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Calculus of Variations

Sharp conditions for the existence of boundary blow-up solutions to the Monge–Ampère equation

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Abstract In this paper we give sharp conditions on $K(x)$ and $f(u)$ for the existence of strictly convex solutions to the boundary blow-up Monge–Ampère problem

$$M[u](x) = K(x)f(u) \quad \text{for } x \in \Omega, \ u(x) \to +\infty \quad \text{as } \mathrm{dist}(x, \partial\Omega) \to 0.$$

Here $M[u] = \det(u_{x_i x_j})$ is the Monge–Ampère operator, and Ω is a smooth, bounded, strictly convex domain in $\mathbb{R}^N$ ($N \geq 2$). Further results are obtained for the special case that Ω is a ball. Our approach is largely based on the construction of suitable sub- and super-solutions.

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1 Introduction

We consider the boundary blow-up problem for the Monge–Ampère equation

$$M[u] = K(x)f(u) \quad \text{in } \Omega, \ u = +\infty \text{ on } \partial\Omega, \tag{1.1}$$

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where $M[u] = \det(u_{x_i x_j})$ is the Monge–Ampère operator, Ω is a smooth, bounded, strictly convex domain in $\mathbb{R}^N (N \geq 2)$, and $K(x)$, $f(u)$ are smooth positive functions. The boundary blow-up condition $u = +\infty$ on $\partial\Omega$ means

$$u(x) \to +\infty \quad \text{as } \operatorname{dist}(x, \partial\Omega) \to 0.$$

Such problems were studied by Cheng and Yau [4,5] with $f(u)$ an exponential function of u, due to their applications in geometry. The case $f(u) = u^p$ $(p > 0)$ and $K(x)$ is a smooth positive function over $\overline{\Omega}$ was considered by Lazer and McKenna [12], and it is proved that in such a case (1.1) has a strictly convex solution if $p > N$, and there is no such solution for $0 < p \leq N$. Further results can be found in [7,10,13–15,21,22].

In this paper, we aim to find sharp conditions on $K(x)$ and $f(u)$ for the existence of a strictly convex solution to (1.1) with $K(x)$ and $f(u)$ chosen from a much larger class of functions than those considered in [12]. More precisely, we will seek sharp conditions for the existence problem for functions $K(x)$ and $f(u)$ which satisfy

(**K**): $K \in C^\infty(\Omega)$ and $K(x) > 0$ in Ω;
(**f**) : there exists $\eta \in \mathbb{R}^1 \cup \{-\infty\}$ such that

(i) $f \in C^\infty(\eta, \infty)$ is positive and strictly increasing in (η, ∞),
(ii) if $\eta \in \mathbb{R}^1$ then additionally $f(\eta) := \lim_{s \to \eta} f(s) = 0$.

To simplify notation, we write $+\infty$ as ∞. Let us note that a function $K(x)$ satisfying (**K**) need not be bounded away from 0 or ∞ near $\partial\Omega$. Examples of functions $f(u)$ satisfying (**f**) clearly include

$$a + e^{bu} (a \geq 0, b > 0), ku^p (k, p > 0).$$

Although various sufficient conditions on $K(x)$ and $f(u)$ satisfying (**K**) and (**f**), respectively, have been found for the existence of solutions to (1.1), none of them is known to be sharp, in the sense that the sufficient condition is also necessary.

For example, suppose that $K \in C^\infty(\overline{\Omega})$ is positive (and hence satisfying (**K**)), and f satisfies (**f**). Then it follows from Matero [13] and Mohammed [14] that

- (1.1) has a strictly convex solution if in addition f satisfies[1]

$$\int^\infty [F(s)]^{-1/(N+1)} ds < \infty; \tag{1.2}$$

- (1.1) has no strictly convex solution if

$$\int^\infty f(s)^{-1/N} ds = \infty. \tag{1.3}$$

Here

$$F(s) = \int_\eta^s f(t)dt \text{ if } \eta \in \mathbb{R}^1, \ F(s) = \int_0^s f(t)dt \text{ if } \eta = -\infty,$$

and $\int^\infty \Phi(s)ds < \infty \ (= \infty)$ means that

$$\int_M^\infty \Phi(s)ds < \infty \ (= \infty) \text{ for all large positive } M.$$

[1] As explained below, when $\eta \in \mathbb{R}^1$, the condition (1.2) alone is actually not sufficient.

If we take $f(u)$ satisfying (**f**) and $f(u) = u^N(\log u)^\alpha$ for all large u, then it is easily checked that $f(u)$ satisfies neither (1.2) nor (1.3) when $\alpha \in (N, N+1]$.

On the other hand, the known results for (1.1) show a clear similarity to that of the corresponding semilinear boundary blow-up problem

$$\Delta u = K(x)f(u) \quad \text{in } \Omega,\ \ u = \infty \text{ on } \partial\Omega. \tag{1.4}$$

By the arguments of Keller [11] and Osserman [16], it is easily checked (see, for example, section 6.1 in [8]) that if $K \in C^\infty(\overline{\Omega})$ is positive, and f satisfies (**f**), then (1.4) has a solution *if and only if*

$$\int^\infty [F(s)]^{-1/2} ds < \infty. \tag{1.5}$$

It will follow from Theorem 1.1 of this paper that, if $K \in C^\infty(\overline{\Omega})$ is positive, and f satisfies (**f**), then (1.2) is also a necessary condition for (1.1) to have a strictly convex solution. Moreover, we will show that, in the case $\eta \in \mathbb{R}^1$, (1.2) alone does not guarantee the existence of a strictly convex solution to (1.1); one needs to require additionally

$$\int_{\eta^+} [F(s)]^{-1/(N+1)} ds = \infty. \tag{1.6}$$

Here $\int_{\eta^+} \Phi(s)ds = \infty$ means that

$$\int_\eta^{\eta+\epsilon} \Phi(s)ds = \infty \text{ for all small positive } \epsilon.$$

Let us observe that if $f(u) = u^p$ with $p > 0$, then (1.2) is equivalent to $p > N$, and (1.6) is equivalent to $p \geq N$.

We would like to emphasize that for (1.4), condition (1.5) is sufficient for the existence problem, whether or not $\eta = -\infty$; but for (1.1), in the case $\eta \neq -\infty$, the condition (1.2) alone is not enough and the extra condition (1.6) is required to guarantee the existence of a strictly convex solution to (1.1). (See Theorem 1.4 for details on the necessity of (1.6).) This difference between the two boundary blow-up problems (1.1) and (1.4) seems overlooked in several previous works, and this paper appears to be the first to notice and demonstrate such a difference.

The first main result of this paper is the following.

Theorem 1.1 *Suppose that $K(x)$ satisfies* (**K**) *and $K \in L^\infty(\Omega)$. Suppose that $f(u)$ satisfies* (**f**)*, and when $\eta \in \mathbb{R}^1$, it satisfies additionally* (1.6)*. Then* (1.1) *has a strictly convex solution if and only if* (1.2) *holds.*

Next we consider more general $K(x)$. Mohammed [14] proved that if $K(x)$ satisfies (**K**) and is such that the Dirichlet problem

$$M[u] = K(x) \quad \text{in } \Omega,\ \ u = 0 \text{ on } \partial\Omega, \tag{1.7}$$

has a strictly convex solution, then (1.1) has a strictly convex solution if f satisfies (**f**) and (1.2)[2].

In [3], Cheng and Yau showed that problem (1.7) has a strictly convex solution if for some $\delta > 0$ and $C > 0$,

$$0 < K(x) < Cd(x)^{\delta-N-1} \quad \text{in } \Omega,\ \text{where } d(x) := \text{dist}(x, \partial\Omega).$$

[2] See footnote 1.

In [15], Mohammed proved that (1.7) has no strictly convex solution if

$$K(x) \geq Cd(x)^{-N-1} \quad \text{in } \Omega \text{ for some } C > 0.$$

These results have been improved by Yang and Chang [21] who showed that for $K(x)$ satisfying (**K**),

(i) (1.7) has no strictly convex solution if

$$K(x) \geq Cd(x)^{-N-1}(-\ln d(x))^{-N} \text{ near } \partial\Omega \text{ for some } C > 0;$$

(ii) (1.7) has a strictly convex solution if

$$K(x) \leq Cd(x)^{-N-1}(-\ln d(x))^{-q} \text{ near } \partial\Omega \text{ for some } q > N \text{ and } C > 0.$$

The second main result of this paper is a correction of the existence result in [14].

Theorem 1.2 *Suppose that $K(x)$ satisfies* (**K**) *and is such that* (1.7) *has a strictly convex solution. Suppose that $f(u)$ satisfies* (**f**)*, and when $\eta \in \mathbb{R}^1$, it satisfies additionally* (1.6). *Then* (1.1) *has a strictly convex solution if* (1.2) *holds.*

Remark 1.3 (i) Let us note that when K satisfies the conditions in Theorem 1.1, by the above mentioned results, (1.7) always has a strictly convex solution. Hence Theorem 1.2 gives a better existence result than Theorem 1.1.

(ii) We suspect that (1.2) is also a necessary condition for (1.1) to have a strictly convex solution under the conditions of Theorem 1.2, but we have failed to find a proof.

The theorem below indicates that without the extra condition (1.6) in Theorems 1.1 and 1.2 in the case $\eta \in \mathbb{R}^1$, (1.1) may have no strictly convex solution.

Theorem 1.4 *Let Ω be a smooth, bounded, strictly convex domain in $\mathbb{R}^N$, $N \geq 2$. Suppose K satisfies* (**K**) *and f satisfies* (**f**) *with $\eta \in \mathbb{R}^1$. If f satisfies* (1.2) *but not* (1.6)*, i.e.,*

$$\int^{\infty} [F(s)]^{-\frac{1}{N+1}} ds < \infty \ \ \text{and} \ \int_{\eta^+} [F(s)]^{-\frac{1}{N+1}} ds < \infty, \tag{1.8}$$

then, for each $K_* > 0$ there exists $R_0 > 0$ depending on K_*, f and N, such that (1.1) *has no strictly convex solution on Ω if $\Omega_{K_*} := \{x \in \Omega : K(x) \geq K_*\}$ contains a ball of radius $R > R_0$.*

Our next result gives conditions on $K(x)$ guaranteeing existence and non-existence of strictly convex solutions to (1.7), which are more general than the ones obtained by Yang and Chang [21] mentioned above.

For a positive function $p(t)$ in $C^1(0, \infty)$ satisfying $p'(t) < 0$ and $\lim_{t \to 0^+} p(t) = \infty$, to distinguish its behavior near $t = 0$ we set $P(\tau) = \int_\tau^1 p(t)dt$. We say such a function $p(t)$ is of class $\mathcal{P}_{finite}$ if

$$\int_{0^+} [P(\tau)]^{\frac{1}{N}} d\tau < \infty,$$

and is of class $\mathcal{P}_\infty$ if

$$\int_{0^+} [P(\tau)]^{\frac{1}{N}} d\tau = \infty.$$

It is easy to check that if $p(t) = t^{-N-1}(-\ln t)^{-q}$ for small $t > 0$, then for $q > N$ one can extend $p(t)$ to a function of class $\mathcal{P}_{finite}$, while for $q \leq N$, one can extend $p(t)$ to a function of class $\mathcal{P}_\infty$.

Theorem 1.5 *Suppose that $K(x)$ satisfies* (**K**). *Then*

(i) (1.7) *has no strictly convex solution if there exists a function $p(t)$ of class $\mathcal{P}_\infty$ such that $K(x) \geq p(d(x))$ near $\partial\Omega$;*
(ii) (1.7) *has a strictly convex solution if there exists a function $p(t)$ of class $\mathcal{P}_{finite}$ such that $K(x) \leq p(d(x))$ near $\partial\Omega$.*

Moreover, in case (ii) *above, if we define*

$$\omega_0(t) := \int_0^t (NP(\tau))^{\frac{1}{N}} d\tau \quad \text{for } t \in (0, b), \tag{1.9}$$

then (1.7) *has a strictly convex solution $u \in C^\infty(\Omega) \cap C(\overline{\Omega})$ such that*

$$-l_0\, \omega_0(d(x)) \leq u(x) < 0 \quad \text{in } \Omega \text{ for some } l_0 > 0. \tag{1.10}$$

Remark 1.6 It is interesting to know what happens to (1.1) if $K(x)$ is such that (1.7) has no strictly convex solution. We will examine some such cases for the radially symmetric situation, and show that (1.1) may have infinitely many strictly convex solutions or no such solution, depending on the behavior of f; see Theorems 5.3 and 5.4 for details.

Remark 1.7 The blow-up rate and uniqueness of solutions are not considered in this paper, and will be discussed in future work. Using more recent regularity results on Monge–Ampère equations in [1,18,20], the smoothness requirements in **(K)** and **(f)** can be considerably relaxed; we leave the details to the interested reader.

The rest of the paper is organized as follows. In Sect. 2 we collect some known results to be used in the subsequent sections. Section 3 is devoted to the proof of Theorem 1.5, while Sect. 4 gives the proof of Theorems 1.1 and 1.2. In Sect. 5, we consider radial solutions and discuss the cases mentioned in Remark 1.6. Section 6 is devoted to the proof of Theorem 1.4.

2 Some preliminary results

In this section, we collect some results for the convenience of later use and reference.

Lemma 2.1 (Lemma 2.1 of [12]) *Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, and let $u^k \in C^2(\Omega) \cap C(\overline{\Omega})$ for $k = 1, 2$. Let $f(x, u)$ be defined for $x \in \Omega$ and u in some interval containing the ranges of u^1 and u^2 and assume that $f(x, u)$ is strictly increasing in u for all $x \in \Omega$. Suppose*

(i) *the matrix $(u^1_{x_i x_j})$ is positive definite in Ω,*
(ii) $M[u^1](x) \geq f(x, u^1(x)), \quad \forall x \in \Omega$,
(iii) $M[u^2](x) \leq f(x, u^2(x)), \quad \forall x \in \Omega$,
(iv) $u^1(x) \leq u^2(x), \quad \forall x \in \partial\Omega$.

Then $u^1(x) \leq u^2(x)$ in Ω.

Remark 2.2 From the proof in [12], it is easily seen that the condition "$f(x, u)$ is strictly increasing in u for all $x \in \Omega$" in Lemma 2.1 can be relaxed to "$f(x, u)$ is nondecreasing in u for all $x \in \Omega$" provided that one of the inequalities in (ii) and (iii) is replaced by a strict inequality. This observation will be used later in the paper.

Lemma 2.3 (Proposition 2.1 of [7]) *Let $u \in C^2(\Omega)$ be such that the matrix $(u_{x_i x_j})$ is invertible for $x \in \Omega$, and let g be a C^2 function defined on an interval containing the range of u. Then*

$$M[g(u)] = M[u]\Big\{[g'(u)]^N + [g'(u)]^{N-1} g''(u)(\nabla u)^T B(u) \nabla u\Big\}, \tag{2.1}$$

where A^T denotes the transpose of the matrix A, $B(u)$ denotes the inverse of the matrix $(u_{x_i x_j})$, and

$$\nabla u = (u_{x_1}, u_{x_2}, \ldots, u_{x_N})^T.$$

The following interior estimate for derivatives of smooth solutions of Monge–Ampère equations is a simple variant of Lemma 2.2 in [12], which follows from [17,19].

Lemma 2.4 *Let Ω be a bounded domain in $\mathbb{R}^N, N \geq 2$, with $\partial\Omega \in C^\infty$. Let $\eta \in [-\infty, +\infty)$ and $f \in C^\infty(\overline{\Omega} \times (\eta, \infty))$ with $f(x,u) > 0$ for $(x,u) \in \overline{\Omega} \times (\eta, \infty)$. Let $u \in C^\infty(\overline{\Omega})$ be a solution of the Dirichlet problem*

$$\begin{cases} M[u](x) = f(x,u), & x \in \Omega, \\ u(x) = c = constant, & x \in \partial\Omega, \end{cases} \tag{2.2}$$

with $\eta < u(x) < c$ in Ω. Let Ω' be a subdomain of Ω with $\overline{\Omega'} \subset \Omega$ and assume that $\eta < a \leq u(x) \leq b$ for $x \in \overline{\Omega'}$ and let $k \geq 1$ be an integer. Then there exists a constant C which depends only on k, a, b, bounds for the derivatives of $f(x,u)$ for $(x,u) \in \overline{\Omega'} \times [a,b]$, and $\text{dist}(\Omega', \partial\Omega)$ *such that*

$$||u||_{C^k(\overline{\Omega'})} \leq C.$$

The existence result below is a variant of Lemma 2.3 in [12], which is a special case of Theorem 7.1 in [2].

Lemma 2.5 *Let Ω be a strictly convex, bounded domain in $\mathbb{R}^N$, $N \geq 2$, with $\partial\Omega \in C^\infty$. Let $f(x,u)$ be a positive C^∞ function on $\overline{\Omega} \times (\eta, c]$, where $c > \eta \geq -\infty$. If there exists a function $u_* \in C^2(\overline{\Omega})$, which is convex on $\overline{\Omega}$, such that $u_* > \eta$ and*

$$\begin{cases} M[u_*](x) \geq f(x, u_*(x)), & x \in \Omega, \\ u_*(x) = c, & x \in \partial\Omega, \end{cases}$$

then there exists a solution u of (2.2) with $u \in C^\infty(\overline{\Omega})$ and u strictly convex. Moreover, $u(x) \geq u_*(x)$ on $\overline{\Omega}$.

Let Ω be a smooth, bounded, strictly convex domain in $\mathbb{R}^N$, by Theorem 1.1 of [2], there exists $u_0 \in C^\infty(\overline{\Omega})$ which is the unique strictly convex solution to

$$M[u_0] = 1 \quad \text{in } \Omega, \ u_0 = 1 \text{ on } \partial\Omega.$$

Set $z(x) := 1 - u_0(x)$. Then $z(x) > 0$ in Ω and it is the unique strictly concave solution to

$$(-1)^N M[z] = 1 \quad \text{in } \Omega, \ z = 0 \text{ on } \partial\Omega. \tag{2.3}$$

Since $(z_{x_i x_j})$ is negative definite on $\overline{\Omega}$, its trace is negative, that is $\Delta z < 0$, and hence one can apply the Hopf boundary lemma to conclude that $|\nabla z| > 0$ for $x \in \partial\Omega$. It follows that there exist positive constants b_1 and b_2 such that

$$b_1 d(x) \leq z(x) \leq b_2 d(x) \quad \text{for } x \in \Omega. \tag{2.4}$$

3 Proof of Theorem 1.5

3.1 Proof of part (i)

Suppose that there exists a function $p(t)$ of class $\mathcal{P}_\infty$ such that $K(x) \ge p(d(x))$ near $\partial\Omega$. We want to show that (1.7) has no strictly convex solution.

We first note that by replacing $p(t)$ by $cp(t)$ with c a suitable small positive constant, we may assume that $K(x) \ge p(d(x))$ in Ω. Secondly, we may modify $p(t)$ for large t and assume that $p(t) = c_0 e^{-t}$ for some positive constant c_0 and all large t, say $t \ge M_0$. Thirdly, with $p(t)$ modified as above, if we define

$$\tilde{P}(\tau) = \int_\tau^\infty p(t)dt,$$

then we still have

$$\int_{0^+} [\tilde{P}(\tau)]^{\frac{1}{N}} d\tau = \infty. \tag{3.1}$$

Moreover,

$$\tilde{P}(t) = c_0 e^{-t},\ \ \tilde{P}(t)/p(t) = 1 \quad \text{for } t \ge M_0,\ \ \tilde{P}(t)/p(t) \to 0 \quad \text{as } t \to 0. \tag{3.2}$$

We now define

$$\sigma(t) = \int_t^\infty (N\tilde{P}(\tau))^{\frac{1}{N}} d\tau \quad \text{for } t > 0. \tag{3.3}$$

By (3.1) we have $\lim\limits_{t\to 0^+} \sigma(t) = \infty$. From (3.3) we obtain

$$(-1)^{N-1}(\sigma'(t))^{N-1}\sigma''(t) = p(t),\ \ -\frac{\sigma'(t)}{\sigma''(t)} = \frac{N\tilde{P}(t)}{p(t)}. \tag{3.4}$$

Define

$$v(x) = l\sigma(cz(x)) - L,\ \ x \in \Omega,$$

where l, L, c are positive constants and $z(x)$ is the same as in (2.3). By (2.1), (2.3) and (3.4), we have

$$\begin{aligned} M[v] &= l^N M[cz]\Big\{[\sigma'(cz)]^N + \sigma''(cz)[\sigma'(cz)]^{N-1}(\nabla(cz))^T B(cz)\nabla(cz)\Big\} \\ &= (-lc)^N\Big\{[\sigma'(cz)]^N + c\sigma''(cz)[\sigma'(cz)]^{N-1}(\nabla z)^T B(z)\nabla z\Big\} \\ &= (lc)^N p(cz)\Big\{-\frac{\sigma'(cz)}{\sigma''(cz)} - c(\nabla z)^T B(z)\nabla z\Big\} \\ &= (lc)^N p(cz)\Big\{\frac{N\tilde{P}(cz)}{p(cz)} - c(\nabla z)^T B(z)\nabla z\Big\}. \end{aligned}$$

By (3.2), we see that $\sup_{t>0} N\tilde{P}(t)/p(t) = C_0 < \infty$. Hence, since $(\nabla z)^T B(z)\nabla z$ is continuous over $\overline{\Omega}$, there exists $m_0 > 0$ such that

$$\frac{N\tilde{P}(cz)}{p(cz)} - c(\nabla z)^T B(z)\nabla z \le C_0 + cm_0 \quad \text{for } x \in \overline{\Omega}.$$

Therefore

$$M[v] \le (C_0 + cm_0)(lc)^N p(cz) \quad \text{in } \overline{\Omega}.$$

Since $z(x) \geq b_1 d(x)$ by (2.4) and $p(t)$ is decreasing, if we choose $c = 1/b_1$ then

$$p(cz(x)) \leq p(d(x)) \leq K(x) \quad \text{for } x \in \Omega.$$

We may then choose $l > 0$ sufficiently small to obtain

$$M[v] < K(x) \quad \text{for } x \in \Omega.$$

Suppose by way of contradiction that (1.7) has a strictly convex solution u. With c and l chosen as above, since $v(x) \to \infty$ as $x \to \partial\Omega$ and $u(x) \to 0$ as $x \to \partial\Omega$, we may use Remark 2.2 (over $\Omega_\delta := \{x \in \Omega : d(x) > \delta\}$ for all small $\delta > 0$) to conclude that $v \geq u$ in Ω. Since $L > 0$ is arbitrary in the definition of v, this clearly is a contradiction. The proof of part (i) of Theorem 1.5 is thus complete.

3.2 Proof of part (ii)

We modify $p(t)$ and define $\tilde{P}(\tau)$ as in the proof of part (i) above, and analogously we still have

$$\int_{0^+} [\tilde{P}(\tau)]^{\frac{1}{N}} d\tau < \infty. \tag{3.5}$$

Set

$$\omega(t) := \int_0^t (N\tilde{P}(\tau))^{\frac{1}{N}} d\tau \quad \text{for } t > 0. \tag{3.6}$$

For l, c positive constants to be determined, and $z(x)$ as given in (2.3), we define

$$w(x) = -l\omega(cz(x)) \quad \text{for } x \in \Omega.$$

Then

$$(\omega'(t))^{N-1}\omega''(t) = -p(t), \quad \frac{\omega'(t)}{\omega''(t)} = -\frac{N\tilde{P}(t)}{p(t)},$$

and by (3.5), $w(x) \to 0$ as $d(x) \to 0$. Moreover, for any $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$ and $x \in \overline{\Omega}$,

$$\sum_{i,j} w_{x_i x_j}\xi_i\xi_j = -lc^2\omega''(cz)\Big(\sum_i z_{x_i}\xi_i\Big)^2 + lc\omega'(cz)\sum_{i,j}(-z)_{x_i x_j}\xi_i\xi_j \geq \sigma_0|\xi|^2$$

for some $\sigma_0 > 0$, since $\omega' > 0$, $\omega'' < 0$ and $-z(x)$ is strictly convex. It follows that $w(x)$ is strictly convex in $\overline{\Omega}$.

By similar calculations to those for $M[v]$ in the proof of part (i) we obtain

$$M[w] = (lc)^N p(cz)\Big\{\frac{N\tilde{P}(cz)}{p(cz)} - c(\nabla z)^T B(z)\nabla z\Big\}.$$

Since $(z_{x_i x_j})$ is negative definite for $x \in \overline{\Omega}$, so is its inverse $B(z)$. Since $|\nabla z| > 0$ near $\partial\Omega$, we obtain

$$-(\nabla z)^T B(z)\nabla z > 0 \quad \text{for } x \in \overline{\Omega} \text{ near } \partial\Omega.$$

For $x \in \Omega$,

$$\frac{N\tilde{P}(cz(x))}{p(cz(x))} > 0$$

and it is bounded away from 0 for $x \in \Omega$ outside any neighborhood of $\partial\Omega$. Hence there exists $\delta_0 > 0$ depending on c such that

$$\frac{N\tilde{P}(cz)}{p(cz)} - c(\nabla z)^T B(z)\nabla z \geq \delta_0 \quad \text{for } x \in \overline{\Omega}. \tag{3.7}$$

It follows that

$$M[w] \geq \delta_0 (lc)^N p(cz) \quad \text{in } \Omega.$$

We may now choose $c = 1/b_2$ and use $z(x) \leq b_2 d(x)$ to deduce

$$p(cz(x)) \geq p(d(x)) \geq K(x) \quad \text{in } \Omega.$$

Therefore, for all large $l > 0$ we have

$$M[w] > K(x) \quad \text{for } x \in \Omega.$$

We now fix c and l as above, and for $\epsilon_n > 0$ decreasing to 0 define

$$\Omega_n := \{x \in \Omega : w(x) < -\epsilon_n\}.$$

Then consider the problem

$$M[u] = K(x) \quad \text{in } \Omega_n, \;\; u = 1 \text{ on } \partial\Omega_n. \tag{3.8}$$

We observe that Ω_n is also a level set of $z(x)$ and hence is strictly convex and smooth. Since $K(x) > 0$ on $\overline{\Omega_n}$, and $w_n(x) := w(x) + 1 + \epsilon_n$ satisfies

$$M[w_n] = M[w] > K(x) \quad \text{in } \Omega_n, \;\; w_n = 1 \text{ on } \partial\Omega_n,$$

and w_n is convex in $\overline{\Omega_n}$, we can apply Lemma 2.5 to conclude that (3.8) has a strictly convex solution u_n and it satisfies $u_n(x) \geq w_n(x) > w(x) + 1$ in Ω_n. Since $u_n = 1$ on $\partial\Omega_n$, the strict convexity of u_n implies $u_n(x) < 1$ in Ω_n. Hence, due to $\Omega_n \subset \Omega_{n+1}$ for $n \geq 1$, we have $u_n = 1 > u_{n+1}$ on $\partial\Omega_n$. For every $\epsilon \in (0, 1 - \max_{\partial\Omega_n} u_{n+1})$, we have

$$M[(1-\epsilon)u_n] = (1-\epsilon)^N K(x) < K(x) = M[u_{n+1}] \quad \text{in } \Omega_n, \;\; (1-\epsilon)u_n \geq u_{n+1} \text{ on } \partial\Omega_n.$$

Hence we can use Remark 2.2 to deduce

$$u_{n+1}(x) \leq (1-\epsilon)u_n(x) \quad \text{for } x \in \Omega_n, \;\; n \geq 1.$$

Letting $\epsilon \to 0$ we obtain

$$w(x) + 1 < u_{n+1}(x) \leq u_n(x) \quad \text{for } x \in \Omega_n, \;\; n \geq 1.$$

It follows that

$$u_0(x) := \lim_{n\to\infty} u_n(x) \text{ exists for } x \in \Omega,$$

and $w(x) + 1 \leq u_0(x) \leq 1$ in Ω.

By Lemma 2.4, for positive integers n and k, there exists $C = C_{n,k}$ independent of m such that

$$\|u_m\|_{C^k(\overline{\Omega}_n)} \leq C \text{ for all } m > n.$$

It follows that the convergence $u_n \to u_0$ also holds in $C^k_{loc}(\Omega)$ for every $k \geq 1$, and $u_0 \in C^\infty(\Omega)$, is strictly convex in Ω, and satisfies

$$M[u_0] = K(x) \quad \text{in } \Omega, \;\; u_0 = 1 \text{ on } \partial\Omega.$$

Clearly $u(x) := u_0(x) - 1$ is a strictly convex solution to (1.7). Moreover,

$$u(x) \geq w(x) = -l\omega(cz(x)) \geq -l\omega(d(x)) \quad \text{for } x \in \Omega.$$

It is easily seen that with $\omega_0(t)$ defined by (1.9), there exists $\epsilon_0 > 0$ small such that

$$\epsilon_0 \omega(d(x)) \leq \omega_0(d(x)) \quad \text{for } x \in \Omega.$$

We thus obtain $u(x) \geq -l_0\, \omega_0(d(x))$ in Ω with $l_0 = l/\epsilon_0$. Since $u(x) = 0$ on $\partial\Omega$ and $u(x)$ is strictly convex, we have $u(x) < 0$ in Ω. Now part (ii) of Theorem 1.5 is also proved.

4 Proof of Theorems 1.1 and 1.2

4.1 Proof of Theorem 1.2 for the case $\eta \in \mathbb{R}^1$

We will need the following lemma whose proof uses results in Sect. 5.1.

Lemma 4.1 *Suppose that D is a bounded domain in $\mathbb{R}^N$ and $K \in C^\infty(\overline{D})$ is positive on $\overline{D}$. Suppose that f satisfies* **(f)** *with $\eta > -\infty$,* (1.2) *and* (1.6)*. Then for any $\delta > 0$ there exists a strictly convex function $u \in C^\infty(\overline{D})$ such that*

$$M[u] \geq K(x) f(u), \;\; \eta + \delta > u(x) > \eta \;\; \textit{in } \overline{D}.$$

Proof By replacing $f(t)$ with $f(t+\eta)$ and u with $u - \eta$, we may assume that $\eta = 0$. Let $K_* := \max_{x \in \overline{D}} K(x)$, and for $\epsilon > 0$ define

$$T_\epsilon := \int_\epsilon^\infty \Big\{(N+1)K_*[F(t) - F(\epsilon)]\Big\}^{-1/(N+1)} dt.$$

Since

$$\big[F(t) - F(\epsilon)\big]^{-1/(N+1)} \leq \left[\frac{1}{2}F(t)\right]^{-1/(N+1)} \quad \text{for all large } t,$$

by (1.2) we see that

$$\int^\infty \big\{(N+1)K_*[F(t) - F(\epsilon)]\big\}^{-1/(N+1)} dt < \infty.$$

We also have

$$F(t) - F(\epsilon) \geq f(\epsilon)(t - \epsilon) \quad \text{for } t > \epsilon.$$

It follows that

$$\int_{\epsilon^+} \big\{(N+1)K_*[F(t) - F(\epsilon)]\big\}^{-1/(N+1)} dt < \infty.$$

Hence T_ϵ is a finite positive number for any $\epsilon > 0$.

On the other hand, due to (1.6) and

$$\big[F(t) - F(\epsilon)\big]^{-1/(N+1)} > \big[F(t)\big]^{-1/(N+1)} \quad \text{for } t > \epsilon,$$

we have

$$T_\epsilon > \int_\epsilon^\infty \big[(N+1)K_* F(t)\big]^{-1/(N+1)} dt \to \infty \quad \text{as } \epsilon \to 0.$$

Therefore we can choose $\epsilon_0 > 0$ sufficiently small such that

$$T_\epsilon > R^{\frac{2N}{N+1}} \quad \text{for } \epsilon \in (0, \epsilon_0],$$

where $R > 0$ is chosen such that $\overline{D} \subset B_R := \{x \in \mathbb{R}^N : |x| < R\}$.

For $\epsilon \in (0, \epsilon_0]$, we define $v(r) = v_\epsilon(r)$ by

$$\int_\epsilon^{v(r)} \big\{(N+1)K_*[F(t) - F(\epsilon)]\big\}^{-1/(N+1)} dt = R^{\frac{N-1}{N+1}} r, \;\; r \in (0, R_\epsilon),$$

with

$$R_\epsilon := T_\epsilon R^{-\frac{N-1}{N+1}} > R.$$

It is easily checked that v is smooth in $(0, R_\epsilon)$,

$$v(0) = \epsilon, \;\; v'(0) = 0, \;\; v'(r) > 0 \quad \text{for } r \in (0, R_\epsilon), \;\; v(r) \to \infty \quad \text{as } r \to R_\epsilon$$

and

$$(v')^{N-1} v'' = R^{N-1} K_* f(v) \ge r^{N-1} K_* f(v) \quad \text{for } r \in (0, R].$$

Moreover, since

$$R^{\frac{N-1}{N+1}} r > \int_\epsilon^{v(r)} \big\{(N+1)K_* F(t)\big\}^{-1/(N+1)} dt,$$

by (1.6) we deduce

$$v(r) \to 0 \text{ uniformly for } r \in [0, R] \quad \text{as } \epsilon \to 0. \tag{4.1}$$

Since $v''(0) = \infty$, to obtain a smooth function u with the required properties we consider the initial value problem

$$(u')^{N-1} u'' = r^{N-1} K_* f(u) \quad \text{for } r > 0, \;\; u(0) = \epsilon/2, u'(0) = 0.$$

By Lemmas 5.1 and 5.2 we see that $u(r)$ is defined for $r \in [0, R]$ and $\epsilon/2 < u(r) < v(r)$ for $r \in (0, R)$, $u''(r) > 0$ for $r \in [0, R]$. Thus

$$M[u(|x|)] = K_* f(u(|x|)) \quad \text{in } B_R.$$

In particular, $u(|x|)$ is a strictly convex function in $C^\infty(\overline{B}_R)$, $u(|x|) \ge \epsilon/2$ in $\overline{B}_R$ and

$$M[u(|x|)] = K_* f(u(|x|)) \ge K(x) f(u(|x|)) \quad \text{in } \overline{D}.$$

By (4.1), for any $\delta > 0$ by shrinking $\epsilon > 0$ further we have $0 < u(|x|) \le v(|x|) < \delta$ for $x \in \overline{D} \subset B_R$. This completes the proof. □

We are now ready to prove the existence of a strictly convex solution to (1.1). We will follow the ideas in the proof of Theorem 3.1 of Mohammed [14], but will make use of Lemma 4.1 above to correct the mistakes there.

Without loss of generality, we again assume that $\eta = 0$. Due to (1.2), we can use Lemma 2.1 of [9] to obtain

$$\lim_{t \to \infty} \frac{F(t)^{1/(N+1)}}{f(t)^{1/N}} = 0.$$

It follows that

$$\gamma(t) := -\int_t^{\infty} [f(s)]^{-1/N} ds \text{ is finite for all } t > 0.$$

Moreover, $\gamma(t)$ is strictly increasing and $\gamma(t) \to 0$ as $t \to \infty$.

Let $u_*(x)$ be a strictly convex solution of (1.7). Since $u_*(x) < 0$ in Ω and $u_*(x) = 0$ on $\partial\Omega$, for all large positive integer k, say $k \geq k_0$,

$$\Omega_k := \{x \in \Omega : u_*(x) < \gamma(k)\}$$

is a smooth strictly convex subdomain of Ω, and

$$\overline{\Omega}_k \subset \Omega_{k+1} \quad \text{for } k \geq k_0, \;\; \Omega = \cup_{k=k_0}^{\infty} \Omega_k.$$

Let w_k be the strictly convex function obtained in Lemma 4.1 with $D = \Omega_k$ satisfying

$$M[w_k] \geq K(x) f(w_k), k > w_k(x) > 0 \quad \text{in } \overline{\Omega}_k.$$

Set

$$\epsilon_k = \min_{x \in \overline{\Omega}_k} w_k(x), \;\; k \geq k_0.$$

We now let $\tilde{f}_k(t)$ be a function satisfying **(f)** with $\eta = -\infty$ and $\tilde{f}(t) = f(t)$ for $t \geq \epsilon_k$. Then we consider the problem

$$M[u] = K(x) \tilde{f}_k(u) \quad \text{in } \Omega_k, \;\; u = k \text{ on } \partial\Omega_k. \tag{4.2}$$

By Theorem 7.1 of [2], (4.2) has a unique strictly convex solution z_k when $K(x)\tilde{f}(u)$ is replaced by $K(x)\tilde{f}(k)$. It follows that

$$M[z_k] = K(x) \tilde{f}_k(k) > K(x) \tilde{f}_k(z_k) \quad \text{in } \Omega_k, \;\; z_k = k \text{ on } \partial\Omega_k.$$

Therefore we can apply Lemma 2.5 to conclude that (4.2) has a strictly convex solution $u_k \in C^{\infty}(\overline{\Omega}_k)$. Since w_k is strictly convex and

$$M[w_k] \geq K(x) f(w_k) = K(x) \tilde{f}_k(w_k) \quad \text{in } \Omega_k, \;\; w_k < k = u_k \text{ on } \partial\Omega_k,$$

by Lemma 2.1 we deduce $u_k \geq w_k$ in Ω_k and in particular, $u_k \geq \epsilon_k$ in Ω_k. Hence $\tilde{f}_k(u_k) = f(u_k)$ in Ω_k and

$$M[u_k] = K(x) f(u_k) \quad \text{in } \Omega_k, \;\; u_k = k \text{ on } \partial\Omega_k.$$

Following [14] we define

$$v_k(x) = \gamma(u_k(x) + \epsilon) \quad \text{for } x \in \overline{\Omega}_k \text{ and small positive constant } \epsilon.$$

This is now well-defined since $\gamma(t)$ is defined for $t \geq 0$ and $u_k(x) + \epsilon > 0$ in $\overline{\Omega}_k$. The same calculation as in [14] yields

$$M[v_k] < K(x) = M[u_*] \quad \text{in } \Omega_k.$$

Since $u_* = \gamma(k) = \gamma(u_k) < v_k$ on $\partial\Omega_k$, by Remark 2.2 we obtain

$$u_*(x) \leq v_k(x) = \gamma(u_k(x) + \epsilon) \quad \text{in } \Omega_k.$$

Letting $\epsilon \to 0$ we obtain

$$u_*(x) \leq \gamma(u_k(x)) \quad \text{for } x \in \overline{\Omega}_k. \tag{4.3}$$

Although it is unclear whether the inverse function γ^{-1} is defined over the entire range of u_*, by the choice of k_0 and the convexity of $u_*(x)$ we know that $\gamma^{-1}(u_*(x))$ is defined over $\Omega \setminus \Omega_{k_0}$. We thus obtain from (4.3) that

$$\gamma^{-1}(u_*(x)) \leq u_k(x) \quad \text{for } x \in \overline{\Omega}_k \setminus \Omega_{k_0}. \tag{4.4}$$

Since

$$u_k = k = \gamma^{-1}(u_*) \leq u_{k+1} \text{ on } \partial\Omega_k \quad \text{for } k \geq k_0.$$

By Lemma 2.1 we obtain

$$u_{k+1}(x) \geq u_k(x) \quad \text{for } x \in \overline{\Omega}_k, k \geq k_0.$$

Combining this with (4.4), we see that there exists $c_0 > 0$ such that

$$u_k(x) \geq c_0 \quad \text{for } x \in \overline{\Omega}_k,\ k \geq k_0.$$

Fix $m \geq k_0$. Since K is C^∞ and positive over $\overline{\Omega}_{m+1}$, by Lemma 2.2 of [14] there exists $h \in C^\infty(\Omega_{m+1})$ such that $u_n \leq h$ in Ω_{m+1} for all $n \geq m+1$. Therefore there exists $C_m > 0$ such that

$$u_n(x) \leq C_m \quad \text{for } x \in \overline{\Omega}_m,\ n \geq m+1.$$

This implies that, for every $x \in \Omega$,

$$u(x) := \lim_{n\to\infty} u_n(x) \text{ exists}$$

and

$$u_m(x) \leq u(x) \leq C_m \quad \text{for } x \in \overline{\Omega}_m,\ m \geq k_0.$$

As we also have $u_n(x) \geq c_0 > 0$ in $\overline{\Omega}_m$ for $n \geq m+1$, and for such n, $\overline{\Omega}_m \subset \Omega_n$,

$$0 < \text{dist}(\overline{\Omega}_m, \partial\Omega_{m+1}) \leq \text{dist}(\overline{\Omega}_m, \partial\Omega_n) < \text{dist}(\Omega_m, \partial\Omega),$$

we are in a position to apply Lemma 2.4 to conclude that, for any fixed integer $k \geq 1$, there exists a constant $C = C_{k,m}$ independent of n such that for all $n > m$,

$$\|u_n\|_{C^k(\overline{\Omega}_m)} \leq C.$$

It follows that the convergence $u_n(x) \to u(x)$ holds in $C^k_{loc}(\Omega)$ for every $k \geq 1$, and $u \in C^\infty(\Omega)$. Moreover, for $x \in \Omega$,

$$M[u](x) = \lim_{n\to\infty} M[u_n](x) = K(x) \lim_{n\to\infty} f(u_n(x)) = K(x) f(u(x)).$$

Since each u_n is strictly convex, $u(x)$ is strictly convex in Ω. By (4.4) we obtain $u(x) \geq \gamma^{-1}(u_*(x))$ on $\Omega \setminus \Omega_{k_0}$, which clearly implies $u = \infty$ on $\partial\Omega$. Thus u is a strictly convex solution of (1.1).

4.2 Proof of Theorem 1.2 for the case $\eta = -\infty$

This case can be proved by a simple modification of the above proof for the case $\eta \in \mathbb{R}^1$. Indeed, it is much simpler; we just follow everything there except that we do not need to modify f to $\tilde{f}_k$ in (4.2), and hence (1.6) and Lemma 4.1 are not required.

4.3 Proof of Theorem 1.1

The sufficiency part already follows from Theorem 1.2. So only the necessity part requires a proof. Assume, contrary to the assertion of the theorem, that there exists $c_0 > 0$ such that

$$G(t) := \int_{c_0}^{t} [(N+1)F(\tau)]^{-\frac{1}{N+1}} d\tau \to \infty \quad \text{as } t \to \infty,$$

and (1.1) has a strictly convex solution u. We aim to derive a contradiction.

Denote by $g(t)$ the inverse of $G(t)$, i.e.,

$$\int_{c_0}^{g(t)} [(N+1)F(\tau)]^{-\frac{1}{N+1}} d\tau = t, \quad \forall t > 0. \tag{4.5}$$

Then

$$g(0) = c_0, \ \lim_{t\to\infty} g(t) = \infty$$

and

$$g'(t) = [(N+1)F(g(t))]^{\frac{1}{N+1}}, \quad g''(t) = \frac{f(g(t))}{[(N+1)F(g(t))]^{\frac{N-1}{N+1}}},$$

$$(g'(t))^{N-1} g''(t) = f(g(t)), \quad \frac{g'(t)}{g''(t)} = \frac{[(N+1)F(g(t))]^{\frac{N}{N+1}}}{f(g(t))}.$$

Take $x_0 \in \mathbb{R}^N \setminus \overline{\Omega}$ so there exists $d_0 > 0$ such that $|x - x_0| \geq d_0$ for $x \in \Omega$. Then define

$$y(x) := \frac{1}{2}|x - x_0|^2 \quad \text{for } x \in \Omega.$$

Clearly

$$[\nabla y(x)]^T = x - x_0, \ (y_{x_i x_j}) \text{ is the identity matrix, and } M[y] = 1.$$

For $c > 0$ define

$$w(x) := g\left(cy(x)\right), \ x \in \Omega.$$

By (2.1) we obtain, for $x \in \Omega$,

$$\begin{aligned} M[w] &= M[cy]\Big\{\big[g'(cy)\big]^N + (g'(cy))^{N-1} g''(cy) (\nabla(cy))^T B(cy) \nabla(cy)\Big\} \\ &= c^N (g'(cy))^{N-1} g''(cy) \left\{ \frac{g'(cy)}{g''(cy)} + c|x - x_0|^2 \right\} \\ &> f(w) c^{N+1} d_0^2, \end{aligned}$$

where we have used

$$\frac{g'(cy(x))}{g''(cy(x))} > 0, \ |x - x_0| \geq d_0 \quad \text{for } x \in \Omega.$$

We thus obtain, in view of $K \in L^\infty(\Omega)$,

$$M[w] > K(x) f(w) \quad \text{in } \Omega$$

provided that c is chosen large enough.

Fix $x_1 \in \Omega$ and by further enlarging c if necessary we may assume that

$$w(x_1) > u(x_1) \text{ and } M[w] > K(x) f(w) \quad \text{in } \Omega.$$

Since $u(x) \to \infty$ as $d(x) \to 0$, while $w(x)$ is continuous on $\overline{\Omega}$, there exists an open connected set D such that

$$x_1 \in D,\ \overline{D} \subset \Omega,\ u(x) < w(x) \quad \text{in } D \text{ and } u(x) = w(x) \text{ on } \partial D.$$

On the other hand, since

$$M[u] = K(x) f(u) \quad \text{in } D \text{ and } w = u \text{ on } \partial D,$$

and the matrix $(w_{x_i x_j})$ is positive definite on $\overline{D}$ (since $y(x)$ is strictly convex in Ω and $g', g'' > 0$), we can apply Lemma 2.1 to conclude that $w(x) \le u(x)$ in D. This contradiction completes our proof.

5 Further results for radial solutions

If $K(x)$ is such that (1.7) has no solution, in general it is difficult to find sharp conditions on $f(u)$ such that (1.1) has a solution. In this section, we consider such a situation in the special case that Ω is a ball and $K = K(|x|)$ is radially symmetric. Our approach in this section is motivated by ideas in [6].

So we consider the problem

$$\begin{cases} M[u] = K(|x|) f(u), & x \in B, \\ u = \infty, & x \in \partial B, \end{cases} \tag{5.1}$$

where B is a ball in $\mathbb{R}^N$ $(N \ge 2)$. For simplicity, and without loss of generality, we assume that B is the unit ball.

By a direct calculation, it is seen (and well-known) that if $v = v(r)$ $(r = |x|)$ is a radially symmetric solution of (5.1), then

$$\begin{cases} (v')^{N-1} v'' = r^{N-1} K(r) f(v),\ r \in (0, 1), \\ v'(0) = 0,\ v(1) = \infty. \end{cases} \tag{5.2}$$

In the radially symmetric setting, the smoothness requirements for K and f can be greatly relaxed. We assume that K and f satisfy, respectively

(K1) : $K \in C([0, 1))$ and $K(r) > 0$ in $[0, 1)$;
(f1) : for some $\eta \in \mathbb{R}^1 \cup \{-\infty\}$, $f(s)$ is locally Lipschitz continuous in (η, ∞), positive and increasing for $s > \eta$.

5.1 The initial value problem and a comparison result

For $v_0 > \eta$, consider the following initial value problem,

$$\begin{cases} (v')^{N-1} v'' = r^{N-1} K(r) f(v),\ r \in (0, 1), \\ v(0) = v_0,\ v'(0) = 0. \end{cases} \tag{5.3}$$

Lemma 5.1 *Assume that K satisfies* **(K1)** *and f satisfies* **(f1)**. *Then for every $v_0 > \eta$,* (5.3) *has a unique solution $v(r)$ over a maximal interval of existence* $[0, a) \subset [0, 1)$. *Moreover, $v' > 0$ in $(0, a)$, $v'' > 0$ in $[0, a)$ and $v(r) \to \infty$ as $r \to a$ if $a < 1$.*

Proof We first show that (5.3) has a unique solution defined over $[0, \delta]$ for $\delta > 0$ sufficiently small. It is easy to see that (5.3) is equivalent to the following integral equation

$$v(r) = v_0 + \int_0^r \left[\int_0^s N t^{N-1} K(t) f(v(t)) dt \right]^{1/N} ds. \tag{5.4}$$

Let $E = C([0,\delta])$ with $\delta > 0$ small to be specified, and define $T : E \to E$ by

$$(Tv)(r) = v_0 + \int_0^r \left[\int_0^s Nt^{N-1}K(t)f(v(t))dt\right]^{1/N} ds.$$

We are going to show that if $\delta > 0$ is sufficiently small, then T is a contraction mapping on a suitable subset of E and hence has a unique fixed point, which gives a unique solution to (5.3) over $[0,\delta]$.

Let $K_* = \max_{r\in[0,1/2]} K(r)$, $k_* = \min_{r\in[0,1/2]} K(r)$ and

$$B_\delta(v_0) = \{v \in E : \|v - v_0\|_E < \delta\}.$$

Fix $\delta_1 \in (0, 1/2)$ such that $v_0 - \delta_1 > \eta$, and let L be the Lipschitz constant of the function $f(u)$ over $[v_0 - \delta_1, v_0 + \delta_1]$:

$$|f(v_1) - f(v_2)| \le L|v_1 - v_2| \quad \text{for } v_1, v_2 \in [v_0 - \delta_1, v_0 + \delta_1].$$

Then

$$m := f(v_0 - \delta_1) \le f(v) \le M := L\delta_1 + f(v_0) \quad \text{for } v \in [v_0 - \delta_1, v_0 + \delta_1].$$

Clearly there exists $\delta_2 \in (0, \delta_1)$ sufficiently small such that

$$\frac{1}{2}\delta^2(K_*M)^{\frac{1}{N}} < \delta \quad \text{for } \delta \in (0, \delta_2].$$

We prove that $T(B_\delta(v_0)) \subset B_\delta(v_0)$ for every $\delta \in (0, \delta_2]$. Indeed, for such δ and any $v \in B_\delta(v_0)$, we have

$$\begin{aligned}|Tv - v_0| &= \int_0^r \left[\int_0^s Nt^{N-1}K(t)f(v(t))dt\right]^{1/N} ds\\ &\le \int_0^r \left[\int_0^s Nt^{N-1}K_*M dt\right]^{1/N} ds\\ &= \frac{1}{2}\delta^2(K_*M)^{\frac{1}{N}} < \delta \quad \text{for } r \in [0,\delta].\end{aligned}$$

Hence $T(B_\delta(v_0)) \subset B_\delta(v_0)$ for every $\delta \in (0, \delta_2]$.

Next we show that T is a contraction mapping on $B_\delta(v_0)$ for all small $\delta > 0$. We first observe that, by the mean value theorem, for $\delta \in (0, \delta_2]$ and $v_1, v_2 \in B_\delta(v_0)$,

$$\begin{aligned}J(s) &:= \left[\int_0^s Nt^{N-1}K(t)f(v_1(t))dt\right]^{1/N} - \left[\int_0^s Nt^{N-1}K(t)f(v_2(t))dt\right]^{1/N}\\ &= \frac{1}{N}\left[\int_0^s Nt^{N-1}K(t)\big[\theta f(v_1) + (1-\theta)f(v_2)\big]dt\right]^{\frac{1}{N}-1}\\ &\quad \int_0^s Nt^{N-1}K(t)\big[f(v_1) - f(v_2)\big]dt,\end{aligned}$$

with $\theta = \theta(s) \in (0, 1)$. Therefore, for $s \in [0,\delta]$,

$$\begin{aligned}|J(s)| &\le \frac{1}{N}\left[\int_0^s Nt^{N-1}k_*m dt\right]^{\frac{1}{N}-1} \cdot \int_0^s Nt^{N-1}K_*L\|v_1 - v_2\|_E dt\\ &= sN^{-1}(k_*m)^{\frac{1}{N}-1}K_*L\|v_1 - v_2\|_E.\end{aligned}$$

It follows that, for $r \in [0, \delta]$,

$$|(Tv_1)(r) - (Tv_2)(r)| = \Big| \int_0^r J(s)ds \Big| \le \delta^2 N^{-1} (k_* m)^{\frac{1}{N}-1} K_* L \|v_1 - v_2\|_E.$$

Hence T is a contraction mapping on $B_\delta(v_0)$ if $\delta \in (0, \delta_2]$ is small enough such that

$$\delta^2 N^{-1} (k_* m)^{\frac{1}{N}-1} K_* L < 1.$$

We fix such a small $\delta > 0$ and have thus proved that (5.3) has a unique solution defined for $r \in [0, \delta]$. Moreover, since

$$(v')^{N-1} v'' = r^{N-1} K(r) f(v) > 0 \quad \text{for } r \in (0, \delta], \text{ and } v'(0) = 0,$$

we further have $v'(r) > 0$, $v''(r) > 0$ for $r \in (0, \delta]$, and $v''(0) := \lim_{r \to 0} v''(r) > 0$.

To extend the solution $v(r)$ to $r > \delta$ we let $v' = u$ and

$$U = \begin{pmatrix} u \\ v \end{pmatrix}.$$

Then we consider the first order ODE system

$$U' = \begin{pmatrix} r^{N-1} K(r) \frac{f(v)}{u^{N-1}} \\ u \end{pmatrix} =: F(r, U), \quad U(\delta) = \begin{pmatrix} v'(\delta) \\ v(\delta) \end{pmatrix}. \tag{5.5}$$

By **(K1)** and **(f1)**, $F(r, U)$ is locally Lipschitz continuous in U in the range $u > 0$ and $v > \eta$, and continuous in $r \in [0, 1)$. Hence (5.5) has a unique solution defined for r in a small neighbourhood of δ. Clearly the v component of U satisfies

$$(v')^{N-1} v'' = r^{N-1} K(r) f(v) > 0, \ v(\delta) > 0, v'(\delta) > 0.$$

It follows that $v'(r) > v'(\delta)$, $v''(r) > 0$ for $r > \delta$. Hence the solution $U(r)$ of (5.5) can be extended to $r > \delta$ until r reaches 1 or until $v(r)$ blows up to ∞. It follows that (5.3) has a unique solution $v(r)$ on some maximal interval of existence $[0, a)$ with $a \le 1$, and $v(r) \to \infty$ as $r \to a$ if $a < 1$. The proof of the lemma is now complete. □

Lemma 5.2 *Assume that K satisfies* **(K1)** *and f satisfies* **(f1)**. *If u_1 and u_2 are functions in $C^1([0, a)) \cap C^2((0, a))$ satisfying $u_1, u_2 > \eta$ when $\eta \in \mathbb{R}^1$,*

$$(u_1')^{N-1} u_1'' \le r^{N-1} K(r) f(u_1), \ (u_2')^{N-1} u_2'' \ge r^{N-1} K(r) f(u_2) \quad \textit{for } r \in (0, a),$$

and $u_1'(0) = u_2'(0) = 0$, $u_1(0) < u_2(0)$. Then $u_1(r) < u_2(r)$ for $r \in [0, a)$.

Proof If $u_1 < u_2$ in $[0, a)$ does not hold, then due to $u_1(0) < u_2(0)$, there exists $\overline{r} \in (0, a)$ such that $u_1(\overline{r}) = u_2(\overline{r})$ and $u_1(r) < u_2(r)$ for $r \in [0, \overline{r})$. Since u_1 and u_2 satisfy (5.4) with the equality sign replaced by inequalities, by the monotonicity of f, we have the following contradiction:

$$\begin{aligned} u_1(\overline{r}) &\le u_1(0) + \int_0^{\overline{r}} \left[\int_0^s N r^{N-1} K(r) f(u_1(r)) dr \right]^{1/N} ds \\ &< u_2(0) + \int_0^{\overline{r}} \left[\int_0^s N r^{N-1} K(r) f(u_2(r)) dr \right]^{1/N} ds \\ &\le u_2(\overline{r}). \end{aligned}$$

The proof is complete. □

5.2 Multiplicity and non-existence results for (5.2)

We examine two cases where K is such that (1.7) has no strictly convex solution.

The theorem below looks at a case with such a function K where f does not satisfy (1.2) and (5.2) has infinitely many solutions.

Theorem 5.3 *Suppose that K satisfies* **(K1)**, *and there exist constants $d_1, d_2 > 0$ and a function $p(t)$ of class $\mathcal{P}_\infty$ such that*

$$d_1 p(1-r) \le K(r) \le d_2 p(1-r) \text{ for all } r < 1 \text{ close to } 1.$$

Suppose that f satisfies **(f1)** *and there exist constants $\alpha \in (0, N)$ and $c_1, c_2 > 0$ such that*

$$c_1 u^\alpha \le f(u) \le c_2 u^\alpha \quad \text{for } u > 0.$$

Then (5.2) has infinitely many strictly convex solutions.

Proof It is obvious that $y(r) = \frac{1}{2}(1-r^2)$ satisfies

$$\begin{cases} (-1)^N y'^{N-1} y'' = r^{N-1}, \ r \in (0,1), \\ y'(0) = 0, \ y(1) = 0. \end{cases}$$

We modify $p(t)$ as in Sect. 3.1 and define $\sigma(t)$ by (3.3). Then we set

$$w(r) := c\big[\sigma(y(r))\big]^{N/(N-\alpha)} \quad \text{for } r \in [0,1) \text{ and some constant } c > 0.$$

We calculate

$$\begin{aligned} w' &= \frac{cN}{N-\alpha} \sigma^{\alpha/(N-\alpha)} \sigma' y', \\ w'' &= \frac{cN}{N-\alpha} \sigma^{\alpha/(N-\alpha)} \left[\sigma' y'' + \sigma'' (y')^2 + \frac{\alpha}{N-\alpha} \frac{(\sigma')^2}{\sigma} (y')^2 \right]. \\ (w')^{N-1} w'' &= \left(\frac{cN}{N-\alpha} \right)^N \sigma^{\frac{N\alpha}{N-\alpha}} (\sigma')^{N-1} \sigma'' (y')^{N-1} y'' \left[\frac{\sigma'}{\sigma''} + \frac{(y')^2}{y''} + \frac{\alpha}{N-\alpha} \frac{(\sigma')^2}{\sigma\sigma''} \frac{(y')^2}{y''} \right]. \end{aligned}$$

Using

$$(\sigma'(t))^{N-1} \sigma''(t) = (-1)^{N-1} p(t), \ \frac{\sigma'(t)}{\sigma''(t)} = -\frac{N \tilde{P}(t)}{p(t)}$$

and

$$y' = -r, \, y'' = -1,$$

we can simplify the above expression to obtain

$$(w')^{N-1} w'' = c^{N-\alpha} \left(\frac{N}{N-\alpha} \right)^N w^\alpha p(y) r^{N-1} \Delta(r),$$

with

$$\Delta(r) := \left[\frac{N \tilde{P}(y)}{p(y)} + r^2 + \frac{\alpha}{N-\alpha} \frac{(\sigma')^2}{\sigma\sigma''} r^2 \right].$$

We have

$$\frac{\sigma'(t)^2}{\sigma(t)\sigma''(t)} = \frac{[N\tilde{P}(t)]^{\frac{N+1}{N}}}{p(t)\int_t^\infty [N\tilde{P}(\tau)]^{1/N}d\tau}$$
$$= \frac{\displaystyle\int_t^\infty (N+1)[N\tilde{P}(s)]^{1/N}p(s)ds}{\displaystyle\int_t^\infty \Big\{-p'(s)\int_s^\infty [N\tilde{P}(\tau)]^{1/N}d\tau + p(s)[N\tilde{P}(s)]^{1/N}\Big\}ds}$$
$$\leq (N+1).$$

It follows that

$$0 \leq \frac{\alpha}{N-\alpha}\frac{\sigma'(y)^2}{\sigma(y)\sigma''(y)}r^2 \leq M_0 := \frac{\alpha}{N-\alpha}(N+1) \quad \text{for } r \in [0,1).$$

Since

$$\lim_{t\to 0}\frac{\tilde{P}(t)}{p(t)} = 0 \text{ and so } \lim_{r\to 1}\frac{\tilde{P}(y(r))}{p(y(r))} = 0,$$

the function

$$\Delta_1(r) := \frac{N\tilde{P}(y(r))}{p(y(r))} + r^2$$

is positive and continuous for $r \in [0, 1)$ with $\Delta_1(r) \to 1$ as $r \to 1$. Therefore we can find positive constants $m_1 < m_2$, depending on the function p, such that

$$m_1 \leq \Delta_1(r) \leq m_2 \quad \text{for } r \in [0,1).$$

It follows that

$$m_1 \leq \Delta(r) \leq m_2 + M_0 \quad \text{for } r \in [0,1).$$

Therefore

$$(w')^{N-1}w'' \leq c^{N-\alpha}\left(\frac{N}{N-\alpha}\right)^N w^\alpha p(y)r^{N-1}(m_2+M_0) \quad \text{for } r \in [0,1). \tag{5.6}$$

$$(w')^{N-1}w'' \geq c^{N-\alpha}\left(\frac{N}{N-\alpha}\right)^N w^\alpha p(y)r^{N-1}m_1 \quad \text{for } r \in [0,1). \tag{5.7}$$

Replacing $p(t)$ by $\epsilon p(2t)$ with $\epsilon > 0$ sufficiently small, we may assume that

$$K(r) \geq p((1-r)/2) \quad \text{for } r \in [0,1).$$

Therefore, due to $y(r) \geq (1-r)/2$, we have

$$p(y(r)) \leq p((1-r)/2) \leq K(r) \quad \text{for } r \in [0,1).$$

It then follows from (5.6) that

$$(w')^{N-1}w'' \leq c^{N-\alpha}\left(\frac{N}{N-\alpha}\right)^N w^\alpha K(r)r^{N-1}(m_2+M_0) \quad \text{for } r \in [0,1).$$

Hence if we take $c = \tilde{c}_1 > 0$ small enough,

$$w_1(r) := \tilde{c}_1[\sigma(y(r))]^{\frac{N}{N-\alpha}}$$

satisfies

$$(w_1')^{N-1}w_1'' \le r^{N-1}K(r)f(w_1) \quad \text{for } r \in [0, 1).$$

Next we construct a function $w_2(r)$ that satisfies the reversed inequality. By replacing $p(t)$ with $Mp(t)$, with $M > 0$ sufficiently large, we may assume that

$$p(1-r) \ge K(r) \quad \text{for } r \in [0, 1).$$

Then, due to $y(r) \le 1 - r$, we have

$$p(y(r)) \ge p(1-r) \ge K(r) \quad \text{for } r \in [0, 1).$$

Thus by (5.7) (with $\sigma(t)$ and m_1 determined by this new function $p(t)$), we have

$$(w')^{N-1}w'' \ge c^{N-\alpha}\left(\frac{N}{N-\alpha}\right)^N w^\alpha K(r)r^{N-1}m_1 \quad \text{for } r \in [0, 1),$$

and if we take $c = \tilde{c}_2$ large enough,

$$w_2(r) := \tilde{c}_2[\sigma(y(r))]^{\frac{N}{N-\alpha}}$$

satisfies

$$w_2(0) > w_1(0),\ (w_2')^{N-1}w_2'' \ge r^{N-1}K(r)f(w_2) \quad \text{for } r \in [0, 1).$$

For any $c \in (w_1(0), w_2(0))$, let v_c denote the unique solution of (5.3) with $v_0 = c$. By Lemma 5.2 we have $w_1(r) < v_c(r) < w_2(r)$ for $r \in [0, 1)$ and such that $v_c(r)$ is defined. Hence we can use Lemma 5.1 to see that $v_c(r)$ is defined for $r \in [0, 1)$ and $v_c'(r) > 0$, $v_c''(r) > 0$ in $(0, 1)$. Since $w_1(r) \to \infty$ as $r \to 1$, we have $v_c(r) \to \infty$ as $r \to 1$. Hence v_c is a strictly convex solution to (5.2). By varying c we thus obtain infinitely many solutions to (5.2). The proof is complete. □

The next theorem gives a case that K is such that (1.7) has no strictly convex solution, f satisfies (1.2), and (5.2) has no solution.

Theorem 5.4 *Suppose that f satisfies* **(f1)** *and there exist $\alpha > N$ and $b > 0$ such that*

$$f(u) \ge bu^\alpha \text{ for all large } u > 0.$$

Suppose that K satisfies **(K1)** *and for some $\beta \ge N + 1$, $c > 0$,*

$$K(r) \ge c(1-r)^{-\beta} \text{ for all } r < 1 \text{ close to } 1.$$

Then (5.2) has no solution.

Proof Suppose (5.2) has a solution $v(r)$. Then $v'(r) > 0$ and $v''(r) > 0$ in $(0, 1)$. Choose $r_0 \in (\frac{1}{2}, 1)$ close to 1 such that

$$f(v(r)) \ge bv^\alpha(r), \quad K(r) \ge c(1-r)^{-\beta} \quad \text{for } r \in [r_0, 1).$$

Then for $r \in [r_0, 1)$, we have

$$(v')^{N-1}v'' \ge bcr^{N-1}(1-r)^{-\beta}v^\alpha \ge bc(1-r_0)^{-\beta}r^{N-1}v^\alpha.$$

Set

$$c_0 := \left[bc(1-r_0)^{N+1-\beta}r_0^{N-1}\right]^{\frac{1}{\alpha-N}}$$

and

$$w(r) := c_0 v(r_0 + (1 - r_0)r), r \in [0, 1).$$

Then clearly

$$w(0) = c_0 v(r_0) > 0, \ w'(0) = c_0(1 - r_0)v'(r_0) > 0$$

and with $s = r_0 + (1 - r_0)r$, $r \in (0, 1)$,

$$\begin{aligned}(w')^{N-1}w'' &= c_0^N (1 - r_0)^{N+1} v'(s)^{N-1} v''(s)\\ &\geq c_0^N (1 - r_0)^{N+1} bc(1 - r_0)^{-\beta} s^{N-1} v^{\alpha}(s)\\ &\geq c_0^N bc(1 - r_0)^{N+1-\beta} (r_0 r)^{N-1} v^{\alpha}(s)\\ &= r^{N-1} w^{\alpha}.\end{aligned}$$

Since $\alpha > N$, by [12], the problem

$$\begin{cases} (W')^{N-1}W'' = r^{N-1}W^{\alpha}, \ r \in (0, \frac{1}{2}),\\ W'(0) = 0, \ W(\frac{1}{2}) = \infty \end{cases}$$

has a positive, strictly convex solution W. We show next that $w \leq W$ in $(0, 1/2)$. Indeed, the function $z(r) := w(r) - W(r)$ satisfies $z'(0) > 0, \ z(\frac{1}{2}) = -\infty$. Hence the maximum of $z(r)$ over $(0, 1/2)$ is achieved at some $r^* \in (0, 1/2)$. It follows that $z'(r^*) = 0$, $z''(r^*) \leq 0$, and so

$$0 < w'(r^*) = W'(r^*), 0 < w''(r^*) \leq W''(r^*).$$

We thus obtain

$$(r^*)^{N-1} w^{\alpha}(r^*) \leq (w'(r^*))^{N-1} w''(r^*) \leq (W'(r^*))^{N-1} W''(r^*) = (r^*)^{N-1} W^{\alpha}(r^*),$$

which leads to $w(r^*) \leq W(r^*)$, and hence $w(r) \leq W(r)$ in $[0, 1/2)$, as we wanted.

From $w(0) \leq W(0)$ and the definition of w we obtain

$$v(r_0) \leq \left[bc(1 - r_0)^{N+1-\beta} r_0^{N-1}\right]^{\frac{1}{N-\alpha}} W(0).$$

Since $\alpha > N$, $\beta \geq N + 1$, it follows that

$$v(r_0) \leq (bc)^{1/(N-\alpha)} 2^{(N-1)/(\alpha-N)} W(0) \text{ for all } r_0 \in (1/2, 1) \text{ close to } 1.$$

But as a solution to (5.2), we have $\lim_{r \to 1} v(r) = \infty$. This contradiction completes the proof. □

6 Proof of Theorem 1.4

Without loss of generality, and for simplicity of notation, we assume that $\eta = 0$. Due to (1.8),

$$\eta_0 := \int_0^{\infty} [(N + 1)K_* F(\tau)]^{-\frac{1}{N+1}} d\tau < \infty.$$

We denote

$$\delta_0 := \left[\frac{\eta_0(N - 1)}{N + 1}\right]^{\frac{N+1}{2N}}, \ R_0 := \eta_0 \delta_0^{-\frac{N-1}{N+1}} + \delta_0.$$

We then define $u_0(r)$ for $r \in [\delta_0, R_0)$ by

$$\int_0^{u_0(r)} [(N+1)K_* F(s)]^{-\frac{1}{N+1}} ds = \delta_0^{\frac{N-1}{N+1}} (r - \delta_0).$$

It is easily checked that $u_0(r)$ satisfies

$$\begin{cases} (u_0')^{N-1} u_0'' = \delta_0^{N-1} K_* f(u_0), \ u_0'(r) > 0 \quad \text{for } r \in (\delta_0, R_0), \\ u_0(\delta_0) = u_0'(\delta_0) = 0, \ u_0(R_0) = \infty. \end{cases}$$

For $\delta \in (0, 1)$, consider the initial value problem

$$\begin{cases} (v')^{N-1} v'' = r^{N-1} K_* f(v) \quad \text{for } r > 0, \\ v(0) = \delta, \ v'(0) = 0. \end{cases} \tag{6.1}$$

By Lemma 5.1, (6.1) has a unique positive solution $v_\delta(r)$ over a maximal interval of existence $[0, R_\delta)$. We prove that $R_\delta \le R_0$.

If $R_\delta \le \delta_0$, then clearly $R_\delta < R_0$. If $R_\delta > \delta_0$, we will show that $R_\delta \le R_0$ and $u_0(r) < v_\delta(r)$ for $r \in (\delta_0, R_\delta)$.

Since

$$(u_0')^{N-1} u_0'' = \delta_0^{N-1} K_* f(u_0) \le r^{N-1} K_* f(u_0) \quad \text{for } r \in (\delta_0, R_\delta),$$

we have

$$u_0(r) \le \int_{\delta_0}^r \left[\int_{\delta_0}^s N t^{N-1} K_* f(u_0(t)) dt \right]^{1/N} ds \quad \text{for } r \in (\delta_0, R_\delta). \tag{6.2}$$

We also have

$$\begin{aligned} v_\delta(r) &= v_\delta(\delta_0) + \int_{\delta_0}^r \left[(v_\delta'(\delta_0))^N + \int_{\delta_0}^s N t^{N-1} K_* f(v_\delta(t)) dt \right]^{1/N} ds \\ &> \int_{\delta_0}^r \left[\int_{\delta_0}^s N t^{N-1} K_* f(v_\delta(t)) dt \right]^{1/N} ds, \ r \in (\delta_0, R_\delta). \end{aligned} \tag{6.3}$$

Assume by way of contradiction that there exists $\overline{r} \in (\delta_0, R_\delta) \cap (0, R_0)$ such that $u_0(\overline{r}) = v_\delta(\overline{r})$. By $u_0(\delta_0) = 0 < v_\delta(\delta_0)$ and the continuity we can find a first such $\overline{r}$, i.e., $u_0(\overline{r}) = v_\delta(\overline{r})$ and $u_0(r) < v_\delta(r)$ for $r \in [\delta_0, \overline{r})$. From (6.2), (6.3) and the monotonicity of f we obtain

$$\begin{aligned} u_0(\overline{r}) &\le \int_{\delta_0}^{\overline{r}} \left[\int_{\delta_0}^s N t^{N-1} K_* f(u_0(t)) dt \right]^{1/N} ds \\ &< \int_{\delta_0}^{\overline{r}} \left[\int_{\delta_0}^s N t^{N-1} K_* f(v_\delta(t)) dt \right]^{1/N} ds \\ &< v_\delta(\overline{r}). \end{aligned}$$

This contradiction shows that $u_0(r) < v_\delta(r)$ for $r \in (\delta_0, R_\delta) \cap (\delta_0, R_0)$, which implies $R_\delta \le R_0$ since $u_0(R_0) = \infty$. We note that necessarily $v_\delta(R_\delta) = \infty$.

Suppose that Ω_{K_*} contains a ball of radius $R > R_0$; without loss of generality we may assume that the ball is $B_R(0)$. We show by a contradiction argument that (1.1) has no strictly convex solution over Ω. So suppose (1.1) has a strictly convex solution u over such a domain Ω. Since $R_\delta \le R_0 < R$, we have $\overline{B}_{R_\delta}(x_0) \subset \Omega_{K_*}$ if $|x_0| < R - R_0$. It follows that $u(x)$ is finite on $\partial B_{R_\delta}(x_0)$. Since $v_\delta(|x - x_0|) \to \infty$ as $x \to \partial B_{R_\delta}(x_0)$, and

$$M[v_\delta(|x - x_0|)] = K_* f(v_\delta(|x - x_0|)) \le K(x) f(v_\delta(|x - x_0|)) \quad \text{in } B_{R_\delta}(x_0),$$

we may now use Lemma 2.1 to deduce

$$u(x) \le v_\delta(|x - x_0|) \quad \text{in } B_{R_\delta}(x_0).$$

It follows that

$$u(x_0) \le v_\delta(0) = \delta, \quad \forall \delta \in (0, 1).$$

Letting $\delta \to 0$, we deduce $u(x_0) \le 0$. On the other hand, since $\eta = 0$ we also have $u(x) \ge 0$ in Ω. Thus we must have

$$u(x) = 0 \quad \text{for all } |x| < R - R_0.$$

This is a contradiction to the assumption that u is strictly convex. The proof is complete. □

Remark 6.1 Let us note that the above proof actually shows that, under the assumptions of Theorem 1.4, if Ω_{K^*} contains a ball of radius $R > R_0$, then there exists no strictly convex function $u \in C^2(\overline{B}_R)$ satisfying

$$M[u] \le K(x) f(u), \;\; u(x) > \eta \quad \text{in } B_R.$$

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