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Calculus of Variations

Finite time blowup of the n-harmonic flow on n-manifolds

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Abstract In this paper, we generalize the no-neck result of Qing and Tian (in Commun Pure Appl Math 50:295–310, 1997) to show that there is no neck during blowing up for the n-harmonic flow as $t \to \infty$. As an application of the no-neck result, we settle a conjecture of Hungerbühler (in Ann Scuola Norm Sup Pisa Cl Sci 4:593–631, 1997) by constructing an example to show that the n-harmonic map flow on an n-dimensional Riemannian manifold blows up in finite time for $n \geq 3$.

Mathematics Subject Classification 58E20 · 53C43

1 Introduction

Let M be an n-dimensional Riemannian manifold without boundary, and let N be another m-dimensional compact Riemannian manifold without boundary (isometrically embedded into $\mathbb{R}^L$). In local coordinates, a smooth Riemannian metric g of M can be represented by

$$g = g_{ij} dx_i \otimes dx_j,$$

where (g_{ij}) is a positive definitive symmetric $n \times n$ matrix. The volume element of $(M; g)$ is defined by

$$dv = \sqrt{|g|}dx \quad \text{with } |g| = \det (g_{ij}).$$

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For a map $u : M \to N \subset \mathbb{R}^L$, the n-energy functional of u is defined by

$$E_n(u; M) = \frac{1}{n}\int_M |\nabla u|^n \, dv,$$

where $|\nabla u|$ is the gradient norm given by

$$|\nabla u(x)|^2 = \sum_{\alpha,i,j} g^{ij}(x)\frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\alpha}{\partial x_j}$$

with $(g^{ij}) = (g_{ij})^{-1}$ the inverse matrix of (g_{ij}). A C^1-map u from M to N is said to be an n-harmonic map if u is a critical point of the n-energy functional; i.e. it satisfies

$$\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x_i}\left[|\nabla u|^{n-2}g^{ij}\sqrt{|g|}\frac{\partial}{\partial x_j}u\right] + |\nabla u|^{n-2}A(u)(\nabla u, \nabla u) = 0 \quad \text{in } M, \tag{1.1}$$

where A is the second fundamental form of N.

In 1964, Eells and Sampson [9] investigated the existence problem of harmonic maps in a homotopic class; i.e. "Given a smooth map $u_0 : M \to N$, is there a harmonic map u, which is homotopic to u_0?" (see [8]).

For the target manifold N with non-positive sectional curvature, Eells and Sampson [9] proved the first existence result of harmonic maps in a homotopic class by introducing the "heat flow method". The heat flow method transforms the existence problem to an evolution problem. Since then, questions on existence and regularity of harmonic maps and their flows have been attracted a great attention (see [8]). One of the key components of the heat flow method for answering the Eells–Sampson question is to prove existence of a global solution to the harmonic map flow. In 1975, Hamilton [11] proved local existence of the heat flow of harmonic map; i.e. the solutions of the heat flow of harmonic map exists locally. If the solution exists only in a finite interval $[0, T_{max})$ with $T_{max} < \infty$ and cannot be extended any further, then we say that the solution blows up in finite time T_{max}. In the two dimensional case (i.e. $n = 2$), Struwe [26] proved global existence of a unique weak solution to the harmonic map flow, where the solution is smooth except for a finite set of point singularities. In 1989, Coron and Ghidaglia [4] constructed the first example to show that for $n \geq 3$, the harmonic map heat flow from S^n into S^n blows up in finite time. However, when $n = 2$, the Dirichlet energy E_2 on the 2-dimensional manifold is conformally invariant on its critical dimension. In addition, Hélein [12] proved that any weak harmonic map from surfaces is smooth. Thus, it was widely believed during the time that the harmonic map heat flow would not blow up in finite time on the 2-dimension manifold. In 1992, Chang et al. [1] made a breakthrough by constructing a counter-example that harmonic map heat flow on S^2 can blow up in finite time.

In higher dimensions (i.e. $n > 2$), E_n is also conformally invariant on the n-dimensional manifold M. Motivated by the Eells–Sampson question on harmonic maps, one can ask whether a given map from an n-dimensional manifold to another manifold can be deformed into an n-harmonic map. Related to this question, Hungerbühler [17] studied the n-harmonic flow in the following setting:

$$\frac{\partial u}{\partial t} = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x_i}\left[|\nabla u|^{n-2}g^{ij}\sqrt{|g|}\frac{\partial}{\partial x_j}u\right] + |\nabla u|^{n-2}A(u)(\nabla u, \nabla u) \tag{1.2}$$

and generalized the result of Struwe [26] by proving that there exists a global weak solution $u : M \times [0, +\infty) \to N$ of (1.2) with initial value u_0 such that $u \in \mathbf{C}^{1,\alpha}(M \times (0, +\infty)\backslash\{\Sigma_k \times T_k\}_{k=1}^L)$ for a finite number of times $\{T_k\}_{k=1}^L$ and a finite number of singular closed sets

$\Sigma_k \subset M$ for $k = 1, \ldots, L$ with an integer L, depending only M and u_0. However, it is still unknown whether the singular set Σ_k of the flow (1.2) at the singular T_k is finite. In order to sort out this issue, the second author [13] introduced a rectified n-harmonic map flow from an n-dimensional from M to N and proved existence of a global solution, which is regular except for a finite number of points, of the rectified n-harmonic map flow.

Based on the fundamental result of Chang et al. [1] for $n = 2$, it is an interesting question whether the n-harmonic flow (1.2) blows up in finite time for $n \geq 3$. Supported by some numerical evidence, Hungerbühler ([16,17]) conjectured the phenomenon of finite time blow-up of the n-harmonic flow for $n \geq 3$. Later, Chen et al. [2] followed the method of Chang–Ding–Ye to construct an example that the n-harmonic flow (1.2) blows up in finite time for $n = 3$. However, due to the nonlinearity and degeneracy of the n-harmonic maps, they [2] also pointed out that their proofs could not be applied to the cases when $n > 3$. Therefore, the conjecture of Hungerbühler for $n > 3$ has remained open since then.

On the other hand, Qing and Tian [24] suggested a program to prove the finite time blow-up of the harmonic map flow for $n = 2$ through an application of the no-neck result for the harmonic map flow as $t \to \infty$ and constructing a special target manifold N with a proper topology. Recently, Chen and Li [3] verified the Qing-Tian program by constructing a special target manifold N with a proper topology to show that the harmonic map flow blows up at finite time for $n = 2$. Later, Liu and Yin [20] successfully applied this idea to construct a proper manifold N to show that the bi-harmonic maps flow on 4-manifolds blows up at finite time.

In this paper, we apply the Qing-Tian program to confirm the conjecture of Hungerbühler on the n-harmonic map flow. Firstly, we define:

Definition 1.1 u is said to be a regular solution to the n-harmonic map flow (1.2) in $M \times (0, T]$ if $u \in C^0(M \times (0, T]; N)$ with $T \leq \infty$ is a solution of (1.2) satisfying

$$\int_0^T \int_M \left|\nabla \left(|\nabla u|^{\frac{n-2}{2}} \nabla u\right)\right|^2 + |\nabla u|^{2n} \, dv \, dt \leq C(T).$$

We generalize the no-neck result of Qing and Tian [24] to the n-harmonic map flow as follow:

Theorem 1 *Let u be a regular solution to the flow (1.2) in $M \times [0, \infty)$ with initial value $u_0 \in C^1(M, N)$. For a sequence $t_i \to \infty$, there is a sub-sequence, still denoted by t_i, such that as $t_i \to \infty$, $u(x, t_i)$ converges to an n-harmonic map u_∞ in $C^{1,\alpha}_{loc}\left(M \backslash \{x^1, \ldots, x^L\}, N\right)$ for some positive $\alpha < 1$, where u_∞ can be extended to $C^{1,\alpha}(M, N)$. Moreover, we have*

i. *(Energy identity) There are a finite number of n-harmonic maps $\omega_{k,l}$ (also called bubbles) on S^n for $k = 1, \ldots, L$ and $l = 1, \ldots, J_k$ such that*

$$\lim_{t_i \nearrow \infty} E_n(u(\cdot, t_i); M) = E_n(u_\infty; M) + \sum_{k=1}^{L} \sum_{l=1}^{J_k} E_n(\omega_{k,l}; S^n).$$

ii. *(No-neck result) There is no neck between the limiting map and bubbles; i.e. the image*

$$u_\infty(M) \cup \bigcup_{k,l} \omega_{k,l}(S^n)$$

is a connected set.

One of the fundamental rules for bubble blowing is the bubble-neck decomposition. During the bubbling procedure, the energy identity implies that the energy is conserved. This means that the loss of energy under the limiting process can be recovered by the energy of a finite number of bubbles. Readers can refer to the pioneering work on the energy identity by Jost [18], Parker [22] regarding the harmonic maps from surfaces and by Ding and Tian [6] for the harmonic map flow. For n-harmonic maps with $n \geq 3$, the isolated singularities are removable due to Duzaar and Fuchs [7] and the energy identity was provided by Wang and Wei [27] for a sequence of approximate n-harmonic maps. In particular, one can use the standard blow-up argument as in Ding and Tian [6] to reduce the multiple bubble problem to the single bubble case. See more details for the bubble-neck decomposition of n-harmonic maps in [13]. These results allow us to construct the bubbling argument in the setting of the n-harmonic maps.

In order to provide an example to show that the n-harmonic flow can blow up in finite time, the key step is to generalize the no-neck result of Qing and Tian [24]. However, those no-neck results in [19] and [24] heavily rely on a key estimate in Ding–Tian's work (Lemma 2.1, [6]) which only works for the case of harmonic maps. To settle this open problem, we generalize the Ding–Tian estimate to the context of n-harmonic maps (Lemma 3.1) and then apply it to prove the no-neck property for the n-harmonic map flow.

Secondly, we apply Theorem 1 to prove the main result of this paper:

Theorem 2 *Let X be any closed manifold of dimension $m > n$ with nontrivial $\pi_n(X)$, and let $N = X \# T^m$ be the connected sum of X with the torus T^m. Then there are infinitely many initial maps $u_0 : S^n \to N$ such that the n-harmonic map flow (1.2) with initial value u_0 blows up in finite time.*

Besides the finite time blow-up result on the harmonic map flow by Chen and Li [3], another related evolution problem to the n-harmonic map flow is the bi-harmonic map flow on 4-dimensional manifolds. Liu and Yin in [21] established the no-neck result of a sequence of biharmonic maps. Later, Liu and Yin [20] generalized the no-neck result to a sequence of approximate biharmonic maps. By combining the no-neck result with a construction of a proper target manifold, they introduced a concept of width of bi-harmonic maps in the covering space to show that the bi-harmonic map flow blows up in finite time. These results provide a skeleton for the proof of Theorem 2.

This paper is organized as follows. In Sect. 2, we show asymptotical behavior of the solution of the n-harmonic flow as $t \to \infty$. In Sect. 3, we generalize Ding–Tian's estimate and apply it to prove the no-neck result for the n-harmonic flow. In Sect. 4, we construct an example to prove Theorem 2 and settle the Hungerbühler conjecture.

2 some estimates and asymptotic behavior of the n-harmonic map flow

In order to study asymptotic behavior of the n-harmonic map flow, we begin with some basic estimates. We recall some results from [17] on the n-harmonic map flow.

Lemma 2.1 *Let $u(t)$ be a regular solution to the n-harmonic map flow (1.2) in $M \times [0, T]$ with initial value $u(0) = u_0$. Then for each s with $0 < s \leq T$, we have*

$$\int_M \frac{1}{n} |\nabla u(s)|^n \, dv + \int_0^t \int_M \left| \frac{\partial u}{\partial t} \right|^2 dv \, dt \leq \int_M \frac{1}{n} |\nabla u_0|^n \, dv.$$

Lemma 2.2 *Let u be a regular solution to the n-harmonic map flow* (1.2)*. Let η be a cut-off function in B_r such that $\eta = 1$ in $B_{\frac{r}{2}}$, $|\nabla\eta| \le \frac{C}{r}$ and $|\eta| \le 1$ in B_r. Then we have*

$$\int_{B_r} |\nabla u|^{2n}\eta^n\,dv \le C\left(\int_{B_r} |\nabla u|^n\,dv\right)^{\frac{2}{n}} \int_{B_r} \left(|\nabla^2 u|^2\,|\nabla u|^{2n-4}\,\eta^n + |\nabla u|^n|\nabla\eta|^n\right)dv. \tag{2.1}$$

and

$$\int_{B_r} |\nabla^2 u|^2|\nabla u|^{2n-4}\eta^n\,dv \le C\int_{B_r} |\nabla u|^{2n}\eta^n + |\nabla u|^n(\eta^n + |\nabla\eta|^n)\,dv. \tag{2.2}$$

Proof By using the Hölder and Sobolev inequalities, we have

$$\begin{aligned}\int_{B_r} |\nabla u|^{2n}\eta^n\,dv &= \int_{B_r} |\nabla u|\left(|\nabla u|^{2n-1}\eta^n\right)dv\\
&\le \left(\int_{B_r} |\nabla u|^n\,dv\right)^{\frac{1}{n}}\left(\int_{B_r}\left(|\nabla u|^{2n-1}\eta^n\right)^{\frac{n}{n-1}}\,dv\right)^{\frac{n-1}{n}}\\
&\le C\left(\int_{B_r} |\nabla u|^n\,dv\right)^{\frac{1}{n}}\int_{B_r} |\nabla(|\nabla u|^{2n-1}\eta^n)|\,dv\\
&\le C\left(\int_{B_r} |\nabla u|^n\,dv\right)^{\frac{1}{n}}\int_{B_r}\left(|\nabla^2 u|\,|\nabla u|^{2n-2}\,\eta^n + |\nabla u|^{2n-1}|\nabla\eta|\,\eta^{n-1}\right)dv.\end{aligned} \tag{2.3}$$

By Young's inequality, we have

$$\begin{aligned}&\left(\int_{B_r} |\nabla u|^n\,dv\right)^{\frac{1}{n}}\int_{B_r} |\nabla^2 u|\,|\nabla u|^{2n-2}\,\eta^n\,dv\\
&\le \left(\int_{B_r} |\nabla u|^n\,dv\right)^{\frac{2}{n}}\int_{B_r} |\nabla^2 u|^2\,|\nabla u|^{2n-4}\,\eta^n dv + \frac{1}{2}\int_{B_r} |\nabla u|^{2n}\,\eta^n dv.\end{aligned} \tag{2.4}$$

Similarly, we have

$$\begin{aligned}\int_{B_r} |\nabla u|^{2n-1}|\nabla\eta|\,\eta^{n-1}\,dv &= \int_{B_r} |\nabla u|^n|\nabla u|^{n-1}|\nabla\eta|\,\eta^{n-1}\,dv\\
&\le C\int_{B_r} |\nabla u|^n|\nabla\eta|^n\eta^{n-1}\,dv + C\int_{B_r} |\nabla u|^{2n}\eta^{n-1}\,dv.\end{aligned} \tag{2.5}$$

Combining (2.3), (2.4) with (2.5), we have

$$\int_{B_r} |\nabla u|^{2n}\eta^n\,dv \le C\left(\int_{B_r} |\nabla u|^n\,dv\right)^{\frac{2}{n}}\int_{B_r}\left(|\nabla^2 u|^2\,|\nabla u|^{2n-4}\,\eta^n + |\nabla u|^n|\nabla\eta|^n\right)dv. \tag{2.6}$$

Using the Ricci identity, we have

$$\nabla_k\nabla_l\left(|\nabla u|^{n-2}\nabla u\right) = \nabla_l\nabla_k\left(|\nabla u|^{n-2}\nabla u\right) + R_M\#\left(|\nabla u|^{n-2}\nabla u\right)$$

with the Riemannian curvature R_M. Integrations by parts twice yield that

$$
\begin{aligned}
&\int_{B_r} \left\langle \nabla_k(|\nabla u|^{n-2}\nabla_k u),\, \nabla_l(|\nabla u|^{n-2}\nabla_l u)\right\rangle \eta^n\, dv \\
&= -\int_{B_r} \left\langle \nabla_l\nabla_k(|\nabla u|^{n-2}\nabla_k u),\, |\nabla u|^{n-2}\nabla_l u\right\rangle \eta^n\, dv \\
&\quad - \int_{B_r} \left\langle \nabla_k(|\nabla u|^{n-2}\nabla_k u),\, |\nabla u|^{n-2}\nabla_l u\right\rangle \nabla_l \eta^n\, dv \\
&= \int_{B_r} \left\langle \nabla_l(|\nabla u|^{n-2}\nabla_k u),\, \nabla_k(|\nabla u|^{n-2}\nabla_l u)\right\rangle \eta^n\, dv \\
&\quad + \int_{B_r} \left\langle R_M\#(|\nabla u|^{n-2}\nabla_k u),\, |\nabla u|^{n-2}\nabla_l u\right\rangle \eta^n\, dv \\
&\quad + \int_{B_r} \left\langle \nabla_l(|\nabla u|^{n-2}\nabla_k u),\, |\nabla u|^{n-2}\nabla_l u\right\rangle \nabla_k \eta^n\, dv \\
&\quad - \int_{B_r} \left\langle \nabla_k(|\nabla u|^{n-2}\nabla_k u),\, |\nabla u|^{n-2}\nabla_l u\right\rangle \nabla_l \eta^n\, dv.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\int_{B_r} \left\langle \nabla_l(|\nabla u|^{n-2}\nabla_k u),\, \nabla_k(|\nabla u|^{n-2}\nabla_l u)\right\rangle \eta^n\, dv \\
&= \int_{B_r} \sum |\nabla u|^{2n-4}|\nabla_{lk} u|^2\, \eta^n + \left\langle \nabla_l(|\nabla u|^{n-2})\nabla_k u,\, \nabla_k(|\nabla u|^{n-2})\nabla_l u\right\rangle \eta^n\, dv \\
&\quad + 2\int_{B_r} \left\langle \nabla_l(|\nabla u|^{n-2})\nabla_k u,\, |\nabla u|^{n-2}\nabla_{kl} u\right\rangle \eta^n\, dv \\
&\geq \int_{B_r} (|\nabla u|^{2n-4}|\nabla^2 u|^2 + |\nabla_k(|\nabla u|^{n-2})\nabla_k u|^2)\eta^n\, dv \\
&\quad + 2(n-2)\int_{B_r} |\nabla u|^{n-4}|\nabla|\nabla u||^2\eta^n\, dv. \qquad (2.7)
\end{aligned}
$$

Combining (2.6) with (2.7), this implies

$$
\begin{aligned}
&\int_{B_r} |\nabla^2 u|^2|\nabla u|^{2n-4}\eta^n\, dv \\
&\leq \int_{B_r} |\nabla\cdot(|\nabla u|^{n-2}\nabla u)|^2\eta^n\, dv + C\int_{B_r} |\nabla u|^{2n-2}\eta^{n-2}(\eta^2 + |\nabla\eta|^2)\, dv \\
&\leq C\int_{B_r} |\nabla u|^{2n}\eta^n + |\nabla u|^n(\eta^n + |\nabla\eta|^n)\, dv. \qquad (2.8)
\end{aligned}
$$

We finish the proof by combining (2.6) with (2.8). □

Lemma 2.3 *There exists a sufficiently small constant $\varepsilon_1 > 0$ such that if u is a regular solution of (1.2) on $B_{2R_0}(x_0) \times \left[t_0 - 2R_0^n, t_0\right]$ satisfying*

$$
\sup_{t_0 - 2R_0^n \leq t \leq t_0} \int_{B_{2R_0}(x_0)} |\nabla u(x,t)|^n\, dv < \varepsilon_1,
$$

we have

$$\int_{t_0-2R_0^n}^{t_0}\int_{B_{R_0}(x_0)} \left|\nabla^2 u\right|^2 |\nabla u|^{2n-4} + |\nabla u|^{2n}\, dvdt$$
$$< C \sup_{t_0-2R_0^n\le t\le t_0}\int_{B_{2R_0}(x_0)} |\nabla u(x,t)|^n\, dv$$

for some constant $C > 0$.

Proof Lemma 2.3 was proved by Hungerbühler by using an extension of the Ladyzhenskaya–Solonnikov–Nikolaevna inequality (see Lemma 5 of [12]). Herewith, we would like to give a slightly different approach by using Lemma 2.2.

Multiplying (1.2) by $\phi^n\nabla\cdot(|\nabla u|^{n-2}\nabla u)$ and using Lemma 2.2 by choosing a sufficiently small ε_1 in above inequalities yields that

$$\begin{aligned}
&\int_{B_{2R_0}(x_0)} \left|\nabla\cdot(|\nabla u|^{n-2}\,\nabla u)\right|^2 \phi^n\, dv\\
&\le \frac{1}{2}\int_{B_{2R_0}(x_0)} \left|\nabla\cdot(|\nabla u|^{n-2}\nabla u)\right|^2 \phi^n\, dv + C\int_{B_{2R_0}(x_0)}\left(\left|\frac{\partial u}{\partial t}\right|^2 + |\nabla u|^{2n}\right)\phi^n\, dv\\
&\le \frac{3}{4}\int_{B_{2R_0}(x_0)} \left|\nabla\cdot(|\nabla u|^{n-2}\nabla u)\right|^2 \phi^n\, dv + C\left(1+\frac{1}{R_0^n}\right)\int_{B_{2R_0}(x_0)} |\nabla u|^n\, dv\\
&\quad + C\int_{B_{2R_0}(x_0)}\left|\frac{\partial u}{\partial t}\right|^2 dv.
\end{aligned}$$

Together with Lemma 2.2, we obtain

$$\begin{aligned}
&\int_{B_{R_0}(x_0)} |\nabla u|^{2n} + |\nabla^2 u|^2|\nabla u|^{2n-4}\, dv\\
&\le C\int_{B_{2R_0}(x_0)} (R_0^{-n}+1)|\nabla u|^n + \left|\frac{\partial u}{\partial t}\right|^2 dv\\
&\le C(R_0^{-n}+1)E_n(u_0) + C\int_{B_{2R_0}(x_0)}\left|\frac{\partial u}{\partial t}\right|^2 dv. \qquad (2.9)
\end{aligned}$$

□

Lemma 2.4 *Let u be a regular solution to (1.2). Then there exists a positive constant* ε_1 *such that if for some* $R_0 > 0$ *the inequality*

$$\sup_{t_0-2R_0^n\le t\le t_0}\int_{B_{2R_0}(x_0)} |\nabla u(x,t)|^n\, dv < \varepsilon_1$$

holds, we have

$$\sup_{[t_0-R_0^n,t_0]\times B_{R_0}(x_0)} |\nabla u|^n\, dv \le CR_0^{-n},$$

where C depending on M is a constant independent of R_0.

Proof The proof is due to Hungerbühler in [17] for $R_0 = 1$. If $R_0 \ne 1$, one can prove it by a re-scaling argument. □

Lemma 2.5 *Let $u : M \to N$ be a regular solution to the Eq. (1.2). Then there is a small constant $\varepsilon_1 > 0$ such that if the inequality*

$$\sup_{t_0 - R_0^n \le t \le t_0} \int_{B_{2R_0}(x_0)} |\nabla u(x,t)|^n < \varepsilon_1,$$

holds for some positive R_0, then $\|u\|_{C^{1,\alpha}([t_0 - \frac{1}{2}R_0^n, t_0] \times B_{R_0}(x_0))}$ is bounded by a constant depending on $E(u_0)$ and R_0.

Proof As pointed out by Hungerbühler in [17], we can apply the result of DiBenedetto-Friedman [5] to obtain a bound of $\|u\|_{C^{1,\alpha}([t_0 - \frac{1}{2}R_0^n, t_0] \times B_{R_0}(x_0))}$. □

Lemma 2.6 (Local energy inequality under small condition) *Let u be a regular solution of (1.2) on $B_{2R_0}(x_0) \times [0, T]$. There exists a sufficiently small constant $\varepsilon_1 > 0$ such that if*

$$\sup_{t_0 - T \le t \le t_0} \int_{B_{2R_0}(x_0)} |\nabla u(x,t)|^n < \varepsilon_1,$$

then we have for every $x \in B_R(x_0)$, any $R \le R_0$ and any two constants τ and s in $(t_0 - T, t_0]$

$$\int_{B_R(x_0)} |\nabla u|^n(\cdot, s)\, dv \le \int_{B_{2R}(x_0)} |\nabla u|^n(\cdot, \tau)\, dv + C \int_s^\tau \int_{B_{2R}(x_0)} |\partial_t u|^2\, dv\, dt$$
$$+ C \left(\frac{(\tau - s)}{R^n} \int_{B_{2R}(x_0)} |\nabla u|^n\, dv \int_s^\tau \int_{B_{2R}(x_0)} |\partial_t u|^2\, dv\, dt \right)^{1/2}$$

for some constant C.

Proof Let ϕ be a cut-off function with support in $B_{2R_0}(x_0)$ such that $\phi = 1$ in $B_{R_0}(x_0)$, $|\nabla \phi| \le C R_0^{-1}$ and $|\phi| \le 1$ in $B_{2R_0}(x_0)$. Multiplying (1.2) by $\phi^n \partial_t u$, we have

$$\int_{B_{2R_0}(x_0)} |\frac{\partial u}{\partial t}|^2 \phi^n\, dv = \int_{B_{2R_0}(x_0)} \left\langle \nabla \cdot (|\nabla u|^{n-2} \nabla u), \frac{\partial u}{\partial t} \right\rangle \phi^n\, dv$$
$$= - \int_{B_{2R_0}(x_0)} \left\langle |\nabla u|^{n-2} \nabla u, \frac{\partial \nabla u}{\partial t} \phi^n + \frac{\partial u}{\partial t} \phi^{n-1} \nabla \phi \right\rangle dv$$
$$\ge -\frac{1}{n} \frac{d}{dt} \int_{B_{2R_0}(x_0)} |\nabla u|^n \phi^n\, dv - C \int_{B_{2R_0}(x_0)} |\nabla u|^{n-1} |\frac{\partial u}{\partial t}| \phi^{n-1} |\nabla \phi|\, dv.$$

Note

$$\int_{B_{2R_0}(x_0)} |\nabla u|^{n-1} \left| \frac{\partial u}{\partial t} \right| \phi^{n-1} |\nabla \phi|\, dv$$
$$\le \left(\int_{B_{2R_0}(x_0)} \left| \frac{\partial u}{\partial t} \right|^2 \phi^n\, dv \right)^{1/2} \left(\int_{B_{2R_0}(x_0)} |\nabla u|^{2n-2} \phi^{n-2} |\nabla \phi|^2\, dv \right)^{1/2}$$

since

$$\int_{B_{2R_0}(x_0)} |\nabla u|^{2n-2} \phi^{n-2} |\nabla \phi|^2\, dv \le C \int_{B_{2R_0}(x_0)} |\nabla u|^{2n} \phi^n + |\nabla u|^n |\nabla \phi|^n\, dv.$$

Therefore, the claim is proved. □

Proposition 2.1 *Let u be a regular solution to (1.2) in $M \times [0, \infty)$. For a sequence $t_i \to \infty$, there is a sub-sequence, still denoted by $t_i \to \infty$, such that $u(\cdot, t_i)$ converges to an n-harmonic maps u_∞ locally in $C^{1,\alpha}(M \setminus \{x^1, \ldots, x^J\}; N)$ with some positive $\alpha < 1$, where u_∞ can be extended regularly on M.*

Proof By Lemma 2.1, we know that $\int_0^\infty \int_M |\partial_t u|^2 \, dv dt$ is finite, so we may choose a sub-sequence $\{t_i\}$ such that as $t_i \to \infty$, $\partial_t u(\cdot, t_i) \to 0$ strongly in $L^2(M)$ and $\int_{t_i - 1}^{t_i} \int_M |\partial_t u(\cdot, t)|^2 \, dv \, dt \to 0$. Moreover, there is a constant $\varepsilon_0 > 0$ such that the singular points $\{x^1, \ldots, x^J\}$ are defined by the condition

$$\limsup_{t_i \to \infty} E_n(u(t_i); B_R(x^k)) \geq \varepsilon_0$$

for any $R \in (0, 2R_0]$ with some fixed $R_0 > 0$.

For each $x_0 \in M \setminus \{x^1, \ldots, x^J\}$, there is a sufficiently small $R_0 > 0$ such that $B_{2R_0}(x_0) \subset M \setminus \{x^1, \ldots, x^J\}$ and for all i,

$$\int_{B_{2R_0}(x_0)} |\nabla u(x, t_i)|^n \, dv < \varepsilon_0 \leq \frac{\varepsilon_1}{2},$$

where ε_1 is the constant defined in Lemma 2.6.

By Lemma 2.6, we have for any $s \in [t_i - 2R_0^n, t_i]$ and for sufficiently large i

$$\begin{aligned} \int_{B_{R_0}(x_0)} |\nabla u|^n(\cdot, s) \, dv &\leq \int_{B_{2R_0}(x_0)} |\nabla u|^n(\cdot, t_i) \, dv + C \int_{t_i - 1}^{t_i} \int_{B_{2R_0}(x_0)} |\partial_t u|^2 \, dv \, dt \\ &+ C \left(\frac{(t_i - s)}{R_0^n} \int_{B_{2R_0}(x_0)} |\nabla u|^n \, dv \int_{t_i - 1}^{t_i} \int_{B_{2R_0}(x_0)} |\partial_t u|^2 \, dv \, dt \right)^{1/2} \\ &< \varepsilon_1. \end{aligned}$$

By Lemma 2.4, we have

$$\sup_{t \in [t_i - R_0^n, t_i], \, x \in B_{R_0}(x_0)} |\nabla u|^n(x, t) \, dv \leq C R_0^{-n}.$$

Then using Lemma 2.5, there is a uniform bound of $\|u(\cdot, t_i)\|_{C^{1,\alpha}(B_{\frac{1}{2}R_0}(x_0))}$, so $u(x, t_i)$ convergence to u_∞ in $C^{1,\beta}(B_{\frac{1}{2}R_0})$ and hence in $C^{1,\beta}_{loc}(M \setminus \{x^1, \ldots, x^J\})$ with $\beta < \alpha$, where $u_\infty \in C^{1,\beta}_{loc}(M \setminus \{x^1, \ldots, x^J\})$ is an n-harmonic map. By the removable singularities of an n-harmonic map, u_∞ can be extended to $C^{1,\alpha}(M)$. □

3 No neck result between the limiting map and bubbles as $t \to \infty$

In this section, we generalize the no-neck result of Qing and Tian [24] to the case of the n-harmonic flow. As suggested by Struwe [26] and Qing [23], the existence of solutions of the heat flow for harmonic maps can be proved by a method of "Palais-Smale sequences" with tension fields $\tau(u) \in L^2$. In the context of n-harmonic maps, the tension field $\tau(u)$ of u is defined as follows:

$$\tau(u) := \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_i} \left[|\nabla u|^{n-2} g^{ij} \sqrt{|g|} \frac{\partial}{\partial x_j} u \right] + |\nabla u|^{n-2} A(u)(\nabla u, \nabla u), \tag{3.1}$$

where A is the second fundamental form of N.

If $\tau(u) = 0$, u is an n-harmonic map. When $\tau(u) \in L^2(M)$ and u satisfies an extra smoothness assumption in (3.1), we define u to be a regular approximated n-harmonic map as follow (see [27] for details):

Definition 3.1 We define a map $u \in W^{1,n}(M; N) \cap C^0(M; N)$ to be a regular approximated n-harmonic map if it satisfies the following conditions:

1. $\nabla \left(|\nabla u|^{\frac{n-2}{2}} \nabla u \right) \in L^2(M)$;
2. There exist $\varepsilon > 0$, $\alpha \in (0, 1)$ and $C > 0$ depending only on M, N and $\|\tau(u)\|_{L^2}$ such that for any $B_{2_r} \in M$ and $E_n(u; B_{2_r}) \leq \varepsilon$, then

$$u \in C^\alpha(B_r; N) \quad \text{and} \quad [u]_{C^\alpha(B_r(x))} \leq C.$$

Let $\{u_i\}$ be a sequence of regular approximated n-harmonic maps with uniform bounds of $E_n(u_i)$ and $\|\tau(u_i)\|_{L^2(M)}$. Wang and Wei [27] proved that $\{u_i\}$ converges to an n-harmonic map u_∞ strongly in $W^{1,q}_{loc}\left(M \backslash \{x^1, \ldots, x^L\}\right)$ for any $q < 2n$, where u_∞ can be extended to $C^{1,\alpha}(M)$. By reducing multi bubbles into a single bubble, they proved that there are a finite number of n-harmonic maps $\omega_{k,l}$ on S^n with $k = 1, \ldots, L$ and $l = 1, \ldots J_k$ such that

$$\lim_{t_i \nearrow \infty} E_n(u_i; M) = E_n(u_\infty; M) + \sum_{k=1}^{L} \sum_{l=1}^{J_k} E_n(\omega_{k,l}, S^n).$$

Then we have

Theorem 3 *Let $\{u_i\}$ be the sequence of regular approximated n-harmonic maps with uniform bounds of $E_n(u_i)$ and $\|\tau(u_i)\|_{L^2(M)}$, and let $\omega_{k,l}$ be the above bubbles. Then there is no neck between the limiting map u_∞ and bubbles $\omega_{k,l}$;*
i.e. the image

$$u_\infty(M) \cup \bigcup_{k,l} \omega_{k,l}(S^n)$$

is a connected set.

We begin with the following ε-regularity estimate for approximated n-harmonic maps. In particular, we generalize the Ding–Tian estimate (see [6], Lemma 2.1), which is a crucial estimate to the proof of no-neck result.

Lemma 3.1 *For $n \geq 2$, let $u \in W^{1,n}(M, N) \cap C^0(M, N)$ be an approximated n-harmonic map. Then there exists a small constant $\varepsilon > 0$ such that if $E_n(u, B_r) \leq \varepsilon$ then*

$$\|u\|_{osc\left(B_{\frac{r}{2}}\right)} \leq C \left(\int_{B_r} |\nabla u|^n \, dv \right)^{\frac{1}{2(n-1)}} + C r^{\frac{n}{2(n-1)}} \left(\int_{B_r} |\tau(u)|^2 \, dv \right)^{\frac{1}{2(n-1)}}. \tag{3.2}$$

Proof Let ϕ be a cut-off function in $C_0^\infty(B_r)$ with $\phi \equiv 1$ in $B_{\frac{r}{2}}$ and $|\nabla \phi| \leq C r^{-1}$ and set $\bar{u} = \frac{1}{|B_{\frac{3}{4}r}|} \int_{B_{\frac{3}{4}r}} u \, dv$. For a sufficient small $a > 0$, we apply Theorem 7.17 in [10] with $p = \frac{2n(n-1)}{n-2+a} > n$, $\gamma = 1 - \frac{n}{p}$, and the Poincaré inequality to obtain

$$\|u\|_{osc(B_{\frac{r}{2}})} = \sup_{x,y\in B_{\frac{r}{2}}} |u(x)-u(y)| \le 2 \sup_{x\in B_{\frac{3}{4}r}} |(u(x)-\overline{u})\,\phi(x)|$$

$$\le Cr^{1-\frac{n-2+a}{2(n-1)}} \left(\int_{B_{\frac{3}{4}r}} |\nabla[(u-\overline{u})\,\phi]|^{\frac{2n(n-1)}{n-2+a}}\,dv\right)^{\frac{n-2+a}{2n(n-1)}}$$

$$\le Cr^{\frac{n-a}{2(n-1)}} \left(\int_{B_{\frac{3}{4}r}} |\nabla u(x)|^{\frac{2n(n-1)}{n-2+a}}\,dv\right)^{\frac{n-2+a}{2n(n-1)}}$$

$$+ Cr^{\frac{n-a}{2(n-1)}} \left(\int_{B_{\frac{3}{4}r}} |(u(x)-\overline{u})\nabla\phi|^{\frac{2n(n-1)}{n-2+a}}\,dv\right)^{\frac{n-2+a}{2n(n-1)}}$$

$$\le Cr^{\frac{n-a}{2(n-1)}} \left(\int_{B_{\frac{3}{4}r}} |\nabla u|^{\frac{2n(n-1)}{n-2+a}}\,dv\right)^{\frac{n-2+a}{2n(n-1)}}$$

$$= Cr^{\frac{n-a}{2(n-1)}} \left(\int_{B_{\frac{3}{4}r}} \left||\nabla u|^{n-2}\nabla u\right|^{\frac{2n}{n-2+a}}\,dv\right)^{\frac{n-2+a}{2n}\frac{1}{(n-1)}}. \tag{3.3}$$

By using the Sobolev–Poincaré inequality on B_1 (page 174 in [10]), we have for $p<n$

$$\left(\int_{B_1} |f-f_{B_1}|^{p^*}\,dv\right)^{1/p^*} \le C\left(\int_{B_1} |\nabla f|^p\,dv\right)^p.$$

Choosing $p=\frac{2n}{n+a}<2$ such that $q=\frac{2n}{n-2+a}=\frac{np}{n-p}=p^*$ and re-scaling from B_1 to B_r and using Hölder's inequality, we have

$$\left(\frac{r^{(n-1)q}}{r^n}\int_{B_r} \left||\nabla u|^{n-2}\,\nabla u\right|^q\,dv\right)^{\frac{1}{q}}$$

$$\le C\left(\frac{r^{2n}}{r^n}\int_{B_r} \left|\nabla(|\nabla u|^{n-2}\,\nabla u)\right|^2\,dv\right)^{1/2} + \frac{C}{r}\int_{B_r} |\nabla u|^{n-1}\,dv. \tag{3.4}$$

By the Hölder inequality, we have

$$\left(\frac{1}{r}\int_{B_r} |\nabla u|^{n-1}\,dv\right)^{\frac{1}{n-1}} \le C\left(\int_{B_{\frac{3}{4}r}} |\nabla u|^n\,dv\right)^{\frac{1}{n}}.$$

Then substituting (3.4) into (3.3), we obtain

$$\|u\|_{osc(B_{\frac{r}{2}})} \le Cr^{\frac{n-a}{2(n-1)}} \left(\int_{B_{\frac{3}{4}r}} \left||\nabla u|^{n-2}\,\nabla u\right|^q\,dv\right)^{\frac{1}{q}\frac{1}{(n-1)}}$$

$$\le Cr^{\frac{n-a}{2(n-1)}} r^{\frac{n}{q(n-1)}-1} r^{\frac{n}{2(n-1)}} \left(\int_{B_r} \left|\nabla(|\nabla u|^{n-2}\,\nabla u)\right|^2\,dv\right)^{\frac{1}{2(n-1)}}$$

$$+ Cr^{\frac{n-a}{2(n-1)}} r^{\frac{n}{q(n-1)}-1} \left(\frac{1}{r}\int_{B_r} |\nabla u|^{n-1}\,dv\right)^{\frac{1}{n-1}}$$

$$
\begin{aligned}
&= Cr^{\frac{n}{2(n-1)}} \left(\int_{B_{\frac{3}{4}r}} \left|\nabla(|\nabla u|^{n-2}\nabla u)\right|^2 dv \right)^{\frac{1}{2(n-1)}} + C\left(\frac{1}{r}\int_{B_r} |\nabla u|^{n-1}\, dv\right)^{\frac{1}{n-1}} \\
&\le C\left(r^n \int_{B_{\frac{3}{4}r}} \left|\nabla(|\nabla u|^{n-2}\nabla u)\right|^2 dv\right)^{\frac{1}{2(n-1)}} + C\left(\int_{B_{\frac{3}{4}r}} |\nabla u|^n\, dv\right)^{\frac{1}{n}} \qquad (3.5)
\end{aligned}
$$

by noting that $\frac{n-a}{2(n-1)} + \frac{n}{q(n-1)} - 1 = 0$ with $q = \frac{2n}{n-2+a}$.

Multiplying (3.1) by $\nabla \cdot (|\nabla u|^{n-2}\nabla u)\, \eta^n$, we have

$$
\int_{B_r} |\nabla \cdot (|\nabla u|^{n-2}\nabla u)|^2 \eta^n\, dv \le \int_{B_r} |\nabla \cdot (|\nabla u|^{n-2}\nabla u)|(\tau(u) + C|\nabla u|^n)\eta^n\, dv.
$$

Now, using Young's inequality, we have

$$
\int_{B_r} |\nabla \cdot (|\nabla u|^{n-2}\nabla u)|^2 \eta^n\, dv \le C\int_{B_r} (|\tau(u)|^2 + |\nabla u|^{2n})\eta^n\, dv.
$$

Using Lemma 2.2 again, it yields that

$$
\int_{B_{\frac{3}{4}r}} |\nabla u|^{2n} + |\nabla^2 u|^2 |\nabla u|^{2n-4}\, dv \le C\int_{B_r} (1 + r^{-n})|\nabla u|^n + |\tau(u)|^2\, dv.
$$

Therefore

$$
\begin{aligned}
&\left(r^n \int_{B_{\frac{3}{4}r}} |\nabla u|^{2n} + |\nabla^2 u|^2 |\nabla u|^{2n-4}\, dv\right)^{\frac{1}{2(n-1)}} \\
&\le C\left(\int_{B_r} |\nabla u|^n + r^n|\tau(u)|^2\, dv\right)^{\frac{1}{2(n-1)}} \\
&\le C\left(\int_{B_r} |\nabla u|^n\, dv\right)^{\frac{1}{2(n-1)}} + Cr^{\frac{n}{2(n-1)}}\left(\int_{B_r} |\tau(u)|^2\, dv\right)^{\frac{1}{2(n-1)}}.
\end{aligned}
$$

We finish the proof by putting these estimates together. □

To analyze the behavior of approximated n-harmonic maps on the neck region, we need the following Pohozaev type inequality, which was proved in [27]:

Lemma 3.2 *For $n \ge 2$, let $u \in W^{1,n}(M, N) \cap C^{1,\alpha}(M, N)$ to be a regular approximated n-harmonic map with tension field $\tau(u) \in L^2(M)$. Then, for any ball $B_r \subset M$, we have*

$$
\int_{\partial B_r} |\nabla u|^n\, ds \le C(n)\left(\int_{\partial B_r} |\nabla_T u|^n\, ds + \int_{B_r} |\tau(u)|\,|\nabla u|\, dv\right), \qquad (3.6)
$$

where $\nabla_T u$ is the tangential gradient on the boundary ∂B_r.

Proof For completeness, we sketch the proof here. Multiplying (3.1) by $x \cdot \nabla u$ and integrating over B_r, we have

$$
\begin{aligned}
&\int_{B_r} \langle \tau(u),\ x \cdot \nabla u \rangle\, dv \\
&= \frac{1}{n} \int_{B_r} \left\langle x\,,\ \nabla(|\nabla\, u|)^n \right\rangle dv + \int_{B_r} |\nabla u|^n\, dv - r \int_{\partial B_r} |\nabla u|^{n-2} \left| \frac{\partial u}{\partial r} \right|^2 ds \\
&= \frac{r}{n} \int_{\partial B_r} |\nabla u|^n\, ds - r \int_{\partial B_r} |\nabla u|^{n-2} \left| \frac{\partial u}{\partial r} \right|^2 ds,
\end{aligned}
$$

where we use the fact that

$$
\int_{B_r} \left\langle x\,,\ \nabla(|\nabla\, u|)^n \right\rangle dv = r \int_{\partial B_r} |\nabla u|^n\, ds - n \int_{B_r} |\nabla u|^n\, dv
$$

and $|\nabla u|^2 = \left|\frac{\partial u}{\partial r}\right|^2 + |\nabla_T u|^2$.

Rearranging the inequality and by adding $(n-1) \int_{\partial B_r} |\nabla u|^{n-2}\, |\nabla_T u|^2\, ds$ to the both sides we have

$$
(n-1) \int_{\partial B_r} |\nabla u|^n\, ds \le n \int_{B_r} |\tau(u)||\nabla u|\, dv + n \int_{\partial B_r} |\nabla u|^{n-2} \left(|\nabla_T u|^2\right) ds.
$$

Then the claim follows from using Young's inequality. □

Now we prove Theorem 3.

Proof By using the standard bubbling arguments as in [6] and [27], one can reduce multiple bubbles to a single bubble. We assume that 0 is the single blowing up point of $\{u_i\}$ and there is only one bubble in B_1. Then, we follow the approach of [24] and [19] to extend the no-neck result to the case of the n-harmonic map flow.

Suppose $r_n\, R = 2^{-j_n}$ and $\delta = 2^{-j_0}$ for any $j_0 < j < j_n$. Then, we denote

$$
L_j = \min\{j - j_0,\ j_n - j\} \quad \text{and} \quad P_{j,t} = B_{2^{t-j}} \backslash B_{2^{-t-j}} \quad \text{for} \quad t \in (0, L_j].
$$

For sufficiently large i, we assume that

$$
E_n(u_i, B_{2^{1-j}} \backslash B_{2^{-j}}) \le \varepsilon^{2(n-1)}, \qquad \text{for any } j_0 \le j \le j_n. \tag{3.7}
$$

Let

$$
h_{i,j,t}(2^{\pm t - j}) = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} u_i(2^{\pm t - j}, \theta)\, d\theta
$$

and

$$
h_{i,j,t}(r) = h_{i,j,t}(2^{t-j}) + \left(h_{i,j,t}(2^{-t-j}) - h_{i,j,t}(2^{t-j}) \right) \frac{\ln(2^{-t+j}\, r)}{-2t\ \ln 2}. \tag{3.8}
$$

Note that the tangential derivative of $h_{i,j,t}(r)$ is zero in n-dimensional spherical coordinates. Therefore, the Laplace operator can be reduced to the following form:

$$
\Delta h_{i,j,t} = \frac{d^2\, h_{i,j,t}}{d^2 r} + \frac{n-1}{r} \frac{dh_{i,j,t}}{dr},
$$

which yields that

$$\begin{aligned}\operatorname{div}(|\nabla h_{i,j,t}|^{n-2}\nabla h_{i,j,t}) &= \left|\frac{d\,h_{i,j,t}}{dr}\right|^{n-2}\left(\frac{d^2 h_{i,j,t}}{d^2 r}+\frac{n-1}{r}\frac{dh_{i,j,t}}{dr}\right)\\ &\quad+\frac{n-2}{2}\left|\frac{d\,h_{i,j,t}}{dr}\right|^{n-4}\frac{dh_{i,j,t}}{dr}\frac{d}{dr}\left|\frac{dh_{i,j,t}}{dr}\right|^2\\ &=(n-1)\left|\frac{d\,h_{i,j,t}}{dr}\right|^{n-2}\left(\frac{d^2 h_{i,j,t}}{d^2 r}+\frac{1}{r}\frac{d\,h_{i,j,t}}{dr}\right)\\ &=0.\end{aligned}$$

This implies that $h_{i,j,t}(r)$ is also a symmetric n-harmonic map for $r\in[2^{-t-j},\ 2^{t-j}]$. By the well-know result of the n-Laplace operator, we note that

$$\begin{aligned}&\int_{P_{j,t}}|\nabla(u_i-h_{i,j,t})|^n\,dv\\ &\le C\int_{P_{j,t}}\left\langle(|\nabla u_i|^{n-2}\nabla u_i-|\nabla h_{i,j,t}|^{n-2}\nabla h_{i,j,t}),\ \nabla(u_i-h_{i,j,t})\right\rangle dv.\end{aligned}$$

for some constant $C>0$.

By integration by parts, we have

$$\begin{aligned}&\int_{P_{j,t}}\left\langle(|\nabla u_i|^{n-2}\nabla u_i-|\nabla h_{i,j,t}|^{n-2}\nabla h_{i,j,t}),\ \nabla(u_i-h_{i,j,t})\right\rangle dv.\\ &=-\int_{P_{j,t}}\left\langle div(|\nabla u_i|^{n-2}\nabla u_i),\ (u_i-h_{i,j,t})\right\rangle dv\\ &\quad+\int_{\partial P_{j,t}}\left\langle(|\nabla u_i|^{n-2}(u_i)_r-|\nabla h_{i,j,t}|^{n-2}(h_{i,j,t})_r),\ (u_i-h_{i,j,t})\right\rangle dv\\ &=\int_{P_{j,t}}\left\langle\left(A(u_i)(\nabla u_i,\nabla u_i)|\nabla u|^{n-2}+\tau(u_i)\right),\ (u_i-h_{i,j,t})\right\rangle dv\\ &\quad+\int_{\partial P_{j,t}}\left\langle\left(|\nabla u_i|^{n-2}(u_i)_r-|\nabla h_{i,j,t}|^{n-2}(h_{i,j,t})_r\right),\ (u_i-h_{i,j,t})\right\rangle dv.\end{aligned}$$

By Lemma 3.1, we obtain

$$\begin{aligned}\|u_i-h_{i,j,t}\|_{C^0(P_{j,t})} &\le \|u_i-h_{i,j,t}(2^{j-t})\|_{C^0(P_{j,t})}+\|u_i-h_{i,j,t}(2^{-j-t})\|_{C^0(P_{j,t})}\\ &\le 2\|u_i\|_{osc(P_{j,t})}\\ &\le C\left(\int_{P_{j-1,t}\cup P_{j,t}\cup P_{j+1,t}}|\nabla u|^n\right)^{\frac{1}{2(n-1)}}\\ &\quad+C\,(2^{t-j+1})^{\frac{n}{2(n-1)}}\left(\int_{B_{2^{t-j+1}}}|\tau(u)|^2\right)^{\frac{1}{2(n-1)}}\\ &\le C\left(\varepsilon+\delta^{\frac{n(t+1)}{2(n-1)}}\right)\le C\varepsilon.\end{aligned}\tag{3.9}$$

By (3.9), we have

$$\begin{aligned}
&\int_{P_{j,t}} |\nabla(u_i - h_{i,j,t})|^n \, dv \\
&\le C \int_{P_{j,t}} \left\langle |\nabla u_i|^{n-2} \nabla u_i - |\nabla h_{i,j,t}|^{n-2} \nabla h_{i,j,t}, \; \nabla(u_i - h_{i,j,t}) \right\rangle dv \\
&\le C \int_{P_{j,t}} \left\langle (A(u_i)(\nabla u_i, \nabla u_i) |\nabla u|^{n-2} + \tau(u_i)), \; (u_i - h_{i,j,t}) \right\rangle dv \\
&\quad + C \int_{\partial P_{j,t}} \left\langle |\nabla u_i|^{n-2} (u_i)_r, \; (u_i - h_{i,j,t}) \right\rangle ds \\
&\le C \left(\varepsilon \int_{P_{j,t}} |\nabla u_i|^n \, dv + \varepsilon \|\tau(u_i)\|_{L^2(P_{j,t})} 2^{(t-j)\frac{n}{2}} \right) \\
&\quad + C \int_{\partial P_{j,t}} |\nabla u_i|^{n-2} |(u_i)_r| \, |u_i - h_{i,j,t}| \, ds \\
&= C(\varepsilon(I_1 + 2^{(t-j)\frac{n}{2}}) + I_2),
\end{aligned}$$

where we set

$$I_1 := f_j(t) = \int_{P_{j,t}} |\nabla u_i|^n \, dv \quad \text{and} \quad I_2 := \int_{\partial P_{j,t}} |\nabla u_i|^{n-2} \, |(u_i)_r| \, |u_i - h_{i,j,t}| \, ds.$$

Using the fact that $\frac{d\,2^x}{dx} = \ln 2(2^x)$, this implies

$$f_j'(t) = \ln 2 \left(2^{t-j} \int_{\{2^{t-j}\} \times S^{n-1}} |\nabla u_i|^n \, ds + 2^{-t-j} \int_{\{2^{-t-j}\} \times S^{n-1}} |\nabla u_i|^n \, ds \right).$$

By the Poincaré inequality and Hölder's inequality, we have

$$\begin{aligned}
I_2 &= \int_{\partial P_{j,t}} (|\nabla u_i|^{n-2} \, |(u_i)_r|) \, |u_i - h_{i,j,t}| \, ds \\
&\le \int_{\{2^{t-j}\} \times S^{n-1}} |\nabla u_i|^{n-1} |u_i - h_{i,j,t}| \, ds + \int_{\{2^{-t-j}\} \times S^{n-1}} |\nabla u_i|^{n-1} |u_i - h_{i,j,t}| \, ds \\
&\le \left(\int_{\{2^{t-j}\} \times S^{n-1}} (|\nabla u_i|^{n-1})^{\frac{n}{n-1}} \, ds \right)^{\frac{n-1}{n}} \left(\int_{\{2^{t-j}\} \times S^{n-1}} |u_i - h_{i,j,t}|^n \, ds \right)^{\frac{1}{n}} \\
&\quad + \left(\int_{\{2^{-t-j}\} \times S^{n-1}} (|\nabla u_i|^{n-1})^{\frac{n}{n-1}} \, ds \right)^{\frac{n-1}{n}} \left(\int_{\{2^{-t-j}\} \times S^{n-1}} |u_i - h_{i,j,t}|^n \, ds \right)^{\frac{1}{n}} \\
&\le \left(\int_{\{2^{t-j}\} \times S^{n-1}} (|\nabla u_i|)^n \, ds \right)^{\frac{n-1}{n}} \left(\int_{\{2^{t-j}\} \times S^{n-1}} |u_i - h_{i,j,t}|^n \, ds \right)^{\frac{1}{n}} \\
&\quad + \left(\int_{\{2^{-t-j}\} \times S^{n-1}} (|\nabla u_i|)^n \, ds \right)^{\frac{n-1}{n}} \left(\int_{\{2^{-t-j}\} \times S^{n-1}} |u_i - h_{i,j,t}|^n \, ds \right)^{\frac{1}{n}} \\
&\le C f_j'(t).
\end{aligned} \tag{3.10}$$

Note that $h_{i,j,t}$ is the average of u_i over S^{n-1}. Then we can estimate the tangential energy by

$$\begin{aligned}\int_{P_{j,t}} |\nabla_T u_i|^n \, dv &\le \int_{P_{j,t}} \left(|(u_i - h_{i,j,t})_r|^2 + |\nabla_T u_i|^2\right)^{\frac{n}{2}} \, dv \\ &= \int_{P_{j,t}} |\nabla (u_i - h_{i,j,t})|^n \, dv \\ &\le C(\varepsilon I_1 + 2^{(t-j)\frac{n}{2}} + I_2). \end{aligned} \tag{3.11}$$

By Lemma 3.2, given a regular approximated n-harmonic map u_i, for all $r \in [\lambda_n R, \delta]$, we have

$$\int_{\partial B_r} |\nabla u_i|^n \, ds \le C(n) \left(\int_{\partial B_r} |\nabla_T u_i|^n \, ds + \int_{B_r} |\tau(u_i)| \, |\nabla u_i| \, dv \right). \tag{3.12}$$

Integrating (3.12) in r from $r = 2^{-t-j}$ to $r = 2^{t-j}$, using Hölder's inequality and (3.11), we obtain

$$\begin{aligned} f_j(t) &= \int_{P_{j,t}} |\nabla u_i|^n \, dv \\ &\le C \left(\int_{P_{j,t}} |\nabla_T u_i|^n \, dv + \int_{2^{-t-j}}^{2^{t-j}} \|\tau(u_i)\|_{L^2(B_r)} \, \|\nabla u_i\|_{L^2(B_r)} \, dr \right) \\ &\le C(\varepsilon(I_1 + 2^{(t-j)\frac{n}{2}}) + I_2)) + C \int_0^{2^{t-j}} \|\tau(u_i)\|_{L^2(B_r)} \, \|\nabla u_i\|_{L^n(B_r)} r^{\frac{n-2}{2}} \, dr \\ &\le C(\varepsilon I_1 + 2^{(t-j)\frac{n}{2}} + I_2) \le C(\varepsilon f_j(t) + 2^{(t-j)\frac{n}{2(n-1)}}) + C f_j'(t). \end{aligned} \tag{3.13}$$

Let $\lambda_n = \frac{n}{2(n-1)} \ln 2$. Choosing ε sufficiently small in (3.13), we have

$$0 \le f_t'(t) - \frac{1}{C} f_j(t) + C e^{\lambda_n (t-j)}.$$

Now, assuming that $\lambda_n > \frac{1}{C}$ for a sufficiently large C, it implies

$$0 \le \left(e^{-\frac{t}{C}} f_j(t) \right)' + C e^{\lambda_n (t-j)} \, e^{-\frac{t}{C}}. \tag{3.14}$$

Integrating (3.14) in t over $[2, L_j]$, this gives

$$\begin{aligned} f_j(2) &\le C \left(e^{\frac{-L_j}{C}} f_j(L_j) + e^{-\lambda_n j} \, e^{\left(\lambda_n - \frac{1}{C}\right) L_j} \right) \\ &\le C \left(e^{\frac{-L_j}{C}} f_j(L_j) + e^{-\lambda_n j} \, e^{\frac{-j}{C}} \right), \end{aligned} \tag{3.15}$$

where we note that

$$P_j = B_{2^{1-j}} \backslash B_{2^{-j}}, \quad P_{j-1} \cup P_j \cup P_{j+1} = B_{2^{2-j}} \backslash B_{2^{-1-j}} \quad \text{and} \quad f_j(2) = \int_{P_{j,2}} |\nabla u_i|^n \, dv.$$

Applying Lemma 3.1 on P_j, we have

$$\begin{aligned}\|u_i\|_{osc(P_j)} &\le C\left(\int_{P_{j-1}\cup P_j\cup P_{j+1}}|\nabla u_i|^n\,dv\right)^{\frac{1}{2(n-1)}}\\ &\quad + (2^{-j})^{\frac{n}{2(n-1)}}\,C\left(\int_{B_{2^{2-j}}}|\tau(u_i)|^2\,dv\right)^{\frac{1}{2(n-1)}}\\ &\le C(f_j^{\frac{1}{2(n-1)}}(2)+e^{-\lambda_n j}).\end{aligned} \tag{3.16}$$

For $j \ge L_j$, under the assumption (3.7) at the beginning of the proof, we can choose a small δ such that $f_j(L_j) \le \varepsilon^{2(n-1)}$ and (3.15) yields that

$$\begin{aligned}f_j^{\frac{1}{2(n-1)}}(2) &\le C\left(e^{\frac{-L_j}{C}}\,f_j(L_j)+e^{-\lambda_n j}\,e^{\left(\lambda_n-\frac{1}{C}\right)L_j}\right)^{\frac{1}{2(n-1)}}\\ &\le C\left(e^{\frac{-L_j}{C}}\,\varepsilon + e^{\frac{-j}{C}}\right).\end{aligned} \tag{3.17}$$

Substituting (3.17) into (3.16) and summing over $j_0 \le j \le j_n$, we have

$$\begin{aligned}\|u_i\|_{osc(B_{2\delta}\setminus B_{2r_nR})} &\le \sum_{j=j_0}^{j_n}\|u_i\|_{osc(P_j)}\\ &\le C\sum_{j=j_0}^{j_n}\left((e^{-\frac{L_j}{C}}\varepsilon + e^{\frac{-j}{C}})+e^{-\lambda_n j}\right)\\ &\le C\left(\sum_{i=0}^{\infty}e^{-\frac{i}{C}}\varepsilon + \sum_{j=j_0}^{\infty}e^{\frac{-j}{C}}\right)\\ &\le C\left(\varepsilon+\delta^{\frac{1}{C}}\right).\end{aligned}$$

Since

$$\|u_i\|_{osc(B_\delta\setminus B_{2r_nR})} = \sup_{x,y\in B_\delta\setminus B_{2r_nR}}|u_i(x)-u_i(y)|$$

is controlled by δ, this implies that

$$u(B_1)\cup\omega_1(\mathbb{R}^n)$$

is a connected set. Thus, there is no neck between the limiting map and the bubbles for regular approximated n-harmonic maps with tension fields bounded in L^2. □

Now we complete the proof of Theorem 1.

Proof of Theorem 1 We briefly describe the procedure of "bubble blowing" by following the idea from Ding and Tian [6]. First, we recall that the removable singularity theorem of n-harmonic maps [7]. Moreover, recall the the gap theorem: there is a constant $\varepsilon_g > 0$ such that if u is an n-harmonic map on S^n satisfying $\int_{S^n}|\nabla u|^n < \varepsilon_g$, then u is a constant on S^n.

Let $u(x,t)$ be a regular solution of the n-harmonic flow in $M\times[0,\infty)$. As $t_i\to\infty$, it was showed in Proposition 2.1 that a subsequence of $u_i := u(t_i)$ converges to an n-harmonic map u_∞ locally in $C^{1,\alpha}(M\backslash\{x^1,\ldots,x^L\})$. Furthermore, there is a constant $\varepsilon_0>0$ such that the singular points (energy concentration points) $\{x^k\}$ are defined by the condition

$$\limsup_{t_i\to\infty} E_n(u(t_i); B_R(x^k)) \geq \varepsilon_0$$

for any $R\in(0,R_0]$, with some fixed $R_0>0$.

Let x^1 be a singular point. Then we find sequences $x_i^1\to x^1$ such that

$$\left|\nabla u(x_i^1)\right| = \max_{B_{R_0}(x^1)} |\nabla u(x,t_i)|, \quad r_i^1 = \frac{1}{\left|\nabla u(x_i^1)\right|}\to 0.$$

In the neighborhood $B_{R_0}(x^1)$ of the singularity x^1, we define the rescaled map

$$u_i^1(x) = u(x_i^1 + r_i^1 x, t_i). \tag{3.18}$$

Then the rescaled map u_i^1 satisfies

$$(r_i^1)^n\frac{\partial u}{\partial t} = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x_i}\left[|\nabla u|^{n-2}g^{ij}\sqrt{|g|}\frac{\partial u}{\partial x_j}\right] + |\nabla u|^{n-2}A(u)(\nabla u,u). \tag{3.19}$$

Now, u_i^1 converge to $u_{1,\infty}$ locally in $\mathbb{R}^n$ as $i\to\infty$, and $u_{1,\infty}$ can be extended to a nontrivial n-harmonic map on S^n (see [7]). We call $\tilde{u}_{1,\infty}$ to be the first bubble, which satisfies

$$E_n(u_{1,\infty};\mathbb{R}^n) = \lim_{R\to\infty}\lim_{t_i\to\infty} E_n(u_i^1; B_R(0)) = \lim_{R\to\infty}\lim_{t_i\to\infty} E_n(u_i; B_{Rr_i^1}(x^1)). \tag{3.20}$$

At each singular point x^k, there are finitely many blow-up points $x_i^{k,l}$ and bubbles $\{\omega_{k,l}\}_{l=1}^{J_k}$ on $\mathbb{R}^n$ (see details in [13]); i.e. at each k, there are sequences $x_i^{k,l}\to p^{k,l}$ for some $p^{k,l}$ and $r_i^{k,l}\to 0$ with $\lim_{i\to\infty}\frac{r_i^{k,l}}{r_i^{k,l-1}}=\infty$ such that passing to a subsequence, $u_i^{k,l}(x) := u_i(x_i^{k,l} + r_i^{k,l}x)$ converges to $\omega^{k,l}$, where $\omega_{k,l}$ is an n-harmonic map in $\mathbb{R}^n$. These mean that there are finite numbers $r_{i,k}$, finite points $x_i^{k,l}$, positive constants $R_{k,l}$, $\delta_{k,l}$ and finitely many number of non-trivial n-harmonic maps $\omega_{k,l}$ on $\mathbb{R}^n$ such that

$$\begin{aligned}
&\lim_{i\to\infty} E_n(u_i; M)\\
&= E_n\left(u_\infty; M\backslash\{x_k\}_{k=1}^L\right) + \sum_{k=1}^L\sum_{l=1}^{J_k} E_n(\omega_{k,l};\mathbb{R}^n)\\
&\quad + \sum_{k=1}^L\sum_{l=1}^{J_k}\lim_{R_{k,l}\to\infty}\lim_{\delta_{k,l}\to 0}\lim_{i\to\infty} E_n(u_i^{k,l}; B_{\delta_{k,l}}\backslash B_{R_{k,l}r_i^{k,l}}(x_i^{k,l})).
\end{aligned} \tag{3.21}$$

Moreover, at each neck region $B_{\delta_{k,l}}\backslash B_{R_{k,l}r_i^{k,l}}(x_i^{k,l})$ in (3.21), for all i sufficiently large, we have

$$\int_{B_{2r}\backslash B_r(x_i^{k,l})} |\nabla u_{k,l,i}|^n dv \leq \varepsilon \tag{3.22}$$

for all $r\in(\frac{R_{k,l}r_i^{k,l}}{4}, 2\delta_{k,l})$, where ε is a fixed constant to be chosen sufficiently small. In fact, (3.22) is a crucial observation by Ding and Tian [6]. This implies that the neck energy can be controlled during bubbling procedure by reducing multiple bubbles to a single bubble case, which leads to the proof of the energy identity for harmonic maps in [6]. For the case of n-harmonic maps, we complete the proof of the energy identity by using a result of Wang and Wei (see Theorem B of [27]) . Now, we can choose a subsequence of time $t_i \to \infty$ such that $\lim_{i\to\infty} \left\| \frac{\partial u}{\partial t}(\cdot, t_i) \right\|_{L^2(M)}$ is bounded. This completes a proof of Theorem 1 by using Theorem 3. □

4 Finite-time blow-up of the *n*-harmonic map flow

As an application of the "no-neck" result, we will construct an example that the n-harmonic flow with initial value u_0 blows up in finite time. The proof here is to use similar ideas in [20]. Due to that there are several modifications for the case of n-harmonic maps, we give a proof for completeness here.

4.1 Width of *n*-harmonic maps in the covering space

We follow the geometric setting as in Sections 3–4 of [20] to construct an example of finite time blowup of the n-harmonic flow. The idea is to construct a proper target manifold N such that we can find infinitely many initial maps $u_0 : S^n \to N$ such that the n-harmonic flow blows up in finite time.

For $m > n$, let the target manifold $N = X \# T^m$ be the connected sum of X with the torus T^m. Here X is a closed m-dimensional manifold with nontrivial $\pi_n(X)$. Thus, there exists a smooth map $h : S^n \to X$ such that it is not homotopic to a constant map. Note that N can be separated into N_1 and N_2 by an embedding sphere $S^{m-1} \subset N$. In particular, $N\backslash N_1$ and $N\backslash N_2$ are homeomorphic to X and T^m respectively. For each $l = 0, 1, 2, \ldots$, let U_l denote a small neighborhood of p_l, which is diffeomorphic to a m-dimensional ball and $V \subset X$ denotes an open set which is diffeomorphic to a ball.

$\mathbb{R}^m$ is the universal cover of T^m with $G = \mathbb{Z}^m$ as the covering transformations group. Now, for any point $p_0 \in \mathbb{R}^m$, its orbit under the transformation group G is the set $\{p_l\}_{l=0}^{\infty} \subset \mathbb{R}^m$. Let U_0 be a small ball in $\mathbb{R}^m$ and its orbit under the transformation group G is a family of balls $\{U_l\}_{l=0}^{\infty} \subset \mathbb{R}^m$. Now, we can find a cover of N by modifying $\mathbb{R}^m$. For each $l = 0, \ldots, \infty$, we remove the small ball U_l from $\mathbb{R}^m$ for $l = 1, 2, \ldots$ by adding a copy of $X\backslash V$, which we identify ∂U_l by the boundary of $X\backslash V$. We denote by X_l the copy of $X\backslash V$ through ∂U_l. This new complete and non-compact manifold is denoted by $\tilde{N}$ and the transformation group G act naturally on $\tilde{N}$. Let $\tilde{N}$ to be a cover of N and $\tilde{g}$ be the corresponding lift metric.

For a continuous map $u : S^n \to N$, we define its "width" of u in a set $S \subset S^{n-1}$ through its lift map $\tilde{u}$ in the covering space $(\tilde{N}, \tilde{g})$ by

$$\mathcal{W}(u; S) := \sup_{x,y\in S} d_{(\tilde{N},\tilde{g})}\left(\tilde{u}(x), \tilde{u}(y)\right). \tag{4.1}$$

We begin with the following lemma that gives an upper bound for the width.

Lemma 4.1 *(Bounded width lemma) If $u : \mathbb{R}^n \to N$ is an n-harmonic map with $E_n(u) < C_1$ for a constant $C_1 > 0$, there exists a constant C_2, depending only on C_1 and N such that the $\mathcal{W}(u; S^n) < C_2$.*

Proof We prove this by contradiction. Suppose that the statement is not true. Then we can find a sequence of n-harmonic maps $\{u_i\}_{i=1}^{\infty}$ with their energy bounded by the constant C_1 such that their width $\mathcal{W}(u_i; S^n)$ can not be bounded as $i \to \infty$.

According the above bubble-neck decomposition, as $t_i \to \infty$, it was showed in Proposition 2.1 that a subsequence of u_i converges to an n-harmonic map u_∞ locally in $C^{1,\alpha}(M\backslash\{x^1, \ldots, x^L\})$.

At each singular point x_k, there are sequences $x_i^{k,l} \to p^{k,l}$ for some $p^{k,l}$ and $r_i^{k,l} \to 0$ with $\lim_{i\to\infty} \frac{r_i^{k,l}}{r_i^{k,l-1}} = \infty$ such that passing to a subsequence, $u_i^{k,l}(x) := u_i(x_i^{k,l} + r_i^{k,l}x)$ converges to $\omega^{k,l}$, where $\omega_{k,l}$ is an n-harmonic map in $\mathbb{R}^n$. These mean that there are finite numbers $r_{i,k}$, finite points $x_i^{k,l}$, positive constants $R_{k,l}$, $\delta_{k,l}$ and finitely many number of non-trivial n-harmonic maps $\omega_{k,l}$ on $\mathbb{R}^n$. Moreover, at each neck region $B_{\delta_{k,l}} \backslash B_{R_{k,l}r_i^{k,l}}(x_i^{k,l})$ in (3.21), for all i sufficiently large, we have

$$\int_{B_{2r}\backslash B_r(x_i^{k,l})} |\nabla u_{k,l,i}|^n dv \le \varepsilon \tag{4.2}$$

for all $r \in (\frac{R_{k,l}r_i^{k,l}}{4}, 2\delta_{k,l})$, where ε is a fixed constant to be chosen sufficiently small. Then

$$\begin{aligned}\lim_{i\to\infty} W(u_i; S^n) &= \lim_{i\to\infty} \sup_{x,y\in S^n} d_{(\tilde{N},\tilde{g})}(\tilde{u_i}(x), \tilde{u_i}(y)) \\ &\le \lim_{\delta\to 0}\lim_{i\to\infty} W\left(u_i; S^n\backslash \cup_{k=1}^L B_\delta(x_k)\right) \\ &\quad + \lim_{\delta_{k,l}\to 0}\lim_{i\to\infty}\sum_{k=1}^L\sum_{\tilde{l}=1}^{\tilde{J}_k} W(u_i^{k,\tilde{l}}; B_{R_{k,\tilde{l}}}(0)\backslash \cup_{j=1}^{L_{k,\tilde{l}}} B_{\delta_{k,l}}(x_i^{k,j})) \\ &\quad + \sum_{k=1}^L\sum_{l=1}^{J_k} \lim_{R_{k,l}\to\infty}\lim_{\delta_{k,l}\to 0}\lim_{i\to\infty} W(u_i^{k,l}; B_{\delta_{k,l}}\backslash B_{R_{k,l}r_i^{k,l}}(x_i^{k,l})),\end{aligned} \tag{4.3}$$

where we note that $\{x_i^{k,l}\}_{l=1}^{J_k} = \cup_{\tilde{l}=1}^{\tilde{J}_k}\{x_i^{k,j}\}_{j=1}^{L_{k,\tilde{l}}}$ is the set of totally blowing points and that $L_{k,\tilde{l}}$ may not exist (This corresponds to the case of a single bubble).

Now we will estimate the width of the above region of bubbling, the neck domain and the base separately. Let $\tilde{u}_i^{k,l}$ denote the lift of $u_i^{k,l}$. Since $u_i^{k,l} \to \omega_{k,l}$ locally in $C^{1,\alpha}(\mathbb{R}^n\backslash\{p_{k,j}\}_{j=1}^{J_l})$, the lift $\tilde{u}_i^{k,l}$ convergence to the lift $\tilde{\omega}_{k,l}$ in the covering space with lift metric $\tilde{g}$ as well, so

$$\lim_{\delta_{k,l}\to 0}\lim_{i\to\infty} \sup_{x\in\mathbb{R}^n\backslash\cup_{l=1}^{J_k} B_{\delta_{k,l}}(x_i^{k,l})\}} d_{(\tilde{N},\tilde{g})}\left(\tilde{u}_i^{k,l}, \tilde{\omega}_{k,l}(x)\right) = 0.$$

By the triangle inequality, we have

$$\lim_{\delta_{k,l}\to 0}\lim_{i\to\infty}\sum_{k=1}^L\sum_{l=1}^{J_k} W\left(u_i^{k,l}; \mathbb{R}^n\backslash \cup_{l=1}^{J_k} B_{\delta_{k,l}}(x_i^{k,l})\right) \le \sum_{k=1}^L\sum_{l=1}^{J_k} \mathcal{W}(\omega_{k,l}; \mathbb{R}^n).$$

Similarly, we have

$$\limsup_{i\to\infty} \sup_{x,y\in\mathbb{R}^n\backslash\cup_{k=1}^l B_\delta(x_k)} d_{(\tilde{N},\tilde{g})}(\tilde{u_i}(x), \tilde{u_i}(y)) \le \mathcal{W}(u_\infty).$$

By the no-neck result in Theorem 1, we have

$$\lim_{R_{k,l}\to\infty}\lim_{\delta_{k,l}\to 0}\lim_{i\to\infty} W(u_i^{k,l}; B_{\delta_{k,l}}\setminus B_{R_{k,l}r_i^{k,l}}(x_i^{k,l}))=0$$

These imply that

$$\lim_{i\to\infty} W(u_i; S^n) \le W\left(u_\infty; S^n\right)+\sum_{k=1}^{L}\sum_{l=1}^{J_k} W(\omega_{k,l}; \mathbb{R}^n),$$

which is contradicted with the assumption. This proves the claim. □

As a consequence, we have

Lemma 4.2 *Let u be a regular solution to (1.2) in $M\times[0,\infty)$ with initial value u_0 satisfying $E_n(u_0) < C_1$ for a constant $C_1 > 0$. Then there is a sequence $t_i \to \infty$ such that $u(\cdot, t_i)$ converges to an n-harmonic maps u_∞ in $C^{1,\alpha}_{loc}(M\backslash\{x^1,\ldots,x^L\})$. Moreover, there exists a constant C_3, depending only on C_1, such that the*

$$\limsup_{i\to\infty} \mathcal{W}(u(\cdot,t_i); S^n)\le C_3.$$

Proof By using Theorem 1, there exists a sequence $t_i \to \infty$ such that $u(t_i)$ converges to an n-harmonic map u_∞ in $C^{1,\alpha}(M\backslash\{x^1,\ldots,x^L\})$ for some positive $\alpha < 1$. Moreover, there are a finite number of n-harmonic maps $\omega_{k,l}$ on S^n with $k = 1,\ldots,L$ and $l = 1,\ldots,J_k$. By applying Lemma 4.1, we have

$$\limsup_{i\to\infty} \mathcal{W}(u_i; S^n)\le \mathcal{W}(u_\infty; S^n)+\sum_{k=1}^{L}\sum_{l=1}^{J_k}\mathcal{W}(\omega_{k,l}; S^n)\le C_3,$$

where C_3 depends on C_2 and total numbers of bubbles. □

With this bounded width lemma, we are now ready to construct the example of the n-harmonic map flow with initial map $u_0 : S^n \to N$ which blows up in finite time. The basic idea is as follows: We construct an initial $u_0 : S^n \to N$ which has finite energy. Then we see if a map u', which is homotopic to u_0, could have a large width.

4.2 Proof of Theorem 2

Since X is a closed manifold of dimension $m > n$ with nontrivial $\pi_n(X)$, we can find a smooth map $h : S^n \to X$ such that

(a) h is non-subjective;
(b) h is not homotopic to any constant map;
(c) $h(S^n)\subset X\backslash\overline{V}$;
(d) h maps the southern hemisphere of S^n to a point $q\in X\backslash V$.

For each $l = 0, 1, \ldots,$ we denote $h_l : S^n \to X_l \subset \tilde{N}$ as a copy of h and $q_l \in X_l$ as a copy of q, and also denote by $\mathcal{S}_p$ the south pole of S^n.

For any large constant $K > 0$, there is a sufficiently large l such that

$$d_{(\tilde{N},\,\tilde{g})}(X_0,\ X_l)\ge K.$$

Let $q_0 \in X_0$ and $q_l \in X_l$ be copies of q. Let Ψ and Φ be two stereographic projections from S^n to $\mathbb{R}^n$ given by

$$\Phi(x^1, \ldots, x^n, x^{n+1}) = \left(\frac{x^1}{1 - x^{n+1}}, \ldots, \frac{x^n}{1 - x^{n+1}}\right), \tag{4.4}$$

$$\Psi(x^1, \ldots, x^n, x^{n+1}) = \left(\frac{x^1}{1 + x^{n+1}}, \ldots, \frac{x^n}{1 + x^{n+1}}\right), \tag{4.5}$$

which map the north pole $\mathcal{N}_p$ and the south pole $\mathcal{S}_p$ of S^n to the infinity respectively.

In order to construct an initial map $u_0 : S^n \to N$, we define a map $\tilde{u}_0 : S^n \to \tilde{N}$ by

$$\tilde{u}_0 = \begin{cases} h_0(x), & \text{for } x \in S^n \backslash B_\sigma(\mathcal{S}_p); \\ \gamma \circ \varphi\left(\frac{\log\sigma - \log|x|}{-\log\sigma}\right), & \text{for } x \in B_\sigma(\mathcal{S}_p) \backslash B_{\sigma^2}(\mathcal{S}_p); \\ h_l \circ \Phi^{-1} \circ (\frac{\Psi(x)}{\sigma^2/2}) & \text{for } x \in B_{\sigma^2}(\mathcal{S}_p). \end{cases} \tag{4.6}$$

Here $\gamma : [0, 1] \to \tilde{N}$ is the shortest geodesics connecting q_0 to q_l in $\tilde{N}$, and φ is a smooth cut-off function on [0, 1] that satisfies:

(1) φ' is non-negative and $|\varphi| \leq 1$;
(2) $\varphi(x) = 0$, for $x \in \left[0, \frac{1}{8}\right]$ and $\varphi(x) = 1$ for $x \in \left[\frac{7}{8}, 1\right]$;
(3) $|\varphi'| \leq C$, where C is a constant.

Under the definition of $\tilde{u}_0$, we can see $\tilde{u}_0|_{\partial B_\sigma(\mathcal{S}_p)} = q_0$. Moreover, for small σ, the metric was flattened out which gives $\tilde{u_0}|_{\partial B_{\sigma^2}(\mathcal{S}_p)} = q_l$.

Given that we have $\tilde{g}$ as the pullback metric for the covering of (N, g), there exists an isometric projection map $\pi : \tilde{N} \to N$. For sufficiently small σ, we can find a smooth $u_0 : S^n \to N$ defined by

$$u_0 = \begin{cases} \pi \circ h_0(x), & \text{for } x \in S^n \backslash B_\sigma(\mathcal{S}_p); \\ \pi \circ \gamma \circ \varphi\left(\frac{\log\sigma - \log|x|}{-\log\sigma}\right), & \text{for } x \in B_\sigma(\mathcal{S}_p) \backslash B_{\sigma^2}(\mathcal{S}_p); \\ \pi \circ h_l \circ \Phi^{-1} \circ (\frac{\Psi(x)}{\sigma^2/2}), & \text{for } x \in B_{\sigma^2}(\mathcal{S}_p). \end{cases} \tag{4.7}$$

Now we claim that there is a constant C_1 depending on h_0 such that

$$E_n(u_0) < E_n(h_l) + E_n(h_0) + 1 = C_1. \tag{4.8}$$

Due to the fact that $E_n(u)$ is conformally invariant, the energy $E_n(u_0)$ over $S^n \backslash B_\sigma(\mathcal{S}_p)$ and $B_{\sigma^2}(\mathcal{S}_p)$ for small σ can be bounded by

$$\int_{S^n \backslash B_\sigma(\mathcal{S}_p)} |\nabla u_0|^n \, dv + \int_{B_{\sigma^2}(\mathcal{S}_p)} |\nabla u_0|^n \, dv \leq E_n(h_0) + E_n(h_l) + \frac{1}{2}. \tag{4.9}$$

Now we have to check if the condition (4.8) is satisfied. We do this by estimating the energy over $B_\sigma(\mathcal{S}_p) \backslash B_{\sigma^2}(\mathcal{S}_p)$, then compare it with (4.9).

Let L be the shortest distance between q_0 and q_l. Since γ is the shortest geodesics connecting q_0 to q_l in $\tilde{N}$, there is a parametrization $\tilde{s}$ such that

$$\int_0^1 \left|(\pi \circ \gamma)'\right| d\tilde{s} = d_{(\tilde{N}, \tilde{g})}(q_0, q_l) = L, \quad \left|(\pi \circ \gamma)'\right| = L.$$

Therefore, we have

$$|\partial_r u_0| \leq |(\pi \circ \gamma)'| \, |\varphi'| \frac{1}{r(-\log\sigma)} \leq \frac{CL}{r(-\log\sigma)}$$

which gives us to estimate the energy of u_0 on the annulus domain; i.e.

$$\begin{aligned}\int_{B_\sigma \backslash B_{\sigma^2}} |\nabla u_0|^n \, dx &\le C \int_{\sigma^2}^{\sigma} |\partial_r u_0|^n r^{n-1} \, dr \\ &\le \frac{CL^n}{(-\log \sigma)^n} \int_{\sigma^2}^{\sigma} \frac{1}{r} \, dr \le \frac{CL^n}{(-\log \sigma)^{n-1}}.\end{aligned}$$

Therefore, the energy on the annulus domain $B_\sigma \backslash B_{\sigma^2}$ can be controlled for any L with a sufficiently small σ. Together with (4.9), we obtained an upper bound C_1 for $E(u_0)$.

Now, for any u' (with a lift $\tilde{u}'$) which is homotopic to u_0, we claim that $\tilde{u}'$ intercepts with X_0 and X_l, which implies

$$\mathcal{W}(u'; S^n) \ge d_{(\tilde{N}, \tilde{g})}(X_0, \; X_l) \ge K > C_3.$$

We prove this claim by contradiction. Assume that $\tilde{u}'$ does not intercept with X_0. Set a continuous map $\overline{\pi} : \tilde{N} \to X$ so that $\overline{\pi}$ maps $\tilde{N} \backslash X_0$ to a single point $p \in X$. Since $u' \cap X_0 = \emptyset$, it follows that $\overline{\pi} \circ \tilde{u}'$ maps to p which is a constant map. However, consider $\overline{\pi} \circ \tilde{u}'$ is homotopic to $\overline{\pi} \circ \tilde{u}_0$ which is homotopic to h_0 as well. This contradicts with the property (b) of the definition of h_0. This shows that $\tilde{u}'$ must intercept with X_0. By a similar argument, $\tilde{u}'$ must intercept with X_l.

Assume that the n-harmonic map flow with initial value u_0 does not blow up in finite time. Let $u(x, t)$ be a regular solution to the flow (1.2) in $M \times [0, \infty)$ with initial value $u_0 \in C^1(M, N)$. By Theorem 1, there is a sub-sequence t_i such that as $t_i \to \infty$, $u(x, t_i)$ converges to an n-harmonic map u_∞ in $C^{1,\alpha}_{loc}(M \backslash \{x^1, \ldots, x^L\})$ for some positive $\alpha < 1$. Since $u_i := u(x, t_i)$ is homotopic to u_0, we have

$$\mathcal{W}(u_i; S^n) \ge K > C_3.$$

On the other hand, by Lemma 4.2, $\limsup_{i\to\infty} \mathcal{W}(u_i; S^n) \le C_3$. This is a contradiction. Therefore, we have constructed initial maps $u_0 : S^n \to N$ such that the n-harmonic map flow with initial value u_0 must blow up in finite time. This completes a proof of Theorem 2. □

Remark It is an interesting question whether the the heat flow for H-systems ([14]) on n-manifolds blows up in finite time for $n \ge 3$.

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