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Calculus of Variations

Local Li–Yau's estimates on $RCD^*(K, N)$ metric measure spaces

Hui-Chun Zhang[1] · Xi-Ping Zhu[1]

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Abstract In this paper, we will study the (linear) geometric analysis on metric measure spaces. We will establish a local Li–Yau's estimate for weak solutions of the heat equation and prove a sharp Yau's gradient for harmonic functions on metric measure spaces, under the Riemannian curvature-dimension condition $RCD^*(K, N)$.

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Contents

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✉ Hui-Chun Zhang
zhanghc3@mail.sysu.edu.cn

Xi-Ping Zhu
stszxp@mail.sysu.edu.cn

1 Department of Mathematics, Sun Yat-sen University, Guangzhou 510275, China

1 Introduction

In the field of geometric analysis, one of the fundamental results is the following Li–Yau's local gradient estimate for solutions of the heat equation on a complete Riemannian manifold.

Theorem 1.1 (Li–Yau [34]) *Let (M^n, g) be an n-dimensional complete Riemannian manifold, and let B_{2R} be a geodesic ball of radius $2R$ centered at $O \in M^n$. Assume that $Ric(M^n) \geq -k$ with $k \geq 0$. If $u(x,t)$ is a smooth positive solution of the heat equation $\Delta u = \partial_t u$ on $B_{2R} \times (0, \infty)$, then for any $\alpha > 1$, we have the following gradient estimate in B_R:*

$$\sup_{x \in B_R} (|\nabla f|^2 - \alpha \cdot \partial_t f)(x,t) \leq \frac{C\alpha^2}{R^2}\left(\frac{\alpha^2}{\alpha^2 - 1} + \sqrt{k}R\right) + \frac{n\alpha^2 k}{2(\alpha - 1)} + \frac{n\alpha^2}{2t} \tag{1.1}$$

where $f := \ln u$ and C is a constant depending only on n.

By letting $R \to \infty$ in (1.1), one gets a global gradient estimate, for any $\alpha > 1$, that

$$|\nabla f|^2 - \alpha \cdot \partial_t f \leq \frac{n\alpha^2 k}{2(\alpha - 1)} + \frac{n\alpha^2}{2t}. \tag{1.2}$$

There is a rich literature on extensions and improvements of the Li–Yau inequality, both the local version (1.1) and the global version (1.2), to diverse settings and evolution equations, for example, in the setting of Riemannian manifolds with Ricci curvature bounded below [9,15,32,33,47], in the setting of weighted Riemannian manifolds with Bakry–Emery Ricci curvature bounded below [7,12,35,43] and some non-smooth setting [10,44], and so on.

Let (X, d, μ) be a complete, proper metric measure space with $\mathrm{supp}(\mu) = X$. The *curvature-dimension condition* on (X, d, μ) has been introduced by Sturm [48] and Lott and Villani [36]. Given $K \in \mathbb{R}$ and $N \in [1, \infty]$, the curvature-dimension condition $CD(K, N)$ is a synthetic notion for "generalized Ricci curvature $\geq K$ and dimension $\leq N$" on (X, d, μ). Bacher and Sturm [6] introduced the *reduced* curvature-dimension condition $CD^*(K, N)$, which satisfies a local-to-global property. On the other hand, to rule out Finsler geometry, Ambrosio et al. [1] introduced the *Riemannian* curvature-dimension condition $RCD(K, \infty)$, which assumes that the heat flow on $L^2(X)$ is linear. Remarkably, Erbar et al. [16] and Ambrosio et al. [5] introduced a dimensional version of Riemannian curvature-dimension condition $RCD^*(K, N)$ and proved that it is equivalent to a Bakry–Emery's Bochner inequality via an abstract Γ_2-calculus for semigroups. In the case of Riemannian geometry, the notion $RCD^*(K, N)$ coincides with the original Ricci curvature $\geq K$ and dimension $\leq N$, and for the case of the weighted manifolds $(M^n, g, e^{\phi} \cdot \mathrm{vol}_g)$, the notion $RCD^*(K, N)$ coincides with the corresponding Bakry–Emery's curvature-dimension condition [36,48]. In the setting of Alexandrov geometry, it is implied by generalized (sectional) curvature bounded below in the sense of Alexandrov [42,52].

Based on the Γ_2-calculus for the heat flow $(H_t f)_{t \geq 0}$ on $L^2(X)$, many important results in geometric analysis have been obtained on a metric measure space (X, d, μ) satisfying $RCD^*(K, N)$ condition. For instance, Li–Yau–Hamilton estimates for the heat flow $(H_t f)_{t \geq 0}$ [17,28,30] and spectral gaps [31,37,44] for the infinitesimal generator of $(H_t f)_{t \geq 0}$.

In this paper, we will study the *locally* weak solutions of the heat equation on a metric measure space (X, d, μ). Let $\Omega \subset X$ be an open set. The $RCD^*(K, N)$ condition implies that the Sobolev space $W^{1,2}(\Omega)$ is a Hilbert space. Given an interval $I \subset \mathbb{R}$, a function $u(x,t) \in W^{1,2}(\Omega \times I)$ is called a *locally weak solution* for the heat equation on $\Omega \times I$ if it satisfies

$$-\int_I \int_\Omega \langle \nabla u, \nabla \phi \rangle d\mu dt = \int_I \int_\Omega \frac{\partial u}{\partial t} \cdot \phi d\mu dt \tag{1.3}$$

for all Lipschitz functions ϕ with compact support in $\Omega \times I$, where the inner product $\langle \nabla u, \nabla \phi \rangle$ is given by polarization in $W^{1,2}(\Omega)$.

Notice that the locally weak solutions $u(x,t)$ do not form a semi-group in general. The method of Γ_2-calculus for the heat flow in the previous works [17,28,31] is no longer be suitable for the problems on locally weak solutions of the heat equation.

To seek an appropriate method to deal with the locally weak solutions for the heat equation, let us recall what is the proof of Theorem 1.1 in the smooth context. There are two main ingredients: the Bochner formula and a maximum principle. The Bochner formula states that

$$\frac{1}{2}\Delta|\nabla f|^2 \geq \frac{(\Delta f)^2}{n} + \langle \nabla f, \nabla \Delta f \rangle + K|\nabla f|^2 \tag{1.4}$$

for any C^3-function f on M^n with Ricci curvature $Ric(M^n) \geq K$ for some $K \in \mathbb{R}$. The maximum principle states that if $f(x)$ is of C^2 on M^n and if it achieves its a local maximal value at point $x_0 \in M^n$, then we have

$$\nabla f(x_0) = 0 \quad \text{and} \quad \Delta f(x_0) \leq 0. \tag{1.5}$$

For simplification, we only consider the special case that $u(x,t)$ is a smooth positive solution for heat equation on a compact manifold M^n with $Ric(M^n) \geq 0$. By using the Bochner formula to $\ln u$, one deduces a differential inequality

$$\left(\Delta - \frac{\partial}{\partial t}\right) F \geq -2\langle \nabla f, \nabla F \rangle + \frac{2}{nt}F^2 - \frac{F}{t},$$

where $f = \ln u$ and $F = t\big(|\nabla f|^2 - \partial_t f\big)$. Then by using the maximum principle to F at one of its maximum points (x_0, t_0), one gets the desired Li–Yau's estimate

$$\max F = F(x_0, t_0) \leq \frac{n}{2}.$$

In this paper, we want to extend these two main ingredients to non-smooth metric measure spaces. Firstly, let us consider the Bochner formula in non-smooth context. Let (X, d, μ) be a metric measure space with $RCD^*(K,N)$. Erbar et al. [16] and Ambrosio et al. [5] proved that $RCD^*(K,N)$ condition is equivalent to a Bakry–Emery's Bochner inequality for the heat flow $(H_t f)_{t\geq 0}$ on X. This provides a global version of Bochner formula for the infinitesimal generator of the heat flow $(H_t f)_{t\geq 0}$ (see Lemma 2.3). On the other hand, a good cut-off function has been obtained in [5,24,40]. By combining these two facts and an argument in [24], one can localize the global version of Bochner formula in [5,16] to a local one.

To state the local version of Bochner formula, it is more convenient to work with a notion of the weak Laplacian, which is a slight modification from [18,20]. Let $\Omega \subset X$ be an open set. Denote by $H^1(\Omega) := W^{1,2}(\Omega)$ and $H^1_0(\Omega) := W^{1,2}_0(\Omega)$. The *weak Laplacian* on Ω is an operator $\mathscr{L}$ on $H^1(\Omega)$ defined by: for each function $f \in H^1(\Omega)$, $\mathscr{L} f$ is a functional acting on $H^1_0(\Omega) \cap L^\infty(\Omega)$ given by

$$\mathscr{L} f(\phi) := -\int_\Omega \langle \nabla f, \nabla \phi \rangle d\mu \quad \forall\, \phi \in H^1_0(\Omega) \cap L^\infty(\Omega).$$

In the case when it holds

$$\mathscr{L} f(\phi) \geq \int_\Omega h \cdot \phi d\mu \quad \forall\, 0 \leq \phi \in H^1_0(\Omega) \cap L^\infty(\Omega)$$

for some function $h \in L^1_{\rm loc}(\Omega)$, then it is well-known [23] that the weak Laplacian $\mathscr{L} f$ can be extended to a signed Radon measure on Ω. In this case, we denote by

$$\mathscr{L} f \geq h \cdot \mu$$

on Ω in the sense of distributions.

Now, the local version of Bochner formula is given as follows.

Theorem 1.2 ([5,24]) *Let (X, d, μ) be a metric measure space with $RCD^*(K, N)$ for some $K \in \mathbb{R}$ and $N \geq 1$. Assume that $f \in H^1(B_R)$ such that $\mathscr{L} f$ is a signed measure on B_R with the density $g \in H^1(B_R) \cap L^\infty(B_R)$. Then we have $|\nabla f|^2 \in H^1(B_{R/2}) \cap L^\infty(B_{R/2})$ and that $\mathscr{L}(|\nabla f|^2)$ is a signed Radon measure on $B_{R/2}$ such that*

$$\frac{1}{2}\mathscr{L}(|\nabla f|^2) \geq \left[\frac{g^2}{N} + \langle \nabla f, \nabla g \rangle + K|\nabla f|^2\right] \cdot \mu$$

on $B_{R/2}$ in the sense of distributions.

Next, we consider to extend the maximum principle (1.5) from smooth Riemannian manifolds to non-smooth metric measure spaces (X, d, μ). A simple observation is that the maximum principle (1.5) on a smooth manifold M^n has the following equivalent form:

Suppose that $f(x)$ is of C^2 on M^n and that it achieves its a local maximal value at point $x_0 \in M^n$. Given any $w \in C^1(U)$ for some neighborhood U of x_0. Then we have

$$\Delta f(x_0) + \langle \nabla f, \nabla w \rangle(x_0) \leq 0.$$

In the following result, we will extend the observation to the non-smooth context. Technically, it is our main effort in the paper.

Theorem 1.3 *Let Ω be a bounded domain in a metric measure space (X, d, μ) with $RCD^*(K, N)$ for some $K \in \mathbb{R}$ and $N \geq 1$. Let $f(x) \in H^1(\Omega) \cap L^\infty_{\rm loc}(\Omega)$ such that $\mathscr{L} f$ is a signed Radon measure with $\mathscr{L}^{\rm sing} f \geq 0$, where $\mathscr{L}^{\rm sing} f$ is the singular part with respect to μ. Suppose that f achieves one of its strict maximum in Ω in the sense that: there exists a neighborhood $U \subset\subset \Omega$ such that*

$$\sup_U f > \sup_{\Omega \setminus U} f.$$

Then, given any $w \in H^1(\Omega) \cap L^\infty(\Omega)$, there exists a sequence of points $\{x_j\}_{j \in \mathbb{N}} \subset U$ such that they are the approximate continuity points of $\mathscr{L}^{\rm ac} f$ and $\langle \nabla f, \nabla w \rangle$, and that

$$f(x_j) \geq \sup_\Omega f - 1/j \quad \text{and} \quad \mathscr{L}^{\rm ac} f(x_j) + \langle \nabla f, \nabla w \rangle(x_j) \leq 1/j.$$

Here and in the sequel of this paper, $\sup_U f$ *means* $\operatorname{ess\,sup}_U f$.

This result is close to the spirit of the Omori–Yau maximum principle [41,51]. It has also some similarity with the approximate versions of the maximum principle developed, for instance by Jensen [26], in the theory of second order viscosity solutions.

A similar parabolic version of the maximum principle, Theorem 4.4, will be given in Sect. 4.

After obtaining the above Bochner formula and the maximum principle (Theorems 1.2, 4.4), we will show the following Li–Yau type gradient estimates for locally weak solutions of the heat equation, which is our main purpose in this paper.

Theorem 1.4 *Let $K \geq 0$ and $N \in [1, \infty)$, and let (X, d, μ) be a metric measure space satisfying $RCD^*(-K, N)$. Let $T_* \in (0, \infty]$ and let B_{2R} be a geodesic ball of radius $2R$ centered at $p \in X$, and let $u(x, t) \in W^{1,2}\big(B_{2R} \times (0, T_*)\big)$ be a positive locally weak solution of the heat equation on $B_{2R} \times (0, T_*)$. Then, given any $T \in (0, T_*)$, we have the following local gradient estimate*

$$\begin{aligned}\sup_{B_R \times (\beta \cdot T, T]} \Big(|\nabla f|^2 - \alpha \cdot \tfrac{\partial}{\partial t} f\Big)(x, t) &\leq \max\Big\{1, \tfrac{1}{2} + \tfrac{KT}{2(\alpha-1)}\Big\} \cdot \tfrac{N\alpha^2}{2T} \cdot \tfrac{1}{\beta^2} \\ &+ \tfrac{C_N \cdot \alpha^4}{R^2(\alpha-1)} \cdot \tfrac{1}{(1-\beta)\beta^2} + C_N \cdot \tfrac{\alpha^2}{\beta^2} \cdot \Big(\tfrac{\sqrt{K}}{R} + \tfrac{1}{R^2}\Big)\end{aligned} \tag{1.6}$$

for any $\alpha > 1$ and any $\beta \in (0, 1)$, where $f = \ln u$, and C_N is a constant depending only on N. Here and in the sequel of this paper, $\sup_{B_R \times [a,b]} g$ *means* $\operatorname{ess\,sup}_{B_R \times [a,b]} g$ *for a function $g(x, t)$.*

The local boundedness and the Harnack inequality for locally weak solutions of the heat equation have been established by Sturm [49,50] in the setting of abstract local Dirichlet form and by Marola and Masson [39] in the setting of metric measure with a standard volume doubling property and supporting a L^2-Poincare inequality. Of course, they are available on metric measure spaces (X, d, μ) satisfying $RCD^*(K, N)$ for some $K \in \mathbb{R}$ and $N \in [1, \infty)$. In particular, any locally weak solutions for the heat equation must be locally Hölder continuous.

As a consequence of Theorem 1.4, letting $R \to \infty$ and $\beta \to 1$, we get the following global gradient estimates.

Corollary 1.5 *Let (X, d, μ) and K, N, T_* be as in the Theorem 1.4. Let $u(x, t)$ is a positive solution of the heat equation on $X \times (0, T_*)$. Then, for almost all $T \in (0, T_*)$, the following gradient estimate holds*

$$\sup_{x \in X} \Big(|\nabla f|^2 - \alpha \cdot \frac{\partial}{\partial t} f\Big)(x, T) \leq \max\Big\{1, \frac{1}{2} + \frac{KT}{2(\alpha-1)}\Big\} \cdot \frac{N\alpha^2}{2T} \leq \Big(1 + \frac{KT}{2(\alpha-1)}\Big) \cdot \frac{N\alpha^2}{2T}$$

for any $\alpha > 1$, where $f = \ln u$.

As another application of the maximum principle, Theorem 1.3, and the Bochner formula, we will deduce a *sharp* Yau's gradient estimate for harmonic functions on metric measure spaces satisfying $RCD^*(-K, N)$ for $K \geq 0$ and $N > 1$.

Let us recall the classical local Yau's gradient estimate in geometric analysis (see [14, 38,51]). Let M^n be an $n(\geq 2)$-dimensional complete non-compact Riemannian manifold with $Ric(M^n) \geq -k$ for some $k \geq 0$. The local Yau's gradient estimate asserts that for any positive harmonic function u on B_{2R}, then

$$\sup_{B_R} |\nabla \ln u| \leq \sqrt{(n-1)k} + \frac{C(n)}{R}. \tag{1.7}$$

In particular, if u is positive harmonic on M^n and $Ric \geq -(n-1)$ on M^n then it follows that $|\nabla \log u| \leq n - 1$ on M^n. This result is sharp, in fact the equality case was characterized in [38]. This means that for $k = n - 1$ in (1.7) the factor $\sqrt{n-1}$ on the right hand side is sharp.

Let (X, d, μ) be a metric measure space satisfying $RCD^*(-K, N)$ for some $K \geq 0$ and $N \in (1, \infty)$. It was proved in [27] the following form of Yau's gradient estimate that, for any positive harmonic function u on $B_{2R} \subset X$, it holds

$$\sup_{B_R} |\nabla \ln u| \leq C(N, K, R). \tag{1.8}$$

In the setting of Alexandrov spaces, by using a Bochner formula and an argument of Nash–Moser iteration, it was proved in [25,53] the following form of Yau's gradient estimate holds: given an n-dimensional Alexandrov space M and a positive harmonic function u on $B_{2R} \subset M$, if the generalized Ricci curvature on $B_{2R} \subset M$ has a lower bound $Ric \geq -k$, $k \geq 0$, in the sense of [52], then

$$\sup_{B_R} |\nabla \ln u| \leq C_1(n)\sqrt{k} + \frac{C_2(n)}{R}.$$

Indeed, by applying Theorem 1.2, the same argument in [25,53] implies this estimate still holds for harmonic function u on a metric measure space (X, d, μ) with $RCD^*(-k, n)$. However, it seems hopeless to improve the fact $C_1(n)$ to the sharp $\sqrt{n-1}$ in (1.7) via a Nash–Moser iteration argument.

The last result in this paper is to establish a sharp local Yau's gradient estimate on metric measure spaces with Riemannian curvature-dimension condition.

Theorem 1.6 *Let $K \geq 0$ and $N \in (1, \infty)$, and let (X, d, μ) be a metric measure space satisfying $RCD^*(-K, N)$. Let B_{2R} be a geodesic ball of radius $2R$ centered at $p \in X$, and let $u(x)$ be a positive harmonic function on B_{2R}. Then the following local Yau's gradient estimate holds*

$$\sup_{B_R} |\nabla \ln u| \leq \sqrt{\frac{1+\beta}{1-\beta} \cdot (N-1)K} + \frac{C(N)}{\sqrt{\beta(1-\beta)} \cdot R} \tag{1.9}$$

for any $\beta \in (0, 1)$.

2 Preliminaries

Let (X, d) be a complete metric space and μ be a Radon measure on X with $\mathrm{supp}(\mu) = X$. Denote by $B_r(x)$ the open ball centered at x and radius r. Throughout the paper, we assume that X is proper (i.e., closed balls of finite radius are compact). Denote by $L^p(\Omega) := L^p(\Omega, \mu)$ for any open set $\Omega \subset X$ and any $p \in [1, \infty]$.

2.1 Reduced and Riemannian curvature-dimension conditions

Let $\mathscr{P}_2(X, d)$ be the L^2-Wasserstein space over (X, d), i.e., the set of all Borel probability measures ν satisfying

$$\int_X d^2(x_0, x) d\nu(x) < \infty$$

for some (hence for all) $x_0 \in X$. Given two measures $\nu_1, \nu_2 \in \mathscr{P}_2(X, d)$, the L^2-Wasserstein distance between them is given by

$$W^2(\nu_0, \nu_1) := \inf \int_{X \times X} d^2(x, y) dq(x, y)$$

where the infimum is taken over all couplings q of ν_1 and ν_2, i.e., Borel probability measures q on $X \times X$ with marginals ν_0 and ν_1. Such a coupling q realizes the L^2-Wasserstein distance is called an *optimal coupling* of ν_0 and ν_1. Let $\mathscr{P}_2(X, d, \mu) \subset \mathscr{P}_2(X, d)$ be the subspace of all measures absolutely continuous w.r.t. μ. Denote by $\mathscr{P}_\infty(X, d, \mu) \subset \mathscr{P}_2(X, d, \mu)$ the set of measures in $\mathscr{P}_2(X, d, \mu)$ with bounded support.

Definition 2.1 Let $K \in \mathbb{R}$ and $N \in [1, \infty)$. A metric measure space (X, d, μ) is called to satisfy the *reduced curvature-dimension condition* $CD^*(K, N)$ if any only if for each pair $\nu_0 = \rho_0 \cdot \mu, \nu_1 = \rho_1 \cdot \mu \in \mathscr{P}_\infty(X, d, \mu)$ there exist an optimal coupling q of them and a geodesic $(\nu_t := \rho_t \cdot \mu)_{t\in[0,1]}$ in $\mathscr{P}_\infty(X, d, \mu)$ connecting them such that for all $t \in [0, 1]$ and all $N' \geq N$:

$$\int_X \rho_t^{-1/N'} d\nu_t \geq \int_{X\times X} \Big[\sigma_{K/N'}^{(1-t)}(d(x_0, x_1))\rho_0^{-1/N'}(x_0) + \sigma_{K/N'}^{(t)}(d(x_0, x_1))\rho_1^{-1/N'}(x_1)\Big] \\ dq(x_0, x_1),$$

where the function

$$\sigma_k^{(t)}(\theta) := \begin{cases} \frac{\sin(\sqrt{k}\cdot t\theta)}{\sin(\sqrt{k}\cdot\theta)}, & 0 < k\theta^2 < \pi^2, \\ t, & k\theta^2 = 0, \\ \frac{\sinh(\sqrt{-k}\cdot t\theta)}{\sinh(\sqrt{-k}\cdot\theta)}, & k\theta^2 < 0, \\ \infty, & k\theta^2 \geq \pi^2. \end{cases}$$

Given a function $f \in C(X)$, the *pointwise Lipschitz constant* [13] of f at x is defined by

$$\mathrm{Lip} f(x) := \limsup_{y\to x} \frac{|f(y) - f(x)|}{d(x, y)} = \limsup_{r\to 0} \sup_{d(x,y)\leq r} \frac{|f(y) - f(x)|}{r},$$

where we put $\mathrm{Lip} f(x) = 0$ if x is isolated. Clearly, $\mathrm{Lip} f$ is a μ-measurable function on X. The *Cheeger energy*, denoted by $\mathrm{Ch} : L^2(X) \to [0, \infty]$, is defined [4] by

$$\mathrm{Ch}(f) := \inf\left\{\liminf_{j\to\infty} \frac{1}{2}\int_X (\mathrm{Lip} f_j)^2 d\mu\right\},$$

where the infimum is taken over all sequences of Lipschitz functions $(f_j)_{j\in\mathbb{N}}$ converging to f in $L^2(X)$. In general, Ch is a convex and lower semi-continuous functional on $L^2(X)$.

Definition 2.2 A metric measure space (X, d, μ) is called *infinitesimally Hilbertian* if the associated Cheeger energy is quadratic. Moreover, (X, d, μ) is said to satisfy *Riemannian curvature-dimension condition* $RCD^*(K, N)$, for $K \in \mathbb{R}$ and $N \in [1, \infty)$, if it is infinitesimally Hilbertian and satisfies the $CD^*(K, N)$ condition.

Let (X, d, μ) be a metric measure space satisfying $RCD^*(K, N)$. For each $f \in D(\mathrm{Ch})$, i.e., $f \in L^2(X)$ and $\mathrm{Ch}(f) < \infty$, it has

$$\mathrm{Ch}(f) = \frac{1}{2}\int_X |\nabla f|^2 d\mu,$$

where $|\nabla f|$ is the so-called minimal relaxed gradient of f (see §4 in [4]). It was proved, according to [4, Lemma 4.3] and Mazur's lemma, that Lipschitz functions are dense in $D(\mathrm{Ch})$, i.e., for each $f \in D(\mathrm{Ch})$, there exist a sequence of Lipschitz functions $(f_j)_{j\in\mathbb{N}}$ such that $f_j \to f$ in $L^2(X)$ and $|\nabla(f_j - f)| \to 0$ in $L^2(X)$. Since the Cheeger energy Ch is a quadratic form, the minimal relaxed gradients bring an inner product as following: given $f, g \in D(\mathrm{Ch})$, it was proved [18] that the limit

$$\langle\nabla f, \nabla g\rangle := \lim_{\epsilon\to 0} \frac{|\nabla(f + \epsilon \cdot g)|^2 - |\nabla f|^2}{2\epsilon}$$

exists in $L^1(X)$. The inner product is bi-linear and satisfies Cauchy–Schwarz inequality, Chain rule and Leibniz rule (see Gigli [18]).

2.2 Canonical Dirichlet form and a global version of Bochner formula

Given an infinitesimally Hilbertian metric measure space (X, d, μ), the energy $\mathscr{E} := 2\mathrm{Ch}$ gives a canonical Dirichlet form on $L^2(X)$ with the domain $\mathbb{V} := D(\mathrm{Ch})$. Let $K \in \mathbb{R}$ and $N \in [1, \infty)$, and let (X, d, μ) be a metric measure space satisfying $RCD^*(K, N)$. It has been shown [1,3] that the canonical Dirichlet form $(\mathscr{E}, \mathbb{V})$ is strongly local and admits a Carré du champ Γ with $\Gamma(f) = |\nabla f|^2$ of $f \in \mathbb{V}$. Namely, the energy measure of $f \in \mathbb{V}$ is absolutely continuous w.r.t. μ with the density $|\nabla f|^2$. Moreover, the intrinsic distance $d_{\mathscr{E}}$ induced by $(\mathscr{E}, \mathbb{V})$ coincides with the original distance d on X.

It is worth noticing that if a metric measure space (X, d, μ) satisfying $RCD^*(K, N)$ then its associated Dirichlet form $(\mathscr{E}, \mathbb{V})$ satisfies the standard assumptions: the local volume doubling property and supporting a local L^2-Poincare inequality (see [45,48]).

Let $\big(\Delta_{\mathscr{E}}, D(\Delta_{\mathscr{E}})\big)$ and $(H_t f)_{t\geq 0}$ denote the infinitesimal generator and the heat flow induced from $(\mathscr{E}, \mathbb{V})$. Let us recall the Bochner formula (also called the *Bakry–Emery condition*) in [16] as following.

Lemma 2.3 *Let (X, d, μ) be a metric measure space satisfying $RCD^*(K, N)$ for $K \in \mathbb{R}$ and $N \in [1, \infty)$, and let $(\mathscr{E}, \mathbb{V})$ be the associated canonical Dirichlet form. Then the following properties hold.*

(i) [16, Theorem 4.8] *If $f \in D(\Delta_{\mathscr{E}})$ with $\Delta_{\mathscr{E}} f \in \mathbb{V}$ and if $\phi \in D(\Delta_{\mathscr{E}}) \cap L^\infty(X)$ with $\phi \geq 0$ and $\Delta_{\mathscr{E}}\phi \in L^\infty(X)$, then we have the Bochner formula:*

$$\frac{1}{2}\int_X \Delta_{\mathscr{E}}\phi \cdot |\nabla f|^2 d\mu \ \geq \frac{1}{N}\int_X \phi(\Delta_{\mathscr{E}} f)^2 d\mu + \int_X \phi\langle\nabla(\Delta_{\mathscr{E}} f), \nabla f\rangle d\mu$$
$$+K\int_X \phi|\nabla f|^2 d\mu. \tag{2.1}$$

(ii) [5, Theorem 5.5] *If $f \in D(\Delta_{\mathscr{E}})$ with $\Delta_{\mathscr{E}} f \in L^4(X) \cap L^2(X)$ and if $\phi \in \mathbb{V}$ with $\phi \geq 0$, then we have $|\nabla f|^2 \in \mathbb{V}$ and the modified Bochner formula:*

$$\int_X \left(-\frac{1}{2}\langle\nabla|\nabla f|^2, \nabla\phi\rangle + \Delta_{\mathscr{E}} f \cdot \langle\nabla f, \nabla\phi\rangle + \phi\cdot(\Delta_{\mathscr{E}} f)^2\right) d\mu$$
$$\geq \int_X \left(K|\nabla f|^2 + \frac{1}{N}(\Delta_{\mathscr{E}} f)^2\right)\cdot\phi d\mu. \tag{2.2}$$

We need the following result on the existence of good cut-off functions on $RCD^*(K, N)$-spaces from [40, Lemma3.1]; see also [5,19,24].

Lemma 2.4 *Let (X, d, μ) be a metric measure space satisfying $RCD^*(K, N)$ for $K \in \mathbb{R}$ and $N \in [1, \infty)$. Then for every $x_0 \in X$ and $R > 0$ there exists a Lipschitz cut-off function $\chi : X \to [0, 1]$ satisfying:*

(i) *$\chi = 1$ on $B_{2R/3}(x_0)$ and $\mathrm{supp}(\chi) \subset B_R(x_0)$;*
(ii) *$\chi \in D(\Delta_{\mathscr{E}})$ and $\Delta_{\mathscr{E}}\chi \in \mathbb{V} \cap L^\infty(X)$, moreover $|\Delta_{\mathscr{E}}\chi| + |\nabla\chi| \leq C(N, K, R)$.*

2.3 Sobolev spaces

Several different notions of Sobolev spaces on metric measure space (X, d, μ) have been established in [13,21,22,46]. They are equivalent to each other on $RCD^*(K, N)$ metric measure spaces (see, for example [2]).

Let (X, d, μ) be a metric measure space satisfying $RCD^*(K, N)$ for some $K \in \mathbb{R}$ and $N \in [1, \infty)$. Fix an open set Ω in X. We denote by $Lip_{\rm loc}(\Omega)$ the set of locally Lipschitz continuous functions on Ω, and by $Lip(\Omega)$ (resp. $Lip_0(\Omega)$) the set of Lipschitz continuous functions on Ω (resp, with compact support in Ω).

Let $\Omega \subset X$ be an open set. For any $1 \le p \le +\infty$ and $f \in Lip_{\rm loc}(\Omega)$, its $W^{1,p}(\Omega)$-norm is defined by

$$\|f\|_{W^{1,p}(\Omega)} := \|f\|_{L^p(\Omega)} + \|{\rm Lip} f\|_{L^p(\Omega)}.$$

The Sobolev spaces $W^{1,p}(\Omega)$ is defined by the closure of the set

$$\left\{f \in Lip_{\rm loc}(\Omega) |\ \|f\|_{W^{1,p}(\Omega)} < +\infty\right\}$$

under the $W^{1,p}(\Omega)$-norm. Remark that $W^{1,p}(\Omega)$ is reflexive for any $1 < p < \infty$ (see [13, Theorem 4.48]). Spaces $W_0^{1,p}(\Omega)$ is defined by the closure of $Lip_0(\Omega)$ under the $W^{1,p}(\Omega)$-norm. We say a function $f \in W^{1,p}_{\rm loc}(\Omega)$ if $f \in W^{1,p}(\Omega')$ for every open subset $\Omega' \subset\subset \Omega$.

The following two facts are well-known for experts. For the convenience of readers, we include a proof here.

Lemma 2.5 (i) *For any* $1 < p < \infty$*, we have* $W^{1,p}(X) = W_0^{1,p}(X)$.
(ii) $W^{1,2}(X) = D(\text{Ch})$.

Proof Given a function $f \in Lip(X) \cap W^{1,p}(X)$, in order to prove (i), it suffices to find a sequence $(f_j)_{j\in\mathbb{N}}$ of Lipschitz functions with compact supports in X such that $f_j \to f$ in $W^{1,p}(X)$.

Consider a family of Lipschitz cut-off χ_j with, for each $j \in \mathbb{N}$, $\chi_j(x) = 1$ for $x \in B_j(x_0)$ and $\chi_j(x) = 0$ for $x \notin B_{j+1}(x_0)$, and $0 \le \chi_j(x) \le 1$, $|\nabla\chi_j|(x) \le 1$ for all $x \in X$. Now $f \cdot \chi_j \in Lip_0(X)$ and $f \cdot \chi_j(x) \to f(x)$ for μ-almost all $x \in X$. Notice that $|f \cdot \chi_j| \le |f| \in L^p(X)$ for all j, the dominated convergence theorem implies that $f \cdot \chi_j \to f$ in $L^p(X)$ as $j \to \infty$. On the other hand, since

$$|\nabla(f \cdot \chi_j)| \le |\nabla f| \cdot \chi_j + |f| \cdot |\nabla \chi_j| \le |\nabla f| + |f| \in L^p(X)$$

for all $j \in \mathbb{N}$, we obtain that the sequence $(f \cdot \chi_j)_{j\in\mathbb{N}}$ is bounded in $W^{1,p}(X)$. By noticing that $W^{1,p}(X)$ is reflexive (see [13, Theorem 4.48]), we can see that $f \cdot \chi_j$ converges weakly to f in $W^{1,p}(X)$ as $j \to \infty$. Hence, by Mazurs lemma, we conclude that there exists a convex combination of $f \cdot \chi_j$ converges strongly to f in $W^{1,p}(X)$ as $j \to \infty$. The proof of (i) is completed.

Let us prove (ii). It is obvious that $W^{1,2}(X) \subset D(\text{Ch})$, since $Lip(X) \cap W^{1,2}(X) \subset D(\text{Ch})$ and $|\nabla f_n| \le Lip(f_n)$. We need only to show $D(\text{Ch}) \subset W^{1,2}(X)$. This follows immediately from the fact that Lipschitz functions are dense in $D(\text{Ch})$. The proof of (ii) is completed. □

3 The weak Laplacian and a local version of Bochner formula

Let (X, d, μ) be a metric measure space satisfying $RCD^*(K, N)$ for some $K \in \mathbb{R}$ and $N \in [1, \infty)$. Fix any open set $\Omega \subset X$. We will denote by the Sobolev spaces $H_0^1(\Omega) := W_0^{1,2}(\Omega)$, $H^1(\Omega) := W^{1,2}(\Omega)$ and $H^1_{\rm loc}(\Omega) := W^{1,2}_{\rm loc}(\Omega)$.

Definition 3.1 (*Weak Laplacian*). Let $\Omega \subset X$ be an open set, the *Laplacian* on Ω is an operator $\mathscr{L}$ on $H^1(\Omega)$ defined as the follows. For each function $f \in H^1(\Omega)$, its Laplacian

$\mathscr{L} f$ is a functional acting on $H_0^1(\Omega) \cap L^\infty(\Omega)$ given by

$$\mathscr{L} f(\phi) := -\int_\Omega \langle \nabla f, \nabla \phi \rangle d\mu \quad \forall\, \phi \in H_0^1(\Omega) \cap L^\infty(\Omega).$$

For any $g \in H^1(\Omega) \cap L^\infty(\Omega)$, the distribution $g \cdot \mathscr{L} f$ is a functional acting on $H_0^1(\Omega) \cap L^\infty(\Omega)$ defined by

$$g \cdot \mathscr{L} f(\phi) := \mathscr{L} f(g\phi) \quad \forall\, \phi \in H_0^1(\Omega) \cap L^\infty(\Omega). \tag{3.1}$$

This Laplacian (on Ω) is linear due to that the inner product $\langle \nabla f, \nabla g \rangle$ is linear. The strongly local property of the inner product $\int_X \langle \nabla f, \nabla g \rangle d\mu$ implies that if $f \in H^1(X)$ and $f = constant$ on Ω then $\mathscr{L} f(\phi) = 0$ for any $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

If, given $f \in H^1(\Omega)$, there exists a function $u_f \in L^1_{\rm loc}(\Omega)$ such that

$$\mathscr{L} f(\phi) = \int_\Omega u_f \cdot \phi d\mu \quad \forall\, \phi \in H_0^1(\Omega) \cap L^\infty(\Omega), \tag{3.2}$$

then we say that "$\mathscr{L} f$ is a function in $L^1_{\rm loc}(\Omega)$" and write as "$\mathscr{L} f = u_f$ in the sense of distributions". It is similar to say that "$\mathscr{L} f$ is a function in $L^p_{\rm loc}(\Omega)$ or $W^{1,p}_{\rm loc}(\Omega)$ for any $p \in [1, \infty]$", and so on.

The operator $\mathscr{L}$ satisfies the following Chain rule and Leibniz rule, which is essentially due to Gigli [18].

Lemma 3.2 *Let Ω be an open domain of a metric measure space (X, d, μ) satisfying $RCD^*(K, N)$ for some $K \in \mathbb{R}$ and $N \in [1, \infty)$.*

(i) (Chain rule) *Let $f \in H^1(\Omega) \cap L^\infty(\Omega)$ and $\eta \in C^2(\mathbb{R})$. Then we have*

$$\mathscr{L}[\eta(f)] = \eta'(f) \cdot \mathscr{L} f + \eta''(f) \cdot |\nabla f|^2. \tag{3.3}$$

(ii) (Leibniz rule) *Let $f, g \in H^1(\Omega) \cap L^\infty(\Omega)$. Then we have*

$$\mathscr{L}(f \cdot g) = f \cdot \mathscr{L} g + g \cdot \mathscr{L} f + 2\langle \nabla f, \nabla g \rangle. \tag{3.4}$$

Proof The proof is given essentially in [18]. For the completeness, we sketch it. We prove only the Chain rule (3.3). The proof of Leibniz rule (3.4) is similar.

Given any $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, we have

$$\mathscr{L}[\eta(f)](\phi) = -\int_\Omega \langle \nabla[\eta(f)], \nabla \phi \rangle d\mu = -\int_\Omega \eta'(f) \cdot \langle \nabla f, \nabla \phi \rangle d\mu,$$

where we have used that (see [18, §3.3]) the inner product $\langle \nabla f, \nabla \phi \rangle$ satisfies the Chain rule, i.e., $\langle \nabla[\eta(f)], \nabla \phi \rangle = \eta'(f) \cdot \langle \nabla f, \nabla \phi \rangle$.

On the other hand, by (3.1), we obtain

$$\begin{aligned}[\eta'(f) \cdot \mathscr{L} f + \eta''(f) \cdot |\nabla f|^2](\phi) &= \mathscr{L} f(\eta'(f) \cdot \phi) + \textstyle\int_\Omega \eta''(f) \cdot |\nabla f|^2 \cdot \phi d\mu \\ &= -\textstyle\int_\Omega \langle \nabla f, \nabla(\eta'(f) \cdot \phi) \rangle d\mu + \int_\Omega \eta''(f) \cdot |\nabla f|^2 \cdot \phi d\mu \\ &= -\textstyle\int_\Omega \langle \nabla f, \nabla \phi \rangle \cdot \eta'(f) d\mu,\end{aligned}$$

where we have used that $\eta'(f) \cdot \phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and that (see [18, §3.3]) the inner product $\langle \nabla f, \nabla g \rangle$ satisfies the Chain rule and Leibniz rule, i.e.,

$$\begin{aligned}\langle \nabla f, \nabla\big(\eta'(f) \cdot \phi\big) \rangle &= \langle \nabla f, \nabla \phi \rangle \eta'(f) + \langle \nabla f, \nabla\big(\eta'(f)\big) \rangle \phi \\ &= \langle \nabla f, \nabla \phi \rangle \eta'(f) + \langle \nabla f, \nabla f \rangle \cdot \eta''(f)\phi.\end{aligned}$$

The combination of the above two equations implies the Chain rule (3.3). The proof is completed. □

To compare the above Laplace operator $\mathscr{L}$ on X with the generator $\Delta_{\mathscr{E}}$ of the canonical Dirichlet form $(\mathscr{E}, \mathbb{V})$, it was shown [18] that the following compatibility result holds.

Lemma 3.3 (Proposition 4.24 in [18]) *The following two statements are equivalent:*

i) $f \in H^1(X)$ *and* $\mathscr{L} f$ *is a function in* $L^2(X)$,
ii) $f \in D(\Delta_{\mathscr{E}})$.

In each of these cases, we have $\mathscr{L} f = \Delta_{\mathscr{E}} f$.

The following regularity result for the Poisson equation has been proved under a Bakry–Emery type heat semigroup curvature condition, which is implied by the Riemannian curvature-dimension condition $RCD^*(K, N)$ (see [16, Theorem 7] and [5, Theorem 7.5]).

Lemma 3.4 ([27,29]) *Let* (X, d, μ) *be a metric measure space satisfying* $RCD^*(K, N)$ *for* $K \in \mathbb{R}$ *and* $N \in [1, \infty)$. *Let* $g \in L^\infty(B_R)$, *where* B_R *is a geodesic ball with radius* R *and centered at a fixed point* x_0. *Assume* $f \in H^1(B_R)$ *and* $\mathscr{L} f = g$ *on* B_R *in the sense of distributions. Then we have* $|\nabla f| \in L^\infty_{\rm loc}(B_R)$, *and*

$$\||\nabla f|\|_{L^\infty(B_{R/2})} \le C(N, K, R) \cdot \left(\frac{1}{\mu(B_R)} \|f\|_{L^1(B_R)} + \|g\|_{L^\infty(B_R)} \right).$$

Proof In the case of $g = 0$, i.e., f is harmonic on B_R, the assertion is proved in [27, Theorem 1.2] (see also [19, Theorem 3.9]). In the general case $g \in L^\infty(\Omega)$, this is proved in [29, Theorem 3.1]. The assertion of the constant $C(N, K, R)$ depending only on N, K, R comes from the fact that both the doubling constant and L^2-Poincare constant on a ball B_R of a $RCD^*(K, N)$-space depend on N, K and R. □

Now we will give a local version of the Bochner formula, Theorem 1.2, by combining the modified Bochner formula (2.2) and a similar argument in [24,28].

Theorem 3.5 ([5,24]) *Let* (X, d, μ) *be a metric measure space satisfying* $RCD^*(K, N)$ *for* $K \in \mathbb{R}$ *and* $N \in [1, \infty)$. *Let* B_R *be a geodesic ball with radius* R *and centered at a point* x_0.

Assume that $f \in H^1(B_R)$ *satisfies* $\mathscr{L} f = g$ *on* B_R *in the sense of distributions with the function* $g \in H^1(B_R) \cap L^\infty(B_R)$. *Then we have* $|\nabla f|^2 \in H^1(B_{R/2}) \cap L^\infty(B_{R/2})$ *and*

$$\frac{1}{2}\mathscr{L}(|\nabla f|^2) \ge \left[\frac{g^2}{N} + \langle \nabla f, \nabla g\rangle + K|\nabla f|^2 \right] \cdot \mu \quad \text{on} \quad B_{R/2} \tag{3.5}$$

in the sense of distributions, i.e.,

$$-\frac{1}{2}\int_{B_{R/2}} \langle \nabla|\nabla f|^2, \nabla\phi\rangle d\mu \ge \int_{B_{R/2}} \phi \cdot \left(\frac{g^2}{N} + \langle \nabla f, \nabla g\rangle + K|\nabla f|^2 \right) d\mu$$

for any $0 \le \phi \in H^1_0(B_{R/2}) \cap L^\infty(B_{R/2})$.

Proof From Lemma 3.4 and that $g \in L^\infty(B_R)$, we know $|\nabla f| \in L^\infty_{\rm loc}(B_R)$.

We take a cut-off χ satisfying (i) and (ii) in Lemma 2.4. Let

$$\widetilde{f}(x) := \begin{cases} f \cdot \chi & \text{if} \quad x \in B_R \\ 0 & \text{if} \quad x \in X \backslash B_R. \end{cases}$$

Then we have $\widetilde{f} \in Lip_0(B_R)$. It is easy to check ${\rm supp}(\mathscr{L}\widetilde{f}) \subset B_R$. In fact, given any $\psi \in H^1_0(X)$ with $\psi = 0$ on B_R, the strongly local property implies that $\int_X \langle \nabla \widetilde{f}, \nabla\psi\rangle d\mu = 0$.

Now we want to calculate $\mathscr{L}\widetilde{f}$ on B_R. By the Leibniz rule (3.4), we have, on B_R,

$$\begin{aligned}\mathscr{L}\widetilde{f} = \mathscr{L}(f\cdot\chi) &= \chi\cdot\mathscr{L}f + f\cdot\mathscr{L}\chi + 2\langle\nabla f,\nabla\chi\rangle\\ &= \chi\cdot g + f\cdot\Delta_{\mathscr{E}}\chi + 2\langle\nabla f,\nabla\chi\rangle \in L^\infty_{\rm loc}(B_R),\end{aligned}$$

where we have used $g\in L^\infty(B_R)$ and $|\nabla f|\in L^\infty_{\rm loc}(B_R)$, and that χ, $|\nabla\chi|$, $\Delta_{\mathscr{E}}\chi\in L^\infty(X)$ in Lemma 2.4. Combining with $\mathrm{supp}(\mathscr{L}\widetilde{f})\subset B_R$, we have $\mathscr{L}\widetilde{f}\in L^2(X)\cap L^\infty(X)$. Therefore, by Lemma 3.3, we get $\widetilde{f}\in D(\Delta_{\mathscr{E}})$ and

$$L^2(X)\cap L^\infty(X)\ni\Delta_{\mathscr{E}}\widetilde{f} = \mathscr{L}\widetilde{f} = \begin{cases}\chi\cdot g + f\cdot\Delta_{\mathscr{E}}\chi + 2\langle\nabla f,\nabla\chi\rangle & \text{if}\quad x\in B_R\\ 0 & \text{if}\quad x\in X\backslash B_R.\end{cases}\tag{3.6}$$

According to Lemma 2.3(ii) and $0\le\phi\in H^1_0(B_{R/2})\subset\mathbb{V}$, we conclude that $|\nabla\widetilde{f}|^2\in\mathbb{V}$ and that

$$\begin{aligned}&\int_X\left(-\frac{1}{2}\langle\nabla|\nabla\widetilde{f}|^2,\nabla\phi\rangle + \Delta_{\mathscr{E}}\widetilde{f}\cdot\langle\nabla\widetilde{f},\nabla\phi\rangle + \phi\cdot(\Delta_{\mathscr{E}}\widetilde{f})^2\right)d\mu\\ &\quad\ge\frac{1}{N}\int_X\phi\cdot(\Delta_{\mathscr{E}}\widetilde{f})^2d\mu + K\int_X\phi|\nabla\widetilde{f}|^2d\mu.\end{aligned}$$

Since $\widetilde{f}=f$ on $B_{R/2}$, we have $|\nabla f|=|\nabla\widetilde{f}|$ for μ-a.e. on $B_{R/2}$. Notice that $|\nabla\widetilde{f}|^2\in\mathbb{V}$ implies that $|\nabla\widetilde{f}|^2\in H^1(B_{R/2})$. Then $|\nabla f|^2\in H^1(B_{R/2})$, and $|\nabla|\nabla f|^2|=|\nabla|\nabla\widetilde{f}|^2|$ in $L^2(B_{R/2})$. By (3.6) and that $|\nabla\chi|=\Delta_{\mathscr{E}}\chi=0$ on $B_{R/2}$ (since $\chi=1$ on $B_{2R/3}$), we have $\Delta_{\mathscr{E}}\widetilde{f}=g$ on $B_{R/2}$. Hence, we obtain

$$\begin{aligned}&\int_{B_{R/2}}\left(-\frac{1}{2}\langle\nabla|\nabla f|^2,\nabla\phi\rangle + g\cdot\langle\nabla f,\nabla\phi\rangle + \phi\cdot g^2\right)d\mu\\ &\quad\ge\frac{1}{N}\int_{B_{R/2}}\phi\cdot g^2d\mu + K\int_{B_{R/2}}\phi|\nabla f|^2d\mu.\end{aligned}\tag{3.7}$$

Noticing that $g\cdot\phi\in H^1_0(B_{R/2})\cap L^\infty(B_{R/2})$ and $\mathscr{L}f=g$ on B_R in the sense of distributions, we have

$$\begin{aligned}\int_{B_{R/2}}g\cdot g\phi d\mu = \mathscr{L}f(g\phi) &= -\int_{B_{R/2}}\langle\nabla f,\nabla(g\phi)\rangle d\mu\\ &= -\int_{B_{R/2}}\langle\nabla f,\nabla g\rangle\cdot\phi d\mu - \int_{B_{R/2}}\langle\nabla f,\nabla\phi\rangle\cdot g d\mu.\end{aligned}$$

By combining this and (3.7), we get the desired inequality (3.5). The proof is finished. □

By using the same argument of [8], one can get an improvement of the above Bochner formula. One can also consult a detailed argument given in [31, Lemma 2.3].

Corollary 3.6 *Let (X,d,μ) be a metric measure space satisfying $RCD^*(K,N)$ for $K\in\mathbb{R}$ and $N\in[1,\infty)$. Let B_R be a geodesic ball with radius R and centered at a fixed point x_0.*

Assume that $f\in H^1(B_R)$ satisfies $\mathscr{L}f=g$ on B_R in the sense of distributions with the function $g\in H^1(B_R)\cap L^\infty(B_R)$. Then we have $|\nabla f|^2\in H^1(B_{R/2})\cap L^\infty(B_{R/2})$ and that the distribution $\mathscr{L}(|\nabla f|^2)$ is a signed Radon measure on $B_{R/2}$. If its Radon–Nikodym decomposition w.r.t. μ is denoted by

$$\mathscr{L}(|\nabla f|^2) = \mathscr{L}^{\rm ac}(|\nabla f|^2)\cdot\mu + \mathscr{L}^{\rm sing}(|\nabla f|^2),$$

then we have $\mathscr{L}^{\rm sing}(|\nabla f|^2)\geq 0$ *and, for* μ*-a.e.* $x\in B_{R/2}$,

$$\frac{1}{2}\mathscr{L}^{\rm ac}(|\nabla f|^2)\geq \frac{g^2}{N}+\langle\nabla f,\nabla g\rangle+K|\nabla f|^2.$$

Furthermore, if $N>1$, *for* μ*-a.e.* $x\in B_{R/2}\cap\{y:\ |\nabla f(y)|\neq 0\}$,

$$\frac{1}{2}\mathscr{L}^{\rm ac}(|\nabla f|^2)\geq \frac{g^2}{N}+\langle\nabla f,\nabla g\rangle+K|\nabla f|^2+\frac{N}{N-1}\cdot\left(\frac{\langle\nabla f,\nabla|\nabla f|^2\rangle}{2|\nabla f|^2}-\frac{g}{N}\right)^2. \quad (3.8)$$

4 The maximum principle

Let $K\in\mathbb{R}$ and $N\in[1,\infty)$ and let (X,d,μ) be a metric measure space satisfying $RCD^*(K,N)$. In this section, we will study the maximum principle on (X,d,μ). Let us begin from the Kato's inequality for *weighted measures*.

4.1 The Kato's inequality

Let Ω be a bounded open set of (X,d,μ). Fix any $w\in H^1(\Omega)\cap L^\infty(\Omega)$, we consider the weighted measure

$$\mu_w:=e^w\cdot\mu\quad\text{on}\ \ \Omega.$$

Since, the density $e^{-\|w\|_{L^\infty(\Omega)}}\leq e^w\leq e^{\|w\|_{L^\infty(\Omega)}}$ on Ω, we know that the associated the Lebesgue space $L^p(\Omega,\mu_w)$ and the Sobolev spaces $W^{1,p}(\Omega,\mu_w)$ are equivalent to the original $L^p(\Omega)$ and $W^{1,p}(\Omega)$, respectively, for all $p\geq 1$. Both the measure doubling property and the L^2-Poincare inequality still hold with respect to this measure μ_w (the constants, of course, depend on $\|w\|_{L^\infty(\Omega)}$).

For this measure μ_w, we defined the associated Laplacian $\mathscr{L}_w$ on $f\in H^1(\Omega)$ by

$$\mathscr{L}_w f(\phi):=-\int_\Omega\langle\nabla f,\nabla\phi\rangle d\mu_w\ \ \left(=-\int_\Omega\langle\nabla f,\nabla\phi\rangle e^w d\mu\right)$$

for any $\phi\in H^1_0(\Omega)\cap L^\infty(\Omega)$. It is easy to check that

$$\mathscr{L}_w f=e^w\cdot\mathscr{L}f+e^w\cdot\langle\nabla w,\nabla f\rangle$$

in the sense of distributions, i.e., $\mathscr{L}_w f(\phi)=\mathscr{L}f(e^w\cdot\phi)+\int_\Omega\langle\nabla w,\nabla f\rangle e^w\cdot\phi d\mu$.

When Ω be a domain of the Euclidean space $\mathbb{R}^N$ with dimension $N\geq 1$, the classical Kato's inequality states that given any function $f\in L^1_{\rm loc}(\Omega)$ such that $\Delta f\in L^1_{\rm loc}(\Omega)$, then Δf_+ is a signed Radon measure and the following holds:

$$\Delta f_+\geq\chi[f\geq 0]\cdot\Delta f$$

in the sense of distributions, where $f_+:=\max\{f,0\}$. Here, $\chi[f\geq 0](x)=1$ for x such that $f(x)\geq 0$ and $\chi[f\geq 0](x)=0$ for x such that $f(x)<0$. In [11], the result was extended to the case when Δf is a signed Radon measure.

In the following, we will extend the Kato's inequality to the metric measure spaces (X,d,μ_w), under assumption $f\in H^1(\Omega)$.

Proposition 4.1 (Kato's inequality) *Let Ω be a bounded open set of (X, d, μ) and let $w \in H^1(\Omega) \cap L^\infty(\Omega)$. Assume that $f \in H^1(\Omega)$ such that $\mathscr{L}_w f$ is a signed Radon measure. Then $\mathscr{L}_w f_+$ is a signed Radon measure and the following holds:*

$$\mathscr{L}_w f_+ \geq \chi[f \geq 0] \cdot \mathscr{L}_w^{\mathrm{ac}} f \cdot \mu_w \quad \text{on} \ \ \Omega, \tag{4.1}$$

in the sense of distributions. In the sequel, we denote the Radon–Nikodym decomposition $\mathscr{L}_w f = \mathscr{L}_w f + \mathscr{L}_w^{\mathrm{sing}} f$.

Proof It suffices to prove the following equivalent property:

$$\mathscr{L}_w |f| \geq \mathrm{sgn}(f) \cdot \mathscr{L}_w f, \tag{4.2}$$

where $\mathrm{sgn}(t) = 1$ for $t > 0$, $\mathrm{sgn}(t) = -1$ for $t < 0$, and $\mathrm{sgn}(t) = 0$ for $t = 0$.

Fix any $\epsilon > 0$ and let

$$f_\epsilon(x) := (f^2 + \epsilon^2)^{1/2} \geq \epsilon.$$

We have $f_\epsilon^2 = f^2 + \epsilon^2$,

$$|\nabla f_\epsilon| = \frac{|f|}{f_\epsilon} |\nabla f| \leq |\nabla f| \tag{4.3}$$

and

$$2 f_\epsilon \cdot \mathscr{L}_w f_\epsilon + 2|\nabla f_\epsilon|^2 = \mathscr{L}_w f_\epsilon^2 = \mathscr{L}_w f^2 = 2f \cdot \mathscr{L}_w f + 2|\nabla f|^2.$$

Thus,

$$\mathscr{L}_w f_\epsilon \geq \frac{f}{f_\epsilon} \cdot \mathscr{L}_w f. \tag{4.4}$$

Notice that $|\nabla f_\epsilon| \leq |\nabla f|$ and $f_\epsilon \to |f|$ in $L^2(\Omega)$ implies that f_ϵ is bounded in $H^1(\Omega)$ and, hence, there exists a subsequence f_{ϵ_j} converging weakly to $|f|$ in $H^1(\Omega)$. Thus, the measures $\mathscr{L}_w(f_{\epsilon_j})$ converges weakly to $\mathscr{L}_w |f|$. On the other hand, notice that $f_\epsilon(x) \to |f(x)|$ for each $x \in \Omega$ and that $|f/f_\epsilon| \leq 1$ on Ω. Letting $\epsilon := \epsilon_j \to 0$ in (4.4), we conclude that

$$\mathscr{L}_w(|f|) \geq \frac{f}{|f|} \cdot \mathscr{L}_w f.$$

This is (4.2), and the proof is completed. □

4.2 Maximum principles

The above Kato's inequality implies the maximum principle Theorem 1.3. Precisely, we have the following.

Theorem 4.2 *Let Ω be a bounded domain. Let $f(x) \in H^1(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ such that $\mathscr{L} f$ is a signed Radon measure with $\mathscr{L}^{\mathrm{sing}} f \geq 0$. Suppose that f achieves one of its strict maximum in Ω in the sense that: there exists a neighborhood $U \subset\subset \Omega$ such that*

$$\sup_U f > \sup_{\Omega \setminus U} f. \tag{4.5}$$

Here and in the sequel of the paper, the notion $\sup_U f$ means always $\operatorname{ess\,sup}_U f$. Then, given any $w \in H^1(\Omega) \cap L^\infty(\Omega)$, for any $\varepsilon > 0$, we have

$$\mu \left\{ x : \ f(x) \geq \sup_\Omega f - \varepsilon \ \text{ and } \ \mathscr{L}^{\mathrm{ac}} f(x) + \langle \nabla f, \nabla w \rangle (x) \leq \varepsilon \right\} > 0. \tag{4.6}$$

In particular, there exists a sequence of points $\{x_j\}_{j\in\mathbb{N}} \subset U$ such that they are the approximate continuity points of $\mathscr{L}^{\rm ac} f$ and $\langle \nabla f, \nabla w\rangle$, and that

$$f(x_j) \geq \sup_\Omega f - 1/j \quad \text{and} \quad \mathscr{L}^{\rm ac} f(x_j) + \langle \nabla f, \nabla w\rangle(x_j) \leq 1/j.$$

Proof Suppose the first assertion (4.6) fails for some sufficiently small $\varepsilon_0 > 0$. Then we have $\big(f - (\sup_\Omega f - \varepsilon_0)\big)_+ \in H^1_0(\Omega)$ (by the maximal property (4.5)) and

$$\mu\left\{x :\ f(x) \geq \sup_\Omega f - \varepsilon_0 \ \text{ and } \ \mathscr{L}^{\rm ac} f + \langle \nabla f, \nabla w\rangle \leq \varepsilon_0\right\} = 0.$$

Then for almost $x \in \{y :\ f(y) \geq \sup_\Omega f - \varepsilon_0\}$ we have

$$\mathscr{L}^{\rm ac}_w f(x)\cdot \mu_w = e^{w(x)}\cdot(\mathscr{L}^{\rm ac} f + \langle \nabla f, \nabla w\rangle)(x)\cdot\mu > e^{-\|w\|_{L^\infty}}\varepsilon_0\cdot\mu > 0.$$

The assumption $\mathscr{L}^{\rm sing} f \geq 0$ implies that $\mathscr{L}^{\rm sing}_w f \geq 0$. By applying the Proposition 4.1 to the function $f - (\sup_\Omega - \varepsilon_0)$, we have

$$\mathscr{L}_w(f - (\sup_\Omega f - \varepsilon_0))_+ \geq \chi[f \geq \sup_\Omega f - \varepsilon_0]\cdot \mathscr{L}^{\rm ac}_w f\cdot\mu_w \geq 0$$

on Ω, in the sense of distributions. Recall that the metric measure space (X, d, μ_w) satisfies a doubling property and supports a L^2-Poincare inequality. Now the weak maximum principle [13, Theorem 7.17] implies that $(f - (\sup_\Omega f - \varepsilon_0))_+ = 0$ on Ω. Thus, $\sup_\Omega f \leq \sup_\Omega f - \varepsilon_0$ on Ω. This is a contradiction, and proves the first assertion (4.6).

The second assertion follows from the first one by taking $\varepsilon = 1/j$. □

Next, let us consider the parabolic version of the maximum principle. We need the following parabolic weak maximum principle.

Lemma 4.3 *Let Ω be a bounded open subset and let $T > 0$. Let $w \in H^1(\Omega_T) \cap L^\infty(\Omega_T)$ with $\partial_t w(x, t) \leq C$ for some constant $C > 0$, for almost all $(x, t) \in \Omega_T$. Suppose that $f(x, t) \in H^1(\Omega_T) \cap L^\infty(\Omega_T)$ with $\lim_{t\to 0}\|f(\cdot, t)\|_{L^2(\Omega)} = 0$ and, for almost all $t \in (0, T)$, that the functions $f(\cdot, t) \in H^1_0(\Omega)$. Assume that, for almost every $t \in (0, T)$, the function $f(\cdot, t)$ satisfies*

$$\mathscr{L}_{w(\cdot,t)} f(\cdot, t) - \frac{\partial}{\partial t} f(\cdot, t)\cdot \mu_{w(\cdot,t)} \geq 0 \quad \text{on } \ \Omega \tag{4.7}$$

Then we have

$$\sup_{\Omega\times(0,T)} f(x, t) \leq 0.$$

Proof The proof is standard via a Gaffney–Davies' method (see also [49, Lemma 1.7]). We include a proof here for the completeness. Since f_+ meets all of conditions in this lemma, by replacing f by f_+, we can assume that $f \geq 0$.

Put

$$\xi(t) := \int_\Omega f^2(\cdot, t)d\mu_{w(\cdot,t)}.$$

Since $\mu_{w(\cdot,t)} = e^w\cdot\mu \leq e^{\|w\|_{L^\infty}}\cdot\mu$ and $f \in H^1(\Omega_T)$, we have, for almost all $t \in (0, T)$,

$$\begin{aligned}\xi'(t) &= \textstyle\int_\Omega \partial_t(f^2)d\mu_{w(\cdot,t)} + \int_\Omega f^2\cdot\partial_t w\cdot d\mu_{w(\cdot,t)}\\ &\leq -2\textstyle\int_\Omega |\nabla f|^2 d\mu_{w(\cdot,t)} + C\cdot\xi(t) \leq C\cdot\xi(t),\end{aligned}$$

where we have used $\partial_t w \le C$ and that the functions $f(\cdot,t) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ for almost all $t \in (0,T)$. By using $\lim_{t\to 0} \xi(t) = 0$ (since $\xi(t) \le e^{\|w\|_{L^\infty}} \cdot \|f(\cdot,t)\|_{L^2(\Omega)}$ and the assumption $\lim_{t\to 0} \|f(\cdot,t)\|_{L^2(\Omega)} = 0$), one can obtain that $\xi(t) \le 0$. This implies $f = 0$ almost all in Ω_T. The proof is finished. □

By using the same argument as in Theorem 4.2, the combination of the Kato's inequality and Lemma 4.3 implies the following parabolic maximum principle.

Theorem 4.4 *Let Ω be a bounded domain and let $T > 0$. Let $f(x,t) \in H^1(\Omega_T) \cap L^\infty(\Omega_T)$ and suppose that f achieves one of its strict maximum in $\Omega \times (0,T]$ in the sense that: there exists a neighborhood $U \subset\subset \Omega$ and an interval $(\delta,T] \subset (0,T]$ for some $\delta > 0$ such that*

$$\sup_{U\times(\delta,T]} f > \sup_{\Omega_T \setminus (U\times(\delta,T])} f.$$

Here $\sup_{U\times(\delta,T]} f$ means $\operatorname{ess\,sup}_{U\times(\delta,T]} f$. Assume that, for almost every $t \in (0,T)$, $\mathscr{L} f(\cdot,t)$ is a signed Radon measure with $\mathscr{L}^{\rm sing} f(\cdot,t) \ge 0$. Let $w \in H^1(\Omega_T) \cap L^\infty(\Omega_T)$ with $\partial_t w(x,t) \le C$ for some constant $C > 0$, for almost all $(x,t) \in \Omega_T$. Then, for any $\varepsilon > 0$, we have

$$(\mu \times \mathcal{L}^1)\left\{(x,t):\ f(x,t) \ge \sup_{\Omega_T} f - \varepsilon \ \text{ and } \ \mathscr{L}^{\rm ac} f(x,t) + \langle \nabla f, \nabla w\rangle(x,t) - \frac{\partial}{\partial t} f(x,t) \le \varepsilon \right\} > 0,$$

where $\mathcal{L}^1$ is the 1-dimensional Lebesgue's measure on $(\delta,T]$.

In particular, there exists a sequence of points $\{(x_j,t_j)\}_{j\in\mathbb{N}} \subset U \times (\delta,T]$ such that every x_j is an approximate continuity point of $\mathscr{L}^{\rm ac} f(\cdot,t_j)$ and $\langle \nabla f, \nabla w\rangle(\cdot,t_j)$, and that

$$f(x_j,t_j) \ge \sup_{\Omega_T} f - 1/j \quad \text{and} \quad \mathscr{L}^{\rm ac} f(x_j,t_j) + \langle \nabla f, \nabla w\rangle(x_j,t_j) - \frac{\partial}{\partial t} f(x_j,t_j) \le 1/j.$$

Proof We will argue by contradiction, which is similar to the proof of Theorem 4.2. Suppose the assertion fails for some small $\varepsilon_0 > 0$. Then, for almost all $(x,t) \in \{(y,s):\ f(y,s) \ge \sup_{\Omega_T} f - \varepsilon_0\}$, we have

$$\mathscr{L}^{\rm ac} f(x,t) + \langle \nabla f, \nabla w\rangle(x,t) - \frac{\partial}{\partial t} f(x,t) \ge \varepsilon_0.$$

Thus, at such (x,t),

$$\begin{aligned}&\left[\mathscr{L}_w^{\rm ac} f(x,t) - \frac{\partial}{\partial t} f(x,t)\right]\cdot \mu_w \\ &\quad \ge \left[\mathscr{L}^{\rm ac} f(x,t) + \langle \nabla f, \nabla w\rangle(x,t) - \frac{\partial}{\partial t} f(x,t)\right]\cdot e^w \cdot \mu \ge \varepsilon_0 \cdot e^w \cdot \mu \ge 0.\end{aligned}$$

The strictly maximal property of f gives that $f_{\varepsilon_0} := \big(f - (\sup_{\Omega_T} f - \varepsilon_0)\big)_+ \in H^1(\Omega_T)$ with $\lim_{t\to 0} \|f_{\varepsilon_0}(\cdot,t)\|_{L^2(\Omega)} = 0$ and, for almost all $t \in (0,T)$, that the functions $f_{\varepsilon_0}(\cdot,t) \in H_0^1(\Omega)$. Notice that $\mathscr{L}^{\rm sing}_{w(\cdot,t)} f(\cdot,t) \ge 0$ by $\mathscr{L}^{\rm sing} f(\cdot,t) \ge 0$. By using the Kato's inequality, we have that, for almost every $t \in (0,T)$,

$$\begin{aligned}\mathscr{L}_w(f - (\sup_{\Omega_T} f - \varepsilon_0))_+ &\ge \chi[f \ge (\sup_{\Omega_T} f - \varepsilon_0)] \cdot \mathscr{L}_w^{\rm ac} f \\ &\ge \chi[f \ge (\sup_{\Omega_T} f - \varepsilon_0)] \cdot \tfrac{\partial f}{\partial t} \cdot \mu_w = \tfrac{\partial}{\partial t}(f - (\sup_{\Omega_T} f - \varepsilon_0))_+ \cdot \mu_w.\end{aligned}$$

Then Lemma 4.3 implies that $(f - (\sup_{\Omega_T} f - \varepsilon_0))_+ = 0$ for almost all $(x, t) \in \Omega_T$. This is a contradiction. □

5 Local Li–Yau's gradient estimates

Let $K \in \mathbb{R}$ and $N \in [1, \infty)$ and let (X, d, μ) be a metric measure space satisfying $RCD^*(K, N)$. In this section, we will prove the local Li–Yau's gradient estimates—Theorem 1.3.

Let $\Omega \subset X$ be a domain. Given $T > 0$, let us still denote

$$\Omega_T := \Omega \times (0, T]$$

the space-time domain, with lateral boundary Σ and parabolic boundary $\partial_P \Omega_T$:

$$\Sigma := \partial\Omega \times (0, T) \quad \text{and} \quad \partial_P \Omega_T := \Sigma \cup (\Omega \times \{0\}).$$

We adapt the following precise definition of *locally weak solution* for the heat equation.

Definition 5.1 Let $T \in (0, \infty]$ and let Ω be a domain. A function $u(x, t)$ is called a *locally weak solution* of the heat equation on Ω_T if $u(x, t) \in H^1(\Omega_T)$ $(= W^{1,2}(\Omega_T))$ and if for any subinterval $[t_1, t_2] \subset (0, T)$ and any geodesic ball $B_R \subset\subset \Omega$, it holds

$$\int_{t_1}^{t_2} \int_{B_R} (\partial_t u \cdot \phi + \langle \nabla u, \nabla \phi \rangle)\, d\mu dt = 0 \tag{5.1}$$

for all test functions $\phi(x, t) \in Lip_0(B_R \times (t_1, t_2)\big)$. Here and in the sequel, we denote always $\partial_t u := \frac{\partial u}{\partial t}$.

Remark 1 The test functions ϕ in this definition can be chosen such that it has to vanish only on the lateral boundary $\partial B_R \times (0, T)$. That is, $\phi \in Lip(B_{R,T})$ with $\phi(\cdot, t) \in Lip_0(B_R)$ for all $t \in (0, T)$.

The local boundedness and the Harnack inequality for locally weak solutions of the heat equation have been established by Sturm [49,50] and Marola and Masson [39]. In particular, any locally weak solutions for the heat equation in Definition 5.1 must be locally Hölder continuous.

Let $u(x, t)$ be a locally weak solution of the heat equation on $\Omega \times (0, T)$. Fubini Theorem implies, for a.e. $t \in [0, T]$, that the function $u(\cdot, t) \in H^1(\Omega)$ and $\partial_t u \in L^2(\Omega)$. Hence, for a.e. $t \in (0, T)$, the function $u(\cdot, t)$ satisfies, in the distributional sense,

$$\mathscr{L} u = \partial_t u \quad \text{on} \quad \Omega. \tag{5.2}$$

Conversely, if a function $u(x, t) \in H^1\big(\Omega_T\big)$ and (5.2) holds for a.e. $t \in [0, T]$, then it was shown [54, Lemma 6.12] that $u(x, t)$ is a locally weak solution of the heat equation on Ω_T.

In the case that $u(x, t)$ is a (globally) weak solution of heat equation on $X \times (0, \infty)$ with initial value in $L^2(X)$, the theory of analytic semigroups asserts that the function $t \mapsto \|u\|_{W^{1,2}(X)}$ is analytic. However, for a locally weak solution of the heat equation on Ω_T, we have not sufficient regularity for the time derivative $\partial_t u$: in general, $\partial_t u$ is only in L^2. This is not enough to use Bochner formula in Theorem 3.5 to (5.2). For overcoming this difficulty, we recall the so-called *Steklov average*.

Definition 5.2 Given a geodesic ball B_R and a function $u(x,t) \in L^1(B_{R,T})$, where $B_{R,T} := B_R \times (0,T)$, the *Steklov average* of u is defined, for every $\varepsilon \in (0,T)$ and any $h \in (0,\varepsilon)$, by

$$u_h(x,t) := \frac{1}{h}\int_0^h u(x,t+\tau)d\tau, \quad t \in (0, T-\varepsilon]. \tag{5.3}$$

From the general theory of L^p spaces, we know that if $u \in L^p(B_{R,T})$, then the Steklov average u_h converges to u in $L^p(B_{R,T-\varepsilon})$ as $h \to 0$, for every $\varepsilon \in (0,T)$.

Lemma 5.3 *If $u \in H^1(B_{R,T}) \cap L^\infty(B_{R,T})$, then we have, for every $\varepsilon \in (0,T)$, that*

$$u_h \in H^1(B_{R,T-\varepsilon}) \cap L^\infty(B_{R,T-\varepsilon}) \quad \text{and} \quad \partial_t u_h \in H^1(B_{R,T-\varepsilon}) \cap L^\infty(B_{R,T-\varepsilon})$$

for every $h \in (0,\varepsilon)$, and that $\|u_h\|_{H^1(B_{R,T-\varepsilon})}$ is bounded uniformly with respect to $h \in (0,\varepsilon)$.

Proof Since $u \in H^1(B_{R,T})$, according to [22], there exists a function $g(x,t) \in L^2(B_{R,T})$ such that

$$|u(x,t) - u(y,s)| \le d_P\big((x,t),(y,s)\big) \cdot (g(x,t) + g(y,s)),$$

for almost all $(x,t),(y,s) \in B_{R,T}$ with respect to the product measure $d\mu \times dt$, where d_P is the product metric on $B_{R,T}$ defined by

$$d_P^2\big((x,t),(y,s)\big) := d^2(x,y) + |t-s|^2.$$

Such a function g is called a Hajłasz-gradient of u on $B_{R,T}$ (see [21, §8]). By the definition of the Steklov average u_h, we have

$$\begin{aligned}
|u_h(x,t) - u_h(y,s)| &\le \frac{1}{h}\int_0^h (g(x,t+\tau) + g(y,s+\tau)) \cdot d_P\big((x,t+\tau),(y,s+\tau)\big)d\tau \\
&= \frac{1}{h}\int_0^h (g(x,t+\tau) + g(y,s+\tau))\, d\tau \cdot d_P\big((x,t),(y,s)\big) \\
&= (g_h(x,t) + g_h(y,s)) \cdot d_P\big((x,t),(y,s)\big)
\end{aligned}$$

for almost all $(x,t),(y,s) \in B_{R,T}$. The fact $g(x,t) \in L^2(B_{R,T})$ implies that $g_h(x,t) \in L^2(B_{R,T-\varepsilon})$ for each $h \in (0,\varepsilon)$ and that the functions g_h converges to g in $L^2(B_{R,T-\varepsilon})$ as $h \to 0$. Then the previous inequality implies that g_h is a Hajłasz-gradient of u_h on $B_{R,T-\varepsilon}$ for all $h \in (0,\varepsilon)$ (see [21]). According to [21, Theorem 8.6], $2g_h$ is a 2-weak upper gradient of u_h. Thus we conclude that $u_h \in W^{1,2}(B_{R,T-\varepsilon})$ and

$$\limsup_{h\to 0}\int_{B_{R,T-\varepsilon}} (|\nabla u_h|^2 + |\partial_t u_h|^2)d\mu dt \le \limsup_{h\to 0}\int_{B_{R,T-\varepsilon}} (2g_h)^2 d\mu dt \le 4\int_{B_{R,T-\varepsilon}} g^2 d\mu dt.$$

Therefore, we get that $\|u_h\|_{H^1(B_{R,T-\varepsilon})}$ is bounded uniformly with respect to $h \in (0,\varepsilon)$ (by combining with $u_h \to u$ in $L^2(B_{R,T-\varepsilon})$ as $h \to 0$).

Lastly, the assertion $u_h \in L^\infty(B_{R,T-\varepsilon})$ follows directly from the definition of u_h and $u \in L^\infty(B_{R,T})$. The assertion of $\partial_t u$ follows from that

$$\partial_t u_h = \frac{u(x,t+h) - u(x,t)}{h}.$$

The proof is completed. □

For a locally weak solution u for the heat equation, we have the following property of u_h.

Lemma 5.4 *Let $u \in H^1(B_{R,T}) \cap L^\infty(B_{R,T})$ be a locally weak solution for the heat equation, and fix any two constants ε, h such that $\varepsilon \in (0, T)$ and $h \in (0, \varepsilon)$. Then for almost all $t \in (0, T - \varepsilon)$*

$$\mathscr{L} u_h = \partial_t u_h$$

on B_R, in the sense of distributions.

Proof The proof is standard. In fact, one can show the assertion for locally Lipschitz function u, and then use an approximating argument to prove the lemma. □

With the aid of the above two lemmas, we will consider firstly the case when a locally weak solution $u \in H^1(B_{R,T}) \cap L^\infty(B_{R,T})$ with $\partial_t u \in H^1(B_{R,T}) \cap L^\infty(B_{R,T})$.

Lemma 5.5 *Given $K \in \mathbb{R}$ and $N \in [1, \infty)$, let (X, d, μ) be a metric measure space satisfying $RCD^*(K, N)$. Let $u(x, t) \in H^1(B_{2R,T}) \cap L^\infty(B_{2R,T})$ be a locally weak solution of the heat equation on $B_{2R,T}$. Assume that $\partial_t u \in H^1(B_{2R,T}) \cap L^\infty(B_{2R,T})$. Then we have $|\nabla u|^2 \in H^1(B_{R,T}) \cap L^\infty(B_{R,T})$.*

Proof Notice that, for almost all $t \in (0, T)$, we have $u(\cdot, t), \partial_t u(\cdot, t) \in H^1(B_{2R}) \cap L^\infty(B_{2R})$ and that $\mathscr{L} u = \partial_t u$ on B_{2R}. By Lemma 3.4, we get

$$\| |\nabla u(\cdot, t)| \|_{L^\infty(B_{3R/2})} \le C(N, K, R) \cdot (|u(\cdot, t)|_{L^\infty(B_{2R})} + |\partial_t u(\cdot, t)|_{L^\infty(B_{2R})}).$$

This implies $|\nabla u|^2 \in L^\infty(B_{3R/2,T})$ and

$$\| |\nabla u(\cdot, \cdot)| \|_{L^\infty(B_{3R/2,T})} \le C(N, K, R) \cdot (|u|_{L^\infty(B_{2R,T})} + |\partial_t u|_{L^\infty(B_{2R,T})}) := C_*.$$

On the other hand, for almost all $t \in (0, T)$, by applying the Bochner formula (3.5) to $\mathscr{L} u = \partial_t u$ on B_{2R}, we conclude that $|\nabla u(\cdot, t)|^2 \in H^1(B_{3R/2}) \cap L^\infty(B_{3R/2})$ and

$$\begin{aligned}\mathscr{L}(|\nabla u|^2) &\ge \Big[2\tfrac{(\partial_t u)^2}{N} + 2\langle \nabla u, \nabla \partial_t u\rangle + 2K|\nabla u|^2\Big] \cdot \mu \\ &\ge -2|\nabla u| \cdot |\nabla \partial_t u| \cdot \mu + 2K|\nabla u|^2 \cdot \mu \ \ge -2\left[C_* \cdot |\nabla \partial_t u| + 2|K|C_*^2\right] \cdot \mu,\end{aligned}$$

on $B_{3R/2}$ in the sense of distributions. By using the Caccioppoli inequality, we conclude that, for almost all $t \in (0, T)$,

$$\| |\nabla |\nabla u|^2(\cdot, t)| \|_{L^2(B_R)} \le C_{N,K,R} \cdot \big(2C_* \cdot \| |\nabla \partial_t u| \|_{L^2(B_{3R/2})} + 2|K| \cdot C_*^2 + \| |\nabla u|^2 \|_{L^2(B_{3R/2})}\big).$$

The integration on $(0, T)$ implies that

$$\| |\nabla |\nabla u|^2 \|_{L^2(B_{R,T})} \le C_{**} \cdot \big(\| |\nabla \partial_t u| \|_{L^2(B_{3R/2,T})} + \| |\nabla u|^2 \|_{L^2(B_{3R/2,T})} + 1\big),$$

for the constants C_{**} depending on N, K, R, T and C_*. Thus, $|\nabla |\nabla u|^2| \in L^2(B_{R,T})$.

Lastly, noting that, for almost all $(x, t) \in B_{R,T}$,

$$|\partial_t |\nabla u|^2|^2 = |\partial_t \langle \nabla u, \nabla u\rangle|^2 = |2\langle \nabla \partial_t u, \nabla u\rangle|^2 \le 4|\nabla \partial_t u|^2 \cdot |\nabla u|^2.$$

Then, by using $|\nabla u|^2 \in L^\infty(B_{3R/2,T})$ and $\partial_t u \in H^1(B_{R,T})$, we get $|\partial_t |\nabla u|^2| \in L^2(B_{3R/2,T})$. By combining with $|\nabla |\nabla u|^2| \in L^2(B_{R,T})$, we conclude $|\nabla u|^2 \in H^1(B_{R,T})$. Now we finish the proof. □

Lemma 5.6 *Given $K \ge 0$ and $N \in [1, \infty)$, let (X, d, μ) be a metric measure space satisfying $RCD^*(-K, N)$. Let $u(x, t) \in H^1(B_{2R,T}) \cap L^\infty(B_{2R,T})$ be the locally weak solution* *of the heat equation on* $B_{2R,T}$. *Assume that* $u \geq \delta > 0$ *and* $\partial_t u \in H^1(B_{2R,T}) \cap L^\infty(B_{2R,T})$. *We put*

$$F(x,t) = t \cdot [|\nabla f|^2 - \alpha \cdot \partial_t f]_+,$$

where $f = \log u$ *and* $\alpha > 1$. *Then, we have*

$$\frac{F}{t} \in H^1(B_{R,T}) \cap L^\infty(B_{R,T}),$$

and that, for almost every $t \in (0,T)$, *the function* $F(\cdot,t)$ *satisfies*

$$\mathscr{L}F - \partial_t F \cdot \mu \geq -2\langle \nabla f, \nabla F\rangle \cdot \mu - \frac{F}{t}\cdot \mu + 2t\left[\frac{1}{N}\big(|\nabla f|^2 - \partial_t f\big)^2 - K|\nabla f|^2\right]\cdot \mu \quad (5.4)$$

on B_R, *in the sense of distributions.*

Proof From Lemma 5.5, we have $|\nabla u|^2 \in H^1(B_{3R/2,T}) \cap L^\infty(B_{3R/2,T})$. By combining with that $\partial_t u \in L^\infty(B_{2R,T}) \cap H^1(B_{2R,T})$ and that $u \geq \delta > 0$, we get that

$$|\nabla f|^2 - \alpha\partial_t f = \frac{|\nabla u|^2}{u^2} - \alpha\frac{\partial_t u}{u} \in H^1(B_{3R/2,T}) \cap L^\infty(B_{3R/2,T}).$$

This implies $F/t = [|\nabla f|^2 - \alpha\partial_t f]_+ \in H^1(B_{3R/2,T}) \cap L^\infty(B_{3R/2,T})$ and proves the first assertion.

By $\partial_t u \in H^1(B_{2R,T})$, we see that $\partial_{tt} u \in L^2(B_{2R,T})$ and that, for almost all $t \in (0,T)$,

$$\mathscr{L}(\partial_t u) = \partial_{tt} u$$

in the sense of distributions. Since $u, \partial_t u \in H^1(B_{2R,T}) \cap L^\infty(B_{2R,T})$ and $u \geq \delta > 0$, by using the chain rule in Lemma 3.2(i) to both u and $\partial_t u$, we have, for almost all $t \in (0,T)$, that the functions $f(\cdot,t), \partial_t f(\cdot,t) \in H^1(B_{2R})$ and

$$\mathscr{L}f = \partial_t f - |\nabla f|^2, \quad \mathscr{L}(\partial_t f) = \partial_{tt} f - 2\langle \nabla f, \nabla\partial_t f\rangle \quad (5.5)$$

on B_{2R} in the sense of distributions.

Consider $F_1(x,t) := t \cdot \partial_t f$. We have, for almost all $t \in (0,T)$, the function $F_1(\cdot,t) \in H^1(B_{2R})$ with

$$\mathscr{L}F_1 = t\mathscr{L}\partial_t f = t \cdot (\partial_{tt} f - 2\langle \nabla f, \nabla\partial_t f\rangle).$$

Noting that

$$\partial_t F_1 = \partial_t f + t\partial_{tt} f \quad \text{and} \quad \langle \nabla f, \nabla F_1\rangle = t\langle \nabla f, \nabla \partial_t f\rangle,$$

we conclude that

$$\mathscr{L}F_1 - \partial_t F_1 = -2\langle \nabla f, \nabla F_1\rangle - \frac{F_1}{t} \quad (5.6)$$

on B_{2R} in the sense of distributions.

Consider $F_2 := t|\nabla f|^2$. Recall that, for almost all $t \in (0,T)$, the function $f(\cdot,t) \in H^1(B_{2R})$ and

$$\partial_t f - |\nabla f|^2 = \frac{\partial_t u}{u} - \frac{|\nabla u|^2}{u^2} \in L^\infty(B_{3R/2}) \cap H^1(B_{3R/2}).$$

Recalling that (X, d, μ) satisfies $RCD^*(-K, N)$, we can apply the Bochner formula (3.5) to $\mathscr{L} f = \partial_t f - |\nabla f|^2$ to conclude that $|\nabla f|^2 \in H^1(B_R)$ and

$$\mathscr{L}(|\nabla f|^2) \geq 2\left[\frac{1}{N}(\partial_t f - |\nabla f|^2)^2 + \langle \nabla f, \nabla(\partial_t f - |\nabla f|^2)\rangle - K|\nabla f|^2\right] \cdot \mu$$

on B_R, in the sense of distributions. Therefore, for almost all $t \in (0, T)$, we get the function $F_2(\cdot, t)$ satisfies

$$\mathscr{L} F_2 - \partial_t F_2 \cdot \mu \geq 2t \cdot \left[\frac{1}{N}(\partial_t f - |\nabla f|^2)^2 - K|\nabla f|^2\right] \cdot \mu - 2\langle \nabla f, \nabla F_2\rangle \cdot \mu - \frac{F_2}{t} \cdot \mu \tag{5.7}$$

on B_R, in the sense of distributions. By combining (5.6) and (5.7), we conclude, for almost all $t \in (0, T)$, that we have, for $\widetilde{F} := F_2 - \alpha \cdot F_1$,

$$\mathscr{L}\widetilde{F} - \partial_t \widetilde{F} \cdot \mu \geq -2\langle \nabla f, \nabla \widetilde{F}\rangle \cdot \mu - \frac{\widetilde{F}}{t} \cdot \mu + 2t\left[\frac{1}{N}(|\nabla f|^2 - \partial_t f)^2 - K|\nabla f|^2\right] \cdot \mu.$$

Now, by using the Kato's inequality to $F = \widetilde{F}_+$, we have the desired estimate (5.4). The proof of this lemma is finished. □

We are ready to prove the following local Li–Yau's estimate under some additional assumptions.

Lemma 5.7 *Given $K \geq 0$ and $N \in [1, \infty)$, let (X, d, μ) be a metric measure space satisfying $RCD^*(-K, N)$. Let $T \in (0, \infty)$ and let $u(x, t) \in H^1(B_{2R,T}) \cap L^\infty(B_{2R,T})$ be a locally weak solution of the heat equation on $B_{2R,T}$. Assume that $u \geq \delta > 0$ and $\partial_t u \in H^1(B_{2R,T}) \cap L^\infty(B_{2R,T})$.*

Then, for any $\alpha > 1$ and any $\beta, \gamma \in (0, 1)$, the following local gradient estimate holds

$$\begin{aligned}\sup_{B_R \times (\gamma \cdot T, T]} (|\nabla f|^2 - \alpha \cdot \frac{\partial}{\partial t} f)(x, t) &\leq \max\left\{1, \frac{1}{2} + \frac{KT}{2(\alpha - 1)}\right\} \cdot \frac{N\alpha^2}{2T} \cdot \frac{1}{(1-\beta)\gamma} \\ &+ \frac{C_N \cdot \alpha^4}{R^2(\alpha - 1)} \cdot \frac{1}{(1-\beta)\beta\gamma} + \left(\frac{\sqrt{K}}{R} + \frac{1}{R^2}\right) \cdot \frac{C_N \cdot \alpha^2}{(1-\beta)\gamma},\end{aligned} \tag{5.8}$$

where $f = \ln u$, and C_N is a constant depending only on N.

Proof From the previous Lemma 5.6, we have $F := t \cdot [|\nabla f|^2 - \alpha \cdot \partial_t f]_+ \in L^\infty(B_{3R/2,T})$. Put

$$M_1 := \sup_{B_{R,T}} F \quad \text{and} \quad M_2 := \sup_{B_{3R/2,T}} F.$$

We can assume $M_1 > 0$. If not, we are done.

Now let us choose $\phi(x) = \phi(r(x))$ to be a function of the distance r to the fixed point x_0 with the following property that

$$\frac{M_1}{2M_2} \leq \phi \leq 1 \text{ on } B_{3R/2}, \quad \phi = 1 \text{ on } B_R, \quad \phi = \frac{M_1}{2M_2} \text{ on } B_{3R/2} \backslash B_{5R/4},$$

and

$$-\frac{C}{R}\phi^{\frac{1}{2}} \leq \phi'(r) \leq 0 \quad \text{and} \quad |\phi''(r)| \leq \frac{C}{R^2} \quad \forall\, r \in (0, 3R/2)$$

for some universal constant C (which is independent of N, K, R). Then we have

$$\frac{|\nabla\phi|^2}{\phi} = \frac{|\phi'|^2|\nabla r|^2}{\phi} \leq \frac{C^2}{R^2} := \frac{C_1}{R^2} \quad \text{on } B_{3R/2}, \tag{5.9}$$

and, by the Laplacian comparison theorem [18, Corollary 5.15] for $RCD^*(-K,N)$ with $N > 1$ and $K > 0$, that

$$\begin{aligned}\mathscr{L}\phi = \phi'\mathscr{L}r + \phi''|\nabla r|^2 &\geq -\tfrac{C}{R}\left(\sqrt{(N-1)K}\coth\left(r\sqrt{\tfrac{K}{N-1}}\right)\right) - \tfrac{C}{R^2}\\ &\geq -\tfrac{C}{R}\left(\sqrt{(N-1)K} + \tfrac{N-1}{R}\right) - \tfrac{C}{R^2} \geq -C_2\left(\tfrac{\sqrt{K}}{R} + \tfrac{1}{R^2}\right)\end{aligned} \tag{5.10}$$

on $B_{3R/2}$, in the sense of distributions, where we have used that

$$\coth\left(r\sqrt{\frac{K}{N-1}}\right) \leq \coth\left(R\sqrt{\frac{K}{N-1}}\right) \leq 1 + \frac{1}{R\sqrt{K/(N-1)}}.$$

We claim that the estimate (5.10) still holds for $RCD^*(-K,N)$ with $N \geq 1$ and $K \geq 0$. Indeed, in the case when $K = 0$ and $N > 1$, the Laplacian comparison theorem states $\mathscr{L}r \leq (N-1)/r$. Then (5.10) still holds. In the case when $N = 1$, since that (X,d,μ) satisfies $RCD^*(-K,N)$ implies that it satisfies $RCD^*(-K,N+1)$, we can use the Laplacian comparison theorem for $RCD^*(-K,N+1)$ to conclude that (5.10) still holds in this case. Therefore, the claim is proved.

Here and in the sequel of this proof, we denote $C_1, C_2, C_3, \ldots$ the various constants which depend only on N. (5.10) implies that the distribution $\mathscr{L}\phi$ is a signed Radon measure (since $\mathscr{L}\phi + C_2(\sqrt{K}/R + 1/R^2)$ is a positive distribution). Then its absolutely continuous part $(\mathscr{L}\phi)^{\mathrm{ac}} \geq -C_2(\sqrt{K}/R + 1/R^2)$ a.e. $x \in B_{3R/2}$ and its singular part $(\mathscr{L}\phi)^{\mathrm{sing}} \geq 0$.

Put $G(x,t) := \phi F$. According to Lemma 5.6 and the Lebiniz rule 3.2(ii), we have $G \in H^1(B_{3R/2,T})$ and, for almost every $t \in (0,T)$, that the function $G(\cdot,t)$ satisfies that

$$\mathscr{L}G = F\mathscr{L}\phi + \phi\mathscr{L}F + 2\langle\nabla\phi, \nabla F\rangle$$

in the sense of distributions. Fix arbitrarily a such $t \in (0,T)$. Then $\mathscr{L}G$ is a signed Radon measure on $B_{3R/2}$ with

$$(\mathscr{L}G)^{\mathrm{sing}} = F(\mathscr{L}\phi)^{\mathrm{sing}} + \phi(\mathscr{L}F)^{\mathrm{sing}} \geq 0 \tag{5.11}$$

and $(\mathscr{L}G)^{\mathrm{ac}} = F(\mathscr{L}\phi)^{\mathrm{ac}} + \phi(\mathscr{L}F)^{\mathrm{ac}} + 2\langle\nabla\phi, \nabla F\rangle$ a.e. $x \in B_{3R/2}$. We have, for almost all $x \in B_{3R/2}$,

$$\begin{aligned}(\mathscr{L}G)^{\mathrm{ac}} - \partial_t G + 2\langle\nabla f, \nabla G\rangle = \phi\left((\mathscr{L}F)^{\mathrm{ac}} - \partial_t F + 2\langle\nabla f, \nabla F\rangle\right)\\ + F(\mathscr{L}\phi)^{\mathrm{ac}} + 2\langle\nabla\phi, \nabla F\rangle + 2\langle\nabla f, \nabla\phi\rangle F.\end{aligned} \tag{5.12}$$

By (5.4) and $G = \phi F$, we have, for almost all $x \in B_{3R/2}$, that, for any fixed $\epsilon > 0$,

$$\begin{aligned}
\text{RHS of (5.12)} \overset{(5.4)}{\geq}\ & \phi\left[-\frac{F}{t} + 2t\left(\frac{1}{N}\left(|\nabla f|^2 - \partial_t f\right)^2 - K|\nabla f|^2\right)\right] \\
& + G\frac{(\mathscr{L}\phi)^{\mathrm{ac}}}{\phi} + 2\langle\nabla\phi\nabla(G/\phi)\rangle + 2\langle\nabla f\nabla\phi\rangle\frac{G}{\phi} \\
\geq\ & -\frac{G}{t} + 2t\phi\left[\frac{1}{N}\left(|\nabla f|^2 - \partial_t f\right)^2 - K|\nabla f|^2\right] \\
& + \frac{G}{\phi}\left[-C_2\left(\frac{\sqrt{K}}{R} + \frac{1}{R^2}\right) - \frac{2C_1}{R^2}\right] + 2\langle\nabla\phi\nabla G\rangle/\phi - 2|\nabla f|\frac{|\nabla\phi|}{\phi}\cdot G \\
\geq\ & -\frac{G}{t} + 2t\phi\left[\frac{1}{N}\left(|\nabla f|^2 - \partial_t f\right)^2 - K|\nabla f|^2\right] \\
& - C_3\frac{G}{\phi}\left(\frac{\sqrt{K}}{R} + \frac{1}{R^2}\right) + 2\langle\nabla\phi\nabla G\rangle/\phi - \epsilon\frac{G^2}{\phi}\frac{C_1}{R^2} - |\nabla f|^2\frac{1}{\epsilon}, \qquad (5.13)
\end{aligned}$$

where we have used (5.9), (5.10) and that, for any $\epsilon > 0$, the following

$$2|\nabla f|\cdot G\frac{|\nabla\phi|}{\phi} \leq \epsilon G^2\frac{|\nabla\phi|^2}{\phi^2} + |\nabla f|^2\frac{1}{\epsilon} \leq \epsilon\frac{G^2}{\phi}\cdot\frac{C_1}{R^2} + |\nabla f|^2\frac{1}{\epsilon}.$$

If we put

$$v = \frac{|\nabla f|^2}{F}$$

then we get $|\nabla f|^2 = F\cdot v$ and

$$F = t(|\nabla f|^2 - \alpha\cdot\partial_t f) = t(F\cdot v - \alpha\cdot\partial_t f).$$

So

$$\partial_t f = \frac{F(vt-1)}{\alpha t}.$$

Therefore we obtain

$$\begin{aligned}
& -\frac{G}{t} + 2t\phi\left[\frac{1}{N}(|\nabla f|^2 - \partial_t f)^2 - K|\nabla f|^2\right] - \epsilon^{-1}|\nabla f|^2 \\
&= -\frac{G}{t} + \phi\frac{2F^2}{N\alpha^2 t}((\alpha-1)vt+1)^2 - 2tK\phi vF - \epsilon^{-1}vF \\
&\geq -\frac{G}{t\phi} + \frac{2G^2}{N\alpha^2 t\phi}((\alpha-1)vt+1)^2 - \frac{2tKvG}{\phi} - \epsilon^{-1}v\frac{G}{\phi}, \qquad (5.14)
\end{aligned}$$

where we have used that $0 < \phi \leq 1$ and $KvG \geq 0$. Denoting by

$$z := (\alpha-1)vt \quad \text{and} \quad A_\epsilon := \frac{2Kt + \epsilon^{-1}}{\alpha - 1},$$

we have

$$\text{RHS of (5.14)} = \frac{1}{\phi}\cdot\left(\frac{2G^2}{N\alpha^2 t}(z+1)^2 - \frac{G}{t}(1 + A_\epsilon z)\right).$$

Finally $z \geq 0$ implies that

$$\frac{1 + A_\epsilon z}{(1+z)^2} \leq \max\left\{1, \frac{1}{2} + \frac{A_\epsilon}{4}\right\} \leq \max\left\{1, \frac{1}{2} + \frac{Kt}{2(\alpha-1)}\right\} + \frac{\epsilon^{-1}}{4(\alpha-1)}.$$

Denote by

$$B_0 := \max\left\{1, \frac{1}{2} + \frac{KT}{2(\alpha-1)}\right\},$$

we have $\frac{1+A_\epsilon z}{(1+z)^2} \leq B_0 + \frac{\epsilon^{-1}}{4(\alpha-1)}$, (since $K \geq 0$ and $t \leq T$) so

$$\text{RHS of (5.14)} \geq \frac{1}{\phi} \cdot \frac{G}{t} \cdot \left(\frac{2G}{N\alpha^2} - B_0 - \frac{\epsilon^{-1}}{4(\alpha-1)}\right) \cdot (z+1)^2.$$

By combining this with (5.12), (5.13) and (5.14), we obtain that

$$\begin{aligned}
&(\mathscr{L}G)^{\mathrm{ac}} - \partial_t G + 2\langle \nabla f, \nabla G\rangle - 2\langle \nabla\phi, \nabla G\rangle/\phi \\
&\geq \frac{1}{\phi} \cdot \frac{G}{t} \cdot \left(\frac{2G}{N\alpha^2} - B_0 - \frac{\epsilon^{-1}}{4(\alpha-1)}\right) \cdot (z+1)^2 \; - C_3 \frac{G}{\phi}\left(\frac{\sqrt{K}}{R} + \frac{1}{R^2}\right) \\
&\quad - \epsilon \frac{G^2}{\phi}\frac{C_1}{R^2}.
\end{aligned} \tag{5.15}$$

From the definition of ϕ and $F/t \in L^\infty(B_{3R/2,T})$ (by Lemma 5.6), we see that G achieves one of its strict maximum in $B_{3R/2,T}$ in the sense of Theorem 4.4. By (5.11), we know that $\mathscr{L}^{\mathrm{sing}} G \geq 0$. Notice also $\partial_t f \in L^\infty(B_{2R,T})$ since $u \geq \delta > 0$ and $\partial_t u \in H^1(B_{2R,T}) \cap L^\infty(B_{2R,T})$ Hence, by using Theorem 4.4 with $w := 2f - 2\ln\phi \in H^1(B_{3R/2,T}) \cap L^\infty(B_{3R/2,T})$, and combining with (5.15), we conclude that there exit a sequence $\{x_j, t_j\}_{j\in\mathbb{N}}$ such that, for each $j \in \mathbb{N}$,

$$G_j := G(x_j, t_j) \geq \sup_{B_{3R/2,T}} G - 1/j \tag{5.16}$$

and that

$$\begin{aligned}
&\frac{G_j}{t_j} \cdot \left(\frac{2G_j}{N\alpha^2} - B_0 - \frac{\epsilon^{-1}}{4(\alpha-1)}\right) \cdot (z(x_j,t_j)+1)^2 - C_3 G_j \cdot \left(\frac{\sqrt{K}}{R} + \frac{1}{R^2}\right) - \epsilon G_j^2 \cdot \frac{C_1}{R^2} \\
&\quad \leq \phi(x_j, t_j) \cdot \frac{1}{j} \leq \frac{1}{j}.
\end{aligned} \tag{5.17}$$

We consider firstly the case when

$$\bar{G} := \sup_{B_{3R/2,T}} G > \frac{N\alpha^2}{2}\left(B_0 + \frac{\epsilon^{-1}}{4(\alpha-1)}\right).$$

In this case, the Eq. (5.16) tells us $G_j \geq \frac{N\alpha^2}{2}\left(B_0 + \frac{\epsilon^{-1}}{4(\alpha-1)}\right)$ for all sufficiently large j. Thus, from (5.17), we have

$$\frac{G_j}{t_j} \cdot \left(\frac{2G_j}{N\alpha^2} - B_0 - \frac{\epsilon^{-1}}{4(\alpha-1)}\right) - C_3 G_j \cdot \left(\frac{\sqrt{K}}{R} + \frac{1}{R^2}\right) - \epsilon G_j^2 \cdot \frac{C_1}{R^2} \leq \frac{1}{j}.$$

Letting $j \to \infty$, we have

$$\frac{\bar{G}}{T} \cdot \left(\frac{2\bar{G}}{N\alpha^2} - B_0 - \frac{\epsilon^{-1}}{4(\alpha-1)} \right) \le C_3 \bar{G} \cdot \left(\frac{\sqrt{K}}{R} + \frac{1}{R^2} \right) + \epsilon \bar{G}^2 \cdot \frac{C_1}{R^2},$$

where we have used $t_j \le T$ for all $j \in \mathbb{N}$. Thus, we have

$$\bar{G} \le \frac{B_0 + \frac{\epsilon^{-1}}{4(\alpha-1)} + C_3 T \cdot \left(\frac{\sqrt{K}}{R} + \frac{1}{R^2} \right)}{\frac{2}{N\alpha^2} - \epsilon T \cdot \frac{C_1}{R^2}}. \tag{5.18}$$

In the case when $\bar{G} \le \frac{N\alpha^2}{2} \left(B_0 + \frac{\epsilon^{-1}}{4(\alpha-1)} \right)$, it is clear that (5.18) still holds.

Fix any $\beta \in (0, 1)$. By choosing $\epsilon = 2\beta R^2/(C_1 \cdot N\alpha^2 T)$. Then we conclude, by (5.18), that

$$\begin{aligned} \bar{G} &\le \frac{B_0 + \frac{C_1 \cdot N\alpha^2 \cdot T}{8(\alpha-1)\cdot\beta R^2} + C_3 T \cdot \left(\frac{\sqrt{K}}{R} + \frac{1}{R^2} \right)}{\frac{2}{N\alpha^2}(1-\beta)} \\ &= B_0 \cdot \frac{N\alpha^2}{2} \cdot \frac{1}{1-\beta} + \left(\frac{C_1 \cdot N^2\alpha^4 \cdot T}{16(\alpha-1)\cdot\beta R^2} + C_3 T \cdot \left(\frac{\sqrt{K}}{R} + \frac{1}{R^2} \right) \cdot \frac{N\alpha^2}{2} \right) \cdot \frac{1}{1-\beta} \\ &\le B_0 \cdot \frac{N\alpha^2}{2} \cdot \frac{1}{1-\beta} + \frac{C_4 \cdot \alpha^4 \cdot T}{(\alpha-1)R^2} \cdot \frac{1}{(1-\beta)\cdot\beta} + C_5 T \cdot \left(\frac{\sqrt{K}}{R} + \frac{1}{R^2} \right) \cdot \frac{\alpha^2}{1-\beta}. \end{aligned} \tag{5.19}$$

Therefore, we have

$$\begin{aligned} \sup_{B_R \times (\gamma \cdot T, T]} F &\le \sup_{B_{R,T}} F \le \sup_{B_{3R/2,T}} G \\ &\le B_0 \cdot \frac{N\alpha^2}{2} \cdot \frac{1}{1-\beta} + \frac{C_4 \cdot \alpha^4 \cdot T}{(\alpha-1)R^2} \cdot \frac{1}{(1-\beta)\cdot\beta} + C_5 T \cdot \left(\frac{\sqrt{K}}{R} + \frac{1}{R^2} \right) \cdot \frac{\alpha^2}{1-\beta}. \end{aligned}$$

By recalling $F = t(|\nabla f|^2 - \alpha \cdot \partial_t f)_+$ and $B_0 = \max\left\{1, \frac{1}{2} + \frac{KT}{2(\alpha-1)}\right\}$, we conclude that the local gradient estimate (5.8) holds, since $t > \gamma \cdot T$. This completes the proof. □

Now, let us remove the additional assumption $\partial_t u \in H^1(B_{2R,T}) \cap L^\infty(B_{2R,T})$ and prove Theorem 1.4.

Proof of Theorem 1.4. Proof of Theorem Let $\alpha > 1$ and $\beta \in (0, 1)$. Without loss of generality, we can assume that $T_* < \infty$. Given any $\delta > 0$, from [50, Theorem 2.2], we have $u + \delta \in L^\infty_{\mathrm{loc}}(B_{2R,T_*})$. Without loss the generality, we can assume that $u + \delta \in L^\infty(B_{2R,T_*})$, since the desired result is a local estimate.

Given any $\varepsilon > 0$, according to Lemmas 5.3 and 5.4, we can use Lemma 5.7 to the Steklov averages $(u + \delta)_h$. Then, by an approximating argument (and taking $\gamma = 1 - \beta$), we have

$$\begin{aligned} \sup_{B_R \times ((1-\beta)T, T]} \left(\frac{|\nabla u|^2}{(u+\delta)^2} - \alpha \cdot \frac{\partial_t u}{u+\delta} \right)(x,t) &\le \max\left\{1, \frac{1}{2} + \frac{KT}{2(\alpha-1)}\right\} \cdot \frac{N\alpha^2}{2T} \cdot \frac{1}{(1-\beta)^2} \\ &+ \frac{C_N \cdot \alpha^4}{R^2(\alpha-1)} \cdot \frac{1}{(1-\beta)^2\beta} + \left(\frac{\sqrt{K}}{R} + \frac{1}{R^2} \right) \cdot \frac{C_N \cdot \alpha^2}{(1-\beta)^2}. \end{aligned}$$

Letting $\delta(\in \mathbb{Q})$ tend to 0^+ and replacing $1-\beta$ by β, we have the desired (1.6). By combining with the arbitrariness of ε, we complete the proof of Theorem 1.4. □

6 A sharp local Yau's gradient estimate

Let $K \geq 0$, $N \in (1, \infty)$ and let (X, d, μ) be a metric measure space satisfying $RCD^*(-K, N)$. Suppose that Ω is a domain in X. In this section, we will prove a sharp local Yau's gradient estimate—Theorem 1.6.

Proof of Theorem 1.4. Fix $\beta \in (0, 1)$. Let u be a positive harmonic function on $B_{2R} := B_{2R}(p)$ and let $f = \log u$. Without loss of generality, we can assume that $u \geq \delta$ for some $\delta > 0$. By the chain rule 3.2(ii), a direct computation shows that

$$\mathscr{L} f = -|\nabla f|^2 \quad \text{on} \quad B_{2R}.$$

Since $|\nabla f| \in L^\infty_{\rm loc}(B_{2R})$, by setting $g := |\nabla f|^2$ and using Corollary 3.6, (noticing that $N > 1$) we know that $g \in H^1(B_{3R/2}) \cap L^\infty(B_{3R/2})$ and $\mathscr{L}^{\rm sing} g \geq 0$ and, for μ-a.e. $x \in \big\{y : g(y) \neq 0\big\} \cap B_{3R/2}$,

$$\begin{aligned} \frac{1}{2}\mathscr{L}^{\rm ac} g &\geq \frac{g^2}{N} - \langle \nabla g, \nabla f\rangle - Kg + \frac{N}{N-1}\cdot\left(\frac{\langle \nabla f, \nabla g\rangle}{2g} + \frac{g}{N}\right)^2 \\ &= \frac{g^2}{N} - \langle \nabla g, \nabla f\rangle - Kg + \frac{N}{N-1}\cdot\left[\left(\frac{\langle \nabla f, \nabla g\rangle}{2g}\right)^2 + \frac{2\langle \nabla f, \nabla g\rangle}{2g}\cdot\frac{g}{N} + \left(\frac{g}{N}\right)^2\right] \\ &\geq \frac{g^2}{N-1} - \frac{N-2}{N-1}\cdot\langle \nabla g, \nabla f\rangle - Kg. \end{aligned} \tag{6.1}$$

Since $g \in L^\infty(B_{3R/2})$, we define

$$M_1 := \sup_{B_R} g \quad \text{and} \quad M_2 := \sup_{B_{3R/2}} g.$$

We assume that $M_1 > 0$ (otherwise, we are done). Now let us choose $\phi(x) = \phi(r(x))$ as above. That is, $\phi(x)$ is a function of the distance r to the fixed point x_0 with the following property that

$$\frac{M_1}{2M_2} \leq \phi \leq 1 \text{ on } B_{3R/2}, \quad \phi = 1 \text{ on } B_R, \quad \phi = \frac{M_1}{2M_2} \text{ on } B_{3R/2}\backslash B_{5R/4},$$

and

$$-\frac{C}{R}\phi^{\frac{1}{2}} \leq \phi'(r) \leq 0 \quad \text{and} \quad |\phi''(r)| \leq \frac{C}{R^2} \quad \forall\, r \in (0, 3R/2)$$

for some universal constant C (which is independent of N, K, R). Then we have, from (5.9) to (5.10), that

$$\frac{|\nabla \phi|^2}{\phi} \leq \frac{C_1}{R^2} \quad \text{and} \quad \mathscr{L}\phi \geq -C_2\left(\frac{\sqrt{K}}{R} + \frac{1}{R^2}\right) \tag{6.2}$$

on $B_{3R/2}$. Then the distribution $\mathscr{L}\phi$ is a signed Radon measure and its absolutely continuous part $(\mathscr{L}\phi)^{\rm ac} \geq -C_2(\sqrt{K}/R + 1/R^2)$ a.e. $x \in B_{3R/2}$, and its singular part $(\mathscr{L}\phi)^{\rm sing} \geq 0$. Here and in the sequel of this proof, we denote $C_1, C_2, C_3, \ldots$ the various constants which depend only on N.

Put $G(x) := \phi \cdot g$. According to the Lebiniz rule 3.2(ii), we have $G \in H^1(B_{3R/2})$ and

$$\mathscr{L} G = g\mathscr{L}\phi + \phi\mathscr{L} g + 2\langle \nabla \phi, \nabla g\rangle$$

in the sense of distributions. Then, by $\mathscr{L}^{\rm sing} g \geq 0$ and $\mathscr{L}^{\rm sing}\phi \geq 0$, we get $\mathscr{L}^{\rm sing} G \geq 0$. The combination of (6.1) and (6.2) implies that

$$\begin{aligned}
\mathscr{L}^{\rm ac} G \geq\ & \phi \mathscr{L}^{\rm ac} g + 2\langle \nabla\phi, \nabla(G/\phi)\rangle + G\frac{(\mathscr{L}\phi)^{\rm ac}}{\phi} \\
\geq\ & 2\phi\left(\frac{g^2}{N-1} - \frac{N-2}{N-1}\cdot\langle\nabla g, \nabla f\rangle - Kg\right) \\
& +2\langle\nabla\phi, \nabla G\rangle/\phi + \frac{G}{\phi}\left[-C_2\left(\frac{\sqrt{K}}{R} + \frac{1}{R^2}\right) - \frac{2C_1}{R^2}\right] \\
\geq\ & \frac{2}{\phi}\cdot\frac{G^2}{N-1} - \frac{2(N-2)}{N-1}\cdot(\langle\nabla G, \nabla f\rangle - G\langle\nabla\phi, \nabla f\rangle/\phi) - 2KG \\
& +2\langle\nabla\phi, \nabla G\rangle/\phi - C_3\cdot\frac{G}{\phi}\left(\frac{\sqrt{K}}{R} + \frac{1}{R^2}\right) \\
\geq\ & \frac{2}{\phi}\cdot\frac{G^2}{N-1} - \frac{2(N-2)}{N-1}\cdot\langle\nabla G, \nabla f\rangle - \frac{2(N-2)}{N-1}\cdot\left(\epsilon\frac{G^2}{\phi}\cdot\frac{C_1}{R^2} + \frac{G}{\phi}\frac{1}{\epsilon}\right) - 2K\frac{G}{\phi} \\
& +2\langle\nabla\phi, \nabla G\rangle/\phi - C_3\cdot\frac{G}{\phi}\left(\frac{\sqrt{K}}{R} + \frac{1}{R^2}\right) \qquad (6.3)
\end{aligned}$$

for any $\epsilon > 0$, where we have used $g = |\nabla f|^2 = G/\phi$, $2KG \leq 2KG/\phi$ and that, for any $\epsilon > 0$, the following

$$-G\langle\nabla\phi, \nabla f\rangle/\phi \leq 2|\nabla f|\cdot G\frac{|\nabla\phi|}{\phi} \leq \epsilon G^2\frac{|\nabla\phi|^2}{\phi^2} + |\nabla f|^2\frac{1}{\epsilon} \leq \epsilon\frac{G^2}{\phi}\cdot\frac{C_1}{R^2} + |\nabla f|^2\frac{1}{\epsilon}.$$

From the definition of ϕ, we know that G achieves one of its strict maximum in $B_{3R/2}$ in the sense of Theorem 4.2. Notice that $\mathscr{L}^{\rm sing} G \geq 0$. Hence, according to Theorem 4.2 for $w := 2\frac{N-2}{N-1}f - 2\ln\phi \in H^1(B_{3R/2}) \cap L^\infty(B_{3R/2})$ (since $u \geq \delta > 0$), and by combining with (6.3), we conclude that there exit a sequence $\{x_j\}_{j\in\mathbb{N}}$ such that, for each $j \in \mathbb{N}$,

$$G_j := G(x_j) \geq \sup_{B_{3R/2}} G - 1/j \qquad (6.4)$$

and that (noticing that $\phi \in (0, 1]$)

$$\begin{aligned}
& 2\frac{G_j^2}{N-1} - \frac{2(N-2)}{N-1}\cdot\left(\epsilon G_j^2\cdot\frac{C_1}{R^2} + G_j\frac{1}{\epsilon}\right) - 2KG_j - C_3\cdot G_j\left(\frac{\sqrt{K}}{R} + \frac{1}{R^2}\right) \\
& \leq \phi(x_j)\cdot\frac{1}{j} \leq \frac{1}{j} \qquad (6.5)
\end{aligned}$$

for any $\epsilon > 0$. Letting $j \to \infty$ and denoting $\bar{G} := \sup_{B_{3R/2}} G = \lim_j G_j$, we obtain

$$\left(\frac{1}{N-1} - \frac{(N-2)\epsilon\cdot C_1}{(N-1)R^2}\right)\cdot\bar{G} \leq K + \frac{N-2}{(N-1)\epsilon} + \frac{C_3}{2}\left(\frac{\sqrt{K}}{R} + \frac{1}{R^2}\right) \qquad (6.6)$$

for any $\epsilon > 0$.

In the case when $N > 2$, by choosing $\epsilon = \frac{\beta \cdot R^2}{(N-2)\cdot C_1}$, we obtain from (6.6) that

$$\begin{aligned} \frac{1-\beta}{N-1}\cdot \bar{G} &\leq K + \frac{C_1\cdot(N-2)^2}{\beta R^2} + \frac{C_3}{2}\left(\frac{\sqrt{K}}{R} + \frac{1}{R^2}\right) \\ &\leq K + \frac{C_1\cdot(N-2)^2}{\beta R^2} + \beta K + \frac{C_3^2}{16\beta R^2} + \frac{C_3}{2R^2}, \end{aligned}$$

where we have used

$$\frac{C_3}{2}\frac{\sqrt{K}}{R} = 2\sqrt{K}\cdot\frac{C_3}{4R} \leq \beta K + \frac{1}{\beta}\frac{C_3^2}{(4R)^2}.$$

Then, we get

$$\begin{aligned} \bar{G} &\leq \frac{1+\beta}{1-\beta}(N-1)K + \frac{N-1}{1-\beta}\cdot\frac{1}{\beta R^2}\left(C_1\cdot(N-2)^2 + \frac{C_3^2}{16} + \frac{C_3\beta}{2}\right) \\ &\leq \frac{1+\beta}{1-\beta}(N-1)K + \frac{C_4}{\beta(1-\beta)\cdot R^2}, \end{aligned} \tag{6.7}$$

where we have used $\beta < 1$.

In the case when $N \in (1, 2]$, from (6.6), we have

$$\frac{1}{N-1}\cdot\bar{G} \leq K + \frac{C_3}{2}\left(\frac{\sqrt{K}}{R} + \frac{1}{R^2}\right) \leq K + \beta K + \frac{C_3^2}{16\beta R^2} + \frac{C_3}{2R^2}.$$

Thus, the estimate (6.7) still holds in this case.

Therefore, the Eq. (6.7) shows that, for any $\beta \in (0, 1)$,

$$\sup_{B_R} g \leq \frac{1+\beta}{1-\beta}(N-1)K + \frac{C_4}{\beta(1-\beta)\cdot R^2}.$$

Now the proof is finished. □

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