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Calculus of Variations

Uniqueness of minimizers of weighted least gradient problems arising in hybrid inverse problems

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Abstract We study the question of uniqueness of minimizers of the weighted least gradient problem

$$\min\left\{\int_{\Omega}|Dv|_a : v\in BV_{loc}(\Omega\backslash S),\ v|_{\partial\Omega}=f\right\},$$

where $\int_{\Omega}|Dv|_a$ is the total variation with respect to the weight function a and S is the set of zeros of the function a. In contrast with previous results, which assume that the weight $a\in C^{1,1}(\Omega)$ and is bounded away from zero, here a is only assumed to be continuous, and is allowed to vanish and also be discontinuous in certain subsets of Ω. We assume instead existence of a C^1 minimizer. This problem arises naturally in the hybrid inverse problem of imaging electric conductivity from interior knowledge of the magnitude of one current density vector field, where existence of a C^1 minimizer is known a priori.

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1 Introduction

Consider the following weighted least gradient problem

$$\min\left\{\int_{\Omega} a|Du| : u \in BV(\Omega), u|_{\partial\Omega} = f\right\}, \tag{1}$$

where Ω is a bounded open set in $\mathbb{R}^n$ ($n \geq 2$) with connected Lipschitz boundary $\partial\Omega$, a is a bounded non-negative function, and $f \in C(\partial\Omega)$. This problem was first studied for the case $a \equiv 1$ in [17,19], where existence of a unique minimizer was proved under the assumption that f is continuous and $\partial\Omega$ has positive mean curvature on a dense subset of $\partial\Omega$ (see the precise hypotheses (3.1) and (3.2) in [17]). As in [6], these assumptions on $\partial\Omega$ are needed in the existence proofs but not for the uniqueness arguments. In [6] it has recently been proved that, if $a \in C^{1,1}(\Omega)$ is positive and bounded away from zero and $f \in C(\partial\Omega)$, then the weighted least gradient problem (1) has at most one minimizer in $BV(\Omega)$. The counterexample to uniqueness in [6] for $a \in C^{1,\alpha}(\Omega)$, $0 < \alpha < 1$, shows that the regularity $a \in C^{1,1}(\Omega)$ is sharp in general. The assumption $a > 0$ in Ω is also crucial for the results in [6].

In contrast with the results in [6,17], here we present uniqueness results that take into account the regularity of an existing minimizer. The main contribution of this paper is to show that if existence is known a priori, then the assumptions on the weight function a can be considerably weakened. The uniqueness result in this paper is based on a calibration argument and takes into account a priori existence of a C^1 minimizer. The following theorem is a consequence of a more general result (Theorem 1.2 below) that will be proved in Sect. 3.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with connected boundary, $f \in C^1(\partial\Omega)$, and $a \in C(\overline{\Omega})$. If $a > 0$ in $\overline{\Omega}$ and (7) has a minimizer $u \in C^1(\overline{\Omega})$ with $|\nabla u| > 0$ in $\overline{\Omega}$, then u is the unique minimizer of (1) in $BV(\Omega)$.*

Our motivation comes from a hybrid inverse problem in medical imaging, which concerns determining the conductivity of a body from knowledge of the magnitude $a = |J|$ (in Ω) of one current density vector field J generated by imposing the voltage f on $\partial\Omega$, see [12]. The interior data $|J|$ can be obtained non-invasively via a magnetic resonance technique pioneered in [7]. In [12] this problem was reduced to the weighted least gradient problem (1), by showing that the voltage potential is a minimizer. More precisely, assume $\Omega \subset \mathbb{R}^n$ is made of conductive materials with conductivity σ. If the voltage f is imposed on $\partial\Omega$, then the corresponding voltage potential u is the solution of the following conductivity equation

$$\begin{cases} \nabla \cdot \sigma \nabla u = 0, & \text{in } \Omega, \\ u = f. & \text{on } \partial\Omega \end{cases} \tag{2}$$

Let $J = -\sigma\nabla u$ be the current density generated by imposing the voltage f on $\partial\Omega$. Then the voltage potential u is a minimizer of the the weighted least gradient problem

$$\min\left\{\int_{\Omega} |J|\,|\nabla u| dx : \ u \in BV(\Omega),\ u|_{\partial\Omega} = f\right\},$$

(see Proposition 1.2 in [12] and the density argument of Proposition 3 in [15]). More generally, if Ω also contains perfectly conducting and insulating inclusions $\mathcal{U}_{\mathcal{P}}$ and $\mathcal{U}_{\mathcal{I}}$. Then the corresponding voltage potential u is the unique solution of the following equation

$$\begin{cases} \nabla \cdot \sigma \nabla u = 0, & \text{in } \Omega \backslash \overline{\mathcal{U}_{\mathcal{P}} \cup \mathcal{U}_{\mathcal{I}}}, \\ \nabla u = 0, & \text{in } \mathcal{U}_{\mathcal{P}}, \\ u|_{+} = u|_{-}, & \text{on } \partial(\mathcal{U}_{\mathcal{P}} \cup \mathcal{U}_{\mathcal{I}}), \\ \int_{\partial \mathcal{U}_{\mathcal{P}}{}^{j}} \sigma \frac{\partial u}{\partial \nu}|_{+} ds = 0, & j = 1, 2, \ldots, \\ \frac{\partial u}{\partial \nu}|_{+} = 0, & \text{on } \mathcal{U}_{\mathcal{I}}, \\ u|_{\partial \Omega} = f, \end{cases} \tag{3}$$

where $\mathcal{U}_{\mathcal{P}} \cap \mathcal{U}_{\mathcal{I}} = \emptyset$ and $\mathcal{U}_{\mathcal{P}} = \bigcup_{j=1} \mathcal{U}_{\mathcal{P}}{}^{j}$ is the partition of $\mathcal{U}_{\mathcal{P}}$ into open connected components (see the appendix in [10] for more details). Moreover, if $\sigma \in C^{\alpha}(\Omega \backslash \overline{\mathcal{U}_{\mathcal{P}} \cup \mathcal{U}_{\mathcal{I}}})$, $f \in C^{1,\alpha}(\partial \Omega)$, and the boundaries of $\mathcal{U}_{\mathcal{P}}, \mathcal{U}_{\mathcal{I}}$, and Ω are regular enough, then it follows from standard elliptic regularity results that $u \in C^{1}(\bar{\Omega} \backslash (\mathcal{U}_{\mathcal{P}} \cup \mathcal{U}_{\mathcal{I}}))$. Under certain assumptions, it is shown in Theorem 2.1 in [10] (see also the density argument of Proposition 3 in [15]) that the solution of the Eq. (3) is a minimizer of (1), where a is the magnitude of the corresponding current density vector field. Once u is determined, the shape and locations of perfectly conducting and insulation inclusion and the conductivity outside of the inclusions can be uniquely identified. Thus existence is known a priori, and the main issue is to prove uniqueness for the variational problem. Indeed if the conductivity to be recovered is C^{α}, then the assumption of Theorems 1.1 and 1.2 are naturally satisfied in our practical setting. See [9,10,13,15] for further results on this problem with only partial data, with insulating or highly conductive inclusions, reconstruction algorithms, and stability, and also [14] for a review. From the point of view of this original application, the uniqueness result in [6] does not apply to the case of embedded insulated and perfectly conductive inclusions described in [10], where the weight a is merely continuous and may vanish in Ω.

The uniqueness results presented here yield global convergence in minimization schemes based on compactness (e.g. [15]) and allow for extending stability results based on the Fredholm's alternative (e.g. in [8]) to the case of vanishing interior data.

Throughout the paper we assume that $\Omega \subset \mathbb{R}^n$ is a bounded open set with connected Lipschitz boundary $\partial \Omega$ and f is continuous on $\partial \Omega$. The following assumptions concern the most general class of weights a we allow. Let $\mathcal{I} \subset\subset \Omega$ be an open set (possibly empty) with finitely many C^1 connected components with finite perimeter, each of which is C^1-diffeomorphic with a ball. In addition, in two dimensions $\mathcal{I}$ is assumed to have at most one such component. We assume that $a = 0$ in $\mathcal{I}$, $a \in C(\overline{\Omega} \backslash \mathcal{I})$, and that a may have at most countable many zeros in $\Omega \backslash \overline{\mathcal{I}}$. In other words, the set of zeros

$$S := \{x \in \overline{\Omega} : a(x) = 0\} \tag{4}$$

satisfy

$$\bar{S} = \overline{\mathcal{I}} \cup \Gamma, \tag{5}$$

where Γ is a countable set of points in $\overline{\Omega} \backslash \overline{\mathcal{I}}$. Note that we do not assume that a vanishes on $\partial \mathcal{I}$. Note also that a may be discontinuous at points on $\partial \mathcal{I}$.

Definition 1 Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with connected boundary. We define $BV_{loc}(\Omega \backslash S)$ to be the space of all functions $u \in L^1(\Omega)$ such that

$$u \in BV(\Omega \backslash S') \textit{ for all open sets } S' \textit{ with } S' \supset\supset S.$$

Recall the following definition from [2]: For any $u \in BV_{loc}(\Omega \backslash S)$ the total variation of u (with respect to the weight a) in Ω is defined as

$$\int_{\Omega} |Du|_a = \sup_{b \in \mathfrak{B}_a} \int_{\Omega} u \nabla \cdot b \; dx, \tag{6}$$

where

$$\mathfrak{B}_a := \left\{b \in L^\infty(\Omega;\mathbb{R}^n):\ \nabla\cdot b \in L^n(\Omega),\ \text{supp}(b) \subset\subset \Omega,\ |b| \le a \text{ a.e. in } \Omega\right\}.$$

This paper considers the question of uniqueness of solutions of the weighted least gradient problem

$$\min\left\{\int_\Omega |Dv|_a : v \in BV_{loc}(\Omega\backslash S),\ v|_{\partial\Omega} = f\right\}, \tag{7}$$

where the boundary condition is in the sense of the trace of functions in $BV(\Omega)$. Note that, since $a \equiv 0$ in the open set $\mathcal{I}$, then $\int_\Omega |Dv|_a$ is independent of $v|_{\mathcal{I}}$.

Our main uniqueness result assumes the existence of a minimizer $u \in C^1(\overline{\Omega}\backslash\mathcal{I})$ of (7), whose set of singularities (possibly empty) satisfy

$$\mathcal{P} := \{x \in \overline{\Omega}\backslash\mathcal{I} : |\nabla u| = 0\} \text{ is the union of countably many } C^1\text{-path-connected sets.} \tag{8}$$

Now we are ready to state the more general result of this paper.

Theorem 1.2 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with connected boundary, $f \in C^1(\partial\Omega)$. Assume that the set of zeros of a satisfy the hypothesis (5) and $a \in C(\overline{\Omega}\backslash\mathcal{I})$. If (7) has a minimizer $u \in C^1(\overline{\Omega}\backslash\mathcal{I})$ that satisfies (8), then u is the unique minimizer of (7) in $BV_{loc}(\Omega\backslash S)$.*

Theorem 1.1 follows from the above theorem by taking $S = \emptyset$. We state other special cases in the next corollaries.

Corollary 1.3 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with connected boundary, $f \in C^1(\partial\Omega)$, and $a \in C(\overline{\Omega})$ with a countable set S of zeros. If (7) has a minimizer $u \in C^1(\overline{\Omega})$ with a countable set of critical points, then u is the unique minimizer of (7) in $BV_{loc}(\Omega\backslash S)$.*

Corollary 1.4 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with connected boundary, $f \in C^1(\partial\Omega)$, and $a \in C(\overline{\Omega})$. If S satisfies (4) and the least gradient problem (7) has a minimizer $u \in C^1(\overline{\Omega})$ such that*

$$\{x \in \overline{\Omega} : |\nabla u(x)| = 0\},$$

is the closure of an open set, then u is the unique minimizer of (7) in $BV_{loc}(\Omega\backslash S)$.

In the original application in [10], one interprets the open subsets of $\{x \in \overline{\Omega} : |\nabla u(x)| = 0\}$ in which $a > 0$ as perfect conductors. To illustrate a simple case with one perfectly conducting inclusion, consider the following example from [18].

Example 1.5 Let $D = \{x \in R^2 :\ \ x^2 + y^2 < 1\}$ be the unit disk, $f(x,y) = x^2 - y^2$, and $\mathcal{P} = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \times (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. It is shown in [18] (see also [10] for a different proof) that

$$u = \begin{cases} 2x^2 - 1, & \text{if } |x| \ge \frac{1}{\sqrt{2}},\ \ |y| \le \frac{1}{\sqrt{2}},\\ 0, & \text{if } (x,y) \in P,\\ 1 - 2y^2, & \text{if } |x| \le \frac{1}{\sqrt{2}},\ \ |y| \ge \frac{1}{\sqrt{2}} \end{cases}$$

is a minimizer of the least gradient problem

$$min\left\{\int_D |\nabla u| dx,\ \ u \in BV(D),\ \ and\ \ u|_{\partial D} = f\right\}. \tag{9}$$

It is easy to see that u satisfies the assumptions of Corollary 1.4 with $S = \mathcal{I} = \emptyset$ and $\mathcal{P} = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \times (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Hence u is the only minimizer in $BV(\Omega)$.

To understand the main ideas of the proofs in Sects. 2 and 3, one can keep in mind the example above and assume $S = \mathcal{I} = \emptyset$ at the first reading, to avoid some of the technicalities.

2 Preliminaries

In this section we recall and present some preliminary results that will be used in the following sections. First we recall a useful representation formula from [2]. For $u \in BV(\Omega)$

$$\int_A |Du|_a = \int_A h(x, v^u)|Du|, \tag{10}$$

where

$$h(x, v^u) = |Du| - \text{ess}\sup_{b\in\mathfrak{B}_{\mathfrak{a}}} (b \cdot v^u)(x) \quad for\ \ |Du| - a.e.\ \ x \in \Omega \tag{11}$$

and v^u denotes the Radon–Nikodym derivative

$$v^u(x) = \frac{d\,Du}{d\,|Du|}. \tag{12}$$

In particular, if $u \in BV(\Omega)$, and the coefficient a is continuous in the Borel measurable subset $A \subset \Omega$, then

$$\int_A |Du|_a = \int_A a|Du|, \tag{13}$$

as shown in [2]. The following lemma provides an extension of this formula.

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open region with Lipschitz boundary. Suppose that u is a minimizer of (7) and satisfies the assumptions of Theorem 1.2. Then*

$$\int_\Omega |Du|_a = \int_\Omega a|\nabla u|dx.$$

Proof Since, $a \in C(\Omega\backslash(\mathcal{I} \cup \mathcal{P}))$. Hence by [2, Proposition 7.1] we have that

$$h(x, v^u) = \begin{cases} a(x) & \text{in } \ \Omega\backslash\overline{\mathcal{I} \cup \mathcal{P}} \\ 0 & \text{in } \ \mathcal{I}. \end{cases} \tag{14}$$

Thus it follows from (10) that

$$\int_\Omega |Du|_a = \int_{\Omega\backslash\overline{\mathcal{I}\cup\mathcal{P}}} a|\nabla u| = \int_\Omega a|\nabla u|dx.$$

□

Let ν denote the outer unit normal vector to $\partial\Omega$ and

$$X := \{b \in L^\infty(\Omega; \mathbb{R}^n) : \ \nabla \cdot b \in L^n(\Omega)\}.$$

Then for every $b \in X$ there exists a unique function $[b \cdot \nu] \in L^\infty_{\mathcal{H}^{n-1}}(\partial\Omega)$ such that

$$\int_{\partial\Omega} [b \cdot \nu]ud\mathcal{H}^{n-1} = \int_\Omega u\nabla \cdot bdx + \int_\Omega b \cdot \nabla udx, \ \ \forall u \in C^1(\overline{\Omega}). \tag{15}$$

Moreover, for $u \in BV(\Omega)$ and $b \in X$, the linear functional $u \mapsto (b \cdot Du)$ gives rise to a Radon measure on Ω, and

$$\int_{\partial\Omega} [b \cdot \nu]ud\mathcal{H}^{n-1} = \int_\Omega u\nabla \cdot bdx + \int_\Omega (b \cdot Du), \ \ \forall u \in BV(\Omega), \tag{16}$$

see [1,3] for a proof. We will need the following generalization of (16) in the proof of our uniqueness result.

Lemma 2.2 *Let S be as defined in (5) and $b \in \mathfrak{B}_a$. If $u \in L^\infty(\Omega)$ and $\int_\Omega |Du|_a < \infty$, then*

$$\int_{\partial\Omega} [b \cdot \nu] u d\mathcal{H}^{n-1} = \int_\Omega u \nabla \cdot b dx + \int_\Omega (b \cdot Du), \tag{17}$$

for some unique function $[b \cdot \nu]$ in $L^\infty_{\mathcal{H}^{n-1}}(\partial\Omega)$.

Proof By hypothesis (5) S has finite perimeter in Ω. Define the set

$$\mathcal{D} := \partial S \cap S.$$

There exists a sequence $\mathcal{D}_i$ of open subsets of Ω with uniformly bounded perimeter such that $\mathcal{D}_{i+1} \subset \mathcal{D}_i$ and $u \in BV(\mathcal{D}_i)$ for all $i \in \mathbb{N}$, and $\cap_{i=1}^\infty \mathcal{D}_i = \mathcal{D}$. Now choose a sequence of cut-off functions φ_i such that $\varphi \equiv 0$ on $\mathcal{D}_i$, $\varphi \equiv 1$ on the compliment of an open set $\mathcal{D}'_i \supset\supset \mathcal{D}_i$ with $\lim_{i\longrightarrow\infty} \mathcal{H}^n(\mathcal{D}'_i) = \mathcal{H}^n(\mathcal{D})$, and $\int_\Omega |D\varphi_i| < c$ for all i. Indeed φ_i could be chosen to be smooth approximations of the characteristic functions $\chi_{\mathcal{D}_i^c}$ in $BV(\mathbb{R}^n)$ and c proportional to the perimeter of $\mathcal{D}_i$. Then $\varphi_i u \in BV(\Omega)$ and hence by (16)

$$\begin{aligned}\int_{\partial\Omega} [b \cdot \nu]\varphi_i u d\mathcal{H}^{n-1} &= \int_\Omega \varphi_i u \nabla \cdot b dx + \int_\Omega (b \cdot D(\varphi_i u)) \\ &= \int_\Omega \varphi_i u \nabla \cdot b dx + \int_\Omega (b \cdot D\varphi_i) u + \int_\Omega \varphi_i (b \cdot Du) \end{aligned} \tag{18}$$

for all $b \in \mathfrak{B}$ and $i \in \mathbb{N}$. Note that $u \in L^\infty(\Omega)$ and $|b| \le a$ a.e. in Ω. It follows from the continuity of a in $\mathcal{D} \subset S = \{x \in \overline{\Omega} : a(x) = 0\}$ that $\lim_{i\to\infty} \| b \|_{L^\infty(D'_i \setminus D_i)} \to 0$. Hence

$$\left| \int_\Omega (b \cdot D\varphi_i) u \right| \le \int_{D'_i \setminus D_i} |(b \cdot D\varphi_i) u| \le \int_{D'_i \setminus D_i} |D\varphi_i| \, \| u \|_\infty \| b \|_{L^\infty(D'_i \setminus D_i)} \longrightarrow 0,$$

as $i \longrightarrow \infty$. Since $\mathcal{I} \subset\subset \Omega$ and Γ [defined in (4)] is countable, by letting $i \longrightarrow \infty$ in (18) we obtain (17). □

The next two results yield a calibration which will be used in the uniqueness proof. Suppose $a \in L^2(\Omega)$ and fix $v_f \in H^1(\Omega)$ with $v_f|_{\partial\Omega} = f$. Consider the weighted least gradient problem

$$(P) \qquad \inf_{v \in H_0^1(\Omega)} \int_\Omega a|\nabla v + \nabla v_f| dx.$$

In [9] it is shown that the dual problem to (P) is

$$(D) \quad \max \left\{ < \nabla v_f, b >: \; b \in L^2(\Omega; \mathbb{R}^n), \; |b(x)| \le a(x) \; a.e. \text{ and } \nabla \cdot b \equiv 0 \right\}.$$

Let $v(P)$ and $v(D)$ be the optimal values of the primal and dual problems. It is shown in [9] that $v(P) = v(D)$ and the dual problem (D) has an optimal solution. The following proposition is an immediate consequence of Proposition 2.1 and Corollary 2.3 in [9].

Proposition 2.1 *Let $a \in L^2(\Omega)$ be a non-negative function and $v_f \in H^1(\Omega)$ with $v_f|_{\partial\Omega} = f$. Then the optimal values of the primal problem (P) and dual problem (D) are equal, and the dual problem (D) has an optimal solution J with $\nabla \cdot J \equiv 0$ in Ω. Moreover, if $v \in H_0^1(\Omega)$ is an optimal solution of the primal problem (P), then*

$$J(x) = a(x)\frac{\nabla(v(x)+v_f(x))}{|\nabla(v(x)+v_f(x))|} \quad \textit{if} \quad |\nabla(v(x)+v_f(x))| \neq 0,$$

for all $x \in \Omega$.

The following result is an immediate consequence of Proposition 2.1.

Corollary 2.3 *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded Lipschitz domain and assume that assumptions of Theorem* 1.2 *are satisfied. Then there exists an optimal solution* $J \in L^2(\Omega;\mathbb{R}^n)$ *of the dual problem (D) such that* $\nabla \cdot J \equiv 0$ *in* Ω, $|J| \leq a$ *a.e. in* Ω. *Moreover*

$$J(x) = \begin{cases} a(x)\frac{\nabla u}{|\nabla u|} & \textit{if} \quad |\nabla u| \neq 0 \\ 0 & \textit{if} \quad |\nabla u| = 0, \end{cases} \tag{19}$$

where u is the solution of (7) *described in the statement of Theorem* 1.2.

3 Uniqueness of the minimizers

In this section we prove Theorem 1.2. To understand the main ideas of the proofs in this section one may assume $S = \mathcal{I} = \emptyset$ at first reading.

Let u be the minimizer of the weighted least gradient problem (7) assumed in the statement of Theorem 1.2, and suppose $u^* \in BV_{loc}(\Omega\backslash S)$ is another minimizer. We will show that $u = u^*$ a.e. in $\Omega\backslash S$. We will prove Theorem 1.2 in four steps. First (*Step I*) we prove that $\frac{\nabla u}{|\nabla u|} = v^{u^*}$ Du^*-a.e. in $\Omega\backslash\overline{\mathcal{I}\cup\mathcal{P}}$, where v^{u^*} is the Radon–Nikodym derivative in (12). In *Step II* we prove that almost every level set of u^* is also a level set of u. In *Step III*, we prove that almost every level set of u^* reaches $\partial\Omega$. Finally in *Step IV*, we show that on almost every level set of u^*, u and u^* take the same values and therefore $u = u^*$ a.e. in $\Omega\backslash S$.

Step I. First notice that u^* is bounded above and below almost everywhere. Indeed if we define

$$\bar{u}(x) = \begin{cases} u^*(x) & \text{if } m_f \leq u^*(x) \leq M_f \\ M_f & \text{if } u^*(x) > M_f, \\ m_f & \text{if } u^*(x) < m_f, \end{cases} \tag{20}$$

where M_f and m_f are the maximum and minimum values of f on $\partial\Omega$, then $\bar{u} \in BV_{loc}(\Omega\backslash S)$ and

$$\int_\Omega |D\bar{u}|_a \leq \int_\Omega |Du^*|_a. \tag{21}$$

Moreover the inequality is strict if $\{x \in \Omega : u^*(x) > M_f\}$ or $\{x \in \Omega : u^*(x) < m_f\}$ has positive measure. Therefore we have range(u^*) = range(f).

Next we prove that

$$\frac{\nabla u}{|\nabla u|} = \frac{dDu^*}{d|Du^*|}$$

$|Du^*| - a.e.$ in $\Omega\backslash\overline{\mathcal{I}\cup\mathcal{P}}$.

Lemma 3.1 *Suppose that the assumptions of Theorem* 1.2 *are satisfied and u is the minimizer of* (7) *assumed in the statement of Theorem* 1.2. *If* u^* *is another minimizer of* (7), *then*

$$\frac{\nabla u}{|\nabla u|} = v^{u^*} \quad |Du^*| - a.e. \ \textit{in} \ \Omega\backslash\overline{\mathcal{I}\cup\mathcal{P}}.$$

Proof It follows from the definition of $h(x, v^{u^*})$ that

$$h(x, v^{u^*}) \geq J \cdot v^{u^*}, \quad |Du^*| - a.e. \text{ in } \Omega,$$

where J is the solution of the dual problem (D) in Corollary 2.3. Hence

$$\begin{aligned} \int_\Omega |Du^*|_a &= \int_\Omega h(x, v^{u^*})|Du^*| \geq \int_\Omega J \cdot v^{u^*}|Du^*| \\ &= \int_\Omega J \cdot Du^* = \int_{\partial\Omega} J \cdot \nu f d\mathcal{H}^{n-1} \\ &= \int_\Omega \nabla u \cdot J dx = \int_\Omega |J||\nabla u| \\ &= \int_\Omega |Du|_a = \int_\Omega |Du^*|_a, \end{aligned}$$

where the third and fifth equalities follow form Lemmas 2.2 and 2.1, respectively. Therefore

$$h(x, v^{u^*}) = J \cdot v^{u^*}, \quad |Du^*| - a.e. \text{ in } \Omega.$$

Since a is continuous in $\Omega \backslash \overline{\mathcal{I} \cup \mathcal{P}}$, as in (14) we have

$$J \cdot v^{u^*} = h(x, v^{u^*}) = a(x), \quad |Du^*| - a.e. \text{ in } \Omega \backslash \overline{\mathcal{I} \cup \mathcal{P}}. \tag{22}$$

Since $|v^{u^*}| = 1$ and $|J| \leq a$, $|Du^*| - a.e.$ in $\Omega \backslash \overline{\mathcal{I} \cup \mathcal{P}}$, we get

$$\frac{J}{|J|} = v^{u^*}, \quad |Du^*| - a.e. \text{ in } \Omega \backslash \overline{\mathcal{I} \cup \mathcal{P}}.$$

On the other hand $|\nabla u| \neq 0$ $|Du^*|$-a.e. on $\Omega \backslash \overline{\mathcal{I} \cup \mathcal{P}}$. Hence it follows from Corollary 2.3 that

$$\frac{\nabla u}{|\nabla u|} = \frac{J}{|J|} = v^{u^*}, \quad |Du^*| - a.e. \text{ in } \Omega \backslash \overline{\mathcal{I} \cup \mathcal{P}}.$$

The proof is now complete. □

Step II For $\lambda \in \text{range}(u^*)$, let

$$E_\lambda = \{x \in \Omega \backslash \overline{\mathcal{I}} : \quad u^*(x) \geq \lambda\}$$

and define

$$E'_\lambda := \left\{ x \in \mathbb{R}^n : \lim_{r \to 0} \frac{\mathcal{H}(B(r, x) \cap E_\lambda)}{\mathcal{H}(B(r))} = 1 \right\}. \tag{23}$$

By changing u^* in a set of measure zero, we may assume that $E_\lambda = E'_\lambda$. Throughout this paper we shall always assume that $E_\lambda = E'_\lambda$. We also define

$$\mathcal{Z} = \{x \in \overline{\Omega} \backslash \mathcal{I} : u(x) \in \overline{\Gamma \cup \mathcal{P}}\}, \tag{24}$$

where Γ is defined in (4) and $\mathcal{P}$ is the set of critical points of u satisfying (8). Notice that if $x \notin \mathcal{Z}$, then $|\nabla u(x)| > 0$.

Let Λ be the set of all $\lambda \in \text{range}(u^*)$ such that every connected component Σ of ∂E_λ with $\Sigma \cap \mathcal{Z} = \emptyset$ is a C^1 hypersurface. In the next lemma we prove that

$$\mathcal{H}^1(\text{range}(u^*) \backslash \Lambda) = 0. \tag{25}$$

Lemma 3.2 *Assume that the assumptions of Theorem 1.2 are satisfied and u is the corresponding minimizer of (7). Let $u^* \in BV_{loc}(\Omega\backslash S)$ be another minimizer and Σ be a connected component of ∂E_λ for some $\lambda \in range(u^*)$. Then for almost every $\lambda \in range(u^*)$ one and only one of the following statements hold:*

(i) $\Sigma \subset \mathcal{Z}$.
(ii) $\Sigma \cap \mathcal{Z} = \emptyset$, *$\Sigma$ is a C^1 hypersurface, and u is constant on Σ.*

Proof By the co-area formula we have

$$0 = \int_{\Omega\backslash\overline{\mathcal{I}\cup\mathcal{P}}} \varphi\left[\frac{\nabla u}{|\nabla u|} - v^{u^*}\right]|Du^*| = \int_0^\infty \int_{\partial^* E_\lambda\cap(\Omega\backslash\overline{\mathcal{I}\cup\mathcal{P}})} \varphi\left[\frac{\nabla u}{|\nabla u|} - v^{u^*}\right] d\mathcal{H}^{n-1} d\lambda \tag{26}$$

for every smooth vector field φ, where $\partial^* E_\lambda$ is the reduced boundary of E_λ. Therefore $\frac{\nabla u}{|\nabla u|} = v^{u^*}$, $\mathcal{H}^{n-1} - a.e.$ in $\partial^* E_\lambda \cap (\Omega\backslash\overline{\mathcal{I}\cup\mathcal{P}})$ for almost every $\lambda \in$ range(u^*). Since $|D\chi_{E_\lambda}|$ is the $(n-1)-$dimensional Hausdorff measure restricted to $\partial^* E_\lambda$ (see Chapter 4 in [4]), for almost every $\lambda \in$ range(u^*) the generalized normal $\nu(x)$ exists for $|D\chi_{E_\lambda}| - a.e.$ $x \in \partial E_\lambda \cap (\Omega\backslash\overline{\mathcal{I}\cup\mathcal{P}})$ and coincides there with the continuous vector field $\frac{\nabla u}{|\nabla u|}$.

Now let $x \in \partial E_\lambda \cap \Omega\backslash\overline{\mathcal{I}\cup\mathcal{P}}$ such that $x \notin \mathcal{Z}$. Since $\mathcal{Z}$ is closed, there exists $\epsilon > 0$ such that $B_\epsilon(x) \cap \mathcal{Z} = \emptyset$. By Theorem 4.8 in [4], $\partial E_\lambda \cap B_\epsilon(x)$ can be represented as the graph of a Lipschitz continuous function g. Thus the derivative of g coincides almost everywhere with a continuous function and therefore g must be C^1. Hence we conclude that if $x \in [\partial E_\lambda \cap (\Omega\backslash\overline{\mathcal{I}\cup\mathcal{P}})]\backslash Z$, then ∂E_λ is a C^1 hyperspace near x for almost every $\lambda \in$ range(u^*).

Next we show that u is constant on every C^1 connected open subset of Σ of $\partial E_\lambda \cap (\Omega\backslash\overline{\mathcal{I}\cup\mathcal{P}})$. Let $\gamma : (-\epsilon, +\epsilon) \to \Sigma$ be an arbitrary C^1 curve. Then

$$\frac{d}{dt}u(\gamma(s)) = |\nabla u(\gamma(s))|\nu(\gamma(s)).\gamma'(s) = 0,$$

because either $|\nabla u(\gamma(s))| = 0$ or $\nu(\gamma(s)).\gamma'(s) = 0$ on Σ. Thus u is constant along γ and hence u is constant on Σ. Therefore it follows from continuity of u and the definition of the set $\mathcal{Z}$ that, for almost every $\lambda \in$ range(u^*), if $\Sigma \not\subset Z$ is a connected component of $\partial E_\lambda \cap (\Omega\backslash\overline{\mathcal{I}\cup\mathcal{P}})$, then Σ is a C^1 hypersurface, u is constant on Σ, and $\Sigma \cap \mathcal{Z} = \emptyset$. The proof is now complete. □

Step III We show next that every connected component of ∂E_λ intersects the boundary $\partial\Omega$.

Proposition 3.1 *Assume that the assumptions of Theorem 1.2 are satisfied and u is the corresponding minimizer of (7). Suppose Σ_λ is a C^1 connected component of $\partial E_\lambda = \partial\{x \in \Omega\backslash\mathcal{I} : \ u^*(x) > \lambda\}$ and $\Sigma_\lambda \cap \mathcal{Z} = \emptyset$. Then*

$$\overline{\Sigma}_\lambda \cap \partial\Omega \neq \emptyset.$$

Proof Assume $\overline{\Sigma}_\lambda \cap \partial\Omega = \emptyset$. Then one of the followings statements hold:

(I) $\overline{\Sigma}_\lambda$ is a manifold without boundary in $\overline{\Omega}\backslash\mathcal{I}$.
(II) $\overline{\Sigma}_\lambda \cap \partial\mathcal{I} \neq \emptyset$.

Case I Assume that $\overline{\Sigma}_\lambda$ is a manifold without boundary in $\overline{\Omega}$. Then, since $\partial\Omega$ is connected, $\partial\Omega \cup \Sigma_\lambda$ is a compact manifold with two connected components. By the Alexander duality theorem for $\partial\Omega\cup\Sigma_\lambda$ (see, e.g., Theorem 27.10 in [5]) we have that $\mathbb{R}^n\backslash(\partial\Omega\cup$

Σ_λ) is partitioned into three open connected components: $\mathbb{R}^n = (\mathbb{R}^n\backslash\overline{\Omega}) \cup O_1 \cup O_2$. Since $\Sigma_\lambda \subset \Omega$ we have $O_1 \cup O_2 = \Omega\backslash\Sigma_\lambda$ and then $\partial O_i \subset \partial\Omega \cup \Sigma_\lambda$, for $i = 1, 2$. We claim that at least one of the ∂O_1 or ∂O_2 is in Σ_λ. Assume not, i.e. for $i = 1, 2$, $\partial O_i \cap \partial\Omega \neq \emptyset$. Since $\partial\Omega$ is connected (by assumption) we have that $O_1 \cup O_2 \cup \partial\Omega$ is connected which implies that $O_1 \cup O_2 \cup (\mathbb{R}^n\backslash\Omega)$ is also connected. Again by applying the Alexander duality theorem for $\Sigma_\lambda \subset \mathbb{R}^n$, we have that $\mathbb{R}^n\backslash\Sigma_\lambda$ has exactly two open connected components, one of which is unbounded: $\mathbb{R}^n\backslash\Sigma_\lambda = O_\infty \cup O_0$. Since $O_1 \cup O_2 \cup (\mathbb{R}^n\backslash\Omega)$ is connected and unbounded, we have that $O_1 \cup O_2 \cup (\mathbb{R}^n\backslash\Omega) \subset O_\infty$, which leaves $O_0 \subset \mathbb{R}^n\backslash(O_1 \cup O_2 \cup (\mathbb{R}^n\backslash\Omega)) \subset \Sigma_\lambda$. This is impossible since O_0 is open and Σ_λ is a hypersurface. Therefore either ∂O_1 or ∂O_2 or both lie in Σ_λ.

Assume $\partial O_1 \subset \Sigma_\lambda$. We claim that u is constant in O_1. Indeed, by Lemma 3.2, $u = c$ on Σ_λ for some c. Hence the new map $\tilde{u}$ defined by

$$\tilde{u} := \begin{cases} u, & x \in \Omega\backslash O_1, \\ c, & x \in \overline{O_1}, \end{cases}$$

is in $BV_{loc}(\Omega\backslash S)$ and decreases the energy, which contradicts the minimality of u. Therefore $u = c$ in O_1. This is a contradiction since we have assumed $\overline{\Sigma}_\lambda \cap \mathcal{Z} = \emptyset$.

Case II Suppose $\overline{\Sigma}_\lambda \cap \partial\mathcal{I} \neq \emptyset$. We first consider the dimensions $n \geq 3$. Let

$$\epsilon^* := min\left\{\min_{i\neq j} dist(\mathcal{I}_i, \mathcal{I}_j), \min_i dist(\mathcal{I}_i, \partial\Omega)\right\},$$

where $\mathcal{I}_i$, $1 \leq i \leq m$, are the open connected components of the set $\mathcal{I}$. For any $0 < \epsilon < \epsilon^*$: define

$$\mathcal{I}^\epsilon = \{x \in \Omega : dist(x, \mathcal{I}) < \epsilon\}.$$

Then $\mathcal{I}^\epsilon$ is an open set with the same number of disjoint open connected components as $\mathcal{I}$. Now let $\Sigma_\lambda^\epsilon = \Sigma_\lambda\backslash\mathcal{I}^\epsilon$ which we know is C^1 on $\Omega\backslash\mathcal{I}^\epsilon$. Since $\partial\Sigma_\lambda^\epsilon \subset \partial\mathcal{I}^\epsilon$ and $\partial\mathcal{I}^\epsilon\backslash\partial\Sigma_\lambda^\epsilon$ is open, each connected component of $\partial\Sigma_\lambda^\epsilon$ is the boundary of an open set in $\partial\mathcal{I}^\epsilon$ with connected boundary. Suppose M is a connected component of $\partial\Sigma_\lambda^\epsilon$. Then $M \subset \partial\mathcal{I}_i^\epsilon$ for some $1 \leq i \leq m$, $\mathcal{I}_i^\epsilon$ is C^1-diffeomorphic image of the unit ball for ϵ small, and M is an orientable manifold without boundary in $\partial\mathcal{I}^\epsilon$. Therefore it follows from Alexander's duality theorem that

$$\partial\mathcal{I}_i^\epsilon\backslash M = V_1 \cup V_2,$$

where V_1, V_2 are disjoint open connected (with respect to the topology of $\partial\mathcal{I}^\epsilon$) sets. Since Σ_λ^ϵ can be extended to a C^1 hypersurface Σ_λ inside $\mathcal{I}^\epsilon\backslash\mathcal{I}$, we can extend Σ_λ^ϵ inside $\mathcal{I}_i^\epsilon$ to obtain a C^1 hypersurface H_λ^ϵ such that

$$H_\lambda^\epsilon \cap (\Omega\backslash\mathcal{I}^\epsilon) = \Sigma_\lambda^\epsilon \cap (\Omega\backslash\mathcal{I}^\epsilon)$$

and $\partial(H_\lambda^\epsilon \cap \mathcal{I}^\epsilon) = M$. Repeating this argument for other connected components of $\partial\Sigma_\lambda^\epsilon$ leads to a C^1 orientable hypersurface $\mathcal{S}_\lambda^\epsilon$ with no boundary such that $\partial\Omega \cap \mathcal{S}_\lambda^\epsilon = \emptyset$ and $\mathcal{S}_\lambda^\epsilon \cap (\Omega\backslash\mathcal{I}^\epsilon) = \partial\Sigma_\lambda^\epsilon$. Now apply Alexander's duality theorem to get the partition

$$\mathbb{R}^n\backslash\mathcal{S}_\lambda^\epsilon = O_0^\epsilon \cup O_\infty^\epsilon,$$

where O_0^ϵ and O_∞^ϵ are open subsets of $\mathbb{R}^n$ and O_∞^ϵ is unbounded. If $\epsilon' < \epsilon$, then $\Sigma_\lambda^\epsilon \subset \Sigma_\lambda^{\epsilon'}$ and $\mathcal{I}^{\epsilon'} \subset \mathcal{I}^\epsilon$. Hence

$$\mathcal{I}^\epsilon \backslash \bar{O}_0^\epsilon \subset \mathcal{I}^{\epsilon'} \backslash \bar{O}_0^{\epsilon'}.$$

Now let

$$O = \cup_{0<\epsilon<\epsilon^*} \left(\mathcal{I}^\epsilon \backslash \bar{O}_0^\epsilon\right).$$

Then O is open and $\partial O \subset \Sigma_\lambda \cup \bar{\mathcal{I}}$. We claim that u is constant in O. Indeed by Lemma 3.2, $u = c$ on Σ_λ for some constant c. Define

$$\tilde{u} := \begin{cases} u, & x \in \Omega \backslash O, \\ c, & x \in O. \end{cases} \tag{27}$$

Then $\tilde{u} \in BV_{loc}(\Omega \backslash S)$ which contradicts the minimality of u. Hence u is constant in O which is a contradiction because we have assumed $\overline{\Sigma}_\lambda \cap \mathcal{Z} = \emptyset$.
Now assume $n = 2$. Since $\overline{\Sigma}_\lambda \cap \partial\Omega = \emptyset$ and $\mathcal{I}$ has only one connected component, there exists two distinct point $a, b \in \bar{\Sigma}_\lambda \cap \partial\mathcal{I}$ such that

$$\partial\mathcal{I} \backslash \{a, b\} = V_1 \cup V_2.$$

Note that $\Sigma_\lambda \cup V_1$ is a continuous closed curve in $\mathbb{R}^2$. By the Jordan Curve Theorem there exists a bounded open set O such that $\partial O = \Sigma_\lambda \cup V_1$. Define $\tilde{u}$ by (27), then with a similar argument we reach a contradiction. In both cases (I) and (II) the contradiction follows from the assumption that $\overline{\Sigma}_\lambda \cap \partial\Omega = \emptyset$. □

Step IV Since $f \in C^1(\partial\Omega)$, f can be extended to a function in $C^1(\mathbb{R}^n \backslash \Omega) \cap BV(\mathbb{R}^n \backslash \Omega)$. We will denote the extension of f to Ω^c by f, again. We will also denote the continuous extension of u^* to $\mathbb{R}^n$ with $u^* = f$ on Ω^c by u^* again. Define

$$F_\lambda = \{x \in \mathbb{R}^n \backslash \bar{\mathcal{I}} : \quad u^*(x) \geq \lambda\}$$

and let the corresponding F'_λ be defines as (23).

Remark 3.3 Let $\Lambda \subset \text{range}(u^*)$ be the set defined by Lemma 3.2 and $\lambda \in \Lambda$. By Lemma 3.2 every connected component of $\partial F'_\lambda \cap (\Omega \backslash \mathcal{Z})$ is a C^1 hypersurface. Since $F_\lambda \cap (\Omega \backslash \mathcal{Z})$ differs from $F'_\lambda \cap (\Omega \backslash \mathcal{Z})$ on a set of measure zero, we may assume that $F_\lambda \cap (\Omega \backslash \mathcal{Z})$ is open.

The proof of the following lemma is very similar to that of Theorem 3.7 in [17]. We include the proof for the convenience of the reader.

Lemma 3.4 *Let Ω be a bounded domain with connected Lipschitz boundary. If $x \in \partial^* F_\lambda \cap \partial\Omega$, where $\partial^* F_\lambda$ is the reduced boundary of F_λ, and*

$$\lim_{r \to 0} \fint_{B_r(x) \cap \Omega} |u^*(y) - f(x)| dy = 0,$$

then $\lambda = f(x)$.

Proof Assume $f(x) < \lambda$. Then

$$\begin{aligned} 0 &= \lim_{r\to 0} \frac{1}{|B_r(x)\cap\Omega|}\left(\int_{B_r(x)\cap\Omega\cap\{u^*<\lambda\}} |u^*(y)-f(x)|dy \right.\\ &\quad \left. + \int_{B_r(x)\cap\Omega\cap\{u^*\geq\lambda\}} |u^*(y)-f(x)|dy\right)\\ &\geq \limsup_{r\to 0} \frac{1}{|B_r(x)\cap\Omega|}\int_{B_r(x)\cap\Omega\cap\{u^*\geq\lambda\}} |u^*(y)-f(x)|dy\\ &\geq (\lambda - f(x))\limsup_{r\to 0}\frac{|B_r(x)\cap\Omega\cap\{u^*\geq\lambda\}|}{|B_r(x)\cap\Omega|}. \end{aligned}$$

Consequently

$$\limsup_{r\to 0}\frac{|B_r(x)\cap\Omega\cap\{u^*\geq\lambda\}|}{|B_r(x)\cap\Omega|} = 0.$$

On the other hand since f is the trace of $u^* \in BV(\mathbb{R}^n\backslash\Omega)$ on $\partial\Omega$, with a similar argument we conclude that

$$\limsup_{r\to 0}\frac{|B_r(x)\cap(\mathbb{R}^n\backslash\Omega)\cap\{u^*\geq\lambda\}|}{|B_r(x)\cap(\mathbb{R}^n\backslash\Omega)|} = 0.$$

Therefore

$$\lim_{r\to 0}\frac{|B_r(x)\cap\{u^*\geq\lambda\}|}{|B_r|} = 0$$

and hence $x \notin \partial^* E_\lambda$ which is a contradiction. Similarly $f(x) > \lambda$ leads to a contradiction. Thus $f(x) = \lambda$. □

Proposition 3.2 *Assume that the assumptions of Theorem 1.2 hold, and let u^* be a the corresponding minimizer of (7). Then for almost every $\lambda \in \Lambda$*

$$u(\partial F_\lambda \cap (\overline{\Omega}\backslash\mathcal{Z})) = \{\lambda\},$$

where Λ and $\mathcal{Z}$ are defined by (25) and (24), respectively.

Proof In view of Remark 3.3 and Proposition 3.1, we may assume that $F_\lambda \cap (\Omega\backslash\mathcal{Z})$ is open and every connected component of $\partial F_\lambda \cap (\Omega\backslash\mathcal{Z})$ is a C^1 hypersurface intersecting $\partial\Omega$. Now let Σ be a connected component of $\partial F_\lambda \cap (\Omega\backslash\mathcal{Z})$. By Proposition 3.1, $\bar{\Sigma}\cap\partial\Omega \neq \emptyset$. Let $x_0 \in \bar{\Sigma}\cap\partial\Omega \neq \emptyset$. Since $x_0 \notin \mathcal{Z}$, $|\nabla u(x_0)| > 0$. On the other hand, note that if $x_0 \in \bar{\Sigma}\cap\partial\Omega\backslash\partial^* F_\lambda$, then x_0 is not a regular point of the function $u^* \in BV(\mathbb{R}^n)$, i.e. u^* is discontinuous at x_0, which is not of jump type (see §4.4 in [16] for a precise definition of regular point of BV functions). Since the set of points which are not regular points of u^* has $(n-1)-$dimensional measure zero (see §4.5 [16]), for almost every $\lambda \in \Lambda$ and $\mathcal{H}^{n-2}$-a.e. $x_0 \in \bar{\Sigma}\cap\partial\Omega$, $x_0 \in \partial^* F_\lambda \cap \partial\Omega$. Thus by Lemma 3.4 we conclude that $u(\Sigma) = \{\lambda\}$. □

It is now straightforward to deduce uniqueness from the results established above. To make the argument rigorous it helps to work with super level sets of the solutions as in [6,17]. Note however that we do not rely on maximum principles for minimum surfaces that are at the core of the uniqueness proofs in [6,17].

Proof of Theorem 1.2 First we prove that $u^* = u$ a.e. in $\Omega\backslash(\mathcal{Z}\cup\bar{\mathcal{I}})$. Suppose this is not true, then without loss of generality we may assume that there exists $\alpha > 0$ such that

$$\mathcal{H}^n(N) > 0,$$

where

$$N := \{x \in \Omega\backslash(\mathcal{Z}\cup\bar{\mathcal{I}}) : \ u^*(x) \geq u(x) + \alpha\},$$

because otherwise the function f in (7) can be replaced by $-f$. Let

$$\lambda^* = \sup\left\{\lambda : \ \mathcal{H}^n(\{x \in \Omega\backslash(\mathcal{Z}\cup\bar{\mathcal{I}}) : \ u(x) \geq \lambda\} \cap N) \geq \frac{\mathcal{H}^n(N)}{2}\right\}.$$

Since $u \in L^1(\Omega\backslash\bar{\mathcal{I}})$, $\lambda^* < \infty$. For $0 < \beta < 1$ define

$$E_1 = \{x \in \Omega\backslash(\mathcal{Z}\cup\bar{\mathcal{I}}) : \ u^*(x) \geq \lambda^* + (1-\beta)\alpha\}.$$

By Lemma 3.2 and Proposition 3.1 there exists $0 < \beta < 1$ such that $\lambda^* + (1-\beta)\alpha \in \Lambda$. Also it follows from the definition of λ^* that $\mathcal{H}^n(K) > 0$, where

$$K := \{x \in \Omega\backslash(\mathcal{Z}\cup\bar{\mathcal{I}}) : \ \lambda^* - \beta\alpha < u(x) < \lambda^*\} \cap N.$$

Now let $E_2 = \{x \in \Omega\backslash(\mathcal{Z}\cup\bar{\mathcal{I}}) : \ u(x) \geq \lambda^*\}$. It is easy to see that $K \subset E_1\backslash\bar{E}_2 \subset \Omega\backslash(\mathcal{Z}\cup\bar{\mathcal{I}})$. On the other hand by Remark 3.3 we may assume that E_1 is open and hence $E_1\backslash\bar{E}_2$ is a non-empty open set. Also

$$\partial(E_1\backslash\bar{E}_2) \subset \left(\partial E_1 \cap \overline{E_2^c}\right) \cup (E_1 \cap \partial E_2)$$

and in particular, $\partial(E_1\backslash\bar{E}_2) \subset \partial E_1 \cup \partial E_2$. Notice that $\partial(E_1\backslash\bar{E}_2) \not\subset \partial E_2$, because otherwise $u = \lambda^*$ in $E_1\backslash\bar{E}_2$ which is in contradiction with the assumption $E_1\backslash\bar{E}_2 \subset (\Omega\backslash\mathcal{Z})$. Let

$$x_0 \in \partial(E_1\backslash\bar{E}_2).$$

Then $x_0 \in \partial E_1 \cap \overline{E_2^c}$. By Proposition 3.2 we have

$$u(x_0) \in u(\partial E_1) = \{\lambda^* + (1-\beta)\alpha\}. \tag{28}$$

On the other hand

$$u(x_0) \in u(\overline{E_2^c}) \subset (-\infty, \lambda^*]$$

which is in contradiction with (28). Hence $u^* = u$ a.e. in $\Omega\backslash(\mathcal{Z}\cup\bar{\mathcal{I}})$.

To finish the proof let Σ be a connected component of $\mathcal{Z}$. Since, $\text{int}(\overline{u(\mathcal{Z})}) = \emptyset$, u is continuous, $u = u^*$ in $\Omega\backslash(\mathcal{Z}\cup\bar{\mathcal{I}})$, and u^* minimizes (7), $u = u^*$ a.e. in Σ. The proof is now complete. □

Remark 3.5 Note that in domension $n = 2$, if the number of components of $\mathcal{I}$ is bigger than one, then there may exists level curves going from one component to the other, and not intersecting $\partial\Omega$. So the uniqueness argument fails. In higher dimensions this can not happen.

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