Calc. Var. (2016) 55:39
DOI 10.1007/s00526-016-0980-0

Calculus of Variations

A spectral condition for Liouville-type result of monostable KPP equation in periodic shear flows

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Received: 14 June 2015 / Accepted: 4 March 2016 / Published online: 28 March 2016

Abstract This paper is devoted to providing a simple condition, in term of spectral theory, that characterizes existence/nonexistence and uniqueness of positive bounded solution to

$$\nabla\cdot[n(y)\nabla u(x,y)]+\alpha(y)\partial_x u+\beta(y)\cdot\nabla_y u+f(x,y,u)=0 \quad (x,y)\in\mathbb{R}\times\mathbb{R}^{N-1}, \tag{0.1}$$

where f is of monostable KPP type nonlinearity and periodic in y. Our contribution answers a conjecture raised by Prof. H. Berestycki: which suitable assumption can impose at infinity that characterizes existence/nonexistence and uniqueness of (0.1) instead of the followings $\liminf_{|z|\to\infty}\partial_u f(z,0) > 0$ as in Berestycki et al. (Ann Mat Pura Appl 186(4):469–507, 2007) and $\limsup_{|z|\to\infty}\partial_u f(z,0) < 0$ as in Berestycki et al. (Bull Math Biol 71:399, 2008) and Berestycki and Rossi (Discret Contin Dyn Syst Ser B 21:41–67, 2008) but allow $\partial_u f(z,0)$ to change sign all the way as $|z|\to\infty$? Our result is simply based on maximum principle and complements to those in Berestycki et al. (Ann Mat Pura Appl 186:469–507, 2007; Bull Math Biol 71:399, 2008), Berestycki and Rossi (Discret Contin Dyn Syst Ser B 21:41–67, 2008) and Vo (J Differ Equ 259:4947–4988, 2015).

Mathematics Subject Classification 35C07 · 35J15 · 35B09 · 35P20 · 92D25

Contents

Communicated by P. Rabinowitz.

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1 Introduction and main results

In this article, we are concerned with an extension Liouville type result for positive bounded solution of semilinear elliptic equation

$$\nabla\cdot[n(y)\nabla u(x,y)]+\alpha(y)\partial_x u+\beta(y)\cdot\nabla_y u+f(x,y,u)=0\quad (x,y)\in\mathbb{R}\times\mathbb{R}^{N-1}, \tag{1.1}$$

where f is of monostable KPP type-nonlinearity. More precisely, we aim at looking for a simple criterion that characterizes existence/nonexistence and uniqueness of positive solution to Eq. (1.1) under a quite general condition of f. This type of equation has a well-known history. From the celebrated works of Kolmogorov–Petrovskii–Piskunov (KPP) [10] and later of Aronson and Weinberger [1], reaction–diffusion equation with KPP nonlinearity becomes a subject of intensive research in mathematical biology, ecology, genetics, medicine and especially in population dynamics. During the time, this model has been proved to be a good model to study the complexity of many natural phenomena, various aspects have been investigated and numerous interesting results were already obtained. More recently, KPP equations with a given forced speed was used to describe the dynamics of a population facing a climate change by Berestycki et al. [2,3,7,8] under the additional condition

$$\limsup_{|z|\to\infty}\partial_u f(z,0)\le -m<0,\quad z=(x,y). \tag{1.2}$$

In [2,3,7,8], assumption (1.2) means that the environment of species is completely unfavorable outside a compact set and it may be favorable inside. This kind of model is newly investigated in one dimensional space by Li et al. [12] under assumption that $\partial_u f(z,0)$ is positive near positive infinity and is negative near negative infinity. The Liouville type result for entirely semilinear elliptic equation

$$a_{ij}(z)\partial_{ij}u(z)+q(z)\cdot\nabla u(z)+f(z,u)=0,\quad z\in\mathbb{R}^N$$

was also studied by Berestycki et al. [4] with the condition

$$\liminf_{|z|\to\infty}(4\underline{\alpha}(z)\partial_u f(z,0)-|q(z)|^2)>0, \tag{1.3}$$

where $\underline{\alpha}(z)=\inf_{\substack{\xi\in\mathbb{R}^N\\|\xi|=1}}a_{ij}(z)\xi_i\xi_j\ge\alpha_*>0$. This condition yields in particular $\liminf_{|z|\to\infty}\partial_u f(z,0)\ge m_1>0$. Currently, the nonlocal dispersal KPP equation has been studied in periodic media by Liang and Shen [11] and in non periodic media by Berestycki, Coville and the author. However, all of the mentioned works require a constant sign of the initial per capita rate of growth near infinity. To the best of our knowledge, the Liouville property for periodic shear flows with $\partial_u f(z,0)$ changes sign up to infinity still remains open as a challenging problem. Before giving the main hypothesis, let us mention the basic assumption

Hypothesis 1

The function $f(x,y,s)\colon\mathbb{R}\times\mathbb{R}^{N-1}\times[0,+\infty)\mapsto\mathbb{R}$, is assumed to be continuous in x, measurable in y, and locally Lipschitz continuous in s. The map $s\mapsto f(z,s)$, $z=(x,y)$, is of class $C^1(0,s_0)$ for some positive constant s_0, uniformly in z and $f(z,0)=0$, $\forall z\in\mathbb{R}^N$.

The functions $n(y), \alpha(y) \colon \mathbb{R}^{N-1} \mapsto \mathbb{R}$, $\beta(y) \colon \mathbb{R}^{N-1} \mapsto \mathbb{R}^{N-1}$ and f are assumed to be periodic in y with the same period $\mathbb{T} = [0, L_1) \times [0, L_2) \times \ldots [0, L_{N-1})$ and $\inf_{\mathbb{T}} n(y) > 0$.

It is conjectured by Prof. H. Berestycki (personal communication) that the Liouville property, namely the existence/nonexistence and uniqueness of positive solution to (1.1), holds under the following assumption:

Hypothesis 2

There exists a periodic function $\mu \in L^\infty(\mathbb{R}^{N-1})$, $\mu \not\equiv 0$ such that

$$\mu(y) = \limsup_{|x|\to\infty} \partial_u f(x, y, 0) \quad \text{and} \quad \lambda_\mu = \lambda_p(-\nabla \cdot [n(y)\nabla] - \beta(y) \cdot \nabla_y - \mu(y), \mathbb{R}^{N-1}) > 0,$$

where $\lambda_p(-\nabla \cdot [n(y)\nabla] - \beta(y) \cdot \nabla_y - \mu(y), \mathbb{R}^{N-1})$ denotes the periodic principal eigenvalue of the following eigenvalue problem

$$\begin{cases} -\nabla \cdot [n(y)\nabla\phi(y)] - \beta(y) \cdot \nabla_y\phi(y) - \mu(y)\phi(y) = \lambda_p\phi(y) & y \in \mathbb{R}^{N-1} \\ \phi(y) > 0 & y \in \mathbb{R}^{N-1} \\ \phi \text{ is } \mathbb{T}\text{-periodic in } \mathbb{R}^{N-1}. \end{cases} \tag{1.4}$$

This paper gives the positive answer for his conjecture.

It is well-known that if n, β, μ are periodic and bounded, there exist a unique eigenvalue λ_p and unique (up to a scalar multiplication) eigenfunction to problem (1.4) (see e.g. [9]). Hypothesis 2 indeed has a realistic ecological interpretation. This means that the environment of the species under investigation is globally unfavorable at infinity. There may have favorable ($\partial_u f(z, 0) > 0$) and unfavorable ($\partial_u f(z, 0) < 0$) patches extending to infinity but only at infinity the unfavorable regions dominate. This situation usually happens when studying the large time behavior of the species under the effect of global warming and therefore it is useful to describe the dynamics of the species facing a climatic metamorphosis [3,7,8,13,14]. If $\mu(y) \equiv -m < 0$, one readily has $\lambda_p = m > 0$, $\phi \equiv 1$ and thus (1.2) is recovered. Indeed, it may be more natural to assume that

$$\mu_\pm(y) = \limsup_{|x|\to\infty} \partial_u f(x, y, 0) \quad \text{and} \quad \lambda_{\mu_\pm} = \lambda_p(-\nabla \cdot [n(y)\nabla] - \beta(y) \cdot \nabla_y - \mu_\pm(y), \mathbb{R}^{N-1}) > 0,$$

however, for sake of presentation, we only consider the case $\mu_+(y) = \mu_-(y)$. An illustration to demonstrate the novelty of this hypothesis is given in Sect. 2. Lastly, the two usual assumptions for KPP-type nonlinearity are also assumed:

Hypothesis 3

$$\exists S > 0 \text{ such that } f(z, s) \le 0 \text{ for } s \ge S, \quad \forall z \in \mathbb{R}^N.$$

Hypothesis 4

$$s \to f(z, s)/s \text{ is nonincreasing a.e in } \mathbb{R}^N \text{ and there exist } D \subset \mathbb{R}^N, |D| > 0$$
$$\text{such that it is strictly decreasing in } D.$$

These two conditions are classical in the context of population dynamics. The first condition means that there is a maximum carrying capacity effect : when the population density is very large, the death rate is higher than the birth rate and the population decreases. The second condition means the intrinsic growth rate decreases when the population density increases. This is due to the intraspecific competition for resources. A simplest typical example for this nonlinearity is

$$f(z, s) = a(z)s - b(z)s^2,$$

where $b(z) \geq 0$ and $a(z)$ satisfies Hypothesis 2. Let us now state the main result

Theorem 1.1 *Assume that f satisfies Hypothesis* 1–4. *Equation* (1.1) *admits a positive solution $u \in (0, S]$ if and only if $\lambda_1 = \lambda_1(-\mathcal{L}_f, \mathbb{R}^N) < 0$, where $\mathcal{L}_f[\phi] = \nabla \cdot [n(y)\nabla\phi] + \alpha(y)\partial_x\phi + \beta(y) \cdot \nabla_y\phi + \partial_u f(x, y, 0)\phi$ and*

$$\lambda_1(-\mathcal{L}_f, \mathbb{R}^N) = \sup\{\lambda \in \mathbb{R} | \exists\phi \in W^{2,N}_{loc}(\mathbb{R}^N), \phi(x, y) > 0 \text{ in } \mathbb{R}^N \text{ such that } (\mathcal{L}_f + \lambda)[\phi] \leq 0 \text{ in } \mathbb{R}^N\} \tag{1.5}$$

Moreover, when it exists, it is unique and $\mathbb{T}$-periodic in y.

The uniqueness of positive solution of Eq. (1.1) is an important issue in many aspects of applied mathematics. It is usually a hard problem and requires a special structure of the nonlinearity. The additional difficulty is due to the fact that we do not a priori require the solutions to be periodic in y and also we do not impose any boundary conditions as $x \to \pm\infty$. The uniqueness is actually proved to hold in the class of nonnegative bounded solutions and thus the unique solution must be $\mathbb{T}$-periodic in y. Indeed, condition (1.2) used in [3,7] is to derive the exponential decay of solution while condition (1.3) is to prove that positive solution of (0.1) must have positive infimum. These properties are crucial in proving the comparison principle, which leads to prove nonexistence and uniqueness. However, the investigation of the uniqueness for more general type of nonlinearities is still an interesting problem. The novelty of this work is that we propose, instead of (1.2) and (1.3), a spectral condition at infinity, Hypothesis 2, which allows $\partial_u f(x, y, 0)$ to change sign all the way $|x| \to \infty$ and only require a spectral condition depending on $\mu(y) = \limsup_{|x|\to\infty} \partial_u f(x, y, 0)$ at infinity. This considerably extends the results of [3,7,8] and carries new mathematical interpretation in the study of the effect of climate change (global warming). Hypothesis 2 plays the central role in our technique to obtain this result. Actually, it helps us to derive the exponential decay for solution (subsolution) of Eq. (1.1), which compensates the lack of compactness of $\mathbb{R}^N$ and therefore it may be useful in other investigation of the problem with non-compact domains. It is worth to mentioning that in [14], the author has considered the case, where $n(y) \equiv 1, \beta(y) \equiv 0, \alpha(y) \equiv constant$ and f depends periodically also in t. Our current result confirms that, in the environment being globally unfavorable at infinity in the sense of Hypothesis 2, the species survive if the unfavorable zone is dominated by the favorable zone, namely $\lambda_1 < 0$, otherwise it must be extinct.

2 An illustration

Before proving the main result, let us provide an illustration of how the main theorem apply and Hypothesis 2 is useful to describe the heterogeneity of habitat of a species. For the sake of simplicity, we only provide the case $\beta \equiv 0$. We consider the family $f_{\sigma,L}(x, y, s) = (\rho_L(x) + \mu_\sigma(y))s - s^2$ in $\mathbb{R} \times \mathbb{R}$, where $\sigma \in (0, 1)$, $L > 0$, periodic in y with the period $T = [0, 1]$ and

$$\rho_L(x) = \begin{cases} 2 & \text{on } [-L, L] \\ \theta + \frac{\sin(x)}{|x|} & \text{outside } [-L, L], \end{cases} \quad \mu_\sigma(y) = \begin{cases} 1 & \text{on } [0, \sigma) \\ -1 & \text{on } [\sigma, 1]. \end{cases}$$

These nonlinearities are discontinuous. However, $\partial_s f_{\sigma,L}(x, y, 0)$ is well defined a.e and all our results apply for zero order coefficient in L^∞ (see [3,7,8] for further discussion of this extension). We see that, for $\theta \in (-1, 1)$, $\partial_s f_{\sigma,L}(x, y, 0)$ is sign-changing as all the way $|x| \to \infty$ and if $\theta = 0$, $\lim_{|x|\to\infty} \partial_s f_{\sigma,L}(x, y, 0) = \mu_\sigma(y)$. Now, for every $\sigma \in (0, 1)$, let $(\lambda_\sigma, \phi_\sigma)$ be the (unique) eigenpair of the eigenvalue problem

$$\begin{cases} -\phi_\sigma'' - \mu_\sigma(y)\phi_\sigma = \lambda_\sigma \phi_\sigma & \text{in } \mathbb{R} \\ \phi_\sigma > 0 \text{ is } T\text{-periodic.} \end{cases}$$

Dividing the equation by ϕ_σ and integrating by part, we get

$$-\int_0^1 \frac{\phi_\sigma'^2}{\phi_\sigma^2} dy - \int_0^1 \mu_\sigma(y) dy = \lambda_\sigma.$$

Hence, $\lambda_\sigma \le -\int_0^1 \mu_\sigma(y)dy = -(2\sigma - 1)$. It is also known that λ_σ is decreasing with respect to σ and that $\sigma \mapsto \lambda_\sigma$ is continuous. Since $\lambda_0 = 1$ we see that there exists a unique $\overline{\sigma}$ such that $\lambda_{\overline{\sigma}} = 0$ and $\lambda_\sigma < 0$ if and only if $\sigma > \overline{\sigma}$.

For $\sigma > \overline{\sigma}$, $\lambda_\sigma < 0$, it is well-known that there exists a unique positive solution (see e.g. [5]) of

$$\begin{cases} -p_\sigma'' - \mu_\sigma(y)p_\sigma + p_\sigma^2 = 0 & \text{in } \mathbb{R} \\ p_\sigma \text{ is } T\text{-periodic.} \end{cases}$$

In this case, the environment is globally favorable at infinity and we conjecture that there is always persistence, namely as $t \to \infty$, $u(t, x, y) \to U(x, y)$, $\forall (x, y) \in \Omega$, where $U(x, y)$ is the unique positive stationary solution of

$$\begin{cases} u_t = \Delta u + \alpha(y)\partial_x u + f_{\sigma,L}(x, y, u) & \text{in } \mathbb{R}^2 \\ \text{u is } T\text{-periodic in} y. \end{cases}$$

If $\liminf_{|x|\to\infty} \partial_s f_{\sigma,L}(x, y, 0) > (\sup_{\mathbb{T}} \alpha)^2/4$ uniformly in y, this conjecture is true and we refer to [4] for its proof. However, this is not our current interest.

For $\sigma < \overline{\sigma}$, the environment is globally unfavorable at infinity. The problem is more subtle and our result applies in this case. Moreover, if $\theta \le -3$, the environment is completely unfavorable near infinity. For instance, we take $\theta = -3$ and $\alpha(y) = conts.$ not too large, says $\alpha(y) \equiv 1$, we claim that there exists a unique threshold value L^* such that the persistence holds if and only if $L > L^*$.

To prove this, let us denote $\mathcal{Q}_L[\phi] = \Delta\phi + \partial_x\phi + \partial_s f_{\sigma,L}(x, y, 0)\phi$, where $\partial_s f_{\sigma,L}(x, y, 0) = \rho_L(x) + \mu_\sigma(y)$ defined in $\mathbb{R}^2$. Since $\mathbb{R}^2$ is unbounded, we cannot define the eigenvalue of $\mathcal{Q}_L$ in the classical sense in $\mathbb{R}^2$. We make use of the definition (1.5). Let us call $\lambda_L = \lambda_1(-\mathcal{Q}_L, \mathbb{R}^2)$ and $\lambda_L' = \lambda_1(-\Delta - \partial_s f_{\sigma,L}(x, y, 0), \mathbb{R}^2)$. Using the Liouville transformation $\varphi(x, y) = e^{\frac{1}{2}x}\phi(x, y)$, one has

$$\lambda_L = \lambda_L' + \frac{1}{4}.$$

Since ρ_L is increasing with respect to L, λ_L is decreasing with respect to L. Moreover, the map $L \mapsto \lambda_L$ is continuous on $[0, \infty]$. Indeed, for any $L \in [0, \infty]$, let $\{L_n\} \in [0, \infty]$ be an arbitrary sequence converging to L, we see that $\|\partial_s f_{\sigma,L_n}(x, y, 0) - \partial_s f_{\sigma,L}(x, y, 0)\|_{L^\infty(\mathbb{R}^2)} \to 0$ as $n \to \infty$. Arguing as in the proof of Proposition 9.2 part (ii) [6], we get $\lim_{n\to\infty} \lambda_{L_n} = \lambda_L$. Moreover, since $\partial_s f_{\sigma,0}(x, y, 0) = -3 + \frac{\sin(x)}{|x|} + \mu_\sigma(y) \le -1$ and $\partial_s f_{\sigma,\infty}(x, y, 0) = 2 + \mu_\sigma(y) \ge 1$, by taking 1 as a test-function, we have $\lambda_0 = \lambda_0' + 1/4 \ge 5/4$ and $\lambda_\infty = \lambda_\infty' + 1/4 \le -3/4$. The claim is proved. As we will see in the next section, for $\sigma < \overline{\sigma}$, namely the environment is globally unfavorable at infinity, the equation

$$\begin{cases} \Delta q + \alpha(y)\partial_x q + f_{\sigma,L}(x,y,q) = 0 & \text{in } \mathbb{R}^2 \\ \text{q is } T\text{-periodic in} y \end{cases} \tag{2.1}$$

admits a unique positive solution if and only if $\lambda_L < 0$. From this result, we can also prove that $u(t,x,y)$ converges as $t \to \infty$ to the unique positive solution $q(x,y)$ of (2.1) if $\lambda_L < 0$ and $u(t,x,y)$ converges to zero if $\lambda_L \geq 0$. Even if the persistence is known, i.e $\lambda_L < 0$, the non-persistence and the uniqueness are still delicate questions.

3 Proof of the main result

Before proving the main result, let us recall the following result, which is proved in [6].

Definition 3.1 (*Maximum principle*) We say that the operator $Lu = a_{ij}(x)\partial_{ij}u + b_i(x)\partial_i u + c(x)u,\ i,j \in (1,2\ldots,N)$, satisfies the maximum principle in Ω if every function $u \in W^{2,N}_{loc}(\Omega)$ such that

$$Lu \geq 0 \text{ a.e in } \Omega, \quad \sup_{\Omega} u < \infty, \quad \forall \xi \in \partial\Omega, \limsup_{x\to\xi} u(x) \leq 0,$$

satisfies $u \leq 0$ in Ω.

Theorem 3.2 [6, Theorem 1.6 (i)] *The operator L satisfies maximum principle in Ω if $\lambda_1''(-L,\Omega) > 0$ and the coefficients of L satisfy*

$$\sup_{\Omega} c < \infty, \quad \limsup_{\substack{x\in\Omega \\ |x|\to\infty}} \frac{a_{ij}(x)}{|x|^2} < \infty, \quad \limsup_{\substack{x\in\Omega \\ |x|\to\infty}} \frac{b(x).x}{|x|^2} < \infty; \tag{3.1}$$

where $b(x) = (b_1(x),\ldots,b_N(x))$ and

$$\lambda_1^{''}(-L,\Omega) = \sup\{\lambda \in \mathbb{R} \colon \exists \phi \in W^{2,N}_{loc}(\Omega), \inf_{\Omega} \phi > 0, (L+\lambda)[\phi] \leq 0 \text{ a.e in } \Omega\}.$$

Lemma 3.3 *Let $u \in W^{2,N}_{loc}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ be a positive subsolution of (1.1) and Hypothesis 2 and 4 hold. There exist $\lambda, \overline{C}, R > 0$ and a periodic $\phi \in W^{2,N-1}_{loc}(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$ with $\inf_{\mathbb{R}^{N-1}} \phi > 0$ such that*

$$u(x,y) \leq \overline{C}e^{-\lambda|x|}\phi(y), \quad \text{as } |x| \geq R, y \in \mathbb{R}^{N-1}.$$

In particular, u decays exponentially as $|x| \to \infty$, uniformly in y.

Proof From Hypothesis 2, for any $\delta > 0$, there exists $R(\delta) > 0$ such that

$$\partial_u f(x,y,0) \leq \mu(y) + \delta \quad |x| \geq R, \quad y \in \mathbb{R}^{N-1}.$$

From Hypothesis 2, there exist a periodic eigenfunction $\phi \in L^\infty(\mathbb{R}^{N-1})$ and $C > 0$ such that

$$\begin{cases} -\nabla\cdot[n(y)\nabla\phi(y)] - \beta(y)\cdot\nabla_y\phi(y) - \mu(y)\phi(y) = \lambda_p\phi(y) & y \in \mathbb{R}^{N-1} \\ \phi(y) \geq C & y \in \mathbb{R}^{N-1} \\ \phi \text{ is } \mathbb{T}\text{-periodic in } \mathbb{R}^{N-1}. \end{cases}$$

From (1.1) and Hypothesis 4, one has

$$\mathcal{L}_\delta[u] = \nabla\cdot[n(y)\nabla u] + \alpha(y)\partial_x u + \beta(y)\cdot\nabla_y u + (\mu(y)+\delta)u \geq 0 \quad |x| \geq R, \quad y \in \mathbb{R}^{N-1}.$$

For any $p > 0$, we consider the function

$$\psi_p(x, y) = e^{(R+p)(\tau-\gamma)}e^{\gamma|x|}\phi(y) + e^{R(\tau+\gamma)}e^{-\gamma|x|}\phi(y)$$

where $\tau, \gamma > 0$ will be chosen later. Let $C_1 = e^{(R+p)(\tau-\gamma)}$ and $C_2 = e^{R(\tau+\gamma)}$, one gets by direct computations

$$\begin{aligned}\mathcal{L}_\delta[\psi_p] &= C_1\left(\gamma^2 n(y) + \gamma\frac{\alpha(y)x}{|x|} + \frac{\nabla\cdot[n(y)\nabla\phi] + \beta(y)\cdot\nabla_y\phi}{\phi} + \mu(y) + \delta\right)e^{\gamma|x|}\phi(y)\\ &\quad + C_2\left(\gamma^2 n(y) - \gamma\frac{\alpha(y)x}{|x|} + \frac{\nabla\cdot[n(y)\nabla\phi] + \beta(y)\cdot\nabla_y\phi}{\phi} + \mu(y) + \delta\right)e^{-\gamma|x|}\phi(y)\\ &\le \left(\overline{n}\gamma^2 + \overline{\alpha}\gamma - \lambda_\mu + \delta\right)\psi_p, \quad |x| \ge R,\ y \in \mathbb{R}^{N-1}\end{aligned}$$

where we use $\overline{n} = \sup_{\mathbb{T}} n(y)$, $\overline{\alpha} = \sup_{\mathbb{R}^{N-1}}|\alpha(y)|$ and $|x| \ge R > 1$ in the last inequality. Take $\delta < \lambda_\mu$ and γ in such the way

$$\gamma \in \left(0, \frac{-\overline{\alpha} + \sqrt{\overline{\alpha}^2 + 4\overline{n}(\lambda_\mu - \delta)}}{2\overline{n}}\right), \tag{3.2}$$

one sees that $\mathcal{L}_\delta[\psi_p] \le 0$, i.e ψ_p is a super solution of $\mathcal{L}_\delta$ in $\{|x| \ge R, y \in \mathbb{R}^{N-1}\}$. Fix γ satisfying (3.2) and $\tau = \gamma/2$, we show that $\psi_p \ge u$ in $B^x_{R+p}\backslash B^x_R \times \mathbb{R}^{N-1}$ for all $p > 0$, where we denote B^x_ρ the ball centered zero and radius ρ in space of x. In fact, since $\inf_{\mathbb{R}^{N-1}}\phi(y) \ge C > 0$ and u is bounded, one can choose and fix R large enough such that

$$\begin{cases}\psi_p(x, y) \ge Ce^{R\tau} \ge u(x, y) & |x| = R,\ y \in \mathbb{R}^{N-1}\\ \psi_p(x, y) \ge Ce^{(R+p)\tau} \ge u(x, y) & |x| = R + p,\ y \in \mathbb{R}^{N-1}.\end{cases}$$

Set $w_p = u - \psi_p$ and $z_p = w_p/\phi$, with γ chosen as in (3.2), one has $\mathcal{L}_\delta[w_p] \ge 0$, $w_p \le 0$ on $\partial(B^x_{R+p}\backslash B^x_R) \times \mathbb{R}^{N-1}$. Direct computations yield

$$\begin{aligned}0 \le \mathcal{L}_\delta[z_p\phi] &= n(y)\phi\Delta z_p + 2n(y)\nabla_y z_p\cdot\nabla_y\phi + \phi\nabla_y n(y)\cdot\nabla_y z + \alpha(y)\phi\partial_x z_p + \phi\beta(y)\cdot\nabla_y z_p\\ &\quad + z_p\nabla\cdot[n(y)\nabla\phi] + z_p\beta(y)\cdot\nabla\phi + z_p(\mu(y) + \delta)\phi\\ &= n(y)\phi\Delta z_p + 2n(y)\nabla_y z_p\cdot\nabla_y\phi + \phi\nabla_y n(y)\cdot\nabla_y z + \alpha(y)\phi\partial_x z_p + \phi\beta(y)\cdot\nabla_y z_p\\ &\quad + (-\lambda_\mu + \delta)\phi z_p.\end{aligned}$$

Let

$$\begin{aligned}\mathcal{L}'_\delta[z_p] = \frac{\mathcal{L}_\delta[z_p\phi]}{\phi} &= n(y)\Delta z_p + 2n(y)\nabla_y z_p\cdot\frac{\nabla_y\phi}{\phi} + \nabla_y n(y)\cdot\nabla_y z + \alpha(y)\partial_x z_p\\ &\quad + \beta(y)\cdot\nabla_y z_p + (-\lambda_\mu + \delta)z_p.\end{aligned}$$

One has $\mathcal{L}'_\delta[z_p] \ge 0$ and $z_p \le 0$ on $\partial(B^x_{R+p}\backslash B^x_R) \times \mathbb{R}^{N-1}$. We shall apply the maximum principle for general unbounded domain (Theorem 3.2) to imply that $z_p \le 0$ in $(B^x_{R+p}\backslash B^x_R) \times \mathbb{R}^{N-1}$, $\forall p > 0$. Let $\mathcal{O}_p = (B^x_{R+p}\backslash B^x_R) \times \mathbb{R}^{N-1}$. Indeed, $\inf_{\mathbb{T}} n(y) > 0$, $\mathcal{L}'_\delta$ is uniformly elliptic and the coefficients of $\mathcal{L}'_\delta$ obviously satisfy condition (3.1). To use Theorem 3.2, we consider

$$\lambda''_1(-\mathcal{L}'_\delta, \mathcal{O}_p) = \sup\left\{\lambda \in \mathbb{R}\colon \exists\phi \in W^{2,N}_{loc}(\mathcal{O}_p), \inf_{\mathcal{O}_p}\phi > 0, (\mathcal{L}'_\delta + \lambda)[\phi] \le 0 \text{ a.e in } \mathcal{O}_p\right\}.$$

Since $-\lambda_\mu + \delta < 0$, we take $\lambda = (\lambda_\mu - \delta)/2 > 0$ and $\phi = 1$ as a test function for λ. One has $(\mathcal{L}'_\delta + \lambda) = 1(\lambda_\mu - \delta)/2 < 0$ and therefore

$$\lambda_1''(-\mathcal{L}'_\delta, \mathbb{R}^N) \geq (\lambda_\mu - \delta)/2 > 0.$$

It follows immediately

$$u(x, y) \leq \psi_p(x, y) = e^{-(R+p)\gamma/2} e^{\gamma|x|}\phi(y) + e^{3R\gamma/2} e^{-\gamma|x|}\phi(y) \quad (x, y) \in \mathcal{O}_p.$$

Letting $p \to \infty$, we finally get

$$u(x, y) \leq e^{3R\gamma/2} e^{-\gamma|x|}\phi(y) \quad \mathbb{R}\backslash B_R^x \times \mathbb{R}^{N-1}.$$

□

Now, we are able to prove the main result.

Proof of Theorem 1.1 We first consider the case $\lambda_1 = \lambda_1(-\mathcal{L}_f, R^N) < 0$. Thanks to [4, Proposition 4.2], we have the limit $\lim_{R\to\infty} \lambda_R = \lambda_1 < 0$, where λ_R is the unique eigenvalue of the problem

$$\begin{cases} -\nabla \cdot [n(y)\varphi_R] - \alpha(y)\partial_x\varphi_R - \beta(y) \cdot \nabla_y\varphi_R - \partial_u f(z, 0)\varphi_R = \lambda_R\varphi_R & z \in B_R \\ \varphi_R(z) > 0 & z \in B_R \\ \varphi_R(z) = 0 & z \in \partial B_R \end{cases} \tag{3.3}$$

and B_R denotes the ball centered zero radius R in $\mathbb{R}^N$. Moreover there exists an eigenfunction $\varphi_\infty \in W^{2,N}_{loc}(\mathbb{R}^N)$ associated with λ_1.

Fix $R > 0$ large enough such that $\lambda_R < 0$, we define $\phi(x)$ as follow :

$$\phi(x) = \begin{cases} \varphi_R(x) & x \in B_R \\ 0 & \text{otherwise.} \end{cases}$$

Since $f(x, s)$ is of class $C^1[0, s_0]$ with respect to s, there exists $\varepsilon_0 > 0$ small enough such that for all $0 < \varepsilon \leq \varepsilon_0$, $(x, y) \in B_R$, we have

$$\begin{aligned} &\nabla \cdot [n(y)\nabla\varepsilon\phi] + \alpha(y)\partial_x(\varepsilon\phi) + \beta(y) \cdot \nabla_y\varepsilon\phi + f(x, y, \varepsilon\phi) \\ &= \varepsilon\phi\left[-\lambda_R + \frac{f(x, y, \varepsilon\phi)}{\varepsilon\phi} - \partial_u f(x, y, 0)\right] > 0. \end{aligned}$$

Hence, $\varepsilon\phi$ is a subsolution of Eq. (1.1). Since ϕ is compactly supported, we can choose ε small such that $\varepsilon \sup \phi \leq S$, where S is a super solution of Eq. (1.1) given by Hypothesis 3. Therefore, by the classical iteration method, there exists a nonnegative solution U satisfying $\varepsilon\phi \leq U \leq S$. Furthermore, thanks to the strong maximum principle, U is strictly positive .

Now let us prove the nonexistence and the uniqueness. They are actually the direct consequences of the following comparison principle.

Let $U, V \in W^{2,N}_{loc}(\mathbb{R}^N)$ respectively be nonnegative bounded super and subsolutions of (1.1). Suppose that for all $r > 0$, $\inf_{\Omega_r} U > 0$ and there exists $C(r) > 0$ such that

$$\forall y \in \mathbb{R}^{N-1}, \quad \|U\|_{W^{2,N}(B_r(0,y))} + \|V\|_{W^{2,N}(B_r(0,y))} \leq C(r), \tag{3.4}$$

where $\Omega_r = (-r, r) \times \mathbb{R}^{N-1}$. There holds

$$V(z) \leq U(z) \quad \forall z \in \mathbb{R}^N. \tag{3.5}$$

Assume for the moment that the comparison principle holds true. Assume by contradiction that (1.1) possesses a positive solution U when $\lambda_1 \geq 0$. Let φ_∞ be a generalized principal eigenfunction associated with λ_1. Without loss of generality, we may assume that $0 < \varphi_\infty(0) < U(0)$. We derive from Hypothesis 4 that

$$\begin{aligned} -\nabla\cdot[n(y)\nabla\varphi_\infty] - \alpha(y)\partial_x\varphi_\infty - \beta(y)\cdot\nabla_y\varphi_\infty &= (\partial_u f(x,y,0)+\lambda_1)\varphi_\infty \\ &\geq f(x,y,\varphi_\infty) \quad \text{in } \mathbb{R}^N. \end{aligned}$$

Thanks to Lemma 3.3, U decays exponentially as $|x| \to \infty$ uniformly in y, the above comparison principle implies that $U(z) \leq \varphi_\infty(z)$ for all $z \in \mathbb{R}^N$. Contradiction!

For the uniqueness, one needs to verify that if $u \not\equiv 0$ is a nonnegative solution of Eq. (1.1), one has $\inf_{\Omega_r} u > 0$. Indeed, the existence result implies that $\lambda_1 < 0$. Fix $r > 0$, φ_R as in (3.3) and ε_0 as above, we may assume, without loss of generality, that $\overline{B}_r^x \times \mathbb{T} \subset B_R$. Let $q \in \mathbb{Z}L_1 \times \mathbb{Z}L_2 \times \cdots \times \mathbb{Z}L_{N-1}$, we define

$$\varepsilon(q) = \inf_{(x,y)\in B_R} \frac{u(x,y+q)}{\varphi_R(x,y)}.$$

Thus, $\varepsilon(q)\varphi_R(x,y) \leq u(x,y+q)$ for $(x,y) \in B_R$ and since $\varphi_R = 0$ on ∂B_R, there exists (x_q, y_q) such that $\varepsilon(q)\varphi_R(x_q,y_q) = u(x_q, y_q+q)$. If there exists q such that $\varepsilon(q) \leq \varepsilon_0$, then $u(x, y+q)$ and $\varepsilon(q)\varphi_R(x,y)$ are respectively solution and subsolution of Eq. (1.1), they must coincide in B_R due to the strong maximum principle. This is impossible since $\varphi_R = 0$ on ∂B_R. Consequently,

$$\forall q \in \mathbb{Z}L_1 \times \mathbb{Z}L_2 \times \cdots \times \mathbb{Z}L_{N-1}, (x,y)\in B_R,\ \ u(x,y+q) \geq \varepsilon(q)\varphi_R(x,y) > \varepsilon_0\varphi_R(x,y).$$

This is done since φ_R has a positive infimum on $\overline{B}_r^x \times \mathbb{T} \subset B_R$. Hence, one can derive the uniqueness by applying directly the comparison principle.

It remains to show (3.5). Since n, β, μ are periodic, there exists a periodic eigenfunction $\phi \in L^\infty(\mathbb{R}^{N-1})$ associated with λ_μ and $C > 0$ such that

$$\begin{cases} -\nabla\cdot[n(y)\nabla\phi(y)] - \beta(y)\cdot\nabla_y\phi(y) - \mu(y)\phi(y) = \lambda_\mu\phi(y) & y \in \mathbb{R}^{N-1} \\ \phi(y) \geq C & y \in \mathbb{R}^{N-1} \\ \phi \text{ is } \mathbb{T}\text{-periodic in } \mathbb{R}^{N-1}. \end{cases}$$

Lemma 3.3 implies that there exists $\gamma, R > 0$ such that

$$V(x,y) \leq e^{3R\gamma/2}e^{-\gamma|x|}\phi(y) \quad |x| \geq R,\ y \in \mathbb{R}^{N-1}.$$

Thus, for any fixed $\varepsilon > 0$, there exists $R(\varepsilon) > 0$ such that

$$V(x,y) \leq e^{3R\gamma/2}e^{-\gamma|x|}\phi(y) \leq \varepsilon\phi(y) \quad |x| \geq R(\varepsilon),\ y \in \mathbb{R}^{N-1}. \tag{3.6}$$

Due to (3.4), $U, V \in C(\mathbb{R}^N) \cap L^\infty(\Omega_r)$ for all $r > 0$. From this and (3.6), we see that the set

$$K_\varepsilon := \{k > 0\colon kU \geq V - \varepsilon\phi \text{ in } \mathbb{R}^N\},$$

is nonempty. Let us call $k(\varepsilon) := \inf K_\varepsilon$. Obviously, the function $k(\varepsilon)\colon \mathbb{R}^+ \to \mathbb{R}$ is nonincreasing. Assume by a contradiction

$$k^* = \lim_{\varepsilon\to 0^+} k(\varepsilon) > 1.$$

Take $0 < \varepsilon < \sup_{\mathbb{R}^N}(V/\phi)$, we have $k(\varepsilon) > 0$, $W^\varepsilon = k(\varepsilon)U - V + \varepsilon\phi \geq 0$. By the definition of $k(\varepsilon)$, there exists a sequence $(x_n^\varepsilon, y_n^\varepsilon)$ in $\mathbb{R}^N$ such that

$$\left(k(\varepsilon) - \frac{1}{n}\right)U(x_n^\varepsilon, y_n^\varepsilon) < V(x_n^\varepsilon, y_n^\varepsilon) - \varepsilon\phi(y_n^\varepsilon). \tag{3.7}$$

Fix $\varepsilon > 0$, letting $n \to \infty$, one gets $\lim_{n\to\infty} W^\varepsilon(x_n^\varepsilon, y_n^\varepsilon) = 0$. For fixed ε, we deduce, from (3.6) and (3.7), that (x_n^ε) is bounded. Now, we use the periodicity of f in y to show that (y_n^ε) is also bounded. Let (z_n^ε) be the sequence in $\mathbb{Z}L_1 \times \mathbb{Z}L_2 \times \cdots \times \mathbb{Z}L_{N-1}$ such that $y_n^\varepsilon - z_n^\varepsilon$ belongs to the periodic cell $[0, L_1) \times [0, L_2) \times \ldots [0, L_{N-1})$. For all $n > 0$, we define the functions

$$U_n(x, y) = U(x, y + z_n^\varepsilon) \quad V_n(x, y) = V(x, y + z_n^\varepsilon).$$

Since f is periodic in y, the functions U_n, V_n satisfy the same differential inequalities as U, V. Thanks to (3.4), U_n and V_n converge, as $n \to \infty$, up to subsequences, respectively to U_∞ and V_∞ locally uniformly in $\mathbb{R}^N$ and they also satisfy the same inequalities as U, V. For fixed ε, $(x_n^\varepsilon, y_n^\varepsilon)$ is bounded, it converges up to subsequence as $n \to \infty$ to $(x(\varepsilon), y(\varepsilon))$ solving

$$W_\infty^\varepsilon(x(\varepsilon), y(\varepsilon)) = 0, \tag{3.8}$$

where $W_\infty^\varepsilon = k(\varepsilon)U_\infty - V_\infty + \varepsilon\phi \geq 0$.

The case that there exists x_0 such that $|x_0| = \liminf_{\varepsilon\to 0^+} |x(\varepsilon)| < \infty$ is ruled out. Indeed, thanks to the periodicity of f in y, arguing as above, we may assume that $y(\varepsilon)$ converges up to a subsequence to $y_0 \in [0, L_1) \times [0, L_2) \times \ldots [0, L_{N-1})$. From (3.8), $k^* < \infty$, the function $W_\infty = k^* U_\infty - V_\infty$ is nonnegative and vanishes at (x_0, y_0). Since f is Lipschitz continuous with respect to second variable and $k^* > 1$, we have

$$\begin{aligned} -\nabla \cdot [n(y)\nabla W_\infty] - \alpha(y)\partial_x W_\infty - \beta(y) \cdot \nabla_y W_\infty &\geq k^* f(z, U_\infty) - f(z, V_\infty) \\ &\geq f(z, k^* U_\infty) - f(z, V_\infty) \geq \xi(z) W_\infty, \end{aligned}$$

where $z = (x, y)$ and some function $\xi(z) \in L^\infty_{loc}(\mathbb{R}^N)$. The strong maximum principle implies that $W_\infty \equiv 0$ in $\mathbb{R}^N$. However, due to Hypothesis 4, this inequality holds strictly in $D \subset \mathbb{R}^N$, with $|D| > 0$. This is a contradiction. Now, we consider the case $\lim_{\varepsilon\to 0^+} |x(\varepsilon)| = \infty$. Recall that $W_\infty^\varepsilon = k(\varepsilon)U_\infty - V_\infty + \varepsilon\phi$ is nonnegative and vanishes at $(x(\varepsilon), y(\varepsilon))$. Thus there exists $r > 0$ such that $k(\varepsilon)U_\infty < V_\infty$ in $B_r(x(\varepsilon), y(\varepsilon))$. For ε small enough, $k(\varepsilon) > 1$, we derive from (3.5) for $B_r(x(\varepsilon), y(\varepsilon))$

$$\begin{aligned} \nabla \cdot [n(y)\nabla W_\infty^\varepsilon] + \alpha(y)\partial_x W_\infty^\varepsilon + \beta(y) \cdot \nabla_y W_\infty^\varepsilon &\leq f(x, V_\infty) - k(\varepsilon) f(x, U_\infty) - (\mu(y) + \lambda_\mu)\varepsilon\phi \\ &\leq f(x, V_\infty) - f(x, k(\varepsilon)U_\infty) - (\mu(y) + \lambda_\mu)\varepsilon\phi \\ &\leq -\frac{f(x, k(\varepsilon)U_\infty)}{k(\varepsilon)U_\infty}(k(\varepsilon)U_\infty - V_\infty + \varepsilon\phi) - \frac{\lambda_\mu}{2}\varepsilon\phi \\ &\quad - \left(\frac{\lambda_\mu}{2} + \mu(y) - \frac{f(x, y, k(\varepsilon)U_\infty)}{k(\varepsilon)U_\infty}\right)\varepsilon\phi. \end{aligned} \tag{3.9}$$

Take $0 < \varepsilon \ll 1$, then $|x(\varepsilon)| \gg 1$, we have

$$\frac{f(x, y, k(\varepsilon)U_\infty)}{k(\varepsilon)U_\infty} < \mu(y) + \frac{\lambda_\mu}{2}, \quad \forall (x, y) \in B_r(x(\varepsilon), y(\varepsilon)),$$

choosing r smaller if necessary. Since $\lambda_\mu > 0$, we get from (3.9)

$$\begin{aligned} &-\nabla \cdot [n(y)\nabla W_\infty^\varepsilon] - \alpha(y)\partial_x W_\infty^\varepsilon - \beta(y) \cdot \nabla_y W_\infty^\varepsilon - \varrho(x) W_\infty^\varepsilon \\ &> \frac{\lambda_\mu}{2}\varepsilon\phi > 0 \quad \text{in } B_r(x(\varepsilon), y(\varepsilon)), \end{aligned}$$

where $\varrho(x) = \frac{f(x, k(\varepsilon)U_\infty)}{k(\varepsilon)U_\infty}$ is bounded. This is a contradiction since the strong maximum principle implies $W_\infty^\varepsilon \equiv 0$ in $\mathbb{R}^N$. As a consequence

$$k^* = \lim_{\varepsilon\to 0^+} k(\varepsilon) \leq 1.$$

Letting $\varepsilon \to 0^+$, therefore

$$V \le \lim_{\varepsilon \to 0^+} (k(\varepsilon)U + \varepsilon\varphi) \le U \quad \text{in } \mathbb{R}^N.$$

This ends the proof. □

Acknowledgments The author is thankful to the anonymous referee for his/her constructive and helpful remark, which improves the paper.

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