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Multiplicity results for some nonlinear elliptic problems with asymptotically p-linear terms

A. M. Candela[1] · **G. Palmieri**[1]

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Abstract Taking any $p > 1$, we consider the asymptotically p-linear problem

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u)) + A_t(x,u,\nabla u) \;=\; \lambda^\infty |u|^{p-2}u + g^\infty(x,u) & \text{in } \Omega, \\ u \;=\; 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, $A(x,t,\xi)$ is a real function on $\Omega \times \mathbb{R} \times \mathbb{R}^N$ which grows with power p with respect to ξ and has partial derivatives $A_t(x,t,\xi) = \frac{\partial A}{\partial t}(x,t,\xi)$, $a(x,t,\xi) = \nabla_\xi A(x,t,\xi)$. If $A(x,t,\xi) \to A^\infty(x,t)$ and $\frac{g^\infty(x,t)}{|t|^{p-1}} \to 0$ as $|t| \to +\infty$, suitable assumptions, variational methods and either the cohomological index theory or its related pseudo-index one, allow us to prove the existence of multiple nontrivial bounded solutions in the *non-resonant case*, i.e. if λ^∞ is not an eigenvalue of the operator associated to $\nabla_\xi A^\infty(x,\xi)$. In particular, while in [14] the model problem $A(x,t,\xi) = \mathcal{A}(x,t)|\xi|^p$ with $p > N$ is studied, here our goal is twofold: extending such results not only to a more general family of functions $A(x,t,\xi)$, but also to the more difficult case $1 < p \leq N$.

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✉ A. M. Candela
annamaria.candela@uniba.it

G. Palmieri
giuliana.palmieri.uniba@gmail.com

1 Dipartimento di Matematica, Università degli Studi di Bari Aldo Moro, Via E. Orabona 4, 70125 Bari, Italy

1 Introduction

Let us consider the nonlinear problem

$$(GP) \qquad \begin{cases} -\operatorname{div}(a(x,u,\nabla u)) + A_t(x,u,\nabla u) \;=\; f(x,u) & \text{in } \Omega, \\ u \;=\; 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $N \geq 2$, $f(x,t)$ is a given real function on $\Omega \times \mathbb{R}$ and $A(x,t,\xi)$ is a real function on $\Omega \times \mathbb{R} \times \mathbb{R}^N$, with $A_t(x,t,\xi) = \frac{\partial A}{\partial t}(x,t,\xi)$, $a(x,t,\xi) = \nabla_\xi A(x,t,\xi)$.

If we set $F(x,t) = \int_0^t f(x,s)ds$, problem (GP) can be associated to the functional

$$\mathcal{J}(u) \;=\; \int_\Omega A(x,u,\nabla u)dx \;-\; \int_\Omega F(x,u)\,dx. \tag{1.1}$$

If $A(x,t,\xi)$ depends on t, the derivative $d\mathcal{J}$ is not defined in the Sobolev space $W_0^{1,p}(\Omega)$ and its natural domain contains $X := W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ where it is also continuous (see Proposition 3.5). Moreover, u is a weak solution of (GP) if

$$\int_\Omega (a(x,u,\nabla u)\cdot\nabla v + A_t(x,u,\nabla u)v)dx - \int_\Omega f(x,u)v dx = 0 \quad \text{for all } v \in X;$$

thus, we prove that u belongs to X and is a critical point of $\mathcal{J}$. Hence, in order to solve (GP), we can use variational tools.

Model problems can be written by considering

$$A_2(x,t,\xi) \;=\; \sum_{i,j=1}^N a_{i,j}(x,t)\xi_i\xi_j \quad \text{or} \quad A_p(x,t,\xi) \;=\; (A_2(x,t,\xi))^{\frac{p}{2}},$$

where $(a_{i,j}(x,t))_{1\leq i,j\leq N}$ is an elliptic matrix.

An example is given by $A(x,t,\xi) = \mathcal{A}(x,t)|\xi|^p$ with $p > 1$, so that the equation in (GP) is reduced to the quasi-p-linear equation

$$-\operatorname{div}(\mathcal{A}(x,u)\,|\nabla u|^{p-2}\,\nabla u) + \frac{1}{p}\,\mathcal{A}_t(x,u)\,|\nabla u|^p \;=\; f(x,u) \qquad \text{in } \Omega, \tag{P}$$

which is studied in [13,14] if $p > N$. In this setting, the related functional is

$$J_{\mathcal{A}}(u) \;=\; \frac{1}{p}\int_\Omega \mathcal{A}(x,u)\,|\nabla u|^p\,dx \;-\; \int_\Omega F(x,u)\,dx. \tag{1.2}$$

Roughly speaking, we say that problem (GP) is asymptotically p-linear, if both $A(x,t,\xi)$ and $F(x,t)$ admit the limit as $|t| \to +\infty$, so that, taking

$$A^\infty(x,\xi) = \lim_{|t|\to+\infty} A(x,t,\xi) \quad \text{and} \quad F(x,t) = \frac{\lambda^\infty}{p}|t|^p + G^\infty(x,t), \tag{1.3}$$

$A^\infty(x,\xi)$ is a positively p-homogeneous function with respect to ξ equivalent to $|\xi|^p$, while $G^\infty(x,t)$ is at worst an infinity of lower order with respect to $|t|^p$ (for more details, see Sect. 3).

Thus, our aim is to investigate the existence of weak solutions of the nonlinear elliptic problem (GP) when it is asymptotically p-linear but in the non-resonant case, i.e., when λ^∞ in (1.3) is not an eigenvalue of the operator associated to $\nabla_\xi A^\infty(x,\xi)$.

For $A(x,t,\xi) = |\xi|^p$ or, at worst, for $A(x,t,\xi) = \bar{A}(x)|\xi|^p$, i.e., for $\mathcal{A}(x,t) \equiv 1$ or $\mathcal{A}(x,t) \equiv \bar{A}(x)$ independent of t, a variational approach was first used for $p = 2$ in the asymptotically linear case (see the seminal papers [1,5]) and then if $p \neq 2$ (see, e.g., [2,4,6,16,17,20–22,25], or the survey in the book [24]). Furthermore, in the model case (P) some multiplicity results have already been proved if $p > N$ (see [13,14]).

We want to prove that the multiplicity results in [14] can also be stated in the general case (GP) and not only if $p > N$, but also in the complementary condition $1 < p \leq N$ (for the complete statements, see Sect. 5).

We note that if $1 < p \leq N$ and λ^∞ is not an eigenvalue of the operator associated to $\nabla_\xi A^\infty(x,\xi)$, in quite general suitable assumptions, a Palais–Smale sequence of $\mathcal{J}$ in X can have subsequences converging in $W_0^{1,p}(\Omega)$, but not in $L^\infty(\Omega)$ (see Example 4.3). Therefore, the classical Palais–Smale condition does not hold. This is why the geometric conditions are given by making use of the topology of $W_0^{1,p}(\Omega)$. As typical of such a problem, we consider subsets of neighbourhoods of zero and of infinity, but with respect to the norm in $W_0^{1,p}(\Omega)$ and not to that in X. Hence, in both cases we have no information about the L^∞-norm for the elements of such sets.

In any case, we prove the existence of multiple nontrivial solutions according to the behaviour of $F(x,t)$, both in zero and at infinity, and by considering $A^0(x,\xi) = A(x,0,\xi)$ for the geometric conditions in zero and the limit function $A^\infty(x,\xi)$ for those at infinity (see Theorems 5.6, 5.7 and 5.8).

Finally, let us point out that no global p-homogeneity assumption on function $A(x,t,\xi)$ is required, but only that $A^0(x,\xi)$ and $A^\infty(x,\xi)$ have to be positively p-homogeneous with respect to ξ. Moreover, also in the non-resonant case, the proof of the boundedness of the Palais–Smale sequences is rather hard and our results imply the previous ones obtained when the term $A(x,t,\xi)$ does not depend on t.

This paper is organized as follows. In Sect. 2 we introduce the weak Palais–Smale condition and prove the related abstract multiplicity results, both with the cohomological index and the related pseudo-index. In Sect. 3 we introduce the hypotheses for (GP) and prove the first properties of $\mathcal{J}$ in X, while the weak Palais–Smale condition is proved in Sect. 4. In Sect. 5 the main results are stated (see Theorems 5.6, 5.7 and 5.8) and, once the geometric conditions have been checked, their proofs are given in Sect. 6, for solutions with negative critical levels via the index theory, and in Sect. 7, for solutions with positive critical levels via the related pseudo-index.

2 The abstract variational setting

We denote $\mathbb{N} = \{1, 2, \dots\}$ and, throughout this section, let us assume that:

- $(X, \|\cdot\|_X)$ is a Banach space with dual $(X', \|\cdot\|_{X'})$,
- $(W, \|\cdot\|_W)$ is another Banach space such that $X \hookrightarrow W$ continuously, i.e. $X \subset W$ and a constant $\sigma_0 > 0$ exists such that
$$\|u\|_W \leq \sigma_0 \|u\|_X \qquad \text{for all } u \in X, \tag{2.1}$$
- $J: \mathcal{D} \subset W \to \mathbb{R}$ and $J \in C^1(X, \mathbb{R})$ with $X \subset \mathcal{D}$.

Furthermore, fixing $\beta, \beta_1, \beta_2 \in \mathbb{R}$ and a set $\mathcal{C} \subset X$, let us denote

- $K^J = \{u \in X : dJ(u) = 0\}$ the set of the critical points of J in X,
- $K^J_\beta = \{u \in X : J(u) = \beta,\ dJ(u) = 0\}$ the set of the critical points of J in X at level β,
- $J^\beta = \{u \in X : J(u) \leq \beta\}$ the sublevel of J with respect to level β,

- $J_{\beta_1}^{\beta_2} = \{u \in X : \beta_1 \le J(u) \le \beta_2\}$ the closed "strip" between β_1 and β_2,
- $I\mathcal{C} = \{su \in X : u \in \mathcal{C}, s \in [0, 1]\}$ the cone with base $\mathcal{C}$,

while, taking $u_0 \in X$, $r > 0$, by pointing out the two different norms $\|\cdot\|_W$ and $\|\cdot\|_X$, for $\ddagger = W$ or $\ddagger = X$ we put

- $B_r^\ddagger(u_0) = \{u \in X : \|u - u_0\|_\ddagger < r\}$,
- $\bar{B}_r^\ddagger(u_0) = \{u \in X : \|u - u_0\|_\ddagger \le r\}$,
- $d_\ddagger(u, \mathcal{C}) = \inf_{v \in \mathcal{C}} \|u - v\|_\ddagger$,
- $N_r^\ddagger(\mathcal{C}) = \{u \in X : d_\ddagger(u, \mathcal{C}) \le r\}$.

In any case, in order to avoid any ambiguity and to simplify, where possible, the notations, from now on we denote by X the space equipped with its given norm while, if a different norm is involved, we write it down explicitely. Accordingly, we denote by $\overline{\mathcal{C}}$ the closure of a set $\mathcal{C} \subset X$ with respect to the norm $\|\cdot\|_X$.

For investigating the number of critical points of a C^1 functional J in the Banach space X, let us introduce suitable variational tools.

For simplicity, taking $\beta \in \mathbb{R}$, we say that a sequence $(u_n)_n \subset X$ is a *Palais–Smale sequence at level* β, briefly $(PS)_\beta$*-sequence*, if

$$\lim_{n \to +\infty} J(u_n) = \beta \quad \text{and} \quad \lim_{n \to +\infty} \|dJ(u_n)\|_{X'} = 0. \tag{2.2}$$

Hence, the functional J satisfies the *Palais–Smale condition at level* β in X, briefly $(PS)_\beta$, if every $(PS)_\beta$-sequence converges in $(X, \|\cdot\|_X)$, up to subsequences (see [23]).

Different versions of the (classical) Palais–Smale condition can be introduced (see, e.g., [9,12,15]). In particular, as in [9], we say that the functional J satisfies the *Brézis–Coron–Nirenberg condition at level* β, if the following statement holds:

"If a $(PS)_\beta$-sequence exists, then β is a critical value".

Unfortunately, in order to find multiple solutions to our model problem (P), both the previous definitions are not useful. In fact, for the Palais–Smale condition, the convergence of a sequence in the intersection space $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ requires the convergence not only in the norm of $W_0^{1,p}(\Omega)$, but also in that of $L^\infty(\Omega)$, which may not hold (see Example 4.3). However, even if the Brézis–Coron–Nirenberg condition allows us to prove some existence results (see [9, Theorem 2]), contrary to the classical Palais–Smale it is not sufficient for finding multiple critical points if they occur at the same critical level.

For this reason, following some ideas developed in [12], in our setting we introduce a new condition, which considers both the involved norms and is weaker than the Palais–Smale but stronger than the Brézis–Coron–Nirenberg.

Definition 2.1 The functional J satisfies a *weak version of the Palais–Smale condition at level* β ($\beta \in \mathbb{R}$), briefly $(wPS)_\beta$, if, for every $(PS)_\beta$-sequence $(u_n)_n$, $u \in X$ exists, such that

(i) $\lim_{n \to +\infty} \|u_n - u\|_W = 0$ (up to subsequences),
(ii) $J(u) = \beta$, $dJ(u) = 0$.

If J satisfies $(wPS)_\beta$ at each level $\beta \in I$, I real interval, we say that J satisfies (wPS) in I.

The following lemmas are direct consequences of Definition 2.1.

Lemma 2.2 *If J satisfies $(wPS)_\beta$ at a level $\beta \in \mathbb{R}$, then K_β^J is compact with respect to $\|\cdot\|_W$.*

Lemma 2.3 *If J satisfies $(wPS)_\beta$ at level $\beta \in \mathbb{R}$, then, for each $\varrho > 0$, some $\varepsilon_\varrho, \mu_\varrho > 0$ exist, such that*

$$u \in J_{\beta-\varepsilon_\varrho}^{\beta+\varepsilon_\varrho}, \quad d_W(u, K_\beta^J) \geq \varrho \qquad \Longrightarrow \qquad \|dJ(u)\|_{X'} \geq \mu_\varrho.$$

Proof Arguing by contradiction, we assume that $\bar{\varrho} > 0$ and a sequence $(u_n)_n \subset X$ exist, so that (2.2) holds and

$$d_W(u_n, K_\beta^J) \geq \bar{\varrho} \quad \text{for all } n \in \mathbb{N}. \tag{2.3}$$

However, by $(wPS)_\beta$ $u \in X$ exists, such that $\|u_n - u\|_W \to 0$ (up to subsequences) and $u \in K_\beta^J$, in contradiction with (2.3). □

Now our aim is to generalize the classical Deformation Lemma (see, e.g., [26, Theorem A.4] or [27, Theorem 3.3.4]), when the Palais–Smale condition is replaced by its weak version in Definition 2.1.

Proposition 2.4 *Let J be a C^1 functional which satisfies (wPS) in $\mathbb{R}$. Taking $\beta \in \mathbb{R}$, for any fixed $\varrho > 0$ and $\varepsilon_0 > 0$ a constant $\varepsilon^* > 0$, $2\varepsilon^* < \varepsilon_0$, exists, such that for each $\varepsilon \in]0, \varepsilon^*]$ a homeomorphism $\Psi : X \to X$ exists which satisfies the following conditions:*

(i) $\Psi(u) = u$ *for all* $u \notin J_{\beta-\varepsilon_0}^{\beta+\varepsilon_0}$;

(ii) $\Psi(J^{\beta+\varepsilon} \backslash N_\varrho^W(K_\beta^J)) \subset J^{\beta-\varepsilon}$ *and* $\Psi(J^{\beta+\varepsilon}) \subset J^{\beta-\varepsilon} \cup N_\varrho^W(K_\beta^J)$.

Furthermore, Ψ is odd if J is even.

Proof The proof is essentially similar to the classical one in [26, Theorem A.4] but checking carefully the change of norm when necessary. Thus, here we just outline the differences with respect to such a proof.

From Lemma 2.3 $\varepsilon_\varrho, \mu_\varrho > 0$ exist, such that

$$u \in J_{\beta-\varepsilon_\varrho}^{\beta+\varepsilon_\varrho}, \quad d_W(u, K_\beta^J) \geq \frac{\varrho}{8} \qquad \Longrightarrow \qquad \|dJ(u)\|_{X'} \geq \mu_\varrho.$$

Moreover, as J is a C^1 functional on X, then $V : X \to X$ pseudogradient vector field of J exists, such that $V(u) = 0$ if $u \in K^J$ and

$$\|dJ(u)\|_{X'} \leq \|V(u)\|_X \leq 2\|dJ(u)\|_{X'}, \quad \langle dJ(u), V(u)\rangle \geq \|dJ(u)\|_{X'}^2 \tag{2.4}$$

for all $u \in X$, and V can be chosen odd if J is even (see [27, Chapter II]).
Now, taking $\varepsilon^* > 0$, such that

$$2\varepsilon^* < \min\left\{\varepsilon_0, \varepsilon_\varrho, \frac{\varrho\, \mu_\varrho}{4\sigma_0}\right\},$$

with σ_0 as in (2.1), for any $\varepsilon \in]0, \varepsilon^*]$, we can define a Lipschitz continuous cut-off function $\chi_\varepsilon : X \to [0, 1]$, such that

$$\chi_\varepsilon(u) \;=\; \begin{cases} 0 \text{ if } u \notin J_{\beta-2\varepsilon}^{\beta+2\varepsilon} \\ 1 \text{ if } u \in J_{\beta-\varepsilon}^{\beta+\varepsilon} \end{cases}. \tag{2.5}$$

On the other hand, taking

$$N^W = N_{\varrho/8}^W(K_\beta^J), \quad C^W = \{u \in X : d_W(u, K_\beta^J) \geq \frac{\varrho}{4}\},$$

(both closed also with respect to $\|\cdot\|_X$) with $N^W \cap C^W = \emptyset$, we can define

$$\vartheta(u) \;=\; \frac{d_W(u, N^W)}{d_W(u, N^W) + d_W(u, C^W)}.$$

By direct computations and from (2.1) it follows that $\vartheta : X \to [0,1]$ is a Lipschitz continuous function, such that

$$\vartheta(u) = \begin{cases} 0 \text{ if } u \in N^W \\ 1 \text{ if } u \in C^W \end{cases}. \tag{2.6}$$

Defining

$$V_\varepsilon(u) = \begin{cases} -\chi_\varepsilon(u)\,\vartheta(u)\,\dfrac{V(u)}{\|V(u)\|_X} & \text{if } u \notin K^J \\ 0 & \text{if } u \in K^J \end{cases}, \tag{2.7}$$

for any "initial point" $u \in X$, we consider the Cauchy problem

$$\begin{cases} \frac{\partial \eta}{\partial s}(s;u) = V_\varepsilon(\eta(s;u)), \\ \eta(0;u) = u. \end{cases} \tag{2.8}$$

By construction, the function V_ε is locally Lipschitz continuous and bounded with respect to $\|\cdot\|_X$; hence, for each $u \in X$ a unique C^1 function $\eta(\cdot;u) : \mathbb{R} \to X$ exists which solves (2.8). Moreover, $\eta(s;\cdot) : X \to X$ is a homeomorphism for each $s \in \mathbb{R}$ and is odd if J is even.

We note that, from definitions (2.5) and (2.6), $\eta(s;u) = u$ not only if $s = 0$ for all $u \in X$ (initial datum in (2.8)), but also for all $s \in \mathbb{R}$, if $u \notin J_{\beta-2\varepsilon}^{\beta+2\varepsilon}$ or $u \in N_{\varrho/8}^W(K_\beta^J)$. In particular, if $u \notin \mathcal{J}_{\beta-\varepsilon_0}^{\beta+\varepsilon_0}$, $\eta(s;u) \equiv u$ for all $s \in \mathbb{R}$.

From (2.4), (2.7) and (2.8) it follows that

$$s \in \mathbb{R} \mapsto J(\eta(s;u)) \in \mathbb{R} \quad \text{is decreasing for any fixed } u \in X, \tag{2.9}$$

thus, $J(\eta(s;u)) \le J(u)$ for all $s \ge 0$.

We point out that, taking any $u \in X$, (2.1) and (2.8) imply that

$$\|\eta(s_1;u) - \eta(s_2;u)\|_W \le \sigma_0 \|\eta(s_1;u) - \eta(s_2;u)\|_X \le \sigma_0 |s_1 - s_2|;$$

whence

$$\|\eta(s;u) - u\|_W \le \sigma_0 s \quad \text{for all } s \ge 0,$$

so, fixing $s^* = \frac{\varrho}{2\sigma_0}$, we have that

$$\|\eta(s^*;u) - u\|_W \le \frac{\varrho}{2}. \tag{2.10}$$

Now, let $u \in J^{\beta+\varepsilon}$. If $u \notin N_\varrho^W(K_\beta^J)$, $s \in [0, s^*]$ exists, such that $\eta(s;u) \in J^{\beta-\varepsilon}$, then (2.9) implies $\eta(s^*;u) \in J^{\beta-\varepsilon}$. On the contrary, if $u \in N_\varrho^W(K_\beta^J)$, either $s \in]0, s^*]$ exists, such that $\eta(s;u) \in N_{\varrho/2}^W(K_\beta^J)$, thus (2.10) implies that $\eta(s^*;u) \in N_\varrho^W(K_\beta^J)$, or $\eta(s^*;u) \in J^{\beta-\varepsilon}$. Hence, we choose $\Psi = \eta(s^*;\cdot)$. □

Now, we assume that J is even and $J(0) = 0$ and, in order to obtain multiple critical points, we quote the main tools on the $\mathbb{Z}_2$-cohomological index on a Banach space X, as introduced by Fadell and Rabinowitz in [18].

Firstly, let us recall the definition and some basic properties of the cohomological index $i(\cdot)$, defined in the Banach space $(X, \|\cdot\|_X)$.

Taking

$$\mathcal{P} = \{P \subset X\backslash\{0\} : \ P \text{ symmetric, i.e.} -u \in P \text{ if } u \in P\}, \tag{2.11}$$

for $P \in \mathcal{P}$ we denote by

- $\tilde{P} = P/\mathbb{Z}_2$ the quotient space of P with each u and $-u$ identified,
- $f : \tilde{P} \to \mathbb{R}\mathrm{P}^\infty$ the classifying map of $\tilde{P}$,
- $f^* : H^*(\mathbb{R}\mathrm{P}^\infty) \to H^*(\tilde{P})$ the induced homomorphism of the Alexander–Spanier cohomology rings.

Then the cohomological index of $P \in \mathcal{P}$ is defined by

$$i(P) = \begin{cases} \sup\left\{m \geq 1 : f^*(\omega^{m-1}) \neq 0\right\} & \text{if } P \neq \emptyset, \\ 0 & \text{if } P = \emptyset, \end{cases}$$

where $\omega \in H^1(\mathbb{R}\mathrm{P}^\infty)$ is the generator of the polynomial ring $H^*(\mathbb{R}\mathrm{P}^\infty) = \mathbb{Z}_2[\omega]$.

Here we list the basic properties of the cohomological index (see, e.g., [24, Proposition 2.12]).

Proposition 2.5 *Index $i : \mathcal{P} \to \mathbb{N} \cup \{0, +\infty\}$ has the following properties:*

(i_1) *Definiteness: taking $P \in \mathcal{P}$, $i(P) = 0$ if and only if $P = \emptyset$;*
(i_2) *Monotonicity: let B be a topological space and let $\eta : X \to B$ be an odd continuous map, then $i(P) \leq i(\eta(P))$ for any $P \in \mathcal{P}$. Hence, the equality holds when the map is an odd homeomorphism. In particular, if $P_1, P_2 \in \mathcal{P}$ are such that $P_1 \subset P_2$, then $i(P_1) \leq i(P_2)$;*
(i_3) *Dimension: taking any finite dimensional space $X_0 \subset X$ and $P \in \mathcal{P}$, such that $P \subset X_0$, then $i(P) \leq \dim X_0$;*
(i_4) *Continuity: If $P \in \mathcal{P}$ is closed, then there is a closed neighbourhood $U \in \mathcal{P}$ of P, such that $i(U) = i(P)$. When P is compact, then U may be chosen to be a ϱ-neighbourhood $N_\varrho^X(P)$;*
(i_5) *Subadditivity: If $P_1, P_2 \in \mathcal{P}$ are closed, then $i(P_1 \cup P_2) \leq i(P_1) + i(P_2)$;*
(i_6) *Stability: taking $P \in \mathcal{P}$, if SP is the suspension of $P \neq \emptyset$, obtained as the quotient space of $P \times [-1, 1]$ with $P \times \{1\}$ and $P \times \{-1\}$ collapsed at different points, then $SP \in \mathcal{P}$ and $i(SP) = i(P) + 1$;*
(i_7) *Piercing property: If $P, P_0, P_1 \in \mathcal{P}$ are closed and $\varphi : P \times [0, 1] \to P_0 \cup P_1$ is a continuous map, such that $\varphi(-u, s) = -\varphi(u, s)$ for all $(u, s) \in P \times [0, 1]$, $\varphi(P \times [0, 1])$ is closed, $\varphi(P \times \{0\}) \subset P_0$, and $\varphi(P \times \{1\}) \subset P_1$, then $i(\varphi(P \times [0, 1]) \cap P_0 \cap P_1) \geq i(P)$;*
(i_8) *Neighbourhood of zero: If U is a bounded closed symmetric neighbourhood of 0 contained in a finite dimensional subspace $X_0 \subset X$, then $\partial U \in \mathcal{P}$ and $i(\partial U) = \dim X_0$.*

Remark 2.6 Since in our setting X is continuously imbedded in W, namely a continuous map $j_W : X \to W$ exists, for simplicity we put $i_W(P) = i(j_W(P))$ if $P \in \mathcal{P}$. Thus, from the monotonicity property (i_2) it follows that

$$i(P) \leq i_W(P) \qquad \text{for all } P \in \mathcal{P}. \tag{2.12}$$

We point out that our problem deals with a functional $J : X \to \mathbb{R}$, which is C^1 with respect to $\|\cdot\|_X$, but cannot satisfy the Palais–Smale condition in the same Banach space, as such a norm is "too strong". Hence, the classical multiplicity theorem in [24, Proposition 3.36] cannot be applied and has to be generalized. Here we state an abstract multiplicity theorem by working with the stronger norm $\|\cdot\|_X$, but assuming (wPS), so that Proposition 2.4 holds.

To this aim, for any integer $k \in \mathbb{N}$ let us define

$$c_k := \inf_{P \in \mathcal{P}_k} \sup_{u \in P} J(u), \tag{2.13}$$

with

$$\mathcal{P}_k = \{P \subset X\backslash\{0\} : \ P \text{ symmetric and compact in } X \text{ with } i(P) \geq k\}.$$

Since $\mathcal{P}_{k+1} \subset \mathcal{P}_k$, then

$$c_k \leq c_{k+1}.$$

Furthermore, for any k-dimensional subspace X_0 of X and $\delta > 0$, from (i_8) we have $\partial B^X_\delta(0) \cap X_0 \in \mathcal{P}_k$, while from the continuity of J in $(X, \|\cdot\|_X)$ we have

$$\sup_{u\in\partial B^X_\delta(0)} J(u) \ \to \ J(0) \quad \text{as } \delta \to 0;$$

hence,

$$c_k \ \leq \ J(0) \ = \ 0. \tag{2.14}$$

Theorem 2.7 *Let $J : X \to \mathbb{R}$ be an even functional of class C^1, such that $J(0) = 0$, which satisfies (wPS) in $\mathbb{R}$. If h, $m \in \mathbb{N}$ exist, such that*

$$-\infty < c_h \leq \ \cdots \leq c_{h+m-1} < 0, \tag{2.15}$$

then J has at least m distinct pairs of nontrivial critical points in X with a negative critical level. Furthermore, if

$$-\infty < c_k < 0 \qquad \textit{for all sufficiently large } k, \tag{2.16}$$

then $c_k \nearrow 0$ and J has infinitely many distinct pairs of nontrivial critical points in X.

Proof The proof can be essentially split into three parts.

Step 1. If $k \in \mathbb{N}$ is such that $-\infty < c_k < 0$, then level c_k is critical. In fact, otherwise Proposition 2.4, with $N^W_\varrho(K^J_\beta) = \emptyset$, $\beta = c_k$, yields a contradiction.

Step 2. If (2.15) holds, from *Step 1.* it is enough to prove that, if $k \in \{h, \ldots, h+m-2\}$ and $j \in \mathbb{N}$ exist, such that $\beta = c_k = \cdots = c_{k+j}$, then the critical point set K^J has infinitely many elements.

From Lemma 2.2, $K^J_\beta \in \mathcal{P}$ is compact in $(W, \|\cdot\|_W)$, then the continuity property (i_4) in Proposition 2.5 implies the existence of a radious $\varrho > 0$, such that

$$i_W(N^W_\varrho(K^J_\beta)) \ = \ i_W(K^J_\beta). \tag{2.17}$$

Fixing $\varepsilon_0 > 0$, such that $\beta + \varepsilon_0 < 0$, from Proposition 2.4 for $\varepsilon \in]0, \varepsilon_0[$ small enough an odd homeomorphism $\Psi : X \to X$ exists, such that (i) and (ii) in Proposition 2.4 hold. Thus, from $c_{k+j} < \beta+\varepsilon$ a set $P_\varepsilon \in \mathcal{P}_{k+j}$ exists, such that $P_\varepsilon \subset J^{\beta+\varepsilon}$ and from (i_5) in Proposition 2.5, (2.12) and (2.17) it follows that

$$\begin{aligned} k+j \ &\leq \ i(P_\varepsilon) \leq i(P_\varepsilon\backslash N^W_\varrho(K^J_\beta)) + i(N^W_\varrho(K^J_\beta)) \\ &\leq \ i(\overline{\Psi(P_\varepsilon\backslash N^W_\varrho(K^J_\beta))}) + i_W(N^W_\varrho(K^J_\beta)) \leq \ k-1+i_W(K^J_\beta), \end{aligned}$$

as $\beta - \varepsilon < c_k$ implies that $\overline{\Psi(P_\varepsilon\backslash N^W_\varrho(K^J_\beta))}$ is a compact symmetric subset of $X\backslash\{0\}$ but it is not in $\mathcal{P}_k$. Hence, $i_W(K^J_\beta) \geq j+1 \geq 2$ and then K^J_β has infinitely many elements.

Step 3. Now, let us assume condition (2.16). Since a large enough $k \in \mathbb{N}$ can be fixed so that condition (2.15) holds for all $m \in \mathbb{N}$, then *Step 2.* implies that J has infinitely many distinct pairs of nontrivial critical points in X. So, we have only to prove that the increasing sequence of critical levels $(c_k)_k$ goes to zero. Arguing by contradiction, we assume that $c_k \nearrow \bar{c}$ with $\bar{c} = \sup_{k\in\mathbb{N}} c_k < 0$. By reasoning as in *Step 1.*, it follows that $\bar{c}$ is also a critical level of J in X; hence, from Lemma 2.2 and property (i_4) in Proposition 2.5, a radious $\varrho > 0$ exists, such that (2.17) holds with $\beta = \bar{c}$. Fixing $\varepsilon_0 > 0$, such that $\bar{c} + \varepsilon_0 < 0$, from Proposition 2.4 for $\varepsilon \in]0, \varepsilon_0[$ small enough an odd homeomorphism $\Psi : X \to X$ exists, such that (i) and (ii) in Proposition 2.4 hold. Moreover, a large enough integer k exists, so that

$$\bar{c} - \varepsilon < c_k \leq c_{k+j} \leq \bar{c} < \bar{c} + \varepsilon < 0 \quad \text{for all } j \in \mathbb{N}.$$

Hence, reasoning as in *Step 2.* with $\beta = \bar{c}$, we prove that $i_W(K^J_\beta) \geq j + 1$ for all $j \in \mathbb{N}$, i.e. $i_W(K^J_{\bar{c}}) = +\infty$ in contradiction with Lemma 2.2. □

Remark 2.8 Theorem 2.7 holds also if the assumption of compactness is weakened, i.e., $\mathcal{P}_k$ is the set of symmetric subsets of $X\setminus\{0\}$, which are closed in X with $i(P) \geq k$.

Since all the critical levels defined as in (2.13), by using the cohomological index, are non-positive (see (2.14)), in order to deal with positive levels we have to replace the cohomological index $i(\cdot)$ with the related pseudo-index introduced by Benci in [7]. So, we recall the definition of the pseudo-index and some of its basic properties (here, we consider the pseudo-index when X is equipped with $\|\cdot\|_X$).

Let $\mathcal{P}^*$ denote the class of symmetric subsets of X, let $\mathcal{M} \in \mathcal{P}$ be closed in X (see (2.11)), and define

$$\mathcal{H} = \{\gamma : X \to X \ : \ \gamma \text{ is an odd homeomorphism, such that } \gamma(u) = u \text{ for all } u \in J^0\}.$$

Then, the *pseudo-index of* $P \in \mathcal{P}^*$ related to $i(\cdot)$, $\mathcal{M}$ *and* $\mathcal{H}$ is defined by

$$i^*(P) = \min_{\gamma\in\mathcal{H}} i(\gamma(P) \cap \mathcal{M}). \tag{2.18}$$

Proposition 2.9 *The pseudo-index* $i^* : \mathcal{P}^* \to \mathbb{N} \cup \{0, +\infty\}$ *has the following properties:*

- (i^*_1) *if* $P_1, P_2 \in \mathcal{P}^*$ *are such that* $P_1 \subset P_2$, *then* $i^*(P_1) \leq i^*(P_2)$;
- (i^*_2) *if* $\eta \in \mathcal{H}$ *and* $P \in \mathcal{P}^*$, *then* $i^*(P) = i^*(\eta(P))$;
- (i^*_3) *if* $P \in \mathcal{P}^*$ *and* $B \in \mathcal{P}$ *are closed, then* $i^*(P \cup B) \leq i^*(P) + i(B)$.

As already pointed out, here we want to apply the pseudo-index theory to our setting and we have to generalize the classical statement in [24, Proposition 3.42].

For any integer $k \geq 1$, such that $k \leq i(\mathcal{M})$, let

$$\mathcal{P}^*_k = \{P \subset X : \ P \text{ symmetric and compact in } X \text{ with } i^*(P) \geq k\}$$

and set

$$c^*_k := \inf_{P\in\mathcal{P}^*_k} \sup_{u\in P} J(u).$$

From $\mathcal{P}^*_{k+1} \subset \mathcal{P}^*_k$, it follows that $c^*_k \leq c^*_{k+1}$.

Theorem 2.10 *Let* $J : X \to \mathbb{R}$ *be an even functional of class* C^1 *which satisfies* (wPS) *in* $\mathbb{R}$ *and is such that* $J(0) = 0$. *If* $h, m \in \mathbb{N}$ *exist, such that*

$$0 < c^*_h \leq \cdots \leq c^*_{h+m-1} < +\infty,$$

then J *has at least* m *distinct pairs of nontrivial critical points in* X *with a positive critical level.*

Proof Firstly, we claim that each $\beta = c_k^*$, $k \in \{h, \dots, h+m-1\}$, is a critical level of J in X. Otherwise, fixing $\varepsilon_0 > 0$ such that $\beta - \varepsilon_0 > 0$, from Proposition 2.4 for small enough $\varepsilon < \varepsilon_0$ a map $\Psi \in \mathcal{H}$ exists, such that $\Psi(J^{\beta+\varepsilon}) \subset J^{\beta-\varepsilon}$. On the other hand, from definition $P_\varepsilon^* \in \mathcal{P}_k^*$ exists, such that $P_\varepsilon^* \in J^{\beta+\varepsilon}$. Hence, the properties of Ψ imply that not only $\Psi(P_\varepsilon^*) \in \mathcal{P}_k^*$, but also $\Psi(P_\varepsilon^*) \subset J^{\beta-\varepsilon}$, i.e. $\beta \le \sup J(\Psi(P_\varepsilon^*)) \le \beta - \varepsilon$: a contradiction.

Now, in order to complete the proof, it is sufficient to investigate what happens if $k \in \{h, \dots, h+m-2\}$ and $j \in \mathbb{N}$ exist, such that $\beta = c_k^* = \dots = c_{k+j}^* > 0$. Accordingly to these assumptions, by reasoning as in the proof of Theorem 2.7, $\varrho > 0$ exists, such that (2.17) holds. Then, fixing $\varepsilon_0 > 0$, such that $\beta + \varepsilon_0 > 0$, from Proposition 2.4 for small enough $\varepsilon \in]0, \varepsilon_0[$ a map $\Psi \in \mathcal{H}$ exists, such that (ii) in Proposition 2.4 holds. Thus, from Proposition 2.9, (2.12) and (2.17) it follows that

$$\begin{aligned} i^*(J^{\beta+\varepsilon}) &\le i^*(J^{\beta+\varepsilon} \setminus N_\varrho^W(K_\beta^J)) + i(N_\varrho^W(K_\beta^J)) \\ &\le i^*(\Psi(J^{\beta+\varepsilon} \setminus N_\varrho^W(K_\beta^J))) + i_W(N_\varrho^W(K_\beta^J)) \\ &\le i^*(\overline{\Psi(J^{\beta+\varepsilon} \setminus N_\varrho^W(K_\beta^J))}) + i_W(K_\beta^J), \end{aligned}$$

and, as in Theorem 2.7, it has to be $i_W(K_\beta^J) \ge 2$. □

Remark 2.11 Theorem 2.10 holds also if $\mathcal{P}_k^*$ is the set of the symmetric subsets which are closed in X with $i^*(P) \ge k$.

3 Hypotheses and first properties

From now on, we investigate the existence of weak solutions of the nonlinear problem (GP), where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $N \ge 2$, so the notations introduced for the abstract setting at the beginning of Sect. 2 are referred to our problem with $(X, \|\cdot\|_X)$ the Banach space defined as

$$X := W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \qquad \|u\|_X = \|u\|_W + |u|_\infty, \tag{3.1}$$

with

$$\|u\|_W = \left(\int_\Omega |\nabla u|^p\, dx\right)^{\frac{1}{p}}, \quad |u|_\infty = \operatorname*{ess\,sup}_{x\in\Omega} |u(x)|$$

(here and in the following, $|\cdot|$ will denote the standard norm on any Euclidean space as the dimension of the considered vector is clear and no ambiguity arises), while $(W, \|\cdot\|_W) = (W_0^{1,p}(\Omega), \|\cdot\|_W)$, and $J = \mathcal{J}$ the functional in (1.1). Moreover, we denote by

- $(W^{-1,p'}(\Omega), \|\cdot\|_{W^{-1}})$ the dual space of $(W_0^{1,p}(\Omega), \|\cdot\|_W)$,
- $L^q(\Omega)$ the Lebesgue space equipped with the canonical norm $|\cdot|_q$ for any $1 \le q \le +\infty$,
- $\text{meas}(\cdot)$ the usual Lebesgue measure in $\mathbb{R}^N$,
- $\Omega_r^u = \{x \in \Omega : |u(x)| > r\}$ if $u : \Omega \to \mathbb{R}$, $r > 0$,

and let us recall that, from the Sobolev Imbedding Theorem, $\sigma_p > 0$ exists, such that

$$\int_\Omega |u|^p dx \le \sigma_p \int_\Omega |\nabla u|^p dx \quad \text{for all } u \in W_0^{1,p}(\Omega) \tag{3.2}$$

and the imbedding $W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$ is compact.

From definition, $X \hookrightarrow W_0^{1,p}(\Omega)$ and $X \hookrightarrow L^\infty(\Omega)$ with continuous imbeddings and (2.1) holds with $\sigma_0 = 1$. Moreover, in the stronger assumption $p > N$, we have

$$|u|_\infty \ \le\ \sigma_\infty \, \|u\|_W \qquad \text{for all } u \in W_0^{1,p}(\Omega); \tag{3.3}$$

hence, in this case $X = W_0^{1,p}(\Omega)$ and the two norms $\|\cdot\|_X$ and $\|\cdot\|_W$ are equivalent.

Here and in the following, let us consider problem (GP), where

$$A : (x, t, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N \mapsto A(x, t, \xi) \in \mathbb{R}$$

is a Carathéodory function of class C^1, i.e. measurable with respect to x in Ω for all $(t, \xi) \in \mathbb{R}\times\mathbb{R}^N$ and C^1 with respect to (t, ξ) in $\mathbb{R}\times\mathbb{R}^N$ for a.e. $x \in \Omega$, with $A_t(x, t, \xi) = \frac{\partial A}{\partial t}(x, t, \xi)$, $a(x, t, \xi) = \nabla_\xi A(x, t, \xi) = (\frac{\partial A}{\partial \xi_1}(x, t, \xi), \dots, \frac{\partial A}{\partial \xi_N}(x, t, \xi))$, which satisfies the following conditions:

(H_1) $p > 1$ exists and some positive continuous functions $\Phi_j : \mathbb{R} \to \mathbb{R}$, $j \in \{0, 1\}$, and $\phi_i : \mathbb{R} \to \mathbb{R}$, $i \in \{0, 1, 2\}$, are such that

$$|A(x, t, \xi)| \le \Phi_0(t) + \phi_0(t)\, |\xi|^p, \tag{3.4}$$

$$|a(x, t, \xi)| \le \Phi_1(t) + \phi_1(t)\, |\xi|^{p-1}, \tag{3.5}$$

$$|A_t(x, t, \xi)| \le \phi_2(t)\, |\xi|^p \tag{3.6}$$

for a.e. $x \in \Omega$ and all $(t, \xi) \in \mathbb{R} \times \mathbb{R}^N$;

(H_2) $\alpha_0 > 0$ exists, such that

$$a(x, t, \xi) \cdot \xi \ge \alpha_0 |\xi|^p \qquad \text{a.e. in } \Omega, \ \text{for all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^N;$$

(H_3) $\alpha_1 > 0$ exists, such that

$$A(x, t, \xi) \ \ge\ \alpha_1 |\xi|^p \qquad \text{a.e. in } \Omega, \ \text{for all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^N;$$

(H_4) a (Carathéodory) function

$$A^\infty : (x, \xi) \in \Omega \times \mathbb{R}^N \ \mapsto\ A^\infty(x, \xi) \in \mathbb{R}$$

exists, which is positively p-homogeneous in ξ for a.e. $x \in \Omega$, and satisfies the following condition: for all $\varepsilon > 0$ a constant $r_\varepsilon > 0$ exists, such that

$$|t| \ge r_\varepsilon \implies \left|A(x, t, \xi) - A^\infty(x, \xi)\right| \le \varepsilon |\xi|^p \text{ for a.e. } x \in \Omega, \ \text{all } \xi \in \mathbb{R}^N;$$

(H_5) a (Carathéodory) vector field

$$a^\infty : (x, \xi) \in \Omega \times \mathbb{R}^N \ \mapsto\ a^\infty(x, \xi) = (a_1^\infty(x, \xi), \dots, a_N^\infty(x, \xi)) \in \mathbb{R}^N$$

exists, which satisfies the following condition: for all $\varepsilon > 0$ a constant $r_\varepsilon > 0$ exists, such that

$$|t| \ge r_\varepsilon \implies \left|a(x, t, \xi) - a^\infty(x, \xi)\right| \le \varepsilon |\xi|^{p-1} \text{ for a.e. } x \in \Omega, \ \text{all } \xi \in \mathbb{R}^N;$$

(H_6) for all $\xi, \xi^* \in \mathbb{R}^N$, $\xi \ne \xi^*$,

$$[a(x, t, \xi) - a(x, t, \xi^*)] \cdot [\xi - \xi^*] > 0 \quad \text{a.e. in } \Omega, \ \text{for all } t \in \mathbb{R}.$$

Remark 3.1 Since $a(x, t, \cdot)$ is continuous for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$, hypothesis (H_2) implies that

$$a(x, t, 0) \ =\ 0 \quad \text{for a.e. } x \in \Omega, \ \text{all } t \in \mathbb{R} \tag{3.7}$$

(it is sufficient to fix any $\xi \neq 0$ and apply (H_2) once to $s\xi$, then to $-s\xi$, and in both passing to the limit as $s \to 0^+$).

Moreover, from (3.5), (H_5) and direct computations, it follows that $M_1, M_2 > 0$ exist, such that

$$\left|a^\infty(x,\xi)\right| \le M_1 + M_2|\xi|^{p-1} \quad \text{for a.e. } x \in \Omega, \text{ all } \xi \in \mathbb{R}^N. \tag{3.8}$$

On the other hand, (3.4), (H_3) and (H_4) imply that $\alpha_2 > 0$ exists, such that

$$\alpha_1|\xi|^p \le A^\infty(x,\xi) \le \alpha_2|\xi|^p \quad \text{for a.e. } x \in \Omega, \text{all } \xi \in \mathbb{R}^N. \tag{3.9}$$

If (3.8) holds, we can consider the nonlinear operator associated to $a^\infty(x,\xi)$, namely,

$$A_p^\infty : u \in W_0^{1,p}(\Omega) \mapsto A_p^\infty u \in W^{-1,p'}(\Omega),$$

such that

$$\langle A_p^\infty u, v\rangle = \int_\Omega a^\infty(x,\nabla u)\cdot\nabla v\,dx \qquad \text{for any } u, v \in W_0^{1,p}(\Omega), \tag{3.10}$$

and denote its spectrum by $\sigma(A_p^\infty)$.

By definition, $\lambda \in \sigma(A_p^\infty)$ if some $u \in W_0^{1,p}(\Omega)$, $u \not\equiv 0$, exist, such that

$$\int_\Omega a^\infty(x,\nabla u)\cdot\nabla\varphi\,dx = \lambda\int_\Omega |u|^{p-2}u\varphi\,dx \qquad \text{for all } \varphi \in W_0^{1,p}(\Omega).$$

Lemma 3.2 *Assume that $A(x,t,\xi)$ and its gradient with respect to ξ, namely $a(x,t,\xi)$, satisfy the growth estimate (3.5) and the hypotheses (H_2), (H_4)–(H_6). Then, the nonlinear operator A_p^∞ defined in (3.10) is:*

(i) *continuous from the reflexive Banach space $W_0^{1,p}(\Omega)$ to its dual $W^{-1,p'}(\Omega)$;*
(ii) *a potential operator such as $a^\infty(x,\xi) = \nabla_\xi A^\infty(x,\xi)$ for a.e. $x \in \Omega$, all $\xi \in \mathbb{R}^N$;*
(iii) *$(p-1)$-homogeneous and it is odd if $A^\infty(x,\cdot)$ is even for a.e. $x \in \Omega$;*
(iv) *uniformly positive as*

$$a^\infty(x,\xi)\cdot\xi \ge \alpha_0|\xi|^p \quad \text{a.e. in } \Omega, \text{ for all } \xi \in \mathbb{R}^N;$$

(v) *of type (S): if $(u_n)_n \subset W_0^{1,p}(\Omega)$ and $u \in W_0^{1,p}(\Omega)$ are such that*

$$u_n \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega), \quad \langle A_p^\infty u_n, u_n - u\rangle \to 0,$$

then $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$, up to subsequences.

Proof (i) The proof follows from the growth estimate (3.8) of the Carathéodory vector field $a^\infty(x,\xi)$ and the properties of the related Nemitskii operator.

(ii) Fixing $i \in \{1,\ldots,N\}$, $\xi \in \mathbb{R}^N$, $h \in \mathbb{R}$, for a.e. $x \in \Omega$, we have that

$$A(x,t,\xi+he_i) - A(x,t,\xi) = h\int_0^1 a_i(x,t,\xi+\theta he_i)d\theta,$$

for all $t \in \mathbb{R}$, where (H_4) implies

$$A(x,t,\xi+he_i) - A(x,t,\xi) \to A^\infty(x,\xi+he_i) - A^\infty(x,\xi) \quad \text{as } |t| \to +\infty,$$

while from (H_5) it follows that

$$a_i(x,t,\xi+\theta he_i) \to a_i^\infty(x,\xi+\theta he_i) \quad \text{as } |t| \to +\infty$$

uniformly with respect to $\theta \in [0, 1]$, hence,

$$\int_0^1 a_i(x, t, \xi + \theta h e_i) d\theta \;\to\; \int_0^1 a_i^\infty(x, \xi + \theta h e_i) d\theta \quad \text{as } |t| \to +\infty.$$

On the other hand, as $A^\infty(x, \cdot)$ is p-homogeneous with $p > 1$, then it is differentiable and $\frac{\partial A^\infty}{\partial \xi_i}(x, \xi)$ exists for a.e. $x \in \Omega$, all $\xi \in \mathbb{R}^N$, while the continuity of $a_i^\infty(x, \cdot)$, (3.8) and the Lebesgue's dominated convergence theorem imply

$$\int_0^1 a_i^\infty(x, \xi + \theta h e_i) d\theta \;\to\; a_i^\infty(x, \xi) \quad \text{as } h \to 0.$$

Hence, $\frac{\partial A^\infty}{\partial \xi_i}(x, \xi) = a_i^\infty(x, \xi)$.

(iii) It follows from (ii) and the properties of homogeneous functions.
(iv) It is a direct consequence of (H_2) and (H_5).
(v) From assumption (H_6), it is a direct consequence of [8, Lemma 5] (see also [10, pp. 27]).

□

Remark 3.3 From assumptions (H_1), (H_4), (H_5), the properties of homogeneous functions and direct computations it follows that some constants $M_0, M_1, M_2 > 0$ exist, such that

$$|a^\infty(x, \xi)| \le M_0 |\xi|^{p-1} \quad \text{for a.e. } x \in \Omega, \text{ all } \xi \in \mathbb{R}^N, \tag{3.11}$$

$$|A(x, t, \xi)| \le M_1 + M_2 |\xi|^p \quad \text{for a.e. } x \in \Omega, \text{ all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^N, \tag{3.12}$$

$$|a(x, t, \xi)| \le M_1 + M_2 |\xi|^{p-1} \quad \text{for a.e. } x \in \Omega, \text{ all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^N. \tag{3.13}$$

Moreover, suppose that

$$f : (x, t) \in \Omega \times \mathbb{R} \mapsto f(x, t) \in \mathbb{R}$$

is a Carthéodory function, i.e. measurable with respect to x in Ω for all $t \in \mathbb{R}$ and continuous with respect to t in $\mathbb{R}$ for a.e. $x \in \Omega$, which satisfies the hypotheses:

(h_1) for any $r > 0$ we have

$$\sup_{|t| \le r} |f(\cdot, t)| \in L^\infty(\Omega);$$

(h_2) $\lambda^\infty \in \mathbb{R}$ and a (Carathéodory) function $g^\infty : \Omega \times \mathbb{R} \to \mathbb{R}$ exist, such that

$$f(x, t) \;=\; \lambda^\infty\, |t|^{p-2}\, t + g^\infty(x, t), \tag{3.14}$$

where

$$\lim_{|t| \to +\infty} \frac{g^\infty(x, t)}{|t|^{p-1}} = 0 \quad \text{uniformly a.e. in } \Omega. \tag{3.15}$$

Now and in the following, we set

$$F(x, t) \;=\; \int_0^t f(x, s) ds.$$

Clearly, from the assumptions on $f(x, t)$, it follows that both the functionals associated to f and F are continuous in $L^p(\Omega)$.

Remark 3.4 From (3.14) it follows that

$$F(x,t) \;=\; \frac{\lambda^\infty}{p}\,|t|^p + G^\infty(x,t), \quad \text{with } G^\infty(x,t) = \int_0^t g^\infty(x,s)ds. \tag{3.16}$$

Furthermore, (h_1) and (3.14) imply that

$$\sup_{|t|\le r} |g^\infty(\cdot,t)| \in L^\infty(\Omega) \qquad \text{for any } r>0; \tag{3.17}$$

while (3.17), respectively (3.15), implies that

$$\sup_{|t|\le r} |G^\infty(\cdot,t)| \in L^\infty(\Omega) \qquad \text{for any } r>0, \tag{3.18}$$

$$\lim_{|t|\to+\infty} \frac{G^\infty(x,t)}{|t|^p} = 0 \;\; \text{uniformly a.e. in } \Omega \tag{3.19}$$

and then

$$\lim_{|t|\to+\infty} \frac{F(x,t)}{|t|^p} \;=\; \frac{\lambda^\infty}{p} \quad \text{uniformly a.e. in } \Omega. \tag{3.20}$$

Hence, (3.15) and (3.17), respectively (3.18) and (3.19), imply that for any $\varepsilon>0$ a constant $L_\varepsilon>0$ exists, such that

$$|g^\infty(x,t)| \le \varepsilon|t|^{p-1} + L_\varepsilon \qquad \text{for a.e. } x\in\Omega,\; \text{all } t\in\mathbb{R}, \tag{3.21}$$

$$|G^\infty(x,t)| \le \varepsilon|t|^{p} + L_\varepsilon \qquad \text{for a.e. } x\in\Omega,\; \text{all } t\in\mathbb{R}; \tag{3.22}$$

so (3.14) and (3.21) imply

$$|f(x,t)| \;\le\; \big(|\lambda^\infty|+\varepsilon\big)\,|t|^{p-1} + L_\varepsilon \qquad \text{for a.e. } x\in\Omega,\; \text{all } t\in\mathbb{R}, \tag{3.23}$$

while (3.16) and (3.22) imply

$$|F(x,t)| \;\le\; \left(\frac{|\lambda^\infty|}{p}+\varepsilon\right)|t|^{p} + L_\varepsilon \qquad \text{for a.e. } x\in\Omega,\; \text{all } t\in\mathbb{R}. \tag{3.24}$$

Hence, in particular, suitable constants $D_1, D_2>0$ exist, such that

$$|F(x,t)| \;\le\; D_1|t|^p + D_2 \qquad \text{for a.e. } x\in\Omega, \text{all } t\in\mathbb{R}. \tag{3.25}$$

Firstly, we need to prove that problem (GP) has a variational structure, since, effectively, its weak solutions are critical points of $\mathcal{J}$ in the Banach space X.

The following proposition can be stated (the proof is essentially the same as in [11, Proposition 3.1]).

Proposition 3.5 *Let us assume that $A(x,t,\xi)$ satisfies the growth estimates (3.5), (3.6) and (3.12), while $f(x,t)$ is such that (h_1)–(h_2) hold. If $(u_n)_n\subset X$, $u\in X$ are such that*

$$\|u_n-u\|_W\to 0 \qquad \text{then} \qquad \mathcal{J}(u_n)\to\mathcal{J}(u) \quad \text{as } n\to+\infty.$$

Furthermore, if $r>0$ exists so that

$$|u_n|_\infty \le r \qquad \text{for all } n\in\mathbb{N},$$

then also

$$\|d\mathcal{J}(u_n)-d\mathcal{J}(u)\|_{X'}\to 0 \quad \text{as} \;\; n\to+\infty.$$

In particular, $\mathcal{J}$ is continuous on X equipped with $\|\cdot\|_W$, while C^1 on X equipped with the stronger norm $\|\cdot\|_X$, and its derivative $d\mathcal{J} : X \to X'$ is such that

$$\langle d\mathcal{J}(u), v\rangle \;=\; \int_\Omega (a(x,u,\nabla u)\cdot\nabla v + A_t(x,u,\nabla u)v)dx - \int_\Omega f(x,u)v dx \tag{3.26}$$

for any $u, v \in X$.

In order to apply variational methods to the study of critical points of $\mathcal{J}$ in the asymptotically p-linear case, we introduce the following further conditions:

(H_7) for all $\varepsilon > 0$ a constant $r_\varepsilon > 0$ exists, such that

$$|t| \geq r_\varepsilon \implies |A_t(x,t,\xi)t| \;\leq\; \varepsilon|\xi|^p \quad \text{for a.e. } x \in \Omega, \text{ all } \xi \in \mathbb{R}^N;$$

(H_8) $\alpha_3 > 0$, $\alpha_3 \leq 1$, exists, such that

$$a(x,t,\xi)\cdot\xi + A_t(x,t,\xi)t \geq \alpha_3\, a(x,t,\xi)\cdot\xi \tag{3.27}$$

for a.e. $x \in \Omega$, all $(t,\xi) \in \mathbb{R}\times\mathbb{R}^N$.

Remark 3.6 From hypothesis (H_4) it follows that $r_0 > 0$ exists, such that

$$A(x,t,0) \;=\; 0 \quad \text{for a.e. } x \in \Omega \text{ if } |t| \geq r_0, \tag{3.28}$$

while (3.6) implies

$$A_t(x,t,0) \;=\; 0 \quad \text{for a.e. } x \in \Omega, \text{ all } t \in \mathbb{R}. \tag{3.29}$$

Furthermore, (3.6) and (H_7) imply that $L > 0$ exists, such that

$$\left|A_t(x,t,\xi)\right| \;\leq\; L|\xi|^p \quad \text{for a.e. } x \in \Omega, \text{ all } (t,\xi) \in \mathbb{R}\times\mathbb{R}^N. \tag{3.30}$$

As useful in the following, taking any $r > 0$ we define the truncation function

$$T_r : \mathbb{R} \to \mathbb{R}, \quad \text{such that} \quad T_r t = \begin{cases} t & \text{if } |t| \leq r \\ r\frac{t}{|t|} & \text{if } |t| > r \end{cases} \tag{3.31}$$

and its remainder

$$R_r t \;=\; t - T_r t \;=\; \begin{cases} 0 & \text{if } |t| \leq r \\ t - r\,\frac{t}{|t|} & \text{if } |t| > r \end{cases}. \tag{3.32}$$

Remark 3.7 The properties of T_r and R_r and direct computations imply that not only their Nemitskii operators are continuous from the Lebesgue space $(L^p(\Omega), |\cdot|_p)$ in itself, but also T_r, $R_r : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ are continuous with respect to $\|\cdot\|_W$; hence, T_r, $R_r : X \to X$ are continuous. Furthermore, if $(u_n)_n \subset W_0^{1,p}(\Omega)$, $u \in W_0^{1,p}(\Omega)$ are such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$, then $T_r u_n \rightharpoonup T_r u$ weakly in $W_0^{1,p}(\Omega)$, too.

4 The weak Palais–Smale condition

In hypotheses (3.5), (3.6), (3.12) and (h_1)–(h_2) we can consider the C^1 functional $\mathcal{J}$ in (1.1) on the Banach space X defined in (3.1) (see Proposition 3.5). Taking any $\beta \in \mathbb{R}$, the aim of this section is to exploit the properties of the $(PS)_\beta$-sequences of $\mathcal{J}$ in X, i.e. sequences $(u_n)_n \subset X$, such that

$$\lim_{n\to+\infty} \mathcal{J}(u_n) = \beta \quad \text{and} \quad \lim_{n\to+\infty} \|d\mathcal{J}(u_n)\|_{X'} = 0. \tag{4.1}$$

Proposition 4.1 *Assume that the hypotheses* (H_1)–(H_8), (h_1)–(h_2) *hold and* $\lambda^\infty \notin \sigma(A_p^\infty)$. *Then, for all* $\beta \in \mathbb{R}$, *each* $(PS)_\beta$*-sequence of* $\mathcal{J}$ *in* X *is bounded in the* $W_0^{1,p}$*-norm. Hence, the critical point set* $K_\beta^{\mathcal{J}}$ *is bounded in* $W_0^{1,p}(\Omega)$.

Furthermore, some strictly positive constants $\bar{R}$, $\bar{\varepsilon}$, $\bar{\mu}$ *exist, such that*

$$u \in \mathcal{J}_{\beta-\bar{\varepsilon}}^{\beta+\bar{\varepsilon}}, \quad \|u\|_W \geq \bar{R} \qquad \Longrightarrow \qquad \|d\mathcal{J}(u)\|_{X'} \geq 2\bar{\mu}. \tag{4.2}$$

Proof Let $(u_n)_n \subset X$ be a $(PS)_\beta$-sequence. Arguing by contradiction, we suppose that $\|u_n\|_W \to +\infty$ and define $v_n = \dfrac{u_n}{\|u_n\|_W}$. As $\|v_n\|_W = 1$ for each $n \in \mathbb{N}$, then $v \in W_0^{1,p}(\Omega)$ exists, such that, up to subsequences, we have that $v_n \rightharpoonup v$ weakly in $W_0^{1,p}(\Omega)$, $v_n \to v$ strongly in $L^s(\Omega)$ for each $s \in [1, p^*[$ and $v_n(x) \to v(x)$ a.e. in Ω. Rearranging conveniently the arguments developed in [14, Proposition 3.5] for the model case (1.2), we prove that

1. $v \not\equiv 0$;
2. a constant $b > 0$ exists, such that for any $\mu > 0$ an integer $n_\mu \in \mathbb{N}$ exists, such that

$$\int_{\Omega\setminus\Omega_\mu^n} |\nabla v_n|^p dx \;\leq\; b\; \max\{\mu, \mu^2\} \qquad \text{for all } n \geq n_\mu,$$

   with $\Omega_\mu^n = \{x \in \Omega : |v_n(x)| > \mu\}$;
3. for all $\varepsilon > 0$ an integer $n_\varepsilon \in \mathbb{N}$ exists, such that for all $n \geq n_\varepsilon$ we have that

$$\left|\int_\Omega a^\infty(x, \nabla v_n)\cdot\nabla\varphi\, dx - \lambda^\infty \int_\Omega |v_n|^{p-2} v_n \varphi\, dx\right| \leq \varepsilon\|\varphi\|_X \text{ for all } \varphi \in X;$$

   hence, fixing any $\varphi \in X$ we obtain

$$\int_\Omega a^\infty(x, \nabla v_n)\cdot\nabla\varphi\, dx - \lambda^\infty \int_\Omega |v_n|^{p-2} v_n \varphi\, dx \;\to\; 0 \quad \text{as } n \to +\infty. \tag{4.3}$$

Now, taking any $r > 0$ and the corresponding truncation function T_r in (3.31), we have that $\|T_r v_n - T_r v\|_X \leq 2r + \|v\|_W$ for all $n \in \mathbb{N}$; hence, from the previous *Step 3* with $\varphi = T_r v_n - T_r v$, it follows that

$$\int_\Omega a^\infty(x, \nabla v_n)\cdot\nabla(T_r v_n - T_r v)\, dx - \lambda^\infty \int_\Omega |v_n|^{p-2} v_n (T_r v_n - T_r v)\, dx \;\to\; 0,$$

which implies

$$\int_\Omega a^\infty(x, \nabla v_n)\cdot\nabla(T_r v_n - T_r v)\, dx \;\to\; 0,$$

as $T_r v_n \to T_r v$ strongly in $L^p(\Omega)$. On the other hand, we note that

$$\begin{aligned}\int_\Omega a^\infty(x, \nabla v_n)\cdot\nabla(T_r v_n - T_r v)\, dx \;&= \int_{\Omega\setminus\Omega_r^n} a^\infty(x, \nabla v_n)\cdot\nabla(v_n - T_r v)\, dx \\ &\quad - \int_{\Omega_r^n\setminus\Omega_r^v} a^\infty(x, \nabla v_n)\cdot\nabla v\, dx,\end{aligned}$$

where (3.11), Hölder inequality and $\text{meas}(\Omega_r^n\setminus\Omega_r^v) \to 0$ imply

$$\int_{\Omega_r^n\setminus\Omega_r^v} a^\infty(x, \nabla v_n)\cdot\nabla v\, dx \;\to\; 0;$$

hence,

$$\int_{\Omega\setminus\Omega_r^n} a^\infty(x, \nabla v_n) \cdot \nabla(v_n - T_r v)\, dx \;\to\; 0.$$

Then, as $v_n = T_r v_n$ in $\Omega\setminus\Omega_r^n$, while $a^\infty(x, \nabla(T_r v_n)) = a^\infty(x, 0) = 0$ in a.a. Ω_r^n, we have

$$\int_\Omega a^\infty(x, \nabla(T_r v_n)) \cdot \nabla(T_r v_n - T_r v)\, dx \;\to\; 0 \quad \text{as } n \to +\infty,$$

with $T_r v_n \rightharpoonup T_r v$ weakly in $W_0^{1,p}(\Omega)$; whence, $T_r v_n \to T_r v$ strongly in $W_0^{1,p}(\Omega)$ from Lemma 3.2*(v)*. From the arbitrariness of $r > 0$ we have that $v_n \to v$ strongly in $W_0^{1,p}(\Omega)$ too, and passing to the limit as $n \to +\infty$ in (4.3) we obtain $\lambda^\infty \in \sigma(A_p^\infty)$ in contradiction with the hypotheses.

Finally, if (4.2) does not hold, a $(PS)_\beta$-sequence $(u_n)_n \subset X$ exists, such that $\|u_n\|_W \to +\infty$, in contradiction with the first part of this proof. □

Proposition 4.2 *Assume $p > N$ and that the hypotheses of Proposition 4.1 hold. Then the functional $\mathcal{J}$ satisfies $(PS)_\beta$ in $W_0^{1,p}(\Omega)$ at each level $\beta \in \mathbb{R}$.*

Proof For the proof, it is sufficient to conveniently rearrange the arguments developed in [14, Proposition 3.6] for the model case (1.2) to our general setting. □

Unfortunately, if $p < N$ the same statement cannot hold as Palais–Smale sequences of $\mathcal{J}$ exist which converge in $\|\cdot\|_W$ but not in $\|\cdot\|_X$.

Example 4.3 Suppose that the hypotheses (H_1), (H_4)–(H_5), (H_7), (h_1)–(h_2) are satisfied and, without loss of generality, assume that the closed unit ball of $\mathbb{R}^N$, namely $\overline{B}_1(0) = \{x \in \mathbb{R}^N : |x| \le 1\}$, is contained in Ω. Taking $u \in X$, such that $d\mathcal{J}(u) = 0$, put $\beta = \mathcal{J}(u)$ and consider a smooth function $v \in C_0^\infty(\mathbb{R}^N)$, such that

$$v(x) \ge 0 \text{ in } \mathbb{R}^N, \quad v(x) = 0 \text{ if } |x| \ge 1, \quad v(0) > 0 \quad \text{and} \quad \int_{B_1(0)} |\nabla v|^p dx = 1.$$

If $p < N$, then $\theta > 0$ exists, such that $\frac{N}{p} - 1 - \theta > 0$ and for each $n \in \mathbb{N}$ we define

$$v_n(x) = n^{\frac{N}{p}-1-\theta} v(nx) \quad \text{and} \quad u_n(x) = u(x) + v_n(x).$$

By definition, $v_n \in W_0^{1,p}(\Omega)$ and $v_n(x) = 0$ for each $x \in \Omega\setminus\{0\}$ if n is large enough; hence, $u_n \in W_0^{1,p}(\Omega)$ and $u_n(x) \to u(x)$ and $\nabla u_n(x) \to \nabla u(x)$ for a.e. $x \in \Omega$. Moreover, direct computations imply that

$$\|v_n\|_W = n^{-\theta}; \quad \text{hence}\ ,\ \|u_n - u\|_W \to 0 \text{ as } n \to +\infty.$$

On the other hand, since the functionals associated to f and F are continuous in $L^p(\Omega)$, from (3.12), (3.13) and the Lebesgue's Dominated Convergence Theorem, it follows that $\mathcal{J}(u_n) \to \beta$ and $\|d\mathcal{J}(u_n)\|_{X'} \to 0$, i.e. $(u_n)_n$ is a $(PS)_\beta$-sequence. In any case,

$$|u_n - u|_\infty \;=\; |v_n|_\infty \;\ge\; n^{\frac{N}{p}-1-\theta} v(0) \;\to\; +\infty,$$

so $(u_n)_n$ has no converging subsequence in X.

As already mentioned, the proof of Proposition 4.2 strongly requires the assumption $p > N$, but if $p \leq N$ we can prove that the weaker condition $(wPS)_\beta$ in Definition 2.1 holds. To this aim, firstly we need to find sufficient conditions for the boundedness of a $W_0^{1,p}$-function.

Lemma 4.4 *Let* $1 < p \leq N$ *and take* $u \in W_0^{1,p}(\Omega)$*. If* $b_0 > 0$ *and* $k_0 \in \mathbb{N}$ *exist, such that the inequality*

$$\int_{\Omega_r^u} |\nabla u|^p dx \leq b_0 \left(r^p \operatorname{meas}(\Omega_r^u) + \int_{\Omega_r^u} |u|^p dx \right) \tag{4.4}$$

holds for all $r \geq k_0$*, then* $u \in L^\infty(\Omega)$*, with* $|u|_\infty$ *bounded from above by a positive constant which can be chosen so that it depends only on* $\operatorname{meas}(\Omega)$*,* N*,* p*,* b_0*,* k_0*,* $\|u\|_W$*.*

Proof It is a direct consequence of [19, Theorem 5.1 in Chapter 2] (see [11, Lemma 4.5]). □

Proposition 4.5 *Let* $p > 1$ *and assume that the hypotheses* (H_1)*–*(H_8)*,* (h_1)*–*(h_2) *hold. Then, if* $\lambda^\infty \notin \sigma(A_p^\infty)$*, functional* $\mathcal{J}$ *satisfies* $(wPS)_\beta$ *in* X *at each level* $\beta \in \mathbb{R}$*.*

Proof Fixing $\beta \in \mathbb{R}$, let $(u_n)_n \subset X$ be a $(PS)_\beta$-sequence of $\mathcal{J}$ in X, i.e. (4.1) holds. Then, from Proposition 4.1 a constant $L > 0$ exists, such that

$$\|u_n\|_W \leq L \quad \text{for all } n \in \mathbb{N}. \tag{4.5}$$

Hence, $u \in W_0^{1,p}(\Omega)$ exists, such that, up to subsequences, we have

$$u_n \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega), \tag{4.6}$$

$$u_n \to u \text{ strongly in } L^p(\Omega), \tag{4.7}$$

therefore,

$$u_n \to u \text{ a.e. in } \Omega \text{ and in measure}, \tag{4.8}$$

and a positive function $\nu \in L^p(\Omega)$ exists, such that

$$|u_n(x)| \leq \nu(x) \quad \text{for a.e. } x \in \Omega, \text{ all } n \in \mathbb{N}. \tag{4.9}$$

For simplicity, our proof is divided into several steps:

1. $u \in L^\infty(\Omega)$;
2. fixing $r \geq |u|_\infty + 1$, we have that

$$\int_{\Omega_r^n} |\nabla u_n|^p dx \to 0 \quad \text{as } n \to +\infty, \tag{4.10}$$

   where, in general, we put $\Omega_\mu^n = \{x \in \Omega : |u_n(x)| > \mu\}$ for any $\mu \geq 0$;
3. taking $r \geq \max\{|u|_\infty + 1, r_0\}$, with $r_0 > 0$ as in (3.28), as $n \to +\infty$ we have

$$\mathcal{J}(T_r u_n) \to \beta \quad \text{and} \quad \|d\mathcal{J}(T_r u_n)\|_{X'} \to 0 \tag{4.11}$$

   and

$$T_r u_n \to u \quad \text{strongly in } W_0^{1,p}(\Omega), \tag{4.12}$$

   with the truncation function T_r as in (3.31);
4. $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$ and $\mathcal{J}(u) = \beta$, $d\mathcal{J}(u) = 0$.

For simplicity, here and in the following we use the notation $(\varepsilon_n)_n$ for any infinitesimal sequence depending only on $(u_n)_n$, while $(\varepsilon_{\mu,n})_n$ for any infinitesimal sequence depending not only on $(u_n)_n$, but also on some fixed real number μ.

Step 1. From the Sobolev Imbedding Theorem, the proof is required only if $p \le N$. So, under this assumption, taking $r > 0$, any $\rho > r$ and considering the truncation function T_ρ, as in (3.31), and the remainder function R_r, as in (3.32), define the new sequence of functions $\varphi^n_{r,\rho}(x) = T_\rho(R_r(u_n(x)))$, namely,

$$\varphi^n_{r,\rho}(x) \;=\; \begin{cases} 0 & \text{if } |u_n(x)| \le r, \\ u_n(x) \;-r\; \frac{u_n(x)}{|u_n(x)|} & \text{if } r < |u_n(x)| \le \rho + r, \\ \rho\; \frac{u_n(x)}{|u_n(x)|} & \text{if } |u_n(x)| > \rho + r. \end{cases} \tag{4.13}$$

By definition

$$\nabla\varphi^n_{r,\rho}(x) \;=\; \begin{cases} 0 & \text{a.e. in } \Omega\backslash\Omega^n_{r,\rho}, \\ \nabla u_n(x) & \text{a.e. in } \Omega^n_{r,\rho}, \end{cases} \tag{4.14}$$

with $\Omega^n_{r,\rho} = \{x \in \Omega : r < |u_n(x)| \le \rho + r\}$; hence, from (4.5) and the properties of T_ρ and R_r it follows that $\varphi^n_{r,\rho} \in X$ with

$$\|\varphi^n_{r,\rho}\|_X \;\le\; L + \rho \qquad \text{for all } n \in \mathbb{N}. \tag{4.15}$$

On the one hand, from (4.1) and (4.15) it follows that

$$|\langle d\mathcal{J}(u_n), \varphi^n_{r,\rho}\rangle| \;\le\; \varepsilon_n(L+\rho) \qquad \text{for all } n \in \mathbb{N}. \tag{4.16}$$

On the other hand, from (3.26), (4.13), (4.14), (H_2), (H_8) with $\alpha_3 \le 1$, and direct computations we prove that

$$\begin{aligned} \langle d\mathcal{J}(u_n), \varphi^n_{r,\rho}\rangle &= \int_{\Omega^n_{r,\rho}} \left(1 - \frac{r}{|u_n|}\right) \big(a(x,u_n,\nabla u_n)\cdot\nabla u_n + A_t(x,u_n,\nabla u_n)u_n\big)dx \\ &\quad + \int_{\Omega^n_{r,\rho}} \frac{r}{|u_n|}\, a(x,u_n,\nabla u_n)\cdot\nabla u_n\, dx + \int_{\Omega^n_{\rho+r}} A_t(x,u_n,\nabla u_n)\rho\frac{u_n}{|u_n|}\, dx \\ &\quad - \int_\Omega f(x,u_n)\varphi^n_{r,\rho} dx \\ &\ge \alpha_3 \int_{\Omega^n_{r,\rho}} a(x,u_n,\nabla u_n)\cdot\nabla u_n\, dx + \int_{\Omega^n_{\rho+r}} \frac{\rho}{|u_n|} A_t(x,u_n,\nabla u_n)u_n\, dx \\ &\quad - \int_\Omega f(x,u_n)\varphi^n_{r,\rho} dx \\ &\ge \alpha_0\alpha_3 \int_{\Omega^n_{r,\rho}} |\nabla u_n|^p\, dx - \int_{\Omega^n_{\rho+r}} |A_t(x,u_n,\nabla u_n)u_n|\, dx \\ &\quad - \int_\Omega f(x,u_n)\varphi^n_{r,\rho} dx. \end{aligned}$$

Hence, by summing up these last estimates and (4.16), for all $\rho > r$ and all $n \in \mathbb{N}$, we have that

$$\begin{aligned} \alpha_0\alpha_3 \int_{\Omega^n_{r,\rho}} |\nabla u_n|^p\, dx \le \varepsilon_n(L+\rho) &+ \int_{\Omega^n_{\rho+r}} |A_t(x,u_n,\nabla u_n)u_n|\, dx \\ &+ \int_\Omega f(x,u_n)\varphi^n_{r,\rho} dx. \end{aligned} \tag{4.17}$$

Now, fix any $\varepsilon > 0$ and $r \geq 1$. From (H_7), a constant $\rho_\varepsilon > r$ exists, such that for any $\rho \geq \rho_\varepsilon$ we have

$$|A_t(x, u_n, \nabla u_n)u_n| \leq \frac{\varepsilon}{L^p}|\nabla u_n|^p \quad \text{for a.e. } x \in \Omega^n_{\rho+r}, \text{ all } n \in \mathbb{N};$$

hence, from (4.5) it follows

$$\limsup_{n\to+\infty} \int_{\Omega^n_{\rho+r}} |A_t(x, u_n, \nabla u_n)u_n|\, dx \leq \varepsilon \quad \text{for any } \rho \geq \rho_\varepsilon. \tag{4.18}$$

Furthermore, taking any $\rho \geq \rho_\varepsilon$, from (4.8), (4.9) and the continuity of the involved maps, the Lebesgue's Dominated Convergence Theorem applies and we have

$$\lim_{n\to+\infty} \int_\Omega f(x, u_n)\varphi^n_{r,\rho} dx = \int_\Omega f(x, u)\varphi_{r,\rho} dx, \tag{4.19}$$

where $\varphi_{r,\rho}(x) = T_\rho(R_r(u(x)))$ and from (3.23) and direct computations a constant $b_0 = b_0(\lambda^\infty) > 0$ exists, such that

$$\left|\int_\Omega f(x, u)\varphi_{r,\rho} dx\right| \leq b_0 \int_{\Omega^u_r} |u|^p dx \quad \text{for all } \rho > r. \tag{4.20}$$

Finally, (4.6) and the weak lower semicontinuity of the norm $\|\cdot\|_W$ imply

$$\int_{\Omega_{r,\rho}} |\nabla u|^p\, dx \leq \liminf_{n\to+\infty} \int_{\Omega^n_{r,\rho}} |\nabla u_n|^p\, dx, \tag{4.21}$$

with $\Omega_{r,\rho} = \{x \in \Omega : r < |u(x)| \leq \rho + r\}$. Hence, summing up, as $n \to +\infty$ in (4.17), from (4.18)–(4.21) it results that

$$\alpha_0\alpha_3 \int_{\Omega_{r,\rho}} |\nabla u|^p\, dx \leq \varepsilon + b_0 \int_{\Omega^u_r} |u|^p dx \quad \text{for all } \rho \geq \rho_\varepsilon, \tag{4.22}$$

thus, as ε is arbitrary small, (4.4) holds for all $r \geq 1$. Then, by applying Lemma 4.4, we have that $u \in L^\infty(\Omega)$ and $|u|_\infty$ is smaller than a constant which depends only on meas(Ω), N, p, α_0, α_3, λ^∞ and $\|u\|_W$.

Step 2. Now, let $r \geq |u|_\infty + 1$ in all the formulae of the proof of *Step 1*. With this choice the limit in (4.19) becomes

$$\lim_{n\to+\infty} \int_\Omega f(x, u_n)\varphi^n_{r,\rho} dx = 0 \quad \text{for any } \rho > r,$$

thus for any $\varepsilon > 0$ a constant $\rho_\varepsilon > r$ exists, such that, by passing to the maximum limit as $n \to +\infty$ in (4.17), from (4.18) it follows that

$$\alpha_0\alpha_3 \limsup_{n\to+\infty} \int_{\Omega^n_{r,\rho}} |\nabla u_n|^p\, dx \leq \varepsilon \quad \text{for all } \rho \geq \rho_\varepsilon;$$

hence, (4.10) holds.

Step 3. As $r > |u|_\infty$, $T_r u = u$; hence, from Remark 3.7 and (4.6) we have that

$$T_r u_n \rightharpoonup u \text{ weakly in } W^{1,p}_0(\Omega) \tag{4.23}$$

and also, from (4.7),

$$T_r u_n \to u \text{ and } T_r u_n - u_n \to 0 \text{ strongly in } L^p(\Omega), \tag{4.24}$$
$$\text{meas}(\Omega_r^n) \to 0,$$
$$T_r u_n \to u \text{ a.e. in } \Omega. \tag{4.25}$$

Since $r \geq r_0$, from (3.28) and the definitions (1.1) and (3.31) it follows that

$$\mathcal{J}(T_r u_n) = \mathcal{J}(u_n) - \int_{\Omega_r^n} A(x, u_n, \nabla u_n)dx - \int_{\Omega_r^n} (F(x, T_r u_n) - F(x, u_n))dx,$$

where (3.12), (4.10) and (4.25) imply that

$$\int_{\Omega_r^n} A(x, u_n, \nabla u_n)dx = \varepsilon_{r,n},$$

while from the continuity of the functional associated to F in $L^p(\Omega)$ and (4.24) we have that

$$\int_{\Omega_r^n} (F(x, T_r u_n) - F(x, u_n))dx = \varepsilon_{r,n}.$$

Hence,

$$\mathcal{J}(T_r u_n) = \mathcal{J}(u_n) + \varepsilon_{r,n}$$

and the first limit in (4.11) follows from (4.1).

Now, taking $\varphi \in X$, from (3.7), (3.29) and direct computations it follows that

$$|\langle d\mathcal{J}(T_r u_n), \varphi\rangle| \leq \|d\mathcal{J}(u_n)\|_{X'} \|\varphi\|_X + \int_{\Omega_r^n} |a(x, u_n, \nabla u_n)||\nabla\varphi|dx + \int_{\Omega_r^n} |A_t(x, u_n, \nabla u_n)||\varphi|dx + \int_{\Omega_r^n} |f(x, T_r u_n) - f(x, u_n)||\varphi|dx.$$

From (3.13), (4.10), (4.25), the Hölder inequality and direct computations it follows that

$$\int_{\Omega_r^n} |a(x, u_n, \nabla u_n)||\nabla\varphi|dx \leq \varepsilon_{r,n}\|\varphi\|_W.$$

Furthermore, (3.30) and (4.10) imply

$$\int_{\Omega_r^n} |A_t(x, u_n, \nabla u_n)||\varphi|dx \leq \varepsilon_{r,n}|\varphi|_\infty,$$

while from the continuity of the functional associated to f in $L^p(\Omega)$ and (4.24) it follows that

$$\int_{\Omega_r^n} |f(x, T_r u_n) - f(x, u_n)||\varphi|dx \leq \varepsilon_{r,n}|\varphi|_\infty.$$

Hence,

$$|\langle d\mathcal{J}(T_r u_n), \varphi\rangle| \leq (\|d\mathcal{J}(u_n)\|_{X'} + \varepsilon_{r,n}) \|\varphi\|_X \quad \text{for all } \varphi \in X,$$

and the second limit in (4.11) follows from (4.1).

Finally, (4.12) follows from (4.10), the second limit in (4.11), (4.23), the given set of hypotheses once we repeat the proof of *Step 4* in [11, Proposition 4.6].

Step 4. Since $u_n = T_r u_n + R_r u_n$, with $\|R_r u_n\|_W^p = \int_{\Omega_r^n} |\nabla u_n|^p dx$, then (4.10) and (4.12) imply $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$. Hence, as $|T_r u_n|_\infty \le r$ for all $n \in \mathbb{N}$, Proposition 3.5 implies

$$\mathcal{J}(T_r u_n) \to \mathcal{J}(u), \quad \|d\mathcal{J}(T_r u_n) - d\mathcal{J}(u)\|_{X'} \to 0.$$

Then the end of the proof follows from (4.11). □

Essentially following some of the ideas introduced in the proof of *Step 1* of Proposition 4.5, but replacing the global condition (H_8) with (3.27), only if t is large enough we can prove a boundedness result similar to [3, Lemma 1.4], but with different hypotheses.

Proposition 4.6 *Assume that the hypotheses* (H_2), (H_7), (h_1), (h_2) *hold and that* $\rho_0 \ge 0$ *exists, such that* (3.27) *holds for a.e.* $x \in \Omega$, *all* $\xi \in \mathbb{R}^N$ *if* $|t| \ge \rho_0$. *If* $u \in W_0^{1,p}(\Omega)$ *is such that*

$$\int_\Omega a(x, u, \nabla u) \cdot \nabla \varphi dx + \int_\Omega A_t(x, u, \nabla u)\varphi dx - \int_\Omega f(x, u)\varphi dx = 0 \tag{4.26}$$

for all $\varphi \in X$, *then a positive constant* $r_u > 0$ *exists, which depends only on* meas(Ω), N, p, α_0, α_3, λ^∞ *and* $\|u\|_W$, *such that* $|u|_\infty \le r_u$. *Hence,* $u \in X$.

Proof From the Sobolev Imbedding Theorem, the proof is required only if $p \le N$. So, by fixing $r \ge \rho_0 + 1$, taking any $\rho > r$, using $\varphi_{r,\rho}(x) = T_\rho(R_r(u(x)))$ as the test function in (4.26), and reasoning as in the proof of *Step 1* in Proposition 4.5, we obtain

$$\alpha_0\alpha_3 \int_{\Omega_{r,\rho}} |\nabla u|^p \, dx \;\le\; \int_{\Omega^u_{\rho+r}} |A_t(x, u, \nabla u)u| \, dx + \int_\Omega f(x, u)\varphi_{r,\rho} dx,$$

with $\Omega_{r,\rho} = \{x \in \Omega : r < |u(x)| \le \rho + r\}$, as

$$a(x, u, \nabla u) \cdot \nabla u + A_t(x, u, \nabla u)u \ge a(x, u, \nabla u) \cdot \nabla u \quad \text{a.e. in } \Omega_{r,\rho}.$$

Thus, (4.22) is satisfied, since (4.20) still holds while from (H_7) for all $\varepsilon > 0$, a constant $\rho_\varepsilon > r$ exists, such that

$$\int_{\Omega^u_{\rho+r}} |A_t(x, u, \nabla u)u| \, dx \;\le\; \varepsilon \quad \text{for any } \rho \ge \rho_\varepsilon.$$

Therefore, (4.4) follows and Lemma 4.4 applies. □

Corollary 4.7 *In the hypotheses of Proposition* 4.1, *if* $\beta \in \mathbb{R}$ *is such that* $K_\beta^{\mathcal{J}} \ne \emptyset$, *then a constant* $r_\beta > 0$ *exists, such that* $|u|_\infty \le r_\beta$ *for all* $u \in K_\beta^{\mathcal{J}}$. *Hence, the critical point set* $K_\beta^{\mathcal{J}}$ *is compact with respect to the* $W_0^{1,p}$*-norm, while it is bounded with respect to the* L^∞*-norm.*

Proof If $p > N$ the statement is a direct consequence of Proposition 4.1 and (3.3). On the other hand, if $p \le N$, from Proposition 4.6 each $u \in K_\beta^{\mathcal{J}}$ is bounded by a constant which depends on $\|u\|_W$, while from Proposition 4.1 it follows that $K_\beta^{\mathcal{J}}$ is bounded with respect to $\|\cdot\|_W$. □

5 Main results

In addition to the hypotheses (H_1)–(H_8), (h_1)–(h_2), we assume that:

(H_9) $A^0(x, \xi)$ is positively p-homogeneous in ξ for a.e. $x \in \Omega$, with

$$A^0(x, \xi) = A(x, 0, \xi);$$

(H_{10}) $A(x, t, \xi)$ is even in (t, ξ) for a.e. $x \in \Omega$;
(h_3) $\lambda^0 \in \mathbb{R}$ and a (Carathéodory) function $g^0 : \Omega \times \mathbb{R} \to \mathbb{R}$ exist, such that

$$f(x, t) \ = \ \lambda^0 \, |t|^{p-2} \, t + g^0(x, t),$$

with

$$\lim_{t \to 0} \frac{g^0(x, t)}{|t|^{p-1}} \ = \ 0 \quad \text{uniformly a.e. in } \Omega;$$

(h_4) $f(x, \cdot)$ is odd for a.e. $x \in \Omega$.

Remark 5.1 Conditions (3.12), (H_3) and (H_9) imply that $\alpha_4 > 0$ exists, such that

$$\alpha_1 |\xi|^p \ \le \ A^0(x, \xi) \ \le \ \alpha_4 |\xi|^p \qquad \text{for a.e. } x \in \Omega, \ \text{all } \xi \in \mathbb{R}^N. \tag{5.1}$$

Moreover, from (3.6) it follows that

$$|A(x, t, \xi) - A^0(x, \xi)| \le \Big| \int_0^t \phi_2(s) ds \Big| \, |\xi|^p \quad \text{for a.e. } x \in \Omega, \ \text{all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^N;$$

hence, for all $\varepsilon > 0$ a constant $r_\varepsilon > 0$ exists, such that

$$|t| \le r_\varepsilon \implies \big| A(x, t, \xi) - A^0(x, \xi) \big| \le \varepsilon |\xi|^p \quad \text{for a.e. } x \in \Omega, \ \text{all } \xi \in \mathbb{R}^N. \tag{5.2}$$

Remark 5.2 From (h_3) it follows that

$$F(x, t) \ = \ \frac{\lambda^0}{p} \, |t|^p + G^0(x, t), \quad \text{with } G^0(x, t) = \int_\Omega g^0(x, s) ds, \tag{5.3}$$

and

$$\lim_{t \to 0} \frac{G^0(x, t)}{|t|^p} \ = \ 0 \text{ uniformly a.e. in } \Omega; \tag{5.4}$$

thus,

$$\lim_{t \to 0} \frac{F(x, t)}{|t|^p} \ = \ \frac{\lambda^0}{p} \quad \text{uniformly a.e. in } \Omega. \tag{5.5}$$

Furthermore, from (h_1), (3.20), (5.5) and direct computations it follows that $D > 0$ exists, such that

$$|F(x, t)| \ \le \ D |t|^p \quad \text{for a.e. } x \in \Omega, \ \text{all } t \in \mathbb{R}. \tag{5.6}$$

Defining $a^0(x, \xi) = a(x, 0, \xi)$, as in (3.10), we introduce the operator

$$A^0_p : u \in W^{1,p}_0(\Omega) \mapsto A^0_p u \in W^{-1,p'}(\Omega)$$

as

$$\langle A^0_p u, v \rangle \ = \ \int_\Omega a^0(x, \nabla u) \cdot \nabla v \, dx \qquad \text{for any } u, v \in W^{1,p}_0(\Omega), \tag{5.7}$$

and denote its spectrum by $\sigma(A^0_p)$.

Remark 5.3 If the hypotheses (3.5), (H_2), (H_6), (H_9)–(H_{10}) hold, from direct computations and [8, Lemma 5] the nonlinear operator A^0_p in (5.7) has the following properties:

(i) it is continuous from the reflexive Banach space $W^{1,p}_0(\Omega)$ to its dual $W^{-1,p'}(\Omega)$,
(ii) it admits a potential operator, as it is $a^0(x,\xi) = \nabla_\xi A^0(x,\xi)$ for a.e. $x \in \Omega$;
(iii) by assumption, it is $(p-1)$-homogeneous and odd;
(iv) it is uniformly positive, as $a^0(x,\xi)\cdot\xi \;\geq\; \alpha_0|\xi|^p$ a.e. in Ω, for all $\xi \in \mathbb{R}^N$;
(v) it is of type (S): if $(u_n)_n \subset X$ and $u \in X$ are such that

$$u_n \rightharpoonup u \text{ weakly in } W^{1,p}_0(\Omega) \quad \text{and} \quad \langle A^0_p u_n, u_n - u\rangle \to 0,$$

then $u_n \to u$ strongly in $W^{1,p}_0(\Omega)$, up to subsequences.

For each $j \geq 1$ let us define:

$$\lambda^0_j = \inf_{Q\in\mathcal{S}_j}\max_{u\in Q}\frac{\displaystyle\int_\Omega A^0(x,\nabla u)dx}{\displaystyle\frac{1}{p}\int_\Omega |u|^p dx}, \tag{5.8}$$

$$\lambda^\infty_j = \inf_{Q\in\mathcal{S}_j}\max_{u\in Q}\frac{\displaystyle\int_\Omega A^\infty(x,\nabla u)dx}{\displaystyle\frac{1}{p}\int_\Omega |u|^p dx}, \tag{5.9}$$

with

$$\mathcal{S}_j = \{Q \subset \mathcal{S}^W :\; Q \text{ symmetric and compact in } W^{1,p}_0(\Omega),\; \text{with } i_W(Q) \geq j\},$$

where

$$\mathcal{S}^W = \{u \in W^{1,p}_0(\Omega) : \|u\|_W = 1\}$$

and $i_W(\cdot)$ is the cohomological index defined on the Banach space $W^{1,p}_0(\Omega)$.

As direct consequence of Lemma 3.2 and Remark 5.3 we have

$$\langle A^\infty_p u, u\rangle \;=\; pA^\infty(x,\nabla u) \text{ and } \langle A^0_p u, u\rangle \;=\; pA^0(x,\nabla u) \quad \text{if } u \in W^{1,p}_0(\Omega).$$

Moreover, [24, Theorem 4.6] applies and the following proposition can be pointed out.

Proposition 5.4 *Assume that the hypotheses* (3.5)*,* (H_2)*,* (H_4)–(H_6) *and* (H_9)–(H_{10}) *hold. Then, taking* $\natural = 0, \infty$*, we have that* $(\lambda^\natural_j)_j$ *is a nondecreasing sequence of eigenvalues of the nonlinear operator* $A^\natural_p$*, such that* $\lambda^\natural_j \nearrow +\infty$*, as* $j \to +\infty$ *and the smallest eigenvalue, called the first eigenvalue, is*

$$\lambda^\natural_1 = \min_{u\in W^{1,p}_0(\Omega)\setminus\{0\}}\frac{\displaystyle\int_\Omega A^\natural(x,\nabla u)dx}{\displaystyle\frac{1}{p}\int_\Omega |u|^p dx} \;>\; 0.$$

Remark 5.5 Since both if $\natural = \infty$ and if $\natural = 0$ the function $A^\natural(x,\xi)$ is positively p-homogeneous in ξ (from (H_4), respectively (H_9)), the eigenvalues $(\lambda^\natural_j)_j$ can be characterized as

$$\lambda_j^{\natural} = \inf_{Q\in\mathcal{W}_j} \max_{u\in Q} \frac{\int_\Omega A^{\natural}(x,\nabla u)dx}{\frac{1}{p}\int_\Omega |u|^p dx}, \tag{5.10}$$

where

$$\mathcal{W}_j = \{Q \subset W_0^{1,p}(\Omega)\setminus\{0\} : Q \text{ symmetric and compact in } W_0^{1,p}(\Omega), \\ \text{with } i_W(Q) \geq j\}.$$

In fact, taking $j \in \mathbb{N}$, firstly we note that $\mathcal{S}_j \subset \mathcal{W}_j$ implies

$$\inf_{Q\in\mathcal{W}_j} \max_{u\in Q} \frac{\int_\Omega A^{\natural}(x,\nabla u)dx}{\frac{1}{p}\int_\Omega |u|^p dx} \leq \lambda_j^{\natural}.$$

On the contrary, since the radial projection

$$\pi : u \in W_0^{1,p}(\Omega)\setminus\{0\} \mapsto \pi(u) = \frac{u}{\|u\|_W} \in S^W$$

is odd and continuous, property (i_2) in Proposition 2.5 implies that if $Q \in \mathcal{W}_j$, then $\pi(Q) \in \mathcal{S}_j$, with

$$\max_{u\in Q} \frac{\int_\Omega A^{\natural}(x,\nabla\pi(u))dx}{\frac{1}{p}\int_\Omega |\pi(u)|^p dx} = \max_{u\in Q} \frac{\int_\Omega A^{\natural}(x,\nabla u)dx}{\frac{1}{p}\int_\Omega |u|^p dx};$$

hence, (5.10) holds.

Now, we are able to state our main results.

Theorem 5.6 *Assume that* (H_1)–(H_{10}) *and* (h_1)–(h_4) *hold for some* $p > 1$. *If* $\lambda^\infty \notin \sigma(A_p^\infty)$ *and* $k, h \in \mathbb{N}$ *exist, such that*

$$\lambda_k^0 < \lambda^0, \qquad \lambda^\infty < \lambda_h^\infty, \qquad \text{with } k > h-1, \tag{5.11}$$

then problem (GP) *has at least* $k-h+1$ *distinct pairs of nontrivial bounded solutions with strictly negative critical levels.*

Theorem 5.7 *Assume that* (H_1)–(H_{10}) *and* (h_1)–(h_4) *hold for some* $p > 1$. *If* $\lambda^\infty \notin \sigma(A_p^\infty)$ *and* $k, h \in \mathbb{N}$ *exist, such that*

$$\lambda^0 < \lambda_k^0, \qquad \lambda_h^\infty < \lambda^\infty, \qquad \text{with } k < h+1, \tag{5.12}$$

then problem (GP) *has at least* $h-k+1$ *distinct pairs of nontrivial bounded solutions with strictly positive critical levels.*

In the same hypotheses of the previous Theorem 5.6, but replacing conditions (h_3) and (5.11) with F super-p-linear at zero, we are able to prove that (GP) has infinitely many solutions.

Theorem 5.8 *Assume that* (H_1)–(H_{10}), (h_1)–(h_2) *and* (h_4) *hold for some* $p > 1$. *If* $\lambda^\infty \notin \sigma(A_p^\infty)$ *and*

$$\lim_{t\to 0} \frac{F(x,t)}{|t|^p} = +\infty \quad \text{uniformly a.e. in } \Omega, \tag{5.13}$$

then problem (GP) *has an infinite number of distinct pairs of nontrivial bounded solutions* $(u_k)_k \subset X$, *with negative critical levels, such that* $\mathcal{J}(u_k) \nearrow 0$.

Before going on with the proofs of our main theorems, we note that in the definitions of the eigenvalues $(\lambda_j^0)_j$ and $(\lambda_j^\infty)_j$ (see (5.8), respectively (5.9)) or in their characterization (5.10), only the space $W_0^{1,p}(\Omega)$ is involved. In any case, our natural setting is X, so we need to characterize them accordingly.

Proposition 5.9 *Assume that* (3.5), (H_2)–(H_6) *and* (H_9)–(H_{10}) *hold. Taking* $\natural = 0, \infty$, *for each* $j \geq 1$ *we define*

$$\tilde{\lambda}_j^\natural = \inf_{Q\in\tilde{\mathcal{W}}_j} \max_{u\in Q} \frac{\displaystyle\int_\Omega A^\natural(x,\nabla u)dx}{\displaystyle\frac{1}{p}\int_\Omega |u|^p dx},$$

with

$$\tilde{\mathcal{W}}_j = \{Q \in \mathcal{W}_j : \exists\, V \text{ subspace of } X, \dim V < +\infty, \text{ such that } Q \subset V\}.$$

Then,

$$\tilde{\lambda}_j^\natural = \lambda_j^\natural.$$

Proof Taking any $j \geq 1$, by definition it is $\tilde{\mathcal{W}}_j \subset \mathcal{W}_j$, then (5.10) implies $\lambda_j^\natural \leq \tilde{\lambda}_j^\natural$. Now, we have just to prove that

$$\tilde{\lambda}_j^\natural \leq \lambda_j^\natural. \tag{5.14}$$

To this aim, fixing $Q \in \mathcal{S}_j$ and any $\varepsilon \in]0,1[$, we split our proof into two steps:

1. $Q_\varepsilon \in \mathcal{W}_j$ exists, such that Q_ε is a bounded subset of X and

$$\max_{v\in Q_\varepsilon} \frac{\displaystyle\int_\Omega A^\natural(x,\nabla v)dx}{\displaystyle\frac{1}{p}\int_\Omega |v|^p dx} \leq \frac{1}{1-\varepsilon} \max_{u\in Q} \frac{\displaystyle\int_\Omega A^\natural(x,\nabla u)dx}{\displaystyle\frac{1}{p}\int_\Omega |u|^p dx}, \tag{5.15}$$

2. $\tilde{Q}_\varepsilon \in \tilde{\mathcal{W}}_j$ exists, such that

$$\max_{w\in \tilde{Q}_\varepsilon} \frac{\displaystyle\int_\Omega A^\natural(x,\nabla w)dx}{\displaystyle\frac{1}{p}\int_\Omega |w|^p dx} \leq \varepsilon \max_{v\in Q_\varepsilon} \frac{\displaystyle\int_\Omega A^\natural(x,\nabla v)dx}{\displaystyle\frac{1}{p}\int_\Omega |v|^p dx}. \tag{5.16}$$

Step 1. Firstly we note that, by definition, Q is a subset of $W_0^{1,p}(\Omega)$ contained in S^W, so from the Sobolev Imbedding Theorem, Q is compact in $L^p(\Omega)$, with $0 \notin Q$; hence, not only the minimum of the L^p-norm in Q is strictly positive, i.e. $b > 0$ exists, such that

$$\frac{1}{p}\int_\Omega |u|^p dx \geq b \quad \text{for all } u \in Q, \tag{5.17}$$

but also from (3.2) and (5.17) a constant $\delta_{1,\varepsilon} > 0$ exists, such that for any measurable subset $E \subset \Omega$ with $\text{meas}(E) < \delta_{1,\varepsilon}$ we have

$$\int_E |u|^p dx \;\le\; \varepsilon |u|_p^p \quad \text{for all } u \in Q. \tag{5.18}$$

On the other hand, from (3.2) we have $|u|_p^p \le \sigma_p$ for all $u \in S^W$; hence, $r_\varepsilon = r_\varepsilon(\delta_{1,\varepsilon}) > 0$ exists, so that

$$\text{meas}(\Omega^u_{r_\varepsilon}) < \delta_{1,\varepsilon} \quad \text{for any } u \in S^W. \tag{5.19}$$

Thus, from (5.18) and (5.19) it follows that

$$\int_{\Omega^u_{r_\varepsilon}} |u|^p dx \;\le\; \varepsilon |u|_p^p \quad \text{for all } u \in Q,$$

which implies that

$$\int_\Omega |T_{r_\varepsilon} u|^p dx \;\ge\; (1-\varepsilon) \int_\Omega |u|^p dx \quad \text{for all } u \in Q, \tag{5.20}$$

with T_{r_ε} as in (3.31).

Now, defining $Q_\varepsilon = T_{r_\varepsilon}(Q)$, by construction it is not only $Q_\varepsilon \subset L^\infty(\Omega)$, but also

$$|v|_\infty \le r_\varepsilon \quad \text{for all } v \in Q_\varepsilon,$$

while the properties of T_{r_ε} and property (i_2) in Proposition 2.5 also imply that $Q_\varepsilon \in \mathcal{W}_j$.

On the other hand, since

$$\int_\Omega A^\natural(x, \nabla(T_{r_\varepsilon} u))dx \;\le\; \int_\Omega A^\natural(x, \nabla u)dx,$$

from (5.20) we have that

$$\frac{\displaystyle\int_\Omega A^\natural(x, \nabla(T_{r_\varepsilon} u))dx}{\displaystyle\frac{1}{p}\int_\Omega |T_{r_\varepsilon} u|^p dx} \;\le\; \frac{1}{1-\varepsilon}\, \frac{\displaystyle\int_\Omega A^\natural(x, \nabla u)dx}{\displaystyle\frac{1}{p}\int_\Omega |u|^p dx} \qquad \text{for all } u \in Q,$$

which implies (5.15).

Step 2. From *Step 1.* two constants L_1, L_2 exist, such that

$$0 < L_1 \le \|v\|_W \le L_2 \quad \text{for all } v \in Q_\varepsilon. \tag{5.21}$$

Moreover, since the map

$$w \;\mapsto\; \frac{\displaystyle\int_\Omega A^\natural(x, \nabla w)dx}{\displaystyle\frac{1}{p}\int_\Omega |w|^p dx}$$

is continuous on $W_0^{1,p}(\Omega)\setminus\{0\}$ and Q_ε is compact in the same space, $\delta_{2,\varepsilon} > 0$ exists (without loss of generality, $\delta_{2,\varepsilon} < L_1$), such that for all $v \in Q_\varepsilon$, $w \in W_0^{1,p}(\Omega)\setminus\{0\}$, we have

$$\left| \frac{\displaystyle\int_\Omega A^\natural(x, \nabla v)dx}{\displaystyle\frac{1}{p}\int_\Omega |v|^p dx} - \frac{\displaystyle\int_\Omega A^\natural(x, \nabla w)dx}{\displaystyle\frac{1}{p}\int_\Omega |w|^p dx} \right| < \varepsilon \quad \text{if } \|v - w\|_W < \delta_{2,\varepsilon}. \tag{5.22}$$

On the other hand, again the compactness of Q_ε in $W_0^{1,p}(\Omega)$ and $Q_\varepsilon \subset X$ imply that $v_1, \dots, v_l \in Q_\varepsilon$ exist, such that

$$Q_\varepsilon \subset \bigcup_{i=1}^{l} B^W_{\delta_{2,\varepsilon}}(v_i). \tag{5.23}$$

Let V be the finite dimensional subspace of X generated by $\{v_1, \dots, v_l\}$. From (5.23) it follows that

$$d_W(v, V) < \delta_{2,\varepsilon} < L_1 \quad \text{for all } v \in Q_\varepsilon. \tag{5.24}$$

We claim that for each $v \in Q_\varepsilon$, one and only one $w(v) \in V\backslash\{0\}$ exists, such that $\|v - w(v)\|_W = d_W(v, V)$.

In fact, from (5.24) we can take a sequence $(w_n)_n \subset V$, such that

$$\|v - w_n\|_W \to d_W(v, V) \quad \text{and} \quad \|v - w_n\|_W \le \delta_{2,\varepsilon} \quad \text{for each } n.$$

Hence, (5.21) implies that $(w_n)_n \subset \{w \in V : \|w\|_W \le \delta_{2,\varepsilon} + L_2\}$ which is compact in V, thus $\bar{w} \in V$ exists, such that

$$\|w_n - \bar{w}\|_W \to 0 \text{ (up to subsequences) and } \|v - \bar{w}\|_W = d_W(v, V).$$

The uniqueness follows from the strong convexity of the space $W_0^{1,p}(\Omega)$. Furthermore, from (5.21) and (5.24), $\|w(v)\|_W \ge L_1 - \delta_{2,\varepsilon} > 0$.

Then, the map

$$\varphi_\varepsilon : v \in Q_\varepsilon \mapsto w(v) \in V\backslash\{0\}$$

is well defined, odd and continuous with respect to $\|\cdot\|_W$. Hence, $\tilde{Q}_\varepsilon = \varphi_\varepsilon(Q_\varepsilon)$ is symmetric and compact both in $W_0^{1,p}(\Omega)$ and in X, with

$$j \le i_W(Q_\varepsilon) \le i_W(\tilde{Q}_\varepsilon)$$

from the monotonicity of the index; thus, $\tilde{Q}_\varepsilon \in \mathcal{W}_j$ and, by construction, $\tilde{Q}_\varepsilon \subset V$. Moreover, (5.22) implies (5.16).

Finally, (5.14) is a direct consequence of the previous steps for (5.8) and (5.9), respectively, and the arbitrariness of ε and $Q \in \mathcal{S}_j$. □

6 Looking for negative critical levels

Throughout this section, we assume that (H_1)–(H_{10}), (h_1)–(h_2) and (h_4) hold. Moreover, suppose $\mathcal{J}(0) = 0$ (true from (H_9) if either (h_3) or (5.13) is satisfied).

Remark 6.1 If $\lambda^\infty \le 0$, from (H_3), (3.2), (3.16) and (3.22) with ε small enough, it follows that

$$\inf \mathcal{J}(X) > -\infty.$$

Since from (H_{10}) and (h_4) it follows that $\mathcal{J}$ is an even functional in X, we can use the cohomological index theory and its related pseudo-index, as stated in Sect. 2. To this aim, for all $j \in \mathbb{N}$, we define

$$\mathcal{P}_j = \{P \subset X\backslash\{0\} : P \text{ symmetric and compact in } X \text{ with } i(P) \ge j\},$$

where $i(\cdot)$ is the cohomological index on $(X, \|\cdot\|_X)$, and, as in (2.13), but with X as in (3.1), $W = W_0^{1,p}(\Omega)$ and $J = \mathcal{J}$, we take

$$c_j = \inf_{P \in \mathcal{P}_j} \max_{u \in P} \mathcal{J}(u). \tag{6.1}$$

Remark 6.2 Taking any $j \in \mathbb{N}$, firstly let us point out that, if $\tilde{\mathcal{W}}_j$ is as in Proposition 5.9, then

$$Q \in \tilde{\mathcal{W}}_j \Longrightarrow Q \in \mathcal{P}_j. \tag{6.2}$$

In fact, if $Q \in \tilde{\mathcal{W}}_j$, a finite dimensional subspace V of X exists, such that $Q \subset V$; hence, $Q \subset X\backslash\{0\}$ is symmetric and compact in X, with $i(Q) = i_W(Q) \geq j$.

On the other hand, either if $\natural = 0$ or if $\natural = \infty$, from (2.12) it follows that, if $P \in \mathcal{P}_j$, then $P \in \mathcal{W}_j$, hence (5.10) implies that

$$\max_{u \in P} \frac{\int_\Omega A^\natural(x, \nabla u)dx}{\frac{1}{p}\int_\Omega |u|^p dx} \geq \lambda_j^\natural \qquad \text{for any } P \in \mathcal{P}_j. \tag{6.3}$$

Finally, denoting

$$\partial B_1^\natural = \{u \in X : \int_\Omega A^\natural(x, \nabla u)dx = 1\}, \tag{6.4}$$

we have that the projection

$$\pi^\natural : u \in X\backslash\{0\} \mapsto \pi^\natural(u) = \frac{u}{\left(\int_\Omega A^\natural(x, \nabla u)dx\right)^{\frac{1}{p}}} \in \partial B_1^\natural \tag{6.5}$$

is odd and continuous; then, from property (i_2) in Proposition 2.5, it follows that

$$P \in \mathcal{P}_j \Longrightarrow \pi^\natural(P) \in \mathcal{P}_j. \tag{6.6}$$

In order to prove Theorems 5.6 and 5.8 by applying Theorem 2.7, firstly we have to show that $c_h > -\infty$ for some $h \in N$ (here c_h is as in (6.1)). To this aim, we note that, fixing any $\tau > 0$ and $P \in \mathcal{P}_h$, from (3.2) and (3.24), direct computations imply the existence of a constant $c(\tau) \in \mathbb{R}$, so that, if $u_0 \in P$ exists with $\|u_0\|_W \leq \tau$, then $\max \mathcal{J}(P) \geq c(\tau)$. Thus, the existence of a uniform lower bound has to be proved only for the maxima of $\mathcal{J}$ on sets P, such that $\|u\|_W > \tau$ for all $u \in P$. In order to achieve this, it is quite natural to approximate $A(x, t, \xi)$ with $A^\infty(x, \xi)$, but such a replacement is allowed only if $|t| > \bar{r}$ with $\bar{r}$ large enough, while for every $u \in W_0^{1,p}(\Omega)$ set $\{x \in \Omega : |u(x)| \leq r\}$ is nontrivial for all $r > 0$. Hence, we have to split Ω into two parts: the set in which $R_r(\pi^\infty(u(x))) \neq 0$ (with the remainder R_r as in (3.32)) and its complementary set. Obviously, taking $\bar{r}$ and a suitable $r > 0$, a related τ can be fixed, so that $\|u\|_W > \tau$ and $|\pi^\infty(u(x))| > r$ imply $|u(x)| > \bar{r}$; hence, the approximating scheme can be used.

More precisely, even if both assumption (h_3) and (5.13) do not hold, the following statement can be proved.

Proposition 6.3 *If $h \in \mathbb{N}$ is such that $\lambda^\infty < \lambda_h^\infty$, then*

$$\inf_{P \in \mathcal{P}_h} \max_{u \in P} \mathcal{J}(u) > -\infty. \tag{6.7}$$

Proof If $\lambda^\infty \leq 0$, then (6.7) follows from Remark 6.1.

Now, we assume that $\lambda^\infty > 0$. As $0 < \lambda^\infty < \lambda_h^\infty$, fix $\varepsilon > 0$, such that

$$\varepsilon < \min\left\{1, \left(\frac{\alpha_1}{\alpha_2 D_1 \operatorname{meas}(\Omega)}\right)^{\frac{1}{p}}\right\}, \tag{6.8}$$

$$\lambda^\infty + \varepsilon p < (1-\varepsilon)\,\lambda_h^\infty \left(1 - \varepsilon p \left(\frac{\alpha_2 D_1 \operatorname{meas}(\Omega)}{\alpha_1}\right)^{\frac{1}{p}}\right), \tag{6.9}$$

with α_1 as in (H_3), α_2 as in (3.9) and D_1 as in (3.25).

Taking $P \in \mathcal{P}_h$, two cases may occur:

(i) $u_0 \in P$ exists, such that $|\pi^\infty(u_0)|_p \leq \left(\frac{\alpha_1}{\alpha_2 D_1}\right)^{\frac{1}{p}}$,

(ii) for all $u \in P$: $|\pi^\infty(u)|_p > \left(\frac{\alpha_1}{\alpha_2 D_1}\right)^{\frac{1}{p}}$,

where π^∞ is as in (6.5).

Case (i) From (H_3), (3.9) and (3.25) it follows that

$$\begin{aligned}\max_{u\in P} \mathcal{J}(u) \geq\ & \mathcal{J}(u_0)\\ \geq\ & \left(\frac{\alpha_1}{\alpha_2} - D_1|\pi^\infty(u_0)|_p^p\right)\int_\Omega A^\infty(x, \nabla u)dx\ - D_2\text{meas}(\Omega)\\ \geq\ & -D_2\text{meas}(\Omega).\end{aligned}$$

Case (ii) In this case, (6.8) and direct computations imply that

$$|v|_\infty\ >\ \varepsilon \qquad \text{for all } v \in \pi^\infty(P), \tag{6.10}$$

while, from (3.2) and (3.9) it follows that

$$\int_\Omega |v|^p dx\ \leq\ \frac{\sigma_p}{\alpha_1} \qquad \text{for all } v \in \pi^\infty(P). \tag{6.11}$$

Now, since from (6.6) we have $\pi^\infty(P) \in \mathcal{P}_h$, from definition (3.32), Remark 3.7 and the monotonicity property (i_2) in Proposition 2.5 it follows that also $R_\varepsilon(\pi^\infty(P)) \in \mathcal{P}_h$; hence, (6.3) implies that

$$\max_{v\in\pi^\infty(P)} \frac{\displaystyle\int_\Omega A^\infty(x, \nabla R_\varepsilon(v))dx}{\displaystyle\frac{1}{p}\int_\Omega |R_\varepsilon(v)|^p dx}\ \geq\ \lambda_h^\infty.$$

Thus, $u_\varepsilon \in P$ exists, such that, if $v_\varepsilon = \pi^\infty(u_\varepsilon)$, we have

$$\int_\Omega A^\infty(x, \nabla R_\varepsilon(v_\varepsilon))dx\ \geq\ \frac{\lambda_h^\infty}{p}\int_\Omega |R_\varepsilon(v_\varepsilon)|^p dx. \tag{6.12}$$

We note that (H_4) and (3.9) imply that a constant $r_\varepsilon > 0$ exists, such that

$$|t| > r_\varepsilon\ \Longrightarrow\ A(x, t, \xi) \geq (1-\varepsilon)A^\infty(x, \xi) \quad \text{for a.e. } x \in \Omega,\ \text{all } \xi \in \mathbb{R}^N. \tag{6.13}$$

Moreover, (3.24) holds for a suitable $L_\varepsilon > 0$, while as $|t|^p$ is a primitive of $p\ |t|^{p-1}t$, direct computations imply that

$$|t|^p\ \leq\ |R_\varepsilon(t)|^p + \varepsilon p\ |t|^{p-1} \quad \text{for all } t \in \mathbb{R}. \tag{6.14}$$

For simplicity, we put $\varrho_\varepsilon := \left(\int_\Omega A^\infty(x, \nabla u_\varepsilon)dx\right)^{\frac{1}{p}}$, so $u_\varepsilon = \varrho_\varepsilon v_\varepsilon$ and two cases may occur:

(a) $\varepsilon\varrho_\varepsilon\ \leq\ r_\varepsilon$,
(b) $\varepsilon\varrho_\varepsilon\ >\ r_\varepsilon$.

Case (a) Since $A(x, t, \xi)$ is positive, from (3.25) and (6.11) it follows that

$$\mathcal{J}(u_\varepsilon)\ \geq\ -D_1|u_\varepsilon|_p^p - D_2\text{meas}(\Omega)\ \geq\ -\frac{D_1\sigma_p}{\alpha_1}\left(\frac{r_\varepsilon}{\varepsilon}\right)^p - D_2\text{meas}(\Omega).$$

Case (b) Taking $\Omega_\varepsilon = \Omega_\varepsilon^{v_\varepsilon} = \{x \in \Omega : \ |v_\varepsilon(x)| > \varepsilon\}$, from (6.10) we have $\text{meas}(\Omega_\varepsilon) > 0$ and

$$|u_\varepsilon(x)| \ > \ r_\varepsilon \quad \text{for all } x \in \Omega_\varepsilon.$$

Therefore, since $A(x, t, \xi)$ is positive, from (3.24), (6.13), and then (H_4), (3.32), and (6.12), (6.14) we have that

$$\begin{aligned}
\mathcal{J}(u_\varepsilon) &\geq (1-\varepsilon) \int_{\Omega_\varepsilon} A^\infty(x, \nabla u_\varepsilon) dx \ - \ \left(\frac{\lambda^\infty}{p} + \varepsilon\right) \int_\Omega |u_\varepsilon|^p dx \ - \ L_\varepsilon \text{meas}(\Omega) \\
&= (1-\varepsilon)\, \varrho_\varepsilon^p \int_\Omega A^\infty(x, \nabla R_\varepsilon(v_\varepsilon)) dx \ - \ \left(\frac{\lambda^\infty}{p} + \varepsilon\right) \varrho_\varepsilon^p \int_\Omega |v_\varepsilon|^p dx - L_\varepsilon \text{meas}(\Omega) \\
&\geq (1-\varepsilon)\, \frac{\lambda_h^\infty}{p}\, \varrho_\varepsilon^p \int_\Omega |R_\varepsilon(v_\varepsilon)|^p dx \ - \ \left(\frac{\lambda^\infty}{p} + \varepsilon\right) \varrho_\varepsilon^p \int_\Omega |v_\varepsilon|^p dx \\
&\quad - L_\varepsilon \text{meas}(\Omega) \\
&\geq (1-\varepsilon)\, \frac{\lambda_h^\infty}{p}\, \varrho_\varepsilon^p \left(\int_\Omega |v_\varepsilon|^p dx - \varepsilon p \int_\Omega |v_\varepsilon|^{p-1} dx \right) \\
&\quad - \left(\frac{\lambda^\infty}{p} + \varepsilon\right) \varrho_\varepsilon^p \int_\Omega |v_\varepsilon|^p dx \ - \ L_\varepsilon \text{meas}(\Omega).
\end{aligned}$$

Summing up, from Hölder inequality, condition *(ii)* and direct computations it follows that

$$\begin{aligned}
\mathcal{J}(u_\varepsilon) \geq \frac{\varrho_\varepsilon^p}{p}\, |v_\varepsilon|_p^p &\left((1-\varepsilon)\lambda_h^\infty \left(1 - \varepsilon p \left(\frac{\alpha_2 D_1 \text{meas}(\Omega)}{\alpha_1} \right)^{\frac{1}{p}} \right) \ - \ \left(\lambda^\infty + \varepsilon p\right) \right) \\
&- L_\varepsilon \text{meas}(\Omega);
\end{aligned}$$

hence, (6.9) implies $\mathcal{J}(u_\varepsilon) \ \geq \ - L_\varepsilon \text{meas}(\Omega)$.

Thus, the thesis follows. □

Remark 6.4 The statement in Proposition 6.3 is optimal as, if $\lambda^\infty > \lambda_h^\infty$, condition (6.7) does not hold (for more details, see Remark 7.4).

Proposition 6.5 *If (h_3) holds and $k \in \mathbb{N}$ is such that $\lambda_k^0 < \lambda^0$, then we have*

$$\inf_{P \in \mathcal{P}_k} \max_{u \in P} \mathcal{J}(u) < 0.$$

Proof Firstly, let us point out that from Lagrange's Theorem and (3.6) it follows that a constant $b_1 > 0$ exists, such that

$$|t| \leq 1 \implies |A(x, t, \xi) - A^0(x, \xi)| \leq b_1 |t| |\xi|^p \text{ for a.e. } x \in \Omega, \text{ all } \xi \in \mathbb{R}^N. \tag{6.15}$$

Now, being $\lambda_k^0 < \lambda^0$, we can choose $\varepsilon > 0$ so small that

$$\left(1 + \varepsilon \frac{b_1}{\alpha_1}\right) (\lambda_k^0 + \varepsilon) + \varepsilon < \lambda^0, \tag{6.16}$$

with α_1 as in (H_3). Then, from (5.4) a constant $r_\varepsilon > 0$ exists, such that

$$|t| \leq r_\varepsilon \quad \implies \quad |G^0(x, t)| \ \leq \ \frac{\varepsilon}{p}\, |t|^p \quad \text{a.e. in } \Omega. \tag{6.17}$$

On the other hand, from Proposition 5.9 a set $Q_0 \in \tilde{\mathcal{W}}_k$ exists, such that

$$\max_{u\in Q_0} \frac{\displaystyle\int_\Omega A^0(x,\nabla u)dx}{\displaystyle\frac{1}{p}\int_\Omega |u|^p dx} < \lambda_k^0 + \varepsilon \tag{6.18}$$

and, since from (6.2) we have $Q_0 \in \mathcal{P}_k$, Q_0 is also compact in X and a constant $\varrho_0 > 0$ exists, such that $|u|_\infty < \varrho_0$ for all $u \in Q_0$. Thus, we can define

$$P_\varepsilon^0 = \{v = \varrho_\varepsilon u \in X \ : \ u \in Q_0\} \quad \text{with} \quad \varrho_\varepsilon = \frac{1}{\varrho_0}\min\{1, \varepsilon, r_\varepsilon\}.$$

Since the map $u \in X \mapsto \varrho_\varepsilon u \in X$ is an odd homeomorphism on $(X, \|\cdot\|_X)$, then property (i_2) in Proposition 2.5 and (6.2) also imply that $P_\varepsilon^0 \in \mathcal{P}_k$.

Taking any $v = \varrho_\varepsilon u \in P_\varepsilon^0$, $u \in Q_0$, from $|v|_\infty < \min\{1, \varepsilon, r_\varepsilon\}$ and (5.3), (6.15), (6.17), hypotheses (H_3), (H_9) and (6.18), it follows that

$$\begin{aligned}
\mathcal{J}(v) &\le \int_\Omega A^0(x,\nabla v)dx + b_1|v|_\infty \int_\Omega |\nabla v|^p dx - \frac{\lambda^0-\varepsilon}{p}\int_\Omega |v|^p dx \\
&\le \left(1 + \frac{b_1}{\alpha_1}\,\varepsilon\right)\varrho_\varepsilon^p \int_\Omega A^0(x,\nabla u)dx - \left(\lambda^0-\varepsilon\right)\frac{\varrho_\varepsilon^p}{p}\int_\Omega |u|^p dx \\
&\le \left(\left(1+\frac{b_1}{\alpha_1}\varepsilon\right)(\lambda_k^0+\varepsilon) - (\lambda^0-\varepsilon)\right)\frac{\varrho_\varepsilon^p}{p}\int_\Omega |u|^p dx.
\end{aligned}$$

Therefore, since the minimum of the L^p-norm in the compact set Q_0 is strictly positive, assumption (6.16) implies that $\max_{u\in P_\varepsilon^0} \mathcal{J}(u) < 0$ and the thesis is true. □

Proof of Theorem 5.6 The hypotheses imply that $\mathcal{J}$ is an even functional, such that $\mathcal{J}(0) = 0$. Moreover, from Propositions 3.5 and 4.5 we have that $\mathcal{J}$ is C^1 in $(X, \|\cdot\|_X)$ and satisfies the (wPS) condition in $\mathbb{R}$. Then, assumption (5.11) allows us to apply Propositions 6.3 and 6.5, so definition (6.1) implies that (2.15) holds with $m = k - h + 1$. Hence, the thesis follows from the first statement of Theorem 2.7. □

Proof of Theorem 5.8 As in the proof of Theorem 5.6, in order to apply the second statement of Theorem 2.7, we have just to prove that $h \in \mathbb{N}$ exists, so that $-\infty < c_k < 0$ for all $k \ge h$, with c_k as in (6.1).

To this aim, firstly we note that, taking any $k \in \mathbb{N}$, if we fix $\bar\lambda > \lambda_k^0$, from (5.13) a constant $\bar r > 0$ exists, such that

$$|t| \le \bar r \implies F(x,t) \ge \bar\lambda |t|^p \quad \text{for a.e. } x \in \Omega.$$

Then, reasoning as in the proof of Proposition 6.5, but with $\bar\lambda$ in the place of λ^0 and $\bar r$ in the place of r_ε, we can find a subset $P_\varepsilon^0 \in \mathcal{P}_k$, such that

$$c_k \le \max_{u\in P_\varepsilon^0} \mathcal{J}(u) < 0.$$

On the other hand, from Proposition 5.4 an integer $h \in \mathbb{N}$ exists, such that $\lambda_h^\infty > \lambda^\infty$. Hence, Proposition 6.3 implies that $c_k \ge c_h > -\infty$ for all $k \ge h$. □

7 Looking for positive critical levels

Throughout this section, we assume that (H_1)–(H_{10}) and (h_1)–(h_4) hold and, in order to find critical points with positive critical level by applying Theorem 2.10, we require some useful information for the pseudo-index theory. To this aim, we need to evaluate the maximum of $\mathcal{J}$ in a family of subsets of X, which are part of a neighbourhood of the origin in $W_0^{1,p}(\Omega)$, so that it is quite natural to approximate $A(x,t,\xi)$ with $A^0(x,\xi)$. Unfortunately, for the L^∞-norm of the elements of such sets no uniform a priori bound is given. So, for a suitable $K > 0$ we split Ω into two parts: one where we can evaluate the truncation map T_K (defined as in (3.31)) to apply the approximating scheme and its complementary set, where the remainder is small enough.

More precisely, the following statement can be proved.

Proposition 7.1 *If $\lambda_h^0 > \lambda^0$, then a suitable $\varepsilon_0 > 0$ can be found, such that for all $\varepsilon \in]0,\varepsilon_0]$, a radius $\tau = \tau(\varepsilon) > 0$ and a constant $c_\varepsilon > 0$ exist, so that*

$$\max_{u\in P} \mathcal{J}(u) \geq c_\varepsilon > 0 \quad \text{for all } P \in \mathcal{P}_h \text{ such that } P \subset \partial B_\tau^0, \tag{7.1}$$

with

$$\partial B_\tau^0 = \left\{u \in X : \int_\Omega A^0(x,\nabla u)dx = \tau^p\right\}.$$

Proof Fix $\varepsilon \in]0,\varepsilon_0]$ with $\varepsilon_0 > 0$ and

$$\varepsilon_0 < \min\left\{1, \frac{\alpha_1}{2\sigma_p + \alpha_1}, \frac{\lambda_h^0 - \lambda^0}{4(\lambda_h^0 + p)}, \frac{2(\lambda_h^0 - \lambda^0)\alpha_1}{3(\lambda_h^0 - \lambda^0)\alpha_1 + 4D\lambda_h^0\sigma_p}\right\}, \tag{7.2}$$

where α_1 is as in (H_3), σ_p as in (3.2), α_4 as in (5.1) and D as in (5.6).

Then, we note that from (5.1) and (5.2), respectively (5.4), a constant $r_\varepsilon > 0$ exists, such that

$$|t| \leq r_\varepsilon \quad \Longrightarrow \quad A(x,t,\xi) \geq (1-\varepsilon)A^0(x,\xi) \quad \text{and} \quad |G^0(x,t)| \leq \varepsilon|t|^p \tag{7.3}$$

for a.e. $x \in \Omega$, all $\xi \in \mathbb{R}^N$.

Moreover, from (5.1) the set ∂B_1^0 is bounded in $W_0^{1,p}(\Omega)$, so it is compact in $L^p(\Omega)$ and $K_\varepsilon > 0$ exists, such that

$$\int_{\Omega_{K_\varepsilon}^v} |v|^p dx < \varepsilon\,\frac{\alpha_1}{2\alpha_4 D} \qquad \text{for all } v \in \partial B_1^0. \tag{7.4}$$

Now, put

$$\tau = \frac{r_\varepsilon}{K_\varepsilon}. \tag{7.5}$$

Taking $P \in \mathcal{P}_h$ such that $P \subset \partial B_\tau^0$, two cases may occur:

(i) $u_0 \in P$ exists such that $|\pi^0(u_0)|_p \leq \left(\frac{\alpha_1}{2\alpha_4 D}\right)^{\frac{1}{p}}$,

(ii) for all $u \in P$: $|\pi^0(u)|_p > \left(\frac{\alpha_1}{2\alpha_4 D}\right)^{\frac{1}{p}}$,

where π^0 is as in (6.5).

Case (i) From definitions (6.4) and (6.5) we have $\pi^0(u_0) \in \partial B_1^0$ with $u_0 = \tau\pi^0(u_0)$, then from (H_3), (5.1) and (5.6), assumption *(i)* implies that

$$\max_{u\in P} \mathcal{J}(u) \geq \mathcal{J}(u_0) \geq \frac{\alpha_1}{\alpha_4}\int_\Omega A^0(x,\nabla u_0)dx - D\int_\Omega |u_0|^p dx \\ = \tau^p\left(\frac{\alpha_1}{\alpha_4} - D\int_\Omega |\pi^0(u_0)|^p dx\right) \geq \tau^p\frac{\alpha_1}{2\alpha_4}. \tag{7.6}$$

Case (ii) Firstly, suppose $\lambda^0 \leq 0$. From (6.4) and (6.5), for each $u \in \partial B_\tau^0$ we have $u = \tau v$ with $v = \pi^0(u) \in \partial B_1^0$. We note that from (5.1) we have

$$\int_\Omega |\nabla v|^p dx \geq \frac{1}{\alpha_4}. \tag{7.7}$$

Moreover, (7.5) implies

$$|u(x)| \leq r_\varepsilon \quad \text{for a.e. } x \in \Omega\backslash\Omega^v_{K_\varepsilon}.$$

From (H_3), (5.3), (5.6), then (7.3), and from (3.2), (7.4) we have

$$\mathcal{J}(u) \geq \alpha_1\int_\Omega |\nabla u|^p dx - \int_{\Omega\backslash\Omega^v_{K_\varepsilon}} G^0(x,u)dx - D\int_{\Omega^v_{K_\varepsilon}} |u|^p dx \\ \geq \alpha_1\tau^p\int_\Omega |\nabla v|^p dx - \varepsilon\tau^p\int_{\Omega\backslash\Omega^v_{K_\varepsilon}} |v|^p dx - D\tau^p\int_{\Omega^v_{K_\varepsilon}} |v|^p dx \\ \geq (\alpha_1 - \varepsilon\sigma_p)\,\tau^p\int_\Omega |\nabla v|^p dx - \varepsilon\tau^p\frac{\alpha_1}{2\alpha_4}.$$

Thus, summing up, estimate (7.2) and (7.7) imply

$$\max_{u\in P} \mathcal{J}(u) \geq \inf_{u\in\partial B_\tau^0} \mathcal{J}(u) \geq \tau^p\frac{\alpha_1}{2\alpha_4}. \tag{7.8}$$

On the contrary, consider $\lambda^0 > 0$. We point out that from (6.5) and (6.6), it follows that $\pi^0(P) \in \mathcal{P}_h$ and $\pi^0(P) \subset \partial B_1^0$. Hence, if we consider the truncation map T_{K_ε} as in (3.31), from Remark 3.7 and property (i_2) in Proposition 2.5 we also have $T_{K_\varepsilon}(\pi^0(P)) \in \mathcal{P}_h$, and (6.3) implies

$$\max_{v\in\pi^0(P)} \frac{\int_\Omega A^0(x,\nabla T_{K_\varepsilon}(v))dx}{\frac{1}{p}\int_\Omega |T_{K_\varepsilon}(v)|^p dx} \geq \lambda_h^0.$$

Thus, $u_\varepsilon \in P$ exists, such that $u_\varepsilon = \tau v_\varepsilon$ with $v_\varepsilon = \pi^0(u_\varepsilon)$, and

$$\int_\Omega A^0(x,\nabla T_{K_\varepsilon}(v_\varepsilon))dx \geq \frac{\lambda_h^0}{p}\int_\Omega |T_{K_\varepsilon}(v_\varepsilon)|^p dx. \tag{7.9}$$

If we define $\Omega_\varepsilon = \Omega^{v_\varepsilon}_{K_\varepsilon} = \{x \in \Omega : |v_\varepsilon(x)| > K_\varepsilon\}$, from (7.5) it follows that

$$|u_\varepsilon(x)| \leq r_\varepsilon \quad \text{for a.e. } x \in \Omega\backslash\Omega_\varepsilon.$$

Furthermore, in case *(ii)*, estimate (7.4) implies

$$\int_{\Omega_\varepsilon} |v_\varepsilon|^p dx < \varepsilon\,|\pi^0(u_\varepsilon)|_p^p. \tag{7.10}$$

Then, since $A(x, t, \xi)$ is positive, from (5.3), (5.6) and (7.3), assumption (H_9) and (3.31) imply that

$$\begin{aligned}\mathcal{J}(u_\varepsilon) &\geq (1-\varepsilon)\int_{\Omega\setminus\Omega_\varepsilon} A^0(x,\nabla u_\varepsilon)dx - \left(\frac{\lambda^0}{p}+\varepsilon\right)\int_{\Omega\setminus\Omega_\varepsilon}|u_\varepsilon|^p dx \\ &\quad - D\int_{\Omega_\varepsilon}|u_\varepsilon|^p dx \\ &\geq (1-\varepsilon)\tau^p\int_\Omega A^0(x,\nabla T_{K_\varepsilon}(v_\varepsilon))dx - \left(\frac{\lambda^0}{p}+\varepsilon\right)\tau^p\int_\Omega |T_{K_\varepsilon}(v_\varepsilon)|^p dx \\ &\quad - D\tau^p\int_{\Omega_\varepsilon}|v_\varepsilon|^p dx.\end{aligned}$$

We note that, from (7.2), inequality (7.9) implies that

$$\begin{aligned}(1-\varepsilon)\int_\Omega A^0(x,\nabla T_{K_\varepsilon}(v_\varepsilon))dx &- \left(\frac{\lambda^0}{p}+\varepsilon\right)\int_\Omega |T_{K_\varepsilon}(v_\varepsilon)|^p dx \\ &\geq \frac{3}{4}\,\frac{\lambda_h^0-\lambda^0}{\lambda_h^0}\int_\Omega A^0(x,\nabla T_{K_\varepsilon}(v_\varepsilon))dx,\end{aligned}$$

where from (5.1) and also (3.2), (3.31) and the characterization of v_ε, it follows that

$$\begin{aligned}\int_\Omega A^0(x,\nabla T_{K_\varepsilon}(v_\varepsilon))dx &\geq \frac{\alpha_1}{\sigma_p}\int_{\Omega\setminus\Omega_\varepsilon}|v_\varepsilon|^p dx \\ &= \frac{\alpha_1}{\sigma_p}\left(|\pi^0(u_\varepsilon)|_p^p - \int_{\Omega_\varepsilon}|v_\varepsilon|^p dx\right).\end{aligned}$$

Hence, summing up, from (7.2), (7.10) and direct computations, it follows that

$$\begin{aligned}\mathcal{J}(u_\varepsilon) &\geq \tau^p\left((1-\varepsilon)\,\frac{3\alpha_1(\lambda_h^0-\lambda^0)}{4\lambda_h^0\sigma_p} - \varepsilon D\right)|\pi^0(u_\varepsilon)|_p^p \\ &\geq \tau^p\,\frac{\alpha_1(\lambda_h^0-\lambda^0)}{4\lambda_h^0\sigma_p}\,|\pi^0(u_\varepsilon)|_p^p;\end{aligned}$$

hence, condition *(ii)* implies

$$\max_{u\in P}\mathcal{J}(u) \geq \mathcal{J}(u_\varepsilon) \geq \tau^p\,\frac{\alpha_1^2(\lambda_h^0-\lambda^0)}{8\alpha_4\lambda_h^0 D\sigma_p}. \tag{7.11}$$

Thus, taking

$$c_\varepsilon = \tau^p\min\left\{\frac{\alpha_1}{2\alpha_4},\,\frac{\alpha_1^2(\lambda_h^0-\lambda^0)}{8\alpha_4\lambda_h^0 D\sigma_p}\right\} > 0,$$

(7.1) follows from (7.6), (7.8) and (7.11). □

In order to obtain *'information at infinity'*, we need the following technical lemma (for more details, see *Step (a)* in the proof of [14, Lemma 4.3]).

Lemma 7.2 *If P is a compact subset of $W_0^{1,p}(\Omega)$, taking any $\varepsilon > 0$ a costant $\rho = \rho(P,\varepsilon) > 0$ exists, such that*

$$\int_{\Omega\setminus\Omega_\rho^u}|\nabla u|^p dx < \varepsilon \quad \textit{for all } u \in P.$$

Proposition 7.3 *If $\lambda_k^\infty < \lambda^\infty$, then a suitable $\varepsilon_\infty > 0$ exists, such that for all $\varepsilon \in]0, \varepsilon_\infty]$ a constant $R_\varepsilon^* > 0$ can be found, such that for each $R \geq R_\varepsilon^*$ a suitable subset P_ε^R exists, so that*

$$\max_{u \in P_\varepsilon^R} \mathcal{J}(u) \leq 0 \quad \text{with } P_\varepsilon^R \in \mathcal{P}_k \text{ such that } P_\varepsilon^R \subset \partial B_R^\infty, \tag{7.12}$$

where

$$\partial B_R^\infty = \left\{u \in X : \int_\Omega A^\infty(x, \nabla u)dx = R^p\right\}.$$

Proof Since $\lambda_k^\infty < \lambda^\infty$, then $\varepsilon_\infty > 0$ exists, such that

$$(\lambda_k^\infty + \varepsilon)(1 + \varepsilon(M_2 + 1)) + \varepsilon p < \lambda^\infty \quad \text{for all } 0 < \varepsilon \leq \varepsilon_\infty, \tag{7.13}$$

with $M_2 > 0$ as in (3.12). Fixing any $0 < \varepsilon \leq \varepsilon_\infty$, from (H_4) and (3.9), a constant $r_\varepsilon > 0$ exists, such that

$$|t| > r_\varepsilon \implies A(x, t, \xi) \leq (1 + \varepsilon)A^\infty(x, \xi) \quad \text{for a.e. } x \in \Omega, \text{ all } \xi \in \mathbb{R}^N. \tag{7.14}$$

Moreover, from Proposition 5.9 a set $Q_\varepsilon \in \tilde{\mathcal{W}}_k$ exists, such that

$$\max_{v \in Q_\varepsilon} \frac{\displaystyle\int_\Omega A^\infty(x, \nabla v)dx}{\displaystyle\frac{1}{p}\int_\Omega |v|^p dx} < \lambda_k^\infty + \varepsilon. \tag{7.15}$$

From (6.2) we have $Q_\varepsilon \in \mathcal{P}_k$ and, for simplicity, we can define $P_\varepsilon = \pi^\infty(Q_\varepsilon) \in \partial B_1^\infty$, with π^∞ as in (6.5). By definition, we have

$$\frac{\displaystyle\int_\Omega A^\infty(x, \nabla v)dx}{\displaystyle\frac{1}{p}\int_\Omega |v|^p dx} = \frac{1}{\displaystyle\frac{1}{p}\int_\Omega |\pi^\infty(v)|^p dx} \quad \text{for all } v \in Q_\varepsilon;$$

hence, (6.6) and (7.15) imply

$$\frac{1}{\lambda_k^\infty + \varepsilon} \leq \frac{1}{p}\int_\Omega |w|^p dx \quad \text{for all } w \in P_\varepsilon, \text{ with } P_\varepsilon \in \mathcal{P}_k. \tag{7.16}$$

Moreover, since P_ε is also compact in $W_0^{1,p}(\Omega)$, from Lemma 7.2 $\rho = \rho(\varepsilon) > 0$ exists, such that

$$\int_{\Omega \setminus \Omega_\rho^w} |\nabla w|^p dx < \varepsilon \quad \text{for all } w \in P_\varepsilon. \tag{7.17}$$

Then, fixing any $R > R_\varepsilon$ with $R_\varepsilon = \frac{r_\varepsilon}{\rho}$, we can define

$$P_\varepsilon^R = \{u = Rw \in X : w \in P_\varepsilon\}. \tag{7.18}$$

Since the map $w \in P_\varepsilon \mapsto Rw \in P_\varepsilon^R$ is an odd homeomorphism from X in itself, from (i_2) in Proposition 2.5 it follows that $P_\varepsilon^R \in \mathcal{P}_k$, while $P_\varepsilon^R \subset \partial B_R^\infty$ from (H_4).

Now, we note that, if $u = Rw \in P_\varepsilon^R$ with $w \in P_\varepsilon$, then

$$|u(x)| > r_\varepsilon \quad \text{if } x \in \Omega_\rho^w,$$

so, (3.12), (3.16), (3.22), (7.14), (H_4) and direct computations imply that

$$\mathcal{J}(u) \leq (1+\varepsilon)R^p \int_{\Omega_\rho^w} A^\infty(x, \nabla w)dx + M_2 R^p \int_{\Omega\setminus\Omega_\rho^w} |\nabla w|^p dx$$
$$- (\lambda^\infty - \varepsilon p)\frac{R^p}{p} \int_\Omega |w|^p dx + b_{1,\varepsilon}.$$

with $b_{1,\varepsilon} = (M_1 + L_\varepsilon)\text{meas}(\Omega)$.

As $P_\varepsilon \subset \partial B_1^\infty$ and (7.13) imply $\lambda^\infty - \varepsilon p > 0$, from (7.16), (7.17) and the last inequality, it follows that

$$\mathcal{J}(u) \leq b_{2,\varepsilon} R^p + b_{1,\varepsilon} \qquad \text{for all } u \in P_\varepsilon^R, \tag{7.19}$$

with $b_{2,\varepsilon} = 1 + \varepsilon(1 + M_2) - \frac{\lambda^\infty - \varepsilon p}{\lambda_k^\infty + \varepsilon}$. Thus, since (7.13) implies $b_{2,\varepsilon} < 0$, a large enough $R_\varepsilon^* > \frac{r_\varepsilon}{\rho}$ can be choosen, so that the second term of (7.19) is negative for all $R \geq R_\varepsilon^*$; hence, (7.12) holds. □

Remark 7.4 If $k \in \mathbb{N}$ is such that $\lambda_k^\infty < \lambda^\infty$, then

$$\inf_{P\in\mathcal{P}_k} \max_{u\in P} \mathcal{J}(u) = -\infty.$$

In fact, reasoning as in the proof of Proposition 7.3 and taking $\varepsilon \leq \varepsilon_\infty$ as in (7.13), for any $R > R_\varepsilon$ we can consider P_ε^R as in (7.18), so that $P_\varepsilon^R \in \mathcal{P}_k$ and inequality (7.19) hold with $b_{2,\varepsilon} < 0$; hence,

$$\lim_{R\to+\infty} \max_{u\in P_\varepsilon^R} \mathcal{J}(u) = -\infty.$$

Proof of Theorem 5.7 The hypotheses imply that $\mathcal{J}$ is an even functional such that $\mathcal{J}(0) = 0$. Moreover, from Proposition 3.5 and Theorem 4.5 we have that $\mathcal{J}$ is C^1 in $(X, \|\cdot\|_X)$ and satisfies the (wPS) condition in $\mathbb{R}$.

Now, in order to prove Theorem 5.7, let $h, k \in \mathbb{N}$ be such that (5.12) holds, so we can fix $0 < \varepsilon \leq \min\{\varepsilon_0, \varepsilon_\infty\}$, with ε_0 as in Proposition 7.1 and ε_∞ as in Proposition 7.3, and take $\tau = \tau(\varepsilon) > 0$ and ∂B_τ^0 as in Proposition 7.1. Then, we consider the pseudo-index $i^*(\cdot)$ related to the cohomological index $i(\cdot)$ on X, the set $\mathcal{M} = \partial B_\tau^0$ (closed in $W_0^{1,p}(\Omega)$ and then in X) and

$$\mathcal{H} = \{\gamma : X \to X : \gamma \text{ odd homeomorphism, such that } \gamma(u) = u \ \forall u \in \mathcal{J}^0\},$$

which is defined as in (2.18) on any symmetric subset $P \subset X$.

Thus, as in Sect. 2, for all $j \in \mathbb{N}$, we define

$$\mathcal{P}_j^* = \{P \subset X : \quad P \text{ symmetric and compact in } X \text{ with } i^*(P) \geq j\}$$

and

$$c_j^* = \inf_{P\in\mathcal{P}_j^*} \max_{u\in P} \mathcal{J}(u).$$

Firstly, taking any $P \in \mathcal{P}_h^*$, since the identity map on X is in $\mathcal{H}$, from definition (2.18), it follows that $h \leq i^*(P) \leq i(P\cap\mathcal{M})$. Thus, the properties of P and $\mathcal{M}$ imply that $P\cap\mathcal{M} \in \mathcal{P}_h$ and $P \cap \mathcal{M} \subset \partial B_\tau^0$; hence, from Proposition 7.1 we have that

$$\max_{u\in P} \mathcal{J}(u) \geq \max_{u\in P\cap\mathcal{M}} \mathcal{J}(u) \geq c_\varepsilon$$

and, from the arbitrariness of P,

$$c_h^* \geq c_\varepsilon > 0. \tag{7.20}$$

Now, take

$$R \ > \ \max \left\{ R_\varepsilon^*, \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{1}{p}} \tau \right\}, \tag{7.21}$$

with $R_\varepsilon^* > 0$ as in Proposition 7.3 and α_1 as in (H_3), α_2 as in (3.9), and consider P_ε^R, such that (7.12) holds. By defining

$$IP_\varepsilon^R \ = \ \{u = sv \in X : \ s \in [0,1], \ v \in P_\varepsilon^R\},$$

the properties of P_ε^R imply that IP_ε^R is a symmetric compact subset of X.

We claim that

$$i^*(IP_\varepsilon^R) \geq k; \qquad \text{hence} \ , IP_\varepsilon^R \in \mathcal{P}_k^*. \tag{7.22}$$

To this aim, let us consider the closed subsets

$$P_0 = \{u \in X : \int_\Omega A^0(x,u)dx \leq \tau^p\}, \quad P_1 = \overline{X \setminus P_0},$$

and, from (3.9), (5.1), (7.12) and (7.21), it follows that

$$P_\varepsilon^R \subset \partial B_R^\infty \subset P_1. \tag{7.23}$$

Taking any $\gamma \in \mathcal{H}$, from (7.12) the restriction of γ to P_ε^R is the identity. Thus, the map

$$\varphi : (v,s) \in P_\varepsilon^R \times [0,1] \mapsto \gamma(sv) \in X = P_0 \cup P_1$$

is continuous, odd in v, and such that

$$\varphi(P_\varepsilon^R \times [0,1]) = \gamma(IP_\varepsilon^R)$$

is closed, while (7.23) implies that

$$\varphi(P_\varepsilon^R \times \{0\}) = \{0\} \subset P_0, \quad \varphi(P_\varepsilon^R \times \{1\}) = \gamma(P_\varepsilon^R) = P_\varepsilon^R \subset P_1.$$

So, the piercing property (i_7) in Proposition 2.5 applies, and then from (7.12) it follows that

$$k \ \leq \ i(P_\varepsilon^R) \ \leq \ i(\varphi(P_\varepsilon^R \times [0,1]) \cap P_0 \cap P_1) \ = \ i(\gamma(IP_\varepsilon^R) \cap \partial B_\tau^0).$$

Thus, (7.22) follows from the arbitrariness of $\gamma \in \mathcal{H}$ and (2.18). Finally, since (7.22) implies

$$c_k^* \leq \max_{u \in IP_\varepsilon^R} \mathcal{J}(u) < +\infty, \tag{7.24}$$

the thesis follows from (7.20), (7.24) and Theorem 2.10, with $m = k + 1 - h$. $\square$

References

1. Amann, H., Zehnder, E.: Nontrivial solutions for a class of nonresonance problems and applications to nonlinear differential equations. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **7**, 539–603 (1980)
2. Anane, A., Gossez, J.P.: Strongly nonlinear elliptic problems near resonance: a variational approach. Commun. Partial Differ. Equ. **15**, 1141–1159 (1990)
3. Arcoya, D., Boccardo, L.: Critical points for multiple integrals of the calculus of variations. Arch. Ration. Mech. Anal. **134**, 249–274 (1996)
4. Arcoya, D., Orsina, L.: Landesman–Lazer conditions and quasilinear elliptic equations. Nonlinear Anal. **28**, 1623–1632 (1997)

5. Bartolo, P., Benci, V., Fortunato, D.: Abstract critical point theorems and applications to some nonlinear problems with "strong" resonance at infinity. Nonlinear Anal. **7**, 981–1012 (1983)
6. Bartolo, R., Candela, A.M., Salvatore, A.: p-Laplacian problems with nonlinearities interacting with the spectrum. Nonlinear Differ. Equ. Appl. **20**, 1701–1721 (2013)
7. Benci, V.: On the critical point theory for indefinite functionals in the presence of symmetries. Trans. Am. Math. Soc. **274**, 533–572 (1982)
8. Boccardo, L., Murat, F., Puel, J.P.: Existence of bounded solutions for nonlinear elliptic unilateral problems. Ann. Mat. Pura Appl. IV Ser. **152**, 183–196 (1988)
9. Brézis, H., Coron, J.M., Nirenberg, L.: Free vibrations for a nonlinear wave equation and a theorem of P. Rabinowitz. Commun. Pure Appl. Math. **33**, 667–689 (1980)
10. Browder, F.E.: Existence theorems for nonlinear partial differential equations. In: Chern, S.S., Smale, S. (eds.) Proceedings of the Symposia in Pure Mathematics, vol. XVI. AMS, Providence, pp. 1–60 (1970)
11. Candela, A.M., Palmieri, G.: Infinitely many solutions of some nonlinear variational equations. Calc. Var. Partial Differ. Equ. **34**, 495–530 (2009)
12. Candela, A.M., Palmieri, G.: Some abstract critical point theorems and applications. In: Hou, X., Lu, X., Miranville, A., Su, J., Zhu, J. (eds.) Dynamical Systems, Differential Equations and Applications, Discrete Contin. Dynam. Syst. Suppl. 2009, pp. 133–142 (2009)
13. Candela, A.M., Palmieri, G.: Multiple solutions for p-Laplacian type problems with an asymptotically p-linear term. In: de Figueiredo, D.G., do Ó, J.M., Tomei, C. (eds.) Analysis and Topology in Nonlinear Differential Equations, *Progr. Nonlinear Differential Equations Appl.*, vol. 85, pp. 175–186 (2014)
14. Candela, A.M., Palmieri, G., Perera, K.: Multiple solutions for p-Laplacian type problems with asymptotically p-linear terms via a cohomological index theory. J. Differ. Equ. **259**, 235–263 (2015)
15. Cerami, G.: Un criterio di esistenza per i punti critici su varietà illimitate. Istit. Lombardo Accad. Sci. Lett. Rend. A **112**, 332–336 (1978)
16. Costa, D.G., Magalhães, C.A.: Existence results for perturbations of the p-Laplacian. Nonlinear Anal. **24**, 409–418 (1995)
17. Drábek, P., Robinson, S.: Resonance problems for the p-Laplacian. J. Funct. Anal. **169**, 189–200 (1999)
18. Fadell, E.R., Rabinowitz, P.H.: Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems. Invent. Math. **45**, 139–174 (1978)
19. Ladyzhenskaya, O.A., Ural'tseva, N.N.: Linear and Quasilinear Elliptic Equations. Academic Press, New York (1968)
20. Liu, S., Li, S.: Existence of solutions for asymptotically 'linear' p-Laplacian equations. Bull. Lond. Math. Soc. **36**, 81–87 (2004)
21. Li, G., Zhou, H.S.: Asymptotically linear Dirichlet problem for the p-Laplacian. Nonlinear Anal. **43**, 1043–1055 (2001)
22. Li, G., Zhou, H.S.: Multiple solutions to p-Laplacian problems with asymptotic nonlinearity as u^{p-1} at infinity. J. Lond. Math. Soc. **65**, 123–138 (2002)
23. Palais, R.S.: Critical point theory and the minimax principle. Proc. Symp. Pure Math. **15**, 185–212 (1970)
24. Perera, K., Agarwal, R.P., O'Regan, D.: Morse Theoretic Aspects of p-Laplacian Type Operators, Math. Surveys Monogr., vol. 161, Am. Math. Soc., Providence RI (2010)
25. Perera, K., Szulkin, A.: p-Laplacian problems where the nonlinearity crosses an eigenvalue. Discrete Contin. Dyn. Syst. **13**, 743–753 (2005)
26. Rabinowitz, P.H.: Minimax Methods in Critical Point Theory with Applications to Differential Equations, CBMS Reg. Conf. Ser. Math., vol. 65, Providence (1986)
27. Struwe, M.: Variational Methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, 4rd edn, Ergeb. Math. Grenzgeb. (4), vol. 34. Springer, Berlin (2008)