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Calculus of Variations

Global solutions of nonlinear Schrödinger equations

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Abstract We study the nonlinear Schrödinger equation in $\mathbb{R}^n$ without making any periodicity assumptions on the potential or on the nonlinear term. This prevents us from using concentration compactness methods. Our assumptions are such that the potential does not change the essential spectrum of the linear operator. This results in $[0, \infty)$ being the absolutely continuous part of the spectrum. If there are an infinite number of negative eigenvalues, they will converge to 0. In each case we obtain nontrivial solutions. We also obtain least energy solutions.

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1 Introduction

We consider the semilinear Schrödinger equation

$$-\Delta u + V(x)u = f(x, u), \quad u \in H^1(\mathbb{R}^n), \tag{1}$$

where $V(x)$ is a given potential. One wishes to find solutions and, in particular, the so called "least energy solutions." These are solutions that minimize the corresponding energy functional. The existence of solutions depends both on the linear operator $\mathcal{A}$ and the nonlinear term $f(x, u)$.

Many authors have studied the problem for the Schrödinger equation (1) under various stipulations (cf., e.g., [1–20,23,25,26,28–32] and references quoted in them). In almost all cases it was required to stipulate that the spectrum of the linear operator $\mathcal{A}u = -\Delta u + V(x)u$ have a gap. This caused writers to make various assumptions on the potential $V(x)$

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to guarantee that this is the case. However, many of these assumptions caused the nature of the spectrum to be far different from that of $-\Delta u$. Thus, any theorem proved for

$$\mathcal{A}u = -\Delta u + V(x)u$$

did not hold for $\mathcal{A}u = -\Delta u$.

Some authors assumed

$$\inf_{\mathbb{R}^n} V(x) > 0; \quad \lim_{|x|\to\infty} V(x) = \infty.$$

Others assumed that there exists a constant B such that $V(x) \leq B$ for all $x \in \mathbb{R}^n$, $V(x) \to B \; as \; |x| \to \infty$ and $\sigma(-\Delta + V(x)) > 0$ together with other assumptions. Another approach assumes that for every $M > 0$ the set $\omega = \{x \in \mathbb{R}^n : V(x) < M\}$ has finite Lebesgue measure. Others assumed that $V(x)$ is in some combination of $L^p(\mathbb{R}^n)$ spaces. In each case the growth of $f(x, t)$ is controlled by the growth of $V(x)$. In most cases the resulting spectrum of $\mathcal{A} = -\Delta + V(x)$ is discrete, consisting only of isolated eigenvalues of finite multiplicity tending to $+\infty$. All of these assumptions cause restrictions on the the nonlinear term depending on V. In most cases the hypothesis

$$\mu F(x, t) \leq t f(x, t), \quad |x| > R,$$

is used, where $\mu > 2$ and

$$F(x, t) = \int_0^t f(x, s)\, ds.$$

A different approach is to assume that the potential is periodic in the coordinates of $\mathbb{R}^n$ and then apply concentration compactness methods. In this case the resulting spectrum of $\mathcal{A} = -\Delta + V(x)$ is absolutely continuous and consists of a finite number of disjoint closed intervals. In order to apply this method, $f(x, t)$ must be periodic in x as well. In the few publications where 0 is permitted to be in $\sigma(-\Delta + V(x))$, an interval of the form $(-\varepsilon, 0)$ is required to be free of the spectrum.

The purpose of the present paper is solve the Eq. (1) under assumptions on $V(x)$ such that the essential spectrum of $\mathcal{A} = -\Delta + V(x)$ is the the same as that of $-\Delta$, i.e., $[0, \infty)$. The situation is different if there are no negative eigenvalues, one negative eigenvalue or two or more negative eigenvalues. If there are no negative eigenvalues, one can solve under the same hypotheses for $f(x, t)$ that can be used for the equation

$$-\Delta u = f(x, u), \quad u \in H^1(\mathbb{R}^n). \tag{2}$$

Otherwise, the hypotheses on $f(x, t)$ need only take the negative eigenvalues into consideration. We can even deal with the case where the negative eigenvalues converge to 0. We do not need an interval of the form $(-\varepsilon, 0)$ to be free of the spectrum. In each of these cases different methods must be employed, requiring different assumptions on the nonlinear term.

Concerning the function $V(x)$ we make the following assumptions:

(V_1)

$$\sup_y \int_{|x-y|<\delta} |V(x)|\, \omega_2(x-y)\, dx \to 0 \; as \; \delta \to 0,$$

and

(V_2)

$$\int_{|x-y|<1} |V(x)|\, \omega_2(x-y)\, dx \to 0 \; as \; |y| \to \infty,$$

where

$$\omega_s(x) = \begin{cases} |x|^{s-n}, & 0 < s < n \\ 1 - \ln|x|^2, & s = n \\ 1, & s > n. \end{cases}$$

These assumptions imply that there is a forms extension $\mathcal{A}$ of the operator

$$-\Delta u(x) + V(x)u(x)$$

on the space $H = H^{1,2}(\mathbb{R}^n)$ having essential spectrum equal to $[0, \infty)$ and a (possibly empty) discrete, countable negative spectrum consisting of isolated eigenvalues of finite multiplicity with a finite lower bound $-L$

$$-\infty < -L \leq \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_l < \cdots < 0. \tag{3}$$

For each $l > 0$, define the subspaces $M = M_l$ and $N = N_l$ of H as

$$N = \bigoplus_{k<l} E(\lambda_k)\,, \quad M = N^{\perp}\,, \quad H = M \oplus N\,.$$

For the operator $\mathcal{A}$ there are three possibilities: (a) it has no negative eigenvalues, (b) it has only one negative eigenvalue, and (c) it has two or more negative eigenvalues. What is interesting is that each of these possibilities must be dealt with differently. We shall study all of them separately.

The following notation will be used throughout the paper:

$$\|u\|_q := \left(\int_{\mathbb{R}^n} |u(x)|^q dx \right)^{1/q}, \quad \|u\| = \|u\|_2,$$

$$(u, v) = \int_{\mathbb{R}^n} u(x)v(x)\, dx, \quad a(u, v) = (\mathcal{A}u, v), \quad a(u) = a(u, u).$$

Let q be any number satisfying

$$\begin{aligned} 2 < q &\leq 2n/(n-2), \quad n > 2 \\ 2 < q &< \infty, \quad n \leq 2 \end{aligned} \tag{3.1.3}$$

and let $f(x, t)$ be a Carathéodory function on $\mathbb{R}^n \times \mathbb{R}$. This means that $f(x, t)$ is continuous in t for a.e. $x \in \mathbb{R}^n$ and measurable in x for every $t \in \mathbb{R}$. We make the following assumptions
(A) The function $f(x, t)$ satisfies

$$|f(x, t)| \leq S(x)^q (|t|^{q-1} + W(x))$$

and

$$f(x, t)/S(x)^q = o\left(|t|^{q-1}\right) \text{ as } |t| \to \infty,$$

where $S(x) > 0$ is a function in $L^q(\mathbb{R}^n)$ satisfying

$$\|Su\|_q \leq C\|u\|_H, \quad u \in H,$$

and W is a function in $L^\infty(\mathbb{R}^n)$. Here

$$\|u\|_q := \left(\int_{\mathbb{R}^n} |u(x)|^q dx\right)^{1/q}.$$

Our other hypotheses depend only on the primitive

$$F(x,t) = \int_0^t f(x,s)\,ds$$

of $f(x,t)$.

Let

$$G(u) = (\mathcal{A}u, u) - 2\int_{\mathbb{R}^n} F(x,u)\,dx, \quad u \in H. \tag{4}$$

It follows that G is a continuously differentiable functional on the whole of H (cf., e.g., [25]). It is easily checked that $u \in H$ is a (weak) solution of (1) iff it is a critical point of $G(u)$. Our methods will make use of this fact. They involve finding linking sets A, B which separate G, i.e., are such that

$$\sup_A G \le \inf_B G.$$

If the spectrum of $\mathcal{A}$ has no gap, i.e., consists only of the interval $[0,\infty)$, the choice of the sets A, B is very limited. If it has one gap, the subspace in the gap is very useful. If it has two gaps, we obtain two useful subspaces. In our situation, if there are no negative eigenvalues, there are no gaps. If there is one negative eigenvalue, there is one gap. If there are two or more negative eigenvalues, there are at least two gaps. Therefore we consider three different situations.

2 The space N_1

We let N_1 be the set of measurable functions $h(x)$ on $\mathbb{R}^n$ satisfying

$$\sup_u \frac{\|hu\|}{\|u\|_H} < \infty.$$

It is clear that bounded functions are in N_1. It can be shown that $L^p(\mathbb{R}^n) \subset N_1$ for $p \ge n$. For a general description of this space, cf. [21]. These functions will be used in our theorems because of the following properties:

Lemma 1 *If $g^{-1} \in N_1$, the following statements are true:*

1. *there is a constant C such that*

$$\|u\|_H \le C\|u\|'_H, \quad u \in H, \tag{5}$$

where

$$(\|u\|'_H)^2 = \|\nabla u\|^2 + \|gu\|^2.$$

2. *If $\mathcal{A}$ has no negative eigenvalues, then there is an $\varepsilon > 0$ such that*

$$a(u) + \|gu\|^2 \ge \varepsilon\|u\|_H^2, \quad u \in H. \tag{6}$$

3. *If $\mathcal{A}$ has only one negative eigenvalue λ_0, then there is an $\varepsilon > 0$ such that*
$$a(u) + \|gu\|^2 \geq \varepsilon \|u\|_H^2, \quad u \in M. \tag{7}$$
4. *If $\mathcal{A}$ has negative eigenvalues λ_{l-1}, λ_l, then there is an $\varepsilon > 0$ such that*
$$a(u) - \lambda_l \|u\|^2 + \|gu\|^2 \geq \varepsilon \|u\|_H^2, \quad u \in M. \tag{8}$$

If $g \in N_1$, then there is a constant C such that
$$\|u\|'_H \leq C\|u\|_H, \quad u \in H. \tag{9}$$
If $g, g^{-1} \in N_1$, then the two norms are equivalent.

Lemma 1 will be proved in Sect. 8.

3 No negative eigenvalues

In this case,
$$a(u) = (\mathcal{A}u, u) \geq 0, \quad u \in H.$$
There are no gaps in the spectrum of $\mathcal{A}$.

We have

Theorem 2 *Assume*

1. *There is a function $g(x)$ such that $g, g^{-1} \in N_1$ and*
$$2F(x, u) \leq -g(x)^2|u|^2 + W(x), \quad u \in \mathbb{R},\ x \in \mathbb{R}^n,$$
where $W(x) \in L^1(\mathbb{R}^n)$.

Then the Eq. (1) has a solution.

Remark 3 It is clear from the equation that the solution obtained will be nontrivial if
$$f(x, 0) \neq 0.$$

To guarantee that a solution will be nontrivial even when $f(x, 0) = 0$, we have

Theorem 4 *Assume*

1. *There are constants $0 < \alpha < 2$, $\delta > 0$, and a function $g(x)$ such that $g, g^{-1} \in N_1$ and*
$$\begin{aligned} 2F(x, u) &\leq -g(x)^2|u|^2, \quad u \in \mathbb{R},\ x \in \mathbb{R}^n,\ |u| < \delta, \\ &\leq -g(x)^2|u|^2 + S(x)^q|u|^\alpha, \quad u \in \mathbb{R},\ x \in \mathbb{R}^n,\ |u| > \delta. \end{aligned}$$
2. *There is a locally bounded function $h(t)$ such that*
$$2F(x, u) \geq -h(u), \quad u \in \mathbb{R},\ x \in \mathbb{R}^n,$$
and
$$c_0 = \sup_{\mathbb{R}} h(u)/u^2 < \infty.$$

Then the Eq. (1) has a nontrivial solution.

4 Only one negative eigenvalue

Let $\lambda_0 < 0$ be the eigenvalue. In this case

$$a(u) \geq \lambda_0 \|u\|^2, \quad u \in H.$$

We can make use of the fact that there is a gap in the spectrum of $\mathcal{A}$. We have

Theorem 5 *Assume*

1.
$$2F(x,u) \geq \lambda_0 |u|^2, \quad u \in \mathbb{R}, \ x \in \mathbb{R}^n.$$

2. *There are constants* $0 < \alpha < 2, \delta > 0$, *and a function* $g(x)$ *such that* $g, g^{-1} \in N_1$ *and*
$$\begin{aligned} 2F(x,u) &\leq -g(x)^2|u|^2, \quad u \in \mathbb{R}, \ x \in \mathbb{R}^n, \ |u| < \delta, \\ &\leq -g(x)^2|u|^2 + S(x)^q|u|^\alpha, \quad u \in \mathbb{R}, \ x \in \mathbb{R}^n, \ |u| > \delta. \end{aligned}$$

Then the Eq. (1) *has a solution.*

To obtain a nontrivial solution we have

Theorem 6 *Assume*

1.
$$2F(x,u) \geq \lambda_0 |u|^2, \quad u \in \mathbb{R}, \ x \in \mathbb{R}^n.$$

2. *There are constants* $0 < \alpha < 2, \delta > 0$, *and a function* $g(x)$ *such that* $g, g^{-1} \in N_1$ *and*
$$\begin{aligned} 2F(x,u) &\leq -g(x)^2|u|^2, \quad u \in \mathbb{R}, \ x \in \mathbb{R}^n, \ |u| < \delta, \\ &\leq -g(x)^2|u|^2 + S(x)^q|u|^\alpha, \quad u \in \mathbb{R}, \ x \in \mathbb{R}^n, \ |u| > \delta. \end{aligned}$$

3. *There is a locally bounded function* $h(t)$ *such that*
$$2F(x,u) \geq -h(u), \quad u \in \mathbb{R}, \ x \in \mathbb{R}^n,$$
and
$$c_0 = \sup_{\mathbb{R}} h(u)/u^2 < \infty.$$

Then the Eq. (1) *has a nontrivial solution.*

5 Two or more negative eigenvalues

Here again we can make use of the fact that there is more than one gap in the spectrum. This allows us to use more complicated linking methods.

Let $\lambda_{l-1}, \ \lambda_l$ be two consecutive negative eigenvalues of $\mathcal{A}$. We have

Theorem 7 *Assume*

1.
$$2F(x,u) \geq \lambda_{l-1} |u|^2, \quad u \in \mathbb{R}, \ x \in \mathbb{R}^n,$$

2. *There are constants* $0 < \alpha < 2, \delta > 0$, *and a function* $g(x)$ *such that* $g, g^{-1} \in N_1$ *and*

$$2F(x,u) \le -g(x)^2|u|^2 + \lambda_l|u|^2, \quad u \in \mathbb{R},\ x \in \mathbb{R}^n,\ |u| < \delta,$$
$$\le -g(x)^2|u|^2 + \lambda_l|u|^2 + S(x)^q|u|^\alpha, \quad u \in \mathbb{R},\ x \in \mathbb{R}^n,\ |u| > \delta.$$

Then the Eq. (1) *has a solution.*

To obtain a nontrivial solution we have

Theorem 8 *Assume*

1.
$$2F(x,u) \ge \lambda_{l-1}|u|^2, \quad u \in \mathbb{R},\ x \in \mathbb{R}^n,$$

2. *There are constants* $0 < \alpha < 2, \delta > 0$, *and a function* $g(x)$ *such that* $g, g^{-1} \in N_1$ *and*

$$2F(x,u) \le -g(x)^2|u|^2 + \lambda_l|u|^2, \quad u \in \mathbb{R},\ x \in \mathbb{R}^n,\ |u| < \delta,$$
$$\le -g(x)^2|u|^2 + \lambda_l|u|^2 + S(x)^q|u|^\alpha, \quad u \in \mathbb{R},\ x \in \mathbb{R}^n,\ |u| > \delta.$$

3. *There is a locally bounded function* $h(t)$ *such that*

$$2F(x,u) \ge -h(u), \quad u \in \mathbb{R},\ x \in \mathbb{R}^n,$$

and

$$c_0 = \sup_{\mathbb{R}} h(u)/u^2 < \infty.$$

Then the Eq. (1) *has a nontrivial solution.*

Remark 9 Note that the hypothesis of Theorem 2 is stronger than hypotheses 1 and 2 of Theorem 5. Hypothesis 1 of Theorem 5 requires

$$2F(x,u) \ge \lambda_l|u|^2, \quad u \in \mathbb{R},\ x \in \mathbb{R}^n,$$

which is stronger than hypothesis 1 of Theorem 7. Consequently, the hypotheses of Theorem 4 are stronger than those of Theorem 6 which are stronger than those of Theorem 8.

6 Least energy solutions

Let $\mathcal{M}$ be the set of all solutions of (1). A solution $\tilde{u}$ is called a "least energy solution" if it minimizes the functional

$$G(u) = a(u) - 2\int_{\mathbb{R}^n} F(x,u)\,dx \tag{10}$$

over the set $\mathcal{M}$.

We have

Theorem 10 *If we add the following hypothesis to Theorems 2–8, then Eq.* (1) *has a least energy solution: The function given by*

$$H(x,u) = uf(x,u) - 2F(x,u) \tag{11}$$

satisfies

$$H(x,u) \ge -W(x) \in L^1(\mathbb{R}^n), \quad u \in \mathbb{R},\ x \in \mathbb{R}^n. \tag{12}$$

We shall prove Theorems 2–10 in Sect. 8. In the next section we describe the construction of the operator $\mathcal{A}$. We obtain the largest self-adjoint extension of $\mathcal{A}_0$ which preserves the essential spectrum.

7 The operator $\mathcal{A}$

The following was proved in [21] (Theorem 10.9, ch. 6, p. 153.)

Theorem 11 *Let P(D) be an elliptic constant coefficient operator of order 2 on $\mathbb{R}^n$, and let V(x) be a function satisfying (V_1) and (V_2). If $\rho(P_0)$ is not empty, then P(D)+V has a forms extension operator $\mathcal{A}$ such that $\sigma_e(\mathcal{A}) = \sigma_e(P_0)$.*

Here P_0 is the closure of the operator $P(D)$ restricted to $C_0^\infty(\mathbb{R}^n)$, and σ_e is the essential spectrum. Any point not in the essential spectrum is either a point in the resolvent or an isolated eigenvalue of finite multiplicity. If $P(D) = -\Delta$, then $\sigma(P_0) = [0, \infty)$. Consequently, in our case, the negative spectrum of $\mathcal{A}$ can have at most a countable number of isolated eigenvalues of finite multiplicity having a finite lower bound. The theorem is proved by showing that the bilinear form

$$a(u, v) = (P(D)u, v) + (Vu, v), \quad u, v \in H$$

is bounded and closed on H. Moreover, if the coefficients of $P(D)$ and the function $V(x)$ are real, then the bilinear form is symmetric and $\mathcal{A}$ is selfadjoint.

8 Proof of Lemma 1.

Proof We have

$$(u, u) = (gu, g^{-1}u) \leq \|gu\| \cdot \|g^{-1}u\| \leq C\|gu\| \cdot \|u\|_H \leq \varepsilon\|u\|_H^2 + K_\varepsilon\|gu\|^2.$$

Hence,

$$\|u\|_H^2 \leq \varepsilon\|u\|_H^2 + \|\nabla u\|^2 + K_\varepsilon\|gu\|^2.$$

To prove (6), assume that there is a sequence $u^{(k)} \in H$ such that $\|u^{(k)}\|'_H = 1$ and $a(u^{(k)}) + \|gu^{(k)}\|^2 \to 0$, where

$$(\|u\|'_H)^2 = \|\nabla u\|^2 + \|gu\|^2.$$

Since

$$\rho_k = \|u^{(k)}\|_H \leq C,$$

there is a renamed subsequence such that $u^{(k)}$ converges to a limit $u \in H$ weakly in H, strongly in $L^2_{loc}(\mathbb{R}^n)$ and a.e. in $\mathbb{R}^n$. Since $a(u^{(k)}) \geq 0$, we have $a(u^{(k)}) \to 0$ and $\|gu^{(k)}\| \to 0$. By our hypothesis on $V(x)$, there is a renamed subsequence of $u^{(k)}$ such that $b(u^{(k)}) \to b(u)$, where $b(u) = (Vu, u)$. Thus, $1 + b(u) = 0$, showing that $u \neq 0$. Since $gu^{(k)} \to gu$ a.e., we have $u = 0$, providing a contradiction. The same proof can be used to prove (7) if we keep in mind that (7) holds only on M. To prove (8), assume that there is a sequence $u^{(k)} \in M$ such that $\|u^{(k)}\|'_H = 1$ and $a(u^{(k)}) - \lambda_l\|u^{(k)}\|^2 + \|gu^{(k)}\|^2 \to 0$. Since

$$\rho_k = \|u^{(k)}\|'_H \leq C,$$

there is a renamed subsequence such that $u^{(k)}$ converges to a limit $u \in H$ weakly in H, strongly in $L^2_{loc}(\mathbb{R}^n)$ and a.e. in $\mathbb{R}^n$. Also, there is a renamed subsequence such that $\|\nabla u^{(k)}\| \to \nu$ and $\|gu^{(k)}\| \to \tau$, where $\nu^2 + \tau^2 = 1$. Since $a(u^{(k)}) - \lambda_l \|u^{(k)}\|^2 \geq 0$ in M, we have $a(u^{(k)}) - \lambda_l \|u^{(k)}\|^2 \to 0$ and $\|gu^{(k)}\| \to 0$. By our hypothesis on $V(x)$, there is a renamed subsequence of $u^{(k)}$ such that $b(u^{(k)}) \to b(u)$, where $b(u) = (Vu, u)$. Since $a(u^{(k)}) - \lambda_l \|u^{(k)}\|^2 \to 0$, and $a(u^{(k)}) \to \nu^2 + b(u)$, we see that $\nu^2 + b(u) \leq 0$. Since $gu^{(k)} \to gu$ a.e. and $\|gu^{(k)}\| \to 0$, we have $\tau = 0$ and $u = 0$. But then, $\nu = 1$ and $1 + b(u) \leq 0$, showing that $u \neq 0$, providing a contradiction. □

9 Proofs of the Theorems

We now give the proof of Theorem 2.

Proof We define

$$G(u) = a(u) - 2\int_{\mathbb{R}^n} F(x, u(x))\, dx, \quad u \in H. \tag{13}$$

By Lemma 1 there is an $\varepsilon > 0$ such that

$$a(u) + \|gu\|^2 \geq \varepsilon \|u\|_H^2, \quad u \in H. \tag{14}$$

Thus

$$G(u) \to \infty, \quad \|u\|_H \to \infty, \tag{15}$$

i.e., $G(u)$ is coercive. Let

$$c = \inf_H G.$$

By Corollary 3.22, p. 29, of [24] there is a sequence $\{u^{(k)}\} \subset H$ such that

$$G(u^{(k)}) = a(u^{(k)}) - 2\int_{\mathbb{R}^n} F\left(x, u^{(k)}(x)\right) dx \to c, \tag{16}$$

$$\left(G'(u^{(k)}), z\right)/2 = a\left(u^{(k)}, z\right) - \int_{\mathbb{R}^n} f\left(x, u^{(k)}\right) \cdot z(x)\, dx \to 0, \quad z \in H \tag{17}$$

and

$$\left(G'(u^{(k)}), u^{(k)}\right)/2 = a\left(u^{(k)}\right) - \int_{\mathbb{R}^n} f(x, u^{(k)}) \cdot u^{(k)}\, dx \to 0. \tag{18}$$

Since $G(u)$ is coercive,

$$\rho_k = \|u^{(k)}\|'_H \leq C,$$

where

$$(\|u\|'_H)^2 = \|\nabla u\|^2 + \|gu\|^2.$$

Thus there is a renamed subsequence such that $u^{(k)}$ converges to a limit $u \in H$ weakly in H, strongly in $L^2_{loc}(\mathbb{R}^n)$ and a.e. in $\mathbb{R}^n$. From (17) we see that

$$(G'(u), z)/2 = a(u, z) - \int_{\mathbb{R}^n} f(x, u(x)) \cdot z(x)\, dx = 0, \quad z \in C_0^\infty(\mathbb{R}^n),$$

from which we conclude easily that u is a solution of (1). □

Proof of Theorem 4 By Lemma 1, there is an $\varepsilon > 0$ such that

$$a(u) + \|gu\|^2 \geq \varepsilon \|u\|_H^2, \quad u \in H. \tag{19}$$

Consequently,

$$G(u) \geq \varepsilon \|u\|_H^2 - \int_{|u|>\delta} S(x)^q |u(x)|^\alpha \geq \varepsilon \|u\|_H^2 - C\|u\|_H^q$$

by Hypothesis (A). As a consequence, there are positive constants η, ρ such that

$$G(u) \geq \eta, \quad \|u\|_H = \rho. \tag{20}$$

Since $0 \in \sigma_e(\mathcal{A})$, there is a $\varphi \in H$ such that $\|\varphi\| = 1$ and $a(\varphi) = (\mathcal{A}\varphi, \varphi) < \eta/2$. Consequently,

$$G(s\varphi) = s^2 a(\varphi) - 2\int F(x, s\varphi) \leq s^2\eta/2 + \int h(s\varphi).$$

Thus,

$$\limsup_{s\to\infty} G(s\varphi)/s^2 \leq \eta/2 + C,$$

since

$$\limsup_{s\to\infty} \int \frac{h(s\varphi)}{s^2\varphi^2}\varphi^2 \leq C\|\varphi\|^2.$$

This implies that there is a sequence $u^{(k)}$ in H such that

$$G(u^{(k)}) \to c \geq \eta/2, \quad G'(u^{(k)})/(\|u^{(k)}\|_H + 1)^2 \to 0 \tag{21}$$

(Theorem 2.7.1 of [22]).

If

$$\rho_k = \|u^{(k)}\|'_H \leq C,$$

there is a renamed subsequence such that $u^{(k)}$ converges to a limit $u \in H$ weakly in H, strongly in $L^2_{loc}(\mathbb{R}^n)$ and a.e. in $\mathbb{R}^n$. In particular we have

$$\left(G'(u^{(k)}), z\right)/2 = a\left(u^{(k)}, z\right) - \int f\left(x, u^{(k)}\right) \cdot z(x)\,dx \to 0, \quad z \in H. \tag{22}$$

From this we see that

$$\left(G'(u), z\right)/2 = a(u, z) - \int f(x, u(x)) \cdot z(x)\,dx = 0, \quad z \in C_0^\infty(\mathbb{R}^n),$$

from which we conclude easily that u is a solution of (1). Moreover, since $\|u^{(k)}\|'_H$ is bounded, (21) implies $G(u) = c \geq \eta/2$ (Theorem 3.4.1 of [22]). Since, $G(0) = 0$, we see that $u \neq 0$.

If

$$\rho_k = \|u^{(k)}\|'_H \to \infty,$$

let $\tilde{u}^{(k)} = u^{(k)}/\rho_k$. Then, $\|\tilde{u}^{(k)}\|'_H = 1$. There is a renamed subsequence such that $\tilde{u}^{(k)}$ converges to a function $\tilde{u}(x) \in H$ weakly in H, strongly in $L^2_{loc}(\mathbb{R}^n)$, a.e. in $\mathbb{R}^n$, and such that $\|\nabla\tilde{u}^{(k)}\| \to \nu$ and $\|g\tilde{u}^{(k)}\| \to \tau$, where $\nu^2 + \tau^2 = 1$. Let $b(u, v) = (Vu, v)$, $b(u) = b(u, u)$.

Since $\tilde{u}^{(k)}$ converges to $\tilde{u}$ weakly in H, we see that $b(\tilde{u}^{(k)}) \to b(\tilde{u})$. Hence, $\|\nabla\tilde{u}^{(k)}\|^2 + b(\tilde{u}^{(k)}) + \|g\tilde{u}^{(k)}\|^2 \to \nu^2 + b(\tilde{u}) + \tau^2$. Note that

$$\int S^q|u|^\alpha \le \left(\int |Su|^{\alpha\cdot(q/\alpha)}\right)^{(\alpha/q)} \left(\int S^{(q-\alpha)\cdot q/(q-\alpha)}\right)^{(q-\alpha)/q} \le \|Su\|_q^\alpha \cdot \|S\|_q^{q-\alpha}.$$

Thus,

$$0 \leftarrow G(u^{(k)})/\rho_k^2 \ge \|\nabla\tilde{u}^{(k)}\|^2 + b(\tilde{u}^{(k)}) + \|g\tilde{u}^{(k)}\|^2 - O(\rho_k^{\alpha-2}) \to \nu^2 + b(\tilde{u}) + \tau^2. \quad (23)$$

Thus, $1 + b(\tilde{u}) = \nu^2 + b(\tilde{u}) + \tau^2 \le 0$. This implies that $b(\tilde{u}) \ne 0$, and consequently $\tilde{u}(x) \not\equiv 0$. But $g\tilde{u}^{(k)} \to g\tilde{u}$ a.e. This means that $\tilde{u} = 0$, a contradiction. Hence, the ρ_k are bounded, and the proof is complete. □

Proof of Theorem 5 We follow the proof of Theorem 4. By Lemma 1 there is an $\varepsilon > 0$ such that (7) holds. This implies that there are positive constants η, ρ such that

$$G(u) \ge \eta, \quad \|u\|_H = \rho, \ u \in M \quad (24)$$

by the argument given there. Note that

$$G(v) = a(v) - 2\int F(x, v) = \lambda_0\|v\|^2 - 2\int F(x, v) \le 0, \quad v \in N$$

by Hypothesis 1. Define $A = M \cap \mathbf{B}_\rho, \quad B = N$, where

$$\mathbf{B}_\rho = \{u \in H : \|u\|_H < \rho\}.$$

Then A links B (Example 2, p. 38 of [22]). If $G_1 = -G$, then

$$\sup_A G_1 \le \inf_B G_1.$$

By Theorem 13.7, p. 259 of [22] or Corollary 3.22, p. 29 of [24] there is a sequence $\{u^{(k)}\} \subset H$ such that

$$G\left(u^{(k)}\right) = a\left(u^{(k)}\right) - 2\int_{\mathbb{R}^n} F\left(x, u^{(k)}(x)\right) dx \to c, \quad (25)$$

$$\left(G'(u^{(k)}), z\right)/2 = a\left(u^{(k)}, z\right) - \int_{\mathbb{R}^n} f\left(x, u^{(k)}\right) \cdot z(x)\, dx \to 0, \quad z \in H \quad (26)$$

and

$$\left(G'(u^{(k)}), u^{(k)}\right)/2 = a\left(u^{(k)}\right) - \int_{\mathbb{R}^n} f\left(x, u^{(k)}\right) \cdot u^{(k)}\, dx \to 0. \quad (27)$$

If

$$\rho_k = \|u^{(k)}\|'_H \le C,$$

where

$$\left(\|u\|'_H\right)^2 = \|\nabla u\|^2 + \|gu\|^2,$$

then there is a renamed subsequence such that $u^{(k)}$ converges to a limit $u \in H$ weakly in H, strongly in $L^2_{loc}(\mathbb{R}^n)$ and a.e. in $\mathbb{R}^n$. From (26) we see that

$$\left(G'(u), z\right)/2 = a(u, z) - \int_{\mathbb{R}^n} f(x, u(x)) \cdot z(x)\, dx = 0, \quad z \in C_0^\infty(\mathbb{R}^n),$$

from which we conclude easily that u is a solution of (1).

If

$$\rho_k = \|u^{(k)}\|'_H \to \infty,$$

let $\tilde{u}^{(k)} = u^{(k)}/\rho_k$. Then, $\|\tilde{u}^{(k)}\|'_H = 1$. There is a renamed subsequence such that $\tilde{u}^{(k)}$ converges to a function $\tilde{u}(x) \in H$ weakly in H, strongly in $L^2_{loc}(\mathbb{R}^n)$, a.e. in $\mathbb{R}^n$, and such that $\|\nabla\tilde{u}^{(k)}\| \to \nu$ and $\|g\tilde{u}^{(k)}\| \to \tau$, where $\nu^2 + \tau^2 = 1$. Let $b(u, v) = (Vu, v),\ b(u) = b(u, u)$. Since $\tilde{u}^{(k)}$ converges to $\tilde{u}$ weakly in H, we see that $b(\tilde{u}^{(k)}) \to b(\tilde{u})$. Hence, $0 \le \|\nabla\tilde{u}^{(k)}\|^2 + b(\tilde{u}^{(k)}) + \|g\tilde{u}^{(k)}\|^2 \to \nu^2 + b(\tilde{u}) + \tau^2$. Since $a(\tilde{u}^{(k)}) \ge 0$, we have $\nu^2 + b(\tilde{u}) \ge 0$. Note that

$$\int S^q|u|^\alpha \le \left(\int |Su|^{\alpha\cdot(q/\alpha)}\right)^{(\alpha/q)} \left(\int S^{(q-\alpha)\cdot q/(q-\alpha)}\right)^{(q-\alpha)/q} \le \|Su\|_q^\alpha \cdot \|S\|_q^{q-\alpha}.$$

Moreover,

$$G\left(u^{(k)}\right)/\rho_k^2 = \|\nabla\tilde{u}^{(k)}\|^2 + b\left(\tilde{u}^{(k)}\right) + \|g\tilde{u}^{(k)}\|^2 + O\left(\rho_k^{\alpha-2}\right) \to \nu^2 + b(\tilde{u}) + \tau^2 = 0.$$

This implies $\nu^2 + b(\tilde{u}) = 0$ and $\tau^2 = 0$. But $g\tilde{u}^{(k)} \to g\tilde{u}$ a.e. This means that $\tilde{u} = 0$ and $\nu = 1$, so that

$$1 + b(\tilde{u}) = 0.$$

But this implies that $b(\tilde{u}) \ne 0$, and consequently $\tilde{u}(x) \not\equiv 0$, a contradiction. Hence, the ρ_k are bounded, and the proof is complete. □

Proof of Theorem 6 We follow the proof of Theorem 4. By Lemma 1 there is an $\varepsilon > 0$ such that (7) holds. This implies that there are positive constants η, ρ such that

$$G(u) \ge \eta, \quad \|u\|_H = \rho,\ u \in M \tag{28}$$

as in the proof of Theorem 4. Note that

$$G(v) = a(v) - 2\int F(x, v) = \lambda_0\|v\|^2 - 2\int F(x, v) \le 0, \quad v \in N$$

by Hypothesis 1. Since $\mathcal{A}$ is not invertible on M, there is a $\varphi \in M$ such that $\|\varphi\| = 1$ and $a(\varphi) = (\mathcal{A}\varphi, \varphi) < \eta$. For $R > \rho$, let

$$\begin{aligned} A_R &= [N \cap \mathbf{B}_R] \cup \{v + s\varphi : s \ge 0,\ \|v + s\varphi\| = R\} \\ B &= M \cap \partial\mathbf{B}_\rho, \end{aligned}$$

where

$$\mathbf{B}_\rho = \{u \in H : \|u\|_H < \rho\}.$$

By Example 3, p.38 of [22], $A_R,\ B$ link each other. Now

$$\begin{aligned} G(v + s\varphi) &\le a(v) + s^2\eta + \int \frac{h(v + s\varphi)}{(v + s\varphi)^2}(v + s\varphi)^2 \\ &\le \lambda_0\|v\|^2 + s^2\eta + c_0\|v + s\varphi\|^2 \\ &\le (c_0 + \lambda_0)\|v\|^2 + (c_0 + \eta)s^2 \\ &\le (2c_0 + \lambda_0 + \eta)R^2. \end{aligned}$$

We can now apply Theorem 2.7.3 of [22] to conclude that there is a sequence $u^{(k)}$ in H such that

$$G\left(u^{(k)}\right) \to c \geq \eta, \quad G'\left(u^{(k)}\right)/\left(\|u^{(k)}\|_H + 1\right)^2 \to 0. \tag{29}$$

We can now follow the proofs of Theorem 5 to reach the desired conclusion. □

Proof of Theorem 7 Define the subspaces M and N of H as before:

$$N = \bigoplus_{k<l} E(\lambda_k)\,, \quad M = N^{\perp}\,, \quad H = M \oplus N\,.$$

Let

$$G(u) = a(u) - 2\int_{\mathbb{R}^n} F(x,u)\,dx. \tag{30}$$

We note that Hypothesis 1 implies

$$G(v) \leq 0, \quad v \in N. \tag{31}$$

In fact, we have

$$G(u) = a(u) - 2\int_{\mathbb{R}^n} F(x,u)\,dx \leq \int_{\mathbb{R}^n} \left[\lambda_{l-1}u^2 - 2f(x,u)\right]dx \leq 0, \quad u \in N.$$

In view of inequality (8), we see that there are positive constants η, ρ such that

$$G(u) \geq \eta, \quad \|u\|_H = \rho,\ u \in M. \tag{32}$$

Take

$$A = \partial\mathbf{B}_\rho \cap M,$$
$$B = N,$$

where

$$\mathbf{B}_\rho = \{u \in H : \|u\|_H < \rho\}.$$

By Example 8, p. 22 of [24], A links B. Moreover,

$$\sup_A[-G] \leq 0 \leq \inf_B[-G]. \tag{33}$$

Hence, we may apply Corollary 2.8.2 of [22] to conclude that that there is a sequence $\{u^{(k)}\} \subset H$ such that

$$G\left(u^{(k)}\right) = a\left(u^{(k)}\right) - 2\int_{\mathbb{R}^n} F\left(x, u^{(k)}(x)\right)dx \to c \leq 0, \tag{34}$$

$$\left(G'(u^{(k)}), z\right)/2 = a\left(u^{(k)}, z\right) - \int_{\mathbb{R}^n} f\left(x, u^{(k)}\right)\cdot z(x)\,dx \to 0, \quad z \in H \tag{35}$$

and

$$\left(G'(u^{(k)}), u^{(k)}\right)/2 = a\left(u^{(k)}\right) - \int_{\mathbb{R}^n} f\left(x, u^{(k)}\right)\cdot u^{(k)}\,dx \to 0. \tag{36}$$

If

$$\rho_k = \|u^{(k)}\|'_H \leq C,$$

there is a renamed subsequence such that $u^{(k)}$ converges to a limit $u \in H$ weakly in H strongly in $L^2_{loc}(\mathbb{R}^n)$ and a.e. in $\mathbb{R}^n$. From (35) we see that

$$\left(G'(u), z\right)/2 = a(u, z) - \int_{\mathbb{R}^n} f(x, u(x)) \cdot z(x)\, dx = 0, \quad z \in C_0^\infty(\mathbb{R}^n),$$

from which we conclude easily that u is a solution of (1). We can now follow the proof of Theorem 4 until we reach (23). In our case this becomes

$$G\left(u^{(k)}\right)/\rho_k^2 \geq \|\nabla\tilde{u}^{(k)}\|^2 + b\left(\tilde{u}^{(k)}\right) + \|g\tilde{u}^{(k)}\|^2 - \lambda_l\|\tilde{u}^{(k)}\|^2 - O\left(\rho_k^{\alpha-2}\right). \tag{37}$$

Since $\lambda_l \leq 0$,

$$G\left(u^{(k)}\right)/\rho_k^2 \to 0,$$

and

$$\|\nabla\tilde{u}^{(k)}\|^2 + b(\tilde{u}^{(k)}) + \|g\tilde{u}^{(k)}\|^2 \to \nu^2 + b(\tilde{u}) + \tau^2,$$

we again have $1 + b(\tilde{u}) \leq 0$, which leads to the desired conclusion. □

Proof of Theorem 8 We follow the proof of Theorem 7. We note that Hypothesis 1 implies

$$G(v) \leq 0, \quad v \in N. \tag{38}$$

In fact, we have

$$G(u) = a(u) - 2\int_{\mathbb{R}^n} F(x, u)\, dx \leq \int_{\mathbb{R}^n} \left[\lambda_{l-1}u^2 - 2f(x, u)\right] dx \leq 0, \quad u \in N.$$

In view of inequality (8), we see that there are positive constants η, ρ such that

$$G(u) \geq \eta, \quad \|u\|_H = \rho, \ u \in M. \tag{39}$$

Let φ be an eigenfunction of $\mathcal{A}$ corresponding to λ_l and satisfying $\|\varphi\| = 1$. For $R > \rho$, let

$$\begin{aligned} A_R &= [N \cap \mathbf{B}_R] \cup \{v + s\varphi : s \geq 0, \ \|v + s\varphi\| = R\} \\ B &= M \cap \partial\mathbf{B}_\rho, \end{aligned}$$

where

$$\mathbf{B}_\rho = \{u \in H : \|u\|_H < \rho\}.$$

By Example 3, p.38 of [22], A_R, B link each other. Now

$$\begin{aligned} G(v + s\varphi) &\leq a(v) + s^2\lambda_l + \int \frac{h(v + s\varphi)}{(v + s\varphi)^2}(v + s\varphi)^2 \\ &\leq \lambda_{l-1}\|v\|^2 + s^2\lambda_l + c_0\|v + s\varphi\|^2 \\ &\leq (c_0 + \lambda_{l-1})\|v\|^2 + (c_0 + \lambda_l)s^2 \\ &\leq (2c_0 + \lambda_{l-1} + \lambda_l)R^2. \end{aligned}$$

We can now apply Theorem 2.7.3 of [22] to conclude that there is a sequence $u^{(k)}$ in H such that

$$G\left(u^{(k)}\right) \to c \geq \eta/2, \quad G'\left(u^{(k)}\right)/\left(\|u^{(k)}\|_H + 1\right)^2 \to 0. \tag{40}$$

We can now follow the proof of Theorem 4 until we reach (23). In our case this becomes

$$G\left(u^{(k)}\right)/\rho_k^2 \geq \|\nabla\tilde{u}^{(k)}\|^2 + b\left(\tilde{u}^{(k)}\right) + \|g\tilde{u}^{(k)}\|^2 - \lambda_l\|\tilde{u}^{(k)}\|^2 - O\left(\rho_k^{\alpha-2}\right). \tag{41}$$

Since $\lambda_l \leq 0$,

$$G\left(u^{(k)}\right)/\rho_k^2 \to 0,$$

and

$$\|\nabla\tilde{u}^{(k)}\|^2 + b\left(\tilde{u}^{(k)}\right) + \|g\tilde{u}^{(k)}\|^2 \to \nu^2 + b(\tilde{u}) + \tau^2,$$

we again have $1 + b(\tilde{u}) \leq 0$, which leads to the desired conclusion. □

Proof of Theorem 10. Let

$$\alpha = \inf_{\mathcal{M}} G(u).$$

There is a sequence $\{u^{(k)}\} \in \mathcal{M}$ such that

$$G\left(u^{(k)}\right) = a\left(u^{(k)}\right) - 2\int_{\mathbb{R}^n} F\left(x, u^{(k)}(x)\right) dx \to \alpha, \tag{42}$$

$$\left(G'(u^{(k)}), z\right)/2 = a\left(u^{(k)}, z\right) - \int_{\mathbb{R}^n} f\left(x, u^{(k)}\right) \cdot z(x)\, dx = 0, \quad z \in H \tag{43}$$

and

$$\left(G'(u^{(k)}), u^{(k)}\right)/2 = a\left(u^{(k)}\right) - \int_{\mathbb{R}^n} f\left(x, u^{(k)}\right) \cdot u^{(k)}\, dx = 0. \tag{44}$$

Thus,

$$\int_{\mathbb{R}^n} H\left(x, u^{(k)}(x)\right) dx = G\left(u^{(k)}\right) \to \alpha.$$

In view of assumption (11), we see that $\alpha \geq -\int W > -\infty$. In view of the arguments given in the proofs of Theorems 2–8, we see that

$$\rho_k = \|u^{(k)}\|_H \leq C.$$

Hence, there is a renamed subsequence such that $u^{(k)}$ converges to a limit $u \in H$, weakly in H, strongly in $L^2_{loc}(\mathbb{R}^n)$ and a.e. in $\mathbb{R}^n$. From (43) we see that

$$\left(G'(u), z\right)/2 = a(u, z) - \int_{\mathbb{R}^n} f(x, u(x)) \cdot z(x)\, dx = 0, \quad z \in C_0^\infty(\mathbb{R}^n),$$

from which we conclude easily that u is a solution of (1). Hence, $u \in \mathcal{M}$. Moreover,

$$G(u) = \int_{\mathbb{R}^n} H(x, u(x))\, dx \leq \liminf \int_{\mathbb{R}^n} H(x, u^{(k)}(x))\, dx = \liminf G(u^{(k)}) = \alpha.$$

Thus, $G(u) = \alpha$. If $\alpha \neq 0$, then $u \neq 0$. This completes the proof.

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