Calc. Var. (2016) 55:100
DOI 10.1007/s00526-016-1026-3

Calculus of Variations

A doubly nonlinear evolution for the optimal Poincaré inequality

Ryan Hynd[1] · Erik Lindgren[2]

Received: 3 February 2016 / Accepted: 27 June 2016 / Published online: 22 July 2016

Abstract We study the large time behavior of solutions of the PDE $|v_t|^{p-2}v_t = \Delta_p v$. A special property of this equation is that the Rayleigh quotient $\int_\Omega |Dv(x,t)|^p dx / \int_\Omega |v(x,t)|^p dx$ is nonincreasing in time along solutions. As t tends to infinity, this ratio converges to the optimal constant in Poincaré's inequality. Moreover, appropriately scaled solutions converge to a function for which equality holds in this inequality. An interesting limiting equation also arises when p tends to infinity, which provides a new approach to approximating ground states of the infinity Laplacian.

Mathematics Subject Classification 35K15 · 39B62 · 35P30 · 47J10 · 35K55

1 Introduction

In this paper, we study solutions $v : \Omega \times (0,\infty) \to \mathbb{R}$ of the PDE

$$|v_t|^{p-2}v_t = \Delta_p v \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain, $p \in (1,\infty)$, and Δ_p is the p-Laplacian

$$\Delta_p \psi := \operatorname{div}(|D\psi|^{p-2}D\psi).$$

Communicated by O. Savin.

Ryan Hynd is partially supported by NSF Grant DMS-1301628. Erik Lindgren is supported by the Swedish Research Council, Grant No. 2012-3124. Partially supported by the Royal Swedish Academy of Sciences.

✉ Erik Lindgren
erik.kristian.lindgren@gmail.com

1 Department of Mathematics, University of Pennsylvania, Philadelphia, USA

2 Department of Mathematics, KTH, Stockholm, Sweden

The p-Laplacian arises in connection with various physical applications. Examples include non-Newtonian fluids, nonlinear elasticity, glacial sliding and capillary surfaces as detailed in [3,7,21,26].

Observe that when $p = 2$, the PDE (1.1) is the heat equation. As a result, we view (1.1) as a nonlinear flow. What separates Eq. (1.1) from typical nonlinear parabolic equations, is the nonlinearity in the time derivative $|v_t|^{p-2}v_t$. This type of equation is known in the literature as a *doubly nonlinear evolution*. Furthermore, we regard (1.1) as special within the class of doubly nonlinear evolutions as it is homogeneous: if v is a solution of (1.1), any multiple of v is also a solution.

Our motivation for studying Eq. (1.1) is its connection with the optimal Poincaré inequality

$$\lambda_p \int_\Omega |\psi|^p dx \le \int_\Omega |D\psi|^p dx, \quad \psi \in W_0^{1,p}(\Omega). \tag{1.2}$$

Here

$$\lambda_p := \inf \left\{ \frac{\int_\Omega |D\psi|^p dx}{\int_\Omega |\psi|^p dx} : \psi \in W_0^{1,p}(\Omega)\setminus\{0\} \right\}$$

is the least p-Rayleigh quotient, and (1.2) is "optimal" as λ_p is the largest constant for which this inequality is valid. A function $\psi \in W_0^{1,p}(\Omega)\setminus\{0\}$ for which equality holds in (1.2) is called a *ground state* of p-Laplacian or simply a *p-ground state*. These functions are easily seen to exist and to satisfy the PDE

$$-\Delta_p \psi = \lambda_p |\psi|^{p-2}\psi \tag{1.3}$$

in Ω. Moreover, λ_p is "simple" in the sense that any two p-ground states are multiples of each other [23,25,28].

In what follows, we prove that a properly scaled solution of the initial value problem

$$\begin{cases} |v_t|^{p-2}v_t = \Delta_p v, & \Omega \times (0,\infty) \\ v = 0, & \partial\Omega \times [0,\infty) \\ v = g, & \Omega \times \{0\} \end{cases} \tag{1.4}$$

converges to a p-ground state as $t \to \infty$. First, we show that (1.4) has a *weak solution* in the sense of a doubly nonlinear evolution, and then derive various global estimates on weak solutions. In particular, we verify that the p-Rayleigh quotient is nonincreasing for each weak solution of (1.4)

$$\frac{d}{dt}\left\{ \frac{\int_\Omega |Dv(x,t)|^p dx}{\int_\Omega |v(x,t)|^p dx} \right\} \le 0.$$

This monotonicity formula and the homogeneity of equation (1.1) are crucial ingredients in establishing the following result.

Theorem 1.1 *Assume $g \in W_0^{1,p}(\Omega)$ and define*

$$\mu_p := \lambda_p^{\frac{1}{p-1}}.$$

Then for any weak solution v of (1.4)*, the limit*

$$\psi := \lim_{t\to\infty} e^{\mu_p t} v(\cdot, t)$$

exists in $W_0^{1,p}(\Omega)$ *and is a* p*-ground state, provided* $\psi \not\equiv 0$*. In this case,* $v(\cdot, t) \not\equiv 0$ *for* $t \geq 0$ *and*

$$\lambda_p = \lim_{t\to\infty} \frac{\int_\Omega |Dv(x,t)|^p dx}{\int_\Omega |v(x,t)|^p dx}.$$

When $p = 2$, a direct proof of Theorem 1.1 can be made by writing the solution of the heat equation in terms of the basis of eigenfunctions for the Dirichlet Laplacian. For $p \neq 2$, no such formulae are available and we must work directly with the equation. It is interesting to compare Theorem 1.1 to other large time asymptotics results for fully nonlinear parabolic equations [5,24] and for nonlinear degenerate flows [1,6,22,30]. Most of these works involve comparison principles and initial conditions which do not change sign. Our main tool in this paper is a compactness property of weak solutions of (1.1) and applies to general initial data.

We also verify that (1.4) has a unique *viscosity solution* when $p \geq 2$. We note it is unknown whether weak solutions are unique or if each weak solution is a viscosity solution. Moreover, the uniqueness of solutions of general doubly nonlinear evolutions is not well understood. Nevertheless, we show there is always one weak solution of (1.4) that arises via the implicit time scheme: $v^0 = g$,

$$\begin{cases} \mathcal{J}_p\left(\frac{v^k - v^{k-1}}{\tau}\right) = \Delta_p v^k, & x \in \Omega \\ v^k = 0, & x \in \partial\Omega \end{cases} \tag{1.5}$$

for $k \in \mathbb{N}$ and $\tau > 0$. Here $\mathcal{J}_p$ is the increasing function

$$\mathcal{J}_p(w) := |w|^{p-2}w, \quad w \in \mathbb{R}.$$

Standard variational methods can be used to show this scheme has a unique weak solution sequence $\{v_\tau^k\}_{k\in\mathbb{N}} \subset W_0^{1,p}(\Omega)$ for each $\tau > 0$. We argue that each v_τ^k is also a continuous viscosity solution and then use viscosity solutions methods to verify the following convergence result.

Theorem 1.2 *Assume that* $p \geq 2$ *and that* $\partial\Omega$ *is smooth. Additionally suppose that* $g \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$ *and that there is a* p*-ground state* φ *such that*

$$-\varphi(x) \leq g(x) \leq \varphi(x), \quad x \in \overline{\Omega}.$$

Denote the solution sequence of the implicit scheme (1.5) *as* $\{v_\tau^k\}_{k\in\mathbb{N}}$ *and set*

$$v_N(\cdot, t) := \begin{cases} g, & t = 0 \\ v_{T/N}^k, & (k-1)T/N < t \leq kT/N, \quad k = 1, \dots, N \end{cases} \tag{1.6}$$

for $N \in \mathbb{N}$ *and* $T > 0$*. Then* $v(\cdot, t) := \lim_{N\to\infty} v_N(\cdot, t)$ *exists in* $L^p(\Omega) \cap C(\overline{\Omega})$ *uniformly in* $t \in [0, T]$*. Moreover,* v *is the unique viscosity solution and a weak solution of the initial value problem* (1.4)*.*

It was previously established that a subsequence of $(v_N)_{N\in\mathbb{N}}$ converges to a weak solution [4,11]. The novelty of Theorem 1.2 is that the full limit exists and that the limit is additionally a viscosity solution. Employing viscosity solutions will also allow us to pass to the limit as the exponent $p \to \infty$ in Eq. (1.1). This idea was inspired by the work of P. Juutinen, P. Lindqvist and J. Manfredi, who first studied the so-called infinity eigenvalue problem and infinity ground states [18]. We view the following result as providing a natural evolution equation for the infinity eigenvalue problem and its ground states.

Theorem 1.3 *Assume $g \in W_0^{1,\infty}(\Omega)$ and let v^p denote a viscosity solution of (1.4) for $p \geq 2$ with initial condition g. There is an increasing sequence $p_k \to \infty$ such that $(v^{p_k})_{k\in\mathbb{N}}$ converges locally uniformly to a viscosity solution of the PDE*

$$\begin{cases} G_\infty(v_t, Dv, D^2v) = 0, & \Omega \times (0, \infty) \\ v = 0, & \partial\Omega \times [0, \infty) \\ v = g, & \Omega \times \{0\} \end{cases} \tag{1.7}$$

as $k \to \infty$. The operator above is defined as

$$G_\infty(\phi_t, D\phi, D^2\phi) := \begin{cases} \min\{-\Delta_\infty\phi, |D\phi| + \phi_t\}, & \phi_t < 0 \\ -\Delta_\infty\phi, & \phi_t = 0\,, \\ \max\{-\Delta_\infty\phi, -|D\phi| + \phi_t\}, & \phi_t > 0 \end{cases}$$

where $\Delta_\infty\phi := D^2\phi D\phi \cdot D\phi$ is the infinity Laplacian.

This paper is organized as follows. In Sect. 2, we discuss the existence theory for weak solutions. In particular, we present a novel compactness result for the doubly nonlinear evolution (1.4). We justify Theorem 1.1 in Sect. 3 and then discuss viscosity solutions and prove Theorem 1.2 in Sect. 4. Finally, we verify Theorem 1.3 in Sect. 5. We thank the Institut Mittag-Leffler for hosting us during the initial phase of this work. We especially thank Peter Lindqvist and Jerry Kazdan for their advice and encouragement.

2 Weak solutions

An important identity for smooth solutions of (1.4) is

$$\frac{d}{dt}\int_\Omega \frac{1}{p}|Dv(x,t)|^p dx = -\int_\Omega |v_t(x,t)|^p dx. \tag{2.1}$$

This identity follows from direct computation. Of course, integrating (2.1) in time yields

$$\int_0^t \int_\Omega |v_t(x,s)|^p dxds + \int_\Omega \frac{1}{p}|Dv(x,t)|^p dx = \int_\Omega \frac{1}{p}|Dg(x)|^p dx \tag{2.2}$$

for $t \geq 0$. This resulting equality leads us to seek solutions defined as follows.

Definition 2.1 Assume $g \in W_0^{1,p}(\Omega)$. We say that a function v satisfying

$$v \in L^\infty([0,\infty); W_0^{1,p}(\Omega)), \quad v_t \in L^p(\Omega \times [0,\infty)) \tag{2.3}$$

is a *weak solution* of (1.4) if for Lebesgue almost every $t > 0$

$$\int_\Omega |v_t(x,t)|^{p-2} v_t(x,t)\phi(x)dx + \int_\Omega |Dv(x,t)|^{p-2} Dv(x,t) \cdot D\phi(x)dx = 0 \tag{2.4}$$

for each $\phi \in W_0^{1,p}(\Omega)$ and

$$v(x,0) = g(x). \tag{2.5}$$

Any v satisfying (2.3) takes values in $L^p(\Omega)$ that are continuous in time, that is

$$v \in C([0,T]; L^p(\Omega)) \quad \text{for any } T > 0.$$

Therefore, we may consider the pointwise values $v(\cdot,t) \in L^p(\Omega)$ of a weak solution and assign the initial condition (2.5). Let us now derive a few properties of solutions.

Lemma 2.2 *Assume v is a weak solution of* (1.4). *Then* $[0, \infty) \ni t \mapsto \int_\Omega |Dv(x,t)|^p dx$ *is absolutely continuous and* (2.1) *holds for almost every* $t > 0$.

Proof Define

$$\Phi(w) := \begin{cases} \int_\Omega \frac{1}{p}|Dw(x)|^p dx, & w \in W_0^{1,p}(\Omega) \\ +\infty, & \text{otherwise} \end{cases}$$

for each $w \in L^p(\Omega)$. Observe that Φ is convex, proper, and lower-semicontinuous. Moreover, by (2.4)

$$\partial\Phi(v(\cdot,t)) = \{-|v_t(\cdot,t)|^{p-2}v_t(\cdot,t)\}$$

for almost every $t > 0$. In view of the integrability of v_t (2.3), it follows that $t \mapsto \Phi(v(\cdot,t))$ is absolutely continuous; for instance, see Corollary 1.4.5 and Remark 1.4.6 of [2] for a detailed proof of this fact. The chain rule now applies, and (2.1) holds for almost every $t > 0$. □

Lemma 2.3 *Assume v is a weak solution of* (1.4). *Then*

$$\int_\Omega |Dv(x,t)|^p dx \le \frac{1}{\mu_p}\int_\Omega |v_t(x,t)|^p dx \tag{2.6}$$

and

$$\frac{d}{dt}\left\{e^{(\mu_p p)t}\int_\Omega |Dv(x,t)|^p dx\right\} \le 0 \tag{2.7}$$

for almost every $t \ge 0$.

Proof Using $v(\cdot,t)$ as a test function in (2.4) and applying Poincaré's inequality (1.2)

$$\begin{aligned} \int_\Omega |Dv(x,t)|^p dx &= \int_\Omega |Dv(x,t)|^{p-2}Dv(x,t)\cdot Dv(x,t)dx \\ &= -\int_\Omega |v_t(x,t)|^{p-2}v_t(x,t)\cdot v(x,t)dx \\ &\le \left(\int_\Omega |v_t(x,t)|^p dx\right)^{1-1/p}\left(\int_\Omega |v(x,t)|^p dx\right)^{1/p} \\ &\le \lambda_p^{-1/p}\left(\int_\Omega |v_t(x,t)|^p dx\right)^{1-1/p}\left(\int_\Omega |Dv(x,t)|^p dx\right)^{1/p}. \end{aligned} \tag{2.8}$$

This proves (2.6). Combining (2.1) and (2.6) gives

$$\frac{d}{dt}\int_\Omega |Dv(x,t)|^p dx \le -p\mu_p\int_\Omega |Dv(x,t)|^p dx. \tag{2.9}$$

Inequality (2.7) follows from (2.9) by direct computation. □

Note that if the initial condition g is a p-ground state, then

$$v(x,t) = e^{-\mu_p t}g(x) \tag{2.10}$$

is a solution of (1.4). Theorem 1.1 asserts all solutions exhibit this "separation of variables" behavior in the limit as $t \to \infty$. Our first clue that this intuition is correct is that the p-Rayleigh quotient is a nonincreasing function of time along the flow. We regard this as a special feature of the PDE (1.1).

Proposition 2.4 *Assume that v is a weak solution of* (1.4) *such that $v(\cdot, t) \neq 0 \in L^p(\Omega)$ for each $t \geq 0$. Then the p-Rayleigh quotient*

$$[0, \infty) \ni t \mapsto \frac{\int_\Omega |Dv(x,t)|^p dx}{\int_\Omega |v(x,t)|^p dx}$$

is nonincreasing.

Proof Employing (2.3), it is not difficult to verify

$$\frac{d}{dt}\int_\Omega \frac{1}{p}|v(x,t)|^p dx = \int_\Omega |v(x,t)|^{p-2} v(x,t) v_t(x,t) dx$$

for almost every time $t > 0$; for instance, it is possible to adapt the proof of Theorem 3 on page 287 of [15]. Suppressing the (x, t) dependence, we compute using (2.1)

$$\begin{aligned} \frac{d}{dt}\frac{\int_\Omega |Dv|^p dx}{\int_\Omega |v|^p dx} &= -p\frac{\int_\Omega |v_t|^p dx}{\int_\Omega |v|^p dx} - p\frac{\int_\Omega |Dv|^p dx}{\left(\int_\Omega |v|^p dx\right)^2}\int_\Omega |v|^{p-2} v v_t dx \\ &= \frac{p}{\left(\int_\Omega |v|^p dx\right)^2}\left\{\int_\Omega |Dv|^p dx \int_\Omega |v|^{p-2} v(-v_t) dx - \int_\Omega |v|^p dx \int_\Omega |v_t|^p dx\right\} \end{aligned} \tag{2.11}$$

which is valid for almost every $t > 0$. By Hölder's inequality

$$\int_\Omega |v|^{p-2} v(-v_t) dx \leq \left(\int_\Omega |v|^p dx\right)^{1-1/p}\left(\int_\Omega |v_t|^p dx\right)^{1/p},$$

and combining this with (2.8) gives

$$\int_\Omega |Dv|^p dx \int_\Omega |v|^{p-2} v(-v_t) dx \leq \int_\Omega |v|^p dx \int_\Omega |v_t|^p dx.$$

From (2.11), we conclude

$$\frac{d}{dt}\frac{\int_\Omega |Dv|^p dx}{\int_\Omega |v|^p dx} \leq 0.$$

□

Corollary 2.5 *Assume g is a p-ground state. The only weak solution of initial value problem* (1.4) *is given by* (2.10).

Proof Let v be a weak solution of (1.4) and assume initially that $v(\cdot, t) \neq 0 \in L^p(\Omega)$ for each $t \geq 0$. By Proposition 2.4,

$$\frac{\int_\Omega |Dv(x,t)|^p dx}{\int_\Omega |v(x,t)|^p dx} \leq \frac{\int_\Omega |Dg(x)|^p dx}{\int_\Omega |g(x)|^p dx} = \lambda_p.$$

Thus, $v(\cdot, t)$ is a p-ground state for each $t \geq 0$. In view of Eq. (1.3)

$$|v_t|^{p-2} v_t = \Delta_p v = -\lambda_p |v|^{p-2} v.$$

In particular,

$$v_t = -\mu_p v \tag{2.12}$$

and therefore, v is given by (2.10).

Otherwise, select the first time $T > 0$ for which $v(\cdot, T) = 0 \in L^p(\Omega)$. By our argument above, $v(\cdot, t)$ is a p-ground state for each $t \in [0, T)$. Moreover, (2.12) holds for almost every $t \in (0, T)$. However, this implies $v(\cdot, T) = e^{-\mu_p T} g \neq 0 \in L^p(\Omega)$. Therefore, there is no such time T and v is given by (2.10). □

Using an implicit time scheme such as (1.5) to solve doubly nonlinear evolutions in reflexive Banach spaces has been carried out with great success; see [2,4,11,12,16,27,31]. In our view, the main insight that makes this approach work is a certain compactness feature of weak solutions that we now explore. Roughly, we verify that any "bounded" sequence of solutions has a subsequence converging to another weak solution. We will also make use of this compactness result in our study of the large time behavior of solutions.

Theorem 2.6 *Assume $\{g^k\}_{k\in\mathbb{N}} \in W^{1,p}_0(\Omega)$ is uniformly bounded in $W^{1,p}_0(\Omega)$, and that for each $k \in \mathbb{N}$, v^k is a weak solution of (1.4) with $v^k(\cdot, 0) = g^k$. Then there is a subsequence $\{v^{k_j}\}_{j\in\mathbb{N}}$ and v satisfying (2.3) such that*

$$v^{k_j} \to v \quad in \quad \begin{cases} C([0, T]; L^p(\Omega)) \\ L^p([0, T]; W^{1,p}_0(\Omega)) \end{cases} \tag{2.13}$$

and

$$v^{k_j}_t \to v_t \quad in \quad L^p(\Omega \times [0, T]) \tag{2.14}$$

as $j \to \infty$, for all $T > 0$. Moreover, v is a weak solution of (1.4) where g is a weak limit of $\{g^{k_j}\}_{k\in\mathbb{N}}$ in $W^{1,p}_0(\Omega)$.

Proof By equation (2.1), we have for each $k \in \mathbb{N}$ and almost every time $t \geq 0$

$$\frac{d}{dt}\int_\Omega \frac{|Dv^k(x,t)|^p}{p}dx = -\int_\Omega |v^k_t(x,t)|^p dx. \tag{2.15}$$

Thus,

$$\int_0^\infty \int_\Omega |v^k_t(x,t)|^p dxdt + \sup_{t\geq 0}\int_\Omega |Dv^k(x,t)|^p dx \leq 2\int_\Omega |Dg^k(x)|^p dx. \tag{2.16}$$

By assumption, the right hand side above is bounded uniformly in $k \in \mathbb{N}$. By the compactness of $W^{1,p}_0(\Omega)$ in $L^p(\Omega)$, the Arzelà–Ascoli theorem as detailed by Simon [29] implies that there is a subsequence $\{v^{k_j}\}_{j\in\mathbb{N}}$ converging uniformly on compact subintervals of $[0, \infty)$ to some v in $L^p(\Omega)$.

The bound (2.16) also ensures

$$Dv^{k_j}(\cdot, t) \rightharpoonup Dv(\cdot, t)$$

in $L^p(\Omega; \mathbb{R}^n)$ for each $t \geq 0$. Moreover, as $\{v^k_t\}_{k\in\mathbb{N}}$ is bounded in $L^p(\Omega \times [0, \infty))$, we may also assume

$$\begin{cases} v^{k_j}_t \rightharpoonup v_t & \text{in } L^p(\Omega \times [0, \infty)) \\ \mathcal{J}_p(v^{k_j}_t) \rightharpoonup \xi & \text{in } L^q(\Omega \times [0, \infty)) \end{cases}.$$

Here $1/p + 1/q = 1$. We claim that in fact

$$\xi = \mathcal{J}_p(v_t) = |v_t|^{p-2}v_t. \tag{2.17}$$

The convexity of the map $\mathbb{R}^n \ni z \mapsto \frac{1}{p}|z|^p$ implies

$$\int_\Omega \frac{1}{p}|Dw(x)|^p dx \geq \int_\Omega \frac{1}{p}|Dv^{k_j}(x,t)|^p dx - \int_\Omega \mathcal{J}_p(v_t^{k_j}(x,t))(w(x) - v^{k_j}(x,t))dx$$

for any $w \in W_0^{1,p}(\Omega)$. Integrating over the interval $t \in [t_0, t_1]$ and sending $j \to \infty$ gives

$$\int_{t_0}^{t_1}\int_\Omega \frac{1}{p}|Dw(x)|^p dxdt \geq \int_{t_0}^{t_1}\left(\int_\Omega \frac{1}{p}|Dv(x,t)|^p dx - \int_\Omega \xi(x,t)(w(x) - v(x,t))dx\right)dt.$$

Therefore,

$$\int_\Omega \frac{1}{p}|Dw(x)|^p dx \geq \int_\Omega \frac{1}{p}|Dv(x,t)|^p dx - \int_\Omega \xi(x,t)(w(x) - v(x,t))dx$$

for almost every time $t \geq 0$. In particular, for each $\phi \in W_0^{1,p}(\Omega)$

$$\int_\Omega \xi(x,t)\phi(x)dx + \int_\Omega |Dv(x,t)|^{p-2}Dv(x,t)\cdot D\phi(x)dx = 0 \tag{2.18}$$

for almost every time $t \geq 0$. As a result, once we verify (2.17), v is then a weak solution of (1.4).

For each interval $[t_0, t_1]$

$$\begin{aligned}\lim_{j\to\infty}\int_{t_0}^{t_1}\int_\Omega |Dv^{k_j}(x,t)|^p dxdt &= \lim_{j\to\infty}\int_{t_0}^{t_1}\int_\Omega |Dv^{k_j}(x,t)|^{p-2}Dv^{k_j}(x,t)\cdot Dv^{k_j}(x,t)dxdt\\ &= -\lim_{j\to\infty}\int_{t_0}^{t_1}\int_\Omega \mathcal{J}_p(v_t^{k_j}(x,t))v^{k_j}(x,t)dxdt\\ &= -\int_{t_0}^{t_1}\int_\Omega \xi(x,t)v(x,t)dxdt\\ &= \int_{t_0}^{t_1}\int_\Omega |Dv(x,t)|^p dxdt.\end{aligned}$$

The last equality is due to (2.18). As a result, $Dv^{k_j} \to Dv$ in $L^p_{\mathrm{loc}}([0,\infty), L^p(\Omega))$. This proves assertion (2.13). And without loss of generality, we assume that

$$\int_\Omega |Dv^{k_j}(x,t)|^p dx \to \int_\Omega |Dv(x,t)|^p dx \tag{2.19}$$

for almost every $t \geq 0$, as $j \to \infty$ (since this occurs for some subsequence of k_j).

Now we will verify (2.17). As in our proof of Lemma 2.2, (2.18) implies

$$\frac{d}{dt}\int_\Omega \frac{1}{p}|Dv(x,t)|^p dx = -\int_\Omega \xi(x,t)v_t(x,t)dx, \quad a.e.\ t \geq 0.$$

Thus for each $t_1 > t_0$

$$\int_{t_0}^{t_1}\int_\Omega \xi(x,s)v_t(x,s)dxds + \int_\Omega \frac{|Dv(x,t_1)|^p}{p}dx = \int_\Omega \frac{|Dv(x,t_0)|^p}{p}dx. \tag{2.20}$$

From (2.15), we may also write

$$\int_{t_0}^{t_1}\int_\Omega \frac{1}{p}|v_t^{k_j}(x,s)|^p + \frac{1}{q}|\mathcal{J}_p(v_t^{k_j}(x,s))|^q dxds + \int_\Omega \frac{|Dv^{k_j}(x,t_1)|^p}{p}dx = \int_\Omega \frac{|Dv^{k_j}(x,t_0)|^p}{p}dx. \tag{2.21}$$

Assuming t_0 and t_1 are times for which the limit (2.19) holds, we let $j \to \infty$ to get

$$\int_{t_0}^{t_1} \int_\Omega \frac{1}{p}|v_t(x,s)|^p + \frac{1}{q}|\xi(x,s)|^q dxds + \int_\Omega \frac{|Dv(x,t_1)|^p}{p} dx \le \int_\Omega \frac{|Dv(x,t_0)|^p}{p} dx$$

by weak convergence. Comparing with (2.20) gives

$$\int_{t_0}^{t_1} \int_\Omega \left(\frac{1}{p}|v_t(x,s)|^p + \frac{1}{q}|\xi(x,s)|^q - \xi(x,s)v_t(x,s) \right) dxds \le 0.$$

Equation (2.17) now follows from the strict convexity of $\mathbb{R} \ni z \mapsto \frac{1}{p}|z|^p$. Substituting $\xi = \mathcal{J}_p(v_t)$ into (2.20) and passing to the limit as $j \to \infty$ in (2.21) also gives

$$\lim_{j\to\infty} \int_{t_0}^{t_1} \int_\Omega |v_t^{k_j}(x,s)|^p dxds = \int_{t_0}^{t_1} \int_\Omega |v_t(x,s)|^p dxds.$$

Thus, we are also able to conclude (2.14). □

Let us briefly discuss how compactness pertains to the existence of weak solutions. To this end, assume $\{v^k\}_{k\in\mathbb{N}}$ is the solution sequence of (1.5) for a given $\tau > 0$. Upon multiplying the PDE in (1.5) by $v^k - v^{k-1}$ and integrating by parts, we obtain

$$\int_\Omega \left(\frac{|v^k - v^{k-1}|^p}{\tau^{p-1}} + \frac{1}{p}|Dv^k|^p \right) dx \le \int_\Omega \frac{1}{p}|Dv^{k-1}|^p dx, \quad k \in \mathbb{N}.$$

Moreover, summing over $k = 1, \dots, j \in \mathbb{N}$ gives

$$\sum_{k=1}^{j} \int_\Omega \frac{|v^k - v^{k-1}|^p}{\tau^{p-1}} dx + \int_\Omega \frac{1}{p}|Dv^j|^p dx \le \int_\Omega \frac{1}{p}|Dg|^p dx, \tag{2.22}$$

which is a discrete analog of the energy identity (2.2).

Let us further assume $\tau = T/N$ and set $\tau_k = k\tau$ for $k = 0, 1, \dots, N \in \mathbb{N}$. It will be useful for us to define the "linear interpolating" approximation as

$$u_N(\cdot,t) := v^{k-1} + \left(\frac{t - \tau_{k-1}}{\tau} \right)(v^k - v^{k-1}), \quad \tau_{k-1} \le t \le \tau_k, \quad k = 1, \dots, N$$

for $t \in [0,T]$ and $N \in \mathbb{N}$. It follows from (2.22) that

$$\int_0^T \int_\Omega |\partial_t u_N(x,t)|^p dxdt + \sup_{0\le t\le T} \int_\Omega |Du_N(x,t)|^p dx \le 2\int_\Omega |Dg(x)|^p dx$$

for all $N \in \mathbb{N}$.

Using the ideas given in the proof of Theorem 2.6, we obtain a subsequence $(u_{N_j})_{j\in\mathbb{N}}$ and weak solution u of

$$\begin{cases} |u_t|^{p-2}u_t = \Delta_p u, & \Omega \times (0,T) \\ u = 0, & \partial\Omega \times [0,T) \\ u = g, & \Omega \times \{0\} \end{cases} \tag{2.23}$$

for which

$$u_{N_j} \to u \quad \text{in} \quad \begin{cases} C([0,T]; L^p(\Omega)) \\ L^p([0,T]; W_0^{1,p}(\Omega)) \end{cases}$$

and

$$\partial_t u_{N_j} \to u_t \quad \text{in} \quad L^p(\Omega \times [0,T]).$$

For $k \in \mathbb{N}$, let u^k be the weak solution of (2.23) just described for $T = k$. Moreover, set $v^k(\cdot, t) = u^k(\cdot, t)$ for $t \in [0, k]$ and $v^k(\cdot, t) := u^k(\cdot, k)$ for $t \in [k, \infty)$. It is immediate that v^k satisfies (2.3). The proof of Theorem 2.6 is also readily adapted to give that $(v^k)_{k \in \mathbb{N}}$ has a subsequence converging as in (2.13) and (2.14) to a global weak solution v of (1.4). We omit the details.

Remark 2.7 We also remark that the subsequence $(v_{N_j})_{j \in \mathbb{N}}$ of the "step function" approximation sequence $(v_N)_{N \in \mathbb{N}}$ defined in (1.6) converges in $C([0, T]; L^p(\Omega))$ to the same weak solution u as the sequence $(u_{N_j})_{j \in \mathbb{N}}$. Indeed, by (2.22)

$$\begin{aligned} \int_\Omega |u_{N_j}(x,t) - v_{N_j}(x,t)|^p dx &\le \max_{1 \le k \le N_j} \int_\Omega |v^k(x) - v^{k-1}(x)|^p dx \\ &\le \left(\frac{T}{N_j}\right)^{p-1} \int_\Omega |Dg(x)|^p dx. \end{aligned}$$

3 Large time limit

This section is dedicated to the proof of Theorem 1.1, which details the large time behavior of solutions of the initial value problem (1.4). Our main tools are the compactness of weak solutions of (1.4) established in Theorem 2.6 and the following lemma, which involves the sign of weak solutions that are close to p-ground states.

Lemma 3.1 *For each positive p-ground state ψ, $C > 0$ and sequence $(s_k)_{k \in \mathbb{N}}$ of positive numbers with $s_k \uparrow \infty$, there is a $\delta = \delta(\psi, C, (s_k)_{k \in \mathbb{N}}) > 0$ with the following property. If v is a weak solution of (1.4) that satisfies*

(i) $\lim_{k \to \infty} e^{\mu_p s_k} v(x, s_k) = \psi$ *in* $W_0^{1,p}(\Omega)$
(ii) $\int_\Omega |v(x, 0)|^p dx \le C$
(iii) $\frac{\int_\Omega |Dv(x,0)|^p dx}{\int_\Omega |v(x,0)|^p dx} \le \lambda_p + \delta$
(iv) $\int_\Omega |v^+(x, 0)|^p dx \ge \frac{1}{2} \int_\Omega |\psi|^p dx,$

then

$$\int_\Omega |e^{\mu_p t} v^+(x,t)|^p dx \ge \frac{1}{2} \int_\Omega |\psi|^p dx. \tag{3.1}$$

for $t \in [0, 1]$.

Proof We argue towards a contradiction. If the result fails, then there exists a triplet $(\psi, C, (s_k)_{k \in \mathbb{N}})$ such that for every $\delta > 0$, there is a weak solution v that satisfies $(i) - (iv)$ while (3.1) fails. Therefore, associated to $\delta_j := 1/j$ $(j \in \mathbb{N})$, there is a weak solution v_j that satisfies (i),

$$\int_\Omega |v_j(x,0)|^p dx \le C, \quad \int_\Omega |v_j^+(x,0)|^p dx \ge \frac{1}{2} \int_\Omega |\psi|^p dx, \quad \frac{\int_\Omega |Dv_j(x,0)|^p dx}{\int_\Omega |v_j(x,0)|^p dx} \le \lambda_p + \frac{1}{j}$$

while

$$\int_\Omega |e^{\mu_p t_j} v_j^+(x, t_j)|^p dx < \frac{1}{2} \int_\Omega |\psi|^p dx \tag{3.2}$$

for some $t_j \in [0, 1]$.

Consequently, the sequence of initial conditions $(v_j(\cdot,0))_{j\in\mathbb{N}}$ is bounded in $W_0^{1,p}(\Omega)$ and has a subsequence (not relabeled) that converges to a positive p-ground state φ in $W_0^{1,p}(\Omega)$. By Theorem 2.6, it also follows that (a subsequence of) the sequence of weak solutions $(v_j)_{j\in\mathbb{N}}$ converges to a weak solution w in $C([0,2],L^p(\Omega))\cap L^p([0,2];W_0^{1,p}(\Omega))$ with $w(\cdot,0)=\varphi$. By Corollary 2.5, $w(\cdot,t)=e^{-\mu_p t}\varphi$.

In addition, we have by (i) and the inequality $\|e^{\mu_p s_k}v_j(\cdot,s_k)\|_{W_0^{1,p}(\Omega)}\le\|v_j(\cdot,0)\|_{W_0^{1,p}(\Omega)}$ that

$$\int_\Omega |Dv_j(x,0)|^p dx \ge \lim_{k\to\infty}\int_\Omega \left|D\left(e^{\mu_p s_k}v_j(\cdot,s_k)\right)\right|^p dx = \int_\Omega |D\psi|^p dx = \lambda_p\int_\Omega |\psi|^p dx \tag{3.3}$$

for all $j\in\mathbb{N}$. Dividing (3.3) by λ_p and letting $j\to\infty$ gives

$$\int_\Omega |\varphi|^p dx = \frac{1}{\lambda_p}\int_\Omega |D\varphi|^p dx = \frac{1}{\lambda_p}\lim_{j\to\infty}\int_\Omega |Dv_j(x,0)|^p dx \ge \int_\Omega |\psi|^p dx.$$

However, letting $j\to\infty$ in (3.2) gives

$$\int_\Omega |\varphi|^p dx \le \frac{1}{2}\int_\Omega |\psi|^p dx.$$

This is a contradiction as $\varphi,\psi\not\equiv 0$. □

Remark 3.2 A similar conclusion holds for v^- provided ψ is a negative ground state and (iv) is replaced with $\int_\Omega |v^-(x,0)|^p dx \ge \frac{1}{2}\int_\Omega |\psi|^p dx$.

Proof of Theorem 1.1 We argue in several steps. We first show that for each sequence $(s_k)_{k\in\mathbb{N}}$ of positive numbers with $s_k\uparrow\infty$, a subsequence of $(e^{\mu_p s_k}v(\cdot,s_k))_{k\in\mathbb{N}}$ has to converge to some p-ground state. This in turn will allow us to prove the convergence of the p-Rayleigh quotient of $v(\cdot,t)$ to the optimal value λ_p. Then we will use the convergence of the p-Rayleigh quotient of $v(\cdot,t)$ and the sign of this p-ground state to derive a crucial lower bound on $L^p(\Omega)$ norm of the same sign of $e^{\mu_p s_k}v(\cdot,s_k)$. Finally, we use this estimate to show that in fact the full sequence converges to this p-ground state.

(1) The following limit

$$S := \lim_{\tau\to\infty}\int_\Omega \left|D\left(e^{\mu_p\tau}v(x,\tau)\right)\right|^p dx \tag{3.4}$$

exists by the monotonicity formula (2.7). If $S=0$, we conclude. So let us assume $S>0$, and suppose $(s_k)_{k\in\mathbb{N}}$ is a sequence of positive numbers increasing to $+\infty$. For each $k\in\mathbb{N}$, define

$$v^k(x,t) := e^{\mu_p s_k}v(x,t+s_k)$$

for $x\in\Omega$ and $t\ge 0$.
Observe, that v^k is a weak solution with $v^k(\cdot,0)=e^{\mu_p s_k}v(\cdot,s_k)$. By (3.4), $(v^k(\cdot,0))_{k\in\mathbb{N}}\subset W_0^{1,p}(\Omega)$ is a bounded sequence. By Theorem 2.8, there is a subsequence $(v^{k_j})_{j\in\mathbb{N}}$ and weak solution w for which v^{k_j} converges to w in $C([0,T];L^p(\Omega))\cap L^p([0,T],W_0^{1,p}(\Omega))$ for all $T>0$; moreover, $v^{k_j}(\cdot,t)$ converges to $w(\cdot,t)$ weakly in $W_0^{1,p}(\Omega)$ for all $t\ge 0$ and strongly for Lebesgue almost every $t\ge 0$. By (3.4), we have

$$S=\lim_{k\to\infty}\int_\Omega \left|D\left(e^{\mu_p(t+s_{k_j})}v(x,t+s_{k_j})\right)\right|^p dx$$

$$
\begin{aligned}
&= e^{p\mu_p t} \lim_{j\to\infty} \int_\Omega |Dv^{k_j}(x,t)|^p dx \\
&= e^{p\mu_p t} \int_\Omega |Dw(x,t)|^p dx
\end{aligned}
$$

for almost every $t \geq 0$. However, as $t \mapsto \int_\Omega |Dw(x,t)|^p dx$ is absolutely continuous (by Lemma 2.2), this equality holds for every $t \geq 0$. Moreover, it must be that $\lim_{j\to\infty} \int_\Omega |Dv^{k_j}(x,t)|^p dx = \int_\Omega |Dw(x,t)|^p dx$ also holds for each $t \geq 0$.

(2) In addition, we have

$$
\begin{aligned}
0 &= \frac{d}{dt} e^{p\mu_p t} \int_\Omega |Dw(x,t)|^p dx \\
&= p e^{p\mu_p t} \left\{ \mu_p \int_\Omega |Dw(x,t)|^p dx - \int_\Omega |w_t(x,t)|^p dx \right\}
\end{aligned} \tag{3.5}
$$

for almost every $t \geq 0$. This computation follows from Lemma 2.2. By the proof of Lemma 2.3, $\mu_p \int_\Omega |Dw(x,t)|^p dx \leq \int_\Omega |w_t(x,t)|^p dx$ for almost every $t \geq 0$ and equality holds only if $w(\cdot,t)$ is a p-ground state for almost every $t \geq 0$. Since $t \mapsto w(\cdot,t) \in L^p(\Omega)$ is continuous and $S = \int_\Omega |D\left(e^{\mu_p t} w(x,t)\right)|^p dx$, there is a single p-ground state ψ for which

$$
w(\cdot,t) = e^{-\mu_p t}\psi.
$$

In summary,

$$
\lim_{j\to\infty} e^{\mu_p(t+s_{k_j})} v(\cdot, t+s_{k_j}) = \psi \tag{3.6}
$$

in $W_0^{1,p}(\Omega)$ for each $t \geq 0$ and in $L^p(\Omega)$ uniformly for each interval $0 \leq t \leq T$. Moreover,

$$
\lim_{t\to\infty} \frac{\int_\Omega |Dv(x,t)|^p dx}{\int_\Omega |v(x,t)|^p dx} = \lim_{j\to\infty} \frac{\int_\Omega |D(e^{\mu_p s_{k_j}} v(x,s_{k_j}))|^p dx}{\int_\Omega |e^{\mu_p s_{k_j}} v(x,s_{k_j})|^p dx} = \frac{\int_\Omega |D\psi|^p dx}{\int_\Omega |\psi|^p dx} = \lambda_p.
$$

(3) As $S = \int_\Omega |D\psi|^p dx > 0$, the p-ground state ψ is determined by its sign. Let us first assume ψ is positive and choose $\delta = \delta(\psi, C, (s_{k_j})_{j\in\mathbb{N}})$ as in Lemma 3.1 where

$$
C := \frac{1}{\lambda_p} \int_\Omega |Dv(x,0)|^p dx.
$$

Note by Poincaré's inequality (1.2) and Lemma 2.3

$$
\int_\Omega |e^{\mu_p t} v(x,t)|^p dx \leq \frac{1}{\lambda_p} \int_\Omega |D\left(e^{\mu_p t} v(x,t)\right)|^p dx \leq C \tag{3.7}
$$

for all $t \geq 0$.

Now fix $j_0 \in \mathbb{N}$ so large that

$$
\int_\Omega |(v^{k_j})^+(x,0)|^p dx \geq \frac{1}{2}\int_\Omega |\psi|^p dx \quad \text{and} \quad \frac{\int_\Omega |Dv^{k_j}(x,0)|^p dx}{\int_\Omega |v^{k_j}(x,0)|^p dx} \leq \lambda_p + \delta
$$

for $j \geq j_0$. Let us additionally fix an $\ell \geq j_0$. By (3.7)

$$
\int_\Omega |v^{k_\ell}(x,0)|^p dx = \int_\Omega |e^{\mu_p s_{k_\ell}} v(x,s_{k_\ell})|^p dx \leq C,
$$

and by (3.6),

$$\lim_{j\to\infty} e^{\mu_p s_{k_j}} v^{k_\ell}(\cdot, s_{k_j}) = \lim_{j\to\infty} e^{\mu_p(s_{k_j}+s_{k_\ell})} v(\cdot, s_{k_j}+s_{k_\ell}) = \psi$$

in $W_0^{1,p}(\Omega)$. Lemma 3.1 then implies

$$\int_\Omega |e^{\mu_p t}(v^{k_\ell})^+(x,t)|^p dx \geq \frac{1}{2}\int_\Omega |\psi|^p dx \tag{3.8}$$

or equivalently,

$$\int_\Omega \left|e^{\mu_p(t+s_{k_\ell})} v^+(x, t+s_{k_\ell})\right|^p dx \geq \frac{1}{2}\int_\Omega |\psi|^p dx. \tag{3.9}$$

for $0 \leq t \leq 1$.
Now set

$$u(x,t) := e^{\mu_p} v^{k_\ell}(x, t+1) = e^{\mu_p(1+s_{k_\ell})} v(x, t+1+s_{k_\ell})$$

for $x \in \Omega$ and $t \geq 0$. Let us verify the hypotheses of Lemma 3.1.

(i) In view of (3.6),

$$\lim_{j\to\infty} e^{\mu_p s_{k_j}} u(\cdot, s_{k_j}) = \lim_{j\to\infty} e^{\mu_p\left(1+s_{k_\ell}+s_{k_j}\right)} v(\cdot, 1+s_{k_\ell}+s_{k_j}) = \psi$$

in $W_0^{1,p}(\Omega)$.

(ii) By (3.7),

$$\int_\Omega |u(x,0)|^p dx = \int_\Omega \left|e^{\mu_p(1+s_{k_\ell})} v(x, 1+s_{k_\ell})\right|^p dx \leq C.$$

(iii) By Proposition 2.4,

$$\begin{aligned}\frac{\int_\Omega |Du(\cdot,0)|^p dx}{\int_\Omega |u(\cdot,0)|^p dx} &= \frac{\int_\Omega \left|Dv(x,1+s_{k_\ell})\right|^p dx}{\int_\Omega \left|v(x,1+s_{k_\ell})\right|^p dx}\\ &\leq \frac{\int_\Omega \left|Dv(x,s_{k_\ell})\right|^p dx}{\int_\Omega \left|v(x,s_{k_\ell})\right|^p dx}\\ &= \frac{\int_\Omega \left|Dv^{k_\ell}(x,0)\right|^p dx}{\int_\Omega \left|v^{k_\ell}(x,0)\right|^p dx}\\ &\leq \lambda_p + \delta.\end{aligned}$$

(iv) Evaluating (3.8) at $t = 1$ gives

$$\int_\Omega |u^+(x,0)|^p dx = \int_\Omega |e^{\mu_p}(v^{k_\ell})^+(x,1)|^p dx \geq \frac{1}{2}\int_\Omega |\psi|^p dx.$$

Then Lemma 3.1 implies

$$\int_\Omega |e^{\mu_p t} u^+(x,t)|^p dx = \int_\Omega |e^{\mu_p(t+1)}\left(v^{k_\ell}\right)^+(x,t+1)|^p dx \geq \frac{1}{2}\int_\Omega |\psi|^p dx.$$

for $0 \leq t \leq 1$. Combining with (3.9) we have that in fact (3.9) holds for $0 \leq t \leq 2$, and by induction, it holds for all $t \geq 0$. Therefore,

$$\int_\Omega \left|e^{\mu_p\left(t+s_{k_j}\right)} v^+(x, t+s_{k_j})\right|^p dx \geq \frac{1}{2}\int_\Omega |\psi|^p dx, \quad t \in [0,\infty) \tag{3.10}$$

for $j \geq j_0$. Finally, if $\psi < 0$, inequality (3.10) holds with v^- replacing v^+.

(4) Now let $(t_\ell)_{\ell \in \mathbb{N}}$ be another sequence of positive numbers increasing to infinity. From our arguments above, t_ℓ has a subequence (that we won't relabel) such that $e^{\mu_p t_\ell} v(\cdot, t_\ell)$ converges to a p-ground state φ in $W_0^{1,p}(\Omega)$ as $\ell \to \infty$. Moreover, φ also satisfies $S = \int_\Omega |D\varphi|^p dx$. By the simplicity of λ_p, $\varphi = \psi$ or $\varphi = -\psi$. Let us assume $\varphi = -\psi$ and without any loss of generality, $\varphi < 0$. As t_ℓ is increasing, we can choose a subsequence $(t_{\ell_j})_{j \in \mathbb{N}}$ for which

$$t_{\ell_j} > s_{k_j}, \qquad j \in \mathbb{N}.$$

Substituting $t = t_{\ell_j} - s_{k_j} > 0$ in (3.10) gives,

$$\int_\Omega \left| e^{\mu_p t_{\ell_j}} v^+(x, t_{\ell_j}) \right|^p dx \geq \frac{1}{2} \int_\Omega |\psi^+|^p dx, \quad j \in \mathbb{N}.$$

However, after letting $j \to \infty$ we find

$$\int_\Omega \left| \varphi^+ \right|^p dx \geq \frac{1}{2} \int_\Omega |\psi^+|^p dx$$

which cannot occur since $\varphi < 0$ and $\psi > 0$ in Ω.

Consequently, for every sequence $(s_k)_{k \in \mathbb{N}}$ of positive numbers increasing to ∞, there is a subsequence of $(e^{\mu_p s_k} v(\cdot, s_k))_{k \in \mathbb{N}}$ converging in $W_0^{1,p}(\Omega)$ to a p-ground state ψ with the same sign that satisfies $S = \int_\Omega |Dw|^p dx$. We appeal to the simplicity of λ_p once again to conclude there is only one such ground state ψ. Therefore, $\lim_{t \to \infty} e^{\mu_p t} v(\cdot, t) = \psi$ in $W_0^{1,p}(\Omega)$, as asserted. $\square$

Remark 3.3 By Morrey's inequality, the family $\{e^{\mu_p t} v(\cdot, t)\}_{t \geq 0}$ is precompact in $C^{0,1-n/p}(\Omega)$ for $p > n$. In this case, $\lim_{t \to \infty} e^{\mu_p t} v(x, t) = \psi(x)$ uniformly in $x \in \Omega$. It would be of great interest to establish uniform convergence for all $p > 1$. It seems to us that the lacking piece of information is a modulus of continuity estimate on solutions of (1.1). Indeed, we have not succeeded in deriving any useful a priori estimates on solutions of (1.1). We hope to do so in forthcoming work.

4 Viscosity solutions

We now turn our attention to proving Theorem 1.2. Therefore, we assume throughout this section that $p \geq 2$, $g \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$, and that there is a p-ground state φ for which

$$-\varphi(x) \leq g(x) \leq \varphi(x), \quad x \in \overline{\Omega}. \tag{4.1}$$

These assumptions will help us verify that (1.4) has a unique *viscosity solution* that is also a weak solution; the reader can find important background material on the theory of viscosity solutions from sources such as [8,13,17]. We remark that we do not consider the "singular" case $p \in (1, 2)$ in order to avoid technicalities and to focus on the new ideas needed to build viscosity solutions of (1.4).

While establishing the uniqueness of viscosity solutions of the initial value problem (1.4) is far from trivial, a standard proof for the comparison of viscosity solutions ($p = 2$) of the heat equation is readily adapted to (1.4). For instance, it is possible to modify the proofs of Theorem 8.2 of [13], Theorem 8.1 of section V.8 in [17], or Theorem 4.7 of [19] to prove the following proposition. The main feature to be exploited is that the term $|v_t|^{p-2}v_t$ is strictly increasing in the time derivative v_t.

Proposition 4.1 *Assume $v \in USC(\overline{\Omega} \times [0, T))$ and $w \in LSC(\overline{\Omega} \times [0, T))$. Suppose the inequality*

$$|v_t|^{p-2}v_t - \Delta_p v \leq 0 \leq |w_t|^{p-2}w_t - \Delta_p w, \quad \Omega \times (0, T)$$

holds in the sense of viscosity solutions and $v(x, t) \leq w(x, t)$ for $(x, t) \in \partial\Omega \times [0, T)$ and for $(x, t) \in \Omega \times \{0\}$. Then

$$v \leq w$$

in $\Omega \times (0, T)$.

Consequently, we will concentrate on confirming the existence of a viscosity solution and showing that this solution is indeed a weak solution. Fortunately, we propose a method that resolves both issues simultaneously. Let us first begin by observing that solutions of the implicit time scheme (1.5) generate viscosity solutions.

Lemma 4.2 *For each $\tau > 0$, the implicit scheme (1.5) generates a solution sequence $\{v^k\}$ of viscosity solutions. Moreover,*

$$\sup_{\Omega} |v^k| \leq \sup_{\Omega} |g|$$

and $v^k \in C^{1,\alpha}_{loc}(\Omega)$ for some $\alpha \in (0, 1]$ and each $k \in \mathbb{N}$.

Proof Consider the implicit scheme (1.5) for $k = 1$

$$\mathcal{J}_p\left(\frac{v^1 - g}{\tau}\right) = \Delta_p v^1, \quad x \in \Omega. \tag{4.2}$$

As $\mathcal{J}_p$ is increasing, this PDE admits a comparison principle for weak sub- and supersolutions. Since the constant function $\sup_\Omega |g|$ is a supersolution, that is nonnegative on $\partial\Omega$, $v^1 \leq \sup_\Omega |g|$. Likewise, $v^1 \geq -\sup_\Omega |g|$, and thus $|v^1| \leq \sup_\Omega |g|$. As the left hand side of the PDE (4.2) is now identified as an $L^\infty(\Omega)$ function, Theorem 2 in [14] implies there is some $\alpha \in (0, 1]$ such that $v^1 \in C^{1,\alpha}_{\text{loc}}(\Omega)$. The assertion for each v^k follows routinely by induction.

Let us now verify that v^1, and similarly each v^k, is a viscosity solution. We will closely follow the argument used to prove Theorem 2.5 in [19]. Assume that $v^1 - \phi$ has a strict local minimum at x_0 where $\phi \in C^\infty(\Omega)$. We are to show

$$\mathcal{J}_p\left(\frac{v^1(x_0) - g(x_0)}{\tau}\right) \geq \Delta_p \phi(x_0). \tag{4.3}$$

If (4.3) doesn't hold, there is a $\delta > 0$ where

$$\begin{cases} \mathcal{J}_p\left(\frac{v^1(x) - g(x)}{\tau}\right) < \Delta_p \phi(x), \\ (v^1 - \phi)(x) > (v^1 - \phi)(x_0) \end{cases}$$

for $x \in B_\delta(x_0)$. Set

$$c := \inf_{\partial B_\delta(x_0)} (v^1 - \phi) > (v^1 - \phi)(x_0),$$

and observe

$$-\Delta_p(\phi + c) < -\mathcal{J}_p\left(\frac{v^1 - g}{\tau}\right) = -\Delta_p v^1, \quad x \in B_\delta(x_0)$$

while $\phi + c \leq v^1$ for $x \in \partial B_\delta(x_0)$. By comparison, $\phi + c \leq v^1$ in $\overline{B}_\delta(x_0)$. In particular,

$$c \leq (v^1 - \phi)(x_0)$$

which is a contradiction. Hence, (4.3) holds and the argument for the subsolution property of v^1 can be made similarly. □

Corollary 4.3 *Let* $\tau > 0$. *Assume* $\{\psi^k\}_{k=0}^\infty \subset C^\infty(\Omega)$ *and* $(x_0, k_0) \in \Omega \times \mathbb{N}$ *is such that*

$$v^k(x) - \psi^k(x) \leq v^{k_0}(x_0) - \psi^{k_0}(x_0) \tag{4.4}$$

for x *in a neighborhood of* x_0 *and* $k \in \{k_0 - 1, k_0\}$. *Then*

$$\mathcal{J}_p\left(\frac{\psi^{k_0}(x_0) - \psi^{k_0-1}(x_0)}{\tau}\right) \leq \Delta_p \psi^{k_0}(x_0).$$

Proof Evaluating the left hand side (4.4) at $k = k_0$ gives

$$\mathcal{J}_p\left(\frac{v^{k_0}(x_0) - v^{k_0-1}(x_0)}{\tau}\right) \leq \Delta_p \psi^{k_0}(x_0)$$

as v^{k_0} is a viscosity solution of (1.5). Evaluating the left hand side of (4.4) at $x = x_0$ and $k = k_0 - 1$ gives $\psi^{k_0}(x_0) - \psi^{k_0-1}(x_0) \leq v^{k_0}(x_0) - v^{k_0-1}(x_0)$. The claim follows from the above inequality and the monotonicity of $\mathcal{J}_p$.

Our candidate for a viscosity solution of (1.4) is $\lim_{N\to\infty} v_N$ where v_N is defined in (1.6). We have already established that a subsequence of $(v_N)_{N\in\mathbb{N}}$ converges to a weak solution in $C([0, T]; L^p(\Omega))$. Therefore, we are left to verify that this sequence converges uniformly to a viscosity solution. Towards this goal, we will employ the half-relaxed limits of v_N

$$\overline{v}(x, t) := \limsup_{\substack{N\to\infty \\ (y,s)\to(x,t)}} v_N(y, s)$$

$$\underline{v}(x, t) := \liminf_{\substack{N\to\infty \\ (y,s)\to(x,t)}} v_N(y, s)$$

for $x \in \overline{\Omega}$ and $t \in [0, T]$. By Lemma 4.2, the sequence $\{v_N\}_{N\in\mathbb{N}}$ is bounded, independently of $N \in \mathbb{N}$. As a result, the above functions are well defined and finite at each $(x, t) \in \overline{\Omega} \times [0, T]$. Moreover, $\overline{v}, -\underline{v}$ are upper semicontinuous and $\overline{v} = \underline{v}$ if and only if v_N converges locally uniformly (see Remark 6.4 of [13]). It is immediate that $\underline{v} \leq \overline{v}$. In order to conclude $\overline{v} \leq \underline{v}$, we will show that $\underline{v}(x, t) = \overline{v}(x, t)$ when $t = 0$ and when $x \in \partial\Omega$ and that $\overline{v}$ and $\underline{v}$ are respective viscosity sub- and supersolutions of the PDE (1.1). We would then be in a position to apply Proposition 4.1.

Lemma 4.4 *Let* φ *be the* p*-ground state in* (4.1). *Then for* $N \in \mathbb{N}$

$$-\varphi(x) \leq v_N(x, t) \leq \varphi(x), \quad (x, t) \in \overline{\Omega} \times [0, T]. \tag{4.5}$$

In particular, for $x_0 \in \partial\Omega$, $\overline{v}(x_0, t) = \underline{v}(x_0, t) = 0$.

Proof Observe that

$$-\Delta_p\varphi + \mathcal{J}_p\left(\frac{\varphi - g}{\tau}\right) = \lambda_p|\varphi|^{p-2}\varphi + \mathcal{J}_p\left(\frac{\varphi - g}{\tau}\right) \geq 0$$

in Ω as $\varphi \geq 0$ and $\varphi \geq g$. Therefore, φ is a supersolution of (4.2). Since $\varphi = v^1 = 0$ on $\partial\Omega$, weak comparison implies $v^1 \leq \varphi$. Likewise, $v^1 \geq -\varphi$. Iterating these bounds for each k yields $-\varphi \leq v^k \leq \varphi$. Consequently, (4.5) holds. Since $\partial\Omega$ is smooth, we have that $\varphi \in C(\overline{\Omega})$ [28]. Thus, for $x_0 \in \partial\Omega$ and $t \in [0, T]$, we can pass to the limit in (4.5) to conclude $\overline{v}(x_0, t) \leq 0 \leq \underline{v}(x_0, t)$.

Lemma 4.5 *For each $x_0 \in \Omega$ and $\varepsilon > 0$, there is a constant $C = C(x_0, \varepsilon)$ such that*

$$|v_N(x, t) - g(x_0)| \leq \varepsilon + C\left(t + \frac{T}{N} + |x - x_0|^{\frac{p}{p-1}}\right) \tag{4.6}$$

for $(x, t) \in \overline{\Omega} \times [0, T]$ and $N \in \mathbb{N}$. In particular, $\overline{v}(x_0, 0) = \underline{v}(x_0, 0) = g(x_0)$.

Proof We first prove there is a constant $C = C(x_0, \varepsilon)$ for which

$$u(x) := g(x_0) + \varepsilon + C\left(\tau + c_p|x - x_0|^{\frac{p}{p-1}}\right), \quad x \in \overline{\Omega}$$

lies above v^1. Here c_p is selected so that $\Delta_p\left(c_p|x - x_0|^{\frac{p}{p-1}}\right) = 1$. Note that since g is continuous on $\overline{\Omega}$, we can find a $\delta > 0$ and $C > 0$ so that

$$|g(x) - g(x_0)| < \varepsilon$$

when $|x - x_0| < \delta$ and

$$\sup_{\Omega} |g| \leq Cc_p|x - x_0|^{\frac{p}{p-1}}$$

when $|x - x_0| \geq \delta$. Indeed, we may choose

$$C = \frac{2\sup_{\Omega}|g|}{c_p\delta^{\frac{p}{p-1}}}.$$

By design, $g(x_0) + \varepsilon + Cc_p|x - x_0|^{\frac{p}{p-1}} - g(x) \geq 0$ for all $x \in \Omega$. Therefore,

$$\begin{aligned}-\Delta_p u + \mathcal{J}_p\left(\frac{u - g}{\tau}\right) &= -C^{p-1} + \mathcal{J}_p\left(\frac{g(x_0) + \varepsilon + Cc_p|x - x_0|^{\frac{p}{p-1}} - g + C\tau}{\tau}\right)\\ &\geq -C^{p-1} + C^{p-1}\\ &= 0.\end{aligned}$$

Choosing C even larger if necessary, we may also assume that $u \geq 0$ on $\partial\Omega$. In this case, weak comparison gives

$$v^1(x) \leq u(x) = g(x_0) + \varepsilon + C\left(\tau + c_p|x - x_0|^{\frac{p}{p-1}}\right)$$

$x \in \Omega$. Similarly, we have

$$v^1(x) \geq g(x_0) - \varepsilon - C\left(\tau + c_p|x - x_0|^{\frac{p}{p-1}}\right).$$

After iterating this procedure k times, we find

$$g(x_0) - \varepsilon - C\left(k\tau + c_p|x - x_0|^{\frac{p}{p-1}}\right) \le v^k(x) \le g(x_0) + \varepsilon + C\left(k\tau + c_p|x - x_0|^{\frac{p}{p-1}}\right).$$

By the definition of v_N (in which $\tau = T/N$), we obtain for $t \in ((k-1)T/N, kT/N]$

$$v_N(x,t) = v^k(x) \le g(x_0) + \varepsilon + C\left(t + \frac{T}{N} + c_p|x - x_0|^{\frac{p}{p-1}}\right).$$

The analogous lower bound holds as well, which implies (4.6). As a result

$$\begin{aligned} &g(x_0) - \varepsilon - C\left(t + c_p|x - x_0|^{\frac{p}{p-1}}\right) \\ &\quad \le \underline{v}(x,t) \le \overline{v}(x,t) \le g(x_0) + \varepsilon + C\left(t + c_p|x - x_0|^{\frac{p}{p-1}}\right), \end{aligned}$$

and therefore

$$g(x_0) - \varepsilon \le \underline{v}(x_0, 0) \le \overline{v}(x_0, 0) \le g(x_0) + \varepsilon.$$

These inequalities conclude the proof, as $\varepsilon > 0$ is arbitrary. □

The following lemma will allow us to exploit the discrete viscosity solutions property of solutions sequences of (1.5) as described in Corollary 4.3. We note this statement is an analog of Lemma A.3 in [9] and is inspired by other works of G. Barles and B. Perthame such as [10].

Lemma 4.6 *Assume $\phi \in C^\infty(\Omega \times (0,T)) \cap C(\overline{\Omega} \times [0,T])$. For $N \in \mathbb{N}$ define*

$$\phi_N(x,t) := \begin{cases} \phi(x,0), & (x,t) \in \Omega \times \{0\}, \\ \phi(x,\tau_k), & (x,t) \in \Omega \times (\tau_{k-1}, \tau_k] \end{cases}, \quad k = 1, \dots, N.$$

Suppose $\overline{v} - \phi$ ($\underline{v} - \phi$) has a strict local maximum (minimum) at $(x_0, t_0) \in \Omega \times (0,T)$. Then there are sequences $(x_j, t_j) \to (x_0, t_0)$ and $N_j \to \infty$, as $j \to \infty$, such that $v_{N_j} - \phi_{N_j}$ has local maximum (minimum) at (x_j, t_j) for each $j \in \mathbb{N}$.

Proof First note that ϕ_N converges to ϕ uniformly on $\Omega \times [0,T]$. Thus,

$$(\overline{v} - \phi)(x,t) := \limsup_{\substack{N\to\infty \\ (y,s)\to(x,t)}} (v_N - \phi_N)(y,s).$$

Consequently, without of loss of generality, we may prove the claim for $\phi \equiv 0$. Another important observation for us is that for any nonempty, compact subset $D \subset \Omega$ and any nonempty, subinterval $I \subset [0,T]$, v_N will achieve a maximum value on $D \times I$. This follows from the continuity of v^k as

$$\sup_{D\times I} v_N = \max\left\{\max_D v^k(x) : k = 1, \dots, N \text{ such that } I \cap (\tau_{k-1}, \tau_k] \ne \emptyset\right\}. \tag{4.7}$$

Now assume that there is $r > 0$ such that

$$\overline{v}(x,t) < \overline{v}(x_0, t_0), \quad (x,t) \in Q_r, \tag{4.8}$$

where $Q_r := B_r(x_0) \times (t_0 - r, t_0 + r) \subset \Omega \times (0,T)$. By definition, we may select a maximizing sequence $\overline{v}(x_0, t_0) = \lim_{j\to\infty} v^{N_j}(y_j, s_j)$ where $(y_j, s_j) \to (x_0, t_0)$ and $N_j \to \infty$. Without loss of generality, we may assume $(y_j, s_j) \in Q_r$ for all $j \in \mathbb{N}$. By the equality (4.7), we may assume there is an $(x_j, t_j) \in \overline{Q_r}$ maximizing v_{N_j} over Q_r. By compactness, we may also assume that up to a subsequence $(x_j, t_j) \to (x_1, t_1) \in \overline{Q_r}$ as $j \to \infty$. Hence,

$$\begin{aligned}\overline{v}(x_0, t_0) &= \limsup_{j\to\infty} v^{N_j}(y_j, s_j)\\ &\le \limsup_{j\to\infty} v^{N_j}(x_j, t_j)\\ &\le \overline{v}(x_1, t_1).\end{aligned}$$

By (4.8), $(x_1, t_1) = (x_0, t_0)$ and the claim follows. □

Proof of Theorem 1.2 It suffices to show that $\overline{v}$ is a viscosity subsolution and $\underline{v}$ is a supersolution of (1.1). Recall that Lemmas 4.4 and 4.5 assert that $\overline{v}$ and $\underline{v}$ agree on $\partial\Omega$ and at $t = 0$, which would allow us to apply Proposition 4.1 to conclude $\overline{v} \le \underline{v}$. In this case, $\overline{v} = \underline{v}$ and $v_N \to v$ uniformly in $\overline{\Omega} \times [0, T]$. Assume that $\phi \in C^\infty(\Omega \times (0, T))$ and $\overline{v} - \phi$ has a strict local maximum at $(x_0, t_0) \in \Omega \times (0, T)$. By Lemma 4.6, there are points (x_j, t_j) converging to (x_0, t_0) and $N_j \in \mathbb{N}$ tending to $+\infty$, as $j \to \infty$, such that $v_{N_j} - \phi_{N_j}$ has a local maximum at (x_j, t_j). Observe that for each $j \in \mathbb{N}$, $t_j \in (\tau_{k_j - 1}, \tau_{k_j}]$ for some $k_j \in \{0, 1, \dots, N_j\}$. Hence, by the definition of v_{N_j} and ϕ_{N_j},

$$\Omega \times \{0, 1, \dots, N_j\} \ni (x, k) \mapsto v^k(x) - \phi(x, \tau_k)$$

has a local maximum at $(x, k) = (x_j, k_j)$. By Lemma 4.3,

$$\mathcal{J}_p\left(\frac{\phi(x_j, \tau_{k_j}) - \phi(x_j, \tau_{k_j - 1})}{T/N_j}\right) \le \Delta_p \phi(x_j, \tau_{k_j}).$$

As $\tau_{k_j - 1} = \tau_{k_j} - T/N_j$ and $|t_j - \tau_{k_j}| \le T/N_j$ for $j \in \mathbb{N}$, we can appeal to the smoothness of ϕ and send $j \to \infty$ to arrive at

$$\mathcal{J}_p(\phi_t(x_0, t_0)) \le \Delta_p \phi(x_0, t_0).$$

Consequently, $\underline{v}$ is a viscosity subsolution of (1.1). By the homogeneity of Eq. (1.1), the same argument applied to $-\overline{v}$ yields that $\underline{v}$ is a supersolution. □

We conclude this section by arguing that when $g \in C^2(\overline{\Omega})$, viscosity solutions of (1.4) satisfy $x \mapsto v(x, t) \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ for almost every $t > 0$ and $|v_t| \le C$.

Proposition 4.7 *Assume v is a viscosity solution of* (1.4) *and that there is a constant $C \ge 0$ such that*

$$|C|^{p-2}C \ge \Delta_p g(x), \quad x \in \Omega. \tag{4.9}$$

Then for each $t \ge s$ and $x \in \Omega$

$$v(x, t) \le v(x, s) + C(t - s).$$

In particular $v_t \le C$. Likewise, if v is a viscosity solution of (1.4) *and there is $C \le 0$ such that*

$$|C|^{p-2}C \le \Delta_p g(x), \quad x \in \Omega.$$

Then $v_t \ge C$.

Proof By assumption (4.9), $(x,t) \mapsto g(x) + Ct$ is a supersolution of (1.1) that is at least as large as v on $\partial\Omega$ and when $t = 0$. By Proposition 4.1, $v(x,t) \le g(x) + Ct$. Now assume $\tau > 0$ is fixed and set $w_1(x,t) := v(x, t+\tau)$ and $w_2(x,t) := v(x,t) + C\tau$. Observe that w_1 and w_2 are viscosity solutions of (1.1) and $w_1(x,t) \le w_2(x,t)$ when either $(x,t) \in \partial\Omega \times [0,T)$ or when $x \in \Omega$ and $t = 0$. By Proposition 4.1, $w_1 \le w_2$ and so $v(x, t+\tau) \le v(x,t) + C\tau$. We may argue similarly for the other assertion. □

Corollary 4.8 *Assume v is a viscosity solution of* (1.4) *and there is $C \ge 0$ such that*

$$|C|^{p-2}C \ge |\Delta_p g(x)|, \quad x \in \Omega. \tag{4.10}$$

Then $|v_t| \le C$.

Corollary 4.9 *Assume v is a viscosity solution of* (1.4) *and g satisfies* (4.10) *for some $C \ge 0$. Then for almost every $t \ge 0$, $x \mapsto v(x,t) \in C^{1,\alpha}_{loc}(\Omega)$.*

Proof As $\Delta_p v = |v_t|^{p-2} v_t \in L^\infty(\Omega)$, for almost every $t > 0$, the claim follows from Theorem 2 in [14]. □

5 Large *p* limit

We are now prepared to deduce the large p limit of Eq. (1.1) and prove Theorem 1.3. We interpret this assertion as a parabolic analog of a theorem of Juutinen, Lindqvist and Manfredi [18]. We also encourage the reader to compare this Theorem 1.3 with the results of [20].

Proof By (2.2) and the assumption that $g \in W^{1,\infty}_0(\Omega)$, $(v^p_t)_{p>r}$ and $(Dv^p)_{p>r}$ are bounded in $L^r_{\text{loc}}(\Omega \times (0,\infty))$ for each $r \ge 1$. Morrey's inequality then implies $(v^p)_{p>n+1} \subset C^{1-(n+1)/p}_{\text{loc}}(\Omega \times (0,\infty))$ has a subsequence $(v^{p_k})_{k\in\mathbb{N}}$ that converges locally uniformly to a continuous function v on $\Omega \times (0,\infty)$. Now suppose $\phi \in C^\infty(\Omega \times (0,\infty))$ and $v - \phi$ has a strict local maximum at some $(x_0, t_0) \in \Omega \times (0,\infty)$. We aim to show $G_\infty(\phi_t(x_0,t_0), D\phi(x_0,t_0), D^2\phi(x_0,t_0)) \le 0$; that is,

$$0 \ge \begin{cases} \min\{-\Delta_\infty \phi(x_0,t_0), |D\phi(x_0,t_0)| + \phi_t(x_0,t_0)\}, & \phi_t(x_0,t_0) < 0 \\ -\Delta_\infty \phi, & \phi_t(x_0,t_0) = 0 \\ \max\{-\Delta_\infty \phi(x_0,t_0), -|D\phi(x_0,t_0)| + \phi_t(x_0,t_0)\}, & \phi_t(x_0,t_0) > 0 \end{cases}. \tag{5.1}$$

By the uniform convergence of v^{p_k} to v, there is a sequence of points $(x_k, t_k) \to (x_0, t_0)$ such that $v^{p_k} - \phi$ has a local maximum at (x_k, t_k). As v^{p_k} is a viscosity solution of (1.1),

$$|\phi_t(x_k,t_k)|^{p_k-2}\phi_t(x_k,t_k) \le \Delta_{p_k}\phi(x_k,t_k), \quad k \in \mathbb{N}. \tag{5.2}$$

If $\phi_t(x_0,t_0) > 0$, then $\phi_t(x_k,t_k) > 0$ for all k large enough. Moreover, (5.2) implies $-\Delta_{p_k}\phi(x_k,t_k) < 0$ and $|D\phi(x_k,t_k)| \ne 0$ for all k large. Rearranging (5.2) gives

$$\frac{1}{p_k - 2}\left(\frac{|\phi_t(x_k,t_k)|}{|D\phi(x_k,t_k)|}\right)^{p_k-4} \phi_t(x_k,t_k)^3 \le \frac{|D\phi(x_k,t_k)|^2 \Delta\phi(x_k,t_k)}{p_k - 2} + \Delta_\infty \phi(x_k,t_k). \tag{5.3}$$

It follows that $-\Delta_\infty\phi(x_0,t_0) \le 0$ in the limit as $k \to \infty$. And as the right hand side of (5.3) is bounded, it must be that $\phi_t(x_k,t_k) \le |D\phi(x_k,t_k)|$ for all k large enough. Hence, $-|D\phi(x_0,t_0)| + \phi_t(x_0,t_0) \le 0$; in particular, (5.1) holds. Now assume $\phi_t(x_0,t_0) = 0$. If in addition $|D\phi(x_0,t_0)| = 0$, then clearly $-\Delta_\infty\phi(x_0,t_0) \le 0$. If $|D\phi(x_0,t_0)| \ne 0$, then

$|D\phi(x_k,t_k)| \neq 0$ for all k large and (5.3) implies $-\Delta_\infty\phi(x_0,t_0) \leq 0$ in the limit as $k \to \infty$. In either case, (5.1) holds.

Finally, suppose $\phi_t(x_0,t_0) < 0$. If additionally, $|D\phi(x_0,t_0)| + \phi_t(x_0,t_0) \leq 0$, then clearly (5.1) follows. Otherwise, $|D\phi(x_0,t_0)| + \phi_t(x_0,t_0) > 0$ and in particular, $|D\phi(x_k,t_k)| + \phi_t(x_k,t_k) > 0$ for all k large. Passing to the limit in (5.3) gives $-\Delta_\infty\phi(x_0,t_0) \leq 0$. In either case, again we have (5.1).

It is now routine to verify that (5.1) holds if $v - \phi$ only has a local maximum at (x_0,t_0). Moreover, our proof that v is a subsolution immediately extends to a proof that v is a supersolution since G_∞ is an odd function:

$$G_\infty(-a,-\xi,-X) = -G_\infty(a,\xi,X).$$

for each $a \in \mathbb{R}$, $p \in \mathbb{R}^n$ and symmetric $n \times n$ matrix X. □

In [18], it was shown that $\lambda_\infty := \lim_{p\to\infty} \lambda_p^{1/p}$ exists. We conjecture that for any viscosity solution v of (1.7), the limit $\psi(x) := \lim_{t\to\infty} e^{\lambda_\infty t} v(x,t)$ exists uniformly in $x \in \Omega$ and is an infinity ground state. That is, ψ is a viscosity solution of the PDE

$$\begin{cases} G_\infty(-\lambda_\infty\psi, D\psi, D^2\psi) = 0, & x \in \Omega \\ \psi = 0, & x \in \partial\Omega \end{cases}.$$

In particular, if $\psi > 0$

$$\begin{cases} \min\{-\Delta_\infty\psi, |D\psi| - \lambda_\infty\psi\} = 0, & x \in \Omega \\ \psi = 0, & x \in \partial\Omega \end{cases}.$$

If our intuition is correct, then it is appropriate to interpret the flow (1.7) as a natural parabolic equation associated with the infinity Laplacian.

References

1. Agueh, M., Blanchet, A., Carrillo, J.: Large time asymptotics of the doubly nonlinear equation in the non-displacement convexity regime. J. Evol. Equ. **10**(1), 59–84 (2010)
2. Ambrosio, L., Gigli, N., Savaré, G.: Gradient Flows in Metric Spaces and in the Space of Probability Measures, 2nd edn. Lectures in Mathematics ETH Zürich. Birkhauser Verlag, Basel (2008)
3. Appell, J., De Pascale, E., Vignoli, A.: Nonlinear spectral theory. de Gruyter Series in Nonlinear Analysis and Applications, 10. Walter de Gruyter & Co., Berlin (2004)
4. Arai, T.: On the existence of the solution for $\partial\phi(u'(t)) + \partial\psi(u(t)) \ni f(t)$. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **26**(1), 75–96 (1979)
5. Armstrong, S.N., Trokhimtchouk, M.: Long-time asymptotics for fully nonlinear homogeneous parabolic equations. Calc. Var. Partial Differ. Equ. **38**(3–4), 521–540 (2010)
6. Aronson, D.G., Peletier, L.A.: Large time behaviour of solutions of the porous medium equation in bounded domains. J. Differ. Equ. **39**(3), 378–412 (1981)
7. Aronsson, G., Janfalk, U.: On Hele-Shaw flow of power-law fluids. Eur. J. Appl. Math. (3), 343–366 (1992)
8. Bardi, M., Capuzzo-Dolcetta, I.: Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations. With appendices by Maurizio Falcone and Pierpaolo Soravia. Systems & Control: Foundations & Applications. Birkhuser Boston, Inc., Boston, MA (1997)
9. Barles, G., Perthame, B.: Discontinuous solutions of deterministic optimal stopping problems. RAIRO Modél. Math. Anal. Num. **21**(4), 557–579 (1987)
10. Barles, G., Perthame, B.: Exit time problems in optimal control and the vanishing viscosity method. SIAM J. Control Optim. **26**, 1133–1148 (1988)
11. Colli, P., Visintin, A.: On a class of doubly nonlinear evolution equations. Comm. Partial Differ. Equ. **15**, 737–756 (1990)
12. Colli, P.: On some doubly nonlinear evolution equations in Banach spaces. Japan J. Indust. Appl. Math. **9**, 181–203 (1992)

13. Crandall, M. G., Ishii, H., Lions, P.-L.: User's guide to viscosity solutions of second order partial differential equations. Bull. Amer. Math. Soc. (N.S.) **27**(1), 1–67 (1992)
14. DiBenedetto, E.: $C^{1+\alpha}$-local regularity of weak solutions of degenerate elliptic equations. Nonlinear Anal. **7**, 827–850 (1983)
15. Evans, L.C.: Partial differential equation. AMS Graduate Stud. Math. **19**, (1997)
16. Evans, L.C.: Some methods of nonlinear functional analysis and applications. Unpublished notes
17. Fleming, W., Soner, H.: Controlled Markov processes and viscosity solutions. Second edition. Stochastic Modeling and Applied Probability, 25. Springer, New York (2006)
18. Juutinen, P., Lindqvist, P., Manfredi, J.: The ∞-eigenvalue problem. Arch. Ration. Mech. Anal. **148**(2), 89–105 (1999)
19. Juutinen, P., Lindqvist, P., Manfredi, J.: On the equivalence of viscosity solutions and weak solutions for a quasi-linear equation. SIAM J. Math. Anal. **33**(3), 699–717 (2001)
20. Juutinen, P., Lindqvist, P.: On the higher eigenvalues for the ∞ -eigenvalue problem. Calc. Var. Partial Differ. Equ. **23**(2), 169–192 (2005)
21. Kristály, A., Rădulescu, V., Varga, C.: Variational principles in mathematical physics, geometry, and economics. Qualitative analysis of nonlinear equations and unilateral problems. Encyclopedia of Mathematics and its Applications, 136. Cambridge University Press, Cambridge (2010)
22. Kamin, S., Vázquez, J.L.: Fundamental solutions and asymptotic behaviour for the p-Laplacian equation. Rev. Mat. Iberoamericana **4**(2), 339–354 (1988)
23. Kawohl, B., Lindqvist, P.: Positive eigenfunctions for the p-Laplace operator revisited. Analysis (Munich) **26**(4), 545–550 (2006)
24. Kim, S., Lee, K.: Asymptotic behavior in degenerate parabolic fully nonlinear equations and its application to elliptic eigenvalue problems. J. Differ. Equ. **254**(8), 3259–3306 (2013)
25. Lindqvist, P.: On the equation div $(|\nabla u|^{p-2}\nabla u)+\lambda|u|^{p-2}u=0$. Proc. Am. Math. Soc. **109**(1), 157–164 (1990)
26. Pelissier, M.C.: Sur Quelques Problèmes Non Lineaires en Glaciologie, Thèse, rev edn. Universitè de Paris-Sud; Publ, Math, Orsay (1975)
27. Mielke, A., Rossi, R., Savaré, G.: Nonsmooth analysis of doubly nonlinear evolution equations. Calc. Var. Partial Differ. Equ. **46**(1–2), 253–310 (2013)
28. Sakaguchi, S.: Concavity properties of solutions to some degenerate quasilinear elliptic Dirichlet problems. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **14**(3), 403–421 (1987)
29. Simon, J.: Compact sets in the space $L^p(0, T; B)$. Ann. Mat. Pura Appl. **146**(4), 65–96 (1987)
30. Stan, D., Vázquez, J.L.: Asymptotic behaviour of the doubly nonlinear diffusion equation $u_t = \Delta_p u^m$ on bounded domains. Nonlinear Anal. **77**, 1–32 (2013)
31. Stefanelli, U.: The Brezis-Ekeland principle for doubly nonlinear equations. SIAM J. Control Optim. **47**(3), 1615–1642 (2008)