Calculus of Variations

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The virial theorem and ground state energy estimates of nonlinear Schrödinger equations in $\mathbb{R}^2$ with square root and saturable nonlinearities in nonlinear optics

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Abstract The virial theorem is a nice property for the linear Schrödinger equation in atomic and molecular physics as it gives an elegant ratio between the kinetic and potential energies and is useful in assessing the quality of numerically computed eigenvalues. If the governing equation is a nonlinear Schrödinger equation with power-law nonlinearity, then a similar ratio can be obtained but there seems to be no way of getting any eigenvalue estimates. It is surprising as far as we are concerned that when the nonlinearity is either square-root or saturable nonlinearity (not a power-law), one can develop a virial theorem and eigenvalue estimates of nonlinear Schrödinger (NLS) equations in $\mathbb{R}^2$ with square-root and saturable nonlinearity, respectively. Furthermore, we show here that the eigenvalue estimates can be used to obtain the 2nd order term (which is of order $\ln \Gamma$) of the lower bound of the ground state energy as the coefficient Γ of the nonlinear term tends to infinity.

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1 Introduction

The nonlinear Schrödinger (NLS) equation, a nonlinear variation of the Schrödinger equation, is a universal model in nonlinear science and mathematics. Such an equation can be represented as follows:

$$i\frac{\partial A}{\partial z} + \Delta A + \Gamma f\left(|A|^2\right) A = 0, \tag{1.1}$$

for $z > 0$, $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, where $A = A(x, z) \in \mathbb{C}, \Gamma \in \mathbb{R}, d \geq 1, \Delta = \sum_{j=1}^{d} \partial_{x_j}^2$ and the function f denotes the nonlinearity. Physically, A is the wave function, d is the transverse dimension and Γ is the strength of nonlinearity. Here we study the case of $d = 2$ for Eq. (1.1), which is non-integrable. But note that in the case of $d = 1$, Eq. (1.1) becomes integrable and, thus, can be investigated by different methods of the inverse scattering theory [1,18]. For the nonlinearity in function f, we consider the square-root and saturable nonlinearities in the following forms:

(I) **square-root nonlinearity:** $f(s) = 1 - \frac{1}{\sqrt{1+s}}$ for $s > 0$,
(II) **saturable nonlinearity:** $f(s) = 1 - \frac{1}{1+s}$ for $s > 0$,

which describe *narrow-gap semiconductors* [19,22] and *photorefractive media* [7–9,14–16,20], respectively. Equation (1.1) can be represented as $i\frac{\partial A}{\partial z} = \frac{\delta E[A]}{\delta A}$, where $E[A] = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla A|^2 - \Gamma F\left(|A|^2\right) dx$ and $F(I) = \int_0^I f(s) ds$. Besides, the total energy $E = K + P$ can be denoted as the sum of the kinetic energy K and the potential energy P, where the kinetic energy is

$$K[A] = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla A|^2 dx, \tag{1.2}$$

and the potential energy is

$$P[A] = -\frac{\Gamma}{2}\int_{\mathbb{R}^2} F\left(|A|^2\right) dx. \tag{1.3}$$

To see solitons of Eq. (1.1), we may set $A(x, z) = e^{i\lambda z} u(x)$ for $x \in \mathbb{R}^2$ and $z > 0$, where $\lambda \in \mathbb{R}$ is a constant and $u = u(x)$ is a real-valued function. Then by (1.1), we get the following nonlinear eigenvalue problem:

$$\Delta u + \Gamma f\left(u^2\right) u = \lambda u \quad \text{in } \mathbb{R}^2, \tag{1.4}$$

where λ is an eigenvalue and u is the associated eigenfunction. When the function $f(s) = s^{\frac{p-1}{2}}$, $p > 1$, is a power law nonlinearity [4,24,25], the eigenvalue λ can be apriori chosen as a positive number because we may set $u(x) = \left(\frac{\lambda}{\Gamma}\right)^{\frac{1}{p-1}} U\left(\sqrt{\lambda}x\right)$ and transform Eq. (1.4) into $\Delta U - U + U^p = 0$ in $\mathbb{R}^2$, which has a unique positive solution U. However, when function f is a square-root and saturable nonlinearity, the eigenvalue λ cannot be any positive number. One naïve counterexample is to set $\lambda = \Gamma > 0$ and Eq. (1.4) has only the zero solution because of the standard Liouville theorem. This motivates us to study the estimate of the eigenvalue λ of the ground state of Eq. (1.4) with the square-root and saturable nonlinearity of function f.

The *virial theorem* of linear Schrödinger equations can be formulated as the ratio of the kinetic energy and the potential energy of linear Schrodinger equations. Such a theorem plays a useful role in assessing the quality of numerical solutions of the eigenvalues of linear Schrodinger equations which is important in quantum mechanics (cf. [5]). To develop a virial

theorem for the nonlinear eigenvalue problem (1.4), we consider $\frac{K}{P}$ the ratio of the kinetic energy and the potential energy defined in (1.2) and (1.3), respectively. For the power-law nonlinearity function $f(s) = s^{\frac{p-1}{2}}$, $p > 1$, the ratio is denoted as

$$a = \frac{\frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2}{-\frac{\Gamma}{p+1} \int_{\mathbb{R}^2} |u|^{p+1}},$$

where $u \in H^1(\mathbb{R}^2) \cap L^{p+1}(\mathbb{R}^2)$ satisfies

$$\Delta u + \Gamma |u|^{p-1} u = \lambda u \quad \text{in} \quad \mathbb{R}^2. \tag{1.5}$$

Then by direct calculations, we get

$$\frac{2}{p+1} \Gamma \int_{\mathbb{R}^2} |u|^{p+1} = \lambda \int_{\mathbb{R}^2} u^2, \tag{1.6}$$

and

$$-\int_{\mathbb{R}^2} |\nabla u|^2 + \Gamma \int_{\mathbb{R}^2} |u|^{p+1} = \lambda \int_{\mathbb{R}^2} u^2. \tag{1.7}$$

Here (1.6) is the Pohozaev identity of (1.5) (cf. [21]), and (1.7) is the constraint of the Nehari manifold of (1.5) (cf. [2,3]). Combining (1.6) and (1.7), we have

$$\int_{\mathbb{R}^2} |\nabla u|^2 = \Gamma \left(1 - \frac{2}{p+1}\right) \int_{\mathbb{R}^2} |u|^{p+1},$$

implying that the ratio is $a = \frac{1}{2}(1 - p)$ so we may represent the virial relation of (1.5) as $\frac{K}{P} = a = \frac{1}{2}(1 - p)$. When $1 < p < 3$ (which is the subcritical case and the existence of ground state is proved in [4,24]), the ratio a is located on the interval $(-1, 0)$, which is the same interval (up to boundary points) as the virial theorem of linear Schrödinger equations with Coulomb potentials (see Section 4 of [5]). However, it seems impossible to get any estimates of the eigenvalue λ from identity (1.6) with $p > 1$. This stimulates us to study the different types of nonlinearities such as the square-root and saturable types here.

For the square-root (I) and saturable (II) nonlinearities of function f, we define the ratio as follows

$$\alpha = \frac{\int_{\mathbb{R}^2} |\nabla u|^2}{-\Gamma \int_{\mathbb{R}^2} \left(\sqrt{1 + u^2} - 1\right)^2}, \tag{1.8}$$

and

$$\beta = \frac{\int_{\mathbb{R}^2} |\nabla u|^2}{-\Gamma \int_{\mathbb{R}^2} \left[u^2 - \ln\left(1 + u^2\right)\right]}, \tag{1.9}$$

respectively. Note that the ratio α is for the eigenvalue problem (with square-root nonlinearity)

$$\Delta u + \Gamma \left(1 - \frac{1}{\sqrt{1 + u^2}}\right) u = \tilde{\lambda} u \quad \text{in} \quad \mathbb{R}^2, \tag{1.10}$$

and the ratio β is for the eigenvalue problem (with saturable nonlinearity)

$$\Delta u + \Gamma \left(1 - \frac{1}{1 + u^2}\right) u = \hat{\lambda} u \quad \text{in} \quad \mathbb{R}^2, \tag{1.11}$$

where $\tilde{\lambda}$ and $\hat{\lambda}$ are the respective eigenvalues, and $u \in H^1(\mathbb{R}^2)$ is the associated eigenfunction. In this paper, we first prove that the ratios α defined in (1.8) and β defined in (1.9) must

be located in the interval $(-1, 0)$, which is the same interval (up to boundary points) as the virial theorem of linear Schrödinger equations with suitable potentials (see Section 4 of [5]). Then we use the ratios α and β to derive the eigenvalue estimate of the ground states of (1.10) and (1.11), respectively (see Theorems 1.1 and 1.2).

For the estimates of the ratio α and the ground state eigenvalue $\tilde{\lambda}$, the results are stated as follows.

Theorem 1.1 *Let $u \in H^1\left(\mathbb{R}^2\right)$ be an eigenfunction and $\tilde{\lambda} \in \mathbb{R}$ be the eigenvalue of problem* (1.10). *Suppose $\Gamma > 0$. Then the ratio $\alpha = \dfrac{\int_{\mathbb{R}^2} |\nabla u|^2}{-\Gamma \int_{\mathbb{R}^2}\left(\sqrt{1+u^2}-1\right)^2}$ defined in* (1.8) *satisfies*

$$-1 < \alpha < 0. \tag{1.12}$$

Moreover, the eigenvalue $\tilde{\lambda}$ has the following estimate:

(i)   *If $-\frac{1}{2} \le \alpha < 0$, then $0 < \tilde{\lambda} \le \Gamma\left(1 + \alpha\right)$,*

(ii)   *If $-1 < \alpha < -\frac{1}{2}$, then $0 < \tilde{\lambda} \le \Gamma\left[1 + \alpha + \left(\sqrt{-2\alpha}-1\right)^2\right]$,*

(iii) *If $-1 < \alpha < -\frac{1}{2}$ and $\|u\|_\infty^2 \le (1+2\alpha)^{-2} - 1$, then $0 < \tilde{\lambda} \le \Gamma\left(1+\alpha\right)$, where $\|u\|_\infty = \max\limits_{x \in \mathbb{R}^2} |u(x)|$.*

For the estimates of the ratio β and the ground state eigenvalue $\hat{\lambda}$, the results are stated as follows.

Theorem 1.2 *Let $u \in H^1\left(\mathbb{R}^2\right)$ be the eigenfunction and $\hat{\lambda} \in \mathbb{R}$ be the eigenvalue of problem* (1.11). *Suppose $\Gamma > 0$. Then the ratio $\beta = \dfrac{\int_{\mathbb{R}^2} |\nabla u|^2}{-\Gamma \int_{\mathbb{R}^2}\left[u^2-\ln(1+u^2)\right]}$ and the eigenvalue $\hat{\lambda}$ satisfy*

$$-1 < \beta < 0, \tag{1.13}$$

$$0 < \hat{\lambda} \le \left(1 + \frac{1}{2}\beta\right)^2 \Gamma. \tag{1.14}$$

Remark 1.3 From Theorems 1.1 and 1.2, the ratios α and β satisfy $-1 < \alpha, \beta < 0$ and is located on the same interval (up to boundary points) as the virial theorem of linear Schrödinger equations with Coulomb potentials (see Section 4 of [5]).

With saturable nonlinearity, the ground state of the nonlinear Schrödinger equation (1.1) is defined as the minimizer of the following problem:

$$\text{Minimize}\left\{E\left[u\right] : u \in H^1\left(\mathbb{R}^2\right), \|u\|_2 = 1\right\},$$

and the existence of ground state can be proved using an energy estimate method [12], where $E\left[u\right] = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 - \Gamma\left[u^2 - \ln\left(1+u^2\right)\right] dx$ and $\|u\|_2^2 = \int_{\mathbb{R}^2} u^2 dx$. Such a ground state satisfies Eq. (1.11) and the eigenvalue $\hat{\lambda}$ comes from the Lagrange multiplier corresponding to the L^2-norm constraint $\|u\|_2 = 1$. The result of the existence of ground state is stated as follows.

Theorem A (cf. [12]) *Consider the following minimization problem:*

$$e_\Gamma = \inf\left\{E\left[u\right] : u \in H^1\left(\mathbb{R}^2\right), \|u\|_2 = 1\right\}, \tag{1.15}$$

where $E\left[u\right] = \int_{\mathbb{R}^2} |\nabla u|^2 - \Gamma\left[u^2 - \ln\left(1+u^2\right)\right]$. Let T_1 be the following positive constant:

$$T_1 = \inf_{\substack{w \in H^1\left(\mathbb{R}^2\right) \\ \|w\|_2 = 1}} \frac{\int_{\mathbb{R}^2} |\nabla w|^2}{\int_{\mathbb{R}^2}\left[w^2 - \ln\left(1+w^2\right)\right]}.$$

Then

(i) *If $\Gamma < T_1$, then $e_\Gamma = 0$ can not be attained by a minimizer, i.e., problem (1.15) has no ground state.*

(ii) *If $\Gamma > T_1$, then $e_\Gamma < 0$ and there exists a minimizer of (1.15) denoted as $U = U(r)$ which is radially symmetric and monotone decreasing for $r > 0$.*

The positivity of T_1 comes from the fact that $w^2 - \ln\left(1 + w^2\right) \leq \frac{1}{2}w^4$ for $w \in \mathbb{R}$ and the Gagliardo-Nirenberg inequality (cf. [6,17]):

$$\int_{\mathbb{R}^2} |\nabla w|^2 \geq C_0 \|w\|_2^{-2} \|w\|_4^4 \quad \text{for} \quad w \in H^1\left(\mathbb{R}^2\right), \tag{1.16}$$

for a positive constant C_0. Hereafter, the norm $\|\cdot\|_p$ denotes $\|w\|_p = \left(\int_{\mathbb{R}^2} |w|^p\right)^{1/p}$ for $p > 1$. Using the eigenvalue estimate of Theorem 1.2, we can derive the ground state energy estimate of e_Γ as follows.

Theorem 1.4 *Let e_Γ be the ground state energy defined in (1.15). Then as $\Gamma \to \infty$, $e_\Gamma = -\frac{\Gamma}{2}\left(1 + o_\Gamma\left(1\right)\right)$, where $o_\Gamma\left(1\right)$ is a small quantity tending to zero as Γ goes to infinity. Furthermore, for $\sigma \in (0, 1)$, there exists a positive constant Γ_σ such that*

$$e_\Gamma \geq -\frac{\Gamma}{2} + \sigma \frac{T_1}{2} \ln \Gamma + C_1 \quad for \quad \Gamma > \Gamma_\sigma, \tag{1.17}$$

where C_1 is a constant independent of Γ.

Note that in (1.17), the 2nd order term of the lower bound of e_Γ is of order $\ln \Gamma$ which goes to positive infinity as Γ tends to infinity.

With a square-root nonlinearity, the ground state of the nonlinear Schrödinger equation (1.1) is defined as the minimizer of the following problem:

$$\text{Minimize}\left\{E\left[u\right] : u \in H^1\left(\mathbb{R}^2\right), \|u\|_2 = 1\right\},$$

where $E\left[u\right] = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 - \Gamma\left(\sqrt{1 + u^2} - 1\right)^2 dx$. Such a ground state satisfies Eq. (1.10) and the eigenvalue $\tilde{\lambda}$ comes from the Lagrange multiplier corresponding to the L^2-norm constraint $\|u\|_2 = 1$. We may generalize the argument of [12] to prove the existence of the ground state and obtain the following result.

Theorem B *Consider the following minimization problem:*

$$\tilde{e}_\Gamma = inf\left\{E\left[u\right] : u \in H^1\left(\mathbb{R}^2\right), \|u\|_2 = 1\right\}, \tag{1.18}$$

where $E\left[u\right] = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 - \Gamma\left(\sqrt{1 + u^2} - 1\right)^2 dx$. Let T_2 be the following positive constant:

$$T_2 = \inf_{\substack{w \in H^1\left(\mathbb{R}^2\right) \\ \|w\|_2 = 1}} \frac{\int_{\mathbb{R}^2} |\nabla w|^2}{\int_{\mathbb{R}^2} \left(\sqrt{1 + w^2} - 1\right)^2}.$$

(i) *If $\Gamma < T_2$, then $\tilde{e}_\Gamma = 0$ can not be attained by a minimizer, i.e., problem (1.18) has no ground state.*

(ii) *If $\Gamma > T_2$, then $\tilde{e}_\Gamma < 0$ and there exists a minimizer $U = U(r)$ of (1.18) which is radially symmetric and monotone decreasing for $r > 0$.*

Here the positivity of T_2 comes from (1.16), i.e., the Gagliardo-Nirenberg inequality (cf. [6, 17]) and the fact that $\left(\sqrt{1+w^2}-1\right)^2 \leq \frac{1}{4}w^4$ for $w \in \mathbb{R}$. The proof of Theorem B is similar to that of Theorem A so we need only provide a brief sketch of the proof in "Appendix II". We can use the eigenvalue estimate of Theorem 1.1 to derive the ground state energy estimate of $\tilde{e}_\Gamma$ as follows.

Theorem 1.5 *Let $\tilde{e}_\Gamma$ be the ground state energy defined in (1.18). Then as $\Gamma \to \infty$, $\tilde{e}_\Gamma = -\frac{\Gamma}{2}(1 + o_\Gamma(1))$, where $o_\Gamma(1)$ is a small quantity tending to zero as Γ goes to infinity. Furthermore, there exists a positive constant Γ_0 such that*

$$\tilde{e}_\Gamma \geq -\frac{\Gamma}{2} + \frac{T_2}{2}\ln\Gamma + C_2 \quad \text{for} \quad \Gamma > \Gamma_0, \tag{1.19}$$

where C_2 is a constant independent of Γ.

Note that in (1.19), the 2nd order term of the lower bound of $\tilde{e}_\Gamma$ is also of order $\ln\Gamma$ (same as that of e_Γ) which goes to positive infinity as Γ tends to infinity. On the other hand, the difference between the ground state energy estimate (1.17) and (1.19) comes from that of the eigenvalue estimate (1.14) (see Theorem 1.2) and Theorem 1.1 (i).

The rest of this paper is organized as follows: The proofs of Theorems 1.1 and 1.5 are given in Sects. 2 and 5, respectively. We provide the proofs of Theorems 1.2 and 1.4 in Sects. 3 and 4, respectively. Brief concluding remarks are given in Sect. 6.

2 Proof of Theorem 1.1

We multiply (1.10) by u and integrate it over $\mathbb{R}^2$. Then using integration by parts, we get

$$-\int_{\mathbb{R}^2}|\nabla u|^2 + \Gamma\int_{\mathbb{R}^2}\left(1 - \frac{1}{\sqrt{1+u^2}}\right)u^2 = \tilde{\lambda}\|u\|_2^2, \tag{2.1}$$

where $\|u\|_2^2 = \int_{\mathbb{R}^2} u^2$. On the other hand, we may multiply (1.10) by $x \cdot \nabla u$ and integrate it over $\mathbb{R}^2$, where $x \cdot \nabla u = \sum_{j=1}^2 x_j \partial_j u$ and $\partial_j u = \frac{\partial u}{\partial x_j}$. Then using integration by parts, we can derive the Pohozaev identity as follows

$$\tilde{\lambda}\|u\|_2^2 = \Gamma\int_{\mathbb{R}^2}\left(\sqrt{1+u^2}-1\right)^2. \tag{2.2}$$

Combining (2.1) and (2.2), we have

$$-\int_{\mathbb{R}^2}|\nabla u|^2 + \Gamma\int_{\mathbb{R}^2}\left(1 - \frac{1}{\sqrt{1+u^2}}\right)u^2 = \Gamma\int_{\mathbb{R}^2}\left(\sqrt{1+u^2}-1\right)^2,$$

which implies

$$\int_{\mathbb{R}^2}|\nabla u|^2 = \Gamma\int_{\mathbb{R}^2}\left(1 - \frac{1}{\sqrt{1+u^2}}\right)u^2 - \Gamma\int_{\mathbb{R}^2}\left(\sqrt{1+u^2}-1\right)^2. \tag{2.3}$$

Hence

$$\begin{aligned}
\alpha &= \frac{\int_{\mathbb{R}^2}|\nabla u|^2}{-\Gamma\int_{\mathbb{R}^2}\left(\sqrt{1+u^2}-1\right)^2} \\
&= 1 - \frac{\int_{\mathbb{R}^2}\left(1 - \frac{1}{\sqrt{1+u^2}}\right)u^2}{\int_{\mathbb{R}^2}\left(\sqrt{1+u^2}-1\right)^2}
\end{aligned} \tag{2.4}$$

It is obvious that $\alpha < 0$ because u is nontrivial.

To prove $\alpha > -1$, we define a function $\tilde{f}(s) = \dfrac{\left(1 - \frac{1}{\sqrt{1+s}}\right)s}{\left(\sqrt{1+s}-1\right)^2}$ for $s > 0$. Then by direct calculation, we have $\tilde{f}(s) = \dfrac{s}{1+s-\sqrt{1+s}}$ and $\tilde{f}'(s) = \left(1+s-\sqrt{1+s}\right)^{-2}\dfrac{1}{\sqrt{1+s}}\left(\sqrt{1+s}-1-\frac{1}{2}s\right) < 0$ for $s > 0$, which gives $\tilde{f}'(s) < 0$ for $s > 0$. Here we have used the fact that $\sqrt{1+s} < 1 + \frac{1}{2}s$ for $s > 0$. Besides, $\lim_{s\to 0+}\tilde{f}(s) = 2$ and $\lim_{s\to\infty}\tilde{f}(s) = 1$ are trivial by direct calculation. Consequently, $\tilde{f}(s) < 2$ for $s > 0$ and

$$\int_{\mathbb{R}^2}\left(1 - \frac{1}{\sqrt{1+u^2}}\right)u^2 = \int_{\mathbb{R}^2}\left(\sqrt{1+u^2}-1\right)^2\tilde{f}\left(u^2\right) \le 2\int_{\mathbb{R}^2}\left(\sqrt{1+u^2}-1\right)^2, \quad (2.5)$$

which gives $\alpha \ge -1$. Here we have used (2.4). Note that $\left(1 - \frac{1}{\sqrt{1+u^2}}\right)u^2 \le 2\left(\sqrt{1+u^2}-1\right)^2$ for all $u \in \mathbb{R}$, and the equality holds true if and only if $u = 0$. Consequently, if $\alpha = -1$ and the equality of (2.5) holds true, then $u \equiv 0$ which contradicts to $u \not\equiv 0$. Therefore, we have completed the proof for the case $-1 < \alpha < 0$, i.e., (1.12).

To prove Theorem 1.1 (i) and (iii), we substitute (1.8) into (2.1) and get

$$\tilde{\lambda}\|u\|_2^2 = \alpha\Gamma\int_{\mathbb{R}^2}\left(\sqrt{1+u^2}-1\right)^2 + \Gamma\int_{\mathbb{R}^2}\left(1 - \frac{1}{\sqrt{1+u^2}}\right)u^2 = \Gamma\int_{\mathbb{R}^2}g\left(u^2\right), \quad (2.6)$$

where $g(s) = \alpha\left(\sqrt{1+s}-1\right)^2 + s - \frac{s}{\sqrt{1+s}}$ for $s > 0$. Let $h(s) = g(s) - (1+\alpha)s$ for $s > 0$. Then $h(s) = 2\alpha\left(1 - \sqrt{1+s}\right) - \frac{s}{\sqrt{1+s}}$ for $s > 0$. By a direct calculation, we get

$$h(s) = s\left(\frac{-2\alpha}{1+\sqrt{1+s}} - \frac{1}{\sqrt{1+s}}\right) \quad \text{for} \quad s > 0, \quad (2.7)$$

Suppose $-\frac{1}{2} \le \alpha < 0$. Then (2.7) implies that $h(s) < 0$, i.e. $g(s) < (1+\alpha)s$ for $s > 0$, which can be used in (2.6) to complete the proof of Theorem 1.1 (i). On the other hand, suppose $-1 < \alpha < -\frac{1}{2}$. Then $h(s) \le 0$ for $0 \le s \le s_\alpha$, and $h(s) > 0$ for $s > s_\alpha$, where $s_\alpha > 0$ satisfies $2\alpha\left[\sqrt{1+s_\alpha} - (1+s_\alpha)\right] - s_\alpha = 0$, i.e. $s_\alpha = \frac{1}{(1+2\alpha)^2} - 1 > 0$. Consequently, $h\left(u^2(x)\right) \le 0$, i.e. $g\left(u^2(x)\right) \le (1+\alpha)u^2(x)$ for $x \in \mathbb{R}^2$ if $u^2(x) \le s_\alpha$ for $x \in \mathbb{R}^2$, i.e. $\|u\|_\infty^2 \le s_\alpha$. This completes the proof of Theorem 1.1 (iii).

The rest of the proof of Theorem 1.1 is to show Theorem 1.1 (ii), as follows. Suppose $-1 < \alpha < -\frac{1}{2}$. Then we may transform Eq. (2.7) into

$$\frac{h(s)}{s} = \frac{-2\alpha}{1+\sqrt{1+s}} - \frac{1}{\sqrt{1+s}} = \hat{h}\left(\sqrt{1+s}\right) \quad \text{for} \quad s \ge s_\alpha, \quad (2.8)$$

where

$$\hat{h}(\tau) = \frac{-2\alpha}{1+\tau} - \frac{1}{\tau} \quad \text{for} \quad \tau \ge \sqrt{1+s_\alpha} = \frac{1}{|1+2\alpha|}.$$

Note that $h(s) \le 0$ for $0 \le s \le s_\alpha$, and $h(s) > 0$ for $s > s_\alpha$, where $s_\alpha > 0$ satisfies $2\alpha\left[\sqrt{1+s_\alpha} - (1+s_\alpha)\right] - s_\alpha = 0$, i.e. $s_\alpha = \frac{1}{(1+2\alpha)^2} - 1 > 0$. It is easy to check that $\hat{h}'(\tau) = \frac{2\alpha}{(1+\tau)^2} + \frac{1}{\tau^2}$ for $\tau > \frac{1}{|1+2\alpha|}$. Then $\max_{\tau \ge \frac{1}{|1+2\alpha|}}\hat{h}(\tau) = \left(\sqrt{-2\alpha}-1\right)^2$ and the maximum occurs at $\tau_\alpha = \frac{1}{\sqrt{-2\alpha}-1} > \frac{1}{|1+2\alpha|}$. Therefore, by (2.8), we have $\frac{h(s)}{s} \le \left(\sqrt{-2\alpha}-1\right)^2$ for $s > 0$ and we have completed the proof of Theorem 1.1 (ii).

3 Proof of Theorem 1.2

We multiply (1.11) by u and integrate over $\mathbb{R}^2$. Then using integration by parts, we get

$$-\int_{\mathbb{R}^2} |\nabla u|^2 + \Gamma \int_{\mathbb{R}^2} \left(1 - \frac{1}{1+u^2}\right) u^2 = \hat{\lambda} \|u\|_2^2 , \tag{3.1}$$

where $\|u\|_2^2 = \int_{\mathbb{R}^2} u^2$. On the other hand, we can also multiply (1.11) by $x \cdot \nabla u$ and integrate it over $\mathbb{R}^2$, where $x \cdot \nabla u = \sum_{j=1}^2 x_j \partial_j u$. Then using integration by parts, we derive the Pohozaev identity as follows

$$\hat{\lambda} \|u\|_2^2 = \Gamma \int_{\mathbb{R}^2} \left[u^2 - \ln\left(1 + u^2\right)\right]. \tag{3.2}$$

Combining (3.1) and (3.2), we have

$$-\int_{\mathbb{R}^2} |\nabla u|^2 + \Gamma \int_{\mathbb{R}^2} \left(1 - \frac{1}{1+u^2}\right) u^2 = \Gamma \int_{\mathbb{R}^2} \left[u^2 - \ln\left(1 + u^2\right)\right],$$

implying

$$\int_{\mathbb{R}^2} |\nabla u|^2 = \Gamma \int_{\mathbb{R}^2} \left(1 - \frac{1}{1+u^2}\right) u^2 - \Gamma \int_{\mathbb{R}^2} \left[u^2 - \ln\left(1 + u^2\right)\right]. \tag{3.3}$$

Hence

$$\beta = \frac{\int_{\mathbb{R}^2} |\nabla u|^2}{-\Gamma \int_{\mathbb{R}^2} [u^2 - \ln(1+u^2)]} \\ = 1 - \frac{\int_{\mathbb{R}^2} \left(1 - \frac{1}{1+u^2}\right) u^2}{\int_{\mathbb{R}^2} [u^2 - \ln(1+u^2)]} \tag{3.4}$$

It is obvious that $\beta < 0$ because u is nontrivial. To prove $\beta > -1$, we define a function $\hat{f}(s) = \frac{\left(1 - \frac{1}{1+s}\right)s}{s - \ln(1+s)}$ for $s > 0$. Then we can show that $\hat{f}'(s) < 0$ for $s > 0$, $\lim_{s \to 0+} \hat{f}(s) = 2$ and $\lim_{s \to \infty} \hat{f}(s) = 1$ (see Proposition A.1 in "Appendix I"). Consequently, $\hat{f}(s) < 2$ for $s > 0$ and

$$\int_{\mathbb{R}^2} \left(1 - \frac{1}{1+u^2}\right) u^2 = \int_{\mathbb{R}^2} \left[u^2 - \ln\left(1 + u^2\right)\right] \hat{f}\left(u^2\right) \leq 2 \int_{\mathbb{R}^2} \left[u^2 - \ln\left(1 + u^2\right)\right] \tag{3.5}$$

which gives $\beta \geq -1$. Here we have used (3.4). Note that $\left(1 - \frac{1}{1+u^2}\right) u^2 \leq 2\left[u^2 - \ln\left(1 + u^2\right)\right]$ for $u \in \mathbb{R}$, and the equality holds true if and only if $u = 0$. Consequently, if $\beta = -1$ and the equality of (3.5) holds true, then $u \equiv 0$ which contradicts to $u \not\equiv 0$. Therefore, we have completed the proof of $-1 < \beta < 0$, i.e., (1.13).

To prove (1.14), we substitute (1.9) into (3.1) and get

$$\hat{\lambda} \|u\|_2^2 = \beta \Gamma \int_{\mathbb{R}^2} \left[u^2 - \ln\left(1 + u^2\right)\right] + \Gamma \int_{\mathbb{R}^2} \left(1 - \frac{1}{1+u^2}\right) u^2 = \Gamma \int_{\mathbb{R}^2} g\left(u^2\right), \tag{3.6}$$

where $g(s) = \beta\left[s - \ln\left(1 + s\right)\right] + \frac{s^2}{1+s}$ for $s > 0$. Now we claim $g(s) < \left(1 + \frac{1}{2}\beta\right)^2 s$ for $s > 0$. Let $h(s) = g(s) - \left(1 + \frac{1}{2}\beta\right)^2 s$ for $s > 0$. Then by direct calculation, we get $h(s) = -\beta \ln\left(1 + s\right) + \frac{1}{1+s} - 1 - \frac{1}{4}\beta^2 s$ and

$$h'(s) = \frac{-\beta}{1+s} - \frac{1}{(1+s)^2} - \frac{1}{4}\beta^2$$

$$= -\frac{1}{(1+s)^2}\left[\frac{1}{4}\beta^2(1+s)^2 + \beta(1+s) + 1\right]$$

$$= -\frac{\left[\frac{1}{2}\beta(1+s) + 1\right]^2}{(1+s)^2} < 0,$$

for $s > 0$. Therefore, $h(s) < h(0) = 0$ i.e. $g(s) < \left(1 + \frac{1}{2}\beta\right)^2 s$ for $s > 0$, which can be put into (3.6) to get (1.14) and, thus, complete the proof of Theorem 1.2.

4 Proof of Theorem 1.4

We first prove the upper bound estimate of the ground state energy e_Γ.

Lemma 4.1 *Given the same assumptions as in Theorem* 1.4, *we have* $e_\Gamma \leq -\frac{\Gamma}{2}(1 + o_\Gamma(1))$ *as* $\Gamma \to \infty$., *where* $o_\Gamma(1)$ *is a small quantity tending to zero as* Γ *goes to infinity.*

Proof Let $u_\tau(x) = \frac{1}{\tau}U\left(\frac{x}{\tau}\right)$ for $x \in \mathbb{R}^2$, $\tau > 0$, where $U \in H^1(\mathbb{R}^2)$, $\|U\|_2 = 1$ and $U > 0$ in $\mathbb{R}^2$. Then $\|u_\tau\|_2 = \|U\|_2 = 1$, $\int_{\mathbb{R}^2}|\nabla u_\tau(x)|^2 dx = \tau^{-2}\int_{\mathbb{R}^2}|\nabla U(x)|^2 dx$ and

$$E[u_\tau] = \frac{1}{2}\int_{\mathbb{R}^2}|\nabla u_\tau|^2 - \Gamma\left[u_\tau^2 - \ln\left(1 + u_\tau^2\right)\right]dx$$

$$= \frac{1}{2}\int_{\mathbb{R}^2}|\nabla u_\tau|^2 dx - \frac{\Gamma}{2}\int_{\mathbb{R}^2}u_\tau^2 dx + \Gamma\int_{\mathbb{R}^2}\ln\left(1 + u_\tau^2\right)dx$$

$$= \frac{1}{2}\tau^{-2}\int_{\mathbb{R}^2}|\nabla U|^2 dx - \frac{\Gamma}{2} + \Gamma\int_{\mathbb{R}^2}\tau^2\ln\left(1 + \tau^{-2}U^2(y)\right)dy$$

$$= -\frac{\Gamma}{2}(1 + o_\Gamma(1)),$$

as $\tau \sim (\ln \Gamma)^{-1/2}$ and $\Gamma \to \infty$, where $y = \frac{x}{\tau}$ and $o_\Gamma(1)$ is a small quantity tending to zero as Γ goes to infinity. Here we have used the fact $\tau^2\ln\left(1 + \tau^{-2}U^2\right) \leq CU^2 \in L^1(\mathbb{R}^2)$ for some constant $C > 0$ (independent of U and τ) and $0 < \tau^2\ln\left(1 + \tau^{-2}U^2(y)\right) \leq \tau^2\ln\left(1 + \tau^{-2}\|U\|_\infty^2\right) \to 0$ as $\tau \to 0$ for $y \in \mathbb{R}^2$. Hence, by the Dominated Convergence Theorem,

$$\int_{\mathbb{R}^2}\tau^2\ln\left(1 + \tau^{-2}U^2(y)\right)dy = o_\Gamma(1) \quad \text{as} \quad \Gamma \to \infty.$$

Note that $\tau \sim (\ln \Gamma)^{-1/2}$ as $\Gamma \to \infty$. Therefore, $e_\Gamma \leq E[u_\tau] = -\frac{\Gamma}{2}(1 + o(1))$ as $\Gamma \to \infty$ and we have completed the proof of Lemma 4.1. $\qquad\square$

For the lower bound estimate of e_Γ, it is obvious that

$$e_\Gamma = E[u_\Gamma] = \frac{1}{2}\int_{\mathbb{R}^2}|\nabla u_\Gamma|^2 - \Gamma\left[u_\Gamma^2 - \ln\left(1 + u_\Gamma^2\right)\right]dx$$

$$= \frac{1}{2}\int_{\mathbb{R}^2}|\nabla u_\Gamma|^2 dx - \frac{\Gamma}{2}\int_{\mathbb{R}^2}u_\Gamma^2 dx + \Gamma\int_{\mathbb{R}^2}\ln\left(1 + u_\Gamma^2\right)dx$$

$$\geq -\frac{\Gamma}{2}\int_{\mathbb{R}^2}u_\Gamma^2 dx = -\frac{\Gamma}{2},$$

where u_Γ is the ground state (energy minimizer) of e_Γ under the L^2-norm constraint $\|u_\Gamma\|_2 = 1$. Consequently, by Lemma 4.1, we obtain $e_\Gamma = -\frac{\Gamma}{2}(1 + o_\Gamma(1))$ as $\Gamma \to \infty$. To get a further estimate of e_Γ, we need the following lemmas.

Lemma 4.2 *Under the same assumptions as Lemma* 4.1, *we have that the ratio* $\beta_\Gamma = \frac{\int_{\mathbb{R}^2} |\nabla u_\Gamma|^2 dx}{-\Gamma \int_{\mathbb{R}^2} [u_\Gamma^2 - \ln(1+u_\Gamma^2)]dx} \to 0$ *as* $\Gamma \to \infty$, *where* u_Γ *is the ground state (energy mini-mizer) of* e_Γ.

Proof We prove by contradiction. Suppose that β_Γ does not approach zero as Γ goes to infinity. Then by (1.13) of Theorem 1.2, we may assume $\beta_\Gamma \to -c_0$ as $\Gamma \to \infty$, where $0 < c_0 \leq 1$ is a constant. Hence

$$
\begin{aligned}
e_\Gamma &= \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u_\Gamma|^2 - \frac{\Gamma}{2} \int_{\mathbb{R}^2} [u_\Gamma^2 - \ln(1+u_\Gamma^2)] \\
&= -\frac{\Gamma}{2}(\beta_\Gamma + 1) \int_{\mathbb{R}^2} [u_\Gamma^2 - \ln(1+u_\Gamma^2)] \\
&\geq -\frac{\Gamma}{2}(\beta_\Gamma + 1) \int_{\mathbb{R}^2} u_\Gamma^2 \qquad\qquad (4.1) \\
&= -\frac{\Gamma}{2}(\beta_\Gamma + 1). \qquad\qquad (4.2)
\end{aligned}
$$

Here we have used the fact that $-1 < \beta_\Gamma < 0$ and $\|u_\Gamma\|_2 = 1$. Combining Lemma 4.1 and (4.1), we have $-\frac{1}{2}(1 + o_\Gamma(1)) \geq -\frac{1}{2}(\beta_\Gamma + 1)$, i.e. $\beta_\Gamma \geq 0$ as $\Gamma \to \infty$, which contradicts with $\beta_\Gamma \to -c_0 \in [-1, 0)$ as $\Gamma \to \infty$. Therefore, we have completed the proof of Lemma 4.2. □

Lemma 4.3 *Under the same assumptions as Lemma* 4.1, *we have that* e_Γ *is decreasing to* Γ *for* $\Gamma > T_1$.

Proof Let u_Γ be the energy minimizer (ground state) of e_Γ for $\Gamma > T_2$. Then

$$
\begin{aligned}
2e_{\Gamma_1} &= \int_{\mathbb{R}^2} |\nabla u_{\Gamma_1}|^2 - \Gamma_1 \int_{\mathbb{R}^2} [u_{\Gamma_1}^2 - \ln(1+u_{\Gamma_1}^2)] \\
&\geq \int_{\mathbb{R}^2} |\nabla u_{\Gamma_1}|^2 - \Gamma_2 \int_{\mathbb{R}^2} [u_{\Gamma_1}^2 - \ln(1+u_{\Gamma_1}^2)] \\
&\geq 2e_{\Gamma_2},
\end{aligned}
$$

for $\Gamma_2 > \Gamma_1 > T_1 > 0$. Hence e_Γ is decreasing to Γ and we have completed the proof of Lemma 4.3. □

Lemma 4.4 *Under the same assumptions as Lemma* 4.1, *we have that* $e_{\Gamma+1} - e_\Gamma \geq -\frac{1}{2}\frac{\hat{\lambda}_{\Gamma+1}}{\Gamma+1}$ *for* $\Gamma > T_1$, *where* $u_{\Gamma+1}$ *is the energy minimizer (ground state) of* $e_{\Gamma+1}$ *and* $\hat{\lambda}_{\Gamma+1}$ *is the associated eigenvalue of* $u_{\Gamma+1}$.

Proof It is obvious that for $\Gamma > T_1$,

$$
\begin{aligned}
2e_{\Gamma+1} &= \int_{\mathbb{R}^2} |\nabla u_{\Gamma+1}|^2 - (\Gamma+1) \int_{\mathbb{R}^2} [u_{\Gamma+1}^2 - \ln(1+u_{\Gamma+1}^2)] \\
&= \int_{\mathbb{R}^2} |\nabla u_{\Gamma+1}|^2 - \Gamma \int_{\mathbb{R}^2} [u_{\Gamma+1}^2 - \ln(1+u_{\Gamma+1}^2)] - \int_{\mathbb{R}^2} [u_{\Gamma+1}^2 - \ln(1+u_{\Gamma+1}^2)] \\
&\geq 2e_\Gamma - \int_{\mathbb{R}^2} [u_{\Gamma+1}^2 - \ln(1+u_{\Gamma+1}^2)].
\end{aligned}
$$

Therefore, we may use the Pohozaev identity (3.2) to complete the proof of Lemma 4.4 just as before. □

Now we want to prove (1.17), i.e., $e_\Gamma \geq -\frac{\Gamma}{2} + \sigma \frac{T_1}{2} \ln \Gamma + C_1$ for $\sigma \in (0, 1)$ and $\Gamma > \Gamma_\sigma$ sufficiently large, where Γ_σ is a positive constant depending on σ and C_1 is a constant independent of Γ. By Lemma 4.4 and (1.14) of Theorem 1.2, we have

$$
e_{\Gamma+1} - e_\Gamma \geq -\frac{1}{2}\left(1 + \frac{1}{2}\beta_{\Gamma+1}\right)^2 = -\frac{1}{2}\left[1 + \beta_{\Gamma+1}\left(1 + \frac{1}{4}\beta_{\Gamma+1}\right)\right], \quad \text{for} \quad \Gamma > T_1 .
$$
(4.3)

From Lemma 4.2, $0 > \beta_\Gamma \to 0$ as $\Gamma \to \infty$, implying that for all $\sigma \in (0, 1)$, there exists a positive constant Γ_σ sufficiently large such that $1 + \frac{1}{4}\beta_{\Gamma+1} > \sigma$ for $\Gamma > \Gamma_\sigma$. Hence, (4.3) becomes

$$
e_{\Gamma+1} - e_\Gamma \geq -\frac{1}{2}(1 + \sigma\beta_{\Gamma+1}) \geq -\frac{1}{2}\left(1 - \sigma\frac{T_1}{\Gamma+1}\right) \quad \text{for} \quad \sigma \in (0, 1) \quad \text{and} \quad \Gamma > \Gamma_\sigma .
$$
(4.4)

Here we have used the fact that $\beta_{\Gamma+1} \leq -\frac{T_1}{\Gamma+1}$ due to $\beta_s = -\frac{1}{s}\frac{\int_{\mathbb{R}^2}|\nabla u_s|^2}{\int_{\mathbb{R}^2}[u_s^2 - \ln(1+u_s^2)]} \leq -\frac{T_1}{s}$ for $s > T_1$ and due to the definition of T_1.

Fix $\sigma \in (0, 1)$ arbitrarily and let $N \in \mathbb{N}$ and $N > \Gamma_\sigma$. Then (4.4) gives

$$
e_{k+1} - e_k \geq -\frac{1}{2}\left(1 - \sigma\frac{T_1}{k+1}\right) \quad \text{for} \quad k = N, N+1, N+2, \dots
$$

Hence, for $n \in \mathbb{N}$,

$$
\begin{aligned}
e_{N+n} - e_N &= \sum_{j=0}^{n-1}\left(e_{N+j+1} - e_{N+j}\right) \\
&\geq -\frac{1}{2}\sum_{j=0}^{n-1}\left(1 - \sigma\frac{T_1}{N+j+1}\right) \\
&= -\frac{n}{2} + \sigma\frac{T_1}{2}\sum_{k=1}^{n}\frac{1}{N+k} \\
&\geq -\frac{n}{2} + \sigma\frac{T_1}{2}\int_{N+1}^{N+n}\frac{1}{t}dt \\
&= -\frac{n}{2} + \sigma\frac{T_1}{2}\left[\ln(N+n) - \ln(N+1)\right],
\end{aligned}
$$

yielding $e_{[\Gamma]+1} \geq -\frac{1}{2}[\Gamma] + \sigma\frac{T_1}{2}\ln([\Gamma]+1) + C_N$ for $\Gamma > N$ sufficiently large, where $[\Gamma] = \sup\{k \in \mathbb{N} : k \leq \Gamma\}$ and where we set $[\Gamma] + 1 = N + n$. Consequently, we get $e_\Gamma \geq e_{[\Gamma]+1} \geq -\frac{1}{2}\Gamma + \sigma\frac{T_1}{2}\ln\Gamma + C_1$ and have completed the proof of (1.17) because $[\Gamma] \leq \Gamma \leq [\Gamma] + 1$ and e_Γ is decreasing to Γ for $\Gamma > T_1$. Here C_1 is a constant independent of Γ. Therefore, we have completed the proof of Theorem 1.4.

5 Proof of Theorem 1.5

We first prove the upper bound estimate of the ground state energy $\tilde{e}_\Gamma$.

Lemma 5.1 *Under the same assumptions of Theorem* 1.5, *we have* $\tilde{e}_\Gamma \leq -\frac{\Gamma}{2}(1 + o_\Gamma(1))$ *as* $\Gamma \to \infty$, *where* $o_\Gamma(1)$ *is a small quantity tending to zero as* Γ *goes to infinity.*

Proof Let $u_\tau(x) = \frac{1}{\tau}U\left(\frac{x}{\tau}\right)$ for $x \in \mathbb{R}^2$, $\tau > 0$, where $U \in H^1(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$, $\|U\|_2 = 1$ and $U > 0$ in $\mathbb{R}^2$. Then $\|u_\tau\|_2 = \|U\|_2 = 1$, $\int_{\mathbb{R}^2}|\nabla u_\tau(x)|^2 dx = \tau^{-2}\int_{\mathbb{R}^2}|\nabla U(x)|^2 dx$ and

$$
\begin{aligned}
E[u_\tau] &= \frac{1}{2}\int_{\mathbb{R}^2}|\nabla u_\tau|^2 - \Gamma\left(\sqrt{1+u_\tau^2}-1\right)^2 dx \\
&= \frac{1}{2}\int_{\mathbb{R}^2}|\nabla u_\tau|^2 dx - \frac{\Gamma}{2}\int_{\mathbb{R}^2}u_\tau^2 dx - \Gamma\int_{\mathbb{R}^2}\left(1 - \sqrt{1+u_\tau^2}\right)dx \\
&= \frac{1}{2}\tau^{-2}\int_{\mathbb{R}^2}|\nabla U|^2 dx - \frac{\Gamma}{2} + \Gamma\int_{\mathbb{R}^2}\frac{u_\tau^2}{1+\sqrt{1+u_\tau^2}}dx \\
&= \frac{1}{2}\tau^{-2}\int_{\mathbb{R}^2}|\nabla U|^2 dx - \frac{\Gamma}{2} + \Gamma\tau\int_{\mathbb{R}^2}\frac{U^2}{\tau+\sqrt{\tau^2+U^2}}dx \\
&= -\frac{\Gamma}{2}(1 + o_\Gamma(1))
\end{aligned}
$$

as $\tau \sim (\ln\Gamma)^{-1/2}$ and $\Gamma \to \infty$, where $o_\Gamma(1)$ is a small quantity tending to zero as Γ goes to infinity. Here we have used the fact that $\tau\int_{\mathbb{R}^2}\frac{U^2}{\tau+\sqrt{\tau^2+U^2}}dx \leq \tau\int_{\mathbb{R}^2}U dx \to 0$ because $U \in L^1(\mathbb{R}^2)$ and $\tau \sim (\ln\Gamma)^{-1/2} \to 0$ as $\Gamma \to \infty$. Therefore, we get $\tilde{e}_\Gamma \leq E[u_\tau] = -\frac{\Gamma}{2}(1 + o_\Gamma(1))$ as $\Gamma \to \infty$ and have completed the proof of Lemma 5.1. □

For the lower bound estimate of $\tilde{e}_\Gamma$, it is obvious that

$$
\begin{aligned}
\tilde{e}_\Gamma = E[u_\Gamma] &= \frac{1}{2}\int_{\mathbb{R}^2}|\nabla u_\Gamma|^2 - \Gamma\left(\sqrt{1+u_\Gamma^2}-1\right)^2 dx \\
&= \frac{1}{2}\int_{\mathbb{R}^2}|\nabla u_\Gamma|^2 dx - \frac{\Gamma}{2}\int_{\mathbb{R}^2}u_\Gamma^2 dx + \Gamma\int_{\mathbb{R}^2}\left(\sqrt{1+u_\Gamma^2}-1\right)dx \\
&\geq -\frac{\Gamma}{2}\int_{\mathbb{R}^2}u_\Gamma^2 dx = -\frac{\Gamma}{2},
\end{aligned}
$$

where u_Γ is the ground state (energy minimizer) of $\tilde{e}_\Gamma$ under the L^2-norm constraint $\|u_\Gamma\|_2 = 1$. Consequently, by Lemma 5.1, we obtain $\tilde{e}_\Gamma = -\frac{\Gamma}{2}(1 + o_\Gamma(1))$ as $\Gamma \to \infty$. To get a further estimate of $\tilde{e}_\Gamma$, we need the following lemmas.

Lemma 5.2 *Under the same assumptions of Theorem* 1.5, *we have that the ratio* $\alpha_\Gamma = \frac{\int_{\mathbb{R}^2}|\nabla u_\Gamma|^2 dx}{-\Gamma\int_{\mathbb{R}^2}\left(\sqrt{1+u_\Gamma^2}-1\right)^2 dx} \to 0$ *as* $\Gamma \to \infty$, *where* u_Γ *is the ground state (energy minimizer) of* $\tilde{e}_\Gamma$.

Proof We prove by contradiction. Suppose that α_Γ may not approach zero as Γ goes to infinity. Then by (1.12) of Theorem 1.1, we may assume $\alpha_\Gamma \to -c_0$ as $\Gamma \to \infty$, where $0 < c_0 \leq 1$ is a constant. Hence

$$\tilde{e}_\Gamma = \frac{1}{2}\int_{\mathbb{R}^2}|\nabla u_\Gamma|^2 - \frac{\Gamma}{2}\int_{\mathbb{R}^2}\left(\sqrt{1+u_\Gamma^2}-1\right)^2$$

$$= -\frac{\Gamma}{2}(\alpha_\Gamma+1)\int_{\mathbb{R}^2}\left(\sqrt{1+u_\Gamma^2}-1\right)^2$$

$$= -\frac{\Gamma}{2}(\alpha_\Gamma+1)\left[\int_{\mathbb{R}^2}u_\Gamma^2 + 2\int_{\mathbb{R}^2}\left(1-\sqrt{1+u_\Gamma^2}\right)\right]$$

$$\geq -\frac{\Gamma}{2}(\alpha_\Gamma+1). \tag{5.1}$$

Here we have used the fact that $-1 < \alpha_\Gamma < 0$ and $\|u_\Gamma\|_2 = 1$. Combining Lemma 5.1 and (5.1), we have $-\frac{1}{2}(1+o(1)) \geq -\frac{1}{2}(\alpha_\Gamma+1)$ i.e. $\alpha_\Gamma \geq 0$ as $\Gamma \to \infty$, contradicting $\alpha_\Gamma \to -c_0 \in [-1,0)$ as $\Gamma \to \infty$. □

Remark 5.3 Lemma 5.2 shows that $\alpha_\Gamma \to 0$ as $\Gamma \to \infty$ so the condition $-\frac{1}{2} \leq \alpha_\Gamma < 0$ of Theorem 1.1 (i) can be satisfied as Γ becomes sufficiently large. Consequently, we obtain the eigenvalue estimate

$$0 < \tilde{\lambda}_\Gamma \leq \Gamma(1+\alpha_\Gamma) \quad \text{for} \quad \Gamma > \Gamma_0, \tag{5.2}$$

where Γ_0 is a positive constant and $\tilde{\lambda}_\Gamma$ is the eigenvalue of the ground state u_Γ with ground state energy $\tilde{e}_\Gamma$.

Lemma 5.4 *Under the same assumptions as in Theorem 1.5, $\tilde{e}_\Gamma$ is decreasing to Γ for $\Gamma > T_2$.*

Proof Let u_Γ be the energy minimizer (ground state) of $\tilde{e}_\Gamma$ for $\Gamma > T_2$. Then

$$\tilde{e}_{\Gamma_1} = \frac{1}{2}\int_{\mathbb{R}^2}|\nabla u_{\Gamma_1}|^2 - \Gamma_1\left(\sqrt{1+u_{\Gamma_1}^2}-1\right)^2 dx$$

$$\geq \frac{1}{2}\int_{\mathbb{R}^2}|\nabla u_{\Gamma_1}|^2 - \Gamma_2\left(\sqrt{1+u_{\Gamma_1}^2}-1\right)^2 dx$$

$$\geq \tilde{e}_{\Gamma_2},$$

for $\Gamma_2 > \Gamma_1 > T_2 > 0$. Hence, $\tilde{e}_\Gamma$ is decreasing with respect to Γ and we have completed the proof of Lemma 5.4. □

Lemma 5.5 *Under the same assumptions as in Theorem 1.5, $\tilde{e}_{\Gamma+1} - \tilde{e}_\Gamma \geq -\frac{1}{2}\frac{\tilde{\lambda}_{\Gamma+1}}{\Gamma+1}$ for $\Gamma > T_2$, where $u_{\Gamma+1}$ is the energy minimizer (ground state) of $\tilde{e}_{\Gamma+1}$.*

Proof It is obvious that for $\Gamma > T_2$,

$$\tilde{e}_{\Gamma+1} = \frac{1}{2}\int_{\mathbb{R}^2}|\nabla u_{\Gamma+1}|^2 - (\Gamma+1)\left(\sqrt{1+u_{\Gamma+1}^2}-1\right)^2$$

$$= \frac{1}{2}\int_{\mathbb{R}^2}|\nabla u_{\Gamma+1}|^2 - \Gamma\left(\sqrt{1+u_{\Gamma+1}^2}-1\right)^2 - \frac{1}{2}\int_{\mathbb{R}^2}\left(\sqrt{1+u_{\Gamma+1}^2}-1\right)^2$$

$$\geq \tilde{e}_\Gamma - \frac{1}{2}\int_{\mathbb{R}^2}\left(\sqrt{1+u_{\Gamma+1}^2}-1\right)^2.$$

Therefore, we can use the Pohozaev identity (2.2) to complete the proof of Lemma 5.5. □

Now we are ready to prove (1.19), i.e., $\tilde{e}_\Gamma \geq -\frac{\Gamma}{2} + \frac{T_2}{2} \ln \Gamma + C_2$ for $\Gamma > \Gamma_0$ sufficiently large, where C_2 is a constants independent of Γ. By Lemma 5.5 and (5.2), we have

$$\tilde{e}_{\Gamma+1} - \tilde{e}_\Gamma \geq -\frac{1}{2} (1 + \alpha_{\Gamma+1}) \geq -\frac{1}{2} \left(1 - \frac{T_2}{\Gamma + 1} \right) \tag{5.3}$$

for $\Gamma > \Gamma_0$. Here we have used the fact that $\alpha_{\Gamma+1} \leq -\frac{T_2}{\Gamma+1}$ from

$$\alpha_s = -\frac{1}{s} \frac{\int_{\mathbb{R}^2} |\nabla u_s|^2}{\int_{\mathbb{R}^2} \left(\sqrt{1 + u_s^2} - 1 \right)^2} \leq -\frac{T_2}{s} \quad for \quad s > T_2,$$

due to the definition of T_2. Fix $N \in \mathbb{N}$ and $N > \Gamma_0$. Then (5.3) gives

$$\tilde{e}_{k+1} - \tilde{e}_k \geq -\frac{1}{2} \left(1 - \frac{T_2}{k + 1} \right) \quad \text{for} \quad k = N, N + 1, N + 2, \ldots$$

Consequently, for $n \in \mathbb{N}$,

$$\begin{aligned}
\tilde{e}_{N+n} - \tilde{e}_N &= \sum_{j=0}^{n-1} \left(\tilde{e}_{N+j+1} - \tilde{e}_{N+j} \right) \\
&\geq -\frac{1}{2} \sum_{j=0}^{n-1} \left(1 - \frac{T_2}{N + j + 1} \right) \\
&= -\frac{n}{2} + \frac{T_2}{2} \sum_{k=1}^{n} \frac{1}{N + k} \\
&\geq -\frac{n}{2} + \frac{T_2}{2} \int_{N+1}^{N+n} \frac{1}{t} dt \\
&= -\frac{n}{2} + \frac{T_2}{2} \left[\ln (N + n) - \ln (N + 1) \right],
\end{aligned}$$

yielding $\tilde{e}_{[\Gamma]+1} \geq -\frac{1}{2} [\Gamma] + \frac{T_2}{2} \ln ([\Gamma] + 1) + C_N$ for $\Gamma > N$ sufficiently large, where $[\Gamma] = \sup \{k \in \mathbb{N} : k \leq \Gamma\}$ and where we set $[\Gamma] + 1 = N + n$. Thus we have $\tilde{e}_\Gamma \geq \tilde{e}_{[\Gamma]+1} \geq -\frac{1}{2}\Gamma + \frac{T_2}{2} \ln \Gamma + C_2$ because $[\Gamma] \leq \Gamma \leq [\Gamma] + 1$ and $\tilde{e}_\Gamma$ is decreasing to Γ for $\Gamma > T_2$. Here C_2 is a constant independent of Γ. Therefore, we have obtained (1.19) and completed the proof of Theorem 1.5.

6 Concluding remarks

The *virial theorem* in physics provides a relationship between the time-average of the total kinetic energy and that of the potential energy. For quantum multi-particle systems governed by the linear Schrödinger equation, this often results in an elegant ratio. However, for the Schrödinger equation in optics with non power-law type nonlinearities such as those square-root and saturable types, no virial results were available previously, to the best of our knowledge. Our study has yielded results concerning the virial relation and also the energy estimate of the *ground state*. Still, not too much is known about the higher energy states.

Appendices: Some Technical Propositions and Lemmas

Appendix I

Proposition A.1 *Let* $\hat{f}(s) = \frac{\left(1 - \frac{1}{1+s}\right)s}{s - \ln(1+s)}$ *for* $s > 0$. *Then* $\hat{f}'(s) < 0$ *for* $s > 0$, $\lim\limits_{s \to 0+} \hat{f}(s) = 2$ *and* $\lim\limits_{s \to \infty} \hat{f}(s) = 1$.

Proof By direct calculation,

$$\hat{f}'(s) = (1+s)^{-2}[s - \ln(1+s)]^{-2}\left[2s^2 - (s^2 + 2s)\ln(1+s)\right]$$
$$= (1+s)^{-2}[s - \ln(1+s)]^{-2}\rho(s),$$

for $s > 0$, where $\rho(s) = 2s^2 - (s^2 + 2s)\ln(1+s)$. We claim that $\rho(s) < 0$ for $s > 0$. By direct calculation, $\rho'(s) = s\eta(s)$ and $\eta(s) = 4 - 2\frac{1+s}{s}\ln(1+s) - \frac{2+s}{1+s}$ for $s > 0$. Moreover, $\eta'(s) = -\frac{2}{s^2}\omega(s)$, $\omega(s) = s - \ln(1+s) - \frac{s^2}{2(1+s)^2}$ and $\omega'(s) = \frac{s^2}{(1+s)^2} + \frac{s^2}{(1+s)^3} > 0$ for $s > 0$. Then $\eta'(s) < 0$ for $s > 0$, which gives $\eta(s) < \eta(0) = 0$ for $s > 0$. Hence $\rho'(s) = s\eta(s) < 0$ for $s > 0$, which implies $\rho(s) < \rho(0) = 0$ and $\hat{f}'(s) = (1+s)^{-2}[s - \ln(1+s)]^{-2}\rho(s) < 0$ for $s > 0$ as well. On the other hand, we easily get $\lim\limits_{s \to 0+} \hat{f}(s) = 2$ and $\lim\limits_{s \to \infty} \hat{f}(s) = 1$ by direct calculation. Therefore, we have completed the proof of Proposition A.1. □

Appendix II: Proof of Theorem B

In order to be able to apply the argument of the proof of Theorem A (cf. [12]), we remark that $\left(\sqrt{1+u^2} - 1\right)^2 \leq \frac{1}{4}u^4$ for $u \in \mathbb{R}$ (due to the fact that $0 \leq \sqrt{1+u^2} - 1 \leq \frac{1}{2}u^2$ for $u \in \mathbb{R}$), which is almost same as the crucial inequality $u^2 - \ln(1+u^2) \leq \frac{1}{2}u^4$ for $u \in \mathbb{R}$ in [12]. Then as for Proposition 3.1 and 3.2 in [12], we have the analogs as folllows.

Proposition A.2 *(i)* *Suppose* $\Gamma \in (0, T_3)$, *i.e.,* $0 < \Gamma < T_3$. *Then the value* $\tilde{e}_\Gamma = 0$ *can not be attained by a minimizer.*
(ii) *Suppose* $\Gamma \leq 0$. *Then* $\tilde{e}_\Gamma = 0$ *and the value can not be attained by a minimizer.*

Hence, we can complete the proof of Theorem B (i) by combining Propositions A.2 (i) and (ii).

For the proof of Theorem B (ii), we use the same ideas as in [12] and consider the following problem:

$$\tilde{e}_{\Gamma,\varepsilon} = \inf_{\substack{u \in H_0^1\left(B_{\frac{1}{\varepsilon}}\right) \\ P_\varepsilon[u] = 1}} \tilde{E}_{\Gamma,\varepsilon}[u],$$

where $B_{\frac{1}{\varepsilon}}$ is the ball with radius $\frac{1}{\varepsilon}$ centered at the origin in $\mathbb{R}^2$,

$$\tilde{E}_{\Gamma,\varepsilon}[u] = \frac{1}{2}\int_{B_{\frac{1}{\varepsilon}}} |\nabla u|^2 - \Gamma\left(\sqrt{1+u^2}-1\right)^2,$$

$P_\varepsilon[u] = \int_{B_{\frac{1}{\varepsilon}}} u^2$ for $\varepsilon > 0$ and $u \in H_0^1\left(B_{\frac{1}{\varepsilon}}\right)$. Note that $\left(\sqrt{1+u^2}-1\right)^2 = u^2 + 2\left(1-\sqrt{1+u^2}\right) \le u^2$ and $\Gamma > T_3 > 0$ so the lower bound estimate $\tilde{E}_{\Gamma,\varepsilon}[u] \ge -\frac{1}{2}\Gamma$ holds true. Hence, we may apply a symmetric-decreasing rearrangement (see Chapter 3 in [10]) and the truncation argument (see the proof of Lemma 3.3 in [12]) to prove the following lemmas.

Lemma A.4 *Assume $\Gamma > T_3 > 0$.*

(i) *For $\varepsilon > 0$, $\tilde{e}_{\Gamma,\varepsilon}$ can be achieved by a minimizer $u_\varepsilon = u_\varepsilon(r) \ge 0$ which is a function radially symmetric and monotone decreasing with r.*

(ii) *For $\varepsilon > 0$ sufficiently small, $\tilde{e}_{\Gamma,\varepsilon} \le -c_0$, where c_0 is a positive constant independent of ε.*

Lemma A.5 *Under the same hypothesis of Lemma A.4, minimizer u_ε satisfies*

$$\|u_\varepsilon\|_{H^1\left(B_{\frac{1}{\varepsilon}}\right)} \le K_0,$$

for $\varepsilon > 0$ sufficiently small, where K_0 is a positive constant independent of ε.

The minimizer u_ε satisfies the following equation:

$$\Delta u_\varepsilon + \Gamma\left(1 - \frac{1}{\sqrt{1+u_\varepsilon^2}}\right)u_\varepsilon = \lambda_\varepsilon u_\varepsilon \text{ in } B_{\frac{1}{\varepsilon}}, \tag{6.1}$$

with the zero Dirichlet boundary condition $u_\varepsilon = 0$ on $\partial B_{\frac{1}{\varepsilon}}$, where λ_ε is the associated Lagrange multiplier. Multiply Eq. (6.1) by u_ε and integrate over $B_{\frac{1}{\varepsilon}}$. Then using integration by parts and $\int_{B_{\frac{1}{\varepsilon}}} u_\varepsilon^2 = 1$, we get $\lambda_\varepsilon = \lambda_\varepsilon \int_{B_{\frac{1}{\varepsilon}}} u_\varepsilon^2 = -\int_{B_{\frac{1}{\varepsilon}}} |\nabla u_\varepsilon|^2 + \Gamma\int_{B_{\frac{1}{\varepsilon}}}\left(1 - \frac{1}{\sqrt{1+u_\varepsilon^2}}\right)u_\varepsilon^2$. Hence, by Lemma A.5,

$$|\lambda_\varepsilon| \le K_1, \tag{6.2}$$

where K_1 is a positive constant independent of ε. Here we have used the fact that $0 \le \left(1 - \frac{1}{\sqrt{1+u_\varepsilon^2}}\right)u_\varepsilon^2 \le u_\varepsilon^2$ and $\|u_\varepsilon\|_2 = 1$. Since $u_\varepsilon = u_\varepsilon(r)$ is radially symmetric, Eq. (6.1) and the zero Dirichlet boundary condition can be reduced to a boundary value problem of an ordinary differential equation as follows:

$$\begin{cases} u''_\varepsilon + \frac{1}{r}u'_\varepsilon + \Gamma\left(1 - \frac{1}{\sqrt{1+u_\varepsilon^2}}\right)u_\varepsilon = \lambda_\varepsilon u_\varepsilon \text{ for } 0 < r < \frac{1}{\varepsilon}, \\ u'_\varepsilon(0) = 0, \qquad\qquad\qquad u_\varepsilon\left(\frac{1}{\varepsilon}\right) = 0. \end{cases} \tag{6.3}$$

Then by the uniqueness of ordinary differential equations, we can show the following.

Lemma A.6 *The minimizer $u_\varepsilon = u_\varepsilon(r)$ is positive for $0 < r < \frac{1}{\varepsilon}$.*

We may extend u_ε to the entire plane $\mathbb{R}^2$ by setting $u_\varepsilon(r) = 0$ for $r > \frac{1}{\varepsilon}$. *Note that each u_ε is radially symmetric and of $H_r^1(\mathbb{R}^2)$, where $H_r^1(\mathbb{R}^2)$ is the function space consisting of $H^1(\mathbb{R}^2)$ functions with radial symmetry. Then Lemma* A.5 *gives* $\|u_\varepsilon\|_{H_r^1(\mathbb{R}^2)} \le K_0$, *yielding*

$$u_\varepsilon \to U \text{ weakly in } H_r^1\left(\mathbb{R}^2\right) \tag{6.4}$$

as ε goes to zero (up to a subsequence). Furthermore, by the compact embedding of H_r^1 radial functions in the space of L_r^4 functions (cf. [13]), we have

$$u_\varepsilon \to U \text{ in } L_r^4\left(\mathbb{R}^2\right) \tag{6.5}$$

as ε goes to zero (up to a subsequence). Now, we want to prove that U is nontrivial. Due to $\left(\sqrt{1+u^2} - 1\right)^2 \le \frac{1}{4}u^4$ for $u \in \mathbb{R}$, we get $\left(\sqrt{1+u_\varepsilon^2} - 1\right)^2 \le \frac{1}{4}u_\varepsilon^4$. Hence, by (6.4), (6.5) *and Lemma* A.4 *(ii),*

$$
\begin{aligned}
-c_0 &\ge \liminf_{\varepsilon \to 0+} \tilde{e}_{\Gamma,\varepsilon} \\
&= \liminf_{\varepsilon \to 0+} \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u_\varepsilon|^2 - \Gamma\left(\sqrt{1+u_\varepsilon^2} - 1\right)^2, \\
&\ge \liminf_{\varepsilon \to 0+} \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u_\varepsilon|^2 - \frac{1}{4}\Gamma u_\varepsilon^4 \\
&\ge \frac{1}{2}\int_{\mathbb{R}^2} |\nabla U|^2 - \frac{1}{4}\Gamma U^4,
\end{aligned}
$$

showing that U is nontrivial. Here Fatou's Lemma has been used. Otherwise, $U \equiv 0$ and we would have $0 > -c_0 \ge 0$, a contradiction.

Now we claim that the limit function U satisfies

$$\Delta U + \Gamma\left(1 - \frac{1}{\sqrt{1+U^2}}\right)U = \lambda_0 U \text{ in } \mathbb{R}^2, \tag{6.6}$$

$U = U(r) \ge 0$ *is radially symmetric, and* $\lim_{r \to \infty} U(r) = 0$, *where λ_0 is the limit of λ_ε's (up to a subsequence) since* (6.2) *implies*

$$\lambda_\varepsilon \to \lambda_0 \text{ as } \varepsilon \to 0+ \text{ (up to a subsequence).} \tag{6.7}$$

Let $\phi \in C_0^\infty(\mathbb{R}^2)$ be any test function and let $\varepsilon > 0$ be such that $\operatorname{supp}(\phi) \subset B_{\frac{1}{\varepsilon}}$. Since u_ε satisfies (6.1), *we have*

$$\int_{\mathbb{R}^2} \nabla u_\varepsilon \cdot \nabla\phi - \Gamma\int_{\mathbb{R}^2}\left(1 - \frac{1}{\sqrt{1+u_\varepsilon^2}}\right)u_\varepsilon\phi = -\lambda_\varepsilon\int_{\mathbb{R}^2} u_\varepsilon\phi. \tag{6.8}$$

Hence, (6.4) *and* (6.5) *give*

$$\int_{\mathbb{R}^2} \nabla u_\varepsilon \cdot \nabla\phi \to \int_{\mathbb{R}^2} \nabla U \cdot \nabla\phi,$$

$$\int_{\mathbb{R}^2} u_\varepsilon\phi \to \int_{\mathbb{R}^2} U\phi,$$

and

$$\int_{\mathbb{R}^2} \frac{u_\varepsilon}{\sqrt{1+u_\varepsilon^2}} \phi = \int_{\mathbb{R}^2} (u_\varepsilon - U) \frac{\phi}{\sqrt{1+u_\varepsilon^2}} + \int_{\mathbb{R}^2} \frac{1}{\sqrt{1+u_\varepsilon^2}} U\phi$$

$$= \int_{\mathbb{R}^2} (u_\varepsilon - U) \frac{\phi}{\sqrt{1+u_\varepsilon^2}} + \int_{\mathbb{R}^2} \frac{1}{\sqrt{1+U^2}} U\phi + \int_{\mathbb{R}^2} \left(\frac{1}{\sqrt{1+u_\varepsilon^2}} - \frac{1}{\sqrt{1+U^2}} \right) U\phi$$

$$= \int_{\mathbb{R}^2} (u_\varepsilon - U) \frac{\phi}{\sqrt{1+u_\varepsilon^2}} + \int_{\mathbb{R}^2} \frac{1}{\sqrt{1+U^2}} U\phi$$

$$+ \int_{\mathbb{R}^2} (U - u_\varepsilon) \frac{U + u_\varepsilon}{\sqrt{1+u_\varepsilon^2}\sqrt{1+U^2} \left(\sqrt{1+U^2} + \sqrt{1+u_\varepsilon^2} \right)} U\phi$$

$$\to \int_{\mathbb{R}^2} \frac{1}{\sqrt{1+U^2}} U\phi .$$

Note that $\left\| \frac{\phi}{\sqrt{1+u_\varepsilon^2}} \right\|_{\frac{4}{3}} \leq \|\phi\|_{\frac{4}{3}}$ *and* $\left\| \frac{U+u_\varepsilon}{\sqrt{1+u_\varepsilon^2}\sqrt{1+U^2}\left(\sqrt{1+U^2}+\sqrt{1+u_\varepsilon^2}\right)} U\phi \right\|_{\frac{4}{3}} \leq \|U\phi\|_{\frac{4}{3}} \leq$ $\|U\|_4 \|\phi\|_2$ *as*

$$0 \leq \frac{U + u_\varepsilon}{\sqrt{1+u_\varepsilon^2}\sqrt{1+U^2} \left(\sqrt{1+U^2} + \sqrt{1+u_\varepsilon^2} \right)} \leq 1 .$$

Thus, (6.8) *and* (6.7) *imply*

$$\int_{\mathbb{R}^2} \nabla U \cdot \nabla \phi - \Gamma \int_{\mathbb{R}^2} \left(1 - \frac{1}{\sqrt{1+U^2}} \right) U\phi = -\lambda_0 \int_{\mathbb{R}^2} U\phi ,$$

from which U satisfies (6.6) *and* $\lim_{|x| \to \infty} U(x) = 0$. *Moreover,* $U = U(r) \geq 0$ *is radially symmetric, since each* u_ε *is positive and radially symmetric. Therefore, the equation for U can be written as*

$$\begin{cases} U'' + \frac{1}{r}U' + \Gamma \left(1 - \frac{1}{\sqrt{1+U^2}} \right) U = \lambda_0 U \text{ for } r > 0, \\ U'(0) = 0, \qquad\qquad\qquad\qquad U(\infty) = 0. \end{cases}$$

By the uniqueness of ordinary differential equations, we have obtained the following.

Lemma A.7 $U(r) > 0$ *for* $r \geq 0$.   □

Due to the fact that $\lim_{r \to \infty} U(r) = 0$, there exists $R_1 > 0$ such that $0 < U(r) \leq 1$ for $r \geq R_1$. By Eq. (6.6), $\Delta U = \left(\lambda_0 - \Gamma \left(1 - \frac{1}{\sqrt{1+U^2}} \right) \right) U \in L^4(B_{R_1})$, since $\lambda_0 - \Gamma \left(1 - \frac{1}{\sqrt{1+U^2}} \right) \in L^\infty$. Hence, by the standard regularity theorem of the Poisson equation, $U \in W^{2,4}(B_{R_1})$ and then by the Sobolev embedding $W^{2,4}(B_{R_1}) \subset L^\infty(B_{R_1})$, we have obtained the following.

Lemma A.8 $U(r) \leq K_2$ *for* $r \geq 0$, *where* K_2 *is a positive constant.*   □

Now we prove that λ_0 is positive by contradiction. Suppose $\lambda_0 \leq 0$. Then Eq. (6.6) and Lemma A.7 imply

$$\Delta U = \lambda_0 U - \Gamma \left(1 - \frac{1}{\sqrt{1+U^2}} \right) U \leq 0 \text{ in } \mathbb{R}^2 .$$

Therefore, by Lemma A.8 and the Liouville Theorem, U must be a constant function, i.e., $U \equiv 0$, which is impossible. Therefore, we conclude the following.

Lemma A.9 *The limit λ_0 of λ_ε as $\varepsilon \downarrow 0$ satisfies $\lambda_0 > 0$.* □

By (6.5), u_ε converges to U almost everywhere as $\varepsilon \downarrow 0$ (up to a subsequence). Moreover, since each u_ε is monotone decreasing with r, we have the following.

Lemma A.10 $U = U(r)$ *is monotone decreasing with r.* □

To complete the proof of Theorem B, we now only need to prove that $\|U\|_2 = 1$ if $u_\varepsilon \to U$ strongly in $L^2(\mathbb{R}^2)$ as $\varepsilon \to 0$ (up to a subsequence). Note that each u_ε satisfies $\|u_\varepsilon\|_2 = 1$. Fix $\sigma > 0$ and consider the set $N_{\sigma,\varepsilon} = \{r > 0 : u_\varepsilon(r) > \sigma\}$. Then by Lemmas A.4 (i) and A.6, $N_{\sigma,\varepsilon} = (0, R_{\sigma,\varepsilon})$ and

$$\pi \sigma^2 R_{\sigma,\varepsilon}^2 \le 2\pi \int_0^{R_{\sigma,\varepsilon}} r u_\varepsilon^2(r)\, dr \le 2\pi \int_0^\infty r u_\varepsilon^2(r)\, dr = \|u_\varepsilon\|_2^2 = 1,$$

implying

$$R_{\sigma,\varepsilon} \le \frac{1}{\sigma\sqrt{\pi}}. \tag{6.9}$$

Note that the upper bound of $R_{\sigma,\varepsilon}$ is $\frac{1}{\sigma\sqrt{\pi}}$ independent of ε. Hence, (6.4) gives

$$u_\varepsilon \to U \text{ weakly in } H_r^1\left(B_{\frac{1}{\sigma\sqrt{\pi}}}\right),$$

and the standard Sobolev compact embedding $H^1\left(B_{\frac{1}{\sigma\sqrt{\pi}}}\right) \hookrightarrow L^2\left(B_{\frac{1}{\sigma\sqrt{\pi}}}\right)$ implies

$$u_\varepsilon \to U \quad \text{strongly in } L^2\left(B_{\frac{1}{\sigma\sqrt{\pi}}}\right) \tag{6.10}$$

as ε goes to zero (up to a subsequence). Moreover, (6.9) implies $r \notin N_{\sigma,\varepsilon}$ and $u_\varepsilon(r) \le \sigma$ for $r > \frac{1}{\sigma\sqrt{\pi}}$. Let $\sigma > 0$ be sufficiently small such that $\frac{\lambda_0}{2} > \Gamma\sigma^2$. Then by (6.3), (6.7) and Lemma A.9,

$$u''_\varepsilon + \frac{1}{r}u'_\varepsilon = -\Gamma\left(1 - \frac{1}{\sqrt{1+u_\varepsilon^2}}\right)u_\varepsilon + \lambda_\varepsilon u_\varepsilon$$

$$\ge \left(-\Gamma\sigma^2 + \lambda_\varepsilon\right)u_\varepsilon$$

$$\ge \frac{\lambda_0}{2}u_\varepsilon \qquad \text{for } \frac{1}{\sigma\sqrt{\pi}} < r < \frac{1}{\varepsilon},$$

i.e.,

$$u''_\varepsilon + \frac{1}{r}u'_\varepsilon \ge \frac{\lambda_0}{2}u_\varepsilon \quad \text{for} \quad \frac{1}{\sigma\sqrt{\pi}} < r < \frac{1}{\varepsilon}. \tag{6.11}$$

Here we have used the fact that $1 - \frac{1}{\sqrt{1+u_\varepsilon^2}} = \frac{u_\varepsilon^2}{\sqrt{1+u_\varepsilon^2}\left(\sqrt{1+u_\varepsilon^2}+1\right)} \le u_\varepsilon^2 \le \sigma^2$ for $r > \frac{1}{\sigma\sqrt{\pi}}$. Note that λ_0 is a positive constant independent of ε. Consequently, Eq. (6.11) gives

$$u_\varepsilon(r) \le e^{-\alpha r} \quad \text{for} \quad r > \frac{1}{\sigma\sqrt{\pi}}, \tag{6.12}$$

where $\alpha > 0$ is a constant independent of ε. Therefore, by (6.10), (6.12) and the Dominated Convergence Theorem, we get $u_\varepsilon \to U$ strongly in $L^2 (\mathbb{R}^2)$, so the proof of Theorem B is complete.

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