Calc. Var. (2017) 56:6
DOI 10.1007/s00526-016-1107-3

Calculus of Variations

Spectral density estimates with partial symmetries and an application to Bahri–Lions-type results

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Received: 4 August 2016 / Accepted: 28 November 2016 / Published online: 5 January 2017

Abstract We are concerned with the existence of infinitely many solutions for the problem $-\Delta u = |u|^{p-2}u + f$ in Ω, $u = u_0$ on $\partial\Omega$, where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 3$. This can be seen as a perturbation of the problem with $f = 0$ and $u_0 = 0$, which is odd in u. If Ω is invariant with respect to a closed strict subgroup of $O(N)$, then we prove infinite existence for all functions f and u_0 in certain spaces of invariant functions for a larger range of exponents p than known before. In order to achieve this, we prove Lieb–Cwikel–Rosenbljum-type bounds for invariant potentials on Ω, employing improved Sobolev embeddings for spaces of invariant functions.

To the memory of Abbas Bahri.

Communicated by P. Rabinowitz.

This research was partially supported by CONACYT Grant 237661 and UNAM-DGAPA-PAPIIT Grant IN104315 (Mexico).

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1 Introduction

The question of existence of infinitely many solutions for the boundary value problem

$$\begin{cases} -\Delta u = |u|^{p-2}u + f, & \text{in } \Omega, \\ u = u_0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

is a classical area of research in the theory of semilinear elliptic equations. We suppose that the domain $\Omega \subseteq \mathbb{R}^N$ is bounded and of class $C^{2,\beta}$ for some $\beta \in (0, 1]$, $N \geq 3$, $p > 2$, $f \in L^2(\Omega)$, and $u_0 \in C^0(\partial\Omega)$. Equation (1.1) can be seen as a perturbation of the problem with $f = 0$ and $u_0 = 0$, for which the existence of infinitely many solutions is well known for $p < 2^*$ due to the oddness of the nonlinearity. Here we denote by $2^* := 2N/(N-2)$ the critical Sobolev exponent.

If $u_0 = 0$ (the *homogeneous boundary condition*), Bahri and Lions [1] and Tanaka [2] proved infinite existence for $2 < p < \tilde{2}_N^{\mathrm{BL}}$, where

$$\tilde{2}_N^{\mathrm{BL}} := 2 + \frac{2}{N-2}.$$

In the former article, Weyl asymptotics for the Dirichlet Laplacian on Ω are employed, and in the latter Lieb–Cwikel–Rosenbljum-type (LCR-type) bounds on the number of nonpositive eigenvalues of the linearization of (1.1) at a critical point of the unperturbed problem. This result presents, up to now, the largest general upper bound for allowed exponents p for infinite existence, and was preceded by the works of various authors, see [3–5] and the related result in [6]. Continued interest in this question manifests itself in the articles [7–15].

Under the *non-homogeneous boundary condition*, where $u_0 \neq 0$ is allowed, Bolle, Ghoussoub and Tehrani [16] established infinite existence for $2 < p < \hat{2}_N^{\mathrm{BGT}}$, where

$$\hat{2}_N^{\mathrm{BGT}} := 2 + \frac{2}{N-1},$$

see also Candela and Salvatore [17] for a previous weaker result.

On the other hand, if Ω is a ball and f is radially symmetric, infinite existence was shown by Struwe [18] for the homogeneous boundary condition, $p \in (2, 2^*)$, and $f \in L^\mu(\Omega)$ for some $\mu > N/2$, see also [19]. In fact, in this result the oddness of the nonlinearity is not needed. In a similar vein, Kazdan and Warner [20] treated annular domains. The non-homogeneous boundary condition was treated in the radial case by Candela, Palmieri and Salvatore [21], who showed infinite existence for $p \in (2, 2^*)$ if $N \geq 4$, and on unbounded domains by Barile and Salvatore [22,23].

As the results above for radially symmetric equations suggest, symmetry may improve the allowed range for p for infinite existence in (1.1). Our goal here is to analyze to what extent *partial symmetries* have this effect. To this end, suppose that G is a closed subgroup of $O(N)$ and that $\overline{\Omega}$ is G-invariant. Recall that a subset X of $\mathbb{R}^N$ is *G-invariant* if $gx \in X$ for all $x \in X$ and $g \in G$, and a function $h\colon X \to \mathbb{R}$ is *G-invariant* if $h(gx) = h(x)$ for all $x \in X$ and $g \in G$. We will also use the term *symmetric* for G-invariant subsets of $\mathbb{R}^N$ and for G-invariant functions defined on them.

In general, an extension of the exponent range is expected in the presence of symmetries. This is due to the greater sparsity and hence faster growth of *symmetric* eigenvalues (either of the Dirichlet Laplacian, or of the linearization of (1.1) at a solution), which in turn improves the lower bounds for symmetric critical values of the unperturbed problem. In what follows, by *infinite symmetric existence* we will refer to the statement that an infinite number of symmetric solutions of (1.1) exists for all f, u_0 in certain spaces of symmetric functions.

The first result in this direction was given by Clapp and Hernández-Martínez [24]. Defining

$$M := \max\{\dim Gx \mid x \in \Omega\}$$

they introduced exponents $\tilde{2}^{\text{CH}}_{N,M}$ and $\hat{2}^{\text{CH}}_{N,M}$ and showed that infinite symmetric existence holds for the homogeneous boundary condition if $p \in (2, \tilde{2}^{\text{CH}}_{N,M})$, and for the inhomogeneous boundary condition if $p \in (2, \hat{2}^{\text{CH}}_{N,M})$. These results rest on Weyl asymptotics for G-symmetric eigenvalues of the Dirichlet Laplacian in Ω obtained by Brüning and Heintze [25] and Donnelly [26]. The only other result we are aware of that also treats partially symmetric settings (although on an unbounded domain) is by Barile and Salvatore [27].

In contrast with [24], we use the variational approach to infinite existence developed by Tanaka [2] and combine it with Bolle's method [16,28] and new symmetric LCR-type bounds. Our proof of these bounds combines techniques from Li and Yau [29] and Blanchard, Stubbe and Rezende [30] with improved embeddings of $H^1_{0,G}(\Omega)$, the subspace of $H^1_0(\Omega)$ of G-symmetric functions. For the case of group actions on $\overline{\Omega}$ where every orbit is infinite, these were given by Hebey and Vaugon [31], and for certain cylindrical domains (with fixed points for the action of G) we use a result by Wang [32].

In each of the four cases we consider, we provide a bound A such that infinite symmetric existence holds if $p \in (2, A)$. Defining

$$m := \min\{\dim Gx \mid x \in \overline{\Omega}\},$$

if $0 < m \leq N - 2$ we denote by $\widetilde{A}_{N,m}$ the bound A under a homogeneous boundary condition, and by $\hat{A}_{N,m}$ the bound A under a non-homogeneous boundary condition. Note that in the latter case we impose stronger regularity conditions on u_0 and f than in some results mentioned before. In the case where $m = 0$ we only consider cylindrical domains of the form $\Omega := \Omega_1 \times B^\ell_r$, where $\Omega_1 \subseteq \mathbb{R}^k$ is a smooth bounded domain, $k \geq 1$, $r > 0$, B^ℓ_r denotes the open ball of radius r in $\mathbb{R}^\ell$, centered at 0, and $\ell \geq 2$. In this case we take G to be the orthogonal group $O(\ell)$ acting on the second factor of $\overline{\Omega}$. Under a homogeneous boundary condition the bound A is denoted by $\widetilde{B}_{k,\ell}$, and under a non-homogeneous boundary condition the bound A is denoted by $\hat{B}_{k,\ell}$.

Comparing known bounds on p with our results in the case $m > 0$ we find the following: we always have $\widetilde{A}_{N,m} > \tilde{2}^{\text{BL}}_N$, and we have $\widetilde{A}_{N,m} > \tilde{2}^{\text{CH}}_{N,M}$ if and only if

$$M - m < \frac{N}{2}\left(1 + \frac{1}{N - m - 1}\right).$$

Moreover, $\widetilde{A}_{N,m}$ is always larger than the corresponding limiting exponent in [27]. We always have $\hat{A}_{N,m} > \hat{2}^{BGT}_N$, and we have $\hat{A}_{N,m} > \hat{2}^{\text{CH}}_{N,M}$ if and only if

$$M - m < \frac{N}{2}.$$

The algebraic verification of these comparisons is straightforward, although tedious in some cases. Note that $\widetilde{A}_{N,m} > 2^*$ if and only if $m > (N-2)/2$, and $\hat{A}_{N,m} > 2^*$ if and only if $m > N/2$, i.e., in some cases we are allowing supercritical exponents p.

For the comparison of bounds for p in the case $m = 0$ with the cylindrical domains introduced above, we note that $N = k + \ell$ and $M = \ell - 1$. Then $\widetilde{B}_{k,\ell} > \max\{\tilde{2}^{BL}_N, \tilde{2}^{\text{CH}}_{N,M}\}$ and $\hat{B}_{k,\ell} > \max\{\hat{2}^{BGT}_N, \hat{2}^{\text{CH}}_{N,M}\}$, and these limiting exponents are subcritical. We proved the comparison algebraically in all cases, except when $k \geq 2$ and $\ell \geq 4$. In this case, numerical evidence strongly suggests the claim.

In our results we easily could have considered more general odd nonlinearities instead of the homogeneous function $u \mapsto |u|^{p-2}u$, as was done, e.g., in [2]. We refrained from doing so in order to keep the formulas for our limiting exponents as simple as possible, facilitating the comparison with those from other papers.

Our text is structured as follows: in Sect. 2 we recall the embedding theorems for Sobolev Spaces of symmetric functions. Sect. 3 is devoted to the proofs of LCR-type bounds on Ω for symmetric eigenvalues. In Sects. 4, 5, 6, and 7 we state and prove the Bahri–Lions-type results on infinite symmetric existence for the perturbed problem (1.1). Appendix A provides the justification for some calculations made in Sect. 3.

2 Embeddings of Sobolev Spaces with Symmetries

For any $r \geq 1$ let $L^r_G(\Omega)$ and $H^1_{0,G}(\Omega)$ denote the closure of the space of all G-symmetric functions $\varphi \in C^\infty_c(\Omega)$ with respect to the L^r- and H^1-norms, respectively. Here we provide theorems on compact embeddings of $H^1_{0,G}(\Omega)$ in $L^r_G(\Omega)$, where $r > 2^*$ if $m > 0$, and in weighted Lebesgue spaces if $m = 0$.

We introduce the *higher critical Sobolev exponents* for the exponent 2 as follows:

$$2^*_{N,k} := \begin{cases} \frac{2(N-k)}{N-k-2} & N-k \geq 3 \\ \infty & N-k = 1, 2 \end{cases}$$

Note that $2^*_{N,k} = 2^*_{N-k}$, the usual critical Sobolev exponent in dimension $N-k$.

The following embedding result improves the usual critical Sobolev exponents, assuming the presence of symmetries.

Theorem 2.1 (Hebey and Vaugon [31]) *Suppose that* $m \geq 1$.

(a) *If* $m \geq N-2$ *and* $p \geq 1$, *then* $H^1_{0,G}(\Omega)$ *embeds continuously and compactly in* $L^p(\Omega)$.
(b) *If* $m < N-2$ *and* $p \in [1, 2^*_{N,m}]$, *then* $H^1_{0,G}(\Omega)$ *embeds continuously in* $L^p(\Omega)$. *This embedding is compact if* $p < 2^*_{N,m}$.

The former theorem disallows symmetry groups with $m = 0$. In some special cases, embeddings of Sobolev spaces of symmetric functions into certain weighted Lebesgue spaces with exponents higher than the critical exponent are known even if $m = 0$. To present these, suppose that $k, \ell \in \mathbb{N}$ satisfy $\ell \geq 2$. Let $\Omega_1 \subseteq \mathbb{R}^k$ be a smooth bounded domain and set $\Omega := \Omega_1 \times B^\ell_r$, where $r > 0$ and B^ℓ_r denotes the open ball of radius r in $\mathbb{R}^\ell$, centered at 0. We write $x = (x', x'')$ for elements of $\mathbb{R}^k \times \mathbb{R}^\ell$, where $x' \in \mathbb{R}^k$ and $x'' \in \mathbb{R}^\ell$. Set $N := k + \ell$ and put $G := I \times O(\ell)$. Fix $\nu \geq 0$ and denote by $L^p(\Omega, |x''|^\nu \, \mathrm{d}x)$ the weighted L^p-space on Ω with norm

$$\|u\|_{L^p(\Omega, |x''|^\nu \, \mathrm{d}x)} := \left(\int_\Omega |u(x)|^p |x''|^\nu \, \mathrm{d}x \right)^{1/p}.$$

Moreover, set

$$\tau_{k,\ell} := \begin{cases} \frac{2}{N-2} \min\left\{ \frac{2(\ell-2)}{k}, \ \nu \right\}, & k \geq 2, \ \ell \geq 3, \\ \frac{2}{k} \min\left\{ \frac{1}{k}, \ \nu \right\}, & k \geq 2, \ \ell = 2, \\ \min\left\{ 2, \ \frac{2\nu}{\ell-1} \right\}, & k = 1, \ \ell \geq 2. \end{cases}$$

Theorem 2.2 (Wang [32]) *If* $p \in (1, 2^*_N + \tau_{k,\ell})$, *then* $H^1_{0,G}(\Omega)$ *embeds compactly in* $L^p(\Omega, |x''|^\nu \, \mathrm{d}x)$.

3 Spectral density estimates

For Schrödinger operators $-\Delta + V$ in $\mathbb{R}^N$ the classical spectral estimates of Lieb, Cwikel and Rosenbljum [33–35] state bounds on the number of nonpositive eigenvalues. If V is radially symmetric, analogous bounds were proved earlier by Bargmann [36]. See [29] and [37] for modern treatments of these classic results.

To state similar bounds for *symmetric* eigenvalues if $V \in L_G^{N/2}(\Omega)$, let $N_G(V)$ denote the dimension of the generalized eigenspace corresponding to all nonpositive eigenvalues of the operator $-\Delta + V$ in $L_G^2(\Omega)$ with Dirichlet boundary conditions.

For a real valued function f on some set X we use the notation $f_\pm := \max\{\pm f, 0\}$, so $f = f_+ - f_-$ and $f_\pm \geq 0$.

Theorem 3.1 (a) *If $N - m \geq 3$, then there is a constant $C = C(N, G, \Omega, m)$ such that*

$$N_G(V) \leq C \int_\Omega V_-^{\frac{N-m}{2}} \quad \textit{for all } V \in L_G^{N/2}(\Omega). \tag{3.1}$$

(b) *If $N - m = 2$ and $\varepsilon > 0$, then there is a constant $C = C(\varepsilon, N, G, \Omega)$ such that*

$$N_G(V) \leq C \int_\Omega V_-^{1+\varepsilon} \quad \textit{for all } V \in L_G^{N/2}(\Omega). \tag{3.2}$$

Proof Our proof is an adaptation of the proof of [29, Theorem 2], in conjunction with Theorem 2.1.

Suppose that q is a positive and G-symmetric function of class $C^{0,\beta}$ on $\overline{\Omega}$. This regularity requirement is easily fulfilled below when we take q to be an approximation of V_-. It allows to prove regularity properties of the function H, which is to be introduced shortly (see also Appendix A). Denote by μ_n the n-th G-symmetric eigenvalue of the problem

$$\begin{cases} -\Delta\psi = \mu q \psi & \text{in } \Omega \\ \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Given $r \in (2, 2^*_{N,m}]$ in case (a) and $r > 2$ in case (b) we claim that there exists a positive constant $C = C(N, G, \Omega, r)$ such that

$$\mu_n^{\frac{r}{r-2}} \int_\Omega q^{\frac{r}{r-2}} \geq Cn \qquad \text{for all } n \in \mathbb{N}. \tag{3.4}$$

To prove the claim, we follow the steps of the proof of [29, Theorem 2], working only in spaces of G-symmetric functions. In particular, we consider the function

$$H(x, y, t) := \sum_{n=1}^{\infty} \mathrm{e}^{-\mu_n t} \psi_n(x)\psi_n(y), \qquad x, y \in \Omega,\ t > 0, \tag{3.5}$$

where ψ_n is the G-symmetric eigenfunction of (3.3) corresponding to μ_n. It follows that H is G-symmetric in its first two arguments. The function H is the G-symmetric heat kernel of Δ/q in a q-weighted space of L^2-functions on Ω, see Appendix A for the definitions and properties of H that justify the following calculations.

By Theorem 2.1 and Poincare's inequality

$$\left(\int_\Omega H^r(x, y, t)\,\mathrm{d}y\right)^{\frac{1}{r}} \leq C \left(\int_\Omega |\nabla_y H(x, y, t)|^2\,\mathrm{d}y\right)^{\frac{1}{2}} \qquad \text{for all } x \in \Omega,\ t > 0. \tag{3.6}$$

On the other hand, the argument leading up to (17) in [29] implies

$$\int_\Omega q(x)\Big(\int_\Omega H(x,y,t)q(y)^{\frac{r-1}{r-2}}\,\mathrm{d}y\Big)^2\,\mathrm{d}x \le \int_\Omega q(x)^{\frac{r}{r-2}}\,\mathrm{d}x \tag{3.7}$$

for all $t \ge 0$.

For the function

$$h(t) := \sum_{n=1}^{\infty} \mathrm{e}^{-2\mu_n t} = \int_\Omega\int_\Omega H^2(x,y,t)q(x)q(y)\,\mathrm{d}x\,\mathrm{d}y \tag{3.8}$$

it holds that

$$h'(t) = -2\int_\Omega q(x)\int_\Omega |\nabla_y H(x,y,t)|^2\,\mathrm{d}y\,\mathrm{d}x, \tag{3.9}$$

see (14) in [29]. Applying Hölder's inequality twice we obtain from (3.6), (3.7) and (3.9)

$$\begin{aligned}
h(t) &\le \int_\Omega q(x)\Big(\int_\Omega H(x,y,t)^r\,\mathrm{d}y\Big)^{\frac{1}{r-1}}\Big(\int_\Omega H(x,y,t)q(y)^{\frac{r-1}{r-2}}\,\mathrm{d}y\Big)^{\frac{r-2}{r-1}}\,\mathrm{d}x\\
&\le \Big(\int_\Omega q(x)\Big(\int_\Omega H(x,y,t)^r\,\mathrm{d}y\Big)^{\frac{2}{r}}\,\mathrm{d}x\Big)^{\frac{r}{2(r-1)}}\\
&\quad\times\Big(\int_\Omega q(x)\Big(\int_\Omega H(x,y,t)q(y)^{\frac{r-1}{r-2}}\,\mathrm{d}y\Big)^2\,\mathrm{d}x\Big)^{\frac{r-2}{2(r-1)}}\\
&\le C\Big(\int_\Omega q(x)\int_\Omega|\nabla_y H(x,y,t)|^2\,\mathrm{d}y\,\mathrm{d}x\Big)^{\frac{r}{2(r-1)}}\Big(\int_\Omega q(x)^{\frac{r}{r-2}}\,\mathrm{d}x\Big)^{\frac{r-2}{2(r-1)}}\\
&= C(-h'(t))^{\frac{r}{2(r-1)}}\,I^{\frac{r-2}{2(r-1)}},
\end{aligned}$$

where we have set

$$I := \int_\Omega q(x)^{\frac{r}{r-2}}\,\mathrm{d}x.$$

The last inequality yields

$$h'(t)h(t)^{-\frac{2(r-1)}{r}} \le -CI^{-\frac{r-2}{r}},$$

which in turn yields (3.4) after integrating from 0 to $1/\mu_n$ and using $h(0) = \infty$.

One now follows the steps (i), (ii), (iv), and (v) of the proof of [29, Cor. 2], applied to (3.4), to obtain

$$N_G(V) \le C\int_\Omega V_-^{\frac{r}{r-2}} \qquad \text{for all } V \in L_G^{N/2}(\Omega).$$

In this process we approximate V_- in $L^{N/2}(\Omega)$ by positive functions $q \in C_G^{0,\beta}(\overline{\Omega})$. To prove (a) we set $r = 2^*_{N,m}$. For (b), given $\varepsilon > 0$ it is sufficient to take r large enough such that $r/(r-2) < 1+\varepsilon$. □

Consider now a domain Ω as in Theorem 2.2 and define the intervals

$$I_{k,\ell} := \begin{cases} \big[1+\frac{k}{2}, \frac{N}{2}\big] & k \ge 2,\ \ell \ge 3,\\ \Big[1+\frac{k^2}{2k+1}, \frac{N}{2}\Big] & k \ge 2,\ \ell = 2,\\ \Big[1+\frac{\ell-1}{\ell+1}, \frac{N}{2}\Big] & k = 1,\ \ell \ge 2. \end{cases}$$

Theorem 3.2 *Suppose that $\gamma \in \operatorname{int} I_{k,\ell}$. Then there is a constant*

$$C = C(G, \Omega, k, \ell, \gamma)$$

such that the following holds true: if $V \in L^{\frac{N}{2}}(\Omega)$ is G-symmetric, then

$$N_G(V) \le C \int_\Omega V_-^\gamma(x)|x''|^{2\gamma-N}\,\mathrm{d}x.$$

Proof As in the proof of Theorem 3.1, we consider a positive and G-symmetric function q of class $C^{0,\beta}$ on $\overline{\Omega}$. Define the eigenvalues μ_n of the problem (3.3) as before. We claim that there exists a positive constant $C = C(G, \Omega, k, \ell, \gamma)$ such that

$$\mu_n^\gamma \int_\Omega q^\gamma(x)|x''|^{2\gamma-N}\,\mathrm{d}x \ge Cn \qquad \text{for all } n \in \mathbb{N}. \tag{3.10}$$

To prove the claim we modify the proof of Theorem 3.1 using an idea from [30]. We define H as in (3.5) and consider $h(t)$ as in (3.8). Set

$$p := \frac{2\gamma}{\gamma-1} = 2 + \frac{2}{\gamma-1}, \qquad b := 1 - \frac{N}{2\gamma}, \qquad \nu := -bp,$$

and define $\tau_{k,\ell}$ accordingly, as in Theorem 2.2. Note that the case $\gamma = N/2$ and $b = 0$ corresponds to the proof of Theorem 3.1.

It follows from the choice of γ that $p \in (1, 2_N^* + \tau_{k,\ell})$. Now Theorem 2.2, Poincaré's inequality, and Eq. (14) of [29] yield a positive constant $C = C(G, \Omega, k, \ell, \gamma)$ such that

$$\begin{aligned}
&\int_\Omega q(x)\left(\int_\Omega |y''|^{-bp}|H(x,y,t)|^p\,\mathrm{d}y\right)^{\frac{2}{p}}\mathrm{d}x \\
&\quad\le C\int_\Omega q(x)\int_\Omega |\nabla_y H(x,y,t)|^2\,\mathrm{d}y = -\frac{C}{2}h'(t).
\end{aligned} \tag{3.11}$$

We apply the reasoning employed in [29] to prove their Eq. (17):

$$\begin{aligned}
&\int_\Omega q(x)\left(\int_\Omega H(x,y,t)|y''|^{b\gamma}q(y)^{\frac{p-1}{p-2}}\,\mathrm{d}y\right)^2\mathrm{d}x \\
&\quad\le \int_\Omega q(x)\left(\int_\Omega H(x,y,0)|y''|^{b\gamma}q(y)^{\frac{p-1}{p-2}}\,\mathrm{d}y\right)^2\mathrm{d}x \\
&\quad= \int_\Omega q^\gamma(x)|x''|^{2b\gamma}\,\mathrm{d}x \\
&\quad=: I.
\end{aligned} \tag{3.12}$$

By the choice of b and γ this last integral is always finite. We calculate with Hölder's inequality, using the triplets of exponents (p, γ, p) and $(2, 2\gamma, p)$:

$$\begin{aligned}
h(t) &= \int_\Omega q(x)\int_\Omega H^2(x,y,t)q(y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_\Omega q(x)\int_\Omega |y''|^{-b}H\left(Hq^{\frac{p-1}{p-2}}|y''|^{b\gamma}\right)^{\frac{1}{\gamma}}(H^2q)^{\frac{1}{p}}\,\mathrm{d}y\,\mathrm{d}x \\
&\le \int_\Omega q(x)\left(\int_\Omega |y''|^{-bp}H^p\,\mathrm{d}y\right)^{\frac{1}{p}}\left(\int_\Omega Hq^{\frac{p-1}{p-2}}|y''|^{b\gamma}\,\mathrm{d}y\right)^{\frac{1}{\gamma}}
\end{aligned}$$

$$
\begin{aligned}
&\times \left(\int_\Omega H^2 q \, \mathrm{d}y \right)^{\frac{1}{p}} \, \mathrm{d}x \\
\leq &\left(\int_\Omega q(x) \left(\int_\Omega |y''|^{-bp} H^p \, \mathrm{d}y \right)^{\frac{2}{p}} \, \mathrm{d}x \right)^{\frac{1}{2}} \\
&\times \left(\int_\Omega q(x) \left(\int_\Omega H q^{\frac{p-1}{p-2}} |y''|^{b\gamma} \, \mathrm{d}y \right)^2 \, \mathrm{d}x \right)^{\frac{1}{2\gamma}} \\
&\times \left(\int_\Omega q(x) \int_\Omega H^2 q(y) \, \mathrm{d}y \, \mathrm{d}x \right)^{\frac{1}{p}}.
\end{aligned}
$$

Therefore, (3.11) and (3.12) imply, with varying positive constants

$$C = C(G, \Omega, k, \ell, \gamma),$$

that

$$h^2(t) \leq -Ch'(t) I^{\frac{1}{\gamma}} h(t)^{\frac{2}{p}}$$

and hence

$$h^{-1-\frac{1}{\gamma}}(t) h'(t) \leq -C I^{-\frac{1}{\gamma}}$$

for $t > 0$. After an integration from 0 to $1/\mu_n$ this yields

$$\mu_n^\gamma I \geq C \sum_{i=1}^{\infty} \mathrm{e}^{-2\mu_i/\mu_n} \geq C \sum_{i=1}^{n} \mathrm{e}^{-2} = Cn$$

and hence (3.10). From here the proof proceeds as for Theorem 3.1, continuing from (3.4). □

Remark 3.3 Theorem 3.2 does not yield an LCR bound for Schrödinger operators on $\mathbb{R}^N$ as in (iii) of the proof of [29, Cor. 2] because the embedding constant here depends on Ω.

4 Bahri–Lions-type results

With the notation from the introduction consider the problem

$$
\begin{cases}
-\Delta u = |u|^{p-2} u + f, & \text{in } \Omega, \\
u = u_0, & \text{on } \partial\Omega, \\
u(gx) = u(x) & \forall g \in G, \ x \in \Omega.
\end{cases}
\qquad (P_G)
$$

We suppose that $p \in (2, 2^*_{N,m})$, $u_0 \in C^*2_G(\partial\Omega)$ and $f \in L^2_G(\Omega)$.

By $u_0 \in C^2_G(\overline{\Omega})$ we also denote the unique harmonic extension of u_0 to Ω. Setting $u = v + u_0$, problem (P_G) is equivalent to

$$
\begin{cases}
-\Delta v = |v + u_0|^{p-2}(v + u_0) + f, & \text{in } \Omega, \\
v = 0, & \text{on } \partial\Omega, \\
v(gx) = v(x) & \forall g \in G, \ x \in \Omega.
\end{cases}
\qquad (P'_G)
$$

We denote by

$$\|v\| := \left(\int_\Omega |\nabla v|^2\right)^{1/2} \tag{4.1}$$

the norm on $H_0^1(\Omega)$, and, using Theorem 2.1 if $m > 0$, we define the class-C^2 energy functional $J : H^1_{0,G}(\Omega) \to \mathbb{R}$ by

$$J(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p}\int_\Omega |u + u_0|^p - \int_\Omega fu.$$

Note that if $p > 2^*$, J cannot be defined naturally on $H_0^1(\Omega)$, and the principle of symmetric criticality cannot be invoked. Nevertheless, we show in Sect. 7 that in the cases considered here, a critical point of J indeed yields a weak solution of (P_G).

For the first existence result we define the exponent

$$\widetilde{A}_{N,m} := 2 + \frac{2}{N - m - 2},$$

where we use the convention $1/0 := \infty$. We then obtain

Theorem 4.1 *If $u_0 = 0$, $0 < m \le N - 2$, $p \in (2, \widetilde{A}_{N,m})$, and $f \in L^2_G(\Omega)$, then there is an unbounded sequence $(u_n) \subseteq H^1_0(\Omega)$ of G-invariant weak solutions to problem (P_G) such that $J(u_n) \to \infty$ as $n \to \infty$.*

For the second existence result we define

$$\hat{A}_{N,m} := 2 + \frac{2}{N - m - 1}$$

and obtain

Theorem 4.2 *If $0 < m \le N-2$, $p \in (2, \hat{A}_{N,m})$, and $f \in C^{0,\alpha}_G(\overline{\Omega})$ for some $\alpha \in (0, 1]$, then there is an unbounded sequence $(u_n) \subseteq H^1_0(\Omega)$ of G-invariant weak solutions to problem (P_G) such that $J(u_n) \to \infty$ as $n \to \infty$.*

Remark 4.3 (a) As mentioned in the introduction, for some values of N and m it holds that $\widetilde{A}_{N,m} > 2^*$ or $\hat{A}_{N,m} > 2^*$. Nevertheless, we always have $\widetilde{A}_{N,m}, \hat{A}_{N,m} < 2^*_{N,m}$.
(b) The Hölder condition on f and the C^2-regularity of u_0 in Theorem 4.2 are needed to verify condition (H3) in [16, Lemma 4.3], see the proof of our Lemma 6.2.

If Ω has the cylindrical symmetry described just before Theorem 2.2, then we can also allow $m = 0$ for the symmetry group G. To state the result we introduce a new limiting exponent. Define functions $\tilde{h}_{k,\ell}$ on $I_{k,\ell}$ by

$$\tilde{h}_{k,\ell}(\gamma) := \min\left\{2 + \frac{1}{\gamma - 1}, \frac{2\gamma\ell}{\gamma(\ell - 2) + k}\right\},$$

and exponents

$$\widetilde{B}_{k,\ell} := \max_{\gamma \in I_{k,\ell}} \tilde{h}_{k,\ell}(\gamma).$$

Note that we always have

$$\widetilde{B}_{k,\ell} < 2^*$$

since the map

$$\gamma \mapsto \frac{2\gamma \ell}{\gamma(\ell - 2) + k}$$

is strictly increasing and takes the value 2^* in $\gamma = N/2$, and since the strictly decreasing map

$$\gamma \mapsto 2 + \frac{1}{\gamma - 1}$$

takes the value $\tilde{2}_N^{BL} < 2^*$ in $\gamma = N/2$.

Theorem 4.4 *If $u_0 = 0$, $p \in (2, \widetilde{B}_{k,\ell})$, and $f \in L^2_G(\Omega)$, then there is an unbounded sequence $(u_n) \subseteq H^1_0(\Omega)$ of G-invariant weak solutions to problem (P_G) such that $J(u_n) \to \infty$ as $n \to \infty$.*

Analogously, for the problem with nonhomogeneous boundary value we define functions $\hat{h}_{k,\ell}$ on $I_{k,\ell}$ by

$$\hat{h}_{k,\ell}(\gamma) := \min \left\{ 2 + \frac{2}{2\gamma - 1}, \ \tilde{h}_{k,l}(\gamma) \right\}$$

and the exponents

$$\hat{B}_{k,\ell} := \max_{\gamma \in I_{k,\ell}} \hat{h}_{k,\ell}(\gamma) \leq \widetilde{B}_{k,\ell}.$$

Theorem 4.5 *If $p \in (2, \hat{B}_{k,\ell})$ and $f \in C^{0,\alpha}_G(\overline{\Omega})$ for some $\alpha \in (0, 1]$, then there is an unbounded sequence $(u_n) \subseteq H^1_0(\Omega)$ of G-invariant weak solutions to problem (P_G) such that $J(u_n) \to \infty$ as $n \to \infty$.*

5 Lower bounds for critical values of the unperturbed functional

In this section we only assume that $p \in (2, 2^*_{N,m})$. Set $X := H^1_{0,G}(\Omega)$ and consider the C^2-functional $E_0 \colon X \to \mathbb{R}$ given by

$$E_0(u) := \frac{1}{2}\|u\|^2 - \frac{1}{p}\int_\Omega |u|^p.$$

Choose an increasing sequence of subspaces (X_n) of X such that $\dim X_n = n$ for each $n \in \mathbb{N}$ and

$$X = \overline{\bigcup_{n=1}^{\infty} X_n}.$$

Put

$$\Gamma := \big\{ \varphi \in C(X, X) \mid \varphi \text{ is odd and there is } R > 0 \text{ such that } \varphi|_{X \setminus B_R(0)} = \mathrm{id}_{X \setminus B_R(0)} \big\}$$

and define

$$c_n := \inf_{\varphi \in \Gamma} \sup_{u \in X_n} E_0(\varphi(u)). \tag{5.1}$$

The compact embedding $X \hookrightarrow L^p(\Omega)$ given in Theorem 2.1 implies by standard arguments that each c_n is a positive critical value of E_0 and that (c_n) is increasing and converges to ∞.

Proposition 5.1 *If* $0 < m \leq N-3$, *then*

$$\inf_{n\in\mathbb{N}} \frac{c_n}{n^s} > 0, \tag{5.2}$$

where $s := \frac{2}{N-m} \cdot \frac{p}{p-2}$. *If* $m = N-2$ *and* $\varepsilon \in (0, \frac{2}{p-2})$, *then* (5.2) *holds with* $s := \frac{1}{1+\varepsilon} \cdot \frac{p}{p-2}$.

Proof Our proof follows the ideas of the proofs of Theorem 1 and Lemma 2.2 in [2].

By [2, Theorem B] there exist critical points $v_n \in X$ of E_0 such that

$$E_0(v_n) \leq c_n \tag{5.3}$$

and such that its large Morse index is greater than or equal to n, i.e., such that

$$-\triangle - (p-1)|v_n|^{p-2}$$

has at least n non-positive eigenvalues. Note that the hypotheses for the cited theorem are satisfied by standard arguments.

Let us first suppose that $m \leq N-3$. Since $p < 2^*_{N,m}$ we have

$$1 < \frac{2p}{(p-2)(N-m)}.$$

From Theorem 3.1, applied to $V := -(p-1)|v_n|^{p-2}$, and from Hölder's inequality we obtain varying positive constants $C > 0$, that do not depend on n, such that

$$n \leq C \int_\Omega (|v_n|^{p-2})^{(N-m)/2} \leq C|v_n|_p^{\frac{(p-2)(N-m)}{2}}. \tag{5.4}$$

Here we denote the norm in $L^p(\Omega)$ by $|\cdot|_p$. On the other hand, since v_n is a critical point of E_0,

$$\|v_n\|^2 = |v_n|_p^p.$$

Hence (5.3) and (5.4) imply that

$$c_n \geq E_0(v_n) = \frac{1}{2}\|v_n\|^2 - \frac{1}{p}|v_n|_p^p = \left(\frac{1}{2} - \frac{1}{p}\right)|v_n|_p^p \geq Cn^s$$

with $s = \frac{2}{N-m} \cdot \frac{p}{p-2}$.

If $m = N-2$, then by the condition on ε it holds true that $(p-2)(1+\varepsilon) < p$. Theorem 3.1 yields, together with Hölder's inequality,

$$n \leq C \int_\Omega \left(|v_n|^{p-2}\right)^{1+\varepsilon} \leq C|v_n|_p^{(p-2)(1+\varepsilon)}$$

and hence, as before, that

$$c_n \geq Cn^s \qquad \text{for } n \text{ large enough,}$$

with $s := \frac{1}{1+\varepsilon} \cdot \frac{p}{p-2}$. □

For the next result recall the setting of Theorem 2.2.

Proposition 5.2 *If* $\gamma \in \operatorname{int} I_{k,\ell}$ *and* $p < \tilde{h}_{k,\ell}(\gamma)$, *then* (5.2) *holds true with* $s := \frac{p}{\gamma(p-2)}$.

Proof From $p < 2 + 1/(\gamma - 1)$ it follows that

$$\frac{p}{\gamma(p-2)} > 1 \quad \text{and} \quad \frac{p}{p-\gamma(p-2)} > 1. \tag{5.5}$$

In particular, $p - \gamma(p-2) > 0$. As in the proof of Proposition 5.1 we find critical points $v_n \in X$ of E_0 such that $E_0(v_n) \le c_n$ and such that its large Morse index is greater than or equal to n. By Theorem 3.2 and Hölder's inequality, using the expressions from (5.5) as conjugate exponents, we obtain

$$n \le C \int_\Omega |v_n|^{\gamma(p-2)} |x''|^{2b\gamma} \, dx \le C |v_n|_p^{\gamma(p-2)} \left(\int_\Omega |x''|^{\frac{2b\gamma p}{p-\gamma(p-2)}} \right)^{\frac{p-\gamma(p-2)}{p}}, \tag{5.6}$$

where we have set $b := 1 - N/(2\gamma)$. The choice of p implies that

$$\frac{2b\gamma p}{p-\gamma(p-2)} > -\ell$$

and therefore the rightmost integral in (5.6) is finite.

As in the proof of Proposition 5.1 we now induce from (5.6) that

$$c_n \ge E_0(v_n) = \left(\frac{1}{2} - \frac{1}{p}\right) |v_n|_p^p \ge C n^{\frac{p}{\gamma(p-2)}}. \qquad \square$$

6 Bolle's method and upper bounds for critical values

In this section we assume the hypotheses from the beginning of Sect. 4, and we continue using the notation from Sect. 5. We will apply Bolle's method using a Theorem of Bolle, Ghoussoub and Tehrani, [16, Theorem 2.2]. To this end we need to introduce further notation. Consistently with the definition of the functional E_0 we define a family of functionals $E_t := E(\cdot, t)$ on X, where $E := X \times [0, 1] \to \mathbb{R}$ is defined by

$$E(u, t) := \frac{1}{2}\|u\|^2 - \frac{1}{p} \int_\Omega |u + tu_0|^p - t \int_\Omega f u.$$

Hence $J = E_1$.

In the following two lemmas we present the results from [16] in a form that is convenient for our purposes. To this end, recall the definition of the critical values c_n given in (5.1).

Lemma 6.1 *If $u_0 = 0$ and if the set of critical levels of J is bounded, then*

$$\sup_{n\in\mathbb{N}} \frac{c_n}{n^{p/(p-1)}} < \infty. \tag{6.1}$$

Proof It is proved in [16, Section 4] that E satisfies the hypotheses (H1), (H2) and (H4') of Theorem 2.2, loc. cit. Since $u_0 = 0$, it is easy to see that (H3) from [16] holds true for E with $f_2(t, s) = a_0(s^2 + 1)^{1/2p} = -f_1(t, s)$ for some $a_0 > 0$, c.f. [38]. Defining continuous functions $\psi_i : [0, 1] \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{cases} \frac{\partial}{\partial t}\psi_i(t, s) = f_i(t, \psi_i(t, s)), & t \in (0, 1], \ s \in \mathbb{R}, \\ \psi_i(0, s) = s, & s \in \mathbb{R}, \end{cases}$$

and using $f_2 \geq 0$ we obtain $\psi_2(1,s) \geq s$ for all $s \in \mathbb{R}$. Since the set of critical values of $J = E_1$ is bounded from above, [16, Theorem 2.2] implies that

$$\left(\frac{c_{n+1}-c_n}{a_0(c_{n+1}^2+1)^{1/2p}+a_0(c_n^2+1)^{1/2p}+1}\right)_n \tag{6.2}$$

remains bounded as $n \to \infty$. The classical argument from [3, Lemma 5.3] or [39, Proposition 10.46] does not apply directly in this setting to show (6.1). Therefore we present the details needed in this case. Put

$$d_n := \frac{c_n}{4a_0(c_n^2+1)^{1/2p}}.$$

In what follows denote by C various positive constants that are independent of n. We obtain from (6.2) that

$$\begin{aligned} d_{n+1}-d_n &\leq \frac{c_{n+1}-c_n}{2a_0\big((c_{n+1}^2+1)^{1/2p}+(c_n^2+1)^{1/2p}\big)} \\ &\leq \frac{c_{n+1}-c_n}{a_0\big((c_{n+1}^2+1)^{1/2p}+(c_n^2+1)^{1/2p}\big)+1} \\ &\leq C \end{aligned}$$

for all n. This implies that $d_n \leq Cn$ and yields in turn that

$$c_n \leq Cn^{p/(p-1)}$$

for all n. □

Lemma 6.2 *If the set of critical levels of J is bounded, then*

$$\sup_{n\in\mathbb{N}} \frac{c_n}{n^2} < \infty. \tag{6.3}$$

Proof It was proved in [16, Lemma 4.3] that E satisfies (H3), loc. cit., with $f_2(t,s) = a_0(s^2+1)^{1/4} = -f_1(t,s)$. Since the set of critical values of $J = E_1$ is bounded we obtain, replacing p by 2 in the proof of Lemma 6.1, a positive constant C that is independent of n such that

$$c_n \leq Cn^2$$

for all n. □

7 Comparing upper and lower bounds

In this section we prove the theorems from Sect. 4, using the notation from Sects. 4–6. In addition, we denote by $\langle\cdot,\cdot\rangle$ the scalar product in $H_0^1(\Omega)$ compatible with the norm defined in (4.1).

It is straightforward to check that under the hypotheses of any of the Theorems 4.1, 4.2, 4.4, and 4.5

$$\frac{2^*_{N,m}}{p-1} \geq \frac{2^*}{2^*-1}$$

holds true. Consequently, if $u \in X$ is a critical point of J in X, then by the embedding $X \hookrightarrow L^{2^*_{N,m}}(\Omega)$ and by the boundedness of Ω the function $v := |u+u_0|^{p-2}(u+u_0)$ lies in $L^{2^*/(2^*-1)}(\Omega) \hookrightarrow H^{-1}(\Omega)$. Therefore, the map

$$X \to \mathbb{R}, \qquad z \mapsto \int_\Omega vz$$

extends to a bounded linear functional on $H_0^1(\Omega)$, an element of $H^{-1}(\Omega)$. By Riesz's Representation Theorem there is $w \in H_0^1(\Omega)$ such that

$$\int_\Omega vz = \langle w, z\rangle \qquad \text{for all } z \in H_0^1(\Omega).$$

In other words, w is the unique weak solution of

$$\begin{cases} -\Delta w = v, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega. \end{cases}$$

Since $u, u_0 \in X$, v is invariant, and the uniqueness implies that w is also invariant, i.e., $w \in X$.

For any $z \in H_0^1(\Omega)$ denote by z_G the orthogonal projection in $H_0^1(\Omega)$ onto X (with respect to $\langle\cdot,\cdot\rangle$), and put $z_\perp := z - z_G$. Since u is a critical point of J in X we have $\langle u - w, z_G\rangle = J'(u)z_G = 0$. On the other hand, $\langle u, z_\perp\rangle = \langle w, z_\perp\rangle = 0$ since $u, w \in X$. It follows that

$$\int_\Omega \nabla u \cdot \nabla z - \int_\Omega vz = \langle u - w, z\rangle = \langle u - w, z_G\rangle + \langle u - w, z_\perp\rangle = 0.$$

Hence u is a weak solution of (P_G) and it suffices to prove in all cases that there is a sequence of critical points $(u_n) \subseteq X$ of J such that $J(u_n) \to \infty$.

Proof of Theorem 4.1 Assume (6.1) to be true. By Lemma 6.1 it is sufficient to reach a contradiction.

If $m \le N-3$, then (6.1) and Proposition 5.1 imply that

$$\frac{2}{N-m} \cdot \frac{p}{p-2} \le \frac{p}{p-1},$$

in contradiction with $p < \widetilde{A}_{N,m}$. If $m = N-2$, then (6.1) and Proposition 5.1 imply that

$$\frac{1}{1+\varepsilon} \cdot \frac{p}{p-2} \le \frac{p}{p-1}$$

for all positive and sufficiently small ε, which is impossible. □

Proof of Theorem 4.2 Assume (6.3) to be true. By Lemma 6.2 it is sufficient to reach a contradiction.

If $m \le N-3$, then (6.3) and Proposition 5.1 imply that

$$\frac{2}{N-m} \cdot \frac{p}{p-2} \le 2,$$

in contradiction with $p < \hat{A}_{N,m}$. If $m = N-2$, then (6.3) and Proposition 5.1 imply that

$$\frac{1}{1+\varepsilon} \cdot \frac{p}{p-2} \le 2$$

for all positive and sufficiently small ε, in contradiction with $p < \hat{A}_{N,m}$. □

Proof of Theorem 4.4 Pick $\gamma \in \operatorname{int} I_{k,\ell}$ such that $p < \tilde{h}_{k,\ell}(\gamma)$. Assume (6.1) to be true. Together with Proposition 5.2 it follows that

$$\frac{p}{\gamma(p-2)} \leq \frac{p}{p-1},$$

in contradiction with $p < 2 + \frac{1}{\gamma-1}$. Hence Lemma 6.1 implies the result. □

Proof of Theorem 4.5 Pick $\gamma \in \operatorname{int} I_{k,\ell}$ such that $p < \hat{h}_{k,\ell}(\gamma)$. It follows in particular that $p < \tilde{h}_{k,\ell}(\gamma)$. Assume (6.3) to be true. Together with Proposition 5.2 it follows that

$$\frac{p}{\gamma(p-2)} \leq 2,$$

in contradiction with $p < 2 + \frac{2}{2\gamma-1}$. Hence Lemma 6.2 implies the result. □

Acknowledgements We thank the anonymous referee for carefully reading the manuscript and for suggesting several improvements.

A The heat kernel in a weighted space

This Appendix provides background material for Sect. 3. We continue to assume that $\Omega \subseteq \mathbb{R}^N$ is bounded and of class $C^{2,\beta}$ and prove the needed properties of the full heat kernel, i.e., including nonsymmetric functions. These properties remain true trivially when restricting to G-symmetric functions.

Suppose that $q \in C^1\left(\overline{\Omega}\right)$ is positive on $\overline{\Omega}$. In this section denote by $L^2_q(\Omega)$ the q-weighted space $L^2(\Omega)$ with scalar product

$$\langle v, w\rangle_q := \int_\Omega vwq$$

and associated norm $|\cdot|_{2,q}$, which is equivalent to the original norm $|\cdot|_2$ in $L^2(\Omega)$. From now on all function spaces are over Ω unless otherwise noted.

In the Hilbert space L^2_q consider the strongly elliptic operator $A := -\Delta/q$ with domain $\mathcal{D}(A) := H^2 \cap H^1_0$. Then A is a densely defined symmetric operator. The standard theory for the Dirichlet Laplacian implies that for every $v \in L^2$ there is $u \in \mathcal{D}(A)$ such that

$$-\Delta u = qv,$$

or, in other words, that A is surjective. It follows from [40, Theorem 13.11(d)] that A is self-adjoint. Moreover, since Ω is bounded, the theorem of Rellich–Kondrakov implies that its resolvent is compact. The quadratic form $\mathcal{Q}_A$ of A is given on $\mathcal{D}(A)$ by

$$\mathcal{Q}_A(v, w) := \langle Av, w\rangle_q = \int_\Omega \nabla v \cdot \nabla w,$$

which coincides with the quadratic form of $-\Delta$, and whose form closure has domain H^1_0. This is true in L^2_q and L^2 since the norms are equivalent. Moreover, C^∞_c is a form core for $\mathcal{Q}_A$. In what follows we set

$$\mathcal{Q}_A(u) := \mathcal{Q}_A(u, u) = \int_\Omega |\nabla u|^2 \geq 0$$

for $u \in H^1_0$. Hence A is a positive operator, $A \geq 0$.

Now consider $\varphi \in C_c^\infty$. It follows that $|\varphi| \in H_0^1$ and that

$$\mathcal{Q}_A(|\varphi|) = \int_\Omega |\nabla|\varphi||^2 = \int_\Omega |\nabla\varphi|^2 = \mathcal{Q}_A(\varphi).$$

Moreover, defining a function φ^* by

$$\varphi^*(x) := \begin{cases} \varphi(x), & \varphi(x) \in [0, 1], \\ 0, & \varphi(x) < 0, \\ 1, & \varphi(x) > 1, \end{cases}$$

we obtain $\varphi^* \in H_0^1$ and

$$\mathcal{Q}_A(\varphi^*) = \int_{\varphi^{-1}([0,1])} |\nabla\varphi|^2 \le \int_\Omega |\nabla\varphi|^2 = \mathcal{Q}_A(\varphi).$$

Hence, by [41, Lemma 1.3.4], the semigroup e^{-tA} is a *symmetric Markov semigroup* as defined on page 22 *loc. cit.*

As for the Dirichlet Laplacian in L^2 one proves that A is sectorial in L_q^2. Hence e^{-tA} is also an infinitely differentiable strongly continuous semigroup in L_q^2: for every $u \in L_q^2$ the map $U\colon [0, \infty) \to L_q^2$ given by

$$U(t) := \mathrm{e}^{-tA}u$$

is continuous, and it is infinitely differentiable in $(0, \infty)$. For $t > 0$ it holds true that $U(t) \in \mathcal{D}(A)$,

$$U'(t) = -AU(t),$$

and $U(0) = u$. By the standard theory of linear parabolic equations, the function $\bar{u}(\cdot, t) := U(t)$ is a classical solution of

$$\begin{cases} q(x)\frac{\partial}{\partial t}\bar{u}(x, t) = \Delta\bar{u}(x, t), & (x, t) \in \Omega \times (0, \infty), \\ \bar{u}(x, 0) = u(x), & x \in \Omega, \\ \bar{u}(x, t) = 0, & (x, t) \in \partial\Omega \times (0, \infty). \end{cases}$$

Moreover, using Nash's inequality one shows exactly as in the proof of [41, Theorem 2.4.6] that e^{-tA} is *ultracontractive*, in the sense that it provides a bounded linear operator from L_q^2 into L^∞ for every $t > 0$. Here one only needs to replace $|\cdot|_2$ by $|\cdot|_{2,q}$. Consequentially, [41, Theorems 2.1.2 and 2.1.4] apply and yield the following: e^{-tA} is of trace class for all $t > 0$, that is, if (μ_i) is the increasing sequence of eigenvalues of A, repeated according to multiplicity, then

$$\sum_{i=1}^\infty \mathrm{e}^{-t\mu_i} < \infty \qquad \text{for all } t > 0. \tag{7.1}$$

Moreover, if Ψ_i denotes the eigenfunction corresponding to μ_i, normalized to $|\Psi_i|_{2,q} = 1$, then $\Psi_i \in L^\infty(\Omega)$,

$$H(x, y, t) := \sum_{i=1}^\infty \mathrm{e}^{-t\mu_i}\Psi_i(x)\Psi_i(y)$$

converges absolutely and uniformly on $\overline{\Omega} \times \overline{\Omega} \times [\kappa, 0)$ for every $\kappa > 0$, and H is the *heat kernel* of A in the sense that $H(\cdot, \cdot, t)$ is the integral kernel of e^{-tA} for $t > 0$:

$$\forall u \in L^2_q \; \forall x \in \Omega\colon (\mathrm{e}^{-tA}u)(x) = \int_\Omega H(x, y, t)u(y)q(y)\,\mathrm{d}y.$$

Moreover, $H \geq 0$ and, by Mercer's Theorem in the form of [42, Theorem 3.a.1], it holds true that

$$\sup_{i\in\mathbb{N}} |\Psi_i|_\infty < \infty. \tag{7.2}$$

Note that the set $\{\Psi_i \mid i \in \mathbb{N}\}$ is orthonormal in L^2_q.

We claim that there is $C > 0$ such that $\Psi_i \in C^2(\overline{\Omega})$ and

$$\|\Psi_i\|_{C^2(\overline{\Omega})} \leq C(1 + \mu_i^2), \qquad \text{for all } i \in \mathbb{N}. \tag{7.3}$$

In fact, since Ψ_i is a weak solution of

$$\begin{cases} -\Delta\Psi_i = \mu_i q\Psi_i, & \text{in } \Omega, \\ \Psi_i = 0, & \text{on } \partial\Omega, \end{cases} \tag{7.4}$$

it is well known by standard regularity estimates that $\Psi_i \in C^{2,\beta}(\overline{\Omega})$ is a classical solution of (7.4). From Sobolev's embedding and [43, Lemma 9.17] we obtain

$$\|\Psi_i\|_{C^{0,\beta}(\overline{\Omega})} \leq C\|\Psi_i\|_{W^{2,N}} \leq C\mu_i|\Psi_i|_N \leq C\mu_i|\Psi_i|_\infty \leq C\mu_i,$$

where C is independent of i by (7.2). And, last but not least, [43, Theorem 6.6] implies that

$$\|\Psi_i\|_{C^{2,\beta}(\overline{\Omega})} \leq C(|\Psi_i|_\infty + \mu_i\|q\|_{C^{0,\beta}(\overline{\Omega})}\|\Psi_i\|_{C^{0,\beta}(\overline{\Omega})}) \leq C(1 + \mu_i^2),$$

where C is independent of i. This proves the claim.

From (7.1) it follows easily that

$$\sum_{i=1}^\infty \mu_i^s \mathrm{e}^{-t\mu_i} < \infty \qquad \text{for all } s, t > 0.$$

Together with (7.2) and (7.3) this implies for all $j, k \in \{1, 2, \ldots, n\}$ that

$$\sum_{i=1}^\infty \mu_i \mathrm{e}^{-t\mu_i}\Psi_i(x)\Psi_i(y), \qquad \sum_{i=1}^\infty \mathrm{e}^{-t\mu_i}\Psi_i(x)\partial_j\Psi_i(y), \qquad \text{and} \qquad \sum_{i=1}^\infty \mathrm{e}^{-t\mu_i}\Psi_i(x)\partial_j\partial_k\Psi_i(y)$$

converge absolutely and uniformly on $\overline{\Omega}\times\overline{\Omega}\times[\kappa, 0)$ for every $\kappa > 0$. Hence H is continuously differentiable in t and twice continuously differentiable in y, and by (7.4)

$$\begin{cases} \left(\frac{\partial}{\partial t} - \frac{\Delta_y}{q}\right) H = 0, & \text{in } \Omega \times \Omega \times (0, \infty), \\ H = 0, & \text{in } \partial\Omega \times \Omega \times (0, \infty) \cup \Omega \times \partial\Omega \times (0, \infty). \end{cases}$$

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