ii}\kappa_i^2\eta_i^2$$

$$+ \left(\beta h' - C - \frac{C_\delta}{\log P}\right)\sum_i\sigma_{n-1}^{ii} - C(\beta, N, K) - \frac{C}{\kappa_1}. \tag{6.11}$$

From (6.1) and the Cauchy–Schwarz inequality, we have

$$\frac{\delta}{2}\sigma_{n-1}^{ii}\frac{P_i^2}{(P \log P)^2} = \frac{\delta}{2}\sigma_{n-1}^{ii}\left(\frac{N}{\tau}\kappa_i - \beta\right)^2\eta_i^2 \leq C\delta\left(N^2\sigma_{n-1}^{ii}\kappa_i^2 + \beta^2\sum_i\sigma_{n-1}^{ii}\right). \tag{6.12}$$

Therefore, by Lemma 2.6, (6.11), (6.12), we obtain

$$0 \geq \left(N - 1 - C\delta N^2\right)\sigma_{n-1}^{ii}\kappa_i^2 + \left(\beta h' - C - \frac{C_\delta}{\log P} - C\delta\beta^2\right)$$

$$\sum_i\sigma_{n-1}^{ii} - C(\beta, N, K) - \frac{C}{\kappa_1}. \tag{6.13}$$

Since $\sigma_{n-1}^{ii}\kappa_i^2 \geq C_1\kappa_1$, we only need to choose the constants N, β, δ carefully. At first, we take constant N satisfying

$$(N - n - 1)C_1 - C(K) \geq 1.$$

Secondly, we choose constant β satisfying

$$\beta h' - 2C - 2 \geq C_2\beta - 2C - 2 \geq 0.$$

Thirdly, we let constant δ satisfying

$$\max\{CC_1\delta N^2, C\delta\beta^2\} \leq 1.$$

At last, we take sufficiently large κ_1 satisfying

$$\frac{C_\delta}{\log P} \leq \frac{C_\delta}{\kappa_1} \leq 1.$$

Finally, by (6.13), we obtain the upper bound of κ_1.

7 The existence results

We are in the position to give the proof of the existence Theorems.

Proof of Theorem 1.1 We use continuity method to solve the existence result. For parameter $0 \le s \le 1$, according to [8,17], we consider the following family of functions

$$\psi^s(V, \nu) = s\psi(t, u) + (1 - s)\varphi(t)\sigma_k(\kappa(t)).$$

where $\kappa(t) = h'/h$ and φ is a positive function defined on I satisfying (i) $\varphi > 0$; (ii) $\varphi(t) > 1$ for $t \le t_-$; (iii) $\varphi(t) < 1$ for $t \ge t_+$; and (iv) $\varphi'(t) < 0$. It is obvious that there exists a unique point $t_0 \in (t_-, t_+)$ such that $\varphi(t_0) = 1$. By [8], $z = t_0$ is the unique hypersurface satisfying problem (1.2) and one can check directly that ψ^s also satisfies (a), (b), (c) in Theorem 1.1. The height estimate can be easily obtained by comparison principle.

The openness and uniqueness are also similar to [17,30]. In view of Evans–Krylov theory, we only need height, gradient and C^2 estimates to complete the closeness part which has been done in Sects. 3, 5 and 6. We complete our proof. □

In what following, we discuss the constant rank theorem in space forms according to [24,26,29]. We rewrite our Eq. (1.2) to be

$$F(a) = -\sigma_k^{-1/k}(a) = -\psi^{-1/k}(X, \nu). \tag{7.1}$$

Proposition 7.1 *Suppose the ambient space* $(\bar{M}, \bar{g}) = (\mathbb{S}^{n+1}, \bar{g})$ *is the sphere with the metric defined by* (1.1) *and* $h(t)$ *is given by* (1.4). *Suppose some compact hypersurface* Σ *satisfies* (7.1) *and its second fundamental form is non-negative definite. Let* X, Y *be two vector fields in the ambient space and* $\bar{\nabla}$ *be the covariant derivative of the ambient space. If the function* ψ *locally satisfies*

$$\bar{\nabla}_X \bar{\nabla}_Y(\psi^{-1/k}) + \lambda\psi^{-1/k}\bar{g}_{X,Y} \ge 0,$$

at any $(u, z) \in \Sigma$, *then the hypersurface* Σ *is of constant rank.*

Proof According to [24], suppose P_0 is the point where the second fundamental form is of the minimal rank l. Let O be some open neighborhood of P_0. If O is sufficiently small, we can pick some constant A as in [24]. Then we use the auxiliary function $\varphi = \sigma_{l+1}(a) + A\sigma_{l+2}(a)$ to establish a differential inequality.

Now we choose a local orthonormal frame $\{e_1 \ldots, e_n\}$ in the hypersurface Σ. Since $\bar{M}$ is the sphere with sectional curvature λ, we obviously have

$$\bar{R}_{abcd} = \lambda(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}).$$

By Lemma 2.2, we have

$$
\begin{aligned}
\varphi_j &= (\sigma_{l+1}^{ii} + A\sigma_{l+2}^{ii})a_{iij}, \\
\varphi_{jj} &= (\sigma_{l+1}^{ii} + A\sigma_{l+2}^{ii})a_{iijj} + (\sigma_{l+1}^{pq,rs} + A\sigma_{l+2}^{pq,rs})a_{pqj}a_{rsj} \\
&= (\sigma_{l+1}^{ii} + A\sigma_{l+2}^{ii})[a_{jjii} - a_{im}(a_{mj}a_{ji} - a_{mi}a_{jj}) - a_{mj}(a_{mj}a_{ii} - a_{mi}a_{ij}) \\
&\quad -2a_{mi}\lambda(\delta_{mj}\delta_{ij} - \delta_{mi}\delta_{jj}) + a_{ji}\lambda\delta_{00}\delta_{ji} + a_{ii}\lambda(-\delta_{00}\delta_{jj}) \\
&\quad -2a_{mj}\lambda(\delta_{mj}\delta_{ii} - \delta_{mi}\delta_{ij}) + a_{jj}\lambda\delta_{00}\delta_{ii} - a_{ij}\lambda\delta_{00}\delta_{ij}] + (\sigma_{l+1}^{pq,rs} + A\sigma_{l+2}^{pq,rs})a_{pqj}a_{rsj}.
\end{aligned}
\tag{7.2}
$$

Thus, we have

$$
\begin{aligned}
F^{jj}\varphi_{jj} &= F^{jj}(\sigma_{l+1}^{ii} + A\sigma_{l+2}^{ii})[a_{jjii} + a_{ii}^2 a_{jj} - a_{jj}^2 a_{ii} + \lambda a_{ii}\delta_{jj} - \lambda a_{jj}\delta_{ii}] \\
&\quad + F^{jj}(\sigma_{l+1}^{pq,rs} + A\sigma_{l+2}^{pq,rs})a_{pqj}a_{rsj} \\
&= (\sigma_{l+1}^{ii} + A\sigma_{l+2}^{ii})[(-\psi^{-1/k})_{ii} - \lambda\psi^{-1/k}\delta_{ii}] - (\sigma_{l+1}^{ii} + A\sigma_{l+2}^{ii})F^{pq,rs}a_{pqi}a_{rsi} \\
&\quad + F^{jj}(\sigma_{l+1}^{pq,rs} + A\sigma_{l+2}^{pq,rs})a_{pqj}a_{rsj}.
\end{aligned}
\tag{7.3}
$$

Since the second fundamental form still satisfies Codazzi property in space forms, the process of dealing with the third order terms is same as [24], We also have

$$
(\psi^{-1/k})_{,ii} = (\psi^{-1/k})_{ii} - a_{ii}(\psi^{-1/k})_{\nu},
$$

where the comma in the first term means taking covariant derivative with respect to the metric of the ambient space. Thus, since the index i is a bad index, the third term is useless. We have our results. □

Now, we can prove Theorem 1.3.

Proof of Theorem 1.3 The proof also use the degree theory by modifying the proof in [30]. We consider the auxiliary equation

$$
\sigma_k(\kappa(X)) = \psi^s = \left(s\psi^{-1/k}(X, \nu) + (1-s)\bar\varphi^{-1/k}\right)^{-k},
\tag{7.4}
$$

where $\bar\varphi$ is defined by $\bar\varphi = C_n^k \varphi \kappa^k(t)$. We claim that, for the sphere,

$$
(\bar\varphi^{-1/k})_{,ij} + \lambda\bar\varphi^{-1/k}\bar g_{ij} \geq 0.
\tag{7.5}
$$

where $\{\bar e_0, \ldots, \bar e_n\}$ is the local orthonormal frame on $\bar M$. If the claim holds, by our condition, it is obvious that the ψ^s satisfies condition (c) for parameter $0 \leq s \leq 1$. By Proposition 7.1, the strictly convexity is preserved along the flow ψ^s.

Now, let's discuss Claim (7.5). Define $\alpha(t) = (C_n^k \varphi)^{1/k}$. Since $\bar\varphi$ is some constant on every slice, we have, for $i, j = 1, \ldots, n$

$$
\begin{aligned}
(\bar\varphi^{-1/k})_{,ij} &= (\bar\varphi^{-1/k})_{ij} - a_{ij}(\bar\varphi^{-1/k})_t = -h^2(t)\kappa(t)\frac{\alpha'(t)\kappa(t) + \alpha(t)\kappa'(t)}{\alpha^2(t)\kappa^2(t)}g_{ij}' \\
&= -h^2(t)\frac{\alpha'(t)\kappa(t) + \alpha(t)\kappa'(t)}{\alpha^2(t)\kappa(t)}g_{ij}'.
\end{aligned}
$$

Thus, in space forms, we have

$$
\frac{h''(t)}{h(t)} = \kappa^2(t) - \frac{1}{h^2(t)}, \quad \kappa'(t) = \frac{h''(t)}{h(t)} - \kappa^2(t) = -\frac{1}{h^2(t)}.
$$

Then, we have

$$
(\bar\varphi^{-1/k})_{,ij} = -h^2(t)\frac{\alpha'(t)\kappa(t) + \alpha(t)\kappa'(t)}{\alpha^2(t)\kappa(t)}g_{ij}' = \frac{\alpha(t) - \alpha'(t)h^2(t)\kappa(t)}{\alpha^2(t)\kappa(t)}g_{ij}' > 0,
$$

since $\alpha' < 0$.

For the unit (namely, $\lambda = 1$) sphere, it is easy to see that

$$
(\bar\varphi^{-1/k})_{tt} = \frac{\partial^2(\bar\varphi^{-1/k})}{\partial t^2} = -\frac{\partial}{\partial t}\frac{\alpha'(t)\kappa(t) + \alpha(t)\kappa'(t)}{\alpha^2(t)\kappa^2(t)} = \frac{\partial}{\partial t}\frac{\alpha - \alpha'\sin t\cos t}{\alpha^2\cos^2 t}.
$$

Thus, we have

$$
\begin{aligned}
(\bar{\varphi}^{-1/k})_{tt} &= \frac{2\alpha' \sin^2 t - \alpha'' \sin t \cos t}{\alpha^2 \cos^2 t} - \frac{\alpha - \alpha' \sin t \cos t}{\alpha^4 \cos^4 t} \\
&\qquad \left[(-2\cos t \sin t)\alpha^2 + 2\alpha\alpha' \cos^2 t\right] \\
&\geq \frac{2\alpha' \sin^2 t}{\alpha^2 \cos^2 t} + 2\frac{\alpha - \alpha' \sin t \cos t}{\alpha^2 \cos^4 t} \cos t \sin t \\
&= \frac{2\alpha \cos t \sin t}{\alpha^2 \cos^4 t} \\
&> 0,
\end{aligned}
$$

if we require $\alpha' < 0$ and $\alpha'' > 0$. Thus, Claim (7.5) holds for unit sphere. Since it is rescaling invariant, then (7.5) holds for any $\lambda > 0$.

Now we can give the requirements for functions $\varphi(t)$ to satisfying $\alpha' < 0$ and $\alpha'' > 0$. It is a straightforward calculation that

$$
k\alpha^{k-1}\alpha' = C_n^k \varphi' \quad \text{and} \quad k\alpha^{k-1}\alpha'' + k(k-1)\alpha^{k-2}(\alpha')^2 = C_n^k \varphi'',
$$

which implies that

$$
\varphi' < 0 \quad \text{and} \quad \varphi\varphi'' > \frac{k-1}{k}(\varphi')^2. \tag{7.6}
$$

We further need that (i) $\varphi > 0$; (ii) $\varphi(t) > 1$ for $t \leq t_-$; (iii) $\varphi(t) < 1$ for $t \geq t_+$. There is a lot of functions satisfying (i) (ii) (iii) and (7.6), for example

$$
\varphi(t) = \exp(\frac{t_- + t_+}{2} - t).
$$

Thus, the initial surfaces satisfy the condition of constant rank theorem and the height estimate comes from comparison principle. The curvature estimate has been obtained in Sect. 5. The rest part of this proof is similar to convex case in the Euclidean space, where we only need to replace the constant rank theorem in [30] by Proposition 8.1 here. □

Remark 7.2 In hyperbolic space, the problem is that the slice spheres do not satisfy the constant rank theorem: Proposition 7.1. It may be an interesting problem to find some other nontrivial initial family of hypersurfaces to satisfy Proposition 7.1.

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