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Calculus of Variations

Compactness results for static and dynamic chiral skyrmions near the conformal limit

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Abstract We examine lower order perturbations of the harmonic map problem from $\mathbb{R}^2$ to $\mathbb{S}^2$ including chiral interaction in form of a helicity term that prefers modulation, and a potential term that enables decay to a uniform background state. Energy functionals of this type arise in the context of magnetic systems without inversion symmetry. In the almost conformal regime, where these perturbations are weighted with a small parameter, we examine the existence of relative minimizers in a non-trivial homotopy class, so-called chiral skyrmions, strong compactness of almost minimizers, and their asymptotic limit. Finally we examine dynamic stability and compactness of almost minimizers in the context of the Landau–Lifshitz–Gilbert equation including spin-transfer torques arising from the interaction with an external current.

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1 Introduction and main results

Isolated chiral skyrmions are homotopically nontrivial field configurations $\boldsymbol{m} : \mathbb{R}^2 \to \mathbb{S}^2$ occurring as relative energy minimizers in magnetic systems without inversion symmetry. In such systems the leading-order interaction is Heisenberg exchange in terms of the Dirichlet energy

$$D(\boldsymbol{m}) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla \boldsymbol{m}|^2 \, dx.$$

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Chiral interactions, in magnetism known as antisymmetric exchange or Dzyaloshinskii–Moriya interactions, are introduced in terms of Lifshitz invariants, the components of the tensor $\nabla \boldsymbol{m} \times \boldsymbol{m}$. A prototypical form is obtained by taking the trace, which yields the helicity functional

$$H(\boldsymbol{m}) = \int_{\mathbb{R}^2} \boldsymbol{m} \cdot (\nabla \times \boldsymbol{m}) \, dx,$$

well-defined for moderately smooth $\boldsymbol{m}$ that decay appropriately to a uniform background state. Extensions to the canonical energy space will be discussed later.

Chiral interactions are sensitive to independent rotations and reflections in the domain $\mathbb{R}^2$ and the target $\mathbb{S}^2$, and therefore select specific field orientations. The helicity prefers curling configurations. The uniform background state $\boldsymbol{m}(x) \to \hat{\boldsymbol{e}}_3$ as $|x| \to \infty$ is fixed by a potential energy $V(\boldsymbol{m}) = V_p(\boldsymbol{m})$ depending on a power $2 \le p \le 4$ with

$$V_p(\boldsymbol{m}) = \frac{1}{2^p} \int_{\mathbb{R}^2} |\boldsymbol{m} - \hat{\boldsymbol{e}}_3|^p \, dx.$$

The borderline case $p = 2$ corresponds to the classical Zeeman interaction with an external magnetic field. The case $p = 4$ turns out to play a particular mathematical role in connection with helicity. From the point of view of physics, since $\frac{1}{4}|\boldsymbol{m}-\hat{\boldsymbol{e}}_3|^4 = |\boldsymbol{m}-\hat{\boldsymbol{e}}_3|^2 + (\boldsymbol{m}\cdot\hat{\boldsymbol{e}}_3)^2 - 1$, the case $p = 4$ features a specific combination of Zeeman and in-plane anisotropy interaction. Upon scaling, the governing energy functional

$$E_\varepsilon(\boldsymbol{m}) = D(\boldsymbol{m}) + \varepsilon\big(H(\boldsymbol{m}) + V(\boldsymbol{m})\big)$$

only depends on one coupling constant $\varepsilon > 0$. For $p = 2$ variants of this functional have been examined in physics literature, see e.g. [3,4,11], predicting the occurrence of specific topological defects, so-called chiral skyrmions, arranged in a regular lattice or as isolated topological soliton. In our scaling, tailored towards an asymptotic analysis, the parameter ε corresponds to the inverse of the renormalized strength of the applied field. The almost conformal regime $0 < \varepsilon \ll 1$ features the ferromagnetic phase of positive energies, where H is dominated by D and V, i.e. $E_\varepsilon(\boldsymbol{m}) \gtrsim D(\boldsymbol{m}) + \varepsilon V(\boldsymbol{m})$. In this case the configuration space

$$\mathcal{M} = \{\boldsymbol{m} : \mathbb{R}^2 \to \mathbb{S}^2 : D(\boldsymbol{m}) + V(\boldsymbol{m}) < \infty\},$$

admits the structure of a complete metric space (see below). In the ferromagnetic regime, $\boldsymbol{m} \equiv \hat{\boldsymbol{e}}_3$ is the unique global energy minimizer, while chiral skyrmions are expected to occur as relative energy minimizers in a nontrivial homotopy class. In the case $p = 2$ and for $0 < \varepsilon \ll 1$ this has been proven in [21].

Homotopy classes are characterized by the topological charge (Brouwer degree)

$$Q(\boldsymbol{m}) = \frac{1}{4\pi} \int_{\mathbb{R}^2} \boldsymbol{m} \cdot (\partial_1 \boldsymbol{m} \times \partial_2 \boldsymbol{m}) \, dx \in \mathbb{Z},$$

which decomposes the configuration space into its path-connected components, the topological sectors. In view of the background state $\hat{\boldsymbol{e}}_3$, the specific topological charge $Q(\boldsymbol{m}) = -1$ is energetically selected by the presence of helicity interaction. In fact, for all $2 \le p \le 4$ we have

$$\inf \{E_\varepsilon(\boldsymbol{m}) : \boldsymbol{m} \in \mathcal{M} \text{ with } Q(\boldsymbol{m}) = -1\} < 4\pi \quad \text{for} \quad \varepsilon > 0,$$

less than the classical topological lower bound for the Dirichlet energy, while

$$\inf \{E_\varepsilon(\boldsymbol{m}) : \boldsymbol{m} \in \mathcal{M} \text{ with } Q(\boldsymbol{m}) \notin \{0, -1\}\} > 4\pi \quad \text{for} \quad \varepsilon \ll 1,$$

a consequence of the energy bounds provided in Sect. 2.

These properties are in contrast to two-dimensional versions of the classical Skyrme functional (see e.g. [2,24]) featuring full rotation and reflection symmetry. Here, the helicity term is replaced by the the Skyrme term

$$S(\boldsymbol{u}) = \frac{1}{4}\int_{\mathbb{R}^2} |\partial_1 \boldsymbol{u} \times \partial_2 \boldsymbol{u}|^2 \, dx,$$

a higher order perturbation of $D(\boldsymbol{u})$, which prevents a finite energy collapse of the topological charge due to concentration effects. In particular, the energy functional $D(\boldsymbol{u}) + \lambda S(\boldsymbol{u}) + \mu V(\boldsymbol{u})$, for positive coupling constants λ, μ, has an energy range above 4π in every non-trivial homotopy class. In the case $p = 4$, the attainment of least energies for unit charge configurations and topologically non-trivial configurations has been examined in [16–18] and [18], respectively. Explicit minimizers arise for $p = 8$, see [24]. We shall recover this situation in the chiral case for $p = 4$.

Our first result confirms existence of (global) minimizers of E_ε in $\mathcal{M}$, subject to the constraint $Q = -1$, extending the result in [21] for $p = 2$ to the whole range $2 \le p \le 4$ of exponents:

Theorem 1 (Existence of minimizers) *Suppose $2 \le p \le 4$ and $0 < \varepsilon \ll 1$. Then the infimum of E_ε in $\mathcal{M}$ subject to the constraint $Q = -1$ is attained by a continuous map $\boldsymbol{m}_\varepsilon$ in this homotopy class such that*

$$4\pi(1 - 4\varepsilon) \le E_\varepsilon(\boldsymbol{m}_\varepsilon) \le 4\pi\big(1 - 2(p-2)\varepsilon\big). \tag{1}$$

For $p = 2$ and $0 < \varepsilon \ll 1$, we have, more precisely,

$$E_\varepsilon(\boldsymbol{m}_\varepsilon) \le 4\pi\left(1 - (4 + o(1))\,\frac{\varepsilon}{|\ln \varepsilon|}\right).$$

If $p = 4$, minimizers are characterized by the equation

$$\mathcal{D}_1 \boldsymbol{m} + \boldsymbol{m} \times \mathcal{D}_2 \boldsymbol{m} = 0 \quad \textit{where} \quad \mathcal{D}_i \boldsymbol{m} = \partial_i \boldsymbol{m} - \tfrac{1}{2}\hat{\boldsymbol{e}}_i \times \boldsymbol{m}. \tag{2}$$

For $2 \le p < 4$, Theorem 1 is obtained by a concentration-compactness argument similar to [17,21]: Provided "vanishing" holds, we prove that the helicity functional becomes negligible, so that the energy of a minimizing sequence approaches 4π, which contradicts the upper bound coming from Lemma 3 below. If "dichotomy" holds, the cut-off result Lemma 8 (see "Appendix 1") yields a comparison function with an energy well below the global minimum in its homotopy class. Hence, neither vanishing nor dichotomy appear.

The case $p = 4$ is special in the sense that vanishing can no longer be ruled out within our approach. However, upper and lower energy bounds match, so that an explicit energy-minimizer in form of a specifically adapted stereographic map $\boldsymbol{m}_0$ is available. It follows that $\boldsymbol{m}_0$ belongs to the class

$$\mathcal{C} := \left\{\boldsymbol{m} : \mathbb{R}^2 \to \mathbb{S}^2 : D(\boldsymbol{m}) = 4\pi,\ Q(\boldsymbol{m}) = -1,\ \lim_{x \to \infty} \boldsymbol{m}(x) = \hat{\boldsymbol{e}}_3\right\}$$

consisting of anti-conformal (harmonic) maps of minimal energy. Recall that harmonic maps on $\mathbb{R}^2$ with finite energy extend to harmonic maps on $\mathbb{S}^2$ (cf. [25]) with a well-defined limit as $x \to \infty$.

Anti-conformal maps are characterized by the equation $\partial_1 \boldsymbol{m} - \boldsymbol{m} \times \partial_2 \boldsymbol{m} = 0$, a geometric version of the Cauchy–Riemann equation. Hence, identifying $\mathbb{R}^2 \simeq \mathbb{C}$, the moduli space of $\mathcal{C}$ is $\mathbb{C}\backslash\{0\} \times \mathbb{C}$. More precisely, $\mathcal{C}$ agrees with the two-parameter family of maps $\boldsymbol{m}_0(z) = \Phi(az + b)$ for $z \in \mathbb{C}$, where $(a, b) \in \mathbb{C}\backslash\{0\} \times \mathbb{C}$ and $\Phi \colon \mathbb{R}^2 \simeq \mathbb{C} \to \mathbb{S}^2$ is a stereographic map of negative degree with $\lim_{x\to\infty}\Phi(x) = \hat{\boldsymbol{e}}_3$, cf. [5, Lemma A.1]. Note that $\mathcal{C}\cap\mathcal{M}$ is empty in the limit case $p = 2$.

In the context of the energies E_ε, the degeneracy of a map $\boldsymbol{m}_0 \in \mathcal{C}$ with respect to the complex scaling parameter a is lifted if

(i) it safisfies the Bogomolny type equation (2), i.e. is also an energy minimizer subject to $Q = -1$ for $p = 4$ and $\varepsilon > 0$ arbitrary, or
(ii) it is obtained from a family of chiral skyrmions $\{\boldsymbol{m}_\varepsilon\}_{\varepsilon\ll 1}$, which we prove for $2 < p < 4$ and conjecture in the limit cases $p \in \{2, 4\}$:

Theorem 2 (Compactness of almost minimizers) *Suppose* $2 < p < 4$ *and* $\{\boldsymbol{m}_\varepsilon\}_{\varepsilon\ll 1} \subset \mathcal{M}$ *is a family such that*

$$Q(\boldsymbol{m}_\varepsilon) = -1 \quad \textit{and} \quad E_\varepsilon(\boldsymbol{m}_\varepsilon) \le 4\pi - C_0\varepsilon$$

for some constant $C_0 > 0$. *Then, we have:*

(i) There exists $\boldsymbol{m}_0 \in \mathcal{C}$ *so that for* $\varepsilon \to 0$, *up to translations and a subsequence,*

$$\nabla \boldsymbol{m}_\varepsilon \to \nabla \boldsymbol{m}_0 \quad \textit{strongly in } L^2(\mathbb{R}^2)$$

and

$$\boldsymbol{m}_\varepsilon - \boldsymbol{m}_0 \rightharpoonup 0 \quad \textit{weakly in } L^p(\mathbb{R}^2).$$

(ii) If $\{\boldsymbol{m}_\varepsilon\}_{\varepsilon\ll 1}$ *satisfies the more restrictive upper bound*

$$E_\varepsilon(\boldsymbol{m}_\varepsilon) \le 4\pi + \varepsilon \min_{\boldsymbol{m}\in\mathcal{C}}\big(H(\boldsymbol{m}) + V(\boldsymbol{m})\big) + o(\varepsilon) \quad \textit{for } \varepsilon \to 0,$$

then, modulo translations, the whole family converges to a unique limit $\boldsymbol{m}_0 \in \mathcal{C}$, *which is determined by*

$$H(\boldsymbol{m}_0) + V(\boldsymbol{m}_0) = \min_{\boldsymbol{m}\in\mathcal{C}}\big(H(\boldsymbol{m}) + V(\boldsymbol{m})\big) = -8\pi(p-2),$$

such that $\boldsymbol{m}_\varepsilon - \boldsymbol{m}_0 \to 0$ *strongly in* $L^p(\mathbb{R}^2)$. *Moreover,*

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}(E_\varepsilon(\boldsymbol{m}_\varepsilon) - 4\pi) = \min_{\boldsymbol{m}\in\mathcal{C}}\big(H(\boldsymbol{m}) + V(\boldsymbol{m})\big).$$

We also have weak or strong $L^{\frac{p}{2}}$ subconvergence of $1 - m_{\varepsilon,3}$ in claim (i) and (ii), respectively. Theorem 2 applies in particular to the family $\{\boldsymbol{m}_\varepsilon\}_{\varepsilon>0}$ of minimizers that has been constructed in Theorem 1. Fixing the adapted stereographic map

$$\Phi\colon \mathbb{R}^2 \to \mathbb{S}^2, \quad \Phi(x) = \left(\frac{2x^\perp}{1+|x|^2}, -\frac{1-|x|^2}{1+|x|^2}\right), \tag{3}$$

so that $Q(\Phi) = -1$ and $\lim_{x\to\infty}\Phi(x) = \hat{\boldsymbol{e}}_3$, we have

$$\boldsymbol{m}_0(x) = \Phi\left(\frac{x}{2(p-2)}\right) \quad \text{for } x \in \mathbb{R}^2.$$

It remains an open question whether for positive ε the minimizers $\boldsymbol{m}_\varepsilon$ of E_ε in the homotopy class $\{Q = -1\}$ are actually unique (up to translations) and axially symmetric. As a first step and for $2 < p < 4$, Theorem 2 implies that $\boldsymbol{m}_\varepsilon$ is at least close in $\dot{H}^1$ and L^p to the unique, axially symmetric vector field $\boldsymbol{m}_0$ given above.

Similar to the existence of minimizers of E_ε, Theorem 2 is proven by means of P. L. Lions' concentration-compactness principle. However, since the minimal energy tends to 4π

as $\varepsilon \to 0$, the argument of Theorem 1 needs to be modified in a suitable way. In fact, in order to rule out "dichotomy", we will use the boundedness of the lower-order correction $H + V$ to the Dirichlet energy D, which comes from the matching upper and lower a-priori bounds to the minimal energy and is preserved by the cut-off result Lemma 8. As a consequence, we obtain a comparison vector-field of non-zero degree with Dirichlet energy strictly below 4π, contradicting the classical topological lower bound $D(\boldsymbol{m}) \geq 4\pi|Q(\boldsymbol{m})|$. "Vanishing", on the other hand, would imply that the helicity functional becomes negligible along a sequence of (almost-)minimizers, which is again ruled out by the a-priori bounds.

The second part of this paper addresses the dynamic stability of spin-current driven chiral skyrmions in the almost conformal regime $\varepsilon \ll 1$. This is ultimately a question of regularity for the Landau–Lifshitz–Gilbert equation, for which finite time blow-up, typically accompanied by topological changes, has to be expected if energy accumulates to the critical threshold of 4π. In the presence of an in-plane spin-velocity $v \in \mathbb{R}^2$ the Landau–Lifshitz–Gilbert equation is given by

$$\partial_t \boldsymbol{m} + (v \cdot \nabla)\boldsymbol{m} = \boldsymbol{m} \times \left[\alpha\, \partial_t \boldsymbol{m} + \beta(v \cdot \nabla)\boldsymbol{m} - \boldsymbol{h}_\varepsilon(\boldsymbol{m})\right] \tag{4}$$

where α and β are positive constants and

$$\boldsymbol{h}_\varepsilon(\boldsymbol{m}) = -\mathrm{grad}\, E_\varepsilon(\boldsymbol{m})$$

is the effective field, see [8,14,15,22,26,28] for a mathematical account. In the Galilean invariant case $\alpha = \beta$ traveling wave solutions are obtained by transporting equilibria $\boldsymbol{m} \times \boldsymbol{h}_\varepsilon = 0$ along $c = v$. In the conformal case $\varepsilon = 0$, as observed in [14], traveling wave solutions are obtained for arbitrary α and β by transporting conformal or anti-conformal equilibria of unit degree along $c \in \mathbb{R}^2$ determined by the free Thiele equation

$$(c - v)^\perp = \alpha c - \beta v.$$

We are interested in the regime $0 < \varepsilon \ll 1$ in the case $p = 4$, where the effective field

$$\boldsymbol{h}_\varepsilon(\boldsymbol{m}) = \Delta \boldsymbol{m} - \varepsilon\,(2\nabla \times \boldsymbol{m} + \boldsymbol{f}(\boldsymbol{m}))$$

admits the smooth potential term

$$\boldsymbol{f}(\boldsymbol{m}) = \frac{1}{4}|\boldsymbol{m} - \hat{\boldsymbol{e}}_3|^2(\boldsymbol{m} - \hat{\boldsymbol{e}}_3).$$

Taking into account the asymptotic behavior of almost minimizers, it is natural to pass to the moving frame

$$\boldsymbol{m}(x,t) \mapsto \boldsymbol{m}(x + ct, t) \quad \text{where} \quad (c - v)^\perp = \alpha c - \beta v. \tag{5}$$

After a rigid rotation in space (see "Appendix 3"), this yields the pulled back equation

$$(\partial_t - \nu \partial_z)\boldsymbol{m} = \boldsymbol{m} \times \left[\alpha(\partial_t - \nu\partial_z)\boldsymbol{m} - \boldsymbol{h}_\varepsilon(\boldsymbol{m})\right] \tag{6}$$

with effective coupling parameter

$$\nu = \frac{2(\alpha - \beta)v}{1 + \alpha^2},$$

where $v > 0$ is now the intensity of the spin current, and with the Cauchy–Riemann operator

$$\partial_z \boldsymbol{m} = \frac{1}{2}\,(\partial_1 \boldsymbol{m} - \boldsymbol{m} \times \partial_2 \boldsymbol{m})$$

revealing the conformal character of (4).

Observe that any $\boldsymbol{m} \in \mathcal{C}$, which is also an equilibrium for the energy, is a static solution of the pulled back dynamic equation, i.e. a traveling wave profile for (4). For $\varepsilon = 0$, the pure Heisenberg model, every $\boldsymbol{m} \in \mathcal{C}$ is a minimizer, hence an equilibrium, recovering the observation from [14]. For $p = 4$ and $\varepsilon > 0$ the matching upper energy bound characterizes $\boldsymbol{m}(x) = \Phi(x/4)$ with Φ given by (3) not only as explicit energy minimizer within the class $\{Q = -1\}$ but also as an explicit static solution of (6), i.e. an explicit traveling wave profile of (4).

Theorem 3 (Existence, stability, compactness) *Suppose* $p = 4$ *and* $0 < \varepsilon \ll 1$.

(i) There exists $\boldsymbol{m} \in \mathcal{C}$ *independent of* ε*, which minimizes the energy in its homotopy class and is a static solution of* (6) *and therefore a traveling wave profile for* (4).

(ii) Suppose $\{\boldsymbol{m}_\varepsilon^0\}_{\varepsilon \ll 1} \subset \mathcal{M}$ *is a family of initial data with* $\nabla \boldsymbol{m}_\varepsilon^0 \in H^2(\mathbb{R}^2)$ *and such that for a constant* $c > 0$ *independent of* ε

$$Q(\boldsymbol{m}_\varepsilon^0) = -1 \quad \text{and} \quad E_\varepsilon(\boldsymbol{m}_\varepsilon^0) \leq 4\pi - c\varepsilon.$$

Then there exists a unique family $\{\boldsymbol{m}_\varepsilon\}_{\varepsilon \ll 1}$ *of local solutions of* (6) *with initial data* $\boldsymbol{m}_\varepsilon(t = 0) = \boldsymbol{m}_\varepsilon^0$ *such that* $\boldsymbol{m}_\varepsilon \in C^0([0; T]; \mathcal{M}) \cap C^\infty(\mathbb{R}^2 \times (0, T])$ *for every*

$$0 < T < \frac{c\alpha}{32\pi(1 + \alpha^2)\nu^2}.$$

(iii) If $\nabla \boldsymbol{m}_\varepsilon^0 \to \nabla \boldsymbol{m}_0$ *strongly in* $L^2(\mathbb{R}^2)$ *for some* $\boldsymbol{m}_0 \in \mathcal{M}$ *as* $\varepsilon \to 0$*, then* $\boldsymbol{m}_0 \in \mathcal{C}$ *and* $\nabla \boldsymbol{m}_\varepsilon(t) \to \nabla \boldsymbol{m}_0$ *in* $L^2(\mathbb{R}^2)$ *for every* $t \in [0, T]$.

1.1 Outline of the paper

The remainder of the paper is structured as follows: First, in Sect. 2, we prove the upper and lower bounds (1) to the minimal energy E_ε in the homotopy class $\{Q = -1\}$, i.e. Lemmas 2 and 3. In particular, we obtain the Eq. (2) characterizing minimizers in the case $p = 4$.

In Sect. 3, we exploit the energy bounds and derive the first two main results, i.e. Theorems 1 and 2. In fact, both will be rather straightforward corollaries of a separate concentration-compactness result in the spirit of [21], i.e. Proposition 1.

Section 4 contains the proof of Theorem 3. The main point are regularity arguments in the spirit of [29], which exploit the energy bounds to rule out blow-up on a uniform time interval.

Finally, in the "Appendix", we provide a few supplementary, technical results: a cut-off lemma similar to the ones used for example in [17,21], which enters the proof of Proposition 1; the explicit construction of a "stream function" that is needed in the upper-bound construction in Lemma 3 for $p = 2$; and the derivation of (6).

1.2 Notation and preliminaries

Throughout the paper, we shall use the convention

$$\nabla \times \boldsymbol{m} = \begin{pmatrix} \nabla \times m_3 \\ \nabla \times m \end{pmatrix} \quad \text{for} \quad \boldsymbol{m} = \begin{pmatrix} m \\ m_3 \end{pmatrix},$$

where

$$\nabla \times m = \partial_1 m_2 - \partial_2 m_1 \quad \text{and} \quad \nabla \times m_3 = -\nabla^\perp m_3 = \begin{pmatrix} \partial_2 m_3 \\ -\partial_1 m_3 \end{pmatrix}.$$

We equip the space $\mathcal{M} = \{\boldsymbol{m} : \mathbb{R}^2 \to \mathbb{S}^2 \in H^1_{\text{loc}}(\mathbb{R}^2) \text{ with } D(\boldsymbol{m}) + V(\boldsymbol{m}) < \infty\}$ with the metric d given as

$$d(\boldsymbol{m}, \boldsymbol{n}) = \|\nabla(\boldsymbol{m} - \boldsymbol{n})\|_{L^2} + \|\hat{\boldsymbol{e}}_3 \cdot (\boldsymbol{m} - \boldsymbol{n})\|_{L^{\frac{p}{2}}}.$$

Completeness with respect to this metric follows from the fact that by virtue of the geometric constraint $|\boldsymbol{m}|^2 = 1$ we have $1 - m_3 = \frac{1}{2}|\boldsymbol{m} - \hat{\boldsymbol{e}}_3|^2$, so that

$$V(\boldsymbol{m}) = 2^{-\frac{p}{2}} \int_{\mathbb{R}^2} (1 - m_3)^{\frac{p}{2}} \, dx.$$

Depending on the context, it is convenient to use this alternative representation. In order to extend the helicity to the configuration space $\mathcal{M}$ we recall that according to a variant (see e.g. [6]) of the approximation result by Schoen and Uhlenbeck [27]

$$\mathcal{M}_0 = \{\boldsymbol{m} : \mathbb{R}^2 \to \mathbb{S}^2 : \boldsymbol{m} - \hat{\boldsymbol{e}}_3 \in C_0^\infty(\mathbb{R}^2; \mathbb{R}^3)\}$$

is a dense subclass of $\mathcal{M}$ with respect to the metric d. The compact support property can be achieved by a suitable cut-off as in Lemma 8. We have for $\boldsymbol{m} \in \mathcal{M}_0$

$$H(\boldsymbol{m}) = \int_{\mathbb{R}^2} (\boldsymbol{m} - \hat{\boldsymbol{e}}_3) \cdot \nabla \times \boldsymbol{m} \, dx$$

while

$$(\boldsymbol{m} - \hat{\boldsymbol{e}}_3) \cdot \nabla \times \boldsymbol{m} = m \cdot \nabla \times m_3 - (1 - m_3) \nabla \times m.$$

Integration by parts yields for $\boldsymbol{m} \in \mathcal{M}_0$

$$H(\boldsymbol{m}) = -2 \int_{\mathbb{R}^2} (1 - m_3) \nabla \times m \, dx = -2 \int_{\mathbb{R}^2} m \cdot \nabla \times m_3 \, dx. \tag{7}$$

The first integral extends continuously to $\mathcal{M}$ since $L^{\frac{p}{2}}$-convergence implies L^2-convergence for sequences of uniformly bounded functions. A closer inspection shows that this extension can also be expressed in terms of the full helicity density. In fact, the second density in (7) satisfies $|m \cdot \nabla \times m_3| \le (1 - m_3^2)|\nabla \boldsymbol{m}|$ since

$$\frac{|\nabla m_3|^2}{1 - m_3^2} = |\nabla m_3|^2 + |\nabla |m||^2 \le |\nabla \boldsymbol{m}|^2.$$

Therefore $(\boldsymbol{m} - \hat{\boldsymbol{e}}_3) \cdot \nabla \times \boldsymbol{m} \in L^1(\mathbb{R}^2)$ for $\boldsymbol{m} \in \mathcal{M}$ with

$$|(\boldsymbol{m} - \hat{\boldsymbol{e}}_3) \cdot \nabla \times \boldsymbol{m}| \le c\,(1 - m_3)|\nabla \boldsymbol{m}|. \tag{8}$$

The validity of the integration by parts formula and (7) is a consequence of a simple decay estimate for the boundary integrals $\int_{\partial B_R} (1 - m_3) \, d\mathcal{H}^1 \to 0$ for a suitable choice of radii $R \to \infty$. In fact, there are radii $n \le R_n \le 2n$ so that

$$n \int_{\partial B_{R_n}} (1 - m_3)^{\frac{p}{2}} \, d\mathcal{H}^1 \le n \fint_n^{2n} \int_{\partial B_r} (1 - m_3)^{\frac{p}{2}} \, d\mathcal{H}^1 \le \int_{\mathbb{R}^2 \setminus B_n} (1 - m_3)^{\frac{p}{2}} \, dx.$$

As $n \to \infty$ we obtain by Jensen's inequality and $2 \le p \le 4$,

$$\int_{\partial B_{R_n}} (1 - m_3) \, d\mathcal{H}^1 \le R_n \Big(\fint_{\partial B_{R_n}} (1 - m_3)^{\frac{p}{2}} \, dx \Big)^{\frac{2}{p}} \lesssim R_n^{\frac{p-4}{p}} \Big(\int_{\mathbb{R}^2 \setminus B_n} (1 - m_3)^{\frac{p}{2}} \, dx \Big)^{\frac{2}{p}} \to 0.$$

Accordingly the energy $E_\varepsilon(\boldsymbol{m}) = D(\boldsymbol{m}) + \varepsilon\,(H(\boldsymbol{m}) + V(\boldsymbol{m}))$, initially defined on $\mathcal{M}_0$, extends to a continuous integral functional on $\mathcal{M}$

$$E_\varepsilon(\boldsymbol{m}) = \int_{\mathbb{R}^2} e_\varepsilon(\boldsymbol{m})\,dx$$

with integrable density

$$e_\varepsilon(\boldsymbol{m}) = \tfrac{1}{2}|\nabla \boldsymbol{m}|^2 + \varepsilon\Big((\boldsymbol{m} - \hat{\boldsymbol{e}}_3)\cdot\nabla\times\boldsymbol{m} + \tfrac{1}{2^p}|\boldsymbol{m} - \hat{\boldsymbol{e}}_3|^p\Big). \tag{9}$$

For later purpose it will be convenient to introduce the topological charge density

$$\omega(\boldsymbol{m}) = \boldsymbol{m}\cdot(\partial_1\boldsymbol{m}\times\partial_2\boldsymbol{m})$$

entering the definition of topological charge

$$Q(\boldsymbol{m}) = \frac{1}{4\pi}\int_{\mathbb{R}^2}\omega(\boldsymbol{m})\,dx \in \mathbb{Z}$$

for $\boldsymbol{m} \in \mathcal{M}_0$, which uniquely extends to $\mathcal{M}$ by virtue of Wente's inequality [13,32], and satisfies the classical topological lower bound

$$D(\boldsymbol{m}) \geq 4\pi|Q(\boldsymbol{m})| \quad \text{for all} \quad \boldsymbol{m}\in\mathcal{M}.$$

2 Energy bounds

Both the treatments of the static and dynamic problem rely on good upper and lower bounds to the energy E_ε in terms of $0 < \varepsilon \ll 1$. In fact, a major problem in extending our analysis to the physically relevant case $p = 2$ consists in the lack of a lower bound that matches the logarithmic upper bound in Theorem 1. Due to the quadratic decay of the stereographic map Φ for $|x| \gg 1$, which leads to a logarithmically growing potential energy V if $p = 2$, we conjecture the logarithmic upper bound to be optimal in terms of scaling.

From the above representations of H and V it follows

$$\big(H(\boldsymbol{m})\big)^2 \leq 32 D(\boldsymbol{m}) V(\boldsymbol{m}) \qquad \forall \boldsymbol{m}\in\mathcal{M}. \tag{10}$$

By Young's inequality we immediately infer the following lower energy bound:

Lemma 1 (Boundedness in $\mathcal{M}$) *Suppose* $2 \leq p \leq 4$ *and* $\varepsilon > 0$. *Then,*

$$E_\varepsilon(\boldsymbol{m}) \geq (1 - 16\varepsilon)D(\boldsymbol{m}) + \tfrac{\varepsilon}{2}V(\boldsymbol{m}) \qquad \textit{for any } \boldsymbol{m}\in\mathcal{M}.$$

Using the helical derivatives (11), we can further improve the lower bound:

Lemma 2 (Lower bound) *Suppose* $2 \leq p \leq 4$, $\varepsilon > 0$ *and* $\boldsymbol{m} \in \mathcal{M}\backslash\{\boldsymbol{m} \equiv \hat{\boldsymbol{e}}_3\}$. *Then*

$$E_\varepsilon(\boldsymbol{m}) \geq 4\pi\,Q(\boldsymbol{m}) + \varepsilon\Big(1 - 2\varepsilon\,\tfrac{V_4(\boldsymbol{m})}{V_p(\boldsymbol{m})}\Big)V_p(\boldsymbol{m})$$

and

$$E_\varepsilon(\boldsymbol{m}) \geq \Big(1 - 2\varepsilon\,\tfrac{V_4(\boldsymbol{m})}{V_p(\boldsymbol{m})}\Big)D(\boldsymbol{m}) + 8\pi\,\varepsilon\,\tfrac{V_4(\boldsymbol{m})}{V_p(\boldsymbol{m})}\,Q(\boldsymbol{m}).$$

The second lower bound is attained if and only if

$$\mathcal{D}_1^\kappa\boldsymbol{m} + \boldsymbol{m}\times\mathcal{D}_2^\kappa\boldsymbol{m} = 0, \quad \textit{where} \quad \mathcal{D}_i^\kappa\boldsymbol{m} = \partial_i\boldsymbol{m} - \kappa\,\hat{\boldsymbol{e}}_i\times\boldsymbol{m}, \tag{11}$$

holds for $\kappa = \frac{V_4(\boldsymbol{m})}{2V_p(\boldsymbol{m})}$*. In particular, for* $Q(\boldsymbol{m}) = -1$

$$E_\varepsilon(\boldsymbol{m}) \geq D(\boldsymbol{m})\left(1 - 4\varepsilon\frac{V_4(\boldsymbol{m})}{V_p(\boldsymbol{m})}\right) \geq 4\pi(1 - 4\varepsilon).$$

A corresponding upper bound in the homotopy class $Q(\boldsymbol{m}) = -1$ is obtained by rescaling the stereographic map Φ appropriately. For $p = 2$, an additional cut-off procedure is needed.

Lemma 3 (Upper bound) *Suppose* $2 \leq p \leq 4$ *and* $\varepsilon > 0$*. Then, there exists a smooth representative* $\tilde{\boldsymbol{m}} \in \mathcal{M}$ *in the homotopy class* $Q = -1$ *such that*

$$\inf\{E_\varepsilon(\boldsymbol{m}) : \boldsymbol{m} \in \mathcal{M},\ Q(\boldsymbol{m}) = -1\} \leq E_\varepsilon(\tilde{\boldsymbol{m}})$$
$$\begin{cases} = 4\pi\big(1 - 2(p-2)\,\varepsilon\big), & \text{if } 2 < p \leq 4, \\ \leq 4\pi\left(1 - \big(4 + o(1)\big)\frac{\varepsilon}{|\ln\varepsilon|}\right), & \text{if } p = 2 \text{ and } 0 < \varepsilon \ll 1. \end{cases}$$

For $p = 4$, upper and lower bounds match, so that the vector field $\tilde{\boldsymbol{m}}$ actually is a minimizer of E_ε in the homotopy class $Q = -1$.

Proof of Lemma 2 As in [21] we will employ the helical derivatives $\mathcal{D}_i^\kappa$ as given in (11) and appeal to the following relation from [21, Proof of Lemma 3.2]:

Step 1: *For any* $\boldsymbol{m} \in \mathcal{M}$, *we have*

$$\begin{aligned} &\tfrac{1}{2}|\nabla\boldsymbol{m}|^2 - \omega(\boldsymbol{m}) + \kappa\big((\boldsymbol{m} - \hat{\boldsymbol{e}}_3)\cdot\nabla\times\boldsymbol{m} + \tfrac{\kappa}{2}(1 - m_3)^2\big) \\ &\quad = |\mathcal{D}_1^\kappa\boldsymbol{m} + \boldsymbol{m}\times\mathcal{D}_2^\kappa\boldsymbol{m}|^2 \geq 0. \end{aligned}$$

Indeed, using $|\mathcal{D}_1^\kappa\boldsymbol{m} + \boldsymbol{m}\times\mathcal{D}_2^\kappa\boldsymbol{m}|^2 = |\mathcal{D}_1^\kappa\boldsymbol{m}|^2 + |\boldsymbol{m}\times\mathcal{D}_2^\kappa\boldsymbol{m}|^2 + 2\mathcal{D}_1^\kappa\boldsymbol{m}\cdot(\boldsymbol{m}\times\mathcal{D}_2^\kappa\boldsymbol{m})$, the claim immediately follows from

$$|\mathcal{D}_1^\kappa\boldsymbol{m}|^2 + |\boldsymbol{m}\times\mathcal{D}_2^\kappa\boldsymbol{m}|^2 = |\nabla\boldsymbol{m}|^2 + \kappa^2(1 + m_3^2) + 2\kappa\,\boldsymbol{m}\cdot\nabla\times\boldsymbol{m}$$

and

$$\mathcal{D}_1^\kappa\boldsymbol{m}\cdot(\boldsymbol{m}\times\mathcal{D}_2^\kappa\boldsymbol{m}) = -\omega(\boldsymbol{m}) - \kappa^2 m_3 - \kappa\,\hat{\boldsymbol{e}}_3\cdot\nabla\times\boldsymbol{m}.$$

Step 2: *Conclusion.* Recall that for $2 \leq p \leq 4$

$$V(\boldsymbol{m}) = V_p(\boldsymbol{m}) = \int_{\mathbb{R}^2}\left(\tfrac{1}{2}(1 - m_3)\right)^{\frac{p}{2}}dx.$$

Choosing $\kappa = \varepsilon$ in Step 1 and integrating over $\mathbb{R}^2$, the first claim follows as in [21]. With the choice of $\kappa = \frac{V_p(\boldsymbol{m})}{2V_4(\boldsymbol{m})}$ it follows that

$$D(\boldsymbol{m}) - 4\pi Q(\boldsymbol{m}) + \frac{V_p(\boldsymbol{m})}{2V_4(\boldsymbol{m})}\big(H(\boldsymbol{m}) + V_p(\boldsymbol{m})\big) \geq 0,$$

i.e.

$$H(\boldsymbol{m}) + V(\boldsymbol{m}) \geq -2\frac{V_4(\boldsymbol{m})}{V_p(\boldsymbol{m})}\big(D(\boldsymbol{m}) - 4\pi Q(\boldsymbol{m})\big).$$

Hence, we obtain the second lower bound:

$$E_\varepsilon(\boldsymbol{m}) = D(\boldsymbol{m}) + \varepsilon\big(H(\boldsymbol{m}) + V(\boldsymbol{m})\big) \geq D(\boldsymbol{m}) - 2\varepsilon\frac{V_4(\boldsymbol{m})}{V_p(\boldsymbol{m})}\big(D(\boldsymbol{m}) - 4\pi Q(\boldsymbol{m})\big).$$

In particular, Step 1 implies that the inequality is sharp if and only if (11) holds for $\kappa = \frac{V_p(\boldsymbol{m})}{2V_4(\boldsymbol{m})}$.

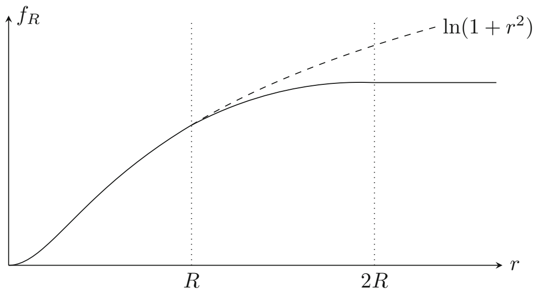

Fig. 1 A sketch of the "stream function" f_R that the upper-bound construction for $p = 2$ is based on

If $Q(\boldsymbol{m}) = -1$, we can use the classical topological lower bound $D(\boldsymbol{m}) \geq 4\pi|Q(\boldsymbol{m})| = 4\pi$ to conclude

$$\begin{aligned} E_\varepsilon(\boldsymbol{m}) &\geq D(\boldsymbol{m}) - 2\varepsilon \tfrac{V_4(\boldsymbol{m})}{V_p(\boldsymbol{m})}\big(D(\boldsymbol{m}) - 4\pi\, Q(\boldsymbol{m})\big) \\ &\geq \Big(1 - 4\varepsilon \underbrace{\tfrac{V_4(\boldsymbol{m})}{V_p(\boldsymbol{m})}}_{\leq 1}\Big) D(\boldsymbol{m}) \geq 4\pi(1-4\varepsilon). \end{aligned}$$

□

Proof of Lemma 3 If $2 < p \leq 4$, we may just define

$$\tilde{\boldsymbol{m}}\colon \mathbb{R}^2 \to \mathbb{S}^2, \quad \tilde{\boldsymbol{m}}(x) := \Phi_\lambda(x) := \Phi(\lambda x),$$

with Φ as in (3) and $\lambda > 0$ yet to be determined. Since $D(\Phi) = 4\pi$, $H(\Phi) = -8\pi$, $V(\Phi) = 2\pi/(p-2)$ and

$$E_\varepsilon(\Phi_{\lambda^*}) = \min_{\lambda>0} E_\varepsilon(\Phi_\lambda) = D(\Phi) - \frac{\varepsilon H(\Phi)^2}{4V(\Phi)}, \quad \lambda^* = -\frac{2V(\Phi)}{H(\Phi)},$$

by a simple scaling argument, we obtain the claim with $\lambda = \lambda^* = (2(p-2))^{-1}$.

For $p = 2$, however, $\Phi \notin \mathcal{M}$, since the potential energy $V(\Phi)$ diverges logarithmically. Thus, Φ needs to be cut off in a suitable way. To this end, for $R \gg 1$ to be chosen later, we fix a smooth function $f_R\colon [0,\infty) \to \mathbb{R}$ (see Fig. 1 and the "Appendix 2" for an explicit construction) so that

$$f_R(r) = \begin{cases} \ln(1+r^2), & \text{for } 0 \leq r \leq R, \\ \text{const.}, & \text{for } r \geq 2R, \end{cases}$$

and, denoting by $0 < C < \infty$ a generic, universal constant, whose value may change from line to line:

$$0 \leq f_R'(r) \leq \tfrac{2r}{1+r^2}, \quad 0 \leq -f_R''(r) \leq \tfrac{C}{1+r^2}, \quad \text{for all } r \geq R.$$

Then, we define a smooth vector field $\Phi_R\colon \mathbb{R}^2 \to \mathbb{S}^2$ via

$$\Phi_R(x) := \left(f_R'(|x|)\frac{x^\perp}{|x|}, \operatorname{sgn}\big(|x|-1\big)\sqrt{1-\big(f_R'(|x|)\big)^2}\right)^T, \quad x \in \mathbb{R}^2.$$

Note that $\Phi_R = \Phi$ on B_R and $\Phi_R = \hat{\boldsymbol{e}}_3$ on $\mathbb{R}^2 \backslash B_{2R}$. On $A_R := B_{2R}\backslash B_R$, we have

$$|\nabla\Phi_R(x)|^2 \leq \tfrac{C}{|x|^4}, \qquad |\Phi_R(x) - \hat{\boldsymbol{e}}_3|^2 \leq \tfrac{C}{|x|^2}, \quad x \in A_R.$$

Hence, we compute in polar coordinates

$$\int_{B_R} \tfrac{1}{2}|\nabla \Phi_R|^2 dx = 4\pi \int_0^R \tfrac{2r}{(1+r^2)^2}\, dr \le 4\pi,$$
$$\int_{B_R} \tfrac{1}{4}|\Phi_R - \hat{\boldsymbol{e}}_3|^2 dx = \pi \int_0^R \tfrac{2r}{1+r^2}\, dr = \pi \ln(1+R^2),$$
$$\int_{A_R} \tfrac{1}{2}|\nabla \Phi_R|^2 dx \le C \int_R^{2R} \tfrac{1}{r^3}\, dr = \tfrac{C}{R^2},$$
$$\int_{A_R} \tfrac{1}{4}|\Phi_R - \hat{\boldsymbol{e}}_3|^2 dx \le C \int_R^{2R} \tfrac{1}{r}\, dr = C.$$

The region $\mathbb{R}^2 \backslash B_{2R}$ does not contribute to the energy. In particular, we have

$$|Q(\Phi) - Q(\Phi_R)| \le C \int_{B_R^C} |\nabla \Phi|^2 + |\nabla \Phi_R|^2\, dx \ll 1 \quad \text{provided } R \gg 1,$$

so that $Q(\Phi_R) = Q(\Phi_R) = -1$.

In order to estimate the contribution from the helicity, we exploit that

$$\begin{pmatrix} \Phi_{1,R} \\ \Phi_{2,R} \end{pmatrix}(x) \cdot \nabla \times \Phi_{3,R}(x) = \operatorname{sgn}\big(|x|-1\big) \frac{\big(f_R'(|x|)\big)^2 f_R''(|x|)}{\sqrt{1-\big(f_R'(|x|)\big)^2}}$$
$$\begin{cases} = -8\frac{|x|^2}{(1+|x|^2)^3}, & \text{for } 0 \le |x| \le R, \\ \le 0, & \text{for } |x| \ge R. \end{cases}$$

Hence, using $\frac{d}{dr}\frac{r^4}{(1+r^2)^2} = 4\frac{r^3}{(1+r^2)^3}$, we find

$$H(\Phi_R) = 2\int_{\mathbb{R}^2} \begin{pmatrix} \Phi_{1,R} \\ \Phi_{2,R} \end{pmatrix} \cdot \nabla \times \Phi_{3,R}\, dx \le -32\pi \int_0^R \big(\tfrac{r}{1+r^2}\big)^3 dr = -8\pi \frac{R^4}{(1+R^2)^2}.$$

Summarizing, for sufficiently large $R \gg 1$, we have obtained

$$D(\Phi_R) \le 4\pi + \tfrac{C}{R^2},$$
$$H(\Phi_R) \le -8\pi \big(\tfrac{R^2}{R^2+1}\big)^2 \le -8\pi + \tfrac{C}{R^2},$$
$$V(\Phi_R) \le \pi \ln(1+R^2) + C.$$

Defining

$$\tilde{\boldsymbol{m}} \colon \mathbb{R}^2 \to \mathbb{S}^2, \quad \tilde{\boldsymbol{m}}(x) = \Phi_R(\lambda x),$$

where $\lambda > 0$ will be chosen below, and rescaling, we arrive at

$$E_\varepsilon(\tilde{\boldsymbol{m}}) = D(\Phi_R) + \varepsilon\lambda^{-1}\big(H(\Phi_R) + \lambda^{-1}V(\Phi_R)\big)$$
$$\le 4\pi + \tfrac{C}{R^2} + \varepsilon\lambda^{-1}\big(-8\pi + \lambda^{-1}\pi\ln(1+R^2) + C(R^{-2} + \lambda^{-1})\big).$$

Now, choose $R = \varepsilon^{-\frac{1}{2}}|\ln\varepsilon|$ and let $\lambda = L|\ln\varepsilon|$ for $L > 0$ fixed and $0 < \varepsilon \ll 1$. Then,

$$E_\varepsilon(\tilde{\boldsymbol{m}}) \le 4\pi + \tfrac{\varepsilon}{|\ln\varepsilon|}\big(-\tfrac{8\pi}{L} + \tfrac{\pi}{L^2} + o(1)\big) \quad \text{for } 0 < \varepsilon \ll 1,$$

which turns into the claim for $L = \frac{1}{4}$. □

3 Compactness and proofs of Theorems 1 and 2

In this section, we prove existence of minimizers $\boldsymbol{m}_\varepsilon$ of E_ε under the constraint $Q = -1$, and their strong convergence to a unique harmonic map $\boldsymbol{m}_0 \in \mathcal{C}$ as $\varepsilon \to 0$. In fact, both results rely on P. L. Lions' concentration-compactness principle. We state the common part as a separate compactness result – Proposition 1 – from which Theorems 1 and 2 can be deduced easily:

Proposition 1 *Suppose* $2 \le p < 4$ *and consider positive numbers* $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ *so that* $\varepsilon_\infty := \lim_{k\to\infty} \varepsilon_k$ *exists and satisfies* $0 \le \varepsilon_\infty \ll 1$. *Define*

$$I := \inf_{\substack{\boldsymbol{m}\in\mathcal{M}\\ Q(\boldsymbol{m})=-1}} E_{\varepsilon_\infty}(\boldsymbol{m}) \quad \begin{cases} = 4\pi, & \text{if } \varepsilon_\infty = 0, \\ < 4\pi, & \text{if } \varepsilon_\infty > 0. \end{cases}$$

Moreover, let $\{\boldsymbol{m}_k\}_k \subset \mathcal{M}$ *be asymptotically minimizing in the homotopy class* $Q = -1$*; that is, suppose that*

$$Q(\boldsymbol{m}_k) = -1 \quad \text{and} \quad \lim_{k\to\infty} E_{\varepsilon_k}(\boldsymbol{m}_k) = I.$$

Finally, assume

$$\liminf_{k\to\infty}\big(-H(\boldsymbol{m}_k)\big) > 0 \quad \text{as well as} \quad \limsup_{k\to\infty}\big(V(\boldsymbol{m}_k) - H(\boldsymbol{m}_k)\big) < \infty.$$

Then, up to translations and a subsequence, there exists $\boldsymbol{m}_\infty \in \mathcal{M}$ *with* $Q(\boldsymbol{m}_\infty) = -1$ *so that*

$$\begin{aligned} \nabla \boldsymbol{m}_k &\rightharpoonup \nabla \boldsymbol{m}_\infty && \text{weakly in } L^2(\mathbb{R}^2), \\ m_k &\rightharpoonup m_\infty && \text{weakly in } L^q(\mathbb{R}^2) \text{ for all } p \le q < \infty, \\ 1 - m_{3,k} &\rightharpoonup 1 - m_{3,\infty} && \text{weakly in } L^q(\mathbb{R}^2) \text{ for all } \tfrac{p}{2} \le q < \infty, \\ \boldsymbol{m}_k &\to \boldsymbol{m}_\infty && \text{strongly in } L^q_{loc}(\mathbb{R}^2) \text{ for all } 1 \le q < \infty, \end{aligned}$$

and

$$\liminf_{k\to\infty} E_{\varepsilon_k}(\boldsymbol{m}_k) \ge E_{\varepsilon_\infty}(\boldsymbol{m}_\infty).$$

In particular, the infimum I *is attained by* $\boldsymbol{m}_\infty \in \mathcal{M}$.

In the case $p = 2$ with $\varepsilon_\infty = 0$, the above result does not apply to families of minimizers $\{\boldsymbol{m}_\varepsilon\}_\varepsilon$ of E_ε, since we are unable to verify the bounds on $-H(\boldsymbol{m}_\varepsilon)$ and $V(\boldsymbol{m}_\varepsilon)$ as $\varepsilon \to 0$ (in fact, in the given scaling, we expect $H(\boldsymbol{m}_\varepsilon) \to 0$ as $\varepsilon \to 0$). For $p = 4$, on the other hand, the proof fails, since we cannot exclude "vanishing" in the concentration-compactness alternative – in the derivation of Theorem 1, we will instead exploit the matching upper and lower bounds to E_ε.

Before turning to the proof of Proposition 1, however, we will deduce both Theorem 1 and Theorem 2:

Proof of Theorem 1 **Step 1** (The case $p = 4$): For $p = 4$, we may appeal to the matching upper and lower bounds Lemmas 2 and 3. That is,

$$\boldsymbol{m}_\varepsilon : \mathbb{R}^2 \to \mathbb{S}^2, \quad \boldsymbol{m}_\varepsilon(x) := \Phi\big(\tfrac{x}{2(p-2)}\big),$$

is a minimizer of E_ε in the homotopy class $Q = -1$. Moreover, by Lemma 2, any minimizer $\tilde{\boldsymbol{m}} \in \mathcal{M}$ of E_ε must satisfy (11) for $\kappa = \frac{V_4(\tilde{\boldsymbol{m}})}{2V_4(\tilde{\boldsymbol{m}})} = \frac{1}{2}$.

Step 2 (The case $2 \le p < 4$): When $V = V_p$ represents the classical Zeeman interaction, that is for $p = 2$, the existence of a minimizer $\boldsymbol{m}_\varepsilon$ of E_ε in the homotopy class $Q = -1$ has been shown in [21]. However, the same approach can be used for the whole range $2 \le p < 4$: Consider a minimizing sequence $\{\boldsymbol{m}_k\}_{k\in\mathbb{N}} \subset \mathcal{M}$ for E_ε with $Q(\boldsymbol{m}_k) = -1$, and let $0 < \varepsilon_k := \varepsilon \ll 1$. Lemma 3 yields for $2 < p < 4$

$$\lim_{k\to\infty} E_\varepsilon(\boldsymbol{m}_k) = \inf\{E_\varepsilon(\boldsymbol{m}) : \boldsymbol{m} \in \mathcal{M},\ Q(\boldsymbol{m}) = -1\} \le 4\pi(1 - 2(p-2)\varepsilon).$$

Hence, using that $D(\boldsymbol{m}_k) + \varepsilon V(\boldsymbol{m}_k) \ge 4\pi$ due to $Q(\boldsymbol{m}_k) = -1$, we obtain

$$\begin{aligned}\liminf_{k\to\infty}\Big(-\varepsilon H(\boldsymbol{m}_k)\Big) &= \liminf_{k\to\infty}\Big(D(\boldsymbol{m}_k) + \varepsilon V(\boldsymbol{m}_k) - E_\varepsilon(\boldsymbol{m}_k)\Big)\\ &\ge 4\pi - 4\pi(1 - 2(p-2)\varepsilon) = 8\pi(p-2)\varepsilon > 0.\end{aligned}$$

If $p = 2$, we can use the upper bound $4\pi\big(1 - \big(4 + o(1)\big)\frac{\varepsilon}{|\ln\varepsilon|}\big) < 4\pi$ to arrive at the same conclusion $\liminf_{k\to\infty}\big(-H(\boldsymbol{m}_k)\big) > 0$.

On the other hand, we may use Lemma 1 to obtain

$$\sqrt{\varepsilon}\limsup_{k\to\infty}|H(\boldsymbol{m}_k)| \overset{(10)}{\lesssim} \limsup_{k\to\infty}\big(D(\boldsymbol{m}_k) + \varepsilon V(\boldsymbol{m}_k)\big) \overset{\text{Lem. 1}}{\lesssim} \limsup_{k\to\infty} E_\varepsilon(\boldsymbol{m}_k) \le 4\pi,$$

i.e.

$$\limsup_{k\to\infty}\big(V(\boldsymbol{m}_k) - H(\boldsymbol{m}_k)\big) < \infty.$$

Hence, we may apply Proposition 1 to obtain convergence (up to a subsequence and translations) of $\{\boldsymbol{m}_k\}_{k\in\mathbb{N}}$ to a limit $\boldsymbol{m}_\infty \in \mathcal{M}$ with $Q(\boldsymbol{m}_\infty) = -1$ and

$$I = \lim_{k\to\infty} E_\varepsilon(\boldsymbol{m}_k) \ge E_\varepsilon(\boldsymbol{m}_\infty) \ge I.$$

Thus, $\boldsymbol{m}_\infty$ minimizes E_ε in the class $\mathcal{M}$, subject to the constraint $Q = -1$. By the H^1 continuity of the topological charge $Q(\boldsymbol{m})$, the constrained minimizer $\boldsymbol{m}_\infty \in \mathcal{M}$ constructed before is a local minimizer of $E_\varepsilon(\boldsymbol{m})$ in $\mathcal{M}$ and as such an almost harmonic map with an L^2 perturbation as considered in [23] (see also [13]). Hence, $\boldsymbol{m}_\infty$ is Hölder continuous. □

Proof of Theorem 2 By the lower bound Lemma 2, we may assume w.l.o.g. that the constant $0 < C_0 < \infty$ satisfies

$$4\pi - C_0^{-1}\varepsilon \le E_\varepsilon(\boldsymbol{m}_\varepsilon) \le 4\pi - C_0\varepsilon. \tag{12}$$

Step 1 (Verification of the assumptions of Proposition 1): *We prove*

$$\lim_{\varepsilon\to 0} D(\boldsymbol{m}_\varepsilon) = 4\pi, \quad \liminf_{\varepsilon\to 0}\big(-H(\boldsymbol{m}_\varepsilon)\big) > 0 \text{ and } \limsup_{\varepsilon\to 0}\big(V(\boldsymbol{m}_\varepsilon) - H(\boldsymbol{m}_\varepsilon)\big) < \infty.$$

Indeed, we have

$$-H(\boldsymbol{m}_\varepsilon) = \frac{1}{\varepsilon}\Big(D(\boldsymbol{m}_\varepsilon) + \varepsilon V(\boldsymbol{m}_\varepsilon) - E_\varepsilon(\boldsymbol{m}_\varepsilon)\Big) \overset{(12)}{\ge} \frac{1}{\varepsilon}\Big(4\pi - (4\pi - C_0\varepsilon)\Big) = C_0,$$

so that $\liminf_{\varepsilon\to 0}\big(-H(\boldsymbol{m}_\varepsilon)\big) > 0$. On the other hand, Lemma 2 and the topological lower bound yield

$$4\pi \le D(\boldsymbol{m}_\varepsilon) \le \frac{1}{1-4\varepsilon} E_\varepsilon(\boldsymbol{m}_\varepsilon) \overset{(12)}{\le} \frac{4\pi - C_0\varepsilon}{1-4\varepsilon} \to 4\pi \quad \text{as } \varepsilon \to 0.$$

Hence, $D(\boldsymbol{m}_\varepsilon) \to 4\pi$ for $\varepsilon \to 0$.

Due to (10), it remains to prove that $V(\boldsymbol{m}_\varepsilon)$ is bounded uniformly in $0 < \varepsilon \ll 1$. Indeed, from Lemma 1, we obtain

$$\frac{\varepsilon}{2} V(\boldsymbol{m}_\varepsilon) \le \underbrace{E_\varepsilon(\boldsymbol{m}_\varepsilon)}_{\overset{(12)}{\le} 4\pi} - 4\pi(1 - 16\varepsilon) \le 64\pi\varepsilon \quad \forall 0 < \varepsilon \ll 1.$$

Thus, $\limsup_{k\to\infty}\big(V(\boldsymbol{m}_k) - H(\boldsymbol{m}_k)\big) < \infty$.

Step 2 (Proof of part i)): By Step 1, we may apply Proposition 1. Hence, there exists $\boldsymbol{m}_0 \in \mathcal{M}$ with $Q(\boldsymbol{m}_0) = -1$ so that in the limit $\varepsilon \to 0$, along a subsequence and up to translations (not relabeled):

$$\begin{aligned} \nabla \boldsymbol{m}_\varepsilon &\rightharpoonup \nabla \boldsymbol{m}_0 && \text{weakly in } L^2(\mathbb{R}^2), \\ m_\varepsilon &\rightharpoonup m_0 && \text{weakly in } L^q(\mathbb{R}^2) \text{ for all } p \le q < \infty, \\ 1 - m_{3,\varepsilon} &\rightharpoonup 1 - m_{3,0} && \text{weakly in } L^q(\mathbb{R}^2) \text{ for all } \tfrac{p}{2} \le q < \infty, \\ \boldsymbol{m}_\varepsilon &\to \boldsymbol{m}_0 && \text{strongly in } L^q_{\text{loc}}(\mathbb{R}^2) \text{ for all } 1 \le q < \infty, \end{aligned}$$

Since, by Step 1 and $Q(\boldsymbol{m}_0) = -1$, we have $4\pi = \liminf_{\varepsilon\to 0} D(\boldsymbol{m}_\varepsilon) \ge D(\boldsymbol{m}_0) \ge 4\pi$, weak convergence $\nabla \boldsymbol{m}_\varepsilon \rightharpoonup \nabla \boldsymbol{m}_0$ upgrades to strong convergence in $L^2(\mathbb{R}^2)$. In particular, $\boldsymbol{m}_0 \in \mathcal{C}$, which proves the first part of the claim.

Step 3 (Proof of part ii)): Assume that

$$E_\varepsilon(\boldsymbol{m}_\varepsilon) \le 4\pi + \varepsilon \min_{\boldsymbol{m}\in\mathcal{C}}\big(H(\boldsymbol{m}) + V(\boldsymbol{m})\big) + o(\varepsilon)$$

holds as $\varepsilon \to 0$, i.e.

$$\limsup_{\varepsilon\to 0} \varepsilon^{-1}\big(E_\varepsilon(\boldsymbol{m}_\varepsilon) - D(\boldsymbol{m}_0)\big) \le \min_{\boldsymbol{m}\in\mathcal{C}}\big(H(\boldsymbol{m}) + V(\boldsymbol{m})\big).$$

By Step 2, we have $\nabla \boldsymbol{m}_\varepsilon \to \nabla \boldsymbol{m}_0$ strongly in $L^2(\mathbb{R}^2)$ and $1 - m_{3,\varepsilon} \rightharpoonup 1 - m_{3,0}$ weakly in $L^{\frac{p}{2}}(\mathbb{R}^2)$ and $L^2(\mathbb{R}^2)$ along a suitable subsequence as $\varepsilon \to 0$. Thus, we obtain

$$\lim_{\varepsilon\to 0} H(\boldsymbol{m}_\varepsilon) = H(\boldsymbol{m}_0), \qquad \liminf_{\varepsilon\to 0} V(\boldsymbol{m}_\varepsilon) \ge V(\boldsymbol{m}_0),$$

and, using that $D(\boldsymbol{m}_\varepsilon) \ge 4\pi = D(\boldsymbol{m}_0)$,

$$\liminf_{\varepsilon\to 0} \varepsilon^{-1}\big(E_\varepsilon(\boldsymbol{m}_\varepsilon) - D(\boldsymbol{m}_0)\big) \ge H(\boldsymbol{m}_0) + V(\boldsymbol{m}_0) \ge \min_{\boldsymbol{m}\in\mathcal{C}}\big(H(\boldsymbol{m}) + V(\boldsymbol{m})\big).$$

Therefore,

$$\varepsilon^{-1}\big(E_\varepsilon(\boldsymbol{m}_\varepsilon) - D(\boldsymbol{m}_0)\big) \to \min_{\boldsymbol{m}\in\mathcal{C}}\big(H(\boldsymbol{m}) + V(\boldsymbol{m})\big) = H(\boldsymbol{m}_0) + V(\boldsymbol{m}_0) \quad \text{as } \varepsilon \to 0.$$

In particular, we obtain

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}\big(D(\boldsymbol{m}_\varepsilon) - D(\boldsymbol{m}_0)\big) = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0} V(\boldsymbol{m}_\varepsilon) = V(\boldsymbol{m}_0).$$

Hence, $\boldsymbol{m}_\varepsilon - \boldsymbol{m}_0 \to 0$ strongly in $L^p(\mathbb{R}^2)$ as $\varepsilon \to 0$, up to translations and a suitable subsequence.

Recall that (with the identification $\mathbb{R}^2 \simeq \mathbb{C}$) $\boldsymbol{m} \in \mathcal{C}$ may be represented as

$$\boldsymbol{m}(x) = \boldsymbol{m}^{(\rho,\varphi)}(x) = \Phi(ax + b) = e^{i\varphi}\Phi(\rho x + \tilde{b})$$

for two complex numbers $a = \rho e^{i\varphi} \neq 0$ and b, with $\tilde{b} = a^{-1}b$. Thus, dropping b due to the translation invariance of the problem, minimization is a finite dimensional problem; in fact, we have

$$H(\boldsymbol{m}^{(\rho,\varphi)}) + V(\boldsymbol{m}^{(\rho,\varphi)}) = \frac{\cos\varphi}{\rho} H(\Phi) + \frac{V(\Phi)}{\rho^2} = -\frac{8\pi\cos\varphi}{\rho} + \frac{2\pi}{\rho^2(p-2)},$$

which obviously is minimized by $\varphi \in 2\pi\mathbb{Z}$ and $\rho = \frac{1}{2(p-2)}$. Hence, up to translation, the unique minimizer of $H + V$ in $\mathcal{C}$ is given by

$$\boldsymbol{m}_0(x) = \Phi(\rho x) \quad \text{with} \quad \rho = \frac{1}{2(p-2)} = -2\frac{V(\Phi)}{H(\Phi)}.$$

In particular, the whole sequence $\{\boldsymbol{m}_\varepsilon\}_{\varepsilon>0}$ converges with respect to d, up to translations, to the unique limit $\boldsymbol{m}_0$. □

It remains to prove Proposition 1:

Proof of Proposition 1 We first remark that in view of (10) and Lemma 1, the assumptions also imply

$$\limsup_{k\to\infty} D(\boldsymbol{m}_k) < \infty \quad \text{and} \quad \liminf_{k\to\infty} V(\boldsymbol{m}_k) > 0.$$

Moreover, we will use the symbol $\lesssim$ to indicate that an inequality holds up to a universal, multiplicative constant that may change from line to line.

Step 1: *We prove:*

$$|H(\boldsymbol{m}_k)| \lesssim \left(\sup_{y\in\mathbb{R}^2} \left(\int_{B_1(y)} |\nabla \boldsymbol{m}_k|^2 \, dx \right)^{\frac{1}{2}} + \sup_{y\in\mathbb{R}^2} \left(\int_{B_1(y)} |\nabla \boldsymbol{m}_k|^2 \, dx \right)^{\frac{2}{p}-\frac{1}{2}} \right) \times \Big(D(\boldsymbol{m}_k) + V(\boldsymbol{m}_k) \Big)$$

for all $k \in \mathbb{N}$. Indeed, choose $\delta > 0$ so that $\cup_{y\in\delta\mathbb{Z}^2} B_1(y) = \mathbb{R}^2$. Then, we have

$$\left| \int_{\mathbb{R}^2} (1 - m_{3,k})(\nabla \times m_k) \, dx \right| \lesssim \sum_{y\in\delta\mathbb{Z}^2} \Big(\int_{B_1(y)} \underbrace{(m_{3,k} - 1)^2}_{=\frac{1}{4}|\boldsymbol{m}_k - \hat{\boldsymbol{e}}_3|^4} \, dx \Big)^{\frac{1}{2}} \Big(\int_{B_1(y)} |\nabla \boldsymbol{m}_k|^2 dx \Big)^{\frac{1}{2}}.$$

Moreover, the Sobolev embedding theorem and Jensen's inequality yield

$$\begin{aligned} \left(\int_{B_1(y)} |\boldsymbol{m}_k - \hat{\boldsymbol{e}}_3|^4 \, dx \right)^{\frac{1}{2}} &\lesssim \int_{B_1(y)} |\nabla \boldsymbol{m}_k|^2 + |\boldsymbol{m}_k - \hat{\boldsymbol{e}}_3|^2 \, dx \\ &\lesssim \int_{B_1(y)} |\nabla \boldsymbol{m}_k|^2 dx + \left(\int_{B_1(y)} \frac{1}{2^p} |\boldsymbol{m}_k - \hat{\boldsymbol{e}}_3|^p \, dx \right)^{\frac{2}{p}}, \end{aligned}$$

so that, using Young's inequality in the last step,

$$\begin{aligned} |H(\boldsymbol{m}_k)| \lesssim & \sum_{y\in\delta\mathbb{Z}^2} \left(\int_{B_1(y)} |\nabla \boldsymbol{m}_k|^2 \, dx \right)^{\frac{3}{2}} \\ & + \sum_{y\in\delta\mathbb{Z}^2} \left(\int_{B_1(y)} |\nabla \boldsymbol{m}_k|^2 \, dx \right)^{\frac{1}{2}} \left(\int_{B_1(y)} \frac{1}{2^p} |\boldsymbol{m}_k - \hat{\boldsymbol{e}}_3|^p \, dx \right)^{\frac{2}{p}} \end{aligned}$$

$$\leq \sup_{y\in\mathbb{R}^2}\Big(\int_{B_1(y)}|\nabla \boldsymbol{m}_k|^2\,dx\Big)^{\frac{1}{2}} D(\boldsymbol{m}_k)$$
$$+ \sup_{y\in\mathbb{R}^2}\Big(\int_{B_1(y)}|\nabla \boldsymbol{m}_k|^2\,dx\Big)^{\frac{2}{p}-\frac{1}{2}}\big(D(\boldsymbol{m}_k)+V(\boldsymbol{m}_k)\big),$$

which is the claim.

Step 2 (Concentration-compactness): We consider the full energy density (9) to define $\rho_k := e_\varepsilon(\boldsymbol{m}_k) \geq 0$. Note that we have

$$\rho_k \gtrsim |\nabla \boldsymbol{m}_k|^2 + \varepsilon_k \tfrac{1}{2^p}|\boldsymbol{m}_k - \hat{\boldsymbol{e}}_3|^p \quad \forall k \in \mathbb{N}$$

and

$$\lim_{k\to\infty}\int_{\mathbb{R}^2}\rho_k\,dx = I > 0.$$

Hence, we may apply the concentration-compactness lemma (see, e.g., [19]) to the sequence $\{\rho_k\}_{k\in\mathbb{N}}$ of non-negative densities and obtain that, up to a subsequence, one of the following holds:

- *Compactness*: There exists a sequence $\{y_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^2$ so that
$$\forall\delta > 0\colon \exists R < \infty\colon \int_{\mathbb{R}^2\setminus B_R(y_k)}\rho_k\,dx \leq \delta.$$
- *Vanishing*: We have
$$\lim_{k\to\infty}\sup_{y\in\mathbb{R}^2}\int_{B_R(y)}\rho_k\,dx = 0 \quad \forall R < \infty.$$
- *Dichotomy*: There exist $a^{(1)}, a^{(2)} > 0$ so that $a^{(1)} + a^{(2)} = I$ and for all $\delta > 0$, there exist $k_0 \in \mathbb{N}$, $\{y_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^2$, $R < \infty$, and a sequence $R_k \to \infty$, so that for $k \geq k_0$:

$$\Big|a^{(1)} - \int_{B_R(y_k)}\rho_k\,dx\Big| + \Big|a^{(2)} - \int_{\mathbb{R}^2\setminus B_{R_k}(y_k)}\rho_k\,dx\Big| + \Big|\int_{B_{R_k}(y_k)\setminus B_R(y_k)}\rho_k\,dx\Big| \leq \delta.$$

In order to conclude, we need to rule out vanishing and dichotomy.

Step 2a (Ruling out "Vanishing"): Suppose vanishing holds. Since ρ_k controls $\frac{1}{2}|\nabla \boldsymbol{m}_k|^2$, while $V(\boldsymbol{m}_k)$ is bounded by assumption, Step 1 yields $\lim_{k\to\infty} H(\boldsymbol{m}_k) = 0$, contradicting the assumption $\liminf_{k\to\infty}\big(-H(\boldsymbol{m}_k)\big) > 0$.

Step 2b (Ruling out "Dichotomy"): Suppose dichotomy holds. In particular, for fixed $0 < \delta \ll 1$ (to be specified later), we have

$$\int_{B_{R_k}\setminus B_R}|\nabla \boldsymbol{m}_k|^2 + \varepsilon_k\tfrac{1}{2^p}|\boldsymbol{m}_k - \hat{\boldsymbol{e}}_3|^p dx \lesssim \int_{B_{R_k}\setminus B_R}\rho_k\,dx \leq \delta.$$

W.l.o.g., we may assume that $R^2\delta^{\frac{p-2}{2}} \geq 1$ and $k \gg 1$, so that $R_k \geq 4R$.

If $\varepsilon_\infty = 0$, we may apply Lemma 8 in "Appendix 1" with $\sigma = 0$, otherwise with $\sigma = 1$, and define $\boldsymbol{m}_k^{(i)} \in \mathcal{M}$, $i = 1, 2$, so that for some constant $C(\delta, R)$ and $c_k \in [R, 2R]$:

$$\boldsymbol{m}_k^{(1)} = \boldsymbol{m}_k \quad \text{on } B_{c_k}, \qquad V(\boldsymbol{m}_k^{(1)}) \lesssim C(\delta, R),$$
$$\boldsymbol{m}_k^{(2)} = \boldsymbol{m}_k \quad \text{on } \mathbb{R}^2\setminus B_{2c_k}, \qquad V(\boldsymbol{m}_k^{(2)}) \lesssim V(\boldsymbol{m}_k) + C(\delta, R),$$

and

$$\begin{aligned}&\int_{\mathbb{R}^2\setminus B_{c_k}} |\nabla \boldsymbol{m}_k^{(1)}|^2 + \sigma\, \varepsilon_k \tfrac{1}{2^p} |\boldsymbol{m}_k^{(1)} - \hat{\boldsymbol{e}}_3|^p \, dx \\ &\quad + \int_{B_{2c_k}} |\nabla \boldsymbol{m}_k^{(2)}|^2 + \sigma\, \varepsilon_k \tfrac{1}{2^p} |\boldsymbol{m}_k^{(2)} - \hat{\boldsymbol{e}}_3|^p \, dx \\ &\lesssim \delta + \sigma (\tfrac{\delta}{R^2})^{2/p} \lesssim \delta.\end{aligned}$$

In particular, we have

$$\begin{aligned}\left|Q(\boldsymbol{m}_k^{(1)}) + Q(\boldsymbol{m}_k^{(2)}) - Q(\boldsymbol{m}_k)\right| &\le \left|\tfrac{1}{4\pi}\int_{B_{2c_k}\setminus B_{c_k}} \omega(\boldsymbol{m}_k)\, dx\right| \\ + \left|Q(\boldsymbol{m}_k^{(1)}) - \tfrac{1}{4\pi}\int_{B_{c_k}} \omega(\boldsymbol{m}_k)\, dx\right| &+ \left|Q(\boldsymbol{m}_k^{(2)}) - \tfrac{1}{4\pi}\int_{\mathbb{R}^2\setminus B_{2c_k}} \omega(\boldsymbol{m}_k)\, dx\right| \lesssim \delta.\end{aligned}$$

Hence, since $Q(\boldsymbol{m}_k) = -1$ and $Q(\boldsymbol{m}_k^{(i)}) \in \mathbb{Z}$, $i = 1, 2$, we obtain

$$Q(\boldsymbol{m}_k^{(1)}) + Q(\boldsymbol{m}_k^{(2)}) = Q(\boldsymbol{m}_k) = -1.$$

Moreover, using the estimate

$$|H(\boldsymbol{m}_k^{(i)})| \overset{(10)}{\lesssim} \Big(D(\boldsymbol{m}_k^{(i)}) \overbrace{V(\boldsymbol{m}_k^{(i)})}^{\lesssim C<\infty}\Big)^{\frac{1}{2}},$$

which also holds localized to B_{2c_k} and $\mathbb{R}^2\setminus B_{c_k}$, respectively, the "dichotomy" condition yields

$$E(\boldsymbol{m}_k^{(i)}) \le a^{(i)} + C\sqrt{\delta} < I \le 4\pi \quad \text{if } \delta \ll 1,\ \text{for } i = 1, 2. \tag{13}$$

If $|Q(\boldsymbol{m}_k^{(i)})| \ge 2$ for some $i \in \{1, 2\}$, Lemma 2 and the inequality $D(\boldsymbol{m}) \ge 4\pi |Q(\boldsymbol{m})|$ imply

$$4\pi > E(\boldsymbol{m}_k^{(i)}) \overset{\text{Lem. } 1}{\ge} (1 - 16\varepsilon) D(\boldsymbol{m}_k^{(i)}) \ge 3\pi |Q(\boldsymbol{m}_k^{(i)})| \ge 6\pi \quad \text{if } 0 < \varepsilon \ll 1,$$

a contradiction. Moreover, $Q(\boldsymbol{m}_k^{(i)}) = 1$ for some $i \in \{1, 2\}$ yields $Q(\boldsymbol{m}_k^{(3-i)}) = -2$, which leads to the same contradiction as above.

Thus, we have $Q(\boldsymbol{m}_k^{(i)}) \in \{-1, 0\}$ for $i = 1, 2$, i.e. there exists $i_0 \in \{1, 2\}$ with $Q(\boldsymbol{m}_k^{(i_0)}) = -1$.

If $\varepsilon_\infty > 0$, we directly obtain a contradiction, since $\boldsymbol{m}_k^{(i_0)}$ is admissible in the variational problem I, hence

$$I \le E(\boldsymbol{m}_k^{(i_0)}) \overset{(13)}{<} I.$$

If $\varepsilon_\infty = 0$, we use that $H(\boldsymbol{m}_k^{(i_0)}) + V(\boldsymbol{m}_k^{(i_0)})$ remains bounded by construction (see Lemma 8 and (10)), and note that R and hence also $C(\delta, R)$ depend on δ, but not on k), and thus

$$4\pi \overset{(13)}{>} a^{(i_0)} + C\sqrt{\delta} \ge \liminf_{k\to\infty} E_{\varepsilon_k}(\boldsymbol{m}_k^{(i_0)}) \ge \liminf_{k\to\infty} D(\boldsymbol{m}_k^{(i_0)}) \ge 4\pi,$$

a contradiction. Therefore, dichotomy cannot occur.

Step 3 (Conclusion): By Step 2, we may assume that compactness holds in the concentration-compactness alternative. W.l.o.g., $y_k = 0$ for all $k \in \mathbb{N}$. By passing to a subsequence and using Rellich's theorem, we may assume that there exists $\boldsymbol{m}_\infty \in \dot{H}^1(\mathbb{R}^2; \mathbb{S}^2)$ such that

$$
\begin{aligned}
\nabla \boldsymbol{m}_k &\rightharpoonup \nabla \boldsymbol{m}_\infty && \text{weakly in } L^2(\mathbb{R}^2),\\
m_k &\rightharpoonup m_\infty && \text{weakly in } L^p(\mathbb{R}^2),\\
1 - m_{3,k} &\rightharpoonup 1 - m_{3,\infty} && \text{weakly in } L^{\frac{p}{2}}(\mathbb{R}^2),\\
\boldsymbol{m}_k &\overset{\star}{\rightharpoonup} \boldsymbol{m}_\infty && \text{weak-* in } L^\infty(\mathbb{R}^2),\\
\boldsymbol{m}_k &\to \boldsymbol{m}_\infty && \text{strongly in } L^p_{\text{loc}}(\mathbb{R}^2) \text{ for } 1 \le p < \infty.
\end{aligned}
$$

Since compactness holds, we have (see [21, Lemma 4.1])

$$
I = \liminf_{k\to\infty}\big(E_{\varepsilon_k}(\boldsymbol{m}_k) + 4\pi Q(\boldsymbol{m}_k)\big) + 4\pi \ge E_{\varepsilon_\infty}(\boldsymbol{m}_\infty) + 4\pi Q(\boldsymbol{m}_\infty) + 4\pi.
$$

If $\varepsilon_\infty > 0$, i.e. $I < 4\pi$, we may immediately exclude $Q(\boldsymbol{m}_\infty) \ge 0$. On the other hand, Lemma 1 in form of the inequality $E_{\varepsilon_\infty}(\boldsymbol{m}_\infty) \ge 4\pi(1 - 16\varepsilon_\infty)|Q(\boldsymbol{m}_\infty)|$ rules out $|Q(\boldsymbol{m}_\infty)| \ge 2$, if ε_∞ is sufficiently small. Hence, we have $Q(\boldsymbol{m}_\infty) = -1$ If $\varepsilon_\infty = 0$, i.e. $I = 4\pi$, we may argue similarly to obtain $Q(\boldsymbol{m}_\infty) \in \{-1, 0\}$. Moreover, if $Q(\boldsymbol{m}_\infty) = 0$, we obtain $E_{\varepsilon_\infty}(\boldsymbol{m}_\infty) = D(\boldsymbol{m}_\infty) = 0$, i.e. $\boldsymbol{m}_\infty = \text{const}$. In particular, using the "compactness" condition and the initial assumption $\limsup_{k\to\infty} V(\boldsymbol{m}_k) < \infty$ to reduce the problem to a bounded set, we obtain $H(\boldsymbol{m}_k) \to 0$, a contradiction. Hence, also for $\varepsilon_\infty = 0$, we have $Q(\boldsymbol{m}_\infty) = -1$. □

4 Regularity of the dynamic problem and proof of Theorem 3

In this section we address existence and regularity issues for the pulled back Landau–Lifshitz–Gilbert equation (6) central for the proof of Theorem 3. We shall keep the discussion of the by now classical methodology brief and rather focus on the substantial new difficulties arising from chiral and spin–torque interactions.

4.1 Local well-posedness

Starting from spatial discretization as in [1,7,30] or spectral truncation as in [20,31] one obtains for initial data $\boldsymbol{m}^0 \in \mathcal{M}$ such that $\nabla \boldsymbol{m}^0 \in H^2(\mathbb{R}^2)$ a local solution $\boldsymbol{m} : \mathbb{R}^2 \times [0, T^*) \to \mathbb{S}^2$ for some terminal time $T^* > 0$, which is bounded below in terms of $\|\nabla \boldsymbol{m}^0\|_{H^2}$, such that for all $T < T^*$

$$
E_\varepsilon(\boldsymbol{m}) \in L^\infty(0, T) \quad \text{and} \quad \nabla \boldsymbol{m} \in L^\infty\left(0, T; H^2(\mathbb{R}^2)\right) \cap L^2\left(0, T; H^3(\mathbb{R}^2)\right).
$$

Initial data $\boldsymbol{m}^0$ and $\nabla \boldsymbol{m}^0$ is continuously attained in $\mathcal{M}$ and $H^2(\mathbb{R}^2)$, respectively, see [31]. As $\nabla \boldsymbol{m} \in W^{1,\infty}\left(0, T; L^2(\mathbb{R}^2)\right)$, interpolation and Sobolev embedding yield uniform Hölder continuity of $\nabla \boldsymbol{m}$ in $\mathbb{R}^2 \times [0, T]$. Uniqueness in this class can be shown by means of a Gronwall argument as in [20,31]. Due to the slow decay of $\boldsymbol{m} - \hat{\boldsymbol{e}}_3$, the conventional L^2-distance is replaced by a suitably weighted L^2-distance, e.g.

$$
\|\boldsymbol{u}\|^2_{L^2_*} := \int_{\mathbb{R}^2} \frac{|\boldsymbol{u}(x)|^2}{1 + |x|^2}\, dx \lesssim \|\boldsymbol{u}\|^2_{L^4}.
$$

As $\nabla \boldsymbol{m}(t) \in H^3(\mathbb{R}^2)$ for almost every $t < T^*$, uniqueness and a bootstrap argument imply $\nabla \boldsymbol{m} \in L^\infty_{\text{loc}}(0, T^*; H^k(\mathbb{R}^2))$ for arbitrary $k \in \mathbb{N}$, in particular $\boldsymbol{m}$ is smooth for positive times. Now one may deduce the following Sobolev estimate from [20] (which equally holds true for approximate equations)

$$\sup_{0\le t\le T} \|\nabla \boldsymbol{m}(t)\|_{H^k}^2 + \int_0^T \|\nabla \boldsymbol{m}(t)\|_{H^{k+1}}^2 \, dt \le c\left(1+\sup_{t\in[0,T]} \|\nabla \boldsymbol{m}(t)\|_{L^\infty}^2\right) \int_0^T \|\nabla \boldsymbol{m}(t)\|_{H^k}^2 \, dt \tag{14}$$

for all $0 \le k \le 2$ and $0 < T < T^*$ (cf. Lemma 4 below). Hence, if $T^* < \infty$, then

$$\limsup_{t\nearrow T^*} \|\nabla \boldsymbol{m}(t)\|_{L^\infty} = \infty.$$

4.2 Local Sobolev estimates for blow-up solutions

Due to lower order perturbations, (6) is translation- but not dilation-invariant. However, with respect to transformations $\tilde{\boldsymbol{m}}(x,t) = \boldsymbol{m}(x_0+\lambda x, t_0+\lambda^2 t)$ the parameters ε and ν exhibit the following scaling behavior $\tilde{\varepsilon} = \lambda\varepsilon$ and $\tilde{\nu} = \lambda\nu$ while $\tilde{\boldsymbol{f}}(\boldsymbol{m}) = \lambda \boldsymbol{f}(\boldsymbol{m})$. Hence, the coefficients of the lower order perturbations are uniformly bounded for $\varepsilon \le \varepsilon_0$ and in the blow-up regime $\lambda \le 1$. In this case we shall call $\tilde{\boldsymbol{m}} = \boldsymbol{m}$ a blow-up solution. We shall need a localized version of (14). For $R > 0$ and a space-time point $z = (x,t)$

$$P_R(z) = \overline{B_R(x)} \times [t-R^2, t]$$

denotes the closed parabolic cylinder and accordingly $P_R = P_R(0)$.

Lemma 4 *Suppose $k \in \mathbb{N}$ and $\boldsymbol{m}$ is a blow-up solution in a neighborhood of P_R for some $R > 0$. Then*

$$\|\nabla \boldsymbol{m}(0)\|_{H^k(B_{R/2})}^2 + \int_{-(R/2)^2}^0 \|\nabla \boldsymbol{m}(t)\|_{H^{k+1}(B_{R/2})}^2 \, dt \le c\left(1+\|\nabla \boldsymbol{m}\|_{L^\infty(P_R)}^2\right) \int_{-R^2}^0 \|\nabla \boldsymbol{m}(t)\|_{H^k(B_R)}^2 \, dt$$

for a constant c, which is independent of $\boldsymbol{m}$ and uniform for $\varepsilon \le \varepsilon_0$ and $\lambda \le 1$.

These estimates are inhomogeneous and depend on R. We shall apply it to blow-up solutions as above with $\varepsilon \le \varepsilon_0$ and $\lambda \le 1$ and radii R in a finite range.

Proof (Sketch of proof) The Landau-Lifshitz form of the equation reads

$$(1+\alpha^2)\,\partial_t \boldsymbol{m} = \alpha\left(\Delta \boldsymbol{m} + |\nabla \boldsymbol{m}|^2 \boldsymbol{m}\right) - \nabla\cdot(\boldsymbol{m}\times\nabla \boldsymbol{m}) + \boldsymbol{F}(\nabla \boldsymbol{m}, \boldsymbol{m}),$$

for a smooth tangent field $\boldsymbol{F}$ that is linear in $\nabla \boldsymbol{m}$. The standard procedure uses test functions $\partial^{\boldsymbol{\nu}}(\phi^2 \partial^{\boldsymbol{\nu}} \boldsymbol{m})$, where $\boldsymbol{\nu}$ is a multi index of length $1 \le |\boldsymbol{\nu}| \le k+1$, and $\phi(x,t) = \varphi(x)\eta(t)$ is an appropriate space-time cut-off function $0 \le \phi \le 1$ where $\varphi \in C_0^\infty(B_1)$ with $\varphi|_{B_{1/2}} = 1$ and $\eta \in C^\infty(\mathbb{R})$ with $\eta(t) = 0$ for $t < -1$ and $\eta(t) = 1$ for $t > -1/4$. In the case $R \ne 1$ one uses suitable rescalings of φ and η. Let us only estimate the contribution from the non-coercive term of second order $\nabla\cdot(\boldsymbol{m}\times\nabla \boldsymbol{m})$:

$$I = \langle \partial^{\boldsymbol{\nu}}(\boldsymbol{m}\times\nabla \boldsymbol{m}), \nabla(\phi^2\partial^{\boldsymbol{\nu}} \boldsymbol{m})\rangle = \langle \left(\boldsymbol{m}\times\partial^{\boldsymbol{\nu}}\nabla \boldsymbol{m} + \boldsymbol{R}_{\boldsymbol{\nu}}\right), \left(\phi^2\partial^{\boldsymbol{\nu}}\nabla \boldsymbol{m} + 2\phi\,\nabla\phi\,\partial^{\boldsymbol{\nu}} \boldsymbol{m}\right)\rangle,$$

which is bounded by

$$\|\phi\,\partial^{\boldsymbol{\nu}}\nabla \boldsymbol{m}\|_{L^2}\left(2\|\nabla\phi\,\partial^{\boldsymbol{\nu}} \boldsymbol{m}\|_{L^2} + \|\phi\,\boldsymbol{R}_{\boldsymbol{\nu}}\|_{L^2}\right) + 2\|\phi\,\boldsymbol{R}_{\boldsymbol{\nu}}\|_{L^2}\|\nabla\phi\,\partial^{\nu} \boldsymbol{m}\|_{L^2},$$

where $|\boldsymbol{R}_{\boldsymbol{\nu}}| \lesssim \sum_{|\ell_1|+|\ell_2|=|\boldsymbol{\nu}|-1} |\nabla^{\ell_1}(\nabla \boldsymbol{m}) \otimes \nabla^{\ell_2}(\nabla \boldsymbol{m})|$. Hence for $t \in [-1, 0]$ fixed

$$\|\phi\, \boldsymbol{R}_{\boldsymbol{\nu}}\|_{L^2} \le \|\phi\, \boldsymbol{R}_{\boldsymbol{\nu}}\|_{L^2(B_1)} \le c\|\nabla \boldsymbol{m}\|_{L^\infty(B_1)}\|\nabla \boldsymbol{m}\|_{H^k(B_1)}.$$

In fact, by Sobolev extension (preserving L^∞ bounds) of $\nabla \boldsymbol{m}|_{B_1}$ to a map $\boldsymbol{g} \in L^\infty \cap H^k(\mathbb{R}^2; \mathbb{R}^6)$ with an equivalent $L^\infty \cap H^k$ bound, Moser's product estimate applies. Hence for arbitrary $\delta > 0$

$$|I| \le \delta \|\phi\, \partial^\nu \nabla \boldsymbol{m}\|_{L^2}^2 + C(\delta)\Big(1 + \|\nabla \boldsymbol{m}\|_{L^\infty(P_1)}^2\Big)\|\nabla \boldsymbol{m}(t)\|_{H^k(B_1)}^2$$

so that the first term can be absorbed for $\delta \lesssim \alpha$. □

4.3 Energy estimates

In proving Theorem 3 we shall argue on the level of energy. We have the following energy inequality for regular solutions $\boldsymbol{m} = \boldsymbol{m}_\varepsilon$ of (6) on a time interval $[0, T]$.

Lemma 5 (Energy inequality) *There exists a universal constant $\lambda > 0$ such that for $\varepsilon \ge 0$, $0 < T < T^*$ and $\varphi : \mathbb{R}^2 \to \mathbb{R}$ smooth with compactly supported gradient*

$$\frac{\alpha}{2}\int_0^T \int_{\mathbb{R}^2} |\partial_t \boldsymbol{m}|^2 \varphi^2 dx dt + \left[\int_{\mathbb{R}^2} e_\varepsilon(\boldsymbol{m}(t))\varphi^2\, dx\right]_{t=0}^T$$
$$\le \frac{\lambda}{\alpha}\int_0^T \int_{\mathbb{R}^2} \Big[(1+\alpha^2)\nu^2 |\partial_z \boldsymbol{m}(t)|^2 \varphi^2 + \big(|\nabla \boldsymbol{m}|^2 + \varepsilon^2 |\boldsymbol{m} - \hat{\boldsymbol{e}}_3|^4\big)\, |\nabla \varphi|^2\Big] dx dt.$$

Proof The claim follows from a standard argument based on the identity

$$\alpha|\partial_t \boldsymbol{m}|^2 - \nu(\alpha \partial_z \boldsymbol{m} + \boldsymbol{m} \times \partial_z \boldsymbol{m}) \cdot \partial_t \boldsymbol{m} = \boldsymbol{h}_\varepsilon(\boldsymbol{m}) \cdot \partial_t \boldsymbol{m},$$

where the right hand side produces the time derivative of the density up to a divergence. The corresponding identity for the helicity term reads

$$(\nabla \times \boldsymbol{m}) \cdot \partial_t \boldsymbol{m} = \partial_t\,[(m_3 - 1)\nabla \times m] - \nabla \times [(m_3 - 1)\partial_t m]\,.$$

Integration by parts and Young's inequality implies the claim. □

If $\varphi \equiv 1$ one can take $\lambda = \frac{1}{2}$ and obtains in the case $Q(\boldsymbol{m}) = -1$

$$\frac{\alpha}{2}\int_0^T \int_{\mathbb{R}^2} |\partial_t \boldsymbol{m}|^2\, dx dt + \Big[E_\varepsilon(\boldsymbol{m}(t))\Big]_{t=0}^T \le \frac{(1+\alpha^2)\nu^2}{4\alpha}\int_0^T \Big[D(\boldsymbol{m}(t)) - 4\pi\Big] dt$$

where we used that

$$2\int_{\mathbb{R}^2} |\partial_z \boldsymbol{m}(t)|^2\, dx = D(\boldsymbol{m}) - 4\pi.$$

Lemma 2 implies for $\varepsilon \le 1/8$ and $E_\varepsilon(\boldsymbol{m}) < 4\pi$ that

$$D(\boldsymbol{m}) - 4\pi < 32\pi\varepsilon.$$

Proposition 2 *Suppose $0 < \varepsilon \le 1/8$ and $E_\varepsilon(\boldsymbol{m}(0)) \le 4\pi - c\varepsilon$, then*

$$E_\varepsilon(\boldsymbol{m}(T)) < 4\pi \quad \textit{for all} \quad 0 < T < \min\left\{\frac{c\alpha}{32\pi(1+\alpha^2)\nu^2}, T^*\right\}.$$

Moreover as $\varepsilon \to 0$

$$\sup_{0 \le t \le T}\int_{\mathbb{R}^2} |\partial_z \boldsymbol{m}(t)|^2\, dx = O(\varepsilon) \quad \textit{and} \quad \int_0^T \int_{\mathbb{R}^2} |\partial_t \boldsymbol{m}|^2\, dx dt = O(\varepsilon).$$

Next we show that the energy density $e_\varepsilon(\boldsymbol{m}(t)) : \mathbb{R}^2 \to [0, \infty)$ remains concentrated along the flow. To this end we invoke Lemma 5 with $\varphi_R(x) = \varphi(x/R)$, where $\varphi(x) = 1$ for $|x| \geq 2$ and $\varphi(x) = 0$ for $|x| \leq 1$. By virtue of Hölder's inequality we obtain the estimate

$$\frac{\alpha}{2} \int_0^T \int_{\mathbb{R}^2} |\partial_t \boldsymbol{m}|^2 \varphi_R^2 dx dt + \left[\int_{\mathbb{R}^2} e_\varepsilon(\boldsymbol{m}(t)) \varphi_R^2 \, dx \right]_{t=0}^T$$
$$\leq c \int_0^T \nu^2 \Big[D(\boldsymbol{m}) - 4\pi \Big] + R^{-2} E_\varepsilon(\boldsymbol{m}) \, dt$$

for generic constants c that only depend on α and φ from which we obtain:

Lemma 6 *There exists a constant $c = c(\alpha)$ such that*

$$\int_{\{|x|>2R\}} e_\varepsilon(\boldsymbol{m}(t)) \, dx \leq \int_{\{|x|>R\}} e_\varepsilon(\boldsymbol{m}(0)) \, dx + c \, T \left(\varepsilon \nu + 1/R^2 \right)$$

for all $0 \leq t \leq T$, $R > 0$ and $\varepsilon > 0$.

4.4 Small energy regularity

The main strategy for proving regularity has been developed in the context of harmonic flows and is well-established [10,12,29]. The terminal time T^* of first blow-up depends on the initial data and the parameters entering the equation. The only possible scenario of finite time blow-up is $|\nabla \boldsymbol{m}(x_k, t_k)| \to \infty$ for some sequence $x_k \in \mathbb{R}^2$ and $t_k \nearrow T^*$. We shall show that for moderately small ε, this scenario can be ruled out as long as $E_\varepsilon(\boldsymbol{m}(t)) < 4\pi$.

Proposition 3 *For $0 < T_0 < \infty$ there exists $\varepsilon_0 > 0$ with the following property: If $0 < \varepsilon < \varepsilon_0$ and $E_\varepsilon(\boldsymbol{m}(t)) < 4\pi$ for all t up to a terminal time $T^* \leq T_0$, then*

$$\limsup_{t \nearrow T^*} E_\varepsilon(\boldsymbol{m}(t)) = 4\pi.$$

It is customary to prove small-energy regularity using Schoen's trick, which is well-established for harmonic maps and flows.

Lemma 7 *There exists $\delta_0 > 0$ such that if $\boldsymbol{m}$ is a blow-up solution in $P_r(z_0)$ with $r \leq 1$, then*

$$\int_{B_r(x_0)} |\nabla \boldsymbol{m}(s)|^2 dy < \delta_0 \quad \textit{for all} \quad s \in [t_0 - r^2, t_0]$$

then

$$|\nabla \boldsymbol{m}| \leq 2/r \quad \textit{in} \quad P_{r/2}(z_0).$$

Proof Since the claim of the Lemma is invariant with respect to the transformation $(x, t) \mapsto (x_0 + rx, t_0 + r^2 t)$, inducing an admissible blow-up solution for $0 < r \leq 1$, we can assume $z_0 = 0$ and $r = 1$. There exists $\rho \in [0, 1)$ such that

$$(1 - \rho)^2 \sup_{P_\rho} |\nabla \boldsymbol{m}|^2 = \max_{\sigma \in [0,1]} (1 - \sigma)^2 \sup_{P_\sigma} |\nabla \boldsymbol{m}|^2.$$

We set $s_0 = \sup_{P_\rho} |\nabla \boldsymbol{m}| = |\nabla \boldsymbol{m}(z_0)|$ for some $z_0 \in P_\rho(0)$ and assume

$$(1 - \rho)^2 s_0^2 \leq 1.$$

Then it follows that $\sup_{P_{1/2}} |\nabla \boldsymbol{m}|^2 \leq 4(1-\rho)^2 s_0^2 \leq 4$, which implies the claim.

If otherwise $(1-\rho)^2 s_0^2 > 1$, then in particular $s_0 > \frac{1}{1-\rho} \geq 1$. So $\lambda = 1/s_0$ is an admissible scaling parameter. For $(x,t) \in P_{1/2}$ we consider the blow-up solution

$$\tilde{\boldsymbol{m}}(x,t) = \boldsymbol{m}(x_0 + s_0^{-1} x, t_0 + s_0^{-2} t),$$

for which

$$\sup_{P_{1/2}} |\nabla \tilde{\boldsymbol{m}}|^2 \leq s_0^{-2} \sup_{P_{1/(2s_0)}(z_0)} |\nabla \boldsymbol{m}|^2 \leq s_0^{-2} \sup_{P_{(1-\rho)/2}(z_0)} |\nabla \boldsymbol{m}|^2 \leq 4.$$

Hence it follows from Sobolev embedding $H^2(B_{1/16}) \hookrightarrow L^\infty(B_{1/16})$ and Lemma 4 applied three times to $\tilde{\boldsymbol{m}}$ (being a blow-up solution) that for a generic constant c

$$1 = |\nabla \tilde{\boldsymbol{m}}(0,0)|^2 \leq c \, \|\nabla \tilde{\boldsymbol{m}}(0)\|^2_{H^2(B_{1/16})} \leq c \int_{-1/4}^{0} \|\nabla \tilde{\boldsymbol{m}}(t)\|^2_{L^2(B_{1/2})} \, dt.$$

But then $1 \leq c \int_{-1}^{0} \|\nabla \boldsymbol{m}(t)\|^2_{L^2(B_1)} \, dt < c \, \delta_0$, impossible for appropriate $\delta_0 > 0$. □

Proof of Proposition 3 Suppose $\boldsymbol{m}$ is a fixed solution up to some terminal time $T^* \leq T_0$ with $E_\varepsilon(\boldsymbol{m}(t)) < 4\pi$ for all $0 \leq t < T^*$. Recall that by (8)

$$|\nabla \boldsymbol{m}|^2 \lesssim e_\varepsilon(\boldsymbol{m}) \lesssim |\nabla \boldsymbol{m}|^2 + |\boldsymbol{m} - \hat{\boldsymbol{e}}_3|^4 \tag{15}$$

uniformly for ε sufficiently small. It follows from Lemma 6 that there exist $\varepsilon_0 > 0$ and $R_0 > 0$, which only depends on $\boldsymbol{m}(0)$ but not explicitly on ε, such that

$$\int_{\{|x|>2R_0\}} |\nabla \boldsymbol{m}(t)|^2 \, dx < \delta_0 \quad \text{for all} \quad 0 < t < T^*$$

if $\varepsilon < \varepsilon_0$. But then, according to Lemma 7, $|\nabla \boldsymbol{m}_\varepsilon(x,t)|$ is uniformly bounded for $|x| > 3R_0$ and $0 < t < T^*$. It follows that blow-up can only occur in a finite domain, and it remains to perform a bubbling analysis as in [29]:

Note that by Lemma 7 the singular set must be finite. Hence after translation and dilation we may assume $\boldsymbol{m} \in C^\infty(P_2 \backslash \{(0,0)\})$ and claim that if ε is sufficiently small and $\boldsymbol{m}$ has a singularity at the origin, then

$$\limsup_{t \nearrow 0} E_\varepsilon(\boldsymbol{m}(t); B_2(0)) \geq 4\pi.$$

If $(0,0)$ is a singularity then by virtue of Lemma 7

$$\int_{B_{r_k}(x_k)} |\nabla \boldsymbol{m}(t_k)|^2 dx = \sup_{x \in B_1} \int_{B_{r_k}(x)} |\nabla \boldsymbol{m}(t_k)|^2 dy = \frac{\delta_0}{4}$$

for suitable sequences $x_k \to 0$, $t_k \nearrow 0$ and $r_k \searrow 0$. Moreover, invoking the local energy inequality in the style of Lemma 6 and (15) we find $0 < \sigma_0 \leq 1/4$ such that

$$\sup_{x \in B_{1/2}(x_k)} \int_{B_{r_k/2}(x)} |\nabla \boldsymbol{m}(t)|^2 dx \leq \frac{\delta_0}{2} \quad \text{for all} \quad t_k - r_k^2 \sigma_0 \leq t \leq t_k$$

for sufficiently large k and small ε. According to Lemma 7, the blow-up solutions

$$\boldsymbol{m}_k(x,t) = \boldsymbol{m}(x_k + r_k x, t_k + r_k^2 t) \quad \text{for} \quad (x,t) \in \mathbb{R}^2 \times [-\sigma_0, 0]$$

admit for $x \in B_{1/2r_k}$ and $t \in [-\sigma_0/2, 0]$ a uniform gradient bound. Local higher order Sobolev bounds then follow from a variant of Lemma 4. We consider $\boldsymbol{m}_k$ as a solution of the perturbed Landau–Lifshitz–Gilbert equation

$$\partial_t \boldsymbol{m}_k = \boldsymbol{m}_k \times (\alpha \partial_t \boldsymbol{m}_k - \Delta \boldsymbol{m}_k) + \boldsymbol{f}_k$$

for a field $\boldsymbol{f}_k \perp \boldsymbol{m}_k$ with

$$|\boldsymbol{f}_k| \lesssim r_k |\nabla \boldsymbol{m}_k| + r_k^2 |\boldsymbol{m}_k - \hat{\boldsymbol{e}}_3|^3$$

hence $\|\boldsymbol{f}_k(t)\|_{L^2} = O(r_k)$ uniformly for all admissible t. It follows from the energy inequality for $\boldsymbol{m}$ that $\int_{-\sigma_0}^0 \int_{\mathbb{R}^2} |\partial_t \boldsymbol{m}_k|^2 \, dx dt \to 0$ as $k \to \infty$, hence $\boldsymbol{v}_k = (\partial_t \boldsymbol{m}_k)(\tau_k)$ and $\boldsymbol{w}_k = \boldsymbol{f}_k(\tau_k)$ converge to zero in $L^2(\mathbb{R}^2)$ for some sequence $\tau_k \nearrow 0$. Note that $\boldsymbol{u}_k = \boldsymbol{m}_k(\tau_k)$ is an almost harmonic map in the sense that

$$\boldsymbol{u}_k \times \Delta \boldsymbol{u}_k = \alpha\, \boldsymbol{u}_k \times \boldsymbol{v}_k - \boldsymbol{v}_k + \boldsymbol{w}_k$$

and subconvergence strongly in $H^1_{\mathrm{loc}}(\mathbb{R}^2)$ to a harmonic map $\boldsymbol{u}$ of finite energy in $\mathbb{R}^2$. To show that $\boldsymbol{u}$ is non-constant we invoke the local energy equality for $\boldsymbol{m}_k$

$$\int_{B_1} |\nabla \boldsymbol{m}_k(0)|^2 \, dx - \int_{B_2} |\nabla \boldsymbol{m}_k(\tau_k)|^2 \, dx \le c \int_{\tau_k}^0 \int_{B_2} \left(|\nabla \boldsymbol{m}_k|^2 + |\boldsymbol{f}_k|^2 \right) dx dt = O(\tau_k),$$

which implies that

$$\int_{B_2} |\nabla \boldsymbol{u}_k|^2 \, dx = \int_{B_2} |\nabla \boldsymbol{m}_k(\tau_k)|^2 dx \ge \frac{\delta}{4} + O(\tau_k).$$

By local strong convergence $\int_{B_2} |\nabla \boldsymbol{u}|^2 \, dx > 0$, and by virtue of the well-known theory about harmonic maps $\frac{1}{2} \int_{\mathbb{R}^2} |\nabla \boldsymbol{u}|^2 \, dx = 4\pi$. According to (8) and Young's inequality, the rescaled energy densities

$$e_{\varepsilon,k}(\boldsymbol{u}) := \frac{|\nabla \boldsymbol{u}|^2}{2} + \varepsilon r_k \left((\boldsymbol{u} - \hat{\boldsymbol{e}}_3) \cdot (\nabla \times \boldsymbol{u}) + \frac{r_k}{16} |\boldsymbol{u} - \hat{\boldsymbol{e}}_3|^4 \right)$$

are non-negative for ε sufficiently small, independently of k. Hence by letting $s_k = t_k + r_k^2 \tau_k \to 0$ we have for arbitrary $R_0 > 0$

$$\int_{B_2(0)} e_\varepsilon(\boldsymbol{m}(s_k)) \, dx \ge \int_{B_1(x_k)} e_\varepsilon(\boldsymbol{m}(s_k)) \, dx = \int_{B_{1/r_k}} e_{\varepsilon,k}(\boldsymbol{u}_k) \, dx \ge \int_{B_{R_0}} e_{\varepsilon,k}(\boldsymbol{u}_k) \, dx$$

for $k > k_0$ depending on R_0, and $\int_{B_{R_0}} e_{\varepsilon,k}(\boldsymbol{u}_k) \, dx = \frac{1}{2} \int_{B_{R_0}} |\nabla \boldsymbol{u}_k|^2 \, dx + O(r_k)$ as $k \to \infty$ which implies the claim by strong $L^2(B_{R_0})$ compactness of $\nabla \boldsymbol{u}_k$. □

Proof of Theorem 3 The first claim has been discussed in the forefront of the theorem. The second follows from Propositions 2 and 3. For the third claim we deduce from Lemma 2 as in the proof of Theorem 2 that $\limsup_{\varepsilon \to 0} V(\boldsymbol{m}_\varepsilon^0) < \infty$ and $\lim_{\varepsilon \to 0} D(\boldsymbol{m}_\varepsilon^0) = 4\pi$, hence $\boldsymbol{m}_0 \in \mathcal{C}$. Moreover, it follows from Proposition 2 that for every sequence $\varepsilon_k \searrow 0$ the corresponding solutions $\boldsymbol{m}_{\varepsilon_k}$ subconverge weakly to a weak solution of $\boldsymbol{m}$ of the standard Landau–Lifshitz–Gilbert equation

$$\partial_t \boldsymbol{m} = \alpha\, \boldsymbol{m} \times \partial_t \boldsymbol{m} - \nabla \cdot (\boldsymbol{m} \times \nabla \boldsymbol{m}) \quad \text{with} \quad \boldsymbol{m}(0) = \boldsymbol{m}_0.$$

Since $\partial_t \boldsymbol{m} = 0$ by Proposition 2, it follows that $\boldsymbol{m} \equiv \boldsymbol{m}_0$. Now for every $t \in [0, T]$ the sequence $\nabla \boldsymbol{m}_{\varepsilon_k}(t)$ converges weakly to $\nabla \boldsymbol{m}_0$ with $\lim_{k \to \infty} D(\boldsymbol{m}_{\varepsilon_k}(t)) = 4\pi$, which implies strong convergence. Finally we deduce convergence of the whole family as $\varepsilon \searrow 0$. □

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Appendix 1: Cut-off lemma

The following cut-off result in the spirit of [9,17,21] is crucial for the proof of Proposition 1:

Lemma 8 *Suppose* $\boldsymbol{m}\colon \mathbb{R}^2 \to \mathbb{S}^2$ *satisfies* $\fint_{\mathbb{R}^2} |\nabla \boldsymbol{m}|^2\, dx < \infty$ *and*

$$\int_{B_{4R}\setminus B_R} |\nabla \boldsymbol{m}|^2\, dx + \sigma \int_{B_{4R}\setminus B_R} \tfrac{1}{2^p} |\boldsymbol{m} - \hat{\boldsymbol{e}}_3|^p\, dx < \delta$$

for some $0 < \delta \ll 1$, $R \geq 1$, $\sigma \in \{0, 1\}$. *Then, there exist*

$$\boldsymbol{m}^{(1)}, \boldsymbol{m}^{(2)}\colon \mathbb{R}^2 \to \mathbb{S}^2 \quad \textit{with} \quad \int_{\mathbb{R}^2} |\nabla \boldsymbol{m}^{(i)}|^2\, dx < \infty \quad \textit{for} \quad i = 1, 2,$$

some $c \in [R, 2R]$ *and a constant* $C = C(\delta, R) < \infty$ *so that*

$$\begin{aligned} &\boldsymbol{m}^{(1)} = \boldsymbol{m} \quad \textit{on } B_c, \qquad V(\boldsymbol{m}^{(1)}) \lesssim C, \\ &\qquad \int_{\mathbb{R}^2\setminus B_c} |\nabla \boldsymbol{m}^{(1)}|^2\, dx + \sigma \int_{\mathbb{R}^2\setminus B_c} \tfrac{1}{2^p} |\boldsymbol{m}^{(1)} - \hat{\boldsymbol{e}}_3|^p\, dx \lesssim \delta + \sigma (\tfrac{\delta}{R^2})^{2/p}, \end{aligned}$$

and

$$\begin{aligned} &\boldsymbol{m}^{(2)} = \boldsymbol{m} \quad \textit{on } \mathbb{R}^2\setminus B_{2c}, \qquad V(\boldsymbol{m}^{(2)}) \lesssim V(\boldsymbol{m}) + C, \\ &\qquad \int_{B_{2c}} |\nabla \boldsymbol{m}^{(2)}|^2\, dx + \sigma \int_{B_{2c}} \tfrac{1}{2^p} |\boldsymbol{m}^{(2)} - \hat{\boldsymbol{e}}_3|^p\, dx \lesssim \delta + \sigma (\tfrac{\delta}{R^2})^{2/p}. \end{aligned}$$

Proof We proceed in several steps. The symbol $\lesssim$ will denote an inequality that holds up to a generic, universal multiplicative constant that may change from line to line.

Step 1 (Choice of radius c): We consider $\boldsymbol{m}$ in polar coordinates and write $\boldsymbol{m}(x) = \boldsymbol{m}(r, \theta)$. Moreover, we define

$$g\colon [R, 4R] \to \mathbb{R}, \quad g(r) := \int_0^{2\pi} \Big(|\partial_r \boldsymbol{m}|^2 + |\tfrac{1}{r}\partial_\theta \boldsymbol{m}|^2 + \sigma \underbrace{\tfrac{1}{2^p}|\boldsymbol{m} - \hat{\boldsymbol{e}}_3|^p}_{=2^{-\frac{p}{2}}(1-m_3)^{\frac{p}{2}}}\Big) d\theta.$$

Poincaré's inequality yields

$$\|\boldsymbol{m}(r,\cdot) - \bar{\boldsymbol{m}}(r)\|_\infty^2 \lesssim \int_0^{2\pi} |\partial_\theta \boldsymbol{m}(r,\theta)|^2 d\theta \quad \forall r > 0,$$

where $\bar{\boldsymbol{m}}(r) := \fint_0^{2\pi} \boldsymbol{m}(r,\theta)\, d\theta$. Hence, we may choose $c \in [R, 2R]$ so that

$$\begin{aligned} \tfrac{\delta}{R} \geq \tfrac{1}{R} \int_{B_{4R}\setminus B_R} |\nabla \boldsymbol{m}|^2 + \sigma \tfrac{1}{2^p} |\boldsymbol{m} - \hat{\boldsymbol{e}}_3|^p\, dx = \tfrac{1}{R} \int_R^{4R} g(r)\, r\, dr \\ \gtrsim \fint_R^{2R} \big(g(r) + g(2r)\big) r\, dr \geq \big(g(c) + g(2c)\big) c. \end{aligned}$$

By definition of g, we obtain

$$1-|\bar{\boldsymbol m}(r)|^2 = \fint_0^{2\pi} |\boldsymbol m(r,\theta)-\bar{\boldsymbol m}(r)|^2\,d\theta \lesssim \|\boldsymbol m(r,\cdot)-\bar{\boldsymbol m}(r)\|_\infty^2$$
$$\lesssim \int_0^{2\pi} |\partial_\theta \boldsymbol m(r,\theta)|^2 d\theta \lesssim Rg(r)r \lesssim \delta \quad \text{for } r=c,2c,$$

and

$$\sigma\big(1-\bar m_3(r)\big) = \sigma^2 \fint_0^{2\pi} \big(1-m_3(r,\theta)\big)\,d\theta \lesssim \sigma\Big(\sigma\int_0^{2\pi}\big(1-m_3(r,\theta)\big)^{\frac{p}{2}}\,d\theta\Big)^{\frac{2}{p}}$$
$$\lesssim \sigma\big(g(r)\big)^{\frac{2}{p}} \lesssim \sigma(\tfrac{\delta}{R^2})^{\frac{2}{p}} \quad \text{for } r=c,2c.$$

In particular, we may assume $|\bar{\boldsymbol m}(c)| \geq \frac{1}{2}$.

Step 2 (Definition of $\boldsymbol m^{(1)}$):
Let

$$\boldsymbol e := \begin{Bmatrix} \frac{\bar{\boldsymbol m}(c)}{|\bar{\boldsymbol m}(c)|}, & \sigma=0 \\ \hat{\boldsymbol e}_3, & \sigma=1 \end{Bmatrix} \in \mathbb{S}^2,$$

so that for $\sigma=0$ we have

$$\|\boldsymbol m(c)-\boldsymbol e\|_\infty^2 \lesssim \underbrace{\|\boldsymbol m(c,\cdot)-\bar{\boldsymbol m}(c)\|_\infty^2}_{\lesssim\delta} + \underbrace{|\bar{\boldsymbol m}(c)-\boldsymbol e|^2}_{=(1-|\bar{\boldsymbol m}(c)|)^2\lesssim\delta^2} \lesssim \delta.$$

If $\sigma=1$, we may modify the second estimate as follows:

$$|\bar{\boldsymbol m}(c)-\hat{\boldsymbol e}_3|^2 \leq \fint_0^{2\pi} \underbrace{|\boldsymbol m(c,\theta)-\hat{\boldsymbol e}_3|^2}_{=2(1-m_3(c,\theta))}\,d\theta \lesssim (1-\bar m_3(c)) \lesssim (\tfrac{\delta}{R^2})^{2/p}.$$

Hence, in either situation,

$$\|\boldsymbol m(c,\cdot)-\boldsymbol e\|_\infty^2 \lesssim \delta + \sigma(\tfrac{\delta}{R^2})^{2/p} \ll 1.$$

We will define $\boldsymbol m^{(1)}\colon \mathbb{R}^2 \to \mathbb{S}^2$ in two steps:

Step 2a (Definition of $\boldsymbol m^{(1)}$ on B_{2c}): Let $\eta\colon \mathbb{R}\to[0,1]$ be a smooth cut-off function with $\eta(s)=1$ for $s\leq 0$ and $\eta(s)=0$ for $s\geq 1$. We define

$$\boldsymbol m^{(1)}(r,\theta) = \begin{cases} \frac{\eta(\frac{r-c}{c})\boldsymbol m(c,\theta)+(1-\eta(\frac{r-c}{c}))\boldsymbol e}{|\eta(\frac{r-c}{c})\boldsymbol m(c,\theta)+(1-\eta(\frac{r-c}{c}))\boldsymbol e|}, & c<r<2c, \\ \boldsymbol m(r,\theta), & 0\leq r\leq c. \end{cases}$$

so that $\boldsymbol m^{(1)}$ has a well-defined trace across ∂B_c. Using the inequality

$$|\partial_i(\rho \boldsymbol m^{(1)})|^2 = \rho^2|\partial_i \boldsymbol m^{(1)}|^2 + |\partial_i\rho|^2 \geq \tfrac{1}{4}|\partial_i \boldsymbol m^{(1)}|^2, \quad i=r,\theta,$$

where

$$\rho = \big|\eta(\tfrac{r-c}{c})\boldsymbol m(c,\theta) + \big(1-\eta(\tfrac{r-c}{c})\big)\boldsymbol e\big| \geq \tfrac{1}{2},$$

we obtain for $c\leq r\leq 2c$ that

$$|\partial_r \boldsymbol m^{(1)}(r,\theta)|^2 \lesssim |\tfrac{1}{c}\eta'(\tfrac{r-c}{c})(\boldsymbol m(c,\theta)-\boldsymbol e)|^2 \lesssim \tfrac{1}{c^2}\|\boldsymbol m(c,\cdot)-\boldsymbol e\|_\infty^2 \lesssim \tfrac{\delta+(\delta R^{-2})^{\frac{2}{p}}}{r^2}.$$

and

$$|\tfrac{1}{r}\partial_\theta \boldsymbol{m}^{(1)}(r,\theta)|^2 \lesssim \left|\tfrac{1}{r}\partial_\theta \boldsymbol{m}(c,\theta)\eta(\tfrac{r-c}{c})\right|^2 \lesssim \tfrac{1}{r^2}|\partial_\theta \boldsymbol{m}(c,\theta)|^2.$$

Hence,

$$\begin{aligned}&\int_c^{2c}\int_0^{2\pi}\big(|\partial_r \boldsymbol{m}^{(1)}(r,\theta)|^2+|\tfrac{1}{r}\partial_\theta \boldsymbol{m}^{(1)}|^2\big)d\theta\, r\, dr\\ &\lesssim \int_c^{2c}\int_0^{2\pi}\big(\tfrac{\delta+(\delta R^{-2})^{\frac{2}{p}}}{r^2}+\tfrac{1}{r^2}|\partial_\theta \boldsymbol{m}(c,\theta)|^2\big)d\theta\, r\, dr\\ &\lesssim \underbrace{\int_c^{2c}\frac{dr}{r}}_{=\ln 2}\Big(\delta+(\tfrac{\delta}{R^2})^{\frac{2}{p}}+\underbrace{\int_0^{2\pi}|\partial_\theta \boldsymbol{m}(c,\theta)|^2 d\theta}_{\lesssim\delta}\Big)\lesssim \delta+(\tfrac{\delta}{R^2})^{\frac{2}{p}}.\end{aligned}$$

Finally, since

$$1=|\boldsymbol{m}|=\big|\eta \boldsymbol{m}+(1-\eta)\boldsymbol{e}+(1-\eta)(\boldsymbol{m}-\boldsymbol{e})\big|\le\big|\eta \boldsymbol{m}+(1-\eta)\boldsymbol{e}\big|+|\boldsymbol{m}-\boldsymbol{e}|$$

implies

$$1-\big|\eta \boldsymbol{m}+(1-\eta)\boldsymbol{e}\big|\le|\boldsymbol{m}-\boldsymbol{e}|,$$

we obtain for ρ as above

$$|\boldsymbol{m}^{(1)}-\boldsymbol{e}|\le\underbrace{|\boldsymbol{m}^{(1)}-\rho \boldsymbol{m}^{(1)}|}_{=1-\rho\le|\boldsymbol{m}(c,\theta)-\boldsymbol{e}|}+\underbrace{\big|\big(\eta \boldsymbol{m}(c,\theta)+(1-\eta)\boldsymbol{e}\big)-\boldsymbol{e}\big|}_{=\eta|\boldsymbol{m}(c,\theta)-\boldsymbol{e}|}\le 2|\boldsymbol{m}(c,\theta)-\boldsymbol{e}|.$$

Hence, in the case $\sigma=1$

$$\begin{aligned}&\int_c^{2c}\int_0^{2\pi}\tfrac{1}{2^p}|\boldsymbol{m}^{(1)}-\hat{\boldsymbol{e}}_3|^p\, d\theta\, r\, dr\lesssim\int_c^{2c}\int_0^{2\pi}|\boldsymbol{m}(c,\theta)-\hat{\boldsymbol{e}}_3|^p\, d\theta\, r\, dr\\ &\lesssim\int_c^{2c}\underbrace{\int_0^{2\pi}\big(1-m_3(c,\theta)\big)^{\frac{p}{2}}\, d\theta}_{\lesssim\frac{\delta}{R^2}}\, c\, dr\lesssim\delta.\end{aligned}$$

Therefore, we have

$$\int_{B_{2c}\setminus B_c}|\nabla \boldsymbol{m}^{(1)}|^2\, dx+\sigma\int_{B_{2c}\setminus B_c}\tfrac{1}{2^p}|\boldsymbol{m}^{(1)}-\hat{\boldsymbol{e}}_3|^p\, dx\lesssim\delta+\sigma\big(\tfrac{\delta}{R^2}\big)^{\frac{2}{p}}$$

Step 2b (Definition of $\boldsymbol{m}^{(1)}$ on $\mathbb{R}^2\setminus B_{2c}$): If $\sigma=1$, there is nothing left to be done and we may just set $\boldsymbol{m}^{(1)}\equiv\hat{\boldsymbol{e}}_3$ on $\mathbb{R}^2\setminus B_{2c}$. Otherwise, we will define $\boldsymbol{m}^{(1)}$ on $(2c,2c+L)$ for some $L\gg 2c$ (to be chosen later) by interpolating $\boldsymbol{e}$ with $\hat{\boldsymbol{e}}_3$. Indeed, let $\gamma\colon[0,1]\to\mathbb{S}^2$ denote a smooth curve that connects $\gamma(0)=\boldsymbol{e}$ with $\gamma(1)=\hat{\boldsymbol{e}}_3$. Assume w.l.o.g. that $|\frac{d}{ds}\gamma(s)|\lesssim 1$ independently of $\boldsymbol{e}\in\mathbb{S}^2$. We introduce a logarithmic cut-off function

$$\eta_L\colon[2c,2c+L]\to[0,1],\quad \eta_L(r):=\frac{\ln(\frac{r}{2c})}{\ln(\frac{2c+L}{2c})},$$

and let

$$\boldsymbol{m}^{(1)}(r,\theta)=\begin{cases}\gamma(\eta_L(r)), & 2c\le r\le 2c+L\\ \hat{\boldsymbol{e}}_3, & 2c+L<r.\end{cases}$$

Then, $\boldsymbol{m}^{(1)}$ has a well-defined trace both across ∂B_{2c} and ∂B_{2c+L}, and

$$\tfrac{d}{dr}\boldsymbol{m}^{(1)}(r) = \frac{(\frac{d}{ds}\gamma)(\eta_L(r))}{r\ln(\frac{2c+L}{2c})}.$$

Hence, $\partial_\theta \boldsymbol{m}^{(1)} = 0$ and

$$\int_{2c}^{2c+L}\int_0^{2\pi} \underbrace{|\partial_r \boldsymbol{m}^{(1)}(r,\theta)|^2}_{\lesssim \frac{1}{r^2}\ln^{-2}(\frac{2c+L}{2c})} d\theta\, r\, dr \lesssim \tfrac{1}{\ln^2(1+\frac{L}{2c})}\int_{2c}^{2c+L}\tfrac{dr}{r} = \tfrac{1}{\ln(1+\frac{L}{2c})} \lesssim \delta,$$

if $L = 2c(e^{\frac{1}{\delta}} - 1)$.

Thus, we may conclude for $\sigma \in \{0, 1\}$:

$$\int_{\mathbb{R}^2\setminus B_{2c}} |\nabla \boldsymbol{m}^{(1)}|^2\, dx + \sigma \int_{\mathbb{R}^2\setminus B_{2c}} \tfrac{1}{2^p}|\boldsymbol{m}^{(1)} - \hat{\boldsymbol{e}}_3|^p\, dx \lesssim \delta + \sigma\big(\tfrac{\delta}{R^2}\big)^{\frac{2}{p}},$$

and

$$V(\boldsymbol{m}^{(1)}) = \int_{B_{2c+L}} \underbrace{\tfrac{1}{2^p}|\boldsymbol{m}^{(1)} - \hat{\boldsymbol{e}}_3|^p}_{\leq 1}\, dx \lesssim (2c+L)^2 =: C(\delta, R).$$

Step 3 (Definition of $\boldsymbol{m}^{(2)}$): In order to define $\boldsymbol{m}^{(2)}$, we proceed as in Step 2. Let

$$\boldsymbol{e} := \tfrac{\bar{\boldsymbol{m}}(2c)}{|\bar{\boldsymbol{m}}(2c)|} \in \mathbb{S}^2.$$

Then

$$\|\boldsymbol{m}(2c,\cdot) - \boldsymbol{e}\|_\infty^2 \lesssim \delta + \sigma\big(\tfrac{\delta}{R^2}\big)^{\frac{2}{p}} \ll 1,$$

and, using the same cut-off function $\eta\colon \mathbb{R} \to [0, 1]$ as before, we may define $\boldsymbol{m}^{(2)}\colon \mathbb{R}^2 \to \mathbb{S}^2$ as

$$\boldsymbol{m}^{(2)}(r,\theta) := \begin{cases} \boldsymbol{e}, & r \leq c,\\ \frac{\eta(\frac{r-c}{c})\boldsymbol{e} + (1-\eta(\frac{r-c}{c}))\boldsymbol{m}(2c,\theta)}{|\eta(\frac{r-c}{c})\boldsymbol{e} + (1-\eta(\frac{r-c}{c}))\boldsymbol{m}(2c,\theta)|}, & c < r < 2c,\\ \boldsymbol{m}(r,\theta), & r \geq 2c, \end{cases}$$

so that $\boldsymbol{m}^{(2)}$ has a well-defined trace across ∂B_c and ∂B_{2c}.

As before, we estimate for $c < r < 2c$

$$|\partial_r \boldsymbol{m}^{(2)}(r,\theta)|^2 \lesssim \tfrac{\delta + (\delta R^{-2})^{\frac{2}{p}}}{r^2} \quad\text{and}\quad |\tfrac{1}{r}\partial_\theta \boldsymbol{m}^{(2)}(r,\theta)|^2 \lesssim \tfrac{1}{r^2}|\partial_\theta \boldsymbol{m}(2c,\theta)|^2,$$

so that

$$\int_c^{2c}\int_0^{2\pi} \big(|\partial_r \boldsymbol{m}^{(2)}(r,\theta)|^2 + |\tfrac{1}{r}\partial_\theta \boldsymbol{m}^{(2)}|^2\big) d\theta\, r\, dr \lesssim \delta + \big(\tfrac{\delta}{R^2}\big)^{\frac{2}{p}}.$$

Moreover, by the same argument as in Step 2, for $\sigma = 1$:

$$\int_c^{2c}\int_0^{2\pi} \tfrac{1}{2^p}|\boldsymbol{m}^{(2)} - \hat{\boldsymbol{e}}_3|^p\, d\theta\, r\, dr \lesssim \delta.$$

Hence, we may conclude for $\sigma \in \{0, 1\}$:

$$\int_{B_{2c}} |\nabla \boldsymbol{m}^{(2)}|^2\, dx + \sigma \int_{B_{2c}} \tfrac{1}{2^p}|\boldsymbol{m}^{(2)} - \hat{\boldsymbol{e}}_3|^p\, dx \lesssim \delta + \sigma\big(\tfrac{\delta}{R^2}\big)^{\frac{2}{p}},$$

and

$$V(\boldsymbol{m}^{(2)}) = \underbrace{\int_{\mathbb{R}^2\setminus B_{2c}} \tfrac{1}{2^p}|\boldsymbol{m}-\hat{\boldsymbol{e}}_3|^p\,dx}_{\le V(\boldsymbol{m})} + \int_{B_{2c}} \underbrace{\tfrac{1}{2^p}|\boldsymbol{m}^{(2)}-\hat{\boldsymbol{e}}_3|^p}_{\le 1}\,dx \lesssim V(\boldsymbol{m}) + \underbrace{(2c)^2}_{=:C(\delta,R)}. \quad \square$$

Appendix 2: Construction of a stream function

Lemma 9 *Given $R > 1$, there exists a smooth function $f_R\colon [0,\infty) \to \mathbb{R}$ so that*

$$f_R(r) = \begin{cases} \ln(1+r^2), & \text{for } 0 \le r \le R,\\ const., & \text{for } r \ge 2R,\end{cases}$$

and

$$0 \le f_R'(r) \le \tfrac{2r}{1+r^2}, \quad 0 \le -f_R''(r) \le \tfrac{C}{1+r^2} \quad \textit{for all } r \ge R.$$

Proof Let $h\colon [0,\infty) \to \mathbb{R}$ be given by (in fact, h is a regularization of the function $y \mapsto \min(y,0)$)

$$h(y) = \int_0^y \eta(s)\,ds,$$

where $\eta\colon \mathbb{R} \to [0,1]$ is a smooth, non-increasing function with

$$\eta(s) = 1 \quad \text{for } s \le 0, \qquad \eta(s) = 0 \quad \text{for } s \ge \tfrac{1}{2}, \quad 0 \le -\eta'(s) \le C \quad \forall s \in \mathbb{R}.$$

Then,

$$f_R(r) := h\left(\ln(1+r^2) - \ln(1+R^2)\right) + \ln(1+R^2), \quad r \ge 0,$$

satisfies the claim.

Indeed, we have $h(y) = y$ for $y \le 0$ and $h(y) = \int_0^\infty \eta(s)\,ds$ for $y \ge \frac{1}{2}$. Since $\ln(1+r^2) - \ln(1+R^2) \le 0$ for $r \le R$, we therefore obtain $f_R(r) = \ln(1+r^2)$. On the other hand, $r \ge 2R \ge 2$ yields $\ln(1+r^2) - \ln(1+R^2) \ge \ln(\frac{1+4R^2}{1+R^2}) \ge \ln(\frac{5}{2}) \ge \frac{1}{2}$, so that $f_R(r) = \int_0^\infty \eta(s)\,ds + \ln(1+R^2)$.

Finally, we have

$$f_R'(r) = \underbrace{\eta\big(\ln(1+r^2) - \ln(1+R^2)\big)}_{\in[0,1]} \tfrac{2r}{1+r^2}$$

and

$$f_R''(r) = \underbrace{\eta'\big(\ln(1+r^2) - \ln(1+R^2)\big)}_{\le 0}\Big(\tfrac{2r}{1+r^2}\Big)^2 + \underbrace{\eta\big(\ln(1+r^2) - \ln(1+R^2)\big)}_{\in[0,1]} \tfrac{2(1-r^2)}{(1+r^2)^2}.$$

In particular, $0 \le f_R'(r) \le \frac{2r}{1+r^2}$ for $r \ge R$ and $0 \le -f_R''(r) \le \frac{C}{1+r^2}$. $\square$

Appendix 3: Pulled back Landau–Lifshitz–Gilbert equation

We shall argue on the level of the Landau–Lifshitz form

$$\begin{aligned}&(1+\alpha^2)\partial_t \boldsymbol{m} + (1+\alpha\beta)(v\cdot\nabla)\boldsymbol{m}\\ &\quad = -\left[(\alpha-\beta)\boldsymbol{m}\times(v\cdot\nabla)\boldsymbol{m} + \boldsymbol{m}\times\boldsymbol{h}_\varepsilon + \alpha\,\boldsymbol{m}\times\boldsymbol{m}\times\boldsymbol{h}_\varepsilon\right],\end{aligned}$$

see e.g. [22], rather than the Gilbert form (4). Solving Thiele's equation we have

$$(1+\alpha^2)c = (1+\alpha\beta)v - (\alpha-\beta)v^\perp.$$

Now we compute

$$\begin{aligned}(1+\alpha^2)\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{m}(x+ct,t) &= (1+\alpha^2)\partial_t\boldsymbol{m} + (1+\alpha^2)(c\cdot\nabla)\boldsymbol{m}\\ &= (1+\alpha^2)\partial_t\boldsymbol{m} + (1+\alpha\beta)(v\cdot\nabla)\boldsymbol{m} - (\alpha-\beta)(v\times\nabla)\boldsymbol{m}\\ &= -(\alpha-\beta)\boldsymbol{\Psi} - (\boldsymbol{m}\times\boldsymbol{h}_\varepsilon + \alpha\,\boldsymbol{m}\times\boldsymbol{m}\times\boldsymbol{h}_\varepsilon),\end{aligned}$$

where with the notation $v\times\nabla = v_1\partial_2 - v_2\partial_1$

$$\begin{aligned}\boldsymbol{\Psi} &= (v\times\nabla)\boldsymbol{m} + \boldsymbol{m}\times(v\cdot\nabla)\boldsymbol{m}\\ &= v_1\,(\partial_2\boldsymbol{m} + \boldsymbol{m}\times\partial_1\boldsymbol{m}) - v_2\,(\partial_1\boldsymbol{m} - \boldsymbol{m}\times\partial_2\boldsymbol{m})\\ &= 2v_1\boldsymbol{m}\times\partial_z\boldsymbol{m} - 2v_2\partial_z\boldsymbol{m}.\end{aligned}$$

where $\partial_z\boldsymbol{m} = \frac{1}{2}(\partial_1\boldsymbol{m} - \boldsymbol{m}\times\partial_2\boldsymbol{m})$. Upon the transformation $\boldsymbol{m}(x+ct,t)\mapsto\boldsymbol{m}(x,t)$ and with effective coupling parameters $\nu_i = \dfrac{2(\alpha-\beta)v_i}{1+\alpha^2}$ this can be written as

$$(1+\alpha^2)\,(\partial_t\boldsymbol{m} + \nu_1\boldsymbol{m}\times\partial_z\boldsymbol{m} - \nu_2\partial_z\boldsymbol{m}) + \boldsymbol{m}\times\boldsymbol{h}_\varepsilon + \alpha\,\boldsymbol{m}\times\boldsymbol{m}\times\boldsymbol{h}_\varepsilon = 0.$$

A rigid rotation yields for $\nu = \sqrt{\nu_1^2+\nu_2^2}$

$$(1+\alpha^2)\,(\partial_t\boldsymbol{m} - \nu\partial_z\boldsymbol{m}) + \boldsymbol{m}\times\boldsymbol{h}_\varepsilon + \alpha\,\boldsymbol{m}\times\boldsymbol{m}\times\boldsymbol{h}_\varepsilon = 0,$$

which easily recasts into (6).

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