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Calculus of Variations

Bifurcation and positive solutions for problem with mean curvature operator in Minkowski space

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Abstract Using bifurcation method, we investigate the existence, nonexistence and multiplicity of positive solutions for the following Dirichlet problem involving mean curvature operator in Minkowski space

$$\begin{cases} -\text{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = \lambda f(|x|, v) & \text{in } B_R(0), \\ v = 0 & \text{on } \partial B_R(0). \end{cases}$$

We managed to determine the intervals of the parameter λ in which the above problem has zero, one or two positive radial solutions corresponding to sublinear, linear, and superlinear nonlinearities f at zero respectively. We also studied the asymptotic behaviors of positive radial solutions as $\lambda \to +\infty$.

Mathematics Subject Classification 35J65 · 34C23 · 35B40

1 Introduction

Consider the following problem

$$\begin{cases} -\text{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = \lambda f(|x|, v) & \text{in } B_R(0), \\ v = 0 & \text{on } \partial B_R(0), \end{cases} \tag{1.1}$$

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where λ is a nonnegative parameter, R is a positive constant and $B_R(0) = \left\{x \in \mathbb{R}^N : |x| < R\right\}$ is the standard open ball in the Euclidean space $\mathbb{R}^N$ ($N \geq 1$) which is centered at the origin and has radius R. Here the nonlinear function $f : [0, R] \times [0, R] \to [0, +\infty)$ is continuous and satisfies the following assumption

$$f(|x|, s) > 0 \quad \text{for any } (|x|, s) \in (0, R] \times (0, R].$$

The aim of this paper is to investigate the existence, nonexistence and multiplicity of positive radial solutions of problem (1.1) according to the different growth conditions of the nonlinearity near zero by using bifurcation analysis and topological methods.

It is well known (see [3]) that the study of spacelike submanifolds of codimension one in the flat Minkowski space $\mathbb{L}^{N+1}$ with prescribed mean extrinsic curvature leads to the type of problems (1.1). Here the flat Minkowski space

$$\mathbb{L}^{N+1} = \left\{(x, t) : x \in \mathbb{R}^N, t \in \mathbb{R}\right\}$$

is endowed with the Lorentzian metric

$$\sum_{i=1}^{N} (dx_i)^2 - (dt)^2,$$

where $(x, t) = (x_1, \ldots, x_N, t)$ are the canonical coordinates in $\mathbb{R}^{N+1}$. This kind of problems is originated from classical relativity. In classical relativity, it is crucial to determine the existence and regularity properties of maximal and constant mean curvature hypersurfaces. These hypersurfaces are spacelike submanifolds of codimension one in the spacetime manifold, with the property that the trace of the extrinsic curvature is zero and constant respectively. The importance of such surfaces lies in that they provide Riemannian submanifolds with properties which reflect those of the spacetime. There are a large amount of papers in the literature on the existence and on qualitative properties of solutions for this type of problems: see [1,12,26] for zero or constant curvature, and [4–7,10,20] for variable curvature.

Recently, using Leray-Schauder degree argument and critical point theory for convex, lower semicontinuous perturbations of C^1-functionals, Bereanu et al. [8] obtained some important existence results for the positive radial solutions of problem (1.1) without parameter λ. In another paper, the same authors [9] successfully established some further nonexistence, existence and multiplicity results for the positive radial solutions of problem (1.1) with $\lambda f(|x|, s) = \lambda\mu(|x|)s^q$, where $q > 1$, $\mu : [0, +\infty) \to \mathbb{R}$ is continuous, strictly positive on $(0, +\infty)$.

Motivated by the interesting studies of Bereanu et al. [8,9] and some earlier works in the literature (see in particular [3] and the references therein), here we continue the investigations on the nonexistence, existence and multiplicity of positive radial solutions of problem (1.1). Our main arguments are based on bifurcation analysis and topological methods. To the best of our knowledge, there are no systematic investigations on problem (1.1) by bifurcation analysis.

As in [8,9], we can easily show that the radially symmetric solutions of problem (1.1) satisfy the following boundary value problem

$$\begin{cases} -\left(r^{N-1}\frac{u'}{\sqrt{1-u'^2}}\right)' = \lambda r^{N-1} f(r, u), & r \in (0, R), \\ u'(0) = u(R) = 0, \end{cases} \tag{1.2}$$

where $r = |x|$ and $u(r) = v(|x|)$.

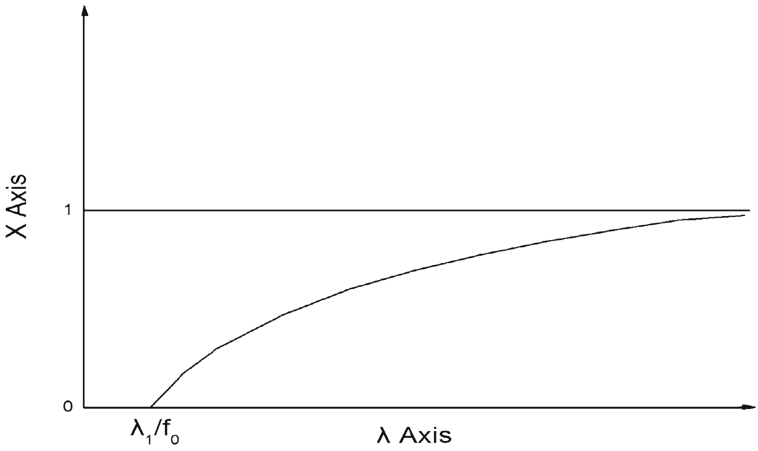

Fig. 1 The case of $f_0 \in (0, +\infty)$

By a solution to problem (1.2), we mean a function $u = u(r) \in C^1[0, R]$ with $\|u'\|_\infty < 1$, such that $r^{N-1}u'/\sqrt{1-u'^2}$ is differentiable and (1.2) is satisfied. Here $\|\cdot\|_\infty$ denotes the usual sup-norm.

Now, we are in a position to state the following hypothesis on the nonlinearity f:

(H_f) $a \in C[0, R]$ is a nonnegative function with $a(r) > 0$ for $r \in (0, R]$ and $f_0 \in [0, +\infty]$ such that

$$\lim_{s\to 0^+} \frac{f(r, s)}{s} = f_0 a(r)$$

uniformly for $r \in [0, R]$.

Let λ_1 be the first eigenvalue for the following linear eigenvalue problem

$$\begin{cases} -\left(r^{N-1}u'\right)' = \lambda r^{N-1} a(r) u, & r \in (0, R), \\ u'(0) = 0 = u(R). \end{cases} \tag{1.3}$$

It is well-known that λ_1 is simple, isolated and the associated eigenfunction has fixed sign in $[0, R)$ (see for example [13] or [27, p. 269]).

Let $X := \{u \in C^1[0, R] : u'(0) = u(R) = 0\}$ with the norm $\|u\| = \|u'\|_\infty$ and $P := \{u \in X : u > 0 \text{ on } [0, R)\}$ be the positive cone in X and $\mathbb{R}^+ = [0, +\infty)$.

Our first main result is the following theorem which deals with the case that f is asymptotically linear near 0, i.e., $f_0 \in (0, +\infty)$.

Theorem 1.1 *Let (H_f) hold with $f_0 \in (0, +\infty)$. The pair $(\lambda_1/f_0, 0)$ is a bifurcation point of problem* (1.2)*. Moreover, there is an unbounded component $\mathscr{C}$ of the set of solution of problem* (1.2) *in $\mathbb{R} \times X$ bifurcating from $(\lambda_1/f_0, 0)$ such that $\mathscr{C} \subseteq \left(\left(\mathbb{R}^+ \times P\right) \cup \{(\lambda_1/f_0, 0)\}\right)$ and $\lim_{\lambda\to+\infty} \|u_\lambda\| = 1$ for $(\lambda, u_\lambda) \in \mathscr{C} \setminus \{(\lambda_1/f_0, 0)\}$. In addition, $(\lambda_1/f_0, 0)$ is the unique bifurcation point on $\mathbb{R}^+ \times \{0\}$ of positive solutions of problem* (1.2).

It follows from Theorem 1.1 that problem (1.1) possesses at least one positive radial solution for any $\lambda \in (\lambda_1/f_0, +\infty)$ (see Fig. 1).

Moreover, the positive radial solution is strictly decreasing thanks to Lemma 1 of [8]. From now on, similarly to that of [24], we add the points $(+\infty, 1)$ and $(+\infty, 0)$ to our space $\mathbb{R} \times X$ so $(+\infty, 1)$ is an element of $\mathscr{C}$ thanks to the asymptotic behavior of positive radial solutions as $\lambda \to +\infty$.

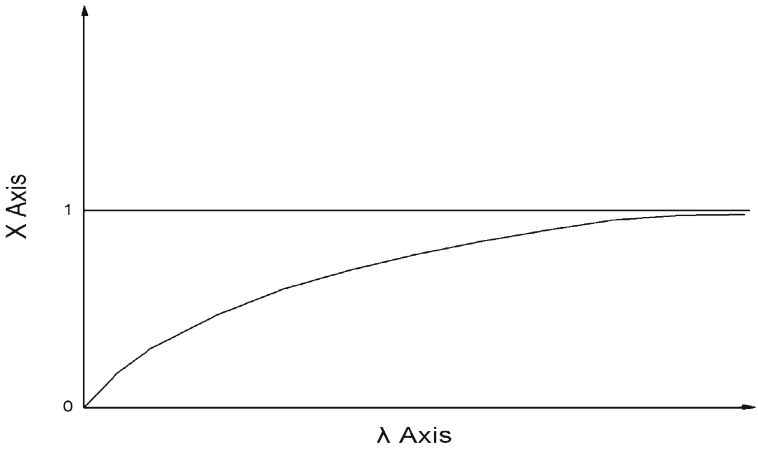

Fig. 2 The case of $f_0 = +\infty$

When $f_0 = +\infty$, f is superlinear with respect to s near 0. Concerning this case, we have the following second main result.

Theorem 1.2 *Assume that (H_f) holds with $f_0 = +\infty$. There is an unbounded component $\mathscr{C}$ of the set of solution of problem (1.2) in $\mathbb{R} \times X$ emanating from $(0, 0)$ such that $\mathscr{C} \subseteq \left(\left(\mathbb{R}^+ \times P\right) \cup \{(0, 0)\}\right)$ and joins to $(+\infty, 1)$.*

By Theorem 1.2, we can see that problem (1.1) has at least one positive radial solution for any $\lambda \in (0, +\infty)$ (see Fig. 2).

In the case of $\lambda = 1$ and f being defined on $[0, R] \times [0, \alpha)$ with $0 < \alpha \leq +\infty$, $f_0 = +\infty$ and $a(r) \equiv 1$, the authors of [8] proved that problem (1.1) has at least one positive radial solution if either $R < \alpha$ or $\alpha = 1 = R$ and the following condition

$$\lim_{s \to 1^-} \sqrt{1 - s^2} f(r, s) = 0 \tag{1.4}$$

holds uniformly for $r \in [0, 1]$. If $\alpha = +\infty$, Theorem 1.2 in particular implies that (1.1) has at least one positive radial solution for any $R < +\infty = \alpha$. For $\alpha \in (0, +\infty)$, our result shows problem (1.1) still has at least one positive radial solution for any $R < +\infty$. Of course our result contains the cases of $R = \alpha \neq 1$ and $\alpha < R$, which are not considered in [8]. Moreover, we do not need the above sublinear condition at 1 of f. In fact, if f also satisfies our conditions at $s = 1$, we always have $\lim_{s \to 1^-} \sqrt{1 - s^2} f(r, s) = 0$ uniformly for $r \in [0, 1]$ because $\lim_{s \to 1^-} \sqrt{1 - s^2} = 0$ and $\lim_{s \to 1^-} f(r, s) = f(r, 1) \geq 0$ for any $r \in [0, 1]$. In general, the conditions of type (1.4) are mainly used to get the *a priori* bounds of solutions with respect to the bounded parameter for the semilinear elliptic equations (see e.g. [15,19,22]). However, here we could obtain $\|u\| < 1$ immediately from problem (1.1), which is different from the classical elliptic equations. Therefore, it is natural to drop this assumption in our current study. From this point of view, Theorem 1.2 extends or complements the corresponding results of [8] even in the case of $\lambda = 1$.

We would also like to point out that the nonlinearity f in the current paper is assumed to be continuous on $[0, R] \times [0, R]$ while in [8] it is continuous in $[0, R] \times [0, \alpha)$. In the latter case, f can have a singularity at α. It is not difficult to verify that $\|u\|_\infty < R$ for any solution u to (1.1). Notice that when $R < \alpha$, the singularity of f at α does not come into play, for

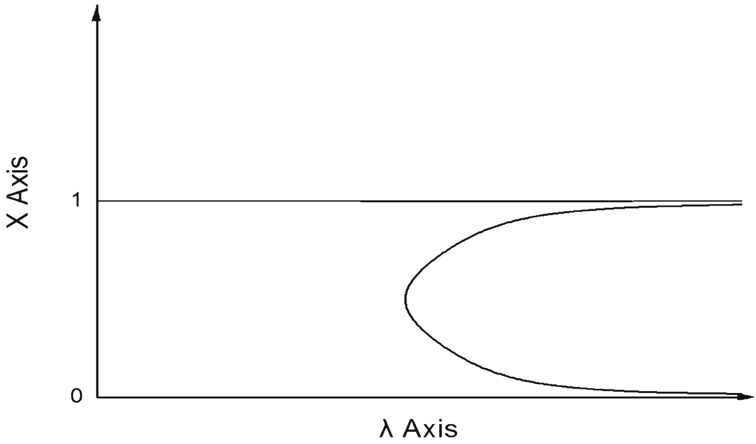

Fig. 3 The case of $f_0 = +\infty$

the second variable cannot reach α at all (see Example 3.2). Indeed, our results cannot cover the corresponding results of [8, Corollary 1] when $R = \alpha = 1$ and f has a singularity at 1. However, here we are able to obtain the asymptotic behavior of positive radial solutions as $\lambda \to +\infty$, which is guaranteed by the continuity of f on $[0, R] \times [0, R]$. Unfortunately, at the time of this writing, we do not know whether this assumption is necessary for the asymptotic behavior of positive radial solutions.

Our third main result deals with the case that the nonlinearity f has sublinear growth rate near 0 (see Fig. 3).

Theorem 1.3 *Assume that* (H_f) *holds with* $f_0 = 0$. *There is an unbounded component* $\mathscr{C}$ *of the set of solution of problem* (1.2) *in* $\mathbb{R} \times X$ *which joins* $(+\infty, 1)$ *to* $(+\infty, 0)$ *such that* $\mathscr{C} \subseteq \left(\mathbb{R}^+ \times P\right)$.

Our last main result in this paper is on the nonexistence of positive radial solutions.

Theorem 1.4 *Assume that there exists a constant* $\varrho > 0$ *such that*

$$\frac{f(r, s)}{s} \leq \varrho a(r)$$

for any $s > 0$ *and* $r \in [0, R]$. *Then there exists* $\varrho_* > 0$ *such that problem (1.2) has no positive solution for any* $\lambda \in (0, \varrho_*)$.

From Theorems 1.1–1.4, we can easily derive the following corollary, which give the ranges of parameter guaranteeing problem (1.1) has zero, one or two positive radial solutions.

Corollary 1.1 *Assume that* (H_f) *holds.*

(a) *If* $f_0 \in (0, +\infty)$, *then there exists* $\mu_1 > 0$ *such that problem* (1.1) *has no positive radial solution for all* $\lambda \in (0, \mu_1)$; *has at least one positive radial solution for all* $\lambda \in (\mu_1, +\infty)$.
(b) *If* $f_0 = +\infty$, *then problem* (1.1) *has at least one positive radial solution for all* $\lambda \in (0, +\infty)$.

(c) *If* $f_0 = 0$, *then there exist* $\mu_2 > 0$ *and* $\mu_3 > 0$ *such that problem* (1.1) *has no positive radial solution for all* $\lambda \in (0, \mu_2)$*; has at least one positive radial solution for all* $\lambda \in [\mu_2, \mu_3]$*; has at least two positive radial solutions for all* $\lambda \in (\mu_3, +\infty)$.

The rest of this paper is organized as follows. In Sect. 2, we prove Theorem 1.1. Section 3 is devoted to proving Theorem 1.2. In the last section, Sect. 4, we present the proofs of Theorems 1.3 and 1.4. In each section, we also give some examples and compare our conclusions with some previous known results in literature.

2 Proof of Theorem 1.1

We first transform problem (1.2) into the form of classical Sturm-Liouville problems. If u is a solution of problem (1.2), then for any $r \in (0, R)$ one has that

$$\begin{aligned}
&\lambda r^{N-1} f(r, u)\\
&= -\left(r^{N-1}u'\right)' \frac{1}{\sqrt{1-u'^2}} - r^{N-1}u'^2u'' \frac{1}{\left(1-u'^2\right)\sqrt{1-u'^2}}\\
&= -(N-1)r^{N-2}u' \frac{1}{\sqrt{1-u'^2}} - r^{N-1}u'' \frac{1}{\sqrt{1-u'^2}} - r^{N-1}u'^2u'' \frac{1}{\left(1-u'^2\right)\sqrt{1-u'^2}}\\
&= -(N-1)r^{N-2}u' \frac{1}{\sqrt{1-u'^2}} - r^{N-1}u'' \frac{1}{\sqrt{1-u'^2}} \frac{1}{1-u'^2}.
\end{aligned}$$

It follows that

$$-u'' = \lambda f(r, u)\left(1-u'^2\right)^{\frac{3}{2}} + \frac{N-1}{r}u'\left(1-u'^2\right).$$

From this we obtain that

$$-\left(r^{N-1}u'\right)' = \lambda r^{N-1} f(r, u)\left(1-u'^2\right)^{\frac{3}{2}} - (N-1)r^{N-2}u'^3.$$

Now, problem (1.2) is equivalent to

$$\begin{cases} -\left(r^{N-1}u'\right)' = \lambda r^{N-1} f(r, u)\left(1-u'^2\right)^{\frac{3}{2}} - (N-1)r^{N-2}u'^3, & r \in (0, R),\\ u'(0) = u(R) = 0. \end{cases} \tag{2.1}$$

Let $\xi : [0, R] \times [0, R] \to \mathbb{R}$ be such that

$$f(r, s) = f_0 a(r)s + \xi(r, s)$$

with

$$\lim_{s\to 0^+} \frac{\xi(r, s)}{s} = 0$$

uniformly for $r \in [0, R]$. Let us consider

$$\begin{cases} -\left(r^{N-1}u'\right)' = \lambda f_0 r^{N-1} a(r)u\left(1-u'^2\right)^{\frac{3}{2}} + \lambda r^{N-1}\xi(r, u)\left(1-u'^2\right)^{\frac{3}{2}} - (N-1)r^{N-2}u'^3,\\ u'(0) = u(R) = 0 \end{cases} \tag{2.2}$$

as a bifurcation problem from the trivial solution axis.

Proof of Theorem 1.1 Let $G(r,s)$ be the Green's function associated with the operator $\mathscr{L}u := -\left(r^{N-1}u'\right)'$ with the same boundary condition as in problem (2.2) (see [23]). Then problem (2.2) can be equivalently written as

$$u = \lambda Lu + H(\lambda, u) := K(\lambda, u), \tag{2.3}$$

where

$$Lu = f_0 \int_0^R G(r,s)s^{N-1}a(s)u(s)\,ds$$

and

$$H(\lambda,u) = \int_0^R G(r,s)s^{N-2}\left[\lambda f_0 a(s)su\left(\left(1-u'^2\right)^{\frac{3}{2}}-1\right)+\lambda\xi(s,u)s\left(1-u'^2\right)^{\frac{3}{2}}-(N-1)u'^3\right]ds.$$

Then it is well known that $L : X \to X$ is linear completely continuous and $H : \mathbb{R}\times X \to X$ is completely continuous (see [23]).

First, we show that $H = o(\|u\|)$ near $u = 0$ uniformly on bounded λ intervals. Let

$$\widetilde{\xi}(r,w) = \max_{0\le s\le w} |\xi(r,s)| \quad \text{for any } r \in [0,R].$$

Then $\widetilde{\xi}$ is nondecreasing with respect to w and

$$\lim_{w\to 0^+} \frac{\widetilde{\xi}(r,w)}{w} = 0. \tag{2.4}$$

Noting that

$$u(r) = \int_R^r u'(t)\,dt,$$

one has that

$$|u(r)| = \left|\int_R^r u'(t)\,dt\right| \le \int_r^R \left|u'(t)\right|\,dt \le \|u\|R$$

for any $r \in [0,R]$. So we have that $\|u\|_\infty \le \|u\|R$. Further it follows from (2.4) that

$$\left|\frac{\xi(r,u)}{\|u\|}\right| \le \frac{\widetilde{\xi}(r,u)}{\|u\|} \le \frac{\widetilde{\xi}\,(r,\|u\|_\infty)}{\|u\|} \le R\frac{\widetilde{\xi}(r,\|u\|R)}{\|u\|R} \to 0 \quad \text{as } \|u\| \to 0 \tag{2.5}$$

uniformly in $r \in [0,R]$. Then we have that

$$\frac{\xi(r,u)\left(1-u'^2\right)^{\frac{3}{2}}}{\|u\|} \to 0 \quad \text{as } \|u\| \to 0$$

uniformly in $r \in [0,R]$. Obviously, we have that

$$\frac{u'^3}{\|u\|} \to 0, \quad \frac{a(r)u\left(\left(1-u'^2\right)^{\frac{3}{2}}-1\right)}{\|u\|} \to 0 \quad \text{as } \|u\| \to 0$$

uniformly in $r \in [0,R]$.

Now, applying Theorem 1.3 of [23] to problem (2.3), we obtain that there exists a continuum $\mathscr{C}$ of solution set of problem (2.3) bifurcating from $(\lambda_1/f_0, 0)$ which is either unbounded or contains a pair $\left(\overline{\lambda}/f_0, 0\right)$ for some $\overline{\lambda}$, eigenvalue of problem (1.3) with $\overline{\lambda} \ne \lambda_1/f_0$. Since (0,0) is the unique solution of problem (2.3) for $\lambda = 0$ (see [9]), so $\mathscr{C} \cap (\{0\}\times X) = \emptyset$. It follows from Lemma 1 of [8] that u is positive on $[0, R)$ and is strictly decreasing for any $(\lambda, u) \in \mathscr{C} \setminus \{(\lambda_1/f_0, 0)\}$. So we have that

$$\mathscr{C} \subseteq \left(\left(\mathbb{R}^+ \times P\right) \cup \{(\lambda_1/f_0, 0)\}\right)$$

and $\mathscr{C}$ is unbounded in $\mathbb{R} \times X$. It is easy to verify that $(\lambda_1/f_0, 0)$ is the unique bifurcation point on $\mathbb{R} \times \{0\}$ of positive solutions of problem (1.2).

Finally, we show the asymptotic behavior of u_λ as $\lambda \to +\infty$ for $(\lambda, u_\lambda) \in \mathscr{C} \setminus \{(\lambda_1/f_0, 0)\}$. To do this, we take any $(\lambda_n, u_n) \in \mathscr{C} \setminus \{(\lambda_1/f_0, 0)\}$ with $\lambda_n \to +\infty$ as $n \to +\infty$. In order to do this, we take any $(\lambda_n, u_n) \in \mathscr{C} \setminus \{(\lambda_1/f_0, 0)\}$ with $\lambda_n \to +\infty$ as $n \to +\infty$. The fact that $(\lambda_n, 0)$ is not a bifurcation point implies that there exists a constant $\delta > 0$ such that $\|u_n\| \geq \delta$ for any $n \in \mathbb{N}$.

Define

$$F_n(r) = \max_{t \in [0,r]} \left(-u_n'(t)\right)$$

for any $r \in [0, R]$ and $n \in \mathbb{N}$. Then it follows from Lemma 1 of [8] that $F_n(r)$ is increasing on $[0, R]$. Clearly, one has $F_n(R) = \|u_n\| \geq \delta$. The fact $u_n'(0) = 0$ implies that $F_n(0) = 0$. Moreover, for any $r_0 \in [0, R)$ and $h \to 0^+$ with $r_0 + h \in [0, R)$, we have that

$$0 \leq F_n\left(r_0 + h\right) - F_n\left(r_0\right) = \max_{t \in [0, r_0+h]} \left(-u_n'(t)\right) - \max_{t \in [0, r_0]} \left(-u_n'(t)\right).$$

Let $r_M \in [0, r_0 + h]$ such that $-u_n'\left(r_M\right) = \max_{t \in [0, r_0+h]} \left(-u_n'(t)\right)$. If $r_M \in [0, r_0]$, we can see that $F_n\left(r_0 + h\right) - F_n\left(r_0\right) = 0$. If $r_M \in (r_0, r_0 + h]$, we have that

$$\max_{t \in [0, r_0+h]} \left(-u_n'(t)\right) - \max_{t \in [0, r_0]} \left(-u_n'(t)\right) \leq -u_n'\left(r_M\right) + u_n'\left(r_0\right) \to 0 \quad \text{as } h \to 0^+.$$

So $F_n(r)$ is right continuous on $[0, R)$. Similarly, we can show that $F_n(r)$ is left continuous on $(0, R]$. Thus we have that $F_n(r)$ is continuous on $[0, R]$.

Therefore, for any $\varepsilon \in (0, \delta/2)$, there exists $\rho_n \in (0, R)$ such that

$$F_n\left(\rho_n\right) = \varepsilon.$$

Let $\rho_* = \liminf_{n \to +\infty} \rho_n$ and $F(r) = \limsup_{n \to +\infty} F_n(r)$. Then up to a subsequence, we have that

$$F\left(\rho_*\right) = \lim_{n \to +\infty} F_n\left(\rho_n\right) = \varepsilon,$$

which implies that $\rho_* > 0$. Indeed, if $\rho_* = 0$, up to a subsequence, we have that

$$\varepsilon = F(0) = \lim_{n \to +\infty} F_n(0) = 0,$$

which is impossible. Similarly, we can show that $\rho_* < R$. Noting that, up to a subsequence, $\lim_{n \to +\infty} F_n\left(\rho_*\right) = F\left(\rho_*\right)$, we have that

$$0 < F_n\left(\rho_*\right) < 2\varepsilon$$

for sufficiently large n. Then for any $r \in [0, \rho_*]$, it follows from the definition of F_n that

$$-u_n'(r) < 2\varepsilon \tag{2.6}$$

for n large enough. Now we have the following claim:

Claim *For any given $\rho \in (0, \rho_*]$, there exists a positive constant τ_0 such that $u_n(\rho) \geq \tau_0$.*

To show the claim, we proceed by contradiction. Assume that $u_n(\rho) \to 0$ as $n \to +\infty$. By virtue of Lemma 1 of [8] one has that $u_n(r) \to 0$ as $n \to +\infty$ for any $r \in [\rho, R]$. It follows that

$$u_n(r) = -\int_r^R u_n'(t)\,dt \to 0 \quad \text{as } n \to +\infty$$

for any $r \in [\rho, R]$. It follows from Lemma 1 of [8] that $-u_n'(r)$ is nonnegative for any $r \in [0, R]$. From the Fatou Lemma, we obtain that

$$\int_r^R \liminf_{n\to+\infty} \left(-u_n'(t)\right)\, dt \le \liminf_{n\to+\infty} \int_r^R \left(-u_n'(t)\right)\, dt = 0$$

for any $r \in [\rho, R]$. It follows that

$$\int_r^R \liminf_{n\to+\infty} \left(-u_n'(t)\right)\, dt = 0$$

for any $r \in [\rho, R]$. In particular, one has that

$$\int_\rho^R \liminf_{n\to+\infty} \left(-u_n'(t)\right)\, dt = 0.$$

So we have that

$$\liminf_{n\to+\infty} \left(-u_n'(r)\right) = 0$$

for any $r \in [\rho, R]$. Thus we get a subsequence of u_n' (which we still denote by u_n' for convenience) such that

$$u_n'(r) \to 0 \quad \text{as } n \to +\infty \tag{2.7}$$

for any $r \in [\rho, R]$. Now combing (2.6) with (2.7), we have that

$$\delta \le \|u_n\| \le 2\varepsilon < \delta$$

for n large enough. This contradiction verifies our claim.

Now integrating the first equation of problem (1.2) with respect to t from 0 to r, one has that

$$r^{N-1} \frac{u_n'}{\sqrt{1-u_n'^2}} = -\lambda_n \int_0^r t^{N-1} f\,(t, u_n)\, dt.$$

It follows that

$$\frac{1}{\sqrt{1-\|u_n\|^2}} \ge \frac{1}{\sqrt{1-u_n'^2}} > \left|\frac{u_n'}{\sqrt{1-u_n'^2}}\right| = \frac{\lambda_n}{r^{N-1}} \int_0^r t^{N-1} f\,(t, u_n)\, dt.$$

It follows immediately that $\|u_n\| < 1$, implying that $\|u_n\|_\infty < R$. So for any $r \in [\rho/4, \rho]$, one has that $u_n(r) \in [\tau_0, R)$. Let $f_1 := \min_{[\rho/4,\rho]\times[\tau_0,R]} f(t,s)$. It is easy to observe that $f_1 > 0$. Now for any $r \in [\rho/2, \rho]$, we have that

$$\begin{aligned}
\frac{1}{\sqrt{1-\|u_n\|^2}} &> \frac{\lambda_n}{r^{N-1}} \int_{\frac{\rho}{4}}^r t^{N-1} f\,(t, u_n)\, dt \\
&\ge \frac{\lambda_n f_1}{r^{N-1}} \int_{\frac{\rho}{4}}^r t^{N-1}\, dt = \frac{\lambda_n f_1}{N r^{N-1}} \left(r^N - \left(\frac{\rho}{4}\right)^N\right) \\
&\ge \frac{\lambda_n f_1}{N \rho^{N-1}} \left(\left(\frac{\rho}{2}\right)^N - \left(\frac{\rho}{4}\right)^N\right) \ge \frac{\lambda_n f_1 \rho}{N 2^N} \left(1 - \frac{1}{2^N}\right),
\end{aligned}$$

which implies

$$\lim_{n\to+\infty}\|u_n\|=1.$$

The proof is completed. □

Example 2.1 For $q>1$, the problem

$$\begin{cases}-\mathrm{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right)=\lambda a(|x|)v+v^q & \text{in } B_R(0),\\ v=0 & \text{on } \partial B_R(0)\end{cases}\tag{2.8}$$

has at least one positive radial solution for any $\lambda\in(\lambda_1,+\infty)$ thanks to Theorem 1.1. Moreover, it follows from Theorem 1.4 that problem (2.8) has no positive radial solution for λ small enough. This appears in contrast to the classical case (see also [8]). When $N\geq 4$, Brezis and Nirenberg [11] showed that the problem

$$\begin{cases}-\Delta v=\lambda v+v^{\frac{N+2}{N-2}} & \text{in } B_R(0),\\ v=0 & \text{on } \partial B_R(0)\end{cases}$$

has a positive solution provided that $\lambda>0$ is sufficiently small. So problems with mean curvature operators in Minkowski space is deviating in a big manner away from the Laplacian.

3 Proof of Theorem 1.2

To prove Theorem 1.2, we need the following topological lemma, which is established in [14].

Lemma 3.1 *Let $\mathcal{X}$ be a normed vector space and let $\{\mathscr{C}_n\}$ be a sequence of unbounded connected subsets in $\mathcal{X}$. Assume that:*

(i) *There exists $z^*\in\liminf_{n\to+\infty}\mathscr{C}_n$ with $\|z^*\|<+\infty$;*
(ii) *For every $R>0$, $\left(\cup_{n=1}^{+\infty}\mathscr{C}_n\right)\cap\overline{B}_R$ is a relatively compact set in $\mathcal{X}$.*

Then the set $\mathcal{D}:=\limsup_{n\to+\infty}\mathscr{C}_n$ is unbounded, closed and connected.

Proof of Theorem 1.2. Define

$$f^n(r,s)=\begin{cases}na(r)s, & s\in\left[0,\frac{1}{n}\right],\\ \left(f\left(r,\frac{2}{n}\right)-a(r)\right)ns+2a(r)-f\left(r,\frac{2}{n}\right), & s\in\left(\frac{1}{n},\frac{2}{n}\right),\\ f(r,s), & s\in\left[\frac{2}{n},+\infty\right).\end{cases}$$

It is easy to see that $\lim_{n\to+\infty}f^n(r,s)=f(r,s)$ and $f_0^n=n$. Consider the following problem

$$\begin{cases}-\left(r^{N-1}\frac{u'}{\sqrt{1-u'^2}}\right)'=\lambda r^{N-1}f^n(r,u), & r\in(0,R),\\ u'(0)=u(R)=0.\end{cases}\tag{3.1}$$

It follows from Theorem 1.1 that there exists a sequence of unbounded continua $\mathscr{C}_n$ of solution set of problem (3.1) emanating from $(\lambda_1/n,0)$ and joining $(+\infty,1)$.

Taking $z^*=(0,0)$, we easily observe that $z^*\in\liminf_{n\to+\infty}\mathscr{C}_n$. The compactness of K implies that $\left(\cup_{n=1}^{+\infty}\mathscr{C}_n\right)\cap B_R$ is pre-compact. Lemma 3.1 implies that $\mathscr{C}=\limsup_{n\to+\infty}\mathscr{C}_n$ is unbounded closed connected such that $z^*\in\mathscr{C}$ and $(+\infty,1)\in\mathscr{C}$.

For any $(\lambda, u) \in \mathscr{C}$, the definition of superior limit (see [28]) shows that there exists a sequence $(\lambda_n, u_n) \in \mathscr{C}_n$ such that $(\lambda_n, u_n) \to (\lambda, u)$ as $n \to +\infty$. Clearly, (λ_n, u_n) satisfies problem (3.1). Noting the compactness of K, letting $n \to +\infty$, we get that

$$u = K(\lambda, u).$$

It follows that u is a solution of problem (1.1). Thus, u is a solution of problem (1.1) for any $(\lambda, u) \in \mathscr{C}$. It is clear that u is nonnegative for any $(\lambda, u) \in \mathscr{C}$ because $u_n \geq 0$.

Next we show that u is a positive solution of problem (1.2) for any $(\lambda, u) \in \mathscr{C}$. Noting Lemma 1 of [8], it is sufficient to show that $\mathscr{C} \cap ((0, +\infty) \times \{0\}) = \emptyset$. Suppose on the contrary that $\mathscr{C} \cap ((0, +\infty) \times \{0\}) \neq \emptyset$. Let $(\mu, 0)$ denote the first intersection of $\mathscr{C} \setminus \{(0, 0)\}$ and $(0, +\infty) \times \{0\}$. Then there exist $(\lambda_n, u_n) \in (\mathscr{C} \setminus \{(0, 0)\})$ with $u_n \not\equiv 0$ such that $\lambda_n \to \mu \in (0, +\infty)$ and $u_n \to 0$ as $n \to +\infty$. It follows from Lemma 1 of [8] that $u_n > 0$ on $[0, R)$ for any $n \in \mathbb{N}$.

Letting $v_n = u_n / \|u_n\|$, we have that

$$\begin{cases} -\left(r^{N-1} v_n'\right)' = \lambda_n r^{N-1} \frac{f(r,u_n)}{u_n} v_n \left(1 - u_n'^2\right)^{\frac{3}{2}} - (N-1) r^{N-2} \frac{u_n'^3}{\|u_n\|}, \ r \in (0, R), \\ v_n'(0) = v_n(R) = 0. \end{cases} \tag{3.2}$$

Let φ_1 be a positive eigenfunction corresponding to λ_1. We multiply the first equation of problem (3.2) by φ_1, and obtain after integrations by parts

$$\lambda_1 \int_0^R v_n r^{N-1} a(r) \varphi_1 \, dr = \int_0^R \left(\lambda_n r^{N-1} \frac{f(r, u_n)}{u_n} v_n \left(1 - u_n'^2\right)^{\frac{3}{2}} - (N-1) r^{N-2} \frac{u_n'^3}{\|u_n\|} \right) \varphi_1 \, dr.$$

On the other hand, the fact $f_0 = +\infty$ implies that $f(r, s) \geq M a(r) s$ for any positive constant M, any $s > 0$ small enough and $r \in [0, R]$. So we have that

$$\lambda_1 \int_0^R v_n r^{N-1} a \varphi_1 \, dr \geq \lambda_n M \int_0^R \left(r^{N-1} a(r) v_n \left(1 - u_n'^2\right)^{\frac{3}{2}} \right) \varphi_1 \, dr.$$

It follows from $\|v_n\| = 1$ immediately that $\{v_n\}$ is uniformly bounded and equicontinuous on $C[0, R]$. From the Arzelà-Ascoli theorem, we get that $v_n \to v$ in $C[0, R]$ (up to a subsequence). So one has that $v_n \to v$ uniformly in $[0, R]$. Letting $n \to +\infty$, one has that

$$\lambda_1 \int_0^R v r^{N-1} a(r) \varphi_1 \, dr \geq \mu M \int_0^R r^{N-1} a(r) v \varphi_1 \, dr.$$

It implies that

$$M \leq \frac{\lambda_1}{\mu},$$

which contradicts the arbitrariness of M. □

Example 3.1 Let $q \in [0, 1)$. For any $\lambda \in (0, +\infty)$, the Dirichlet problem

$$\begin{cases} -\operatorname{div}\left(\frac{\nabla v}{\sqrt{1 - |\nabla v|^2}} \right) = \lambda a(|x|) v^q & \text{in } B_R(0), \\ v = 0 & \text{on } \partial B_R(0) \end{cases}$$

has at least one positive radial solution.

Example 3.2 For any $\alpha > R$ and $\gamma > 0$, both problems

$$\begin{cases} -\operatorname{div}\left(\frac{\nabla v}{\sqrt{1 - |\nabla v|^2}} \right) = \lambda \frac{a(|x|) v^q}{\sqrt{\alpha^2 - v^2}} & \text{in } B_R(0), \\ v = 0 & \text{on } \partial B_R(0) \end{cases}$$

and

$$\begin{cases} -\text{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = \lambda \frac{a(|x|)v^q}{(\alpha-v)^\gamma} & \text{in } B_R(0), \\ v = 0 & \text{on } \partial B_R(0) \end{cases}$$

have at least one positive classical radial solution for any $\lambda > 0$. In addition, for any $\alpha \leq R$, the Dirichlet problem

$$\begin{cases} -\text{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = \lambda a(|x|)v^q(\alpha + v)^\gamma & \text{in } B_R(0), \\ v = 0 & \text{on } \partial B_R(0) \end{cases}$$

has at least one positive classical radial solution for any $\lambda > 0$.

Example 3.3 Let $0 \leq q < 1 \leq p$. The problem

$$\begin{cases} -\text{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = \lambda v^q + v^p & \text{in } B_R(0), \\ v = 0 & \text{on } \partial B_R(0) \end{cases}$$

has at least one positive classical radial solution for any $\lambda > 0$. While for the problem

$$\begin{cases} -\Delta v = \lambda v^q + v^p & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

with $0 < q < 1 < p$, Ambrosetti et al. showed in [2] that it has a positive solution if and only if $0 < \lambda \leq \Lambda$ for some $\Lambda > 0$. Here Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. This example shows that problems with mean curvature operators have a big different to the semilinear elliptic problems (see also [8]).

Example 3.4 Consider the following problem

$$\begin{cases} -\text{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = \lambda a(|x|)p(v) & \text{in } B_R(0), \\ v = 0 & \text{on } \partial B_R(0), \end{cases}$$

where $p : [0, R] \to \mathbb{R}$ is a continuous function such that $p(0) = 0$ and $p(s)s > 0$ for any $s \in (0, R]$. If $\lim_{s\to 0^+} p(s)/s := p_0 = +\infty$, the above problem has at least one positive radial solution for any $\lambda \in (0, +\infty)$.

Rather than under the assumption $p_0 = +\infty$, Bereanu et al. proved in [8] that the above problem with $\lambda = 1$ has at least one positive radial solution under the following assumption

$$R^N < N \int_0^R r^{N-1} a(r) P(R - r)\, dr. \tag{3.3}$$

In fact, condition (3.3) is equivalent to $p_0 = +\infty$ in some special cases. To see this, let $N > 2$, $a(r) \equiv 1$ and $R \leq R_0$ which is to be determined later.

We first claim that condition (3.3) implies $p_0 = +\infty$. Otherwise, there would exist an $M \in (0, +\infty)$ such that $p(s)/s \leq M$ for any $s > 0$. So we could conclude that

$$\begin{aligned} R^N &< N \int_0^R r^{N-1} P(R - r)\, dr = N \int_0^R r^{N-1} \int_0^{R-r} p(s)\, ds\, dr \\ &\leq NM \int_0^R r^{N-1} \int_0^{R-r} s\, ds\, dr = \frac{NM}{2} \int_0^R r^{N-1} (R - r)^2\, d \\ &\leq \frac{NMR^4}{32} \int_0^R r^{N-3}\, dr = \frac{NMR^{N+2}}{32(N-2)}, \end{aligned}$$

which yields that

$$R > \sqrt{\frac{32(N-2)}{MN}} := R_0. \tag{3.4}$$

However, (3.4) contradicts the assumption $R \le R_0$.

On the other hand, $p_0 = +\infty$ implies that there exists $\rho > 0$ such that $p(s)/s \le M$ for any $s \in (0, \rho]$ and $M > 0$. In particular, taking $a(r) \equiv 1$ and

$$M = \frac{2R^N}{\rho^2 \left(R^N - (R-\rho)^N\right)},$$

one can easily show that

$$\begin{aligned} N \int_0^R r^{N-1} P(R-r)\, dr &> N \int_{R-\rho}^R r^{N-1} \int_0^{R-r} p(s)\, ds\, dr \\ &\ge MN \int_{R-\rho}^R r^{N-1} \int_0^{R-r} s\, ds\, dr = \frac{MN}{2} \int_{R-\rho}^R r^{N-1}(R-r)^2\, dr \\ &\ge \frac{MN\rho^2}{2} \int_{R-\rho}^R r^{N-1}\, dr = \frac{M\rho^2}{2}\left(R^N - (R-\rho)^N\right) = R^N, \end{aligned}$$

which is just condition (3.3).

Example 3.5 Given $m \ge 0$ and $q \in (0, 1)$, the Hénon type problem

$$\begin{cases} -\mathrm{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = \lambda |x|^m v^q & \text{in } B_R(0), \\ v = 0 & \text{on } \partial B_R(0) \end{cases} \tag{3.5}$$

has at least one positive radial solution for any $\lambda \in (0, +\infty)$. If $q \ge 1$, it follows from Theorem 1.4 that problem (3.5) has no positive radial solution for any $\lambda \in \left(0, \widetilde{\lambda}\right)$ for some $\widetilde{\lambda} > 0$. While, in the Laplacian case, Hénon proved that the problem

$$\begin{cases} -\Delta v = \lambda |x|^m v^q & \text{in } B_1(0), \\ v = 0 & \text{on } \partial B_1(0) \end{cases} \tag{3.6}$$

has a positive radial solution if

$$q \in \left(1, \frac{N+2m+2}{N-2}\right) \ (N \ge 3, m > 0).$$

Problem (3.6) was introduced by Hénon [16] as a model to study spherically symmetric clusters of stars. Problems of this type have been extensively studied (see for instance [17, 18,21,25] and the references therein). In [8], the author showed that if the inequality

$$1 < \frac{NR^{m+q+1}\Gamma(q+2)\Gamma(N+M)}{(q+1)\Gamma(N+M+q+2)}$$

holds, then problem (3.5) has at least one classical positive radial solution. In fact, this inequality implies $q < 1$ in some special case. To see this clearly, letting $m = 0$, $R = 1$ and supposing on the contrary that $q = 1$, we observe that the above inequality is equivalent to simpler one $(N+2)(N+1) < 1$, which is impossible. So, to guarantee that problem (3.5) has at least one positive radial solution for any $\lambda \in (0, +\infty)$, the condition of $q < 1$ may be optimal.

4 Proof of Theorems 1.3, 1.4

In order to deal with the case that f has sublinear growth near 0, we need the following Whyburn type limit lemmata.

Lemma 4.1 (see [14]) *Let X be a normed vector space and let $\{C_n\}$ be a sequence of unbounded connected subsets in X. Assume that:*

(i) *There exists $z^* \in \liminf_{n\to+\infty} C_n$ with $\|z^*\| = +\infty$;*
(ii) *There exists a homeomorphism $T : X \to X$ such that $\|T(z^*)\| < +\infty$ and $\{T(C_n)\}$ be a sequence of unbounded connected subsets in X;*
(iii) *For every $R > 0$, $\left(\cup_{n=1}^{+\infty} T(C_n)\right) \cap \overline{B}_R$ is a relatively compact set of X.*

Then the set $D := \limsup_{n\to+\infty} C_n$ is unbounded, closed and connected.

Lemma 4.2 (see [14]) Let (X, ρ) be a metric space. If $\{C_i\}_{i\in\mathbb{N}}$ is a sequence of sets whose limit superior is L and there exists a homeomorphism $T : X \to X$ such that for every $R > 0$, $\left(\cup_{i=1}^{+\infty} T(C_i)\right) \cap B_R$ is a relatively compact set, then for each $\epsilon > 0$ there exists an m such that for every $n > m$, $C_n \subset V_\epsilon(L)$, where $V_\epsilon(L)$ denotes the set of all points p with $\rho(p, x) < \epsilon$ for any $x \in L$.

Proof of Theorem 1.3. For any $n \in \mathbb{N}$, define

$$f^n(r,s) = \begin{cases} \frac{1}{n}a(r)s, & s \in \left[0, \frac{1}{n}\right], \\ \left(f\left(r, \frac{2}{n}\right) - \frac{a(r)}{n^2}\right) ns + 2\frac{a(r)}{n^2} - f\left(r, \frac{2}{n}\right), & s \in \left(\frac{1}{n}, \frac{2}{n}\right), \\ f(r,s), & s \in \left[\frac{2}{n}, +\infty\right). \end{cases}$$

To proceed, we consider the following problem

$$\begin{cases} -\left(r^{N-1}\frac{u'}{\sqrt{1-u'^2}}\right)' = \lambda r^{N-1} f^n(r,u), \ \ r \in (0,R), \\ u'(0) = u(R) = 0. \end{cases} \tag{4.1}$$

It is easy to see that $\lim_{n\to+\infty} f_n(r,s) = f(x,s)$ and

$$\lim_{s\to0^+} \frac{f^n(r,s)}{s} = \frac{1}{n} \text{ uniformly in } r \in [0,R].$$

Then Theorem 1.1 implies that there exists a sequence of unbounded continua $\mathscr{C}_n$ of solution set of problem (4.1) emanating from $(\lambda_1 n, 0)$.

Taking $z^* = (+\infty, 0)$, we observe that $z^* \in \liminf_{n\to+\infty} \mathscr{C}_n$ with $\|z^*\|_{\mathbb{R}\times X} = +\infty$. Define a mapping $T : \mathbb{R}\times X \to \mathbb{R}\times X$ such that

$$T(\lambda, u) = \begin{cases} \left(\frac{1}{\lambda}, u\right) & \text{if } \lambda \in (-\infty, 0) \cup (0, +\infty), \\ (0, u) & \text{if } \lambda = \infty, \\ (\infty, u) & \text{if } \lambda = 0. \end{cases}$$

It is easy to verify that T is a homeomorphism and $\|T(z^*)\|_{\mathbb{R}\times X} = 0$. The compactness of K then implies that $\left(\cup_{n=1}^{+\infty} T(\mathscr{C}_n)\right) \cap B_R$ is pre-compact. Lemma 4.1 implies that $\mathscr{C} = \limsup_{n\to+\infty} \mathscr{C}_n$ is unbounded closed connected with $z^* \in \mathscr{C}$.

Next we show that the projection of $\mathscr{C}$ on $\mathbb{R}$ is nonempty. From the argument of Theorem 1.1, we have known that $\mathscr{C}_n$ has unbounded projection on $\mathbb{R}$ for any fixed $n \in \mathbb{N}$. By Lemma 4.2, for each fixed $\epsilon > 0$ there exists an m such that for every $n > m$, $\mathscr{C}_n \subset V_\epsilon(\mathscr{C})$. This implies that

$$(\lambda_1 n, +\infty) \subseteq \text{Proj}\left(\mathscr{C}_n\right) \subseteq \text{Proj}\left(V_\epsilon\left(\mathscr{C}\right)\right),$$

where Proj $(\mathscr{C})$ denotes the projection of $\mathscr{C}$ on $\mathbb{R}$. It follows that the projection of $\mathscr{C}$ is nonempty on $\mathbb{R}$.

By an argument similar to that of Theorem 1.2, we can show that u is a nonnegative solution of problem (1.1) for any $(\lambda, u) \in \mathscr{C}$.

Finally, we show that $\mathscr{C} \subseteq \left(\mathbb{R}^+ \times P\right)$. In view of Lemma 1 of [8], it is enough to show that $\mathscr{C} \cap ([0, +\infty) \times \{0\}) = \emptyset$. Suppose on the contrary that there exists a sequence $\{(\lambda_n, u_n)\} \subseteq \mathscr{C}$ such that $\lim_{n\to+\infty} \lambda_n = \mu$ and $\lim_{n\to+\infty} \|u_n\| = 0$ as $n \to +\infty$. From (2.1) we can easily get that

$$u_n = \int_0^R G(r, s)s^{N-2} \left[\lambda_n f(s, u_n(s)) \left(1 - u_n'^2\right)^{\frac{3}{2}} - (N-1)u_n'^3\right] ds.$$

Letting $v_n = u_n / \|u_n\|$, we have that

$$v_n = \int_0^R G(r, s)s^{N-2} \left[\lambda_n \frac{f(s, u_n(s))}{\|u_n\|} \left(1 - u_n'^2\right)^{\frac{3}{2}} - (N-1)\frac{u_n'}{\|u_n\|}u_n'^2\right] ds.$$

Similar to (2.5), we can show that

$$\lim_{n\to+\infty} \frac{f(r, u_n)}{\|u_n\|} = 0$$

uniformly in $r \in [0, R]$. By the compactness of K, we obtain that for some convenient subsequence $v_n \to v_0$ as $n \to +\infty$. Letting $n \to +\infty$, we obtain that $v_0 \equiv 0$. This contradicts the fact of $\|v_0\| = 1$. □

Proof of Theorem 1.4 Let $\lambda > 0$ and u be a positive solution of problem (1.2). Integrating the first equation of problem (1.2) with respect to t over $[0, r]$, it follows that

$$-r^{N-1}\frac{u'(r)}{\sqrt{1-u'^2}} = \lambda \int_0^r t^{N-1} f(t, u(t))\, dt.$$

Since u is strictly decreasing on $[0, R]$, we deduce that

$$\begin{aligned} -r^{N-1}u'(r) \le -r^{N-1}\frac{u'(r)}{\sqrt{1-u'^2}} &= \lambda \int_0^r t^{N-1} f(t, u)\, dt \\ &\le \lambda\rho \int_0^r t^{N-1}a(t)u(t)\, dt \le \lambda\rho u(0) \int_0^r a(t)t^{N-1}\, dt \le \frac{\lambda\rho a^0 u(0) r^N}{N}, \end{aligned}$$

where $a^0 := \max_{r\in[0,R]} a(r)$. Then integrating over $[0, R]$, we obtain that

$$u(0) \le \frac{\lambda\rho a^0 u(0) R^2}{2N}.$$

It follows that

$$\lambda \ge \frac{2N}{\rho a^0 R^2}.$$

This completes the proof. □

Example 4.1 Consider the following problem

$$\begin{cases} -\text{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = \lambda a(|x|)v^q & \text{in } B_R(0), \\ v = 0 & \text{on } \partial B_R(0). \end{cases} \tag{4.2}$$

If $N \geq 1$ is an integer, $R > 0$ and $q > 1$, then there exist $\lambda_* > 0$ and $\lambda^* > 0$ such that problem (4.2) has at least two positive radial solutions for all $\lambda \in (\lambda^*, +\infty)$, one positive radial solution for all $\lambda \in [\lambda_*, \lambda^*]$, zero positive radial solution for all $\lambda \in (0, \lambda_*)$.

In fact, if $N \geq 2$, the authors of [9] have shown that $\lambda_* = \lambda^* = \Lambda$ for problem (4.2). For the general case when $f_0 = 0$ and $N \geq 1$, we do not know whether this relation still holds. However, our results give a component of positive radial solutions and contains more nonlinearities.

Example 4.2 Consider the following problem

$$\begin{cases} -\text{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = \lambda a(|x|)\left(e^v - v\right) & \text{in } B_R(0), \\ v = 0 & \text{on } \partial B_R(0). \end{cases} \tag{4.3}$$

Theorem 1.3 shows that there exist $\lambda_* > 0$ and $\lambda^* > 0$ such that problem (4.3) has at least two positive radial solutions for all $\lambda \in (\lambda^*, +\infty)$, one positive radial solution for all $\lambda \in [\lambda_*, \lambda^*]$, zero positive radial solution for all $\lambda \in (0, \lambda_*)$. Clearly, this example cannot be contained by problem (4.2).

Example 4.3 For any $p \geq 1$, there exists $\xi^* > 0$ such that

$$\begin{cases} -\text{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = \lambda v^p & \text{in } B_R(0), \\ v = 0 & \text{on } \partial B_R(0) \end{cases} \tag{4.4}$$

has at least one positive solution for any $\lambda \in (\xi^*, +\infty)$.Moreover, there exists $\eta_* > 0$ such that problem (4.4) has no positive solution for any $\lambda \in (0, \eta_*)$. These are in contrast to the classical case (see [8]).

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